Math 361. O.D.E

I Classification.

① First Order O.D.E

$$\boxed{y' = f(x,y)}$$

ⓐ Linear First Order O.D.E

$$\boxed{a(x)\,y' + b(x)\,y = c(x)}$$

$$y' = \frac{c(x) - b(x)\,y}{a(x)}$$

Examples:

Linear $\begin{cases} 3y' + 4y = e^x \\ x^2 y' + (\sin x)\,y = \cos x \end{cases}$

Non Linear $\begin{cases} y' + 4y^2 = e^x \\ 3y\,y' + (\cos x)\,y = \tan x \end{cases}$

b) Separable First Order ODE

$$\boxed{y' = P(x)\,Q(y)}$$

Examples $\begin{cases} y' = y e^x \\ y' = e^{x+y} = e^x e^y \end{cases}$

NO! $y' = \sin(x+y)$

② Second Order O.D.E

$y'' = f(y', y, x)$

$f(y'', y', y, x) = 0$

a) 2nd Order O.D.E w/ const. coeff

$a y'' + b y' + c y = d$

$\boxed{\begin{array}{l} a y'' + b y' + c y = f(x) \\ a y'' + b y' + c y = 0 \end{array}}$ Homg

Homogeneous

Inhomogeneous

b) 2nd Order linear O.D.E.

$$\boxed{a(x)\,y'' + b(x)\,y' + c(x)\,y = f(x)}$$

③ Higher Order Linear O.D.E.

$a_n(x)\,y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + \dots a_0(x)\,y = 0$

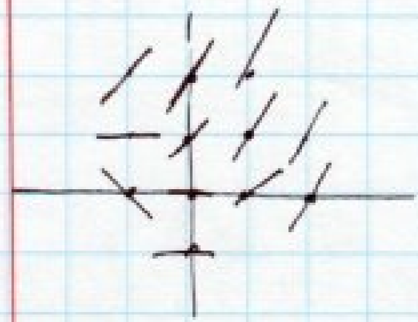
$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = 0$

Direction Fields

Example

$y' = x + y$

x	y	y'
0	0	0
0	1	1
0	2	2
1	0	1
1	1	2
1	2	3



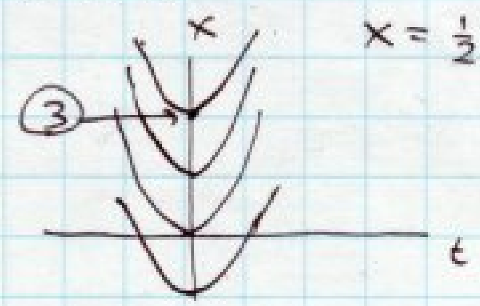
Direction Flow

Integral curves

Example 2

$x = x(t)$ $\quad \frac{dx}{dt} = t$

$x' = t$



$x = \frac{1}{2}t^2 + C$

Initial Conditions

Ex $x' = t \quad x(0) = 3$ (I.C)

$x = \frac{1}{2}t^2 + c$

$3 = 0 + c$

$c = 3$

$x = \frac{1}{2}t^2 + 3$

—o—

Ex $y = f(x)$

$y' = y$

$\cancel{y = e^x + c}$ $\quad y = ce^x$

$\cancel{y' = e^x \neq y}$ $\quad y' = ce^x$ ✓

Linear First Order O.D.E

$$a(x)\,y' + b(x)\,y = c(x) \qquad a(x) \neq 0$$

$$y' + \frac{b(x)}{a(x)}\,y = \frac{c(x)}{a(x)}$$

$$\boxed{y' + P(x)\,y = q(x)}$$

Introduce an integrating factor $\mu(x)$

Multiply by $\mu(x)$

$$\underbrace{\mu y' + \mu P y}_{} = \mu q$$

Compare with

$$(\mu y)' = \mu y' + \mu' y$$

Set $\mu' = \mu P$

$$\frac{d\mu}{dx} = \mu P$$

$$\frac{1}{\mu}\frac{d\mu}{dx} = P$$

$$\int \frac{1}{\mu}\,d\mu = \int P\,dx$$

$$\ln\mu = \int P\,dx$$

$$\boxed{\mu = e^{\int P dx}} \leftarrow$$

$$(\mu y)' = \mu q$$

$$\mu y = \int \mu q$$

$$\boxed{y = \frac{1}{\mu}\int \mu q}$$

Example (2.1 #2)

$$x^2 y' + 3xy = \frac{\sin x}{x}$$

$$\rightarrow y' + \frac{3}{x}\,y = \frac{\sin x}{x^3}$$

$$P = \frac{3}{x} \qquad \int P(x)\,dx = \int P = \int \frac{3}{x}\,dx$$

$$\int P = 3\ln x$$

$$\mu = e^{\int P} = e^{3\ln x} = e^{\ln x^3} = x^3$$

$$x^3 y' + 3x^2 y = \sin x$$

$$[x^3 y]' = \sin x$$

$$x^3 y = \int \sin x\,dx$$

$$x^3 y = -\cos x + c$$

$$y = \frac{c - \cos x}{x^3}$$

—o—

④

$$xy' + 2y = e^x$$

a) $y' + \frac{2}{x}\,y = \frac{e^x}{x}$

$$P = \frac{2}{x} \qquad \int P = 2\ln x$$

$$\mu = e^{2\ln x} = x^2$$

$$x^2 y' + 2xy = x e^x$$

$$[x^2 y]' = x e^x$$

$$x^2 y = \int x e^x\,dx = x e^x - e^x + c$$

$$y = \frac{1}{x}e^x - \frac{1}{x^2}e^x + \frac{c}{x^2}$$

$$\begin{array}{rl|l} + & x & e^x \\ - & 1 & e^x \\ + & 0 & e^x \end{array}$$

Do:

8) $xy' + 2y = \sin x$

*) $x^2 Y' + 3xY = \frac{\sin x}{x}$

$\cancel{x'} =$

$$Y' + \frac{3}{x}\,Y = \frac{\sin x}{x^3}$$

$$P = 3/x \qquad \int P = 3\ln x$$

$$\mu = e^{\int P} = e^{3\ln x} = x^3$$

$$x^3 Y' + 3x^2 Y = \sin x$$

$$(x^3 Y)' = \sin x$$

$$x^3 y = -\cos x + c$$

$$y = \frac{-\cos x + c}{x^3}.$$

Sect 2.2

Given $y' + P(x)\,y = g(x)$

$P(x_0, y_0)$

Theorem: If $P(x)$ and $g(x)$ are continuous functions on $[\alpha < x_0 < \beta]$. Then

$$y' + P(x)\,y = g(x)$$
$$y(x_0) = y_0$$

has a unique solution on (α, β)

First Order Linear O.D.E

$$a(x)\, y' + b(x)\, y = c(x)$$

$$y' + p(x)\, y = q(x)$$

Let $\mu(x) = e^{\int p}$

Multiply by μ

$$[\mu y]' = \mu q$$

$$\mu y = \int \mu q$$

$$\boxed{y = \frac{1}{\mu}\int \mu q}$$

Pb. 2.1 #11

$$y' + y = 5\sin 2x$$

$$p = 1$$

$$\mu = e^{\int 1\,dx} = e^{x}$$

$$e^{x} y' + e^{x} y = 5e^{x}\sin 2x$$

$$[e^{x} y]' = 5e^{x}\sin 2x$$

$$[e^{x} y] = 5\int e^{x}\sin 2x\, dx$$

$$e^{x} y = 5\left[-\tfrac{2}{5}e^{x}\cos 2x + \tfrac{1}{5}e^{x}\sin 2x\right] + C$$

$$y = -2\cos 2x + \sin 2x + Ce^{-x}$$

Pb. 2.1 #5

$$y' - 2y = 3e^{x}$$

$$p = -2$$

$$\mu = e^{\int p} = e^{-2x}$$

$$e^{-2x} y' - 2e^{-2x} y = 3e^{-x}$$

$$[e^{-2x} y]' = 3e^{-x}$$

$$e^{-2x} y = -3e^{-x} + C$$

$$y = -3e^{2x}\cdot e^{-x} + Ce^{2x}$$

$$y = -3e^{x} + Ce^{2x}$$

Pb 2.1 #1

$$y' + 3y = t + e^{-2t}$$

$$p = 3$$

$$\mu = e^{\int p} = e^{3t}$$

$$e^{3t} y' + 3e^{3t} y = te^{3t} + e^{t}$$

$$[e^{3t} y]' = \int (te^{3t} + e^{t})\, dt$$

$$e^{3t} y = \tfrac{1}{3}te^{3t} - \tfrac{1}{9}e^{3t} + e^{t} + C$$

+	t	e^{3t}
−	1	$\frac{1}{3}e^{3t}$
+	0	$\frac{1}{9}e^{3t}$

$$y = \tfrac{1}{3}t - \tfrac{1}{9} + e^{-2t} + Ce^{-3t}$$

Separable First Order O.D.E

General 1^{st}-order O.DE

$$y' = f(x, y)$$

Separable if

$$y' = p(x)\, q(y)$$

$$\frac{dy}{dx} = p(x)\, q(y)$$

$$\boxed{\int \frac{1}{q(y)}\, dy = \int p(x)\, dx}$$

Examples: 2.2 # 13

$$y' = \frac{2x}{y + x^2 y} = \frac{1}{y} \cdot \frac{2x}{1+x^2}$$

$$\int y\, dy = \int \frac{2x}{1+x^2}\, dx$$

$$\frac{1}{2} y^2 = \ln(1+x^2) + C$$

$$y^2 = 2\ln(1+x^2) + C$$

2.2 # 1, 3, 5, 7, 9, 17, 19, 23

19) $\sin 2x\, dx + \cos 3y\, dy = 0$

I.C $\quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{3}$

$$\int \cos 3y\, dy = -\int \sin 2x\, dx$$

$$\frac{1}{3} \sin 3y = \frac{1}{2} \cos 2x + C$$

$$\frac{1}{3} \sin \pi = \frac{1}{2} \cos \pi + C \qquad C = \frac{1}{2}$$

$$\frac{1}{3} \sin 3y = \frac{1}{2} \cos 2x + \frac{1}{2}$$

$$2 \sin 3y - 3\cos 2x = 3 \qquad \text{Ans:}$$

$$f(x,y) = C$$

23) $y' = 2y^2 + xy^2 \qquad y(0) = 1$

$$\frac{dy}{dx} = y^2(2+x)$$

$$\int \frac{1}{y^2}\, dy = \int (2+x)\, dx$$

$$-\frac{1}{y} = 2x + \frac{x^2}{2} + C$$

$$-1 = C$$

$$-\frac{1}{y} = 2x + \frac{x^2}{2} - 1$$

$$-1 = y\left(2x + \frac{x^2}{2} - 1\right)$$

$$y = \frac{1}{1 - 2x - \frac{x^2}{2}} = \frac{2}{2 - 4x - x^2}$$

5) $y' = \cos^2 x \cos^2 2y$

$$\frac{dy}{dx} = \cos^2 x \cos^2 2y$$

$$\int \frac{1}{\cos^2 2y}\, dy = \int \cos^2 x\, dx$$

$$\int \sec^2 2y\, dy = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx$$

$$\frac{1}{2} \tan 2y = \frac{x}{2} + \frac{1}{4} \sin 2x + C$$

$$2 \tan 2y = 2x + \sin 2x + C$$

$$2x + \sin 2x - 2\tan 2y = C$$

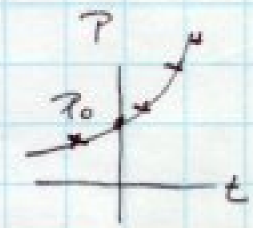
Modeling:

1) Population Model

P = population

Ass: Rate of change of the pop. is proportional to the population

$$\frac{dP}{dt} = kP$$



$$\int \frac{1}{P}\, dP = \int k\, dt$$

$$\ln P = kt + C$$

$$P = e^{kt + C}$$

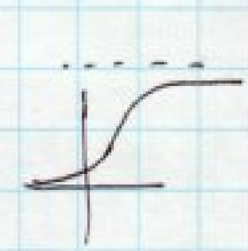
$$= e^C e^{kt} \qquad t = 0$$

$$P = P_0 e^{kt} \qquad P = P_0$$

Malthus.

→ Logistic Model

$$\frac{dP}{dt} = kP(M - P)$$



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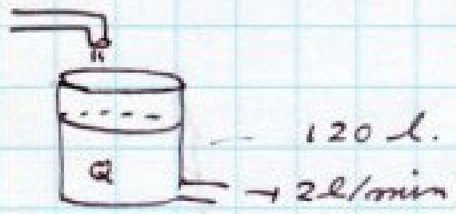
Modeling Process

1 Observable Phenomenum
2. Look for patterns
   Make assumptions
3. Mathematical model
4. Solution compared to data
5. Accept, reject, modify

2.3 # 1, 3, 5, 9, 13, 18, 21, 29

Pollution Model



Originally we have 120 l of H_2O
Water is coming in at a rate of 2 l/min, 3 g/l nitrate
Water is leaking out at the same rate.

