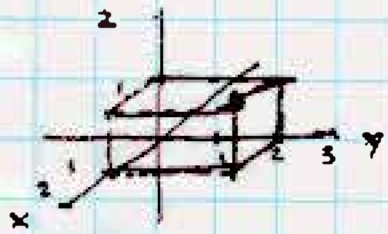
Math 261.

3-d. Coordinate system.

Points

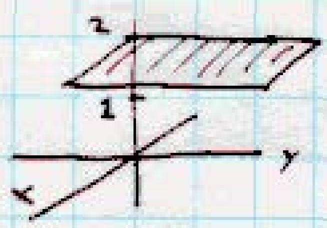
$P(1,2,2)$



Simple planes

Ex

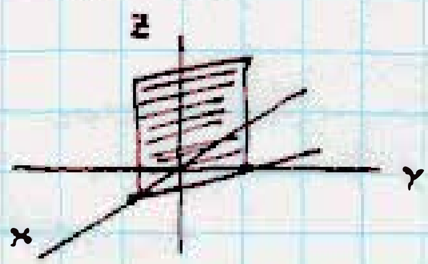
$z = 2$



Ex $\quad x + y = 1$

$y = -x + 1$

x	y
0	1
1	0



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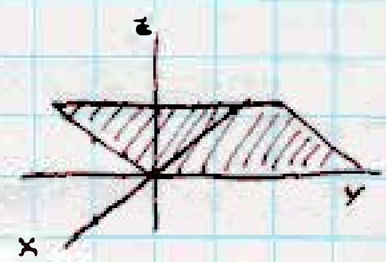


Distance formula

$P(x_1, y_1, z_1)$

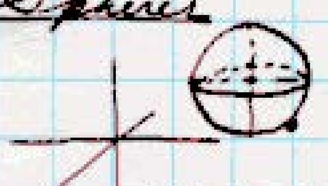
$Q(x_2, y_2, z_2)$

$$d(PQ) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$$

Spheres

~~C (h, k,~~

$C(x_0, y_0, z_0)$



$\bar{x}\,(x, y, z)$

$$\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} = R$$

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = R^2$$

Ex. Find the eq. of a sphere centered at $(0,-1,2)$ with radius 4 $\quad (x, y, z)$

Ans:

$$(x-0)^2+(y+1)^2+(z-2)^2 = 16$$

Ex Find the eq. of a sphere passing thru $(4,3,-1)$ whose center is at $(1,2,3)$

Sol a) find the radius

$$R = \sqrt{(4-1)^2+(3-2)^2+(-1-3)^2}$$
$$= \sqrt{3^2+1^2+(-4)^2}$$
$$= \sqrt{9+1+16}$$
$$= \sqrt{26}$$

b) write the eq.

$$(x-1)^2+(y-2)^2+(z-3)^2 = 26$$

Ex Given: $\qquad$ pb (16)

$$x^2+y^2+z^2 = x+y+z$$

Find the center and radius

Sol

$$(x^2-x \qquad)+(y^2-y \qquad)+(z^2-z) = 0$$

$$(x^2-x+\tfrac{1}{4})+(y^2-y+\tfrac{1}{4})+(z^2-z+\tfrac{1}{4}) = \tfrac{3}{4}$$

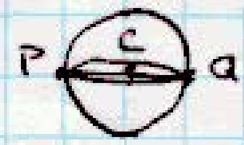
$$(x-\tfrac{1}{2})^2+(y-\tfrac{1}{2})^2+(z-\tfrac{1}{2})^2 = \tfrac{3}{4}$$

$$C(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) \qquad R = \frac{\sqrt{3}}{2}$$

Hw 13.1 # 3, 5, 8, 9, 13, 15, 31, 20

Pb 20

$P(2,1,4) \quad Q(4,3,10)$



$$C\left(\frac{2+4}{2}, \frac{1+3}{2}, \frac{4+10}{2}\right)$$

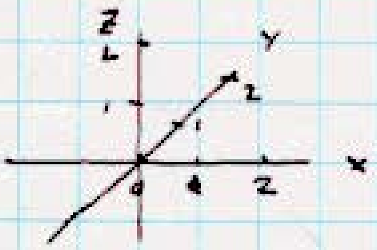
$$C(3, 2, 7)$$

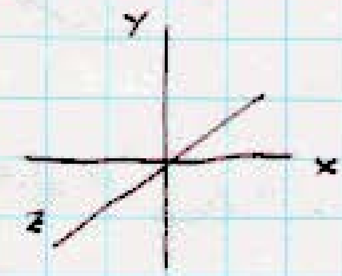
$$R = \sqrt{(3-2)^2+(2-1)^2+(7-4)^2}$$
$$= \sqrt{1+1+9} = \sqrt{11}$$

Ans $(x-3)^2+(y-2)^2+(z-7)^2 = 11$

Math 261. Fall 1999.

3-d. Coordinate system.





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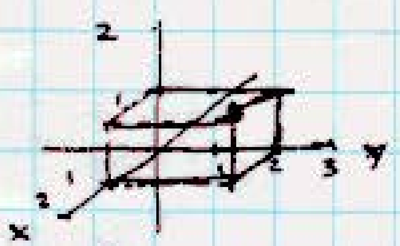

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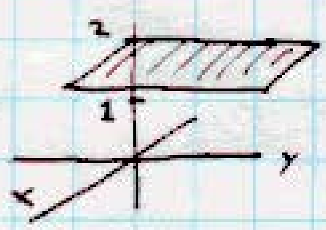

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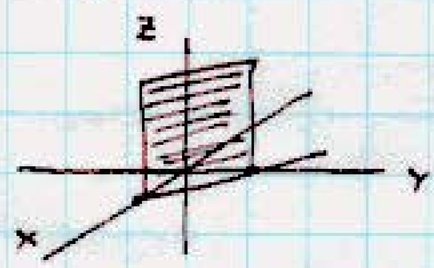

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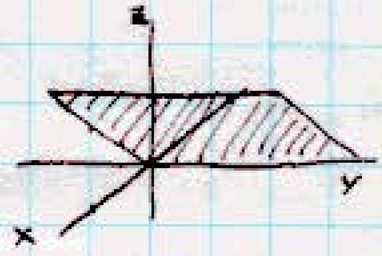
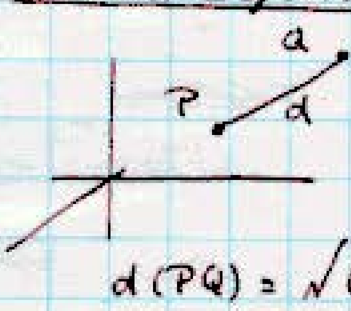


Distance formula



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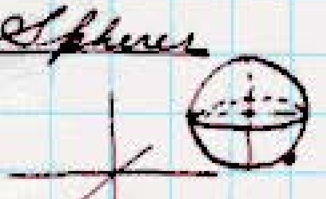
Spheres



~~$C(h, k,$~~

$C(x_0, y_0, z_0)$

$\bar{x}\,(x, y, z)$

$$\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = R$$

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$$x^2 + y^2 + z^2 = x + y + z$$

Find the center and radius

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$$(x^2 - x + \tfrac{1}{4}) + (y^2 - y + \tfrac{1}{4}) + (z^2 - z + \tfrac{1}{4}) = \frac{3}{4}$$

$$(x - \tfrac{1}{2})^2 + (y - \tfrac{1}{2})^2 + (z - \tfrac{1}{2})^2 = \frac{3}{4}$$

$$C(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) \qquad R = \frac{\sqrt{3}}{2}$$

Hw 13.1 # 3, 5, 8, 9, 13, 15, 31, 20

Pb 20

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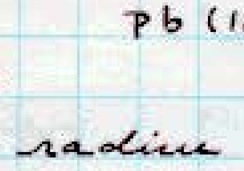


$C\left(\frac{2+4}{2}, \frac{1+3}{2}, \frac{4+10}{2}\right)$

$C(3, 2, 7)$

$$R = \sqrt{(3-2)^2 + (2-1)^2 + (7-4)^2}$$
$$= \sqrt{1 + 1 + 9} = \sqrt{11}$$

Ans $(x - 3)^2 + (y - 2)^2 + (z - 7)^2 = 11$

Vectors

$\mathbb{R}^3 = \{(a_1, a_2, a_3)\}$ a's $\in \mathbb{R}$.

Let $\vec{v}, \vec{w} \in \mathbb{R}^3$ vectors

$k \in \mathbb{R}$ scalars

Give $\mathbb{R}^3$ a Vector Space struct.
ie Define 2 operations:

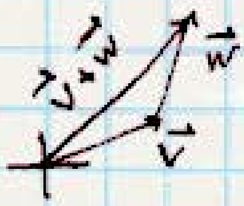
a) $\vec{v} = (a_1, a_2, a_3)$
$\vec{w} = (b_1, b_2, b_3)$
$\vec{v} + \vec{w} = (a_1+b_1, a_2+b_2, a_3+b_3)$

b) $k\vec{v} = (ka_1, ka_2, ka_3)$

Geometry

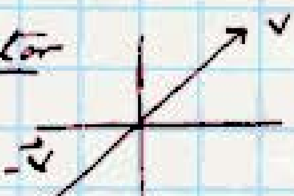
$P(a_1, a_2, a_3)$
$\vec{v} = (a_1, a_2, a_3)$

Sums:



Neg of a vector

$-V$



Ex $\vec{v} = (3, 0, 4)$
$-\vec{v} = (-3, 0, -4)$

Subtraction

$\vec{A} = (3, -1, 2)$
$\vec{B} = (-1, 2, 1)$
$\vec{A} - \vec{B} = (3-(-1), -1-(2), 2-1)$
$= (4, -3, 1)$
$4\vec{A} = (12, -4, 8)$

Picture

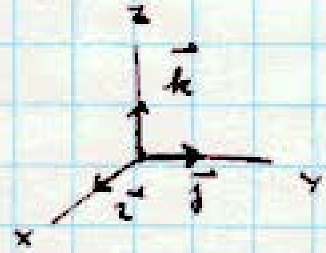
Convention

$\vec{AB}$

Ex $A = (3, 2, 0)$
$B = (1, 0, 1)$
$\vec{AB} = (-2, -2, 1)$

Unit vectors

$\vec{i} = (1, 0, 0)$
$\vec{j} = (0, 1, 0)$
$\vec{k} = (0, 0, 1)$



Ex
$\vec{v} = (3, 2, -1)$
$= 3(1,0,0) + 2(0,1,0) - 1(0,0,1)$
$\vec{v} = 3\vec{i} + 2\vec{j} - \vec{k}$

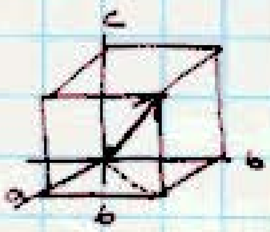
~~$\vec{v} = (a_0, a_1, a_2)$~~

$$\boxed{\begin{aligned}\vec{v} &= (a_1, a_2, a_3)\\ &= a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\end{aligned}}$$

Length (Norm) of a vector

Def $\vec{v} = (a, b, c)$

$|\vec{v}| = \sqrt{a^2+b^2+c^2}$



Unit vectors are vectors whose length is 1

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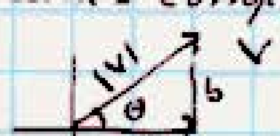
Let
$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \left(\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$

$|u| = 1$

2-dim-components $\mathbb{R}^2$



$\vec{v} = (a, b)$
$= a\vec{i} + b\vec{j}$

$a = |\vec{v}|\cos\theta$
$b = |\vec{v}|\sin\theta$

$\vec{F} = F_x\vec{i} + F_y\vec{j}$
$F_x = |\vec{F}|\cos\theta$
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$\vec{v} = (a_1, a_2, a_3)$

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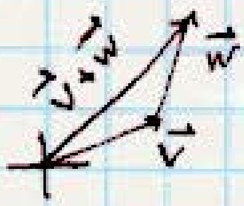

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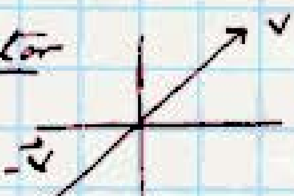

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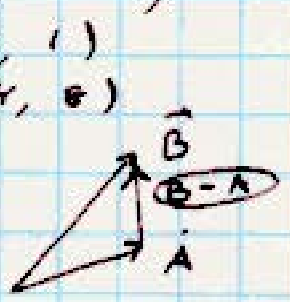

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TOPS FORM 3314

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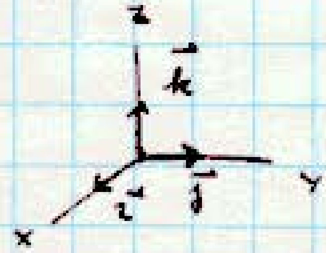
$\vec{k} = (0, 0, 1)$



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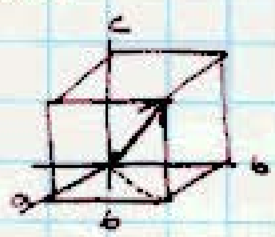

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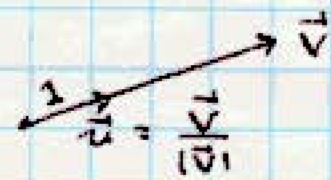
$|u| = 1$



2-dim - components $\mathbb{R}^2$

$\vec{v} = (a, b)$

$= a\vec{i} + b\vec{j}$

$a = |\vec{v}|\cos\theta$

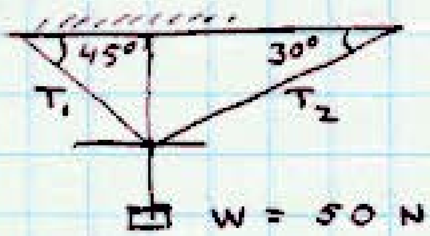
$b = |\vec{v}|\sin\theta$

$\vec{F} = F_x\vec{i} + F_y\vec{j}$

$F_x = |\vec{F}|\cos\theta$

$F_y = |\vec{F}|\sin\theta$.

Example



Find T_1, T_2

Solution

Horizontally: $T_1 \cos 45° = T_2 \cos 30°$

Vertically: $T_1 \sin 45° + T_2 \sin 30° = 50$

Solve:

$$T_1 = T_2 \frac{\cos 30°}{\cos 45°}$$

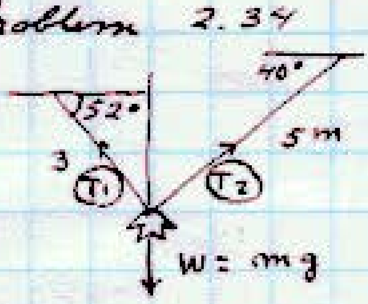
$$T_2 \cos 30° \cdot \frac{\sin 45°}{\cos 45°} + T_2 \sin 30° = 50$$

$$T_2 (\cos 30° \tan 45° + \sin 30°) = 50$$

$$T_2 = \frac{50}{\cos 30° \tan 45° + \sin 30°}$$

DO 3.2 # 1, 5, 19, 25, 33, 34

Problem 2.34



$m = 5$ kg

$g = 9.8$ m/s

$$\begin{cases} T_1 \cos 52° = T_2 \cos 40° \\ T_1 \sin 52° + T_2 \sin 40° = mg \end{cases}$$

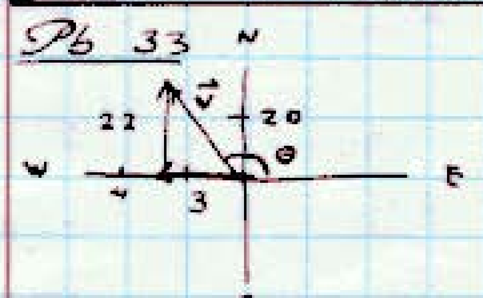
$$T_1 = T_2 \frac{\cos 40°}{\cos 52°}$$

$$T_2 \frac{\cos 40°}{\cos 52°} \cdot \sin 52° + T_2 \sin 40° = mg$$

$$T_2 [\cos 40° \tan 52° + \sin 40°] = mg$$

$$T_2 = \frac{mg}{\cos 40° \tan 52° + \sin 40°}$$

Pb 33



$V_1 = -3\vec{i} + 0\vec{j}$

$V_2 = 0\vec{i} + 22\vec{j}$

$\vec{V} = -3\vec{i} + 22\vec{j}$

$v = \sqrt{3^2 + 22^2}$

$\theta = \tan^{-1} \frac{22}{-3}$

Dot Product

Def $\vec{v} = (a^1, b^1, c^1)$

$\vec{w} = (a^2, b^2, c^2)$

$$\boxed{\vec{v} \cdot \vec{w} = a^1a^2 + b^1b^2 + c^1c^2}$$

Ex $\vec{v} = (1, 3, 0)$

$w = (2, -1, 1)$

$\vec{v} \cdot \vec{w} = 2 - 3 + 0 = -1$

Fact 1. $\boxed{\vec{v} \cdot \vec{v} = |\vec{v}|^2}$

Fact 2. $\boxed{\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos \theta}$

Example

Given $\vec{v} = (-1, 1, 0)$

$\vec{w} = (2, -1, 1)$ Find θ

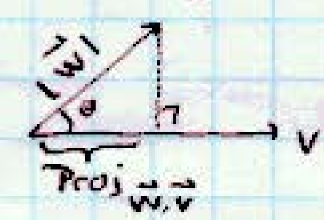
Sol $\cos \theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}| \cdot |\vec{w}|}$

$= \frac{-2 - 1}{\sqrt{2}\sqrt{6}} = \frac{-3}{\sqrt{12}}$

$\theta = \cos^{-1}\left(\frac{-3}{\sqrt{12}}\right)$

Fact 3 $\vec{v} \perp \vec{w}$ iff $\vec{v} \cdot \vec{w} = 0$

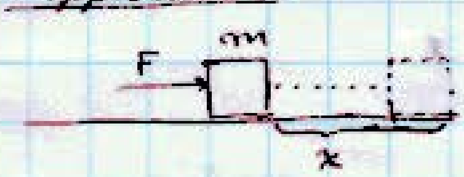
Fact 4



$\text{Proj}_{\vec{w},\vec{v}} = |\vec{w}| \cdot \cos \theta$

$$\boxed{\text{Proj}_{\vec{w},\vec{v}} = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|}}$$

Application

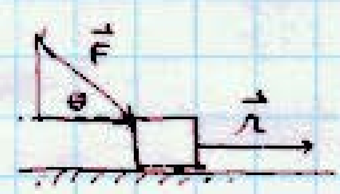


F = Newtons

x = meters

$W = F \cdot x$

W = Joules

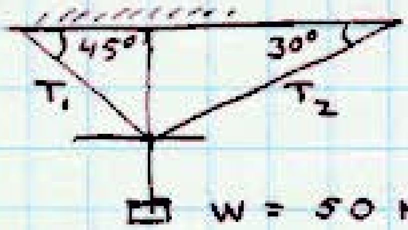


$W = |\vec{F}| \cos \theta \cdot |\vec{r}|$

$W = \vec{F} \cdot \vec{r}$ F = const.

