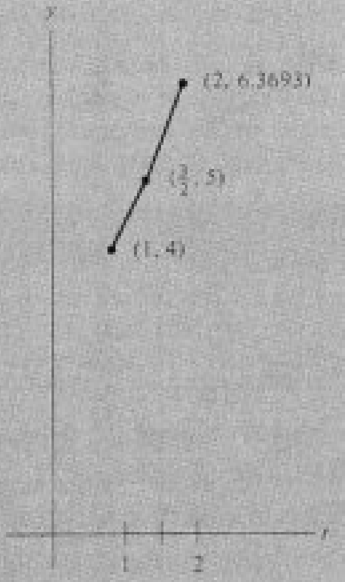
2. To find the polygonal path, plot the points (t_0, y_0), (t_1, y_1), and (t_2, y_2) and join them by line segments. See Fig. 5.

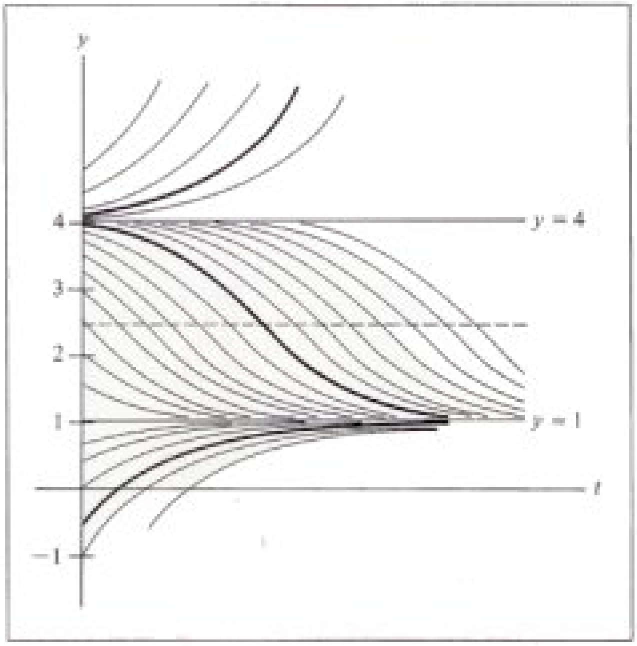


10.4 QUALITATIVE THEORY OF DIFFERENTIAL EQUATIONS

In this section we present a technique for sketching solutions to differential equations of the form $y' = g(y)$ *without having to solve the differential equation.* This technique is valuable for three reasons. First, there are many differential equations for which explicit solutions cannot be written down. Second, even when an explicit solution is available, we still face the problem of determining its behavior. For example, does the solution increase or decrease? If it increases, does it approach an asymptote or does it grow arbitrarily large? Third, and probably most significantly, in many applications the explicit formula for a solution is unnecessary; only a general knowledge of the behavior of the solution is needed. That is, a qualitative understanding of the solution is sufficient.

The theory introduced in this section is part of what is called the *qualitative theory of differential equations.* We shall limit our attention to differential equations of the form $y' = g(y)$. Such differential equations are called *autonomous.* The term "autonomous" here means "independent of time" and refers to the fact that the right-hand side of $y' = g(y)$ depends only on y and not on t. All the applications studied in the next section involve autonomous differential equations.

Throughout this section we consider the values of each solution $y = f(t)$ only for $t \geqq 0$. To introduce the qualitative theory, let us examine the graphs of the various typical solutions of the differential equation $y' = \frac{1}{2}(1 - y)(4 - y)$. The solution curves in Fig. 1 illustrate the following properties.



Property I Corresponding to each zero of $g(y)$ there is a constant solution of the differential equation. Specifically, if $g(c) = 0$, the constant function $y = c$ is a solution. (The constant solutions in Fig. 1 are $y = 1$ and $y = 4$.)

Property II The constant solutions divide the ty-plane into horizontal strips. Each nonconstant solution lies completely in one strip.

Property III Each nonconstant solution is either strictly increasing or decreasing.

Property IV Each nonconstant solution either is asymptotic to a constant solution or else increases or decreases without bound.

It can be shown that Properties I–IV are valid for the solutions of any autonomous differential equation $y' = g(y)$ provided that $g(y)$ is a "sufficiently well-behaved" function. We shall assume these properties in this chapter.

Using Properties I–IV, we can sketch the general shape of any solution curve by looking at the graph of the function $g(y)$ and the behavior of that graph near $y(0)$. The procedure for doing this is illustrated in the following example.