1) Write D.E. - Solve
2) How much nitrate at any time

Solution

Q = # of grams of Nitrate

Question: $\frac{dQ}{dt} = ?$

$$\frac{dQ}{dt} = \text{amt in} - \text{Amt out}$$
$$= \text{rate in} - \text{rate out}$$

Rate in: $= 2\frac{l}{min} \cdot \frac{3g}{l} = 6\frac{g}{min}$

Rate out: $\frac{Q}{120} \cdot \frac{2l}{min} = \frac{Q}{60} \frac{g}{min}$

$$\frac{dQ}{dt} = 6 - \frac{Q}{60} \qquad \boxed{Q(0) = 0}$$

$$\frac{dQ}{dt} + \frac{1}{60} Q = 6 \qquad P = \frac{1}{60}$$

$$\mu = e^{t/60} \qquad \mu = e^{\int P}$$

$$(e^{t/60} Q)' = 6e^{t/60} = (\mu Q)'$$

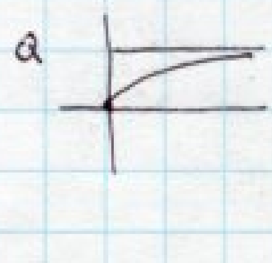
$$e^{t/60} Q = 360 e^{t/60} + C$$

$$Q = 360 + Ce^{-t/60}$$

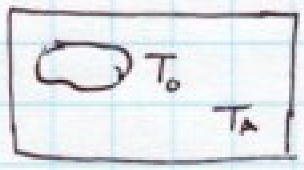
$$Q(0) = 360 + C = 0$$

$$C = -360$$

$$Q = 360(1 - e^{-t/60})$$



Newton's Law of Cooling



$T(t)$ temp of the object

$$\frac{dT}{dt} = k(T_A - T_A)$$

$$\frac{dT}{dt} = k(T - T_A) \qquad T(0) = T_0$$

$$T_0 = 95°C \qquad T_A = 25°C$$

After $t = 10$ minutes
$T = 90°C$

What is the temperature after 20 min

Sol

Given

$$\frac{dT}{dt} = k(T - 25) \qquad T(0) = 95 \checkmark$$
$$T(10) = 90 \checkmark$$
$$T(20) = ?$$

$$\int \frac{1}{T-25}\, dt = \int k\, dt$$

$$\ln(T-25) = kt + C$$

$$T - 25 = e^{kt+c} = e^c e^{kt}$$

$$T - 25 = Ae^{kt}$$

$$T = Ae^{kt} + 25$$

$$t=0 \qquad 95 = A + 25$$

$$A = 95 - 25$$

$$T = (95-25)e^{kt} + 25$$

$$T = (T_0 - T_A)e^{kt} + T_A$$

$$t=10 \qquad 90 = 70e^{10k} + 25$$

$$\frac{90-25}{70} = e^{10k}$$

$$\ln\left(\frac{90-25}{95-25}\right) = 10k$$

$$k = \frac{1}{10}\ln\left(\frac{90-25}{95-25}\right)$$

$$= \frac{1}{10}\ln\left(\frac{T_{10} - T_A}{T_0 - T_A}\right)$$

$$k = \frac{1}{10}\ln\left(\frac{65}{70}\right)$$

$$T = 70\, \exp\left[\frac{t}{10}\ln\frac{65}{70}\right] + 25$$

$$T(20) = 70\exp\left(2\ln\frac{65}{70}\right) + 25$$

$$= 85.4°C$$

Note:

1) $\frac{dy}{dt} = ky$

$\boxed{\frac{dy}{dt} - ky = 0}$ Sol $y_h = Ae^{kt}$

2) $\frac{dy}{dt} = k(y - c)$

$\boxed{\frac{dy}{dt} - ky = B}$ Sol $y = \underbrace{Ae^{kt}}_{y_h} + \underbrace{M}_{y_p}$

① Linear homogeneous O.DE

② Linear inhomogeneous O.DE.

↳ $y = y_h + y_p$

Logistic Model

M - Max population

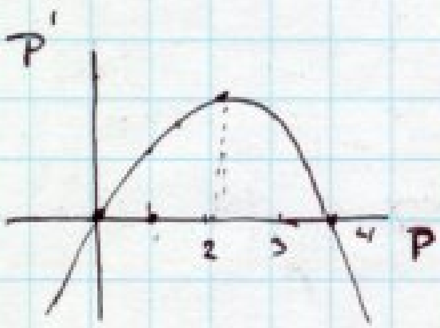
P - Population

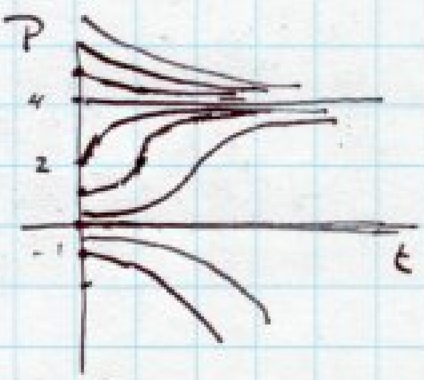
$\frac{dP}{dt} = kP(M-P)$

Example

$\frac{dP}{dt} = 2P(4-P)$

$\boxed{P(0) = 1,}$ $(P(0) = 2)$ $P(0) = 5$





Equil Solutions are

$P(t) = 0$ Unstable

$P(t) = 4$ - Stable

2.5 # Pg 85

1, 3, 8, 9, 10

General Case

$y' = r(1 - \frac{1}{k}y)y$ Pg (76)

$y(0) = y_0$

$\frac{dy}{dt} = ry(1 - \frac{1}{k}y)$

$\int \frac{1}{y(1-\frac{1}{k}y)} dy = \int r\,dt$

$\frac{1}{y(1-\frac{1}{k}y)} = \frac{A}{y} + \frac{B}{1-\frac{1}{k}y}$

$1 = A(1 - \frac{1}{k}y) + By$

$y = 0$ $1 = A$ $A = 1$

$y = k$ $1 = kB$ $B = 1/k$

$\int \frac{1}{y(1-\frac{1}{k}y)} dy = \int (\frac{1}{y} + \frac{1/k}{1-\frac{1}{k}y}) dy = \int r\,dt$

$\ln y - \ln(1 - \frac{1}{k}y) = rt + c$

$\ln \frac{y}{1-(1/k)y} = rt + c$

$\frac{y}{1-\frac{1}{k}y} = e^{rt+c} = e^c e^{rt} = Ae^{rt}$

$y = Ae^{rt} - \frac{1}{k}y\,Ae^{rt}$

$y + \frac{1}{k}y\,Ae^{rt} = Ae^{rt}$

$y[1 + \frac{A}{k}e^{rt}] = Ae^{rt}$

$y = \frac{Ae^{rt}}{1 + \frac{A}{k}e^{rt}}$

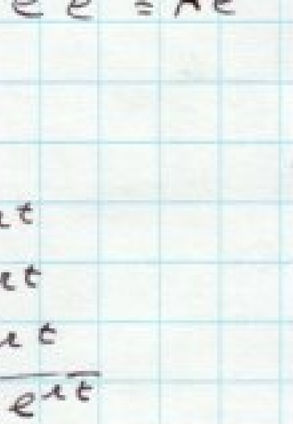
$y = \frac{A}{e^{-rt} + \frac{A}{k}} = \frac{A}{\frac{A}{k} + e^{-rt}}$

I.C $t = 0$ $y = y_0$

$y_0 = \frac{A}{\frac{A}{k} + 1}$ $A = \frac{A}{k}y_0 + y_0$

$A(1 - \frac{1}{k}y_0) = y_0$

$$\boxed{A = \frac{y_0}{1 - y_0/k} \qquad y = \frac{A}{\frac{A}{k} + e^{-rt}}}$$



$y = \frac{k}{1 + \frac{k}{A}e^{-rt}} = \frac{k}{1 + \frac{k(1-y_0/k)}{y_0}e^{-rt}}$

$\boxed{y = \frac{ky_0}{y_0 + (k - y_0)e^{-rt}}}$

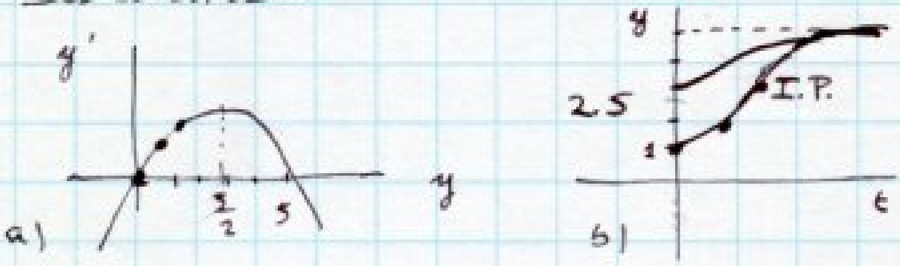
TOPS FORM 3314

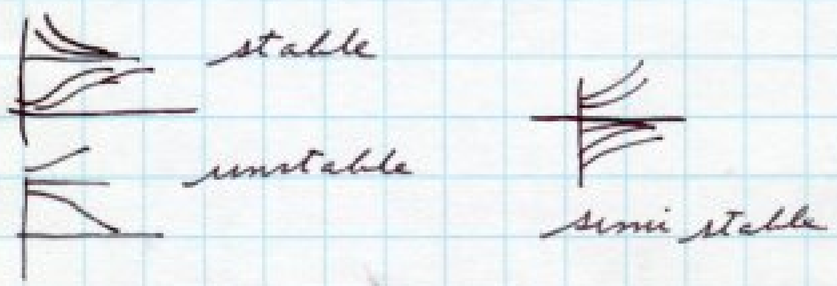
Example

$y' = f(x, y) \qquad y(0) = y_0$

$y' = 3y(5-y) \qquad y(0) = 1$

Sketch sol





Explanation: $y(0) = 1$

- Graph (a) shows the value of the slope y' as a function of y
- Graph (b) shows the values of y v.s t.
- When $y = 1$ we see from (a) that y' is positive, hence y is increasing
- We draw in (b) a graph that begins at $(t, y) = (0, 1)$ with positive slope.
- At some later time, say $t = 1.5$ we see from (a) that y is now bigger than 1 so y' is even more positive.
- We continue drawing the graph in (b) with a larger slope
- From (b) y continues to increase so from (a), the slope gets larger until it reaches the maximum slope at $y = 5/2$
- At this value of t when y reaches ~~the~~ the value $5/2$ we have an inflection point
- As we pass ~~t~~ beyond the inflection point, y continues to increase but with a smaller slope. In other words, the function was concave up before the I.P and now it becomes concave down.
- As y continues to increase, y' continues to become smaller until it approaches 0 at $y = 4$. So the graph levels off approaching the line $y = 4$ asymptotically.
- A similar analysis is applied to other initial conditions.

Exact Equations:

Recall

$f = f(x,y)$

$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$

Suppose $f = \text{konst}$ $\qquad df = 0$

$\boxed{f_x\,dx + f_y\,dy = 0}$

$f_y\,dy = -f_x\,dx \qquad f = k$

$y' = -\frac{f_x}{f_y}$

Def A first order O.D.E of the type

$M\,dx + N\,dy = 0$

is exact if $M = f_x$, $N = f_y$

Here the solution $f = k$

Example:

$2xy^3\,dx + 3x^2y^2\,dy = 0$

$2xy^3\,dx + 3x^2y^2\,dy = 0$

$f = x^2y^3 = k$

Formally:

$M = 2xy^3 \qquad M_y = 6xy^2$

$N = 3x^2y^2 \qquad N_x = 6xy^2$

Equation is exact if $\boxed{M_y = N_x}$

(Think back on Green's Thm)

$f = x^2y^3$

Example:

$\cancel{(2xy^3 + \frac{1}{y})\,dx}$

$\underbrace{(2xy^3 + x^3)}_{M = f_x}dx + \underbrace{(3x^2y^2 + \frac{1}{y})}_{N = f_y}dy = 0$

$M_y = 6xy^2$

$N_x = 6xy^2$

$f = x^2y^3 + \frac{1}{4}x^4 + \ln y = k$

8) $\underbrace{(e^x \sin y + 3y)}_{M}dx - \underbrace{(3x - e^x \sin y)}_{N}dy = 0$

$M = e^x \sin y + 3y \qquad M_y = e^x \cos y + 3$

$N = -(3x - e^x \sin y) \qquad N_x = -3 + e^x \sin y$

Not exact.

2) $(2x + 4y) + (2x - 2y)y' = 0$

$(2x + 4y) + (2x - 2y)\frac{dy}{dx} = 0$

$(2x + 4y)\,dx + (2x - 2y)\,dy = 0$

$M = 2x + 4y \qquad M_y = 4$

$N = 2x - 2y \qquad N_x = 2$

Not exact.

$y' = -\frac{(2x + 4y)}{(2x - 2y)}$ $\qquad$ Oops!

6) $\frac{dy}{dx} = -\left[\frac{ax - by}{bx - cy}\right]$

$(bx - cy)\,dy = -(ax - by)\,dx$

$(ax - by)\,dx + (bx - cy)\,dy = 0$

$M = ax - by \qquad M_y = -b$

$N = bx - cy \qquad N_x = b$

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4) $\underbrace{(2xy^2 + 2y)}_{M} + \underbrace{(2x^2y + 2x)}_{N}y' = 0$

$(2xy^2 + 2y) + (2x^2y + 2x)\frac{dy}{dx} = 0$

$(2xy^2 + 2y)\,dx + (2x^2y + 2x)\,dy = 0$

$M = 2xy^2 + 2y \qquad M_y = 4xy + 2$

$N = 2x^2y + 2x \qquad N_x = 4xy + 2$

$M = f_x \qquad f = \boxed{x^2y^2 + 2xy = k}$

$N = f_y$

9) $(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)\,dx + (xe^{xy}\cos 2x - 3)\,dy = 0$

$\begin{cases} M = ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x \\ N = xe^{xy}\cos 2x - 3 \end{cases}$

$f = e^{xy}\cos 2x - 3y + x^2 = k$

$$\begin{tikzcd} & f \arrow[dl, dash] \arrow[dr, dash] & \\ f_x & & f_y \end{tikzcd}$$

Complications:

Ex $3x^2y^2\,dx + 2x^3y\,dy = 0$

$M = 3x^2y^2 \quad M_y = 6x^2y$

$N = 2x^3y \quad N_x = 6x^2y$

$M = f_x \Rightarrow f = x^3y^2 = k$

$N = f_y$

* $3y\,dx + 2x\,dy = 0$

$M = 3y \quad M_y = 3$

$N = 2x \quad N_x = 2 \quad M_y \neq N_x$

Mult (*) by x^2y - equation (*) becomes exact!