Pb: 13.3 # 3, 9, 15, 19, 25, 30
33, 39, 43, 49, 50, 55

Example



Find $T_1,\ T_2$

Solution

Horizontally: $T_1 \cos 45^\circ = T_2 \cos 30^\circ$

Vertically: $T_1 \sin 45^\circ + T_2 \sin 30^\circ = 50$

Solve:

$$T_1 = T_2 \frac{\cos 30^\circ}{\cos 45^\circ}$$

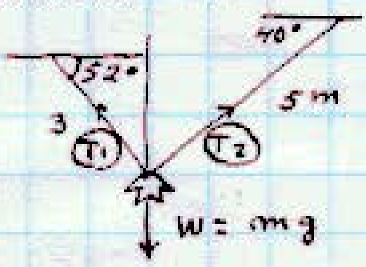
$$T_2 \cos 30^\circ \cdot \frac{\sin 45^\circ}{\cos 45^\circ} + T_2 \sin 30^\circ = 50$$

$$T_2 (\cos 30^\circ \tan 45^\circ + \sin 30^\circ) = 50$$

$$T_2 = \frac{50}{\cos 30^\circ \tan 45^\circ + \sin 30^\circ}$$

DO 3.2 # 1, 5, 19, 25, 33, 34

Problem 2.34



$m = 5$ kg

$g = 9.8$ m/s

$$\begin{cases} T_1 \cos 52^\circ = T_2 \cos 40^\circ \\ T_1 \sin 52^\circ + T_2 \sin 40^\circ = mg \end{cases}$$

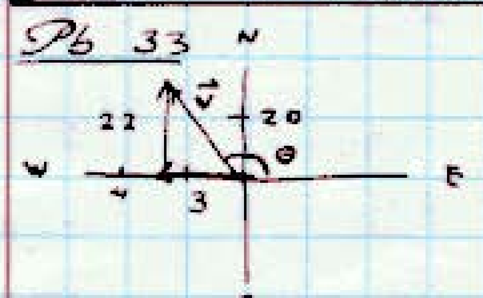
$$T_1 = T_2 \frac{\cos 40^\circ}{\cos 52^\circ}$$

$$T_2 \frac{\cos 40^\circ}{\cos 52^\circ} \cdot \sin 52^\circ + T_2 \sin 40^\circ = mg$$

$$T_2 [\cos 40^\circ \tan 52^\circ + \sin 40^\circ] = mg$$

$$T_2 = \frac{mg}{\cos 40^\circ \tan 52^\circ + \sin 40^\circ}$$

Pb 33



$V_1 = -3\vec{i} + 0\vec{j}$

$V_2 = 0\vec{i} + 22\vec{j}$

$\vec{V} = -3\vec{i} + 22\vec{j}$

$v = \sqrt{3^2 + 22^2}$

$\theta = \tan^{-1} \frac{22}{-3}$

Dot Product

Def $\vec{V} = (a^1, b^1, c^1)$

$\vec{W} = (a^2, b^2, c^2)$

$$\boxed{\vec{V} \cdot \vec{W} = a^1a^2 + b^1b^2 + c^1c^2}$$

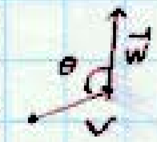
Ex $\vec{V} = (1, 3, 0)$

$W = (2, -1, 1)$

$\vec{V} \cdot \vec{W} = 2 - 3 + 0 = -1$

Fact 1. $\boxed{\vec{V} \cdot \vec{V} = |\vec{V}|^2}$

Fact 2. $\boxed{\vec{V} \cdot \vec{W} = |\vec{V}||\vec{W}| \cos\theta}$



Example

Given $\vec{V} = (-1, 1, 0)$

$\vec{W} = (2, -1, 1)$ Find θ

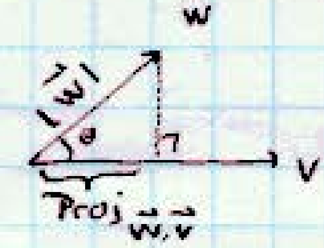
Sol $\cos\theta = \dfrac{\vec{V} \cdot \vec{W}}{|\vec{V}| \cdot |\vec{W}|}$

$$= \frac{-2-1}{\sqrt{2}\sqrt{6}} = \frac{-3}{\sqrt{12}}$$

$$\theta = \cos^{-1}\left(\frac{-3}{\sqrt{12}}\right)$$

Fact 3 $\vec{V} \perp \vec{W}$ iff $\vec{V} \cdot \vec{W} = 0$

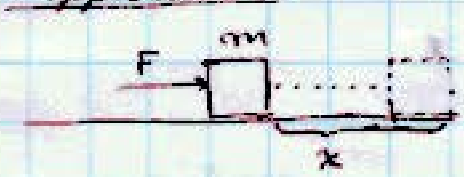
Fact 4



$\text{Proj}_{\vec{W},\vec{V}} = |\vec{W}| \cdot \cos\theta$

$$\boxed{\text{Proj}_{\vec{W},\vec{V}} = \frac{\vec{V} \cdot \vec{W}}{|\vec{V}|}}$$

Application

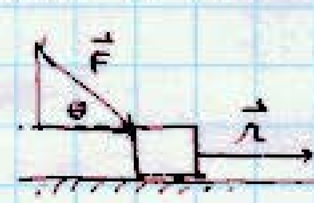


F = Newtons

x = meters

$W = F \cdot x$

W = Joules



$W = |\vec{F}| \cos\theta \cdot |\vec{r}|$

$W = \vec{F} \cdot \vec{r}$

F = const.

Pb: 13.3 # 3, 9, 15, 19, 25, 30
33, 39, 43, 49, 50, 55

8/24

Questions _ Hw 13.3

Pb 50)

$|\vec F| = 20$ lbs

$|\vec r| = 4$ ft

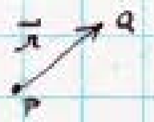
$W = \vec F \cdot \vec r = |\vec F| \cdot |\vec r| \cos\theta$

$= (20)(4)\cos 40° = 80 \cos 40°$

Pb 49.

$\vec F = 10\vec i + 18\vec j - 6\vec k$ $\quad P(2,3,0)$

$\vec r = (4-2)\vec i + (9-3)\vec j + 15\vec k$ $\quad Q(4,9,15)$



$= 2\vec i + 6\vec j + 15\vec k$

$W = \vec F \cdot \vec r = (10, 18, -6) \cdot (2, 6, 15)$

$= 20 + 108 - 90$

$= 385$

Cross Product

A. Determinants

a) 2×2 case:

$$\begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} = (3)(1) - (2)(5) = 3 - 10 = -7$$

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc}$$

b) 3×3 case: Cofactors

$$\begin{vmatrix} 3 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = +3 \cdot \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} - 0 \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= 3(5) - 0(5) + 1(-5)$$

$$= 15 - 5 = 10$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

B. Cross Product

Let $\vec A = a_1\vec i + a_2\vec j + a_3\vec k$

$\vec B = b_1\vec i + b_2\vec j + b_3\vec k$

$$\boxed{\vec A \times \vec B = \begin{vmatrix} \vec i & \vec j & \vec k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$$

$$\vec A \times \vec B = \vec i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Example

$\vec A = 3\vec i - 2\vec j + \vec k$

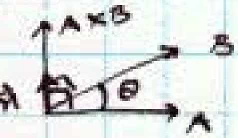
$\vec B = \vec i + 2\vec j + 3\vec k$

$$\vec A \times \vec B = \vec i \begin{vmatrix} -2 & 1 \\ 2 & 3 \end{vmatrix} - \vec j \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + k \begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix}$$

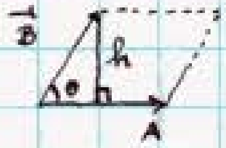
$$= -8\vec i - 8\vec j + 8\vec k$$

Theorem: $\vec A \times \vec B = |\vec A| \cdot |\vec B| \cdot \sin\theta \cdot \vec n$

where is a unit vector ⊥ to $\vec A$ and $\vec B$ in a r.h. sense



Geometry

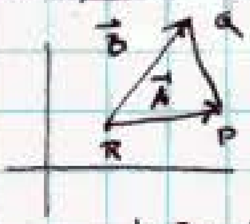


$h = |\vec B| \cdot \sin\theta$

$Area_{▱} = |\vec A| \cdot |\vec B| \cdot \sin\theta$

$|\vec A \times \vec B| = Area_{▱}$

Example: $(\mathbb{R}^2)$



$R(a_1, b_1)$

$P(a_2, b_2)$

$Q(a_3, b_3)$

$$A = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

Using Cross Products

$R(1,2)$

$A = \frac{1}{2} |\vec A \times \vec B|$ $\quad P(3,5)$

$Q(3,8)$

$\vec A = \vec{RP} = (2,3,0) \hookrightarrow \mathbb{R}^3$

$\vec B = \vec{RQ} = (2,6,0)$

$A = |\vec A \times \vec B|$

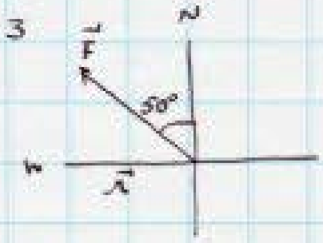
$$\vec A \times \vec B = i \begin{vmatrix} 3 & 0 \\ 6 & 0 \end{vmatrix} - j \begin{vmatrix} 2 & 0 \\ 2 & 0 \end{vmatrix} + k \begin{vmatrix} 2 & 3 \\ 2 & 6 \end{vmatrix}$$

$$= 6\vec k$$

$Area = \frac{1}{2} |\vec A \times \vec B| = 3$

Questions - Hw 13.3

Pb 50)



$|\vec{F}| = 20$ lbs

$|\vec{r}| = 4$ ft

$W = \vec{F}\cdot\vec{r} = |\vec{F}|\cdot|\vec{r}|\cos\theta$

$= (20)(4)\cos 40° = 80\cos 40°$

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$\vec{F} = 10\vec{i} + 18\vec{j} - 6\vec{k}$ $\quad P(2,3,0)$

$\vec{r} = (4-2)\vec{i} + (9-3)\vec{j} + 15\vec{k}$ $\quad Q(4,9,15)$

$= 2\vec{i} + 6\vec{j} + 15\vec{k}$

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$$\begin{vmatrix} 3 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = +3\cdot\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} - 0\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= 3(5) - 0(5) + 1(-5)$$

$$= 15 - 5 = 10$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

B. Cross Product

Let $\vec{A} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{B} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$$\boxed{\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$$

$$\vec{A}\times\vec{B} = \vec{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Example

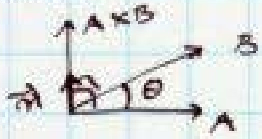
$\vec{A} = 3\vec{i} - 2\vec{j} + \vec{k}$

$\vec{B} = \vec{i} + 2\vec{j} + 3\vec{k}$

$$\vec{A}\times\vec{B} = \vec{i}\begin{vmatrix} -2 & 1 \\ 2 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + k\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix}$$

$$= -8\vec{i} - 8\vec{j} + 8\vec{k}$$

Theorem: $\vec{A}\times\vec{B} = |\vec{A}|\cdot|\vec{B}|\cdot\sin\theta\cdot\vec{n}$ where $\vec{n}$ is a unit vector $\perp$ to $\vec{A}$ and $\vec{B}$ in a r.h. sense



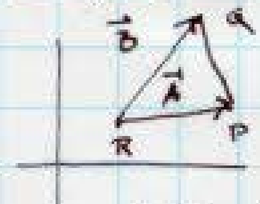
Geometry

$h = |\vec{B}|\cdot\sin\theta$

$Area_{▱} = |\vec{A}|\cdot|\vec{B}|\cdot\sin\theta$

$|\vec{A}\times\vec{B}| = Area_{▱}$

Example: $(\mathbb{R}^2)$



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$A = \frac{1}{2}|\vec{A}\times\vec{B}|$ $\quad R(1,2)$, $P(3,5)$, $Q(3,8)$

$\vec{A} = \vec{RP} = (2,3,0) \hookrightarrow \mathbb{R}^3$

$\vec{B} = \vec{RQ} = (2,6,0)$

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$$= 6\vec{k}$$

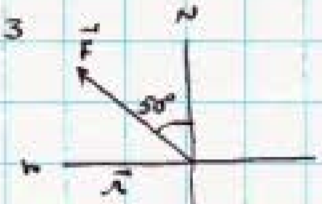
Area $= \frac{1}{2}|\vec{A}\times\vec{B}| = 3$

Questions _ Hw 13.3

Pb 50)

$|\vec{F}| = 20$ lbs

$|\vec{r}| = 4$ ft



$W = \vec{F}\cdot\vec{r} = |\vec{F}|\cdot|\vec{r}|\cos\theta$

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Pb 49.

$\vec{F} = 10\vec{i} + 18\vec{j} - 6\vec{k}$ $\quad P(2,3,0)$

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A. Determinants

a) 2×2 case:

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$$= 3(5) - 0(5) + 1(-5)$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \quad = 15 - 5 = 10$$

B. Cross Product

Let $\vec{A} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{B} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$$\boxed{\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$$

$$\vec{A}\times\vec{B} = \vec{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

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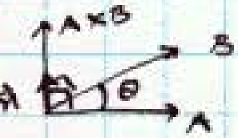
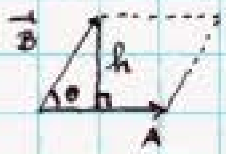


Geometry



$h = |\vec{B}|\cdot\sin\theta$

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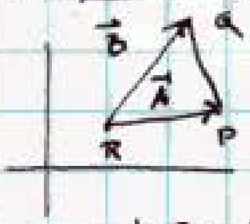
Example: ($\mathbb{R}^2$)



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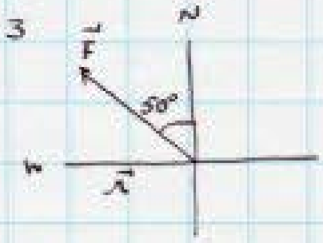
$Area = \frac{1}{2}|\vec{A}\times\vec{B}| = 3$

Questions - Hw 13.3

Pb 50)



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Pb 49.

$\vec{F} = 10\vec{i} + 18\vec{j} - 6\vec{k}$ $\quad P(2,3,0)$

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B. Cross Product

Let $\vec{A} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\quad\vec{B} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$$\boxed{\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$$

$$\vec{A}\times\vec{B} = \vec{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Example

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$\vec{B} = \vec{i} + 2\vec{j} + 3\vec{k}$

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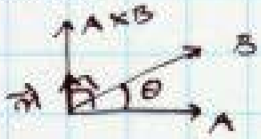
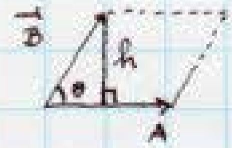
$$= -8\vec{i} - 8\vec{j} + 8\vec{k}$$

Theorem: $\vec{A}\times\vec{B} = |\vec{A}|\cdot|\vec{B}|\cdot\sin\theta\cdot\vec{n}$ where is a unit vector $\perp$ to $\vec{A}$ and $\vec{B}$ in a r.h. sense



Geometry



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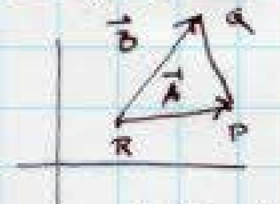
$|\vec{A}\times\vec{B}| = Area_{\square}$

Example: $(\mathbb{R}^2)$



$R(a_1, b_1)$

$P(a_2, b_2)$

$Q(a_3, b_3)$

$$A = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

Using Cross Products

$A = \frac{1}{2}|\vec{A}\times\vec{B}|$ $\quad R(1,2)$, $P(3,5)$, $Q(3,8)$

$\vec{A} = \vec{RP} = (2,3,0) \hookrightarrow \mathbb{R}^3$

$\vec{B} = \vec{RQ} = (2,6,0)$

$A = |\vec{A}\times\vec{B}|$

$$\vec{A}\times\vec{B} = i\begin{vmatrix} 3 & 0 \\ 6 & 0 \end{vmatrix} - j\begin{vmatrix} 2 & 0 \\ 2 & 0 \end{vmatrix} + k\begin{vmatrix} 2 & 3 \\ 2 & 6 \end{vmatrix}$$

$$= 6\vec{k}$$

$Area = \frac{1}{2}|\vec{A}\times\vec{B}| = 3$

Hw: 13.4 # 1, 4, 15, 25, 30, 33, 35, 40

Triple Product.