.MPLE 1 Sketch the solution to $y' = e^{-y} - 1$ that satisfies $y(0) = -2$.

Solution Here $g(y) = e^{-y} - 1$. On a yz-coordinate system we draw the graph of the function $z = g(y) = e^{-y} - 1$ [Fig. 2(a)]. The function $g(y) = e^{-y} - 1$ has a zero when $y = 0$. Therefore, the differential equation $y' = e^{-y} - 1$ has the constant solution $y = 0$. We indicate this constant solution on a ty-coordinate system in Fig. 2(b). To begin the sketch of the solution satisfying $y(0) = -2$, we locate this initial value of y on the (horizontal) y-axis in Fig. 2(a) and on the (vertical) y-axis in Fig. 2(b).

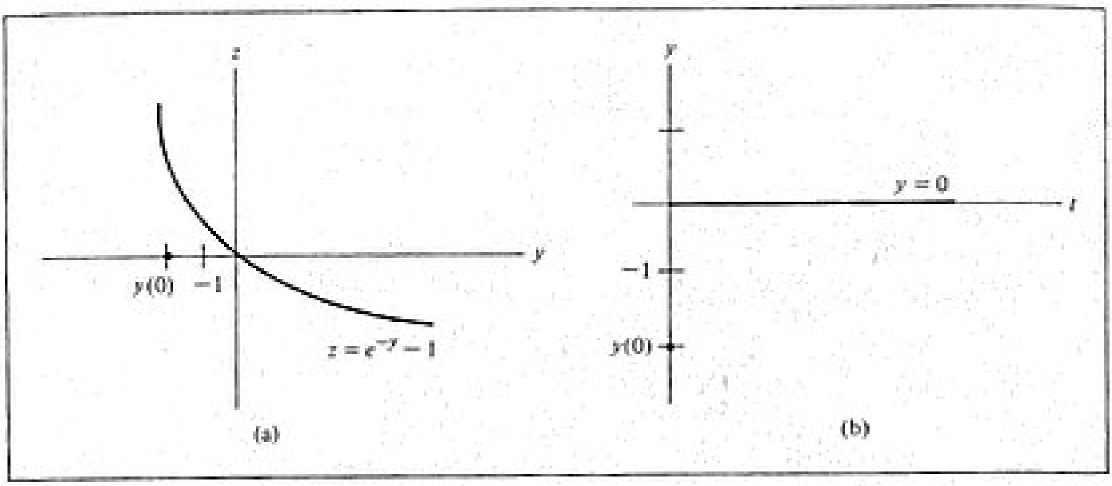
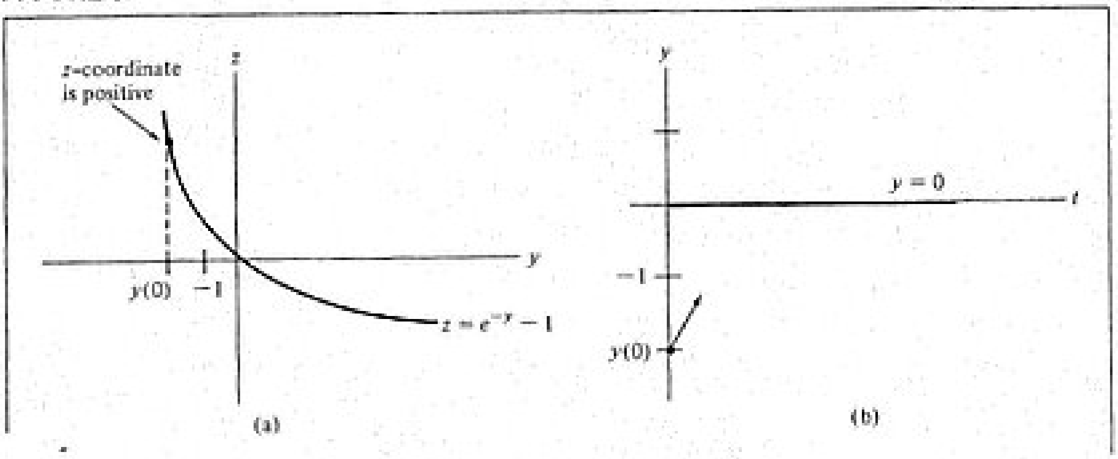


FIGURE 2

To determine whether the solution increases or decreases when it leaves the initial point $y(0)$ on the ty graph, we look at the yz graph and note that $z = g(y)$ is positive at $y = -2$ [Fig. 3(a)]. Consequently, since $y' = g(y)$, the derivative of the solution is positive, which implies that the solution is increasing. We indicate this by an arrow at the initial point in Fig. 3(b). Moreover, the solution

FIGURE 3



y will increase asymptotically to the constant solution $y = 0$, by Properties III and IV of autonomous differential equations.