2.6 # 1-15 odd

Last time:

$M\,dx + N\,dy = 0$

Exact if $M_y = N_x$

If so $M = f_x$

$N = f_y$

$M\,dx + N\,dy = df \Rightarrow f = k$

$= f_x\,dx + f_y\,dy$

Ex

$(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)\,dx + (xe^{xy}\cos 2x - 3)\,dy = 0$

$$\begin{cases} M = ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x \\ N = xe^{xy}\cos 2x - 3 \end{cases}$$

$M_y = (xye^{xy} + e^{xy})\cos 2x - 2xe^{xy}\sin 2x$

$= e^{xy}[xy\cos 2x + \cos 2x - 2x\sin 2x]$

$N_x = x\frac{\partial}{\partial x}(e^{xy}\cos 2x) + e^{xy}\cos 2x$

$= x(ye^{xy}\cos 2x - 2e^{xy}\sin 2x) + e^{xy}\cos 2x$

$= e^{xy}[xy\cos 2x + \cos 2x - 2x\sin 2x]$

$M_y = N_x$

$\int N\,dy = e^{xy}\cos 2x - 3y + c(x)$

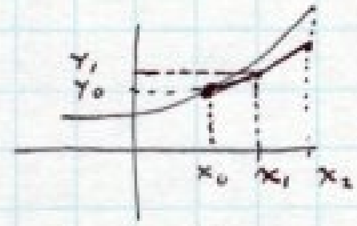
$\frac{\partial}{\partial x}[e^{xy}\cos 2x - 3y] = ye^{xy}\cos 2x - 2e^{xy}\sin 2x$

$f = e^{xy}\cos 2x - 3y + x^2 = k$

Numerical methods

Euler's method:

$y' = f(x,y)$ & $y(x_0) = y_0$



$P(x_0,y_0)$ Slope $= f(x_0,y_0)$

Let $x_1 = x_0 + h$ h small

$x_2 = x_1 + h$

L: $\frac{y-y_0}{x-x_0} = f(x_0,y_0)$

$y - y_0 = (x-x_0)f(x_0,y_0)$

$y = y_0 + (x-x_0)f(x_0,y_0)$

$y_1 = y_0 + hf(x_0,y_0)$

$y_2 = y_1 + hf(x_1,y_1)$

$y_3 = y_2 + hf(x_2,y_2)$

$\vdots$

$y_n = y_{n-1} + hf(x_{n-1},y_{n-1})$

$$\boxed{y_{n+1} = y_n + hf(x_n,y_n)}$$

Ex

$y' = x + y \quad y(0) = 1$

Find $y(1)$ Choose $h = \frac{1}{4}$

Sol $h = 1/4 = 0.25 \quad (x_0,y_0) = (0,1)$

$f(x,y) = x + y$

$x_0 = 0$	$y_0 = 1$	$y_1 = 1 + f(0,1)\cdot h$
$x_1 = 0.25$	$y_1 = 1.25$	$y_2 = 1.25 + hf(0.$
$x_2 = 0.5$	$y_2 = 1.625$	$y_3 = 1.625 +$
$x_3 = 0.75$	$y_3 = 2.156$	etc.
$x_4 = 1.0$	$y_4 =$	

11) $(\underbrace{x\ln y + xy}_{M})dx + (\underbrace{y\ln x + xy}_{N})dy = 0$

$M_y = \frac{x}{y} + x$

$N_x = \frac{y}{x} + y$ not exact ☺

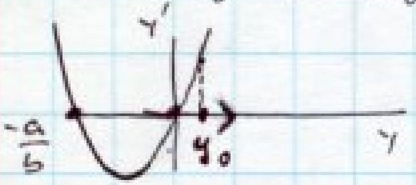
— o —

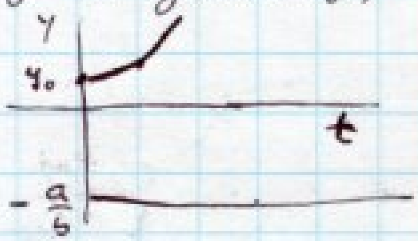
Pg 84 # 1

#1) $\frac{dy}{dt} = ay + by^2$ $a > 0$, $b > 0$

i) Graph $y_0 \geq 0$

y' vs $ay + by^2 = y(a+by)$





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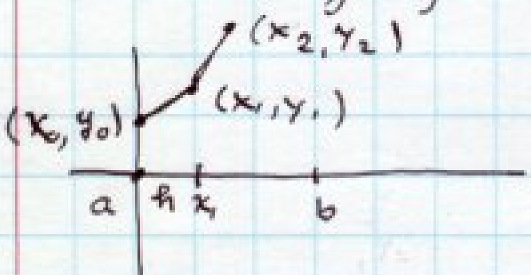
Euler's Method (Review)

Solve $y' = f(x,y)$ $y(0) = y_0$



$n = \#$ of steps

$a = 0$ $x_0 = 0$ $h = \frac{b-a}{n}$

$b = 1$ $y_0 = 1$

$x_1 = a + h$ $m = f(x_0, y_0)$

$\frac{y_1 - y_0}{x_1 - x_0} = f(x_0, y_0)$

$y_1 = y_0 + (x_1 - x_0) f(x_0, y_0)$

$y_1 = y_0 + h f(x_0, y_0)$

$y_2 = y_1 + h f(x_1, y_1)$

$\vdots$

$x_2 = a + 2h$

$x_3 = a + 3h$

$x_k = a + kh$

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

2.7 # 3, 5, 9, 11, 15

Second Order Linear Equations

I. Constant Coeff.

a) Homogeneous

$ay'' + by' + cy = 0$

b) Inhomogeneous

$ay'' + by' + cy = f(x)$

a) Homogeneous

Ex 1 $y'' + 5y' + 6y = 0$

Let $y = e^{rx}$

~~$r^2 e^{rx} + 5ry$~~

$r^2 e^{rx} + 5re^{rx} + 6e^{rx} = 0$

$(r^2 + 5r + 6)e^{rx} = 0$

$r^2 + 5r + 6 = 0$

$(r+3)(r+2) = 0$

$r = -3, -2$

$y = e^{-3x}$ $y = e^{-2x}$

$y = c_1 e^{-3x} + c_2 e^{-2x}$

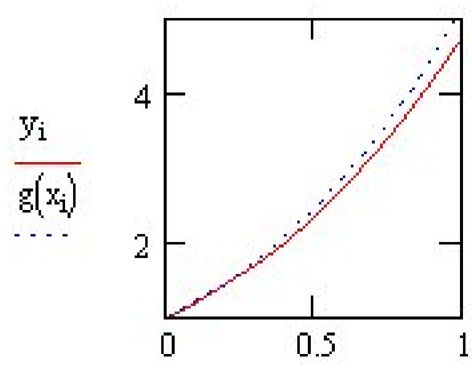
Mathcad

Euler's Method. Solve: $y' = f(x,y) := y + x + 1$

$\Delta x := \frac{b-a}{n}$ $i := 0..n$ $y_0 := 1$

$x_i := a + \Delta x \cdot i$

$g(x) := 3 \cdot e^x - x - 2$ $y_{i+1} := y_i + \Delta x \cdot f(x_i, y_i)$



Interval: $a \equiv 0$ $b \equiv 1$

No. of iterations: $n \equiv 10$

Approx. $y_n = 4.7812$

Exact Sol. $g(1) = 5.1548$

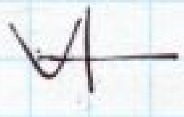
Second Order Linear O.D.E (cont) with Constant Coefficients

Different real roots

Example 1 ($r = r_1, r_2 \quad r_1 \neq r_2$)

$y'' + 5y' + 6y = 0$

Let $y = e^{mx} = e^{rx}$



$r^2 + 5r + 6 = 0$

$(r+3)(r+2) = 0 \quad r = -3, -2$

$y = c_1 e^{-3x} + c_2 e^{-2x}$

Example 2. ($r = r_1, r_1$ real)

$y'' + 6y' + 9y = 0$ Repeated root

Let $y = e^{rx}$

$(r^2 + 6r + 9) e^{rx} = 0$

$(r+3)(r+3) = 0 \quad r = -3, -3$

$y_1 = c_1 e^{-3x}$

Reduction of Order

Try $\boxed{y_2 = u_1(x) e^{-3x}}$

$9y = 9u_1(x) e^{-3x}$

$6y' = 6[u'(x) e^{-3x} - 3u(x) e^{-3x}]$

$\underline{y'' = \boxed{[u'' e^{-3x} - 3u' e^{-3x} - 3u' e^{-3x} + 9u e^{-3x}}}$

$y'' + 6y' + 9y = u'' e^{-3x} = 0$

$\therefore \quad u''(x) = 0$

$u = c_1 + c_2 x$

$y = (c_1 + c_2 x) e^{-3x}$

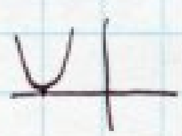
$y = c_1 e^{-3x} + c_2 x e^{-3x}$

In general:

If $r = r_1$ is a repeated root then $y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$

Example

$y'' + 4y' + 4y = 0$



Let $y = e^{rx}$

$r^2 + 4r + 4 = 0$

$(r+2)(r+2) = 0 \quad r = -2, -2$

$y = c_1 \underbrace{e^{-2x}}_{y_1} + c_2 \underbrace{x e^{-2x}}_{y_2}$

$y_1 =$  $y_2 =$ 

Critically damped

Example 4 (Complex Roots)

$y'' + y' + 4y = 0$

Let $y = e^{rx}$

$r^2 + r + 4 = 0$

$r = \frac{-1 \pm \sqrt{1-16}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{15}}{2}$

$y_1 = e^{(-\frac{1}{2} + i\frac{\sqrt{15}}{2})x} = e^{-\frac{x}{2}} \cdot e^{i\frac{\sqrt{15}}{2} \cdot x}$

$y_2 = e^{(-\frac{1}{2} - i\frac{\sqrt{15}}{2})x} = e^{-\frac{x}{2}} e^{-i\frac{\sqrt{15}}{2} x}$

Euler: $e^{i\beta x} = \cos\beta x + i \sin\beta x$

$\beta = 1 \quad \boxed{e^{ix} = \cos x + i \sin x}$

Proof of Euler's Formula:

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$i = \sqrt{-1}, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1$

$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots$

$= \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)}_{\cos x} + i\underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right)}_{\sin x}$

$e^{ix} = \cos x + i \sin x$

Note: $e^{i\pi} = -1$ Q.E.D.

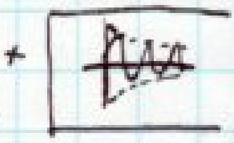
Back to the problem:

$y_1 = e^{-\frac{x}{2}} e^{i\frac{\sqrt{15}}{2} x} \qquad y = c_1 y_1 + c_2 y_2$

$y_2 = e^{-\frac{x}{2}} e^{-i\frac{\sqrt{15}}{2} x}$

$y = c_1 e^{-\frac{x}{2}} \left(\cos\frac{\sqrt{15}}{2}x + i \sin\frac{\sqrt{15}}{2}x\right) + c_2 e^{-\frac{x}{2}} \left(\cos\frac{\sqrt{15}}{2}x - i \sin\frac{\sqrt{15}}{2}x\right)$

$y = (c_1 + c_2) e^{-\frac{x}{2}} \cos\frac{\sqrt{15}}{2}x + i(c_1 - c_2) e^{-\frac{x}{2}} \sin\frac{\sqrt{15}}{2}x$



$y = C_1 e^{-\frac{x}{2}} \cos\frac{\sqrt{15}}{2}x + C_2 e^{-\frac{x}{2}} \sin\frac{\sqrt{15}}{2}x$

$y = e^{-\frac{x}{2}} [C_1 \cos(\frac{\sqrt{15}}{2}x) + C_2 \sin\frac{\sqrt{15}}{2}x]$

Summary:

$$ay'' + by' + cy = 0$$

Let $y = e^{rx} \Rightarrow ar^2 + br + c = 0$

Case 1 $r = r_1, r_2$ real, different

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Case 2: $r = r_1$ double root

$$y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$$

Case 3: $r = \alpha \pm i\beta$

$$y = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$$

Examples: Solve:

1) $y'' + y' + y = 0$ Let $y = e^{rx}$

$r^2 + r + 1 = 0$

$r = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{1}{2} i\sqrt{3}$

$y = e^{-\frac{x}{2}}[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x]$

2) $y'' + 7y' + 10y = 0$

I.C: $y(0) = 1$ $y'(0) = 0$

Sol: Let $y = e^{rt}$

$r^2 + 7r + 10 = 0$

$(r+5)(r+2) = 0$ $r = -5, -2$

$y = c_1 e^{-5t} + c_2 e^{-2t}$

$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 1 \\ y'(0) &= -5c_1 - 2c_2 = 0 \end{aligned}\right\}$$

Solve $\begin{cases} c_1 = -\frac{2}{5}c_2 \\ (1 - \frac{2}{5})c_2 = 1 \\ \frac{3}{5}c_2 = 1 \quad c_2 = \frac{5}{3} \end{cases}$

$c_1 = -\frac{2}{3}$

$y = -\frac{2}{3}e^{-5t} + \frac{5}{3}e^{-2t}$

— o —

$\boxed{3.1 \ \#12}$

$y'' + 3y' = 0$ $y(0) = -2$

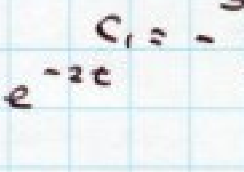
$y = e^{rt}$ $y'(0) = 3$

$r^2 + 3r = 0$

$r(r+3) = 0$ $r = 0, -3$

$y = c_1 e^{0t} + c_2 e^{-3t}$

$y = c_1 + c_2 e^{-3t}$



$\begin{cases} -2 = c_1 + c_2 \\ 3 = -3c_2 \end{cases}$ | $c_2 = -1$, $c_1 = -1$; $y = -1 - e^{-3t}$

3.4 #21

$y'' + y' + \frac{5}{4}y = 0$ $y(0) = 3$

$r^2 + r + \frac{5}{4} = 0$ $y'(0) = 1$

$r = \frac{-1 \pm \sqrt{1-5}}{2} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i$

$y = e^{-\frac{t}{2}}[c_1 \cos t + c_2 \sin t]$

$3 = c_1$

$y = e^{-\frac{t}{2}}[3\cos t + c_2 \sin t]$ $y'(0) = 1$

$y'(0) = -\frac{1}{2} \cdot 3 + c_2 = 1$ $c_2 = \frac{5}{2}$

$y = e^{-\frac{t}{2}}[3\cos t + \frac{5}{2}\sin t]$

— o —

Theory

Operators:

$ay'' + by' + cy = 0$

Let $L = a\frac{d^2}{dt^2} + b\frac{d}{dt} + c$

$LY = 0$ $L(Y) = 0$

Def: An operator is called linear if:

a) $L(Y_1) + L(Y_2) = L(Y_1 + Y_2)$

b) $L(c_1 y_1) = c_1 L(y_1)$

Ex $L = \frac{d}{dt}$ a) $L(y_1 + y_2) = \frac{d}{dt}(y_1 + y_2) = L(y_1) + L(y_2)$

b) $L(c_1 y_1) = \frac{d}{dt}(c_1 y_1) = c_1 L(y_1)$

Ex 2 It is very easy to verify that

$L = a\frac{d^2}{dt^2} + b\frac{d}{dt} + c = 0$

is a linear operator

$0 = LY$ is called a linear D.E.