$\vec{A} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$

$\vec{B} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$

$\vec{C} = c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k}$

Def: $(\vec{A}\vec{B}\vec{C}) = \vec{A} \cdot (\vec{B} \times \vec{C})$

Note:

$$(\vec{A}\vec{B}\vec{C}) = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \bullet \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \bullet \left[\vec{i}\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \vec{j}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + k\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right]$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \cdot \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

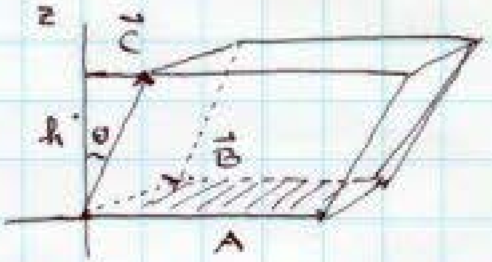
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Result:

$$(\vec{A}\vec{B}\vec{C}) = \vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Geometry:

$\vec{A}, \vec{B} \in$ x-y Plane



$\vec{A} \times \vec{B} = Area \ \vec{k}$

$(\vec{A} \times \vec{B}) \cdot C = |\vec{A} \times \vec{B}| \cdot |\vec{C}| \cdot \cos\theta$

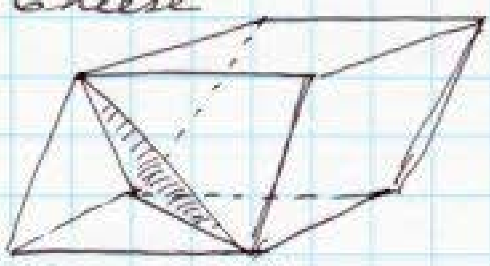
$= Area \cdot |\vec{C}| \cdot \cos\theta$

$= Area \cdot h$

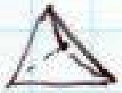
$= Volume(\ )$

Remark $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$

Hw: Cheese

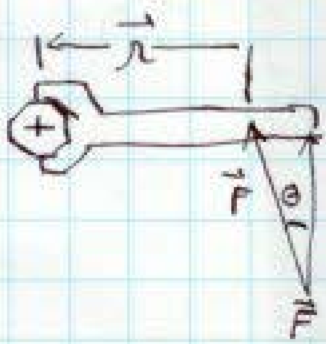


Prove that

1  = 6 

$Vol(1\ ) = \frac{1}{6} Vol(\ )$

Torque



Depends on

$|\vec{F}|, |\vec{r}|, \perp$

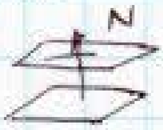
$\tau = |\vec{F}| \sin\theta \cdot |\vec{r}|$

Tao

$\tau = |\vec{r} \times \vec{F}|$

$\vec{\tau} = \vec{r} \times \vec{F}$

26) $\begin{cases} x = 3+2t \\ y = t \\ z = 8-t \end{cases}$ // to Plane $2x+4y+8z=17$



$N = \langle 2, 4, 8 \rangle$

$P\,(3, 0, 8)$

~~$3(x-2)+0(y-4)+$~~

$2(x-3)+4(y-0)+8(z-8)=0$

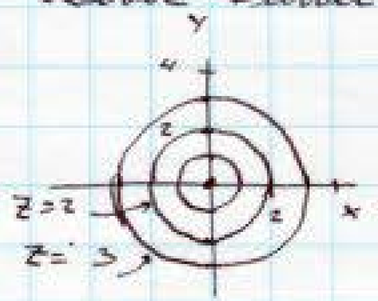
$\boxed{2x+4y+8z=70}$

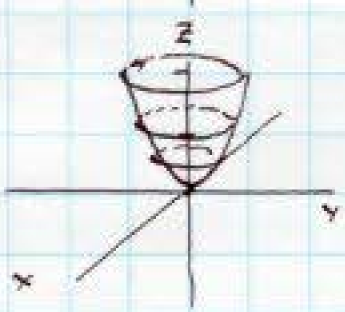
Quadric Surfaces

Example 1 — Level Curves

$z = f(x,y) = x^2+y^2$

z	x,y
0	$x^2+y^2=0$
1	$x^2+y^2=1$
4	$x^2+y^2=4$
9	$x^2+y^2=9$





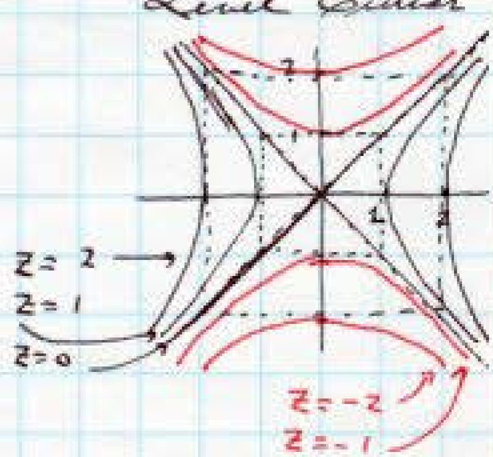
Circular Paraboloid

1. Level Curves
2. Plane Profiles

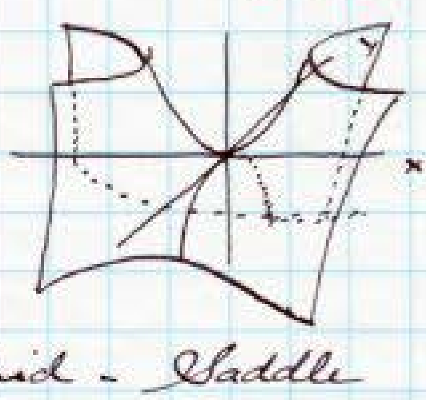
Example 2 — Level Curves

$z = f(x,y) = x^2-y^2$

z	x^2-y^2
-4	$x^2-y^2=-4$
-1	$x^2-y^2=-1$
0	$x^2-y^2=0$
1	$x^2-y^2=1$
4	$x^2-y^2=4$



3-D Graph

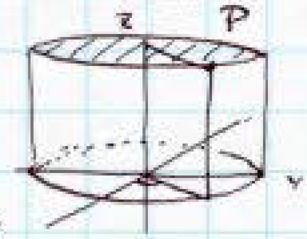


Hyperbolic Paraboloid - Saddle

13.6 # 3, 17, 19, 20, 25, 26, 31, 37

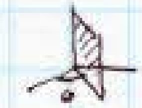
Cylindrical Coordinates:

$P(r, \theta, z)$

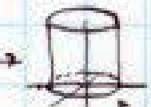


$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$\begin{cases} z = \text{constant} & \text{- Plane} \\ \theta = \text{constant} & \text{- Plane} \\ r = \text{constant} & \text{- Cylinder} \end{cases}$



Eq $x^2+y^2=a^2$



If $x = r\cos\theta$

$y = r\sin\theta$

$r^2\cos^2\theta + r^2\sin^2\theta = a^2$

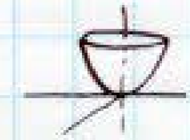
$r^2(\cos^2\theta+\sin^2\theta) = a^2$

$r^2 = a^2$ $\boxed{r=a}$

Question

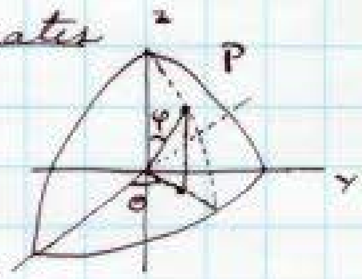
$z = x^2+y^2$

$z = r^2$



Spherical Coordinates

$P(\rho, \theta, \varphi)$



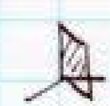
$r = \rho\sin\varphi$

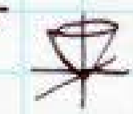
$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$

$\rho = \text{const}$ - Sphere

$\theta = \text{const}$ - Plane

$\varphi = \text{const}$ - Cone

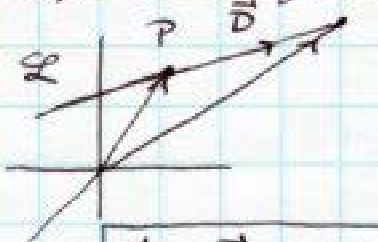




13.7 # 3, 9, 15, 20, 31, 34, 37, 43, 49, 56

Lines and Planes

Equations of Lines:



$P(x_0, y_0, z_0)$

$\vec{D} = \langle a, b, c \rangle$

$\vec{X} = \langle x, y, z \rangle$

$$\boxed{\vec{X} - \vec{P} = k\,\vec{D}}$$

Expand this equation:

$\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle = k\langle a, b, c \rangle$

$\langle x - x_0, y - y_0, z - z_0 \rangle = \langle ka, kb, kc \rangle$

$$\begin{cases} x - x_0 = ka \\ y - y_0 = kb \\ z - z_0 = kc \end{cases} \qquad k = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Three versions of an equation of a line

		Form
V.1	$\vec{X} - \vec{P} = t\,\vec{D}$	Vector
V.2	$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$	Parametric
V.3	$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$	Scalar

Example

Find the equation of the line passing through $P(1, 3, -1)$ and $Q(2, 3, 5)$

$(x_0, y_0, z_0) = (2, 3, 5)$

$\vec{D} = \langle 1, 0, 6 \rangle = \vec{PQ}$

$$\begin{cases} x - 2 = t \cdot 1 \\ y - 3 = t \cdot 0 \\ z - 5 = t \cdot 6 \end{cases} \qquad \begin{aligned} x - 2 &= t \\ y - 3 &= 0 \\ z - 5 &= 6t \end{aligned}$$

or $\frac{x - 2}{1} = \frac{y - 3}{0} = \frac{z - 5}{6}$

Equation of a Plane.

N = Normal

Giv $\begin{cases} \vec{P} = \langle x_0, y_0, z_0 \rangle \\ \vec{N} = \langle A, B, C \rangle \end{cases}$

$\vec{X} = \langle x, y, z \rangle$

$\vec{X} - \vec{P}$ lies on the plane

$$\boxed{(\vec{X} - \vec{P}) \cdot \vec{N} = 0}$$

Expand this equation:

$(\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle) \cdot \langle A, B, C \rangle = 0$

$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle A, B, C \rangle = 0$

$$\boxed{A(x - x_0) + B(y - y_0) + C(z - z_0) = 0}$$

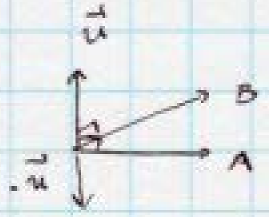
Three versions of an equation of a plane.

		Form
V.1	$(\vec{X} - \vec{P}) \cdot N = 0$	Vector
V.2	$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$	
V.3	$Ax + By + Cz = D$	

13.5 # 2, 7, 11, 15, 19, 23, 26, 30

Questions

13.4 # 15.



$\vec{A} = \langle 1, -1, 1 \rangle$

$\vec{B} = \langle 0, 4, 4 \rangle$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 0 & 4 & 4 \end{vmatrix} = \vec{i}\begin{vmatrix} -1 & 1 \\ 4 & 4 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix} + k\begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix}$$

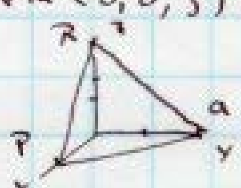
$$= -8\vec{i} - 4\vec{j} + 4\vec{k}$$

$$\vec{u} = \frac{-8\vec{i} - 4\vec{j} + 4\vec{k}}{\sqrt{64+16+16}} = \frac{-8\vec{i} - 4\vec{j} + 4\vec{k}}{\sqrt{96}}$$

$$= \frac{-8\vec{i} - 4\vec{j} + k\vec{k}}{4\sqrt{6}} = \left(\frac{-2\vec{i} - \vec{j} + k}{\sqrt{6}}\right)$$

Ans: $\pm \frac{1}{\sqrt{6}}(-2\vec{i} - \vec{j} + \vec{k})$

#25) $P(1,0,0)$ $Q(0,2,0)$ & $R(0,0,3)$



$\vec{A} = \overrightarrow{PQ} = (-1, 2, 0)$

$\vec{B} = \overrightarrow{PR} = (-1, 0, 3)$

$$\vec{A}\times\vec{B} = \vec{i}\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} -1 & 0 \\ -1 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} -1 & 2 \\ -1 & 0 \end{vmatrix}$$

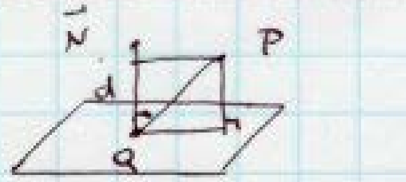
$$= 6\vec{i} + 3\vec{j} + 2\vec{k}$$

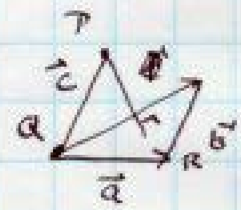
Area $\Delta = \frac{1}{2}|A\times B|$

$$= \frac{1}{2}\sqrt{36+9+4} = \frac{7}{2}$$

---

40)





1) Find the normal $\vec{N}$
2) Find a point Q on the plane
3) Project PQ onto $\vec{N}$

$N = \vec{a}\times\vec{b}$

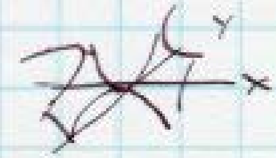
$\overrightarrow{PQ}\cdot\vec{N} = |\overline{PQ}|\cdot|\vec{N}|\cdot \cos\theta$

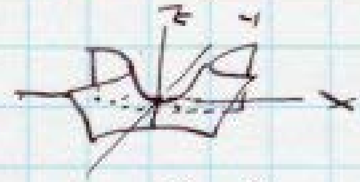
$$d = |\overline{PQ}|\cos\theta = \frac{\overrightarrow{PQ}\cdot\vec{N}}{|\vec{N}|}$$

$$d = \frac{\overrightarrow{PQ}\cdot\vec{N}}{|\vec{N}|} = \frac{\overrightarrow{PQ}\cdot(a\times b)}{|a\times b|} = \frac{|c\cdot(a\times b)|}{|a\times b|}$$

13.7 # 56

$z = x^2 - y^2$





$z = x^2 - y^2$

Cylindrical $\quad x = r\cos\theta$

$y = r\sin\theta$

$z = (r\cos\theta)^2 - (r\sin\theta)^2$

$= r^2\cos^2\theta - r^2\sin^2\theta$

$= r^2(\cos^2\theta - \sin^2\theta)$

$z = r^2\cos 2\theta$

Spherical $\quad x = \rho\sin\varphi\cos\theta$

$y = \rho\sin\varphi\sin\theta$

$z = \rho\cos\varphi$

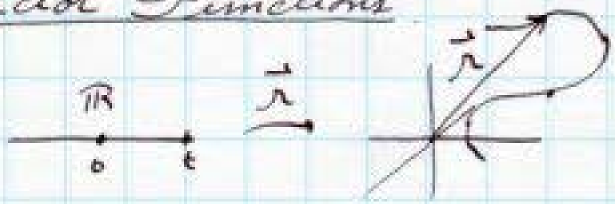
$\rho\cos\varphi = (\rho\sin\varphi\cos\theta)^2 - (\rho\sin\varphi\sin\theta)^2$

$\rho\cos\varphi = \rho^2\sin^2\varphi\,(\cos^2\theta - \sin^2\theta)$

$\cos\varphi = \rho\sin^2\varphi\cos 2\theta$

$$\rho = \frac{\cos\varphi}{\sin^2\varphi\cos 2\theta}$$

Vector Functions



$t \longmapsto (x(t), y(t), z(t))$

Example

$\vec{r}(t) = (1+2t,\ 3-t\ ,\ 5+7t)$

$\vec{r}(t) = (1+2t)\vec{i} + (3-t)\vec{j} + (5+7t)k$

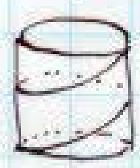
Equivalent

$\begin{cases} x = 1+2t \\ y = 3-t \\ z = 5+7t \end{cases}$ Parametric equation. Line.

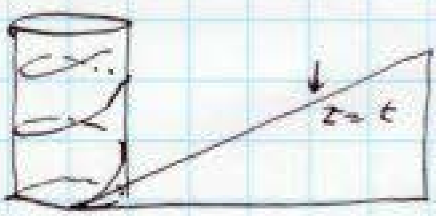
Example

$\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$

$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \\ z(t) = t \end{cases}$



Helix



Continuity

Recall. If $f: \mathbb{R} \to \mathbb{R}$.

then $f(x)$ is cont at $x = a$

if 1) $f(a)$ is defined

2) $\lim_{x\to a} f(x)$ exists

$$\boxed{3)\ \lim_{x\to a} f(x) = f(a)}$$

Def $\vec{r}(t)$ is cont at $t = t_0$ if

$$\lim_{t\to t_0} \vec{r}(t) = \vec{r}(t_0)$$

Example

$\vec{r}(t) = \frac{\sin t}{t}\vec{i} + \frac{t^2-1}{t+1}\vec{j} + \frac{e^t-1}{t}k$

$\lim_{t\to 0} \vec{r}(t) = 1\,\vec{i} + (-1)\vec{j} + \underline{1}\,\vec{k}$

Do 14.1 # 4, 5, 11, 12, 15, 23, 27

14.2 Diff of vector functions

Example

$\vec{r}(t) = t^3\vec{i} - (2t+1)\vec{j} + t^4\vec{k}$

$\frac{d\vec{r}}{dt} = (3t^2)\vec{i} - (2)\vec{j} + 4t^3\vec{k}$

All rules of diff. apply

Ex [Pb 33]

$\int_0^1 (t\vec{i} + t^2\vec{j} + t^3k)\,dt$

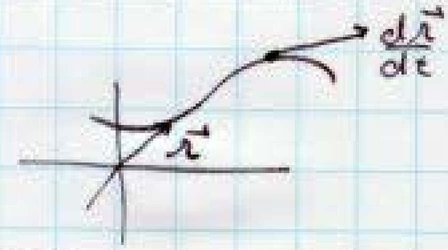
$= \frac{t^2}{2}\vec{i} + \frac{t^3}{3}\vec{j} + \frac{t^4}{4}\vec{k}\Big]_0^1$

$= \frac{1}{2}\vec{i} + \frac{1}{3}\vec{j} + \frac{1}{4}\vec{k}$

Do: 9-14

33-36

Tangents



$\vec{r}(t)$

$\frac{d\vec{r}}{dt}$ - tangent to the curve

If t - time

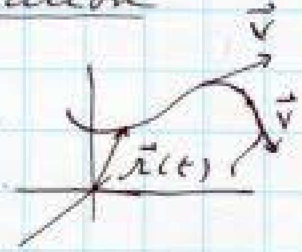
$\vec{r}(t)$ - Position

$\frac{d\vec{r}}{dt} = \vec{v}$. Velocity

Curvature & Acceleration

$\vec{r}(t)$ = Position Vector

t = time



$\frac{d\vec{r}}{dt} = \vec{V}$ Velocity

$\frac{d^2\vec{r}}{dt^2} = \vec{A}$ Acceleration

$$|\vec{V}| = v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Def $v = \frac{ds}{dt}$ Scalar $\left(\frac{m}{s}\right)$

Arc length

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$s = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Def Tangent Vector

$\vec{T} = \frac{\vec{V}}{v}$ $\quad |\vec{T}| = 1$

Note: $\vec{T} = \frac{d\vec{r}}{dt} \Big/ \frac{ds}{dt} = \frac{d\vec{r}}{ds}$

Note: $\vec{T}\cdot\vec{T} = 1$

$$\frac{d}{ds}(\vec{T}\cdot\vec{T}) = \vec{T}\cdot\frac{d\vec{T}}{ds} + \vec{T}\cdot\frac{d\vec{T}}{ds} = 0$$

$2\,\vec{T}\cdot\frac{d\vec{T}}{ds} = 0 \qquad \frac{d\vec{T}}{ds} \perp \vec{T}$

Def $\vec{N} = \frac{d\vec{T}}{ds} \Big/ \left|\frac{d\vec{T}}{ds}\right| \qquad |\vec{N}| = 1$

$\frac{d\vec{T}}{ds} = \left|\frac{d\vec{T}}{ds}\right|\cdot\vec{N} \qquad \kappa = \left|\frac{d\vec{T}}{ds}\right|$

κ = Curvature.

Summary.