Next, we place an arrow in Fig. 4(a) to remind us that y will move from $y = -2$ toward $y = 0$. As y moves to the right toward $y = 0$ in Fig. 4(a), the z-coordinate of points on the graph of $g(y)$ becomes less positive; that is, $g(y)$ becomes less positive. Consequently, since $y' = g(y)$, the slope of the solution curve becomes less positive. Thus the solution curve is concave down, as we have shown in Fig. 4(b). ●

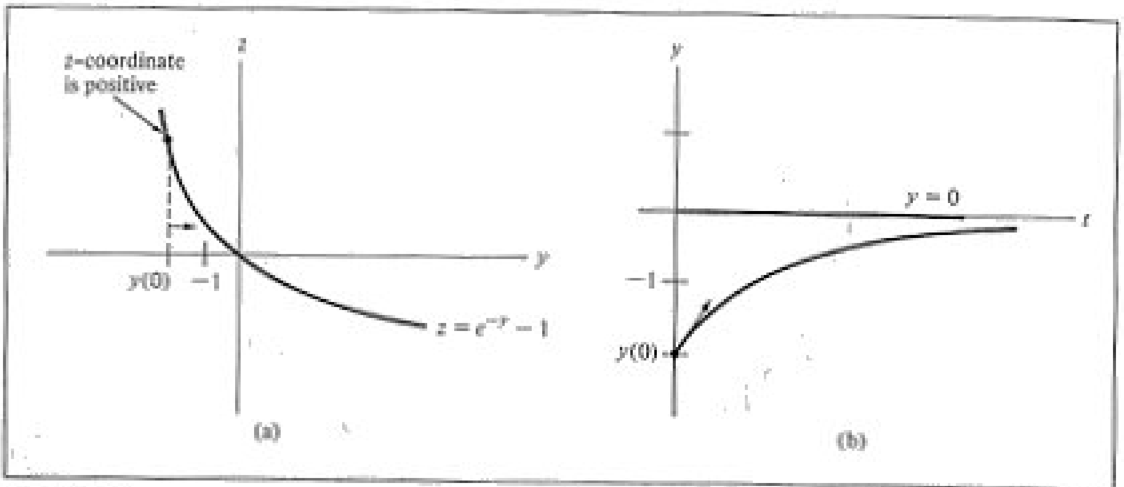


FIGURE 4

An important point to remember when sketching solutions is that z-coordinates on the yz graph are values of $g(y)$, and since $y' = g(y)$, a z-coordinate gives the *slope* of the solution curve at the corresponding point on the ty graph.

EXAMPLE 2 Sketch the graphs of the solutions to $y' = y + 2$ satisfying
(a) $y(0) = 1$,
(b) $y(0) = -3$.

Solution Here $g(y) = y + 2$. The graph of $z = g(y)$ is a straight line of slope 1 and z-intercept 2. [See Fig. 5(a).] This line crosses the y-axis only where $y = -2$. Thus the differential equation $y' = y + 2$ has one constant solution, $y = -2$. [See Fig. 5(b).]

(a) We locate the initial value $y(0) = 1$ on the y-axes of both graphs in Fig. 5. The corresponding z-coordinate on the yz graph is positive; therefore, the solution on the ty graph has positive slope and is increasing as it leaves the initial point. We indicate this by an arrow in Fig. 5(b). Now, Property IV of autonomous differential equations implies that y will increase without bound from its initial value. As we let y increase from 1 in Fig. 6(a), we see that the z-coordinates [i.e., values of $g(y)$] increase. Consequently, y' is