Principle of Superposition

Let L be a linear Diff Operator. Then, if

$LY_1 = 0$, $LY_2 = 0$

$L(c_1 y_1 + c_2 y_2) = 0$

Proof

$$L(c_1y_1 + c_2y_2) = L(c_1y_1) + L(c_2y_2)$$
$$= c_1L(y_1) + c_2L(y_2)$$
$$= c_1 \cdot 0 + c_2 \cdot 0$$
$$= 0 \qquad \text{Q.E.D}$$

This means that if y_1, y_2 are solutions of $Ly=0$, so is $y = \underbrace{c_1y_1 + c_2y_2}_{\text{linear combination}}$

Linear Independence

Two functions (vectors) y_1 and y_2 are l.i if

$$c_1y_1 + c_2y_2 = 0 \Rightarrow c_1 = c_2 = 0$$

Note: $c_1y_1' + c_2y_2' = 0$

Def: $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ Wronskian

If $W = 0$ $\quad y_1, y_2$ are l.d

$W \neq 0$ $\quad y_1, y_2$ are l.i

Ex: $y_1 = t \qquad y_2 = t^2$

$$W = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t^2 \qquad \text{l.i} \quad t>0$$

Ex $y_1 = e^{2x} \qquad y_2 = xe^{2x}$

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix}$$
$$= e^{4x}(2x + 1 - 2x)$$
$$= e^{4x} \neq 0 \qquad \text{l.i}$$

Theorem: Let y_1 and y_2 be solutions of a 2nd order linear O.D.E ($Ly=0$), Then $W(y_1, y_2)$ is either identically equals to 0 or is never 0.

Proof

$$Ly = ay'' + by' + cy = 0$$

a, b, c could depend on x

Proof

Given:
$$\left.\begin{array}{l} ay_1'' + by_1' + cy_1 = 0 \\ ay_2'' + by_2' + cy_2 = 0 \end{array}\right\} *$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

Rewrite (*) as

$$\begin{cases} y_1'' + py_1' + qy_1 = 0 & p = b/a \\ y_2'' + py_2' + qy_2 = 0 & q = c/a \end{cases}$$

$$W' = y_1y_2'' + y_1'y_2' - y_2y_1'' - y_2'y_1'$$
$$W' = y_1y_2'' - y_2y_1''$$
$$\underline{pW = py_1y_2' - py_2y_1'}$$

$$W' + pW = y_1(y_2'' + py_2') - y_2(y_1'' + py_1')$$
$$= y_1(-qy_2) - y_2(-qy_1)$$
$$W' + pW = 0$$
$$W = ce^{-\int p} \begin{cases} = 0 & c = 0 \\ \neq 0 & c \neq 0 \end{cases}$$

Abel's Formula for the Wronskian $\boxed{W = ce^{-\int p}}$

Reduction of Order.

Note: $\left(\frac{y_2}{y_1}\right)' = \frac{y_1y_2' - y_2y_1'}{y_1^2}$

$$\left(\frac{y_2}{y_1}\right)' = \frac{W}{y_1^2}$$

$$\frac{y_2}{y_1} = \int \frac{W}{y_1^2}$$

$$\boxed{y_2 = y_1 \int \frac{W}{y_1^2}}$$

3.2 # 1, 3, 5, 6
13, 21, 23, 25

Example:

$$y'' + 4y' + 4y = 0$$

Let $y = e^{rt}$

$$r^2 + 4r + 4 = 0$$
$$(r+2)(r+2) = 0 \qquad r = -2, -2$$
$$y_1 = e^{-2t}$$

$P = 4$

$W = c\,e^{-\int 4\,dt} = c\,e^{-4t}$

$y_2 = y_1 \int \frac{W}{y_1^2} = e^{-2t} \int \frac{e^{-4t}}{e^{-4t}}\,dt$

$= e^{-2t} \int 1\,dt = t e^{-2t}$

$y = c_1 e^{-2t} + c_2 t e^{-2t}$

Find the Wronskian of

$y_1 = e^{-2t} \qquad y_2 = t e^{-2t}$

$$W = \begin{vmatrix} e^{-2t} & t e^{-2t} \\ -2e^{-2t} & e^{-2t} - 2t e^{-2t} \end{vmatrix}$$

$= e^{-2t}\left[e^{-2t} - 2t\,e^{-2t}\right] + 2t e^{-4t}$

$= e^{-4t} \neq 0$

P. 168 23) $t^2 y'' - 4t y' + 6y = 0 \qquad y_1 = t^2$

$y_2 = y_1 \int \frac{W}{y_1^2} \qquad W = c\,e^{-\int P}$

$P = -\frac{4}{t} \qquad W = c\,e^{+4\ln t} = c\,t^{+4}$

$y_2 = y_1 \int \frac{W}{y_1^2} = t^2 \int \frac{t^{+4}}{t^4}\,dt = t^3$

$\cancel{= t^2 \int t^{-8}\,dt = \left(t^2\left(\frac{t^{-7}}{-7}\right)\right)}$

$\cancel{= -\frac{1}{7} t^{-5}}$

$y = c_1 t^2 + c_2 t^3$

Verify

$6\,y_2 = t^3\,(6) = 6t^3$

$-4t\,y_2' = 3t^2(-4t) = -12t^3$

$t^2 y_2'' = 6t \cdot t^2 = 6t^3$

$L y_2 = 0$

Terminology

$y = c_1 y_1 + c_2 y_2$ General Sol

$\{y_1, y_2\}$ - Fundamental Sol

- Basis for the Kernel

$c_1 y_1 + c_2 y_2$ - Linear Combination

Ex: $y'' + 2y' + 8y = 0 \qquad y = e^{mt}$

$y(0) = 4 \qquad m^2 + 2m + 8 = 0$

$y'(0) = 2 \qquad m = -1 \pm \sqrt{-7}$

$m = -1 \pm i\sqrt{7}$

$y = e^{-t}\left[c_1 \cos\sqrt{7}\,t + c_2 \sin\sqrt{7}\,t\right]$

$y(0) = c_1 = 4$

$y = e^{-t}\left[4\cos\sqrt{7}\,t + c_2 \sin\sqrt{7}\,t\right]$

$y'(0) = -4 + c_2\sqrt{7} = 2$

$c_2 = 6/\sqrt{7}$

$y = e^{-t}\left[4\cos\sqrt{7}\,t + \frac{6}{\sqrt{7}} \sin\sqrt{7}\,t\right]$

Recall:

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$

$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$

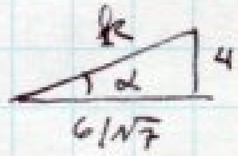
$4\cos\sqrt{7}\,t + \frac{6}{\sqrt{7}}\sin\sqrt{7}\,t$

$\sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\beta = \sqrt{7}\,t$

$\sin\alpha = 4/k$

$\cos\alpha = \frac{6}{\sqrt{7}}/k$



$\tan\alpha = \frac{4\cdot\sqrt{7}}{6} = \frac{2\sqrt{7}}{3} \qquad \sqrt{16 + \frac{36}{7}}$ k

$\beta = \sqrt{7}\,t$

$4\cos\sqrt{7}\,t + \frac{6}{\sqrt{7}}\sin t = \sin(\sqrt{7}\,t + \alpha)$

$y = e^{-t}\left[\sin(\sqrt{7}\,t + \alpha)\right]$

$= e^{-t}\sin(\sqrt{7}\,t + \alpha)$

TOPS FORM 3314

DO 3.4 # 7, 9, 11, 17, 19, 21

3.5 # 1, 5, 9, 11, 13

Ex 14 sect 3.5

$$y'' + 4y' + 4y = 0 \qquad y(-1) = 2$$

Let $y = e^{mx}$ $\qquad y'(-1) = 1$

$$m^2 + 4m + 4 = 0$$

$$(m+2)^2 = 0 \qquad m = -2, -2$$

$$y = c_1 e^{-2x} + c_2 x e^{-2x}$$

$$y(-1) = c_1 e^2 - c_2 e^2 = 2 \qquad (1)$$

$$y' = -2c_1 e^{-2x} + c_2 [-2x e^{-2x} + e^{-2x}]$$

$$y'(-1) = -2c_1 e^2 + c_2 (+2e^2 + e^2) = 1 \qquad (2)$$

$$c_1 e^2 - c_2 e^2 = 2$$

$$-2e^2 c_1 + (2e^2 + e^2) c_2 = 1$$

$$\begin{bmatrix} e^2 & -e^2 \\ -2e^2 & 3e^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} e^2 & -e^2 \\ -2e^2 & 3e^2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(e^2)^2} \begin{bmatrix} 7e^2 \\ 5e^2 \end{bmatrix} = \begin{bmatrix} 7e^{-2} \\ 5e^{-2} \end{bmatrix}$$

$$y = 7e^{-2} e^{-2x} + 5e^{-2} x e^{-2x}$$

$$y = 7e^{-2(x+1)} + 5x e^{-2(x+1)}$$

Verify in Mathcad:

$$y := 7 \cdot e^{-2 \cdot (t+1)} + 5 \cdot t \cdot e^{-2 \cdot (t+1)}$$

$$\frac{d^2}{dt^2} y + 4 \cdot \frac{d}{dt} y + 4 \cdot y \rightarrow 0$$

3.2 #5 $Y_1 = e^t \sin t \qquad Y_2 = e^t \cos t$

$$W(Y_1, Y_2) = \begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t \cos t + e^t \sin t & -e^t \sin t + e^t \cos t \end{vmatrix}$$

$$W = -e^{2t}\sin^2 t + e^{2t}\sin t \cos t - (e^{2t}\cos^2 t + e^{2t}\sin t \cos t)$$

$$W = -e^{2t}\sin^2 t - e^{2t}\cos^2 t = -e^{2t}(\sin^2 t + \cos^2 t) = -e^{-2t}$$

— o —

Summary:

1. Linear 2^{nd} Order O.D.E (Homog)

$$a(x)y'' + b(x)y' + c(x)y = 0$$

$$y'' + p(x)y' + q(x)y = 0$$

2. Constant Coeff. (Homog)

$$ay'' + by' + cy = 0$$

Let $y = e^{rx}$

$$ar^2 + br + c = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Case 1

$r = r_1, r_2$ real different

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Case 2

$r = r_1, r_1$ Repeated root

$$y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$$

Case 3

$$r = \alpha \pm i\beta$$

$$y = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$$

3. Reduction of Order.

Given: $y'' + Py' + qy = 0$

$y = y_1$ is a solution

Try $y = c(x) y_1$ find 2^{nd} solution

4. Euler's Formula

$$e^{ix} = \cos x + i \sin x$$

5) Wronskian

$$W(Y_1, Y_2) = \begin{vmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{vmatrix} = Y_1 Y_2' - Y_2 Y_1'$$

a) $W \neq 0$ Y_1, Y_2 are l.i

b) Y_1, Y_2 are sol. of $LY = 0$

$$W(Y_1, Y_2) \neq 0 \Rightarrow \boxed{Y = c_1 Y_1 + c_2 Y_2}$$

(5) Cont y_1, y_2 fundamental solutions.. (Basis)

General sol $y = c_1 y_1 + c_2 y_2$ is a linear combination

Solution is a Vector Space of dim 2

(6) $LY = Y'' + PY' + qY = 0$

Y_1, Y_2 - fund. sol

$$W' + PW = 0$$

$$W = c e^{-\int P}$$ Abel's Formula

— o —

Ex $y'' + 2y' + 2y = 0$

Let $y = e^{rt}$

$$r^2 + 2r + 2 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(2)}}{2} = \frac{-2 \pm 2\sqrt{1-2}}{2}$$

$$r = -1 \pm i$$

$$y_1 = e^{-x}\cos x \qquad y_2 = e^{-x}\sin x$$

$$W = c e^{-\int 2dx} = c e^{-2x}$$

— o —

Euler's Formula

① $i = \sqrt{-1}$ $\quad i^5 = i$

$i^2 = -1$ $\quad i^6 = -1$

$i^3 = -i$ $\quad i^7 = -i$

$i^4 = 1$ $\quad i^8 = 1$

② $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \ldots$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} \ldots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!}$$

$$= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots)$$

$$e^{ix} = \cos x + i \sin x$$

$$e^{i\pi} = -1$$

Inhomogeneous 2nd order O.D.E.

$$L = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$$

$$Ly = f(x)$$

$$y'' + p(x)y' + q(x)y = f(x)$$

Recall that

$Ly = 0$ has a solution space of the form

$$y = c_1 y_1 + c_2 y_2, \quad y_1, y_2 \text{ l.i.}$$

(Solution space is a V.S. of dimension 2, y_1, y_2 basis)

Notation. We will call the solution

$$y_h = c_1 y_1 + c_2 y_2$$

the homogeneous solution.