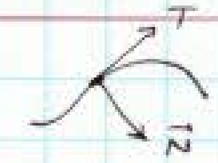
$\vec{T} = \frac{\vec{V}}{v} \Rightarrow \frac{d\vec{r}}{dt} = v.\vec{T}$

$\frac{d\vec{T}}{ds} = \kappa\vec{N}$

$$\frac{d^2\vec{r}}{dt^2} = \frac{dv}{dt}.\vec{T} + v.\frac{d\vec{T}}{dt}$$

$$= \frac{dv}{dt}.\vec{T} + v.\frac{d\vec{T}}{ds}.\frac{ds}{dt}$$

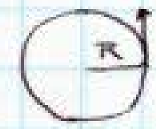
$$\frac{d^2\vec{r}}{dt^2} = \frac{dv}{dt}\vec{T} + v^2.\frac{d\vec{T}}{ds}$$



$$\frac{d^2\vec{r}}{dt^2} = \frac{dv}{dt}\vec{T} + (v^2\kappa)\vec{N}$$

$$\boxed{\vec{A} = \frac{dv}{dt}\vec{T} + (v^2\kappa)\vec{N}} \qquad v^2\kappa = \frac{v^2}{R}$$

Will Prove



$\kappa = \frac{1}{R}$

Summary:

$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = v\vec{T}$ $\qquad |\vec{T}| = 1$

$\qquad |\vec{N}| = 1$

$\ddot{\vec{r}} = \frac{d^2\vec{r}}{dt^2} = \frac{dv}{dt}\vec{T} + v^2\kappa\vec{N}$ $\qquad \vec{T}\cdot\vec{N} = 0$

$\dot{\vec{r}} \times \ddot{\vec{r}} = v^3\kappa\,(\vec{T}\times\vec{N})$

$= v^3\kappa\cdot\vec{B}$

$|\dot{\vec{r}}\times\ddot{\vec{r}}| = v^3\kappa\cdot\overset{=1}{|\vec{B}|}$

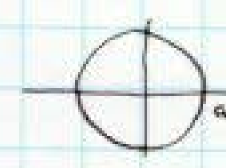
$$\boxed{\kappa = \left|\frac{\dot{\vec{r}}\times\ddot{\vec{r}}}{v^3}\right|}$$

Example:

$\vec{r}(t) = (a\cos t)\,\vec{i} + (a\sin t)\,\vec{j}$

This means:

$$\begin{cases} x(t) = a\cos t \\ y(t) = a\sin t \end{cases}$$



Note

$x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t$

$= a^2(\cos^2 t + \sin^2 t)$

$x^2 + y^2 = a^2$

Velocity

$\vec{V} = \frac{d\vec{r}}{dt} = -a\sin t\,\vec{i} + a\cos t\,\vec{j}$

$v = |\vec{V}|$

$= \sqrt{(-a\sin t)^2 + (a\cos t)^2}$

$= \sqrt{a^2\sin^2 t + a^2\cos^2 t}$

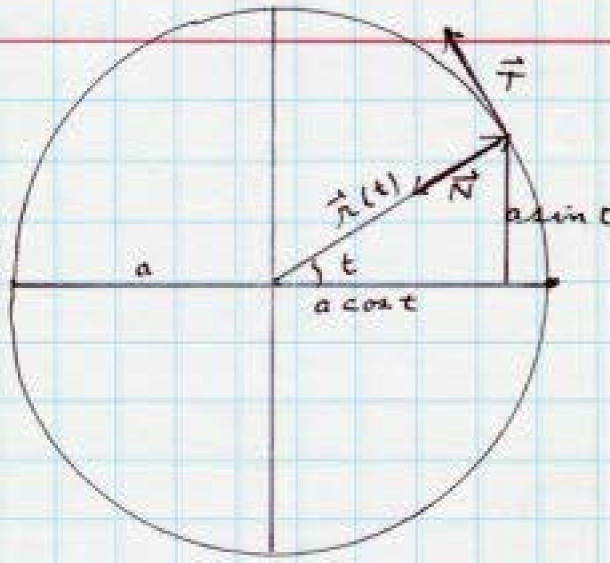
$= \sqrt{a^2(\sin^2 t + \cos^2 t)}$

$= \sqrt{a^2}$

$v = a \qquad \vec{T} = \frac{\vec{V}}{v} =$

$\vec{T} = -\sin t\,\vec{i} + (\cos t)\,\vec{j}$

Sept 9



$\frac{ds}{dt} = v = a \qquad s = at + c$

$ds = a\,dt \qquad t=0 \quad s=0$

$s = \int a\,dt \qquad s = at$

$\vec{N} = \frac{d\vec{T}}{ds} \Big/ \left|\frac{dT}{ds}\right| \qquad \kappa = \left|\frac{dT}{ds}\right|$

$\vec{T} = -\sin(\frac{s}{a})\,\vec{i} + \cos(\frac{s}{a})\,\vec{j}$

$\frac{d\vec{T}}{ds} = -\frac{1}{a}\cos\frac{s}{a}\,\vec{i} + (-)\sin(\frac{s}{a})\cdot\frac{1}{a}\,\vec{j}$

$\frac{d\vec{T}}{ds} = -\frac{1}{a}\left[\cos(\frac{s}{a})\vec{i} + \sin(\frac{s}{a})\vec{j}\right]$

$\left|\frac{d\vec{T}}{ds}\right| = \frac{1}{a} \qquad K = \frac{1}{a}$

$\vec{N} = -\left[\cos\frac{s}{a}\,\vec{i} + \sin(\frac{s}{a})\vec{j}\right]$

$\vec{N} = -(\cos t\,\vec{i} + \sin t\,\vec{j})$

Recall $\kappa = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{v^3} \qquad v = a$

$\dot{\vec{r}} = -a\sin t\,\vec{i} + a\cos t\,\vec{j} + 0\,\vec{k}$

$\ddot{\vec{r}} = -a\cos t\,\vec{i} - a\sin t\,\vec{j} + 0\,\vec{k}$

$\dot{\vec{r}} \times \ddot{\vec{r}} = \vec{i}\cdot 0 - \vec{j}\cdot 0 + \vec{k}\,a^2(\sin^2 t + \cos^2 t)$

$= a^2\vec{k}$

$|\dot{\vec{r}} \times \ddot{\vec{r}}| = a^2$

$\kappa = \frac{a^2}{a^3} = \frac{1}{a}$

DO: 14.3 # 1, 3, 7, 13, 18, 33

14.4 # 7, 11, 28, 29, 31

14.2

35) $\int_0^{\pi/4} (\cos 2t\,\vec{i} + \sin 2t\,\vec{j} + t\sin t\,\vec{k})\,dt$

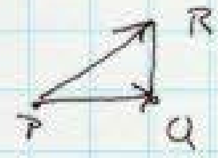
$= \frac{1}{2}\sin 2t\,\vec{i} - \frac{1}{2}\cos 2t\,\vec{j} + (-t\cos t + \sin t)\,\vec{k}$

$= \frac{1}{2}(1)\,\vec{i} - \frac{1}{2}(0-1)\,\vec{j} + (-\frac{\pi}{4}\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2})\,\vec{k}$

<u>Pb</u> P(1,0,1)

Q(1,1,0)

R(2,1,1)



$\vec{A} = \vec{PQ} = \langle 0, 1, -1 \rangle$

$\vec{B} = \vec{PR} = \langle 1, 1, 0 \rangle$

$\vec{A}\times\vec{B} = \vec{i}\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} + k\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$

$= +\vec{i} - \vec{j} - k$

$+1(x-1) - 1(y-0) - 1(z-1) = 0$

$\cancel{-x + y + z = -2}$

$\boxed{\cancel{x - y - z = 2}}$

$x - y - z = 0$

14.3 #7.

$\vec{r}(t) = e^t \sin t\,\vec{i} + e^t\cos t\,\vec{j}$

$\vec{v}(t) = (e^t\cos t + e^t \sin t)\,\vec{i} +$

$(-e^t\sin t + e^t\cos t)\,\vec{j}$

$v = \left[(e^t\cos t + e^t\sin t)^2 + (-e^t\sin t + e^t\cos t)^2\right]^{\frac{1}{2}}$

$= e^t\left[(\cos t + \sin t)^2 + (-\sin t + \cos t)^2\right]^{1/2}$

$= e^t\left[(1 + 2\cos t\sin t) + (1 - 2\sin t\cos t)\right]^{1/2}$

$v = \sqrt{2}\,e^t = \frac{ds}{dt}$

$s = \int v\,dt$

$= \int \sqrt{2}\,e^t\,dt \qquad e^t = \frac{1}{\sqrt{2}}s$

$s = \sqrt{2}\,e^t \qquad t = \ln(s/\sqrt{2})$

$\vec{r}(s) = \frac{1}{\sqrt{2}}s\left[\sin(\ln\frac{s}{\sqrt{2}})\,\vec{i} + \cos(\ln\frac{s}{\sqrt{2}})\,\vec{j}\right]$

<u>Do</u>: <u>*Group Work*</u>:

1) $x = 2 - 3\sin 2t$ Find Cartesian

$y = 4\cos 2t - 1$ eq. (eliminate t)

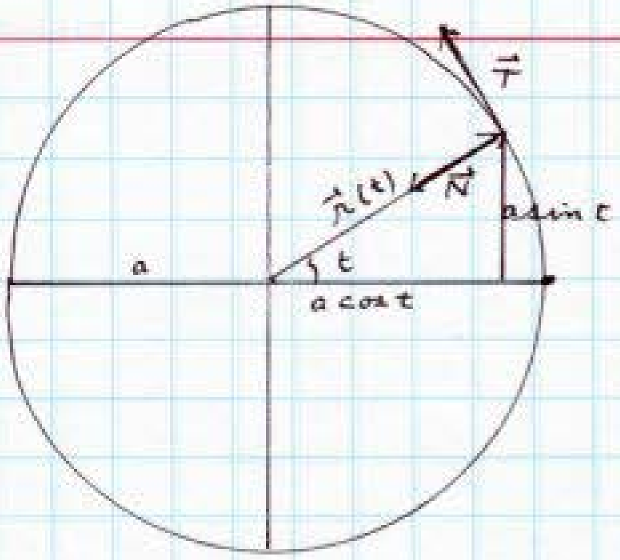
2) $\vec{A} = \langle 1, -2, 4 \rangle$ $\vec{B} = \langle 2, -2, 1 \rangle$

a) $\|\vec{A}\|$, $\|\vec{B}\|$, b) $\cos\theta$, c) Area ▱

3) P(0,0,0), Q(1,-1,1), R(1,2,3).

Equation of plane

Sept 9



$$\frac{ds}{dt} = v = a \qquad s = at + c$$

$$ds = a\,dt \qquad t = 0 \quad s = 0$$

$$s = \int a\,dt \qquad s = at$$

$$\vec{N} = \frac{d\vec{T}}{ds} / \left|\frac{dT}{ds}\right| \qquad \kappa = \left|\frac{dT}{ds}\right|$$

$$\vec{T} = -\sin\left(\frac{s}{a}\right)\vec{i} + \cos\left(\frac{s}{a}\right)\vec{j}$$

$$\frac{d\vec{T}}{ds} = -\frac{1}{a}\cos\frac{s}{a}\,\vec{i} + (-)\sin\left(\frac{s}{a}\right)\cdot\frac{1}{a}\,\vec{j}$$

$$\frac{d\vec{T}}{ds} = -\frac{1}{a}\left[\cos\left(\frac{s}{a}\right)\vec{i} + \sin\left(\frac{s}{a}\right)\vec{j}\right]$$

$$\left|\frac{d\vec{T}}{ds}\right| = \frac{1}{a} \qquad K = \frac{1}{a}$$

$$\vec{N} = -\left[\cos\frac{s}{a}\,\vec{i} + \sin\left(\frac{s}{a}\right)\vec{j}\right]$$

$$\vec{N} = -(\cos t\,\vec{i} + \sin t\,\vec{j})$$

Recall $\quad \kappa = \dfrac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{v^3} \qquad v = a$

$$\dot{\vec{r}} = -a\sin t\,\vec{i} + a\cos t\,\vec{j} + 0\vec{k}$$

$$\ddot{\vec{r}} = -a\cos t\,\vec{i} - a\sin t\,\vec{j} + 0\vec{k}$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \vec{i}\cdot 0 - \vec{j}\cdot 0 + \vec{k}\,a^2(\sin^2 t + \cos^2 t)$$

$$= a^2\vec{k}$$

$$|\dot{\vec{r}} \times \ddot{\vec{r}}| = a^2$$

$$K = \frac{a^2}{a^3} = \frac{1}{a}$$

DO: 14.3 # 1, 3, 7, 13, 18, 33

14.4 # 7, 11, 28, 29, 31

4) Find κ: $\vec{r}(t) = \langle 4t, -5\cos 2t, 5\sin 2t \rangle$

5) $P(3,2,4)$, $Q(1,-1,5)$.. Eq of line

6) $P(1,-3,4)$, $x+2y-2z=2$. Distance from P to plane

14.3 13) $\vec{r} = \langle \frac{1}{3}t^3, t^2, 2t \rangle$

$$\vec{V} = \langle t^2, 2t, 2 \rangle$$

$$v = \sqrt{t^4 + 4t^2 + 4}$$

$$\frac{ds}{dt} = \sqrt{(t^2+2)^2} = t^2 + 2$$

$$s = \frac{1}{3}t^3 + 2t$$

$$T = \frac{\vec{V}}{v} = \left\langle \frac{t^2}{t^2+2}, \frac{2t}{t^2+2}, \frac{2}{t^2+2} \right\rangle$$

$$N = T'(t) / |T'(t)|$$

$$\left(\frac{t^2}{t^2+2}\right)' = \frac{(t^2+2)(2t) - t^2(2t)}{(t^2+2)^2}$$

$$= \frac{2t\,[t^2+2-t^2]}{(t^2+2)^2} = \frac{4t}{(t^2+2)^2}$$

$$\left(\frac{2t}{t^2+2}\right)' = \frac{(t^2+2)(2) - 2t(2t)}{(t^2+2)^2}$$

$$= \frac{4-2t^2}{(t^2+2)^2}$$

$$\left(\frac{2}{t^2+2}\right)' = \frac{-4t}{(t^2+2)^2}$$

$$T'(t) = \left\langle \frac{4t}{(t^2+2)^2}, \frac{(4-2t^2)}{(t^2+2)^2}, \frac{-4t}{(t^2+2)^2} \right\rangle$$

$$|T'(t)| = \frac{1}{(t^2+2)^2}\left[16t^2 + (4-2t^2)^2 + 16t^2\right]^{\frac{1}{2}}$$

$$= \frac{1}{(t^2+2)^2}\left[32t^2 + 16 - 16t^2 + 4t^4\right]^{\frac{1}{2}}$$

$$= \frac{1}{(t^2+2)^2}\left[16 + 16t^2 + 4t^4\right]^{\frac{1}{2}}$$

$$= \frac{1}{(t^2+2)^2}\left[(4+2t^2)^2\right]^{\frac{1}{2}}$$

$$= \frac{4+2t^2}{(t^2+2)^2}$$

$$N = \frac{\langle 4t, 4-2t^2, -4t \rangle}{4+2t^2}$$

18) $\vec{r}(t) = \langle e^t\cos t, e^t\sin t, t \rangle$

$$\vec{V}(t) = \langle -e^t\sin t + e^t\cos t, e^t\cos t + e^t\sin t, 1 \rangle$$

$$v(t) = \left[(-e^t\sin t + e^t\cos t)^2 + (e^t\cos t + e^t\sin t)^2 + 1\right]^{1/2}$$

$$v(t) = \left[e^{2t}\left[(-\sin t + \cos t)^2 + (\cos t + \sin t)^2\right] + 1\right]^{\frac{1}{2}}$$

$$= \left[e^{2t}(1 - 2\sin t\cos t) + (1 + 2\cos t\sin t) + 1\right]^{\frac{1}{2}}$$

$$= \left[e^{2t}(2) + 1\right]^{1/2}$$

$$= (2e^{2t} + 1)^{1/2}$$

$$\vec{T} = \frac{\vec{V}}{v} = \frac{\langle e^t(-\sin t + \cos t), e^t(\cos t + \sin t), 1 \rangle}{\sqrt{2e^{2t}+1}}$$

$$\ddot{r} = \langle -2e^t\sin t, 2e^t\cos t, 0 \rangle$$

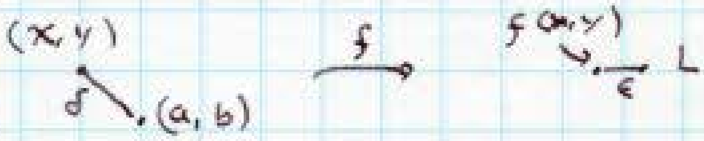
Continuity & Limits

Def: $\lim_{(x,y)\to(a,b)} f(x,y) = L$

if for every $\epsilon > 0$, $\exists$ a $\delta > 0$ such that

$|f(x,y) - L| < \epsilon$

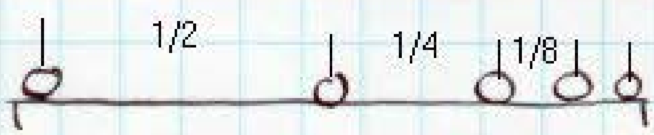
whenever $d((x,y),(a,b)) < \delta$



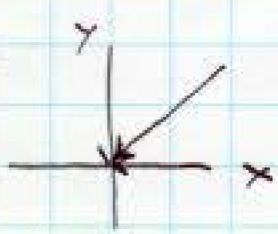
Example

Ex 1. $f(x,y) = x^2 + y$

$$\lim_{(x,y)\to(1,2)} f(x,y) = 1^2 + 2 = 3$$



Ex $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} =$



$Y = 0$

$$\lim_{(x,y)\to(0,0)} f = \lim_{(x,y)\to(0,0)} \frac{0}{x^2} = 0$$

$Y = X$

$$\lim_{(x,y)\to(0,0)} = \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+x^2} = \lim_{(x,y)\to(0,0)} \frac{1}{2} = \frac{1}{2}$$

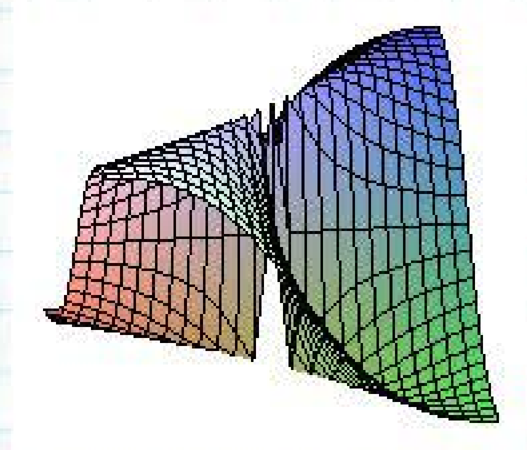
diff

Along

$Y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+m^2(x^2)} = \frac{1}{1+m^2}$$

Lim at $(x,y)\to(0,0)$ does not exist! $f(x,y)$ is not cont. at $(0,0)$



f(x) = xy / (x^2 + y^2)
not continuous at (0,0)

Partial Derivatives (15.3)