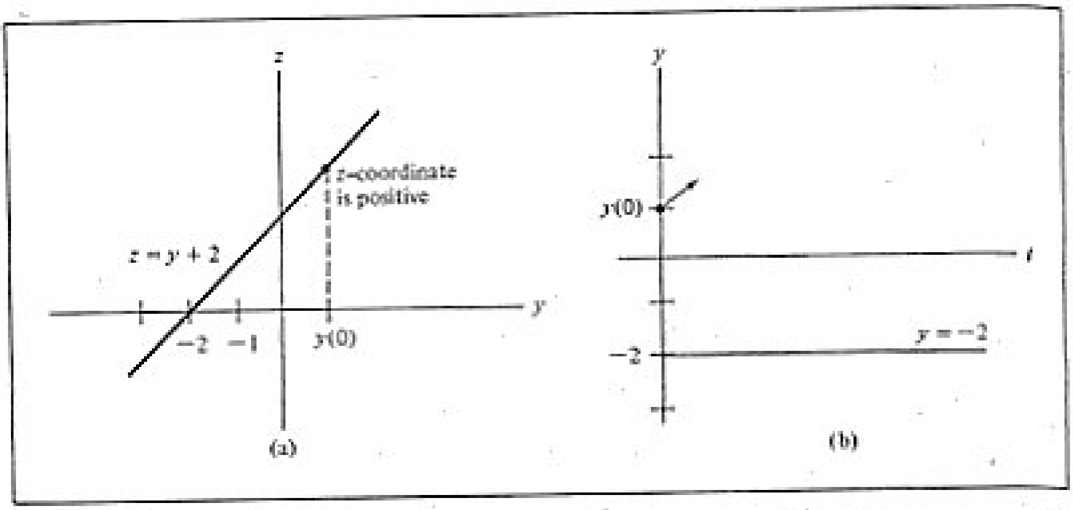


FIGURE 5

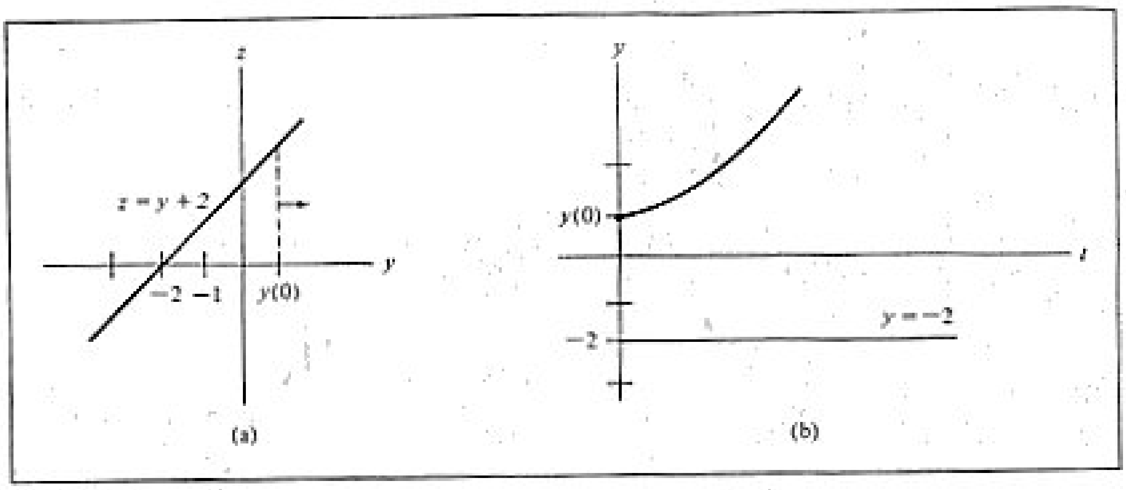
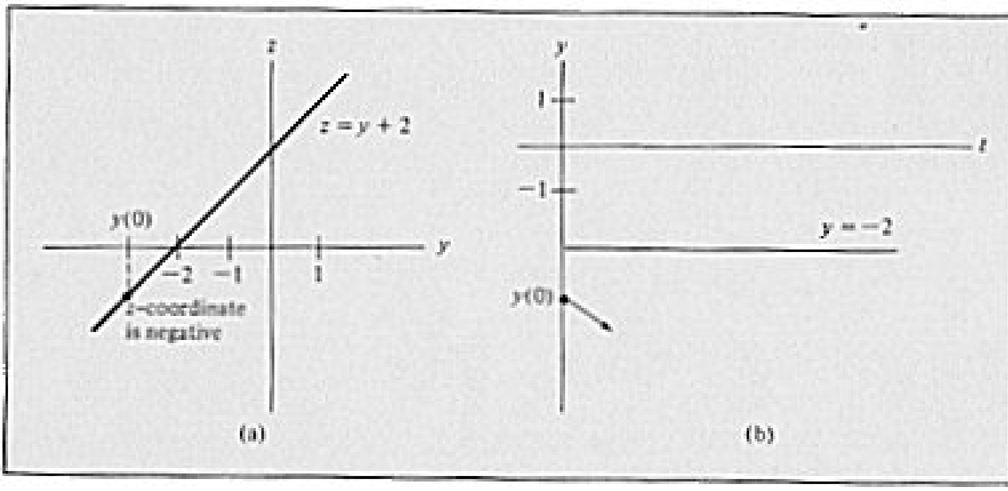
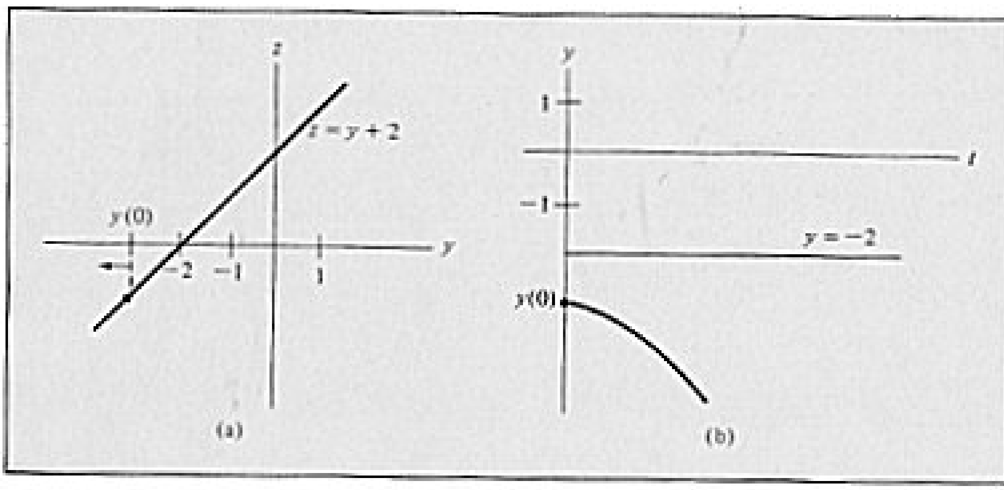