Theorem: If y_p is any solution of $Ly = f(x)$ then the general solution of the inhomogeneous equation is

$$y = y_h + \underbrace{y_p}_{\text{Particular solution}}$$

Particular solutions

1) Undetermined Coeff.
2) Variation of Parameters

Example: Solve

$$y'' + 4y = 5\cos 2x$$

Consider $Ly = 0$

$$y = e^{rx} \quad r^2 + 4 = 0 \quad r = \pm 2i$$

$$y = c_1 \cos 2x + c_2 \sin 2x$$

$$4\, y_p = [Ax\cos 2x + Bx\sin 2x]\ 4$$

$$y_p'' = A[-4x\cos 2x - 2^2\sin 2x] + B[-4x\sin 2x + 2^2\cos 2x]$$

$$Ly_p = (4A - 4A)x\cos 2x + (4B - 4B)x\sin 2x - 4A\sin 2x + 4B\cos 2x = 5\cos 2x$$

$$-4A\sin 2x + 4B\cos 2x = 5\cos 2x$$

$$4B = 5$$

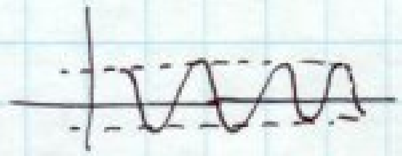
$$B = 5/4 \quad A = 0$$

$$y_p = \frac{5}{4}x\sin 2x$$

$$y = c_1\cos 2x + c_2\sin 2x + \frac{5}{4}x\sin 2x$$

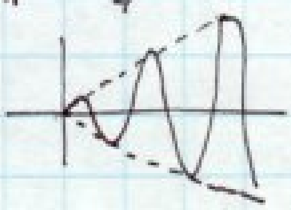
Note:

$$y'' + 4y = 0 \quad y = c_1\cos 2x + c_2\sin 2x$$

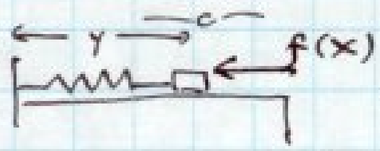


$$y'' + 4y = 5\cos 2x$$

$$y = y_h + \frac{5}{4}x\sin 2x$$



Resonance



$$\vec{F} = m\vec{a}$$

$$F = my'' = -ky$$

$$my'' + ky = 0$$

$$y'' + \frac{k}{m}y = 0$$

$$\boxed{y'' + \omega_0 y = 0}$$

Summary

To solve

$$Ly = f(x) \qquad f(x) = \begin{cases}\cos\alpha x \\ \sin\alpha x\end{cases}$$

$$y_p = x^k[A\cos\alpha x + B\sin\alpha x]$$

where k is the smallest positive integer such that no term in y_p is contained in the homogeneous solution

Undetermined Coeff. (Cont)

Ex:

$y'' + 36y = 5e^{3x} + 8x\cos 6x$

$y = y_h + y_p \qquad y_h = e^{mx}$

$m^2 + 36 = 0$

$m = \pm 6i$

$y_h = c_1\cos 6x + c_2 \sin 6x$

$\boxed{\times 36}$ $y_p = Ae^{3x} + x(B\cos 6x + C\sin 6x)$

$y_p'' = 9Ae^{3x} +$
$B(-36x\cos 6x - 6\sin x) +$
$C(-36x\sin 6x + 6\cos x)$

$Ly_p = Ae^{3x}(36+9) +$
$B(-6\sin x) + C(6\cos x)$ No

$y_p = Ae^{3x} +$
$x[(Cx+D)\cos 6x + (Ex+G)\sin 6x]$

$45A = 5 \quad \boxed{A = \frac{1}{9}}$

Examples

$y'' + 2y' + 5y = 0$

$m^2 + 2m + 5 = 0$

$m = -1 \pm \sqrt{1-5} = -1 \pm 2i$

$y_h = e^{-x}(c_1\cos 2x + c_2\sin 2x)$

$y'' + 2y' + 5y = 3x^2 + xe^{-x}$

$y_p = (A_2x^2 + A_1x + A_0) + \text{--}$

$y_p = (A_2x^2 + A_1x + A_0) +$
$(B_1x + B_0)e^{-x}$

$y'' + 2y' + 5y = 3x^2e^{-x}\cos 2x$

$y_p = x[(A_2x^2 + A_1x + A_0)e^{-x}\cos 2x +$
$(B_2x^2 + B_1x + B_0)e^{-x}\sin 2x]$

$y'' + 5y' + 6y = 0 = Ly$

$m^2 + 5m + 6 = 0 \qquad m = -2, -3$

$(m+2)(m+3) = 0$

$y_h = c_1e^{-2x} + c_2e^{-3x}$

$Ly = y'' + 5y' + 6y = 2x + xe^{-x}$

$y_p = (A_1x + A_0) + (B_1x + B_0)e^{-x}$

Ex: $y'' + 5y' + 6y = x^2e^{-2x} + \sin 5x$

$y_h = c_1e^{-2x} + c_2e^{-3x}$

$y_p = x(A_2x^2 + A_1x + A_0)e^{-2x} +$
$B\sin 5x + C\cos 5x$

Ex

$y'' + 6y' + 9y = x^2e^{-3x}$

$m^2 + 6m + 9 = 0$

$(m+3)^2 = 0 \qquad m = -3$

$y_h = c_1e^{-3x} + c_2xe^{-3x}$

$y_p = x^2(A_2x^2 + A_1x + A_0)e^{-3x}$

3.6 # 1, 3, 7, 9, 13
19, 21, 23, 25 $\leftarrow y_p$

3) $y'' - 2y' - 3y = -3te^{-t}$

$m^2 - 2m - 3 = 0$

$(m-3)(m+1) = 0 \qquad m = 3, -1$

$y_h = c_1e^{3t} + c_2e^{-t}$

$y_p = t(a_1t + a_0)e^{-t}$

$\rightarrow$ $y_p = (a_1t^2 + a_0t)e^{-t}$

$y_p' = -(a_1t^2 + a_0t)e^{-t} + e^{-t}(2a_1t + a_0)$

$\rightarrow$ $y_p' = (-a_1t^2 - a_0t + 2a_1t + a_0)e^{-t}$

$y_p'' = -(-a_1t^2 - a_0t + 2a_1t + a_0)e^{-t} +$
$(-2a_1t - a_0 + 2a_1)e^{-t}$

$y_p'' = (a_1t^2 + a_0t - 2a_1t - a_0$
$- 2a_1t - a_0 + 2a_1)e^{-t}$

$\rightarrow$ $y_p'' = e^{-t}(a_1t^2 + a_0t - 4a_1t - 2a_0 + 2a_1)$

$-3y_p = (-3a_1t^2 - 3a_0t)e^{-t}$

$-2y_p' = (2a_1t^2 + 2a_0t - 4a_1t - 2a_0)e^{-t}$

$\underline{y_p'' = (a_1t^2 + a_0t - 4a_1t - 2a_0 + 2a_1)e^{-t}}$

$Ly_p = [-8a_1t]e^{-t} + (2a_1 - 4a_0)e^{-t}$
$= -3te^{-t}$

$-8a_1 = -3 \qquad a_1 = 3/8$

$2a_1 - 4a_0 = 0 \qquad a_0 = \frac{1}{2}a_1 = \frac{3}{16}$

$y_p = \frac{3}{8}t^2e^{-t} + \frac{3}{16}te^{-t}$

$y = c_1e^{-t} + c_2e^{+3t} + e^{-t}[\frac{3}{8}t^2 + \frac{3}{16}t)$

Variation of Parameters

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$
$$Ly = f$$

$$y = y_h + y_p$$
$$y_h = c_1 y_1 + c_2 y_2$$

$$y_p = u_1(x)\,y_1 + u_2(x)\,y_2$$

$$\begin{cases} q\,y_p = q u_1 y_1 + q u_2 y_2 \\ p\,y_p' = p u_1 y_1' + p u_2 y_2' + \overbrace{(u_1' y_1 + u_2' y_2)}^{=0} \\ y_p'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2' \end{cases}$$

$$L y_p = u_1 L y_1 + u_2 L y_2 + u_1' y_1' + u_2' y_2'$$
$$= 0 + 0 + u_1' y_1' + u_2' y_2' = f$$

Hence

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = f \end{cases}$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-f y_2}{W}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{W} = \frac{f y_1}{W}$$

Conclusion

$$\boxed{\begin{array}{l} y_p = u_1 y_1 + u_2 y_2 \\ \text{where} \\ u_1 = \int \frac{-f y_2}{W}, \quad u_2 = \int \frac{f y_1}{W} \end{array}}$$

Example:

1) $y'' - 5y' + 6y = 2e^t$

$y_h = e^{rt}$

$r^2 - 5r + 6 = 0$

$(r-3)(r-2) = 0 \quad r = 2, 3$

$y_h = c_1 e^{2t} + c_2 e^{3t}$

$f = 2e^t$

$$W = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = 1\,e^{5t}$$

$$u_1 = \int \frac{-2e^t e^{3t}}{e^{5t}}\,dt = -2\int e^{-t}\,dt$$
$$= 2e^{-t} + c_1$$
$$u_2 = \int \frac{2e^t e^{2t}}{e^{5t}}\,dt = 2\int e^{-2t}\,dt$$
$$= -e^{-2t} + c_2$$

$$y = (2e^{-t} + c_1)\,e^{2t} + (-e^{-2t} + c_2)\,e^{3t}$$
$$= c_1 e^{2t} + c_2 e^{3t} + 2e^t - e^t$$
$$= \underbrace{c_1 e^{2t} + c_2 e^{3t}}_{y_h} + \underbrace{e^t}_{y_p}$$

3.7 #1, 3, 5, 9, 13, 15, 17, 20, 22*

3) $y'' + 2y' + y = 3e^{-t}$

$y_h = e^{rt}$

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0 \quad r = -1, -1$

$y_h = c_1 e^{-t} + c_2 t e^{-t}$

A) Undetermined Coeff

$y_p = t^2 A e^{-t} = A t^2 e^{-t}$

$$\begin{cases} y_p = A t^2 e^{-t} \\ 2y_p' = 2A\,[2te^{-t} - t^2 e^{-t}] \\ y_p'' = A\,[2e^{-t} - 2(2t)e^{-t} + t^2 e^{-t}] \end{cases}$$

$$L y_p = 2Ae^{-t} = 3e^{-t}$$
$$A = \frac{3}{2}$$
$$y = c_1 e^{-t} + c_2 t e^{-t} + \frac{3}{2} t^2 e^{-t}$$

B) Variation of Parameters

$f = 3e^{-t}$

$y_1 = e^{-t}$

$y_2 = te^{-t}$

$$W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} = e^{-2t}$$

$$y = u_1 y_1 + u_2 y_2$$

$$y_p = e^{-t}\int \frac{-3e^{-t}\,te^{-t}}{e^{-2t}}\,dt + te^{-t}\int \frac{3e^{-t}e^{-t}}{e^{-2t}}\,dt$$
$$= e^{-t}\int -3t\,dt + te^{-t}\int 3\,dt$$
$$= -\frac{3}{2} t^2 e^{-t} + 3t^2 e^{-t}$$
$$y_p = \frac{3}{2} t^2 e^{-t}$$

$$y = y_h + y_p = c_1 e^{-t} + c_2 t e^{-t} + \frac{3}{2} t^2 e^{-t}$$

Last time:

Given: $Ly = y'' + P(x)y' + q(x)y = f(x)$

$Ly_1 = 0$

1) Reduction of order (Abel's)

$W = Ae^{-\int P}$

$y_2 = y_1 \int \frac{W}{y_1^2}\,dx$

2) $y = c_1y_1 + c_2y_2$ $\quad c_1 = \int -\frac{y_2 f}{W}\,dx$

Var. of Par $\quad c_2 = \int \frac{y_1 f}{W}\,dx$

— o —

Pb: 178 # 3

$y'' - 2y' - 3y = -3te^{-t}$

Let $y = e^{rt}$

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$r = 3, -1 \quad y_1 = e^{3t}, \quad y_2 = e^{-t}$

$W = -4e^{2t}$

$c_1 = \int \frac{-e^{-t}(-3te^{-t})}{-4e^{2t}}\,dt$

$= -\frac{3}{4}\int \frac{te^{-2t}}{e^{2t}}\,dt = -\frac{3}{4}\int te^{-4t}\,dt$

$\begin{array}{rr|l} + & t & e^{-4t} \\ - & 1 & -\frac{1}{4}e^{-4t} \\ + & 0 & \frac{1}{16}e^{-4t} \end{array}$

$c_1 = -\frac{3}{4}\left[-\frac{1}{4}te^{-4t} - \frac{1}{16}e^{-4t}\right] + a_1$

$c_1 = \frac{3}{16}e^{-4t}\left[t + \frac{1}{4}\right] + a_1$

$c_2 = \int \frac{te^{3t}(-3e^{-t})}{-4e^{2t}}\,dt$

$c_2 = \frac{3}{4}\int t\,dt = \frac{3}{4}\frac{t^2}{2} + a_2$

$y = \left[\frac{3}{16}e^{-4t}(t+\frac{1}{4}) + a_1\right]e^{3t} + \left[\frac{3}{4}\frac{t^2}{2} + a_2\right]e^{-t}$

$y = a_1e^{3t} + a_2e^{-t} + \frac{3}{16}e^{-t}(t+\frac{1}{4}) + \frac{3}{4}\cdot\frac{1}{2}t^2e^{-t}$

$y = a_1e^{3t} + a_2e^{-t} + e^{-t}\left[\frac{3}{8}t^2 + \frac{3}{16}t + \frac{3}{64}\right]$

$y = a_1e^{3t} + a_2e^{-t} + \frac{3}{64}e^{-t}\left[8t^2 + 6t + 1\right]$

3.7 * 5, 9, 13, 15, 17, 20, 22*

184 # 7

$y'' + 4y' + 4y = t^{-2}e^{-2t}$

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0 \quad r = -2$

$y_1 = e^{-2t} \quad y_2 = te^{-2t}$

$W = Ae^{-4t}$

$W = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & -2te^{-2t} + e^{-2t} \end{vmatrix}$

$= -2te^{-4t} + e^{-4t} + 2te^{-4t}$

$W = e^{-4t}$

$c_1 = \int \frac{-te^{-2t}t^{-2}e^{-2t}}{e^{-4t}}\,dt$

$c_1 = \int -t^{-1}\,dt = -\int \frac{1}{t}\,dt =$

$c_1 = -\ln t + a_1$

$c_2 = \int \frac{e^{-2t}t^{-2}e^{-2t}}{e^{-4t}}\,dt$

$c_2 = \int t^{-2}\,dt = -\frac{1}{t} + a_2$

$y = c_1y_1 + c_2y_2$

$= (-\ln t + a_1)e^{-2t} + (-\frac{1}{t} + a_2)te^{-2t}$

$y = a_1e^{-2t} + a_2te^{-2t} + e^{-2t}(-\ln t - \frac{t}{t})$

$y = a_1e^{-2t} + a_2te^{-2t} - e^{2t}(1 + \ln t)$

Mass Spring Systems


m

$F = -kx \qquad k = 8.8\,\frac{N}{m}$

$m\cdot\frac{d^2x}{dt^2} = -kx$

$m = 0.25$

$m\frac{d^2x}{dt^2} + kx = 0$

$x = e^{rt}$

$mr^2 + k = 0$

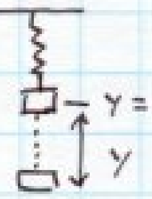
$r^2 = -\frac{k}{m} \qquad r = \pm i\sqrt{\frac{k}{m}}$