Def Let $z = f(x,y)$

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,\,y) - f(x,y)}{h}$$

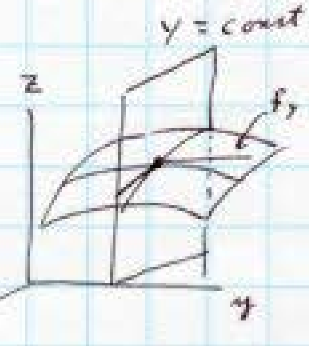
$$\frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x,\,y+k) - f(x,y)}{k}$$

Geometry

$z = f(x,y)$

$\frac{\partial f}{\partial x}$ = slope in x-dir

$\frac{\partial f}{\partial y}$ = slope in y-dir



Notation

$\frac{\partial f}{\partial x} = f_x = D_x f = f_{,1}$

$\frac{\partial f}{\partial y} = f_y = D_y f = f_{,2}$

Examples

Ex 1 $z = f(x,y) = x^3 + xy^2 + y^3$

$\frac{\partial f}{\partial x} = 3x^2 + y^2$

$\frac{\partial f}{\partial y} = 2xy + 3y^2$

Ex 2 $f(x,y) = x\sin(xy)$

$\frac{\partial f}{\partial x} = x \cdot \frac{\partial}{\partial x}\sin(xy) + \sin xy \cdot \frac{\partial}{\partial x} x$

$= xy\cos xy + \sin xy$

$\frac{\partial f}{\partial y} = x^2\cos(xy)$

Ex 3 $f(x,y) = e^{xy^2} + \ln(x+y^3)$

$f_x = e^{xy^2} \cdot y^2 + \frac{1}{x+y^3}$

$f_y = e^{xy^2} \cdot 2yx + \frac{3y^2}{x+y^3}$

Notation:

Let $z = f(x,y)$

$\frac{\partial^2 f}{\partial x^2} = f_{xx} = D_x D_x f = D_{xx} f = f_{,11}$

$\frac{\partial^2 f}{\partial y^2} = f_{yy} = D_y D_y f = D_{yy} f = f_{,22}$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} = D_y D_x f = f_{,12}$

$\boxed{15.3}$
Do # 13, 21, 22, 35, 39, 41, 45, 49, 59, 66
71, 76

Ex $f(x,y) = x^2y^3 + e^{x^2y}$

Compute $f_{xx}, f_{xy}, f_{yx}, f_{yy}$

$$\begin{cases} f_x = 2xy^3 + 2xy \cdot e^{x^2y} \\ f_y = 3x^2y^2 + x^2 e^{x^2y} \end{cases}$$

$f_{xx} = 2y^3 + 2xy \cdot 2xy\, e^{x^2y} + 2y\, e^{x^2y}$

$f_{xx} = 2y^3 + e^{x^2y}(4x^2y^2 + 2y)$

$$\begin{cases} f_{xx} = 2y^3 + 2ye^{x^2y}(2x^2y+1) \\ f_{xy} = 6xy^2 + 2xe^{x^2y}(1+x^2y) \\ f_{yx} = 6xy^2 + 2xe^{x^2y}(1+x^2y) \\ f_{yy} = 6x^2y + x^4e^{x^2y} \end{cases}$$

Example

$PV = nRT$ $\quad n, R$ = const.

Pb Compute $\frac{\partial P}{\partial V}, \frac{\partial P}{\partial T}$

Sol

$P = \frac{nRT}{V}$ $\quad \frac{\partial P}{\partial V} = -\frac{nRT}{V^2}$

$\frac{\partial P}{\partial T} = \frac{nR}{V}$

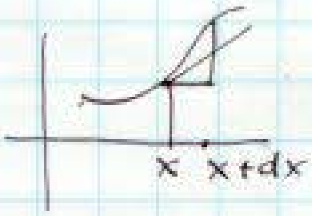
What if we do not hold one of the variables constant?

Differentials

Review $y = f(x)$

$f'(x)$ = derivative

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$



$\frac{dy}{dx} = f'(x)$ $\qquad dy = f'(x)\,dx$
$= \frac{\partial f}{\partial x} dx$

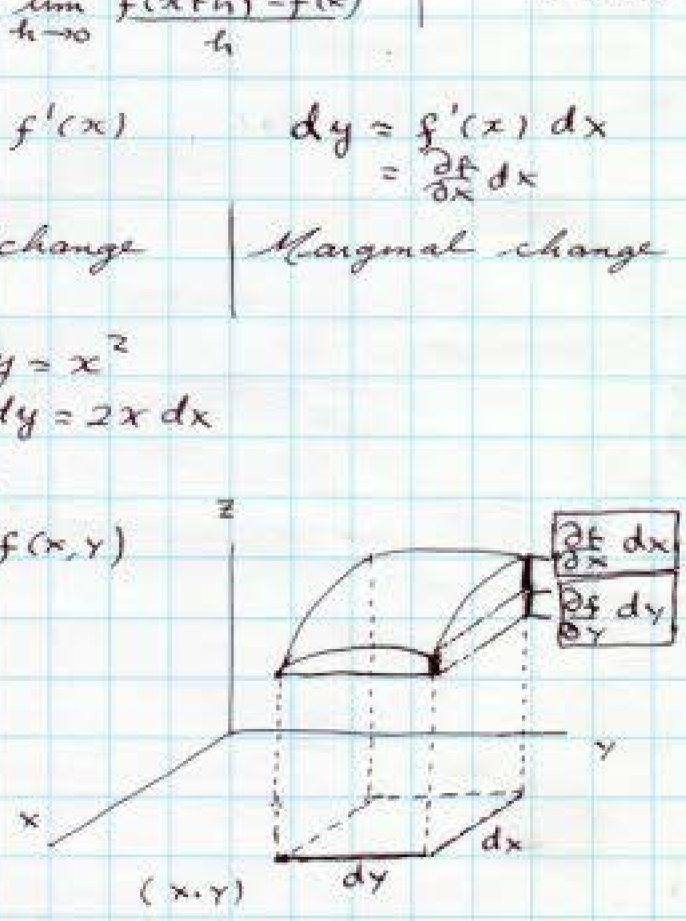
Rate of change | Marginal change

Ex $y = x^2$

$dy = 2x\,dx$

New

$z = f(x,y)$



$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

Ex $z = x^2 + 3xy^3$

$dz = (2x + 3y^3)dx + (9xy^2)dy$

Ex (Implicit Differentiation)

Pb 40 $\boxed{15.3}$

$xyz = \cos(x+y+z)$ $\quad \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$z = f(x,y)$

$d(xyz) = d(\cos(x+y+z))$

$xy\,dz + xz\,dy + yz\,dx =$

$-\sin(x+y+z)(dx + dy + dz)$

$xy\,dz + \sin(x+y+z)\,dz = -xz\,dy - yz\,dx$

$- \sin(x+y+z)dx - \sin(x+y+z)dy$

$[xy + \sin(x+y+z)]\,dz = (\ \ )dx + (\ \ )\underline{dy}$

$dz = \frac{-yz - \sin(x+y+z)}{xy + \sin(x+y+z)} dx - \frac{xz + \sin(x+y+z)}{xy + \sin(x+y+z)}$

$\frac{\partial z}{\partial x} = \frac{-yz - \sin(x+y+z)}{xy + \sin(x+y+z)}$

$\frac{\partial z}{\partial y} = -\frac{xz + \sin(x+y+z)}{xy + \sin(x+y+z)}$

$\left\{ \begin{array}{l} \text{a) Take "d" of equation} \\ \text{b) solve for } dz \end{array} \right.$

Do #39 pg 940

Implicit Differentiation

Fact $z = f(x,y)$

$$\boxed{dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy}$$

Example $x^2 + y^2 + z^2 = 9$

$2x\,dx + 2y\,dy + 2z\,\underline{dz} = 0$

$2z\,dz = -2x\,dx - 2y\,dy$

$dz = -\frac{x}{z} dx - \frac{y}{z} dy$

$\frac{\partial z}{\partial x} = -\frac{x}{z} \qquad \frac{\partial z}{\partial y} = -\frac{y}{z}$

Chain Rule

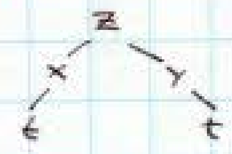
$y = f(x)$ $\qquad \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$x = x(t)$

Suppose $z = f(x,y)$

$x = x(t)$

$y = y(t)$ $\qquad z = f(x(t), y(t))$



$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

Example

$z = xy \qquad x = t^3 \quad y = 3t^2$

$\frac{dz}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}$

$= 3t^2 \cdot 3t^2 + t^3 \cdot 6t = 15t^4$

Check

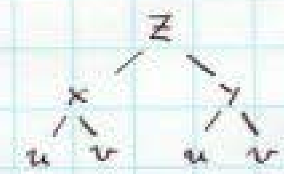
$z = t^3 \cdot 3t^2 = 3t^5$

$\frac{dz}{dt} = 15t^4$

Suppose $z = f(x,y)$

$x = x(u,v)$

$y = y(u,v)$



$$\left\{ \begin{array}{l} \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} \end{array} \right.$$

22) F.T.C

Ver 1. $\int_a^b f(x)\,dx = F(b) - F(a)$

where $F'(x) = f(x)$

Ver 2 $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

$f(x,y) = \int_y^x \cos t^2\,dt$

$\frac{\partial f}{\partial x} = \cos x^2$

$\frac{\partial f}{\partial y} = -\cos y^2$

Def $f(x) = \int_0^x \sin t^2\,dt$

39) $xy + yz = xz$

$d(xy) + d(yz) = d(xz)$

$x\,dy + y\,dx + y\,dz + z\,dy = x\,dz + z\,dx$

$(y-x)\,dz = (z-y)\,dx - (x+z)\,dy$

$dz = \frac{(z-y)}{(y-x)} dx - \frac{(x+z)}{(y-x)} dy$

$z_x = \frac{z-y}{y-x} \qquad z_y = -\frac{x+z}{y-x}$

35 $f(x,y,z) = \frac{x}{(y+z)}$ $\quad f_z(3,2,1)$

$f_z = \frac{-x}{(y+z)^2}$ $\quad f_z(3,2,1) = \frac{-3}{(2+1)^2} = -\frac{1}{3}$

66. $u_{xx} + u_{yy} = 0$

$u = \ln\sqrt{x^2+y^2} = \ln(x^2+y^2)^{1/2}$

$= \frac{1}{2}\ln(x^2+y^2)$

$\begin{cases} u_x = \frac{2x}{2(x^2+y^2)} = \frac{x}{x^2+y^2} \\ u_y = \frac{y}{x^2+y^2} \end{cases}$

$u_{xx} = \frac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$ $\quad \begin{cases} u_{xx}+u_{yy} = 0 \\ \underline{\text{Laplace}} \end{cases}$

__Chain Rule__

$\begin{cases} z = x^2+y^2 \\ x = u+3v \\ y = 7u-v \end{cases}$

$z_u,\ z_v$ $\quad (u,v) = (1,0)$

$\quad (x,y) = (1,7)$

$\begin{cases} \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \end{cases}$

$z_u = 2x\cdot 1 + 2y\cdot 7$

$z_v = 2x\cdot 3 + 2y\cdot(-1)$

__If__ $z = (u+3v)^2 + (7u-v)^2$

Let $\begin{cases} x = u+3v \\ y = 7u-v \end{cases}$

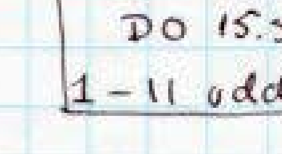
$\boxed{\text{DO 15.5 } 1-11 \text{ odd}}$

15.5 5) $w = xe^{y/z}$

$x = t^2$

$y = 1-t$

$z = 1+2t$ $\quad w_t$



$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$

$= e^{y/z}\cdot 2t + xe^{y/z}\frac{1}{z}(-1) + xe^{\frac{y}{z}}\cdot\left(\frac{-y}{z^2}\right)\cdot 2$

$= e^{y/z}\left(2t - \frac{x}{z} - \frac{2xy}{z^2}\right)$

15.5

Do # 19, 25, 29, 39, 41, 45, 49

__Directional Derivative__

Let $z = f(x,y)$

$\vec{v}$ = __unit__ vector

$\vec{x}_0 = (x_0, y_0)$

$D_{\vec{v}} f(x_0,y_0) = \lim_{t\to 0} \frac{f(\vec{x}_0 + t\vec{v}) - f(\vec{x}_0)}{t}$

Def $\vec{\nabla} f$ - Gradient of f

$\vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$

Let $\vec{v} = v_1\vec{i} + v_2\vec{j}$

Thm

$\boxed{D_{\vec{v}} f(\vec{x}_0) = \vec{\nabla} f(x_0)\cdot\vec{v}}$

$D_{\vec{v}} f(x_0) = v^1\frac{\partial f}{\partial x} + v^2\frac{\partial f}{\partial y}$

__Example__

Let $f(z) = 4 - x^2 - y^2$

Find $D_{\vec{v}} f(1,2)$, $\vec{v} = \langle\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$

Sol

1) $\vec{\nabla} f = -2x\vec{i} - 2y\vec{j}$

2) $D_v f = \vec{\nabla} f\cdot\vec{v}$

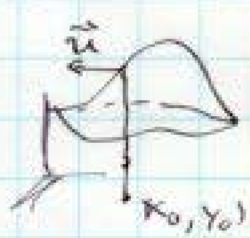
$= -\frac{2x}{\sqrt{2}} - \frac{2y}{\sqrt{2}}$

$D_v f(1,2) = \frac{-2}{\sqrt{2}} - \frac{4}{\sqrt{2}} = -\frac{6}{\sqrt{2}}$

Prop

Let $z = f(x,y)$

Let $\vec{x}_0 = (x_0, y_0)$



The surface has max rate of change in the direction of the gradient!

Pf

$D_{\vec{u}} f(x_0) = \vec{\nabla} f(x_0)\cdot\vec{u}$ $\quad |u| = 1$

$D_{\vec{u}} f(x_0) = |\vec{\nabla} f(x_0)|\cdot|\vec{u}|\cdot\cos\theta$

Largest when $\vec{u} \parallel \nabla f(x_0)$

Smallest $\boxed{=0}$ $\vec{u} \perp \nabla f(x_0)$

21) $z = \ln(x+\sqrt{x^2+y^2})$

$$\frac{\partial z}{\partial x} = \frac{1}{x+\sqrt{x^2+y^2}}\cdot\frac{\partial}{\partial x}\left(x+\sqrt{x^2+y^2}\right)$$

$$= \frac{1}{x+\sqrt{x^2+y^2}}\cdot\left(1+\frac{1}{2}\,\frac{2x}{\sqrt{x^2+y^2}}\right)$$

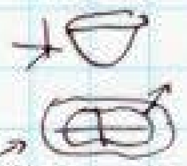
$$= \frac{1+2x}{(x+\sqrt{x^2+y^2})\sqrt{x^2+y^2}}$$

2-dim vs 3-dim Gradients

1) $z = f(x,y)$

$\nabla z = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$

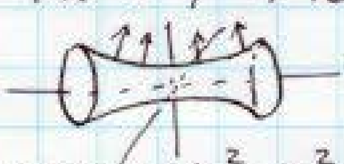
$z = \boxed{c = f(x,y)}$



2) $F(x,y,z) = c$

$\vec{\nabla} F = \frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}k$

Ex $4x^2 - y^2 + 16z^2 = 1$

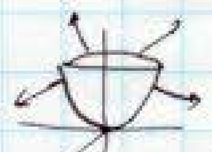


$F(x,y,z) = 4x^2 - y^2 + 16z^2 = 1$

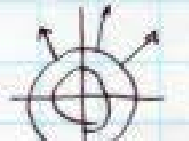
$\nabla F = 8x\vec{i} - 2y\vec{j} + 32z\vec{k}$

Consider

$z = x^2 + y^2 = f$



a) $\nabla f = \underbrace{2x\vec{i} + 2y\vec{j}}$



b) $x^2 + y^2 - z = 0$

$F(x,y,z) = x^2 + y^2 - z = 0$

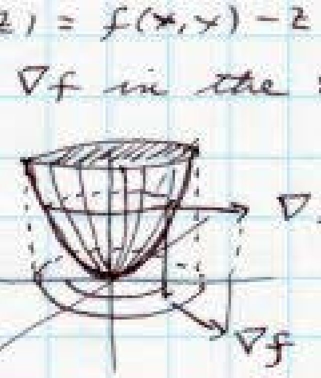
$\vec{\nabla} F = \underbrace{2x\vec{i} + 2y\vec{j}} - \vec{k}$

Fact If $z = f(x,y)$

$F(x,y,z) = f(x,y) - z$

$\vec{\nabla} F$ projects to ∇f in the x-y plane

$z = f(x,y)$



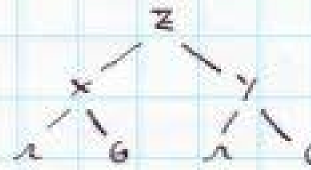
Do # 15.6 # 3, 7, 10, 11, 17, 23, 30

Sugg # 12, 28, 32

15.5 # 41

$z = f(x,y)$ $\quad x = r\cos\theta$

$y = r\sin\theta$



$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}r\cos\theta$$

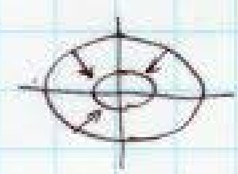
$$\frac{1}{r}\cdot\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x}\sin\theta + \frac{\partial z}{\partial y}\cos\theta$$

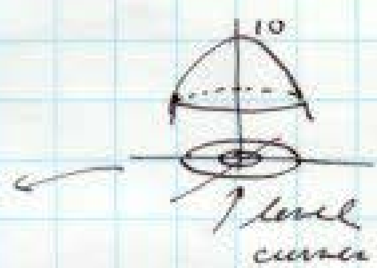
$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

2-dim vs 3-dim Gradient

1) $z = f(x,y) = 10 - x^2 - 2y^2$

$\nabla f = -2x\vec{i} - 4y\vec{j}$

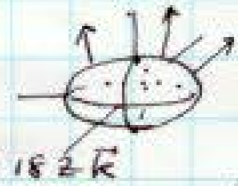




2) $x^2 + 4y^2 + 9z^2 = 1$

$F(x,y,z) = c$

$\nabla F = 2x\vec{i} + 8y\vec{j} + 18z\vec{k}$



3) $z = f(x,y)$

$F = f(x,y) - z$

$F = 10 - x^2 - 2y^2 - z = 0$

$\nabla F = \underbrace{-2x\vec{i} - 4y\vec{j}}_{\nabla f} - k$

Proj of ∇F onto (x-y) plane is ∇f

$\nabla\left(f(x,y) - z\right) =$

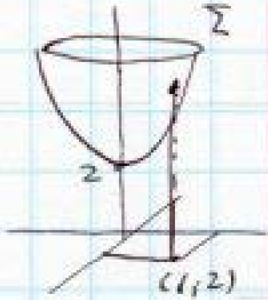
$\underbrace{f_x\vec{i} + f_y\vec{j}}_{\nabla f} - k$

Ex Let $z = 4x^2 + 9y^2 + 2 = f$

Elliptic Paraboloid

$P(1,2)$.