FIGURE 6

increasing, so the graph of the solution must be concave up. We have sketched the solution in Fig. 6(b).

(b) Next we graph the solution for which $y(0) = -3$. From the graph of $z = y + 2$, we see that z is negative when $y = -3$. This implies that the solution is decreasing as it leaves the initial point. (See Fig. 7.) It follows that the values of y will continue to decrease without bound and become more and more negative. This means that on the yz graph y must move to the *left* [Fig. 8(a)]. We now examine what happens to $g(y)$ as y moves to the left. (This is the opposite of the ordinary way to read a graph.) The z-coordinate becomes more negative; hence the slopes on the solution curve will become more negative. Thus the solution curve must be concave down, as in Fig. 8(b).





From the preceding examples we can state a few rules for sketching a solution to $y' = g(y)$ with $y(0)$ given:

1. Sketch the graph of $z = g(y)$ on a yz-coordinate system. Find and label the zeros of $g(y)$.
2. For each zero c of $g(y)$ draw the constant solution $y = c$ on the ty-coordinate system.
3. Plot $y(0)$ on the y-axes of the two coordinate systems.
4. Determine whether the value of $g(y)$ is positive or negative when $y = y(0)$. This tells us whether the solution is increasing or decreasing. On the ty graph, indicate the direction of the solution through $y(0)$.
5. On the yz graph, indicate which direction y should move. (*Note:* If y is moving *down* on the ty graph, y moves to the *left* on the yz graph.) As y moves in the proper direction on the yz graph, determine whether $g(y)$ becomes more positive, less positive, more negative, or less negative. This tells us the concavity of the solution.

6. Beginning at $y(0)$ on the ty graph, sketch the solution, being guided by the principle that the solution will grow (positively or negatively) without bound unless it encounters a constant solution. In this case, it will approach the constant solution asymptotically.

.MPLE 3 Sketch the solutions to $y' = y^2 - 4y$ satisfying $y(0) = 4.5$ and $y(0) = 3$.

Solution Refer to Fig. 9. Since $g(y) = y^2 - 4y = y(y - 4)$, the zeros of $g(y)$ are 0 and 4; hence the constant solutions are $y = 0$ and $y = 4$. The solution satisfying $y(0) = 4.5$ is increasing, because the z-coordinate is positive when $y = 4.5$ on the yz graph. This solution continues to increase without bound. The solution satisfying $y(0) = 3$ is decreasing because the z-coordinate is negative when $y = 3$ on the yz graph. This solution will decrease and approach asymptotically the constant solution $y = 0$.