$x = A\cos\sqrt{\frac{k}{m}}\,t + B\sin\sqrt{\frac{k}{m}}\,t$

$= A\cos\omega t + B\sin\omega t$

$\omega = \sqrt{\frac{k}{m}} \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

Damped Mass / Spring System



$F_1 = -ky$

$F_2 = \beta \frac{dy}{dt}$

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$

$y = e^{rt} \qquad mr^2 + \beta r + k = 0$

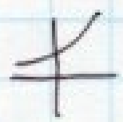
$$r = \frac{-\beta \pm \sqrt{\beta^2 - 4km}}{2m}$$

$$r = -\frac{\beta}{2m} \pm \sqrt{\frac{\beta^2 - 4km}{4m^2}}$$

$$r = -\frac{\beta}{2m} \pm \sqrt{\frac{\beta^2}{4m^2} - \frac{k}{m}}$$

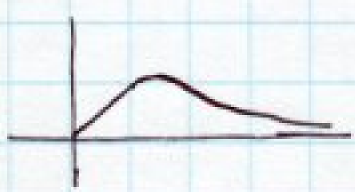
Case I $\quad \beta^2 - 4km > 0$

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$



Case II $\quad \beta^2 - 4km = 0$

$$y = c_1 t e^{-\frac{\beta}{2m}t} + c_2 e^{-\frac{\beta}{2m}t}$$



Critically Damped

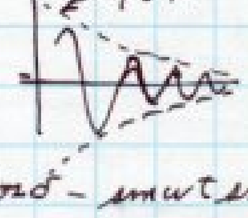
Case III $\quad \beta^2 - 4km < 0$

$$y = e^{-\frac{\beta}{2m}t}\left[c_1 \cos\sqrt{-\frac{\beta^2}{4m^2} + \frac{k}{m}}\,t + c_2 \sin\sqrt{-\frac{\beta^2}{4m^2} + \frac{k}{m}}\,t\right]$$

$$y = e^{-\frac{\beta}{2m}t}\left[c_1 \cos\omega t + c_2 \sin\omega t\right]$$

$$\omega = \sqrt{-\frac{\beta^2}{4m^2} + \frac{k}{m}} = \sqrt{\frac{k}{m} - \frac{\beta^2}{4m^2}}$$

$$\boxed{y = A e^{-\frac{\beta}{2m}t}\cos(\omega t + \delta)}$$



$$A\cos(\omega t + \delta) = A(\cos\omega t\cos\delta - \sin\omega t \sin\delta)$$
$$= \underbrace{(A\cos\delta)}_{c_1}\cos\omega t - \underbrace{A\sin\delta}_{c_2}\sin\omega t$$

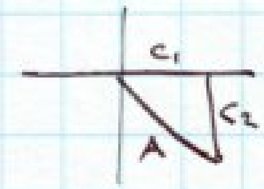
$c_1 = A\cos\delta$

$c_2 = -A\sin\delta$

$A = \sqrt{c_1^2 + c_2^2}$

$-\tan\delta = c_2/c_1$

$\delta = -\tan^{-1}\left(\frac{c_2}{c_1}\right)$



—o—

Forced Harmonic Oscillator

$$my'' + \beta y' + ky = F_0\cos\omega t$$

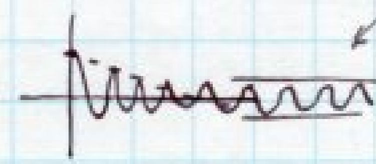
$$y_h = A e^{-\frac{\beta}{2m}t}\cos(\omega_N t - \delta)$$

$$\omega_N = \sqrt{\frac{k}{m} - \frac{\beta^2}{2m}}$$

Case 1 $\quad \omega \neq \omega_N$

$$y_p = B_1\cos\omega t + B_2\sin\omega t$$

steady state

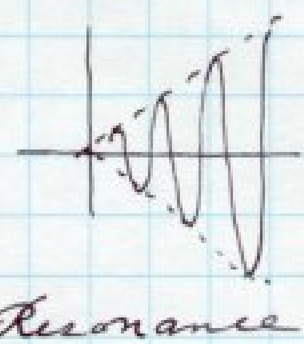


Case 2 $\quad \omega = \omega_N$

$\beta = 0$

$$y_h = A\cos(\omega t - \delta)$$

$$y_p = t(B_1\cos\omega t + B_2\sin\omega t)$$



Resonance

Case 3 – General

$$my'' + \beta y' + ky = F_0\cos\omega t$$

$$y_p = A\cos\omega t + B\sin\omega t$$

Recall $\quad e^{i\omega t} = \cos\omega t + i\sin\omega t$

$y_p = Ae^{i\omega t}$ – Real Part

$$my'' + \beta y' + ky = F_0 \operatorname{Re}(e^{i\omega t})$$

$$-mA\omega^2 + \beta A i\omega + kA = F_0$$

$$A = \frac{F_0}{(k - m\omega^2) + i\beta\omega}$$

Forced Damped Oscillator

$$my'' + \beta y' + ky = F_0 \cos \omega t$$

$$m\ddot{y} + \beta \dot{y} + ky = F_0 \cos \omega t$$

$$y_h = e^{rt}$$

$$mr^2 + \beta r + k = 0$$

$$r = \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m}$$

Assume

$$r = -\frac{\beta}{2m} \pm \sqrt{\frac{\beta^2}{4m^2} - \frac{k}{m}} \qquad \frac{k}{m} > \frac{\beta^2}{4m^2}$$

$$r = -\frac{\beta}{2m} \pm i\sqrt{\frac{k}{m} - \frac{\beta^2}{4m^2}} \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

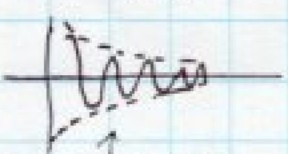
$$y_h = e^{-\frac{\beta}{2m}t}\left[c_1 \cos\sqrt{\frac{k}{m} - \frac{\beta^2}{4m^2}}\; t + c_2 \sin\sqrt{\frac{k}{m} - \frac{\beta^2}{4m^2}}\; t\right]$$

Note

$$A\cos(\alpha t + \delta) = A[\cos \alpha t \cos\delta - \sin\alpha t \sin\delta]$$

$$y_h = e^{-\frac{\beta}{2m}t} \cdot A\cos\left[\sqrt{\frac{k}{m} - \frac{\beta^2}{4m^2}}\; t - \delta\right]$$

$$\begin{cases} c_1 = A\cos\delta \\ c_2 = A\sin\delta \end{cases}$$



Transient sol.

Particular solution

$$my'' + \beta y' + ky = F_0 e^{i\omega t}$$

$$my'' + \beta y' + ky = F_0 e^{i\omega t} \quad \boxed{Re}$$

$$y_p = Be^{i\omega t}$$

$$\begin{cases} k y_p = kBe^{i\omega t} \\ \cdot \beta y_p' = B(i\omega)e^{i\omega t} \cdot \beta \\ m y_p'' = -B\omega^2 e^{i\omega t} \cdot m \end{cases}$$

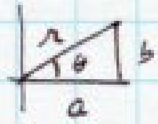
$$L y_p = Be^{i\omega t}[k + i\beta\omega - m\omega^2] = F_0 e^{i\omega t}$$

$$B = \frac{F_0}{(k - m\omega^2) + i\beta\omega}$$

$$y_p = \frac{F_0}{(k - m\omega^2 + i\beta\omega)} e^{i\omega t}$$

Recall

$$z = a + ib = re^{i\theta} = r\cos\theta + ir\sin\theta$$



$$z = \underbrace{(k - m\omega^2)}_{a} + i\underbrace{\beta\omega}_{b}$$

$$z = \sqrt{(k - m\omega^2)^2 + \beta^2\omega^2}\; e^{i\theta}$$

$$\theta = \tan^{-1}\left[\frac{k\omega}{k - m\omega^2}\right]$$

$$y_p = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \beta^2\omega^2}} e^{i\omega t} \cdot e^{-i\theta}$$

$$y_p = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \beta^2\omega^2}} \cos(\omega t - \theta)$$

$$y_p = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \beta^2\omega^2}} \cos(\omega t - \theta)$$

Infinite Series Solutions

Power Series:

1. A power series is a function of the form
$$f(x)=\sum_{n=0}^{\infty} a_n (x-c)^n$$

Examples:

a) $f(x)=e^x=\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1+x+\frac{x^2}{2!}+\dots$

b) $f(x)=\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x-\frac{x^3}{3!}+\dots$

c) $f(x)=\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1-\frac{x^2}{2!}+\dots$

d) $f(x)=\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x+\frac{x^3}{3!}+\dots$

e) $f(x)=\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots$

f) $f(x)=\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1+x+x^2 \quad \boxed{|x|<1}$

g) $f(x)=-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$

Note

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1-x+x^2-x^3+\dots$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1-x^2+x^4-x^6+\dots$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x-\frac{x^3}{3}+\frac{x^5}{5}-\dots$$

$$\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\dots$$

If $f(x)$ is analytic $\Rightarrow$ ~~sold~~
$$f(x)=\sum_{n=0} \frac{f^n(0)}{n!} x^n$$

Analytic $\Rightarrow$ infinitely diff

$f(x)$ is analytic at $x=c$ if $\exists\, a_n$ such that
$$f(x)=\sum_{n=0} a_n (x-c)^n$$

Convergence:

Ratio test

$s=\sum A_n$

Let $\rho = \lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| \begin{cases} <1 & C \\ >1 & D \\ =1 & ? \end{cases}$

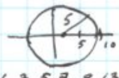
Example
$$f(x)=\sum_{n=0}^{\infty} x^n$$
$$\rho = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| = |x|<1 \quad C$$

Example:
$$f(x)=\sum^{\infty} \frac{(x-2)^n}{5^n}$$
$$\rho=\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{5^{n+1}}\cdot\frac{5^n}{(x-2)^n}\right|<1$$
$$=\left|\frac{(x-2)}{5}\right|<1$$

$|x-2|<5$

$-5<x-2<5$

$-3<x<7$



$5=$ radius of convergence

5.1 # 1, 3, 5, 7, 9, 13, 15

Shifting Indices:

Example:

1) $\sum_{k=1}^{\infty} k^2 = 1^2+2^2+3^2+\dots$

$\sum_{n=1}^{\infty} n^2 = 1^2+2^2+3^2$

2) Given $\sum_{k=1} k^2 = S$

Let $n=k-2$

$S=\sum_{n=-1}^{\infty}(n+2)^2 = 1^2+2^2+3^2$

$$\sum_{k=1}^{\infty} k^2 = \sum_{n=-1}^{\infty}(n+2)^2$$
$$=\sum_{k=-1}(k+2)^2$$
$$=\sum_{k=2}(k-1)^2$$

$$\sum_{k=1}^{\infty} k^3-k = \sum_{k=-2}^{\infty}(k+3)^3-(k+3)$$

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=3} a_{n-2} x^{n-2}$$

Infinite Series Solutions

Example:

$y'' - y = 0$

Let $y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$

$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$

$\underline{y} = \underline{\sum_{n=0}^{\infty} a_n x^n}$

$y'' - y = \sum_{n=0} n(n-1) a_n x^{n-2} - \sum_{n=0} a_n x^n$

$y'' - y = \sum_{n=-2} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0} a_n x^n$

$= \sum_{n=0} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0} a_n x^n$

$= \sum_{n=0} [(n+2)(n+1) a_{n+2} - a_n] x^n = 0$

$(n+2)(n+1) a_{n+2} - a_n = 0$

$$\boxed{a_{n+2} = \frac{a_n}{(n+2)(n+1)}} \quad n = 0, 1, 2, 3 \ldots$$

$n = 0 \quad a_2 = \frac{a_0}{2 \cdot 1}$

$n = 2 \quad a_4 = \frac{a_2}{4 \cdot 3} = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} a_0$

$a_4 = \frac{1}{4!} a_0$

$n = 4 \quad a_6 = \frac{1}{6!} a_0$

$y = \sum_{n=0} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$y = [a_0 + \frac{a_0}{2!} x^2 + \frac{a_0}{4!} x^4 + \cdot\,] + [a_1 + \cdots]$

$y = a_0 [1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots] + a_1 [x + \frac{x^3}{3!} + \cdot$

$y = a_0 \cosh x + a_1 \sinh x$

$\cosh x = \frac{e^x + e^{-x}}{2} \quad \sinh x = \frac{e^x - e^{-x}}{2}$

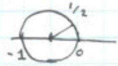
5) $f(x) = \sum \frac{(2x+1)^n}{n^2}$

$\rho = \lim_{n \to \infty} \left| \frac{(2x+1)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(2x+1)^n} \right|$

$= \lim_{n \to \infty} \left| (2x+1) \frac{n^2}{(n+1)^2} \right|$

$= \quad |2x+1| < 1$

$|x + \frac{1}{2}| < \frac{1}{2}$



$-1 < 2x+1 < 1$

$-2 < 2x < 0 \qquad |x - \rho| < a$

$-1 < x < 0$

13) $f(x) = \ln x \quad x = 1 \quad c = 1$

$f(x) = \sum_{n=0} \frac{f^{n}(c)\,(x-c)^n}{n!}$

$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$

$f(x) = \ln x \qquad f(1) = 0$

$f'(x) = x^{-1} \qquad f'(1) = 1$

$f''(x) = -x^{-2} \qquad f''(1) = -1$

$f'''(x) = 2x^{-3} \qquad f'''(1) = 2$

$f^{iv}(x) = -3 \cdot 2 x^{-4} \qquad f^{iv}(1) = -3!$

$\ln x = (x-1) + \frac{1}{2}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \cdots$

$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$

Ex

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdot$

$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} \cdots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$

$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} \cdots$

Infinite Series Solutions (cont)

Prob 13, 5§2

$$2y'' + xy' + 3y = 0 = Ly$$

$$y = \sum_{n=0} a_n x^n$$

$$\begin{cases} 3y = \sum_{n=0} 3a_n x^n \\ xy' = \sum_{n=0} n a_n x^{n-1}\cdot x \\ 2y'' = \sum_{n=0} n(n-1) a_n x^{n-2}\cdot 2 \end{cases}$$

$$\begin{cases} 3y = \sum_{n=0} 3a_n x^n \\ xy' = \sum_{n=0} n a_n x^n \\ 2y'' = 2\sum_{n=0} (n+2)(n+1) a_{n+2} x^n \end{cases}$$

$$Ly = \sum_{n=0} [(3+n)a_n + 2(n+2)(n+1)a_{n+2}]x^n = 0$$

$$(n+3)a_n + 2(n+1)(n+2)a_{n+2} = 0$$

$$a_{n+2} = \frac{-(n+3)}{2(n+1)(n+2)} a_n$$

$n=0$ $\quad a_2 = \frac{-3}{2(1\cdot 2)} a_0 = \frac{-3}{2\cdot 2!} a_0$

$n=2$ $\quad a_4 = \frac{-5}{2(3\cdot 4)} a_2 = \frac{(-5)(-3)}{2^2\, 4!} a_0$

$n=4$ $\quad a_6 = \frac{-7}{2(5\cdot 6)} a_4 = \frac{(-7)(-5)(-3)}{2^3\, 6!} a_0$

$$a_{2k} = \frac{(-1)^k\, 3\cdot 5\cdot 7\cdots(2k+1)}{2^k (2k)!} a_0$$

$$y_1(x) = \sum_{k=0} \frac{(-1)^k\, 3\cdot 5\cdot 7\cdots(2k+1)}{2^k(2k)!} x^{2k}$$

$$y_1 = \sum_{k=0} (-1)^k \frac{(1)(2\cdot 4\cdot 6\cdots 2k)\,3\cdots(2k+1)\, x^{2k}}{2^k(2k)!\,(2\cdot 4\cdot 6\cdots 2k)}$$

$$= \sum \frac{(-1)^k\,(2k+1)!}{2^k(2k)!\,2^k\cdot k!} x^{2k}$$

$$y_1 = \sum_{k=0} \frac{(-1)^k\,(2k+1)!}{2^{2k}(2k)!\,k!} x^{2k}$$

Regular Points.