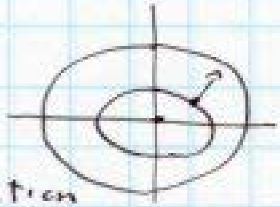
In which direction is ~~the~~ z steepest. What is the slope in that direction.



Sol

$\nabla z = 8x\vec{i} + 18y\vec{j}$

$\nabla f(x_0) = 8\vec{i} + 36\vec{j}$



1) $\vec{u} = \dfrac{\langle 8, 36\rangle}{\sqrt{8^2+36^2}}$ Direction

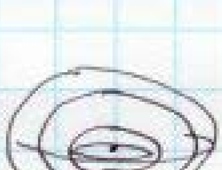
2) Slope $= \sqrt{8^2+36^2} =$

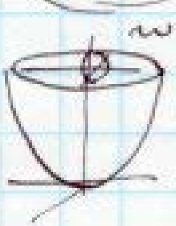
3-d. Gradient

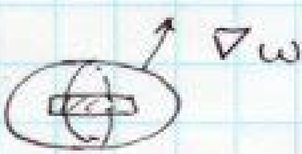
Given

$$w = x^2 + 3y^2 + 8z^2$$

w	$x^2+3y^2+8z^2$
0	$x^2+3y^2+8z^2 = 0$
1	$x^2+3y^2+8z^2 = 1$
4	$x^2+3y^2+8z^2 = 4$







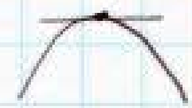
$\nabla w \perp$ to level surfaces!

Max-Min

I) $f: \mathbb{R} \to \mathbb{R}$ $\quad z = f(x)$

1) $f_x = 0$

Solve. Find the critical points x_c



2) $f_{xx}(x_c) \begin{cases} > 0 & \text{min} \\ < 0 & \text{Max} \\ = 0 & \text{fails} \end{cases}$

If $f''(x) = 0$ IP?

II $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad z = f(x,y)$

$(x,y) \to f(x,y) = z$

Discussion

a) Max

$f_x = 0,\ f_y = 0$

$f_{xx},\ f_{yy} < 0$

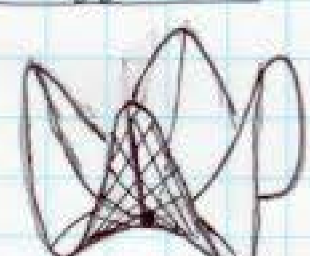
b) Min

$f_x = 0,\ f_y = 0$

$f_{xy},\ f_{yy} \geq 0$

Necessary!

Not sufficient!



$z = xy\,\dfrac{x^2 - y^2}{x^2 + y^2}$

$f_{xy}(0,0) \neq f_{yx}(0,0)$

Second Derivative Test

Given $z = f(x,y)$

1) $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$ Solve. Find the critical Points (x_c, y_c)

2) Find Hessian $H_{xy} = f_{xx} \cdot f_{yy} - f_{xy}^2$

$H_{xy}(x_c, y_c) \begin{cases} > 0 \cdot f_{xx} \begin{cases} > 0 & \text{min} \\ < 0 & \text{Max} \end{cases} \\ < 0 \quad \text{Saddle} \\ = 0 \quad \text{Fails} \end{cases}$

Ex $f(x,y) = x^2 - y^2$

1) $f_x = 2x = 0 \qquad (x_c, y_c) = (0,0)$

$f_y = -2y = 0$

2) $f_{xx} = 2 \qquad f_{xy} = 0 \qquad H_{xy} = -4 < 0$

$f_{yx} = 0 \qquad f_{yy} = -2 \qquad$ Saddle

Note. $H_{xy} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

Pb #6 15.7

$f(x,y) = x^3 y + 12x^2 - 8y$

1) $f_x = 3x^2 y + 24x = 0$

$f_y = x^3 - 8 = 0 \qquad x = 2$

If $x = 2 \quad 12y + (24)(2) = 0 \quad y = -4$

$(x_c, y_c) = (2, -4)$

2) $f_{xx} = 6xy + 24 \qquad f_{xy} = 3x^2$

$f_{yx} = 3x^2 \qquad f_{yy} = 0$

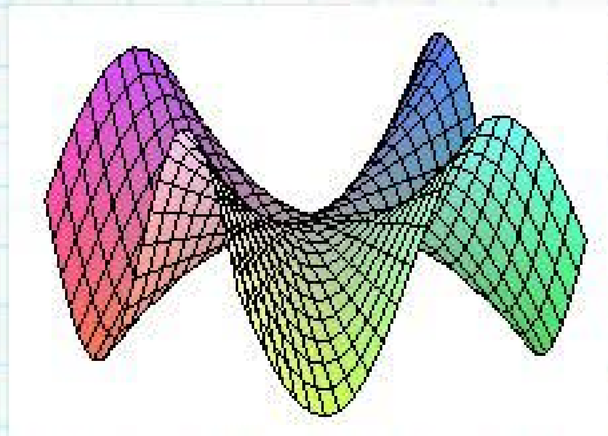
$H = -9x^4$

$H(2,-4) < 0$ Saddle

15.7 # 5−13 odd

15.6 # 3, 7, 10, 11, 17, 23, 30

Sugg 12, 28, 32

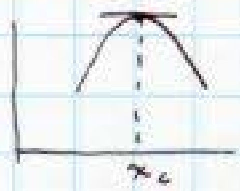


Max - Min

Review - 1-dim

$f : \mathbb{R} \to \mathbb{R}$ $y = f(x)$

1) $f_x = 0$ Solve
find c.p x_c



2) $f_{xx} \begin{cases} > 0 & \cup \\ < 0 & \cap \\ = 0 & \text{fails . IP?} \end{cases}$

Max-Min for surfaces

$f : \mathbb{R}^2 \to \mathbb{R}$ $z = f(x,y)$

1) $\left.\begin{array}{l} f_x = 0 \\ f_y = 0 \end{array}\right\}$ solve

2) $H_{xy} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \begin{cases} > 0 \ \begin{cases} f_{xx} > 0 \\ f_{xx} < 0 \end{cases} \\ < 0 \\ = 0 \ \text{Fails} \end{cases}$

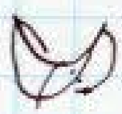
$H_{xy} = f_{xx} f_{yy} - f_{xy}^2$

Examples

Ex 1 $z = xy = f(x,y)$

1) $f_x = y = 0$
$f_y = x = 0$ $(x_c, y_c) = (0,0)$

2) $f_{xx} = 0$ $f_{xy} = 1$
$f_{yx} = 1$ $f_{yy} = 0$



$H_{xy} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0$

Ex 2 76 10

$f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$

1) $f_x = 6x^2 + y^2 + 10x = 0$
$f_y = 2xy + 2y = 0$
$2y(x+1) = 0$
$y = 0$ $x = -1$

$y = 0$ $6x^2 + 10x = 0$ $x = 0$
$2x(3x+5) = 0$ $x = -5/3$

if $x = -1$ $6x^2 + y^2 + 10x = 0$
$6 + y^2 - 10 = 0$
$y^2 = 4$ $y = \pm 2$

$(x_c, y_c) = (0,0), (-\frac{5}{3}, 0)$
$(-1,2), (-1,-2)$

② $f_{xx} = 12x + 10$ $f_{xy} = 2y$
$f_{yx} = 2y$ $f_{yy} = 2x + 2$

$H = (12x+10)(2x+2) - 4y^2$

Check $(0,0)$
$H(0,0) = 20 > 0$ local min at $(0,0)$
$f_{xx}(0,0) = 10 > 0$

Check $(-1,2)$
$H(-1,2) = (2)(0) - 4 \cdot 4 < 0$ Saddle
$H(-1,-2) = 0 - 16 < 0$ Saddle

5 - 13 odd

---

13) Oct 5

$f(x,y) = \dfrac{x^2y^2 - 8x + y}{xy}$

$f(x,y) = xy - \frac{8}{y} + \frac{1}{x}$

1) $f_x = y - \frac{1}{x^2} = 0$ $y = 1/x^2$
$f_y = x + \frac{8}{y^2} = 0$ $x + 8x^4 = 0$
$x(1 + 8x^3) = 0$

$x = 0$, $x = -\frac{1}{2}$
$y = \infty$ $y = 4$

2) $f_{xx} = \frac{2}{x^3}$ $f_{xy} = 1$
$f_{yx} = 1$ $f_{yy} = -\frac{16}{y^3}$

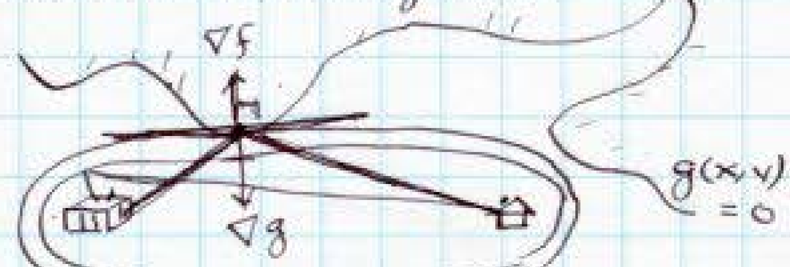
$H = -\frac{32}{x^3y^3} - 1$

$H(-\frac{1}{2}, 4) = \frac{-32}{-\frac{1}{8} \cdot 64} - 1 = 3 > 0$

$f_{xx}(-\frac{1}{2}, 4) = -16 < 0$

Do. $f(x,y) = x^4 - 5x^2 + y^2 + 3x + 2$

Lagrange Multipliers
Constrained Optimization



Given $f(x,y)$, find the extrema subject to the constraint $g(x,y) = 0$

∇f is parallel to ∇g

$\boxed{\nabla f = -\lambda \nabla g}$ Solve

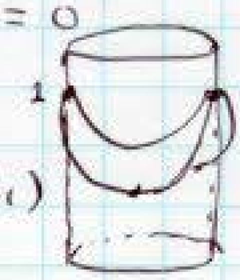
alternatively.

Define
$$\boxed{\begin{aligned} &F(x,y,\lambda) = f + \lambda g \\ &\vec{\nabla} F = 0 \\ &F_x = F_y = F_\lambda = 0 \end{aligned}}$$

Example: Pb3 15.8 {Lagrange Mult.

3) $f(x,y) = x^2 - y^2$

$g(x,y) = x^2 + y^2 - 1 = 0$



Let

$F = f + \lambda g$

$F = (x^2 - y^2) + \lambda(x^2 + y^2 - 1)$

$$\begin{cases} F_x = 2x + 2x\lambda = 0 \\ F_y = -2y + 2y\lambda = 0 \\ F_\lambda = x^2 + y^2 - 1 = 0 \end{cases}$$

$2x(1+\lambda) = 0 \qquad x = 0 \quad \lambda = -1$

$2y(-1+\lambda) = 0 \qquad y = 0 \quad \lambda = 1$

$x = 0 \quad y^2 - 1 = 0 \quad y = \pm 1 \;\Big|\; y = 0 \quad x^2 - 1 = 0 \quad x = \pm 1$

Sol $(0,1), (0,-1), (1,0)(-1,0)$

$f(0,1) = -1 \qquad f(1,0) = 1$

$f(0,-1) = -1$ min $\quad f(-1,0) = 1$ Max

Find Extrema of $f(x,y)$

$g(x,y) = 0, \; h(x,y) = 0$

$F(x,y,\lambda,\eta) = f + \lambda g + \eta h$

$F_x = 0 \quad F_\lambda = 0 \;\Big|\;$ Do: 15.8 #

$F_y = 0 \quad F_\eta = 0 \;\Big|\;$ 3, 5, 7, 9, 15

15.8 #15

$f(x,y,z) = x + 2y$

$g(x,y,z) = x + y + z - 1 = 0$

$h(x,y,z) = y^2 + z^2 - 4 = 0$

Let $F = f + \lambda g + \eta h$

$F = (x+2y) + \lambda(x+y+z-1) + \eta(y^2+z^2-4)$

$$\begin{cases} F_x = 1 + \lambda = 0 & \lambda = -1 \\ F_y = 2 + \lambda + 2y\eta = 0 = 1 + 2y\eta \\ F_z = \lambda + 2z\eta = 0 = -1 + 2z\eta \\ F_\lambda = x + y + z - 1 = 0 \\ F_\eta = y^2 + z^2 - 4 = 0 \end{cases}$$

$\eta = -\frac{1}{2y} = \frac{1}{2z} \Rightarrow -z = y$

$x + y - y - 1 = 0 \qquad x = 1$

$y^2 = 4 \qquad y = \pm\sqrt{2}$

$(1, \sqrt{2}, \sqrt{2})$ M $\qquad z = \mp\sqrt{2}$

$(1, -\sqrt{2}, \sqrt{2})$ m

$(1, \sqrt{2}, -\sqrt{2})$ M

$(1, -\sqrt{2}, -\sqrt{2})$ m $\quad (x+2y) = f$

Max $f_{max} = 1 + 2\sqrt{2}$

$f_{min} = 1 - 2\sqrt{2}$

9) $f(x,y,z) = xyz$ Optimize

$g = x^2 + 2y^2 + 3z^2 - 6 = 0$ } const

Let $F = f + \lambda g$

$F = xyz + \lambda(x^2 + 2y^2 + 3z^2 - 6)$

$$\begin{cases} F_x = yz + 2x\lambda = 0 \\ F_y = xz + 4y\lambda = 0 \\ F_z = xy + 6z\lambda = 0 \\ F_\lambda = x^2 + 2y^2 + 3z^2 - 6 = 0 \end{cases}$$

$\lambda = -yz/2x \qquad -\frac{yz}{2x} = -\frac{xz}{4y}$

$\lambda = -xz/4y \qquad y^2 = \frac{x^2}{2}$

$\lambda = -xy/6z$

$-\frac{xy}{6z} = -\frac{xz}{4y} \qquad \boxed{2y^2 = x^2} \quad \boxed{4y^2 = 6z^2}$

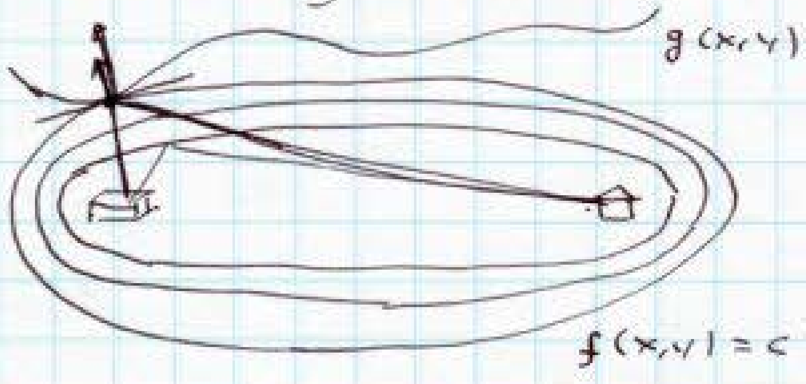
$4y^2 = 6z^2 \qquad 2y^2 = 3z^2$

$2y^2 + 2y^2 + 2y^2 - 6 = 0$

$6y^2 = 6$

$y = \pm 1, \; x = \pm\sqrt{2}, \; z = \pm\sqrt{\frac{2}{3}}$

Chicken-Turkey Problem



Farmer Chuck Jones
Extremum of a function with a constraint
$\nabla f \parallel \nabla g$

Pb Find the extrema of $f(x,y)$ subject to the constr. $g(x,y) = c$

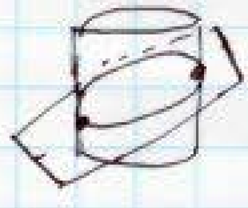
Sol $\boxed{\vec{\nabla} f = \lambda \vec{\nabla} g}$

or Let $\boxed{F(x,y,\lambda) = f + \lambda g}$

$\vec{\nabla} F = 0$

$$\begin{cases} F_x = 0 \\ F_y = 0 \\ F_\lambda = 0 \end{cases} \quad \text{Solve}$$

Example # 4 pg 991



$f(x,y) = 4x + 6y$
$g(x,y) = x^2 + y^2 = 13$

Sol Let

$F = f + \lambda g = (4x + 6y) + \lambda(x^2 + y^2 - 13)$

$$\vec{\nabla} F : \begin{cases} F_x = 4 + 2x\lambda = 0 \\ F_y = 6 + 2y\lambda = 0 \\ F_\lambda = x^2 + y^2 - 13 = 0 \quad \text{(Constr)} \end{cases}$$

$2x\lambda = -4 \qquad \lambda = -\frac{2}{x} \qquad x = -\frac{2}{\lambda}$

$2y\lambda = -6 \qquad \lambda = -\frac{3}{y} \qquad y = -\frac{3}{\lambda}$

$\frac{4}{\lambda^2} + \frac{9}{\lambda^2} - 13 = 0 \qquad \lambda^2 = 1$

$4 + 9 - 13\lambda^2 = 0 \qquad \lambda = \pm 1$

$13\lambda^2 = 13$

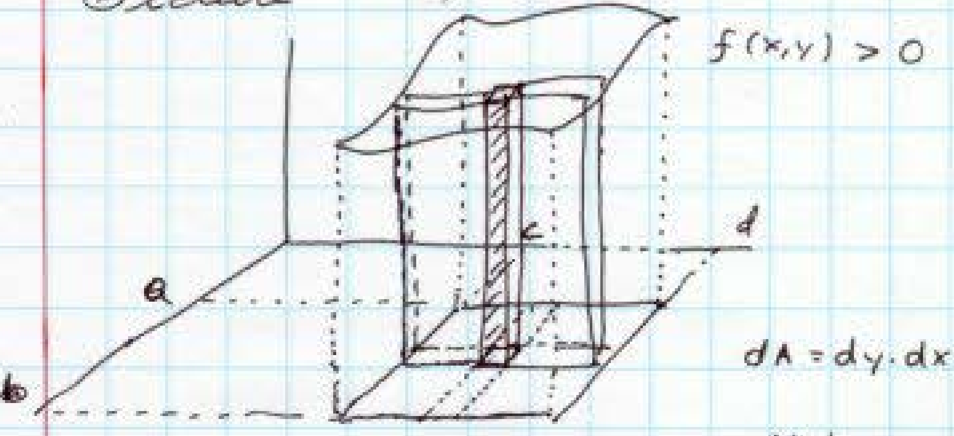
$\lambda = 1 \quad (-2,-3) \qquad f(-2,-3) = -26$ min