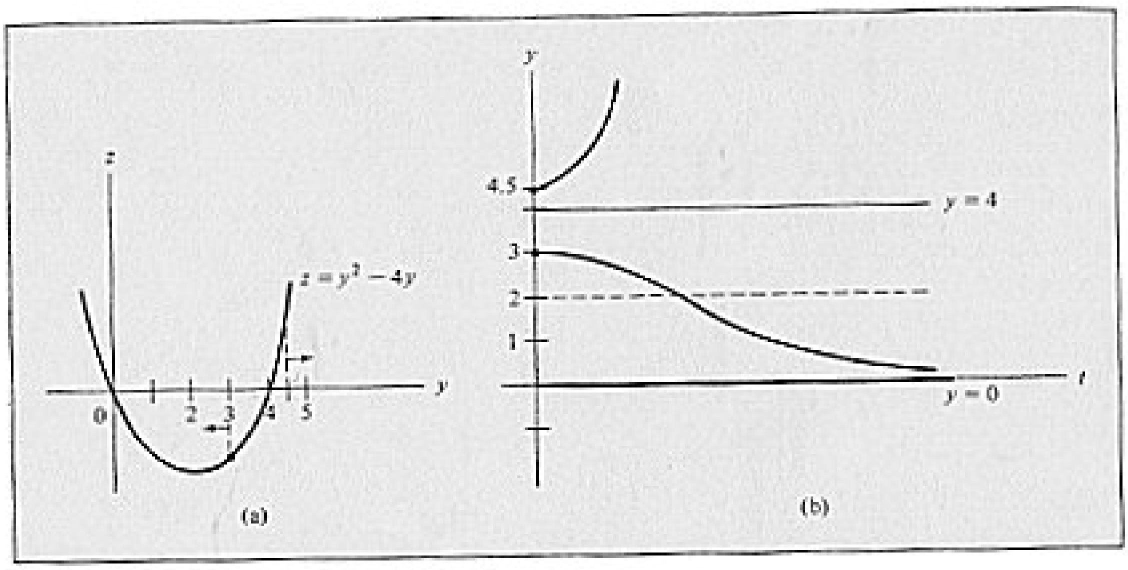


FIGURE 9

An additional piece of information about the solution satisfying $y(0) = 3$ may be obtained from the graph of $z = g(y)$. We know that y decreases from 3 and approaches 0. From the graph of $z = g(y)$ in Fig. 9 it appears that at first the z-coordinates become more negative until y reaches 2 and then become less negative as y moves on toward 0. Since these z-coordinates are slopes on the solution curve, we conclude that as the solution moves downward from its initial point on the ty-coordinate system, its slope becomes more negative until the y-coordinate is 2 and then the slope becomes less negative as the y-coordinate approaches 0. Hence the solution is concave down until $y = 2$ and then is concave up. Thus there is an inflection point at $y = 2$, where the concavity changes. ●

We saw in Example 3 that the inflection point at $y = 2$ was produced by the fact that $g(y)$ had a minimum at $y = 2$. A generalization (see below) of the argument in Example 3 shows that inflection points of solution curves occur at each value of y where $g(y)$ has a nonzero relative maximum or minimum point. Thus we may formulate an additional rule for sketching a solution of $y' = g(y)$.

7. On the ty-coordinate system draw dashed horizontal lines at all values of y at which $g(y)$ has a *nonzero* relative maximum or minimum point. A solution curve will have an inflection point whenever it crosses such a dashed line.

It is useful to note that when $g(y)$ is a quadratic function, as in Example 3, its maximum or minimum point occurs at a value of y halfway between the zeros of $g(y)$. This is because the graph of a quadratic function is a parabola, which is symmetric about a vertical line through its vertex.

EXAMPLE 4 Sketch a solution to $y' = e^{-y}$ with $y(0) > 0$.

Solution Refer to Fig. 10. Since $g(y) = e^{-y}$ is always positive, there are no constant solutions to the differential equation and every solution will increase without bound. When drawing solutions that asymptotically approach a horizontal line, we have no choice as to whether to draw it concave up or concave down. This decision will be obvious from its increasing or decreasing nature and from knowledge of inflection points. However, for solutions that grow without bound, we must look at $g(y)$ in order to determine concavity. In this example, as t increases, the values of y increase. As y increases, $g(y)$ becomes less positive. Since $g(y) = y'$, we deduce that the slope of the solution curve becomes less positive; therefore, the solution curve is concave down. ●

FIGURE 10