Problem: Solve

$$a(x)y'' + b(x)y' + c(x)y = 0$$

$$(*) \quad y'' + p(x)y' + q(x)y = 0$$

$$p = \frac{b}{a} \qquad q = \frac{c}{a}$$

$x = c$ is called a regular point if $p(x)$ and $q(x)$ are analytic at c

ie: $p(x) = \sum a_n (x-c)^n$

$\quad q(x) = \sum b_n (x-c)^n$

Then $y = \sum A_n (x-c)^n$ will yield solutions to $(*)$

Example:

$$(1-x^2)y'' - 2xy' + ky = 0$$

$$p = -\frac{2x}{1-x^2} \qquad q = \frac{k}{1-x^2}$$

$x=0$ is a regular point $\Rightarrow y = \sum a_n x^n$

$x = \pm 1$ are not regular points

Don't try $y = \sum a_n (x-1)^n$

$\quad y = \sum a_n (x+1)^n$

Solve 5.3 # 22 (Legendre)

$$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$$

Let $y = \sum_{n=0} a_n x^n$

$$\begin{cases} \ell(\ell+1)y = \sum_{n=0} a_n \ell(\ell+1)x^n \\ -2xy' = \sum_{n=0} n a_n x^{n-1}(-2x) \\ (1-x^2)y'' = \sum_{n=0} n(n-1)a_n x^{n-2}(1-x^2) \end{cases}$$

$$Ly = \sum_{n=0} a_n \ell(\ell+1)x^n + \sum_{n=0} -2n a_n x^n + \sum_{n=0} n(n-1)a_n x^{n-2} + \sum_{n=0} -n(n-1)a_n x^n$$

$$Ly = \sum_{n=0} a_n[\ell(\ell+1) - 2n - n(n-1)]x^n + \sum_{n=0} (n+2)(n+1)a_{n+2} x^n$$

$(n+2)(n+1)\, a_{n+2} + [\ell(\ell+1) - 2n - n^2 + n]\, a_n = 0$

$(n+2)(n+1)\, a_{n+2} + [\ell(\ell+1) - n^2 - n]\, a_n = 0$

$(n+2)(n+1)\, a_{n+2} + [\ell(\ell+1) - (n)(n+1)]\, a_n = 0$

- Note:

If ℓ = integer, then the sum terminates at $n = \ell$

→ Get a polynomial of degree equals to ℓ.

- Quantization

$(n+2)(n+1)\, a_{n+2} + [\ell^2 - n^2 + \ell - n]\, a_n = 0$

$(n+2)(n+1)\, a_{n+2} + [(\ell+n)(\ell-n) + (\ell-n)]\, a_n = 0$

$(n+2)(n+1)\, a_{n+2} + [(\ell-n)(\ell+n+1)]\, a_n = 0$

$$\boxed{a_{n+2} = -\frac{(\ell-n)(\ell+n+1)}{(n+2)(n+1)}\, a_n}$$

— o —

$n=0 \qquad a_2 = \frac{-\ell(\ell+1)}{2!}\, a_0$

$n=2 \qquad a_4 = \frac{-(\ell-2)(\ell+3)}{4\cdot 3}\, a_2$

$a_4 = \frac{(-1)^2\, \ell(\ell-2)\cdot(\ell+1)(\ell+3)}{4!}\, a_0$

Suppose $\ell = 0 \Rightarrow a_2 = 0,\ a_4 = 0$

$y = 1a_0 \qquad P_0 = 1$

$\ell = 2$

$a_2 = \frac{-2(3)a_0}{2!} = -3a_0$

$a_4 = 0$

$y = a_0 - 3a_0 x^2 = a_0(1-3x^2)$

$P_2 = (1-3x^2)$

$\ell = 4$

$a_2 = -\frac{4(5)}{2!}\, a_0 = -10\, a_0$

$a_4 = \frac{(4)(2)(5)(7)}{4!}\, a_0 = \frac{35}{3}\, a_0$

$y = a_0 - 10a_0 x^2 + \frac{35}{3} a_0 x^4$

$= a_0 \left[1 - 10x^2 + \frac{35}{3} x^4\right]$

Summary

$P_0 = 1$

$P_2 = 1 - 3x^2$

$P_4 = 1 - 10x^2 + \frac{35}{3} x^4$

Graph $\boxed{x = \cos\theta}$

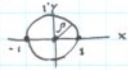
Radius of Convergence

$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$

$p(x) = \frac{-2x}{1-x^2} \qquad q(x) = \frac{\ell(\ell+1)}{1-x^2}$

$x^2 = 1$

$x = \pm 1$



$y = \sum a_n x^n$ Taylor Series about $x = 0$

$\rho = 1 \qquad |x| < 1$

$x = \pm 1$ are called singular points

Example:

Given $(1+x^2)y'' - 5y' + xy = 0$

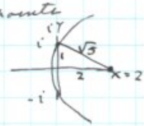
Find the radius of convergence of $y = \sum a_n (x-2)^n$

$P = \frac{-5}{1+x^2} \qquad q = \frac{x}{1+x^2}$

$x^2 = -1$

$x = \pm i$ - singular points

- poles



$|x-2| < \sqrt{5}$

Example:

1) $x^2y'' + 3xy' - 5 = 0$

2) $x^2y'' + xy' + (4-x^2)y = 0$

1) Euler Eq $\quad p = \frac{3}{x}, \; q = \frac{-5}{x^2}$

2) Bessel Eq $\quad p = \frac{1}{x}, \; q = \frac{4-x^2}{x^2}$

Here $y = \sum a_n x^n$ will not do.

Next. Regular Singular Points.

Poles:

Example:

$f(x) = \frac{\sin x}{x^3}$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$

$\frac{1}{x^3}\sin x = \frac{1}{x^2} - \frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} \dots$

$f(x) = \frac{a_{-2}}{x^2} + \frac{a_{-1}}{x} + a_0 + a_1 x + a_2 x^2 + \quad \boxed{\text{Laurent}}$

Has a pole of order 2 at $x=0$

$f(x) = x^p \sum_{n=0}^{\infty} a_n x^n \quad \boxed{p = -2}$

Def Given

$y'' + p(x)y' + q(x)y = 0$

$x = c$ is called a regular singular point if

i) $p(x)$ has a pole of order 1 or less

ii) $q(x)$ has a pole of order 2

at $x = c$

Example: (Bessel)

$x^2y'' + xy' + (n^2 - x^2)y = 0$

$p(x) = \frac{1}{x}$ pole of order 1

$q(x) = \frac{n^2 - x^2}{x^2}$ pole of order 2

at $x=0$. $x=0$ is a r.s.p.

Equivalent Definition

$x=c$ is a r.s.p of

$y'' + p(x)y' + q(x)y = 0$

if i) $\lim_{x\to c} (x-c)p(x)$ exists

ii) $\lim_{x\to c} (x-c)^2 q(x)$ exists

Examples.

$(1-x^2)y'' - 2xy' + ky = 0$

$p(x) = \frac{-2x}{1-x^2} \quad q(x) = \frac{k}{1-x^2}$

at $x=1$ ① $\lim_{x\to 1} (x-1)p(x) = $

$= \lim_{x\to 1} \frac{(x-1)(-2x)}{(1-x)(1+x)} = 1$ ✓

ii) $\lim_{x\to 1} (x-1)^2 q(x) = 0$

$x=1$ is a r.s.p.

Solving O.D.E. w/ r.s.p.

Prototype: (Euler)

$ax^2y'' + bxy' + cy = 0$

Reg. Sing. Pt at $x=0$

5.5 # 13

① $2x^2y'' + xy' - 3y = 0$

Let $y = x^p$

$-3y = -3x^p$

$xy' = xpx^{p-1}$

$2x^2y'' = 2p(p-1)x^{p-2}\cdot x^2$

$L y = x^p[-3 + p + 2p(p-1)] = 0$

$2p(p-1) + p - 3 = 0 \quad$ Indicial

$2p^2 - 2p + p - 3 = 0$

$2p^2 - p - 3 = 0 \quad p = -1$

$(2p-3)(p+1) = 0 \quad p = 3/2$

$y = c_1 x^{-1} + c_2 x^{3/2}$

②

~~Re~~ Last time: $y'' + P(x)y' + q(x)y = 0$

1) $x = c$ is a regular point
$y = \sum a_n (x-c)^n$ yields sol.

2) $x = c$ is a regular singular point
$y = x^p \sum a_n (x-c)^n$ Yields sol

3) Special case of (2) is
$ax^2y'' + bxy' + cy = 0$
$P = \frac{b}{ax}$, $q = \frac{c}{ax^2}$ $x = 0$ r.s.p

$y = x^p$ - Euler Eq.

Today: Euler Equations

Given $ax^2y'' + bxy' + cy = 0$
Let $y = x^p = x^r$

$Ly = x^r[ar(r-1) + br + c] = 0$
$F(r) = ar^2 - ar + br + c = 0$
$= ar^2 + (b-a)r + c = 0$

Case 1 $r = r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$
$y = c_1 x^{r_1} + c_2 x^{r_2}$

Case 2 $r = r_1, r_1$ $r_1 \in \mathbb{R}$
$y = c_1 x^{r_1} + c_2 x^{r_1} \ln x$

Case 3 $r = \alpha \pm i\beta$
$y = c_1 x^{\alpha} \cos(\beta \ln x) + c_2 x^{\alpha} \sin(\beta \ln x)$

$x^{\alpha \pm i\beta} = x^{\alpha} \cdot x^{i\beta} = x^{\alpha} e^{\pm(\beta \ln x)i}$
$= x^{\alpha}[\cos(\beta \ln x) \pm i \sin(\beta \ln x)]$

Examples pbs 5.5

6) $(x-1)^2 y'' + 8(x-1)y' + 12y = 0$
$x = 1$ is a r.s.p
Let $y = (x-1)^r$
$12y = 12(x-1)^r$
$8(x-1)y' = 8r(x-1)^{r-1} \cdot (x-1)$
$(x-1)^2 y'' = r(r-1)(x-1)^{r-2} \cdot (x-1)^2$
$Ly = [r(r-1) + 8r + 12](x-1)^r = 0$
$r^2 - r + 8r + 12 = 0$
$r^2 + 7r + 12 = 0$
$(r+4)(r+3) = 0$ $r = -4, -3$
$y = c_1 (x-1)^{-4} + c_2 (x-1)^{-3}$

11) $x^2 y'' + 2xy' + 4y = 0$
Let $y = x^p$
$4y = 4x^p$
$2xy' = 2px^{p-1} \cdot x$
$x^2 y'' = p(p-1)x^{p-2} \cdot x^2$
$Ly = [p(p-1) + 2p + 4]x^p = 0$
$\Rightarrow p^2 + p + 4 = 0$
$p = \frac{-1 \pm \sqrt{1-16}}{2} = -\frac{1}{2} \pm \frac{\sqrt{15}}{2} i$

$y = c_1 x^{-\frac{1}{2}} \cos(\frac{\sqrt{15}}{2} \ln x) +$
$c_2 x^{-1/2} \sin(\frac{\sqrt{15}}{2} \ln x)$

Regular Singular Points (General)

5.6 # 1

$2xy'' + y' + xy = 0$
$P = \frac{1}{2x}$ $\lim_{x \to 0} (x-0)P = \frac{1}{2}$ ✓
$q = \frac{1}{2}$ $\lim_{x \to 0} (x-0)^2 q = 0$ ✓

Let $y = x^p \sum_{n=0} a_n x^n = \sum_{n=0} a_n x^{n+p}$

$xy = \sum_{n=0} a_n x^{n+p+1}$
$y' = \sum_{n=0} a_n (n+p) x^{n+p-1}$
$2xy'' = \sum_{n=0} a_n (n+p)(n+p-1) x^{n+p-2}\, 2x$

$xy = \sum_{n=2} a_{n-2} x^{n+p-1}$
$y' = \sum_{n=0} a_n x^{n+p-1}$
$2xy'' = \sum_{n=0} 2a_n (n+p)(n+p-1) x^{n+p-1}$

$Ly = a_0 + 2a_0 p(p-1) x^{p-1} +$
$a_1 + 2a_1 (p+1)p x^p +$
$\sum_{n=2} a_{n-2} + a_n[1 + 2(n+p)(n+p-1)] x^{n+p-1}$

$a_0[1 + 2p(p-1)] = 0$ Indicial
$a_1[1 + 2p(p+1)] = 0$
$a_{n-2} + a_n[1 + 2(n+p)(n+p-1)] = 0$

$2p(p-1) + 1 = 0$ $(2p \quad)(p \quad)$
$2p^2 - 2p + 1 = 0$
$p = \frac{1 \pm \sqrt{1-2}}{2} = \frac{1 \pm i}{2}$ Yikes!