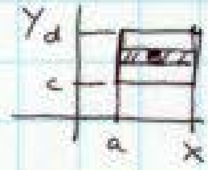
$\lambda = -1 \quad (2,3) \qquad f(2,3) = 26$ Max

Volumes by Double Integrals

$z = f(x,y)$. Find volume under the surface bounded by $a \le x \le b$, $c \le y \le d$

Picture





$dV = f(x,y) \cdot dy\,dx$

$V = \int_a^b \int_c^d f(x,y)\,dy\,dx$

Example 16 Sect 15.2

$f(x,y) = x e^{xy} \qquad R = [0,1] \times [0,1]$

$I = \int_0^1 \int_0^1 x e^{xy}\,dy\,dx$

$I = \int_0^1 e^{xy} \Big]_{y=0}^{y=1} dx$

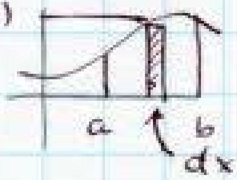
$= \int_0^1 (e^x - 1)\,dx = e^x - x \Big]_0^1 =$

$= e - 1 - (1) = e - 2$

Integration

1) Review

$y = f(x)$



$$A_{[a,b]} = \int_a^b f(x)\,dx$$

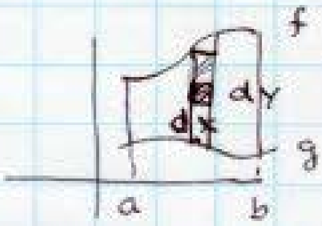
$$= F(x)\Big]_a^b = F(b) - F(a) \qquad F' = f \qquad \text{FTC}$$

2) Multiple integration
   a) Double Integrals
   b) Triple Integrals
   c) Change Coordinates

Double Integrals

Areas

$dA = dy\,dx$



$$A = \int_a^b \int_{g(x)}^{f(x)} 1\,dy\,dx$$

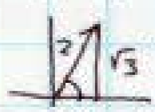
$$= \int_a^b y\Big]_{g(x)}^{f(x)} dx$$

$$= \int_a^b [f(x) - g(x)]\,dx$$

15.6 #3

$f(x,y) = x^2y^3 + 2x^4y \qquad P(1,-2)$

$\theta = \frac{\pi}{3} \qquad \vec{u} = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$



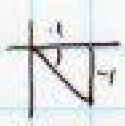
$\nabla f = \langle 2xy^3 + 8x^3y,\ 3x^2y^2 + 2x^4 \rangle$

$\nabla f = \langle -16 - 16,\ 12 + 2 \rangle = \langle -32, 14 \rangle$

$D_u f(P) = \langle -32, 14 \rangle \cdot \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

$= -16 + 7\sqrt{3}$

15.6 # 17



$g(x,y,z) = x \tan^{-1} \frac{y}{z}$

$P(1, 2, -2)$

$\vec{v} = \langle 1, 1, -1 \rangle \qquad \vec{u} = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \rangle$

$$\nabla g = \left\langle \tan^{-1}\frac{y}{z},\ \frac{x \cdot 1/z}{1 + (\frac{y}{z})^2},\ \frac{x(-\frac{y}{z^2})}{1 + (\frac{y}{z})^2} \right\rangle$$

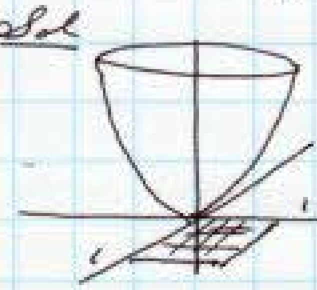
$\nabla g(P) = \langle -\frac{\pi}{4}, -\frac{1}{2}, -\frac{1}{2} \rangle$

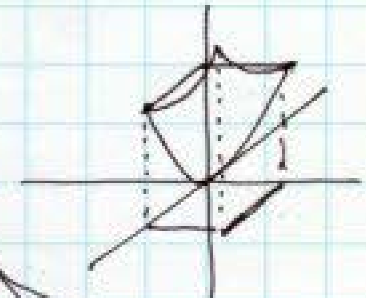
$D_v g(P) = \nabla g(P) \cdot \vec{u} = \boxed{\frac{-\pi}{4\sqrt{3}}} + \cancel{\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}}$

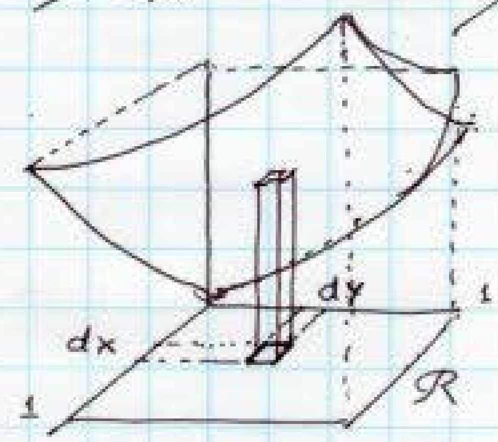
Volumes by Double Integrals

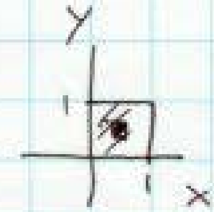
Pb Find the volume under the surface $z = x^2 + y^2$ for $0 \le x \le 1$, $y \in [0,1]$

Sol









$dV = f(x,y) \cdot dy\,dx$

$V = \iint_R f(x,y)\,dy\,dx$

$V = \int_0^1 \int_0^1 (x^2 + y^2)\,dy\,dx$

$= \int_0^1 [x^2y + \frac{1}{3}y^3]_{y=0}^{y=1}\,dx$

$= \int_0^1 [x^2 + \frac{1}{3}]\,dx$

$= \frac{x^3}{3} + \frac{x}{3}\Big]_0^1$

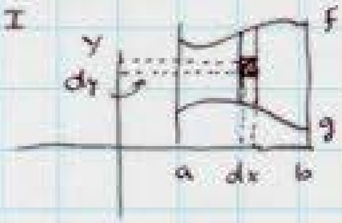
$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

Read 16.1, 2

Do # 3-17 odd 16.2

Double Integrals over General Reg.

Type I

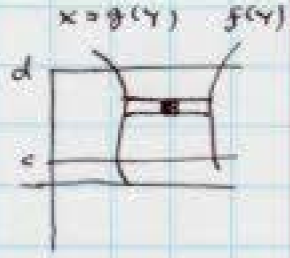


$z = f(x,y)$

$$I = \iint_R f\underbrace{(x,y)\,dy\,dx}_{dV}$$

$$= \int_a^b \int_{g(x)}^{f(x)} f(x,y)\,dy\,dx$$

Type II

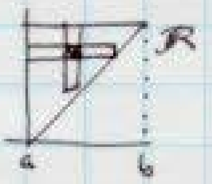


$$I = \iint_R f(x,y)\,dA$$

$$= \int_c^d \int_{g(y)}^{f(y)} f(x,y)\,dx\,dy$$

$\boxed{dA = dx\,dy}$

Some regions can be classified as type I as well as type II

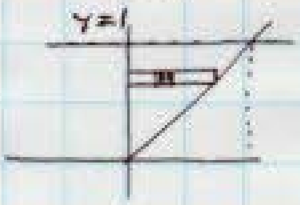


$$\iint f(x,y)\,dy\,dx = \iint f\,dx\,dy$$

Fubini's Theorem

Pb 10, (16.3)

$$\iint_D e^{y^2}\,dA \qquad D = \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le y\}$$



$$I = \int_0^1 \int_0^y e^{y^2}\,dx\,dy$$

$$I = \int_0^1 x e^{y^2}\Big]_{x=0}^{x=y} dy$$

$$= \int_0^1 y e^{y^2}\,dy$$

$$= \tfrac12 e^{y^2}\Big]_0^1 = \tfrac12 (e-1)$$

Do # $\boxed{16.3}$

1, 5, 8, 11, 13, 19, 22, 24, 26

31, 39, 41, 42, 43

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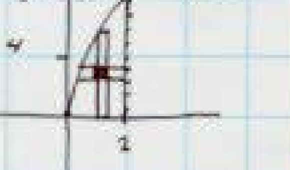
44) $\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy = I$



$x = 2 \qquad x = \sqrt[3]{y}$

$y = 0 \qquad y = 8$

$$I = -\int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx$$

$$= -\int_0^2 x^3 e^{x^4}\,dx = -\tfrac14 e^{x^4}\Big]_0^2$$

$$= -\tfrac14 (e^{16} - 1).$$

16.2 # 9)

$$I = \int_1^4 \int_1^2 \left(\frac{y}{x} + \frac{x}{y}\right) dy\,dx$$

$$= \int_1^4 \frac{y^2}{2x} + x \ln y \Big]_{y=1}^{y=2} dx$$

$$= \int_1^4 \left(\frac{2}{x} + x\ln 2 - \frac{1}{2x}\right) dx$$

$$= 2\ln x + \frac{x^2}{2}\ln 2 - \frac12 \ln x\Big]_1^4$$

$$= \frac32 \ln x + \frac12 x^2 \ln 2\Big]_1^4$$

$$= \frac32 \ln 4 + 8\ln 2 - \left(\tfrac12 \ln 2\right)$$

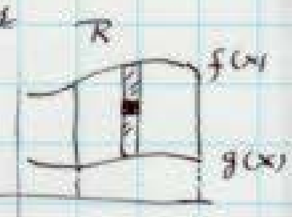
$$= 3\ln 2 + 8\ln 2 - \tfrac12 \ln 2$$

$$= 2\tfrac12 \ln 2$$

More on Double integrals

$z = f(x,y)$

I



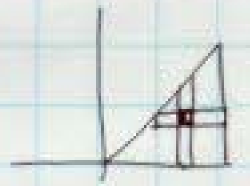
$$I = \iint_R f(x,y)\,dA$$

$$I = \int_a^b \int_{g(x)}^{f(x)} f(x,y)\,dy\,dx$$

II

$$I = \int_c^d \int_{g(y)}^{f(y)} f\,dx\,dy$$

Hybrid regions
Integrate in any order.



Fubini's Theorem

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dy\,dx$$

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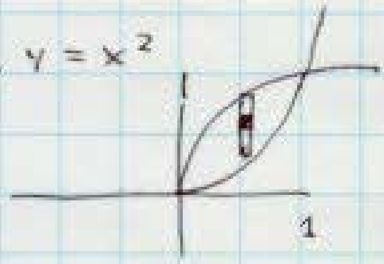
14 (16.3)

$$\iint_D (x+y)\,dA \qquad y = \sqrt{x},\ y = x^2$$



$$I = \int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)\,dy\,dx$$

$$= \int_0^1 \left(xy + \tfrac12 y^2\right)\Big|_{x^2}^{\sqrt{x}} dx$$

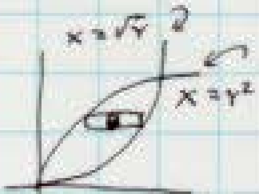
$$= \int_0^1 \left[x\sqrt{x} + \tfrac12 x - \left(x^3 + \tfrac12 x^4\right)\right] dx$$

$$= \int_0^1 \left(x^{3/2} - \tfrac12 x - x^3 + \tfrac12 x^4\right) dx$$

$$= \frac25 - \frac14 - \frac14 + \frac1{10} = 3/10$$

= [illegible] $\frac{23}{60}$

$$I = -\int_0^1 \int_{\sqrt{y}}^{y^2} (x+y)\,dx\,dy$$

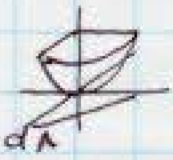


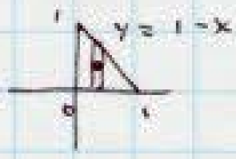
$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (x+y)\,dx\,dy$$

16.3 # 22

$z = x^2+y^2+4 \qquad x=0,\ y=0,\ z=0$

$x+y = 1$





$I = \iint_R (x^2+y^2+4)\, dA$

$I = \int_0^1 \int_0^{1-x} (x^2+y^2+4)\, dy\, dx$

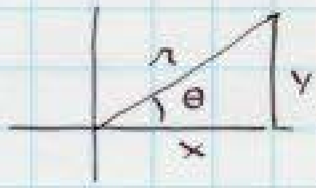
$I = \int_0^1 \int_0^{1-y} (x^2+y^2+4)\, dx\, dy$

$= \int_0^1 \left[\tfrac{1}{3}x^3 + xy^2 + 4x\right]_0^{1-y} dy$

$= \int_0^1 \left[\tfrac{1}{3}(1-y)^3 + y^2(1-y) + 4(1-y)\right] dy$

Polar Coordinates - Areas, Volumes

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

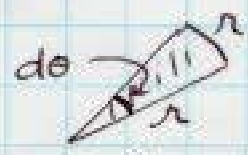


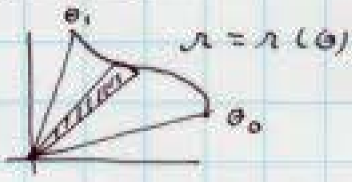
$z = z$

$r^2 = x^2+y^2$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Areas by single & Integrals





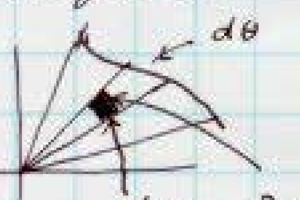
$dA = \frac{1}{2} r^2\, d\theta$

$A = \int_{\theta_0}^{\theta_1} \frac{1}{2} r^2\, d\theta$

Areas by Double Integrals.

$dA = r\, dr\, d\theta$

$A = \int_{\theta_0}^{\theta_1} \int_0^r r\, dr\, d\theta$



$= \int_{\theta_0}^{\theta_1} \frac{1}{2} r^2\, d\theta$

$\boxed{16.4}$

7, 9 11, 13

19, 20, 25,

27, 29

Examples:

$x = r\cos\theta$

$y = r\sin\theta$ Abuse

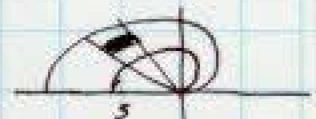
$\iint_R f(x,y)\, dx\, dy = \iint f(r,\theta)\, r\, dr\, d\theta$

Use Polar coord. where there is symmetry w.r.t z axis (Look for (x^2+y^2))

13) $\iint_D (x^2+y^2)\, dA \qquad r = \theta$

$\theta \in [0, 2\pi] \quad r = 2\theta$

$dA = r\, dr\, d\theta$



$I = \int_0^{2\pi} \int_\theta^{2\theta} r^2 \cdot r\, dr\, d\theta$

$= \int_0^{2\pi} \frac{1}{4} r^4 \Big]_\theta^{2\theta} d\theta = \int_0^{2\pi} \frac{1}{4}(16\theta^4 - \theta^4)\, d\theta$

$= \frac{15}{4} \int_0^{2\pi} \theta^4\, d\theta = \frac{15}{4} \frac{\theta^5}{5}\Big]_0^{2\pi}$

$= \frac{3}{4}(2\pi)^5 = 24\pi^5$

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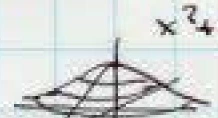
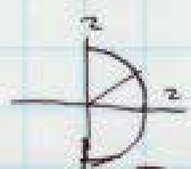
11) $\iint e^{-x^2-y^2}\, dA \qquad x = \sqrt{4-y^2}$

$f(x,y) = e^{-(x^2+y^2)} \qquad x^2+y^2 = 4$





$I = 2\int_0^{\frac{\pi}{2}} \int_0^2 e^{-r^2}\, r\, dr\, d\theta$

$= 2\int_0^{\frac{\pi}{2}} -\frac{1}{2} e^{-r^2}\Big]_0^2 d\theta$

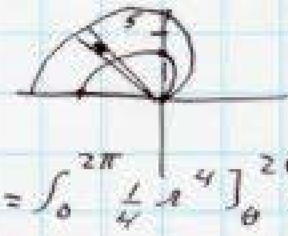
$= -\int_0^{\pi/2} (e^{-4} - 1)\, d\theta = -\frac{\pi}{2}(e^{-4} - 1)$

$= \frac{\pi}{2}(1 - e^{-4})$

13. $\iint (x^2+y^2)\, dA \qquad r=\theta \qquad r=2\theta$

$dA = r\,dr\,d\theta \qquad 0 \le \theta \le 2\pi$

$I = \int_0^{2\pi}\int_\theta^{2\theta} r^2 \cdot r\, dr\, d\theta$



$I = \int_0^{2\pi}\int_\theta^{2\theta} r^3\, dr\, d\theta = \int_0^{2\pi} \frac{1}{4} r^4 \Big]_\theta^{2\theta} d\theta$

$= \frac{1}{4}\int_0^{2\pi} (16\theta^4 - \theta^4)\, d\theta$

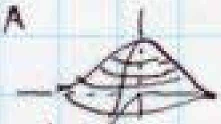
$= \frac{1}{4}\int_0^{2\pi} 15\theta^4\, d\theta = \frac{15}{4}\frac{\theta^5}{5}\Big]_0^{2\pi}$

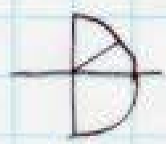
$= \frac{3}{4}(2\pi)^5 = 24\pi^5$

11) $\iint e^{-x^2-y^2}\, dA \qquad x = \sqrt{4-y^2}$

$I = \iint e^{-(x^2+y^2)}\, dA \qquad x^2+y^2 = 4$

$= 2\int_0^{\pi/2}\int_0^2 e^{-r^2}\, dA$





$I = 2\int_0^{\frac{\pi}{2}}\int_0^2 e^{-r^2} r\, dr\, d\theta$

$= 2\int_0^{\pi/2} -\frac{1}{2} e^{-r^2}\Big]_0^2\, d\theta$

$= 2\left(-\frac{1}{2}\right)\int_0^{\pi/2} (e^{-4} - e^0)\, d\theta$

$= (-1)\,\frac{\pi}{2}(e^{-4} - 1) = \frac{\pi}{2}(1 - e^{-4})$

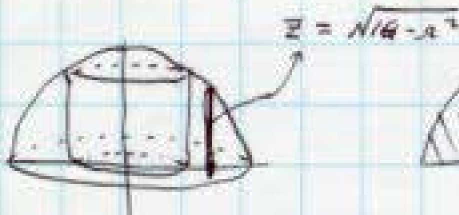
Do 7, 9, 11, 13
19, 20, 25, 27, 29

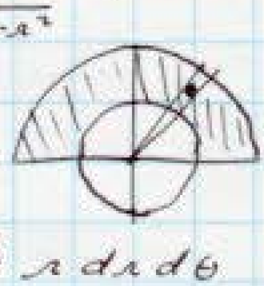
Oct 14 16.4 # 20

Volume inside $x^2+y^2+z^2 = 16$

Outside $x^2+y^2 = 4$

$z = \sqrt{16-r^2}$





$I = 2\int_0^{2\pi}\int_2^4 \sqrt{16-r^2}\; r\, dr\, d\theta$

$= 2\int_0^{2\pi} -\frac{1}{3}(16-r^2)^{3/2}\Big]_2^4\, d\theta$

$= -\frac{2}{3}\cdot 2\pi\,[-12^{3/2}] = \frac{4\pi}{3}$

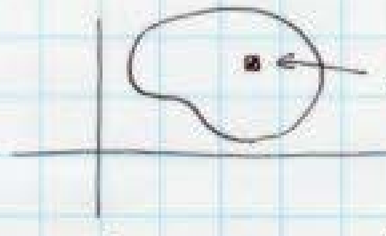
$= \frac{4}{3}\pi\, 12^{3/2}$

Means (Averages)

Course	Grade	
3	A − 4	3 × 4
3	B − 3	3 × 3
5	A − 4	5 × 4
1	D − 1	1 × 1
4	C − 2	4 × 2

$Ave = \dfrac{(3\times4)+(3\times3)+(5\times4)+(1\times1)+(4\times2)}{3+3+5+1+4}$

$Ave = \boxed{\dfrac{\sum_{i=1}^{m} m_i x_i}{\sum m_i}}$



2-dim plate

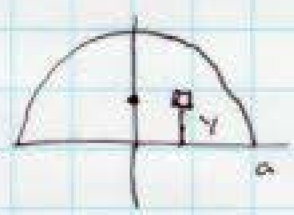
$dm = \sigma\, dA$

density − $\sigma = \frac{dm}{dA}$

$\boxed{\bar{x} = \dfrac{\int x\, dm}{\int dm}} \qquad dm = \sigma\, dA$

$\boxed{\bar{y} = \dfrac{\int y\, dm}{\int dm}}$

Example.