1 $2xy'' + y' + xy = 0$

$x = 0$ is a r.s.p:

$$y = x^p \sum_{n=0} a_n x^n = \sum_{n=0} a_n x^{n+p}$$

$$\begin{cases} xy = \sum a_n x^{n+p+1} \\ y' = \sum_{n=0} a_n (n+p) x^{n+p-1} \\ 2xy'' = \sum_{n=0} a_n (n+p)(n+p-1) x^{n+p-2} (2x) \end{cases}$$

$$xy = \sum_{n=2} a_{n-2} x^{n+p-1}$$

$$y' = \sum_{n=0} a_n (n+p) x^{n+p-1}$$

$$2xy'' = \sum_{n=0} 2 a_n (n+p)(n+p-1) x^{n+p-1}$$

$$\begin{aligned} Ly = & [a_0 p + 2a_0 (p)(p-1)] x^{p-1} + \\ & [a_1 (p+1) + 2a_1 (p+1) p] x^p + \\ & \sum_{n=2} a_{n-2} + a_n [n+p + 2(n+p) \cdot \\ & (n+p-1)] x^{n+p-1} \qquad = 0 \end{aligned}$$

$$a_0 [p + 2p(p-1)] = 0$$

$$a_1 [(p+1) + 2p(p+1)] = 0$$

$$a_{n-2} + a_n [(n+p) + 2(n+p)(n+p-1)] = 0$$

$$p[1 + 2(p-1)] = p(2p-1) = 0$$

$$p = 0, \quad p = \tfrac{1}{2}$$

$$a_{n-2} + a_n [(n+p)(1 + 2n + 2p - 2)] = 0$$

$$a_{n-2} + a_n (n+p)(2n + 2p - 1) = 0$$

$$a_n = \frac{-a_{n-2}}{(n+p)(2n+2p-1)} \qquad p = \tfrac{1}{2}$$

$$a_n = \frac{-a_{n-2}}{(n+\frac{1}{2})(2n)} = \frac{-a_{n-2}}{(2n+1)n}$$

$n = 2 \quad a_2 = -\dfrac{a_0}{(2)5}$

$n = 4 \quad a_4 = -\dfrac{a_2}{4(9)} = \dfrac{(-1)^2 a_0}{(2 \cdot 4)(5 \cdot 9)}$

$n = 6 \quad a_6 = \dfrac{-a_4}{6(13)} = \dfrac{(-1)^3 a_0}{(2 \cdot 4 \cdot 6)(5 \cdot 9 \cdot 13)}$

$$y_1 = a_0 \left[1 - \frac{1}{2 \cdot 5} x^2 + \frac{(-1)^2 x^4}{(2 \cdot 4)(5 \cdot 9)} + \text{etc} \right)$$

Do 5.6 # 1, 3, 4, 5

Laplace Transforms
(Heavyside)

1) $\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$

Ex
$\mathcal{L}[\sin at] = \int_0^\infty e^{-st} \sin at\, dt$

$\int e^{-st} \sin at\, dt = -e^{-st} \frac{[a\cos at + s\sin at]}{s^2+a^2}$

$\int_0^\infty e^{-st} \sin at\, dt = 0 + \frac{a}{s^2+a^2}$

Build table of Laplace Transforms

2) $\mathcal{L}[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt$

Let $u = e^{-st}$ $\quad du = -s e^{-st}$

$dv = f'(t)dt \quad v = f(t)$

$\mathcal{L}[f'(t)] = e^{-st} f(t)\Big]_0^\infty + \int_0^\infty s e^{-st} f(t)\, dt$

$= -f(0) + \mathcal{L}(s f(t))$

$\mathcal{L}[f'(t)] = s\mathcal{L}(f(t)) - f(0)$

$\mathcal{L}[f''(t)] = s^2\mathcal{L}(f(t)) - sf(0) - f'(0)$

Example: #21 6.2

$y'' - 2y' + 2y = \cos t$

$y(0) = 1$

$y'(0) = 0$

$\begin{cases} y = f(t) \\ \mathcal{L}(f(t)) = F(s) = Y(s) \end{cases}$

$\mathcal{L}[y'' - 2y' + 2y] = \mathcal{L}(\cos t)$

$s^2 Y - s - 2(sY - 1) + 2Y = \frac{s}{s^2+1}$

$Y(s^2 - 2s + 2) - s + 2 = \frac{s}{s^2+1}$

$Y(s^2 - 2s + 2) = \frac{s}{s^2+1} + s - 2$

$Y = \frac{s}{(s^2+1)(s^2-2s+2)} + \frac{s-2}{s^2-2s+2}$

Final Exam Review.

I First Order Equations.

a) Linear

$$a(x)\,y' + b(x)\,y = c(x)$$

$$\mu = e^{\int P} \quad P = \frac{b}{a}$$

b) Separable:

$$y' = P(x)\cdot g(y)$$

$$\int \frac{1}{g}\,dy = \int P\,dx$$

c) Exact.

$$M\,dx + N\,dy = 0 \qquad y' = -\frac{M}{N}$$

$$M = f_x,\; N = f_y \Rightarrow \boxed{M_y = N_x}$$

↑ Test

—o—

1) $t\frac{dx}{dt} + 3x = \frac{e^{-t^2}}{t} \qquad x(1) = 0$

$(*)\; x' + \frac{3}{t}x = \frac{e^{-t^2}}{t^2}$

$P = \frac{3}{t} \quad \int P = 3\ln t \quad \mu = e^{3\ln t}$

$\mu = t^3$

$t^3x' + 3t^2x = te^{-t^2}$

$(t^3x)' = te^{-t^2}$

$t^3x = \int te^{-t^2}dt$

$t^3x = -\frac{1}{2}e^{-t^2} + c$

$x = -\frac{1}{2t^3}e^{-t^2} + \frac{c}{t^3}$

$0 = -\frac{1}{2}e^{-1} + c \qquad c = \frac{1}{2}e^{-1}$

$x = -\frac{1}{2t^3}e^{-t^2} + \frac{e^{-1}}{2t^3}$

—o—

2) $(x\cos xy)\,y' = -y\cos xy + 2x$

$y(1) = 2$

$x\cos xy\,\frac{dy}{dx} = -y\cos xy + 2x$

$(y\cos xy - 2x)\,dx + (x\cos xy)\,dy = 0$

$M_y = \cos xy - yx\sin xy$

$N_x = \cos xy - xy\sin xy \quad \boxed{M_y = N_x}$ ✓

$f_x = y\cos xy - 2x$

$f_y = x\cos xy \qquad df = 0$

$f = \cos xy - x^2 = C$

$\cos 2 - 1 = C$

$\cos xy - x^2 = \cos 2 - 1$

3) $T_0 = 95\,°C \quad t = 1 \quad T = 80\,°C$

$T_A = 20\,°C \quad t = 3 \quad T = ?$

$\frac{dT}{dt} = -k(T_A - T) = k(T - T_A)$

$\int \frac{1}{T - T_A}\,dt = \int k\,dt$

$\ln(T - T_A) = kt + c$

$T - T_A = Ce^{kt}$

$T = Ce^{kt} + T_A$

$95 = C + 20 \qquad C = 75$

$T = 75e^{kt} + 20$

$80 = 75e^{k} + 20$

$60 = 75e^{k}$

$\frac{60}{75} = e^{k} \qquad k = \ln\left(\frac{60}{75}\right)$

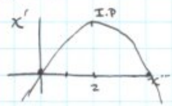
$T = 75e^{t\ln\frac{60}{75}} + 20$

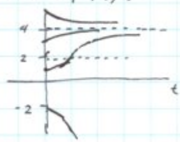
$T(3) = 75e^{3\ln(60/75)} + 20$

$= 58.4$

—o—

4) $x' = 2x(4 - x) \qquad x = -2, 1, 3, 5$





$x = 0$, Unstable

$x = 4$, Stable.

Exact sol

$\int \frac{1}{2x(4-x)}\,dx = \int dt \quad$ etc.

—o—

5) $y'' - 4y' + 4y = e^{2x} + 4$

$y_h = e^{rx} \qquad \underline{y = y_h + y_p}$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0 \quad r = 2, 2$

$\boxed{y_h = c_1e^{2x} + c_2xe^{2x}}$

$y_p = Ax^2e^{2x} + B$

$4y_p = 4Ax^2e^{2x} + 4B$

$-4y_p' = [2Axe^{2x} + 2Ax^2e^{2x}]\cdot -4$

$y_p'' = 2Ae^{2x} + 2\cdot 2Ax\cdot 2e^{2x} + 4Ax^2e^{2x}$

$Ly_p = 2Ae^{2x} + 4B = e^{2x} + 4$

$A = \frac{1}{2} \quad B = 1 \qquad \boxed{y_p = \frac{1}{2}x^2e^{2x} + 1}$

5 (6) $y''' + 4y'' + 13y' = x + 5xe^{-2x}\cos 3x$

$y_h = e^{rx}$

$r^3 + 4r^2 + 13r = 0$

$r(r^2 + 4r + 13) = 0$

$r = 0 \quad r = -2 \pm \sqrt{4-13} = -2 \pm 3i$

$y_h = c_1 + c_2 e^{-2x}[\cos 3x] + c_3 e^{-2x}\sin 3x$

$y_p = x(A_1x + A_0) + xe^{-2x}[(B_1x + B_0)\cos 3x + (C_1x + C_0)\cdot \sin 3x)$

$y_p = x(A_1x + A_0) + xe^{-2x}[(B_1x + B_0)\cos 3x + (C_1x + C_0)\sin 3x]$

— o —

6. $y'' + y = \sec x \tan x$

$y_h = c_1\cos x + c_2 \sin x$

$y_1 = \cos x \qquad W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$

$y_2 = \sin x$

$y_p = y_1u_1 + y_2u_2$

$u_1 = \int \frac{-f\,y_2}{W} = -\int \sec x \tan x \sin x\,dx$

$u_1 = -\int \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}\cdot \sin x\,dx$

$= -\int \frac{\sin^2 x}{\cos^2 x}\,dx = -\int \tan^2 x\,dx$

$= -\int (\sec^2 x - 1)\,dx$

$u_1 = -\tan x + x$

$u_2 = \int \frac{f\,y_1}{W} = \int \sec x \tan x \cos x\,dx$

$u_2 = \int \tan x\,dx = \ln|\sec x + \tan x|$

$y_p = \cos x[x - \tan x] + \sin x[\ln(\sec x + \tan x)]$

— o —

7. $xy'' - (1+x)y' + y = 0 \qquad y_1 = e^x$

$P = -\frac{1+x}{x} = -\frac{1}{x} - 1$

$y_2 = y_1 \int \frac{W}{y_1^2}$

$W = e^{-\int P} = e^{-[-\ln x - x]} = xe^x$

$y_2 = e^x \int \frac{xe^x}{e^{2x}}\,dx$

$$\begin{array}{c|c} +\ x & e^{-x} \\ -\ 1 & -e^{-x} \\ +\ 0 & e^{-x} \end{array}$$

$= e^x \int xe^{-x}\,dx$

$= e^x[-xe^{-x} - e^{-x}]$

$= -x - 1 \qquad y = c_1e^x + c_2(x+1)$

8) $x^2y'' - 2xy' + 2y = 2x^3$

$2y_h = x^p \cdot 2$

$-2xy_h' = px^{p-1}(-2x)$

$x^2y_h'' = p(p-1)x^{p-2}\cdot x^2$

$Ly_p = [2 - 2p + p(p-1)]x^p = 0$

$p^2 - 3p + 2 = 0$

$(p-2)(p-1) = 0 \quad p = 1, 2$

$y_h = c_1x + c_2x^2$

$y_1 = x \qquad W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$

$y_2 = x^2$

$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = \underbrace{2x}_{f}$!!!!

$u_1 = \int -\frac{f\,y_2}{W} = \int -\frac{2x\,x^2}{x^2}\,dx = -x^2$

$u_2 = \int \frac{f\,y_1}{W} = \int \frac{2x\cdot x}{x^2}\,dx = 2x$

$y_p = x(-x^2) + x^2(2x) = x^3$

$\boxed{y = c_1x + c_2x^2 + x^3}$

9) $y'' + 4y = 6\sin 2x$

$y(0) = y'(0) = 0$

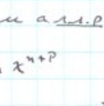
$\mathcal{L}[y'] = sY - y(0)$

$\mathcal{L}[y''] = s^2Y - sy(0) - y'(0)$

$\mathcal{L}[y'' + 4y] = \mathcal{L}[6\sin 2t]$

$s^2Y + 4Y = \frac{12}{s^2+4}$

$Y = \frac{12}{(s^2+4)^2}$



$y = -\frac{3}{2}t\cos 2t + \frac{3}{4}\sin 2t$

10) $x^2y'' + xy' + x^2y = 0$

$P = \frac{1}{x} \quad q = 1 \qquad x = 0$ is a r.s.p

$y = x^p \sum_{n=0} a_nx^n = \sum_{n=0} a_n x^{n+p}$

$x^2y = \sum_{n=0} a_n x^{n+p+2} = \sum_{n=2} a_{n-2}x^{n+p}$

$xy' = \sum_{n=0} a_n(n+p)x^{n+p-1}\cdot x$

$x^2y'' = \sum_{n=0} a_n(n+p)(n+p-1)x^{n+p-2}\cdot x^2$

$Ly = a_0[p + p(p-1)]x^p + a_1[\cdots]x^{p+1} + \sum_{n=2}[a_{n-2} + a_n(n+p)^2]x^{n+p}$

$P + P(P-1) = 0 \quad P^2 = 0 \quad P = 0$

$a_n = \frac{-a_{n-2}}{(n+P)^2} \qquad a_n = \frac{-a_{n-2}}{n^2}$

$n = 2 \qquad a_2 = -\frac{a_0}{2^2}$

$n = 4 \qquad a_4 = \frac{-1\, a_2}{4^2} = \frac{(-1)^2}{4^2 \cdot 2^2}\, a_0$

$y_1 = a_0 \left[1 - \frac{1}{2^2} x^2 + \frac{(-1)^2}{4^2 \cdot 2^2} x^4 + \cdots \right]$