$x^2+y^2 = a^2$

$\sigma = 1$

$y = \sqrt{a^2-x^2}$

$dm = \sigma \cdot dy\, dx$

$\bar{x} = \dfrac{\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} x\, \overset{1}{\sigma}\, dy\, dx}{\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} \sigma\, dy\, dx} = 0$

$\bar{y} = \dfrac{\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} y\, dy\, dx}{\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} dy\, dx} = \frac{4}{3}\frac{a}{\pi}$

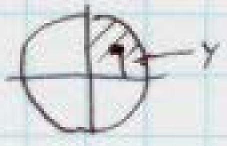
$\bar{r} = \dfrac{\int r\, dm}{\int dm} \qquad dm = \sigma \begin{cases} dA \\ dV \end{cases}$

$I_x = \int r_x^2\, dm$ | #7, 9, 11, 15, 17

Oct 15

16.5 # 11

$x^2+y^2 \le 1 \qquad y=\sqrt{1-x^2}$



$\rho = ky$

$$\bar{x} = \frac{\int_0^1\int_0^{\sqrt{1-x^2}} x\,ky\,dy\,dx}{\int_0^1\int_0^{\sqrt{1-x^2}} ky\,dy\,dx} =$$

$$\bar{y} = \frac{\int_0^1\int_0^{\sqrt{1-x^2}} y\,\overbrace{(ky\,dy\,dx)}^{dm=\rho dA}}{\int_0^1\int_0^{\sqrt{1-x^2}} (ky\,dy\,dx)} =$$

Surface Area

$z = f(x,y)$

$\vec{a} = \Delta x\,\vec{i} + f_x \Delta x\,\vec{k}$

$\vec{b} = \Delta y\,\vec{j} + f_y \Delta y\,\vec{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ dx & 0 & f_x dx \\ 0 & dy & f_y dy \end{vmatrix}$$

$= -f_x\,dx\,dy\,\vec{i} - f_y\,dx\,dy\,\vec{j} + dx\,dy\,\vec{k}$

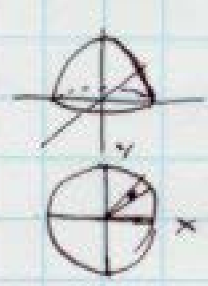
$= dx\,dy\,(-f_x, -f_y, 1)$

$$\boxed{dS = \sqrt{1+f_x^2+f_y^2}\,dx\,dy}$$

Ex 6 $\quad z = 4-x^2-y^2$ above $z=0$

Find Surface area

$f(x,y) = 4-x^2-y^2$



$f_x = -2x$

$f_y = -2y$

$dS = \sqrt{1+4x^2+4y^2}\,dA$

$S = \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\,r\,dr\,d\theta$

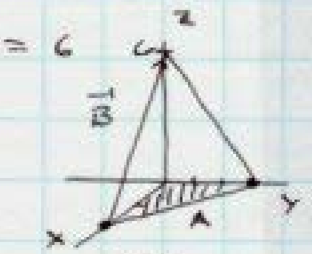
$= \int_0^{2\pi} \frac{1}{12}(1+4r^2)^{3/2}\Big]_0^2\,d\theta$

$= \frac{1}{12}\left[17^{3/2}-1\right]\cdot 2\pi$

Do # 1, 3, 5, 6, 10

Oct 18

#3, 16.6 $\qquad 3x+2y+z=6$

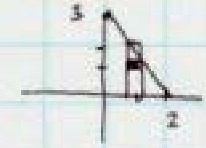


$z = 6-3x-2y = f(x,y)$

$3x+2y=6$

$2y = 6-3x$

$y = 3-\frac{3}{2}x = g$



$$\boxed{dS = \sqrt{1+f_x^2+f_y^2}\,dA}$$

$f_x = -3 \qquad f_y = -2 \qquad A = \langle -2, 3, 0\rangle$

$dS = \sqrt{1+9+4}\,dy\,dx \qquad B = \langle -2, 0, 6\rangle$

$S = \int_0^2\int_0^g \sqrt{14}\,dy\,dx$

$S = \int_0^2 \sqrt{14}\,(3-\frac{3}{2}x)\,dx$

$= \sqrt{14}\,(3x-\frac{3}{4}x^2)\Big]_0^2$

$= \sqrt{14}\,(6-3) = 3\sqrt{14}$

$A = \langle -2, 3, 0\rangle, \quad B = \langle -2, 0, 6\rangle$

$\vec{A}\times\vec{B} = i\begin{vmatrix} 3 & 0 \\ 0 & 6\end{vmatrix} - j\begin{vmatrix} -2 & 0 \\ -2 & 6\end{vmatrix} + k\begin{vmatrix} -2 & 3 \\ -2 & 0\end{vmatrix}$

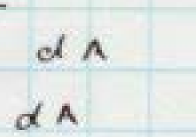
$= \langle 18, 12, 6\rangle$

$= 6\langle 3, 2, 1\rangle$

$\frac{1}{2}|A\times B| = 3\sqrt{9+4+1} = 3\sqrt{14}$

5) $y^2+z^2=9$

$z^2 = 9-y^2$



$z = \sqrt{9-y^2} = f(x,y)$

$f_x = 0$

$f_y = \frac{-2y}{2\sqrt{9-y^2}} = \frac{-y}{\sqrt{9-y^2}}$

$dS = \sqrt{1+0^2+\frac{y^2}{9-y^2}}\,dA$

$= \sqrt{\frac{9-y^2}{9-y^2}+\frac{y^2}{9-y^2}}\,dA$

$dS = \frac{3}{\sqrt{9-y^2}}\,dA$

$S = \int_0^4\int_0^2 \frac{3}{\sqrt{9-y^2}}\,dy\,dx$

$= \int_0^4 3\sin^{-1}\frac{y}{9}\Big]_0^2\,dx$

$= 12\sin^{-1}\frac{2}{9}$

Quiz

1) $\iint_D \frac{4y}{x^3+2}\,dA \qquad 1 \le x \le 2 \qquad 0 \le y \le 2x$

2) Volume inside the sphere $x^2+y^2+z^2=16$ and outside the cylinder $x^2+y^2=4$

Oct 18

Triple Integrals

$dv = dz\,dy\,dx$

$V = \iiint dv$

$V = \int_a^b \int_{f(x)}^{g(x)} \int_{g(x,y)}^{f(x,y)} 1\,dv$

$V = \int_a^b \int_{f(x)}^{g(x)} \int_{g(x,y)}^{f(x,y)} 1\,dz\,dy\,dx$

Ex 4)

$I = \int_1^2 \int_0^x \int_0^{1-y} (x^3y^2z)\,dz\,dy\,dx$

$= \int_1^2 \int_0^x \frac{1}{2}x^3y^2z^2\Big]_0^{1-y} dy\,dx$

$= \int_1^2 \int_0^x \frac{1}{2}x^3y^2(1-y)^2\,dy\,dx$

$= \int_1^2 \int_0^x \frac{1}{2}x^3y^2(1-2y+y^2)\,dy\,dx$

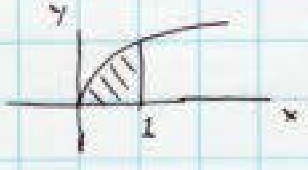
$= \int_1^2 \int_0^x \frac{1}{2}x^3(y^2-2y^3+y^4)\Big]\,dy\,dx$

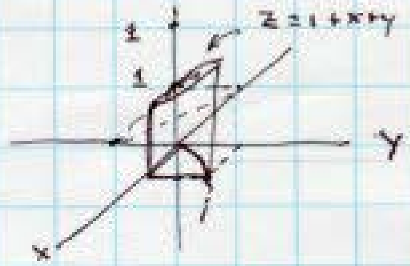
$= \int_1^2 \frac{1}{2}x^3\left[\frac{1}{3}y^3-\frac{2}{4}y^4+\frac{1}{5}y^5\right]_0^x dx$

$= \int_1^2 \frac{1}{2}\left(\frac{1}{3}x^6-\frac{1}{2}x^7+\frac{1}{5}x^8\right)dx$

etc.

9) $\iiint_E 6xy\,dV \qquad z=1+x+y \qquad y=\sqrt{x},\ y=0,\ x=1$





$I = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} (6xy)\,dz\,dy\,dx$

16.7 Do # 3, 7, 11, 13, 17

Oct 19 (16.7)

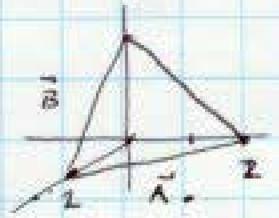
11) $\iiint_E xy\,dV \qquad (0,0,0) \quad (0,2,0) \quad (1,0,0) \quad (0,0,3)$

Find eq. of plane

$\vec{A} = \langle -1, 2, 0\rangle$

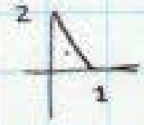
$\vec{B} = \langle -1, 0, 3\rangle$

$\vec{A}\times\vec{B} = \langle 6, 3, 2\rangle = \vec{N}$



$6(x-1)+3y+2z=0$

$6x+3y+2z=6$



$z = -3x - \frac{3}{2}y + 3$

$y = -2x + 2$

$I = \int_0^1 \int_0^{-2x+2} \int_0^{-3x-\frac{3}{2}y+3} xy\,dz\,dy\,dx$

$I = \frac{1}{10}$

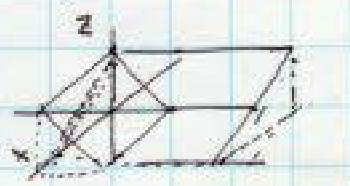
13) $\iiint_E z\,dV \qquad x=0,\ y=0,\ z=0 \qquad y+z=1,\ x+z=1$

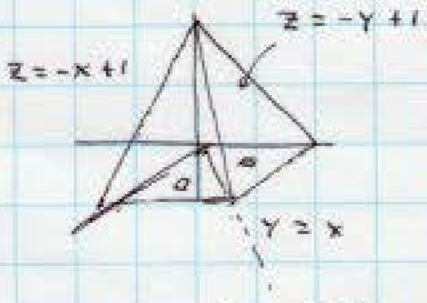
$z = -y+1$

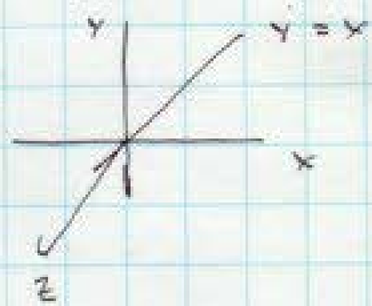
$z = -x+1$

$-y+1 = -x+1$

$y = x$







$I = \int_0^1 \int_0^x \int_0^{-x+1} \underbrace{z}_{\rho}\,dz\,dy\,dx + \int_0^1 \int_0^y \int_0^{-y+1} z\,dz\,dx\,dy$

$I =$

Center of mass

$\rho = \frac{dm}{dv} \qquad \boxed{dm = \rho\,dV} \quad \boxed{dm = \rho\,dA}$

$\bar{x} = \frac{\iiint x\,dm}{\iiint dm} = \frac{\int x\,dm}{\int dm}$

$\bar{y} = \frac{\int y\,dm}{\int dm} \qquad \bar{z} = \frac{\int z\,dm}{\int dm}$

Ex Volume bounded by $x=0, y=0, z=0$, $y+z=1$, $x+z=+1$. Suppose $\rho = 1$. Find $\bar{z}$

Sol

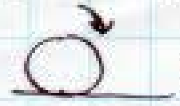
$\bar{z} = \dfrac{\int_0^1\int_0^x\int_0^{-x+1} z\,dz\,dy\,dx + \int_0^1\int_0^y\int_0^{-y+1} z\,dz\,dx\,dy}{\int_0^1\int_0^x\int_0^{-x+1} dz\,dy\,dx + \int_0^1\int_0^y\int_0^{-y+1} dz\,dx\,dy}$

$\boxed{\bar{z} = \frac{1}{4}}$

Moment of Inertia of a Sphere

R = radius

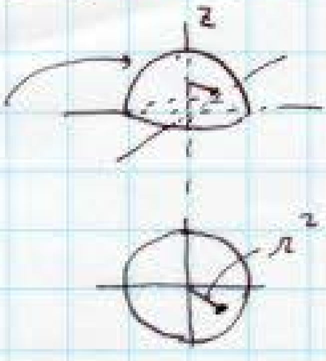
m = mass



$KE_R = \frac{1}{2} I \omega^2$

$$x^2+y^2+z^2=R^2$$

$$z=\sqrt{R^2-(x^2+y^2)}$$



$$I_z = \iiint \underbrace{(x^2+y^2)}\, dm$$

$dm = \rho\, dV$

$$I_z = \int_0^{2\pi}\int_0^R\int_0^{\sqrt{R^2-r^2}} r^2 \rho\, dz \cdot r\, dr\, d\theta$$

$$m = \rho \cdot \frac{4}{3}\pi R^3$$

$$I = \frac{m \cdot 2\int_0^{2\pi}\int_0^R\int_0^{\sqrt{R^2-r^2}} r^3 \rho\, dz\, dr\, d\theta}{\rho \cdot \frac{4}{3}\pi R^3}$$

$$I = \frac{2}{5} m R^2$$

Moment of Inertia of cylinder

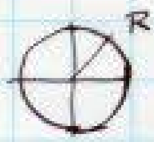
R = radius

h = height

ρ = density

m = mass

$z = h$



$$I_z = \iiint r^2\, dm$$

$$I_z = \frac{m\int_0^{2\pi}\int_0^R\int_0^h r^2 \rho\, dz \cdot r\, dr\, d\theta}{\rho \cdot \pi R^2 h}$$

$$= \frac{m \cdot 2\pi h \rho \int_0^R r^3\, dr}{\rho \pi R^2 h}$$

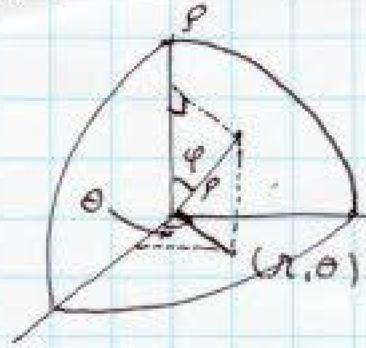
$$= \frac{m \cdot 2\pi h \rho R^4}{4 \rho \pi R^2 \cdot h}$$

$$I_z = \frac{1}{2} m R^2$$

Do 16.8

7, 9, 11, 17, 21, 23

Integrals in Spherical Coordinates



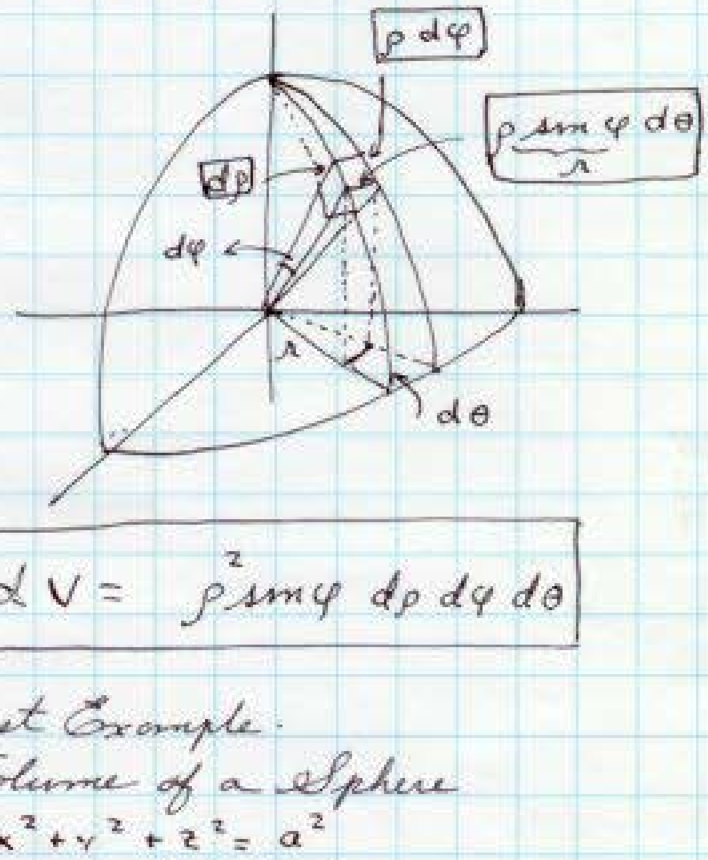
13.7

$$\rho \sin\varphi = r$$

$$\begin{cases} x = r\cos\theta = \rho\sin\varphi\cos\theta \\ y = r\sin\theta = \rho\sin\varphi\sin\theta \\ z = = \rho\cos\varphi \end{cases}$$

$$\begin{cases} \rho^2 = x^2+y^2+z^2 \\ \varphi = \cos^{-1}(z/\rho) \\ \theta = \tan^{-1}(y/x) \end{cases}$$

Differential of Volume



$$\boxed{dV = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta}$$

First Example

Volume of a Sphere

$$x^2+y^2+z^2=a^2$$

$$\boxed{\rho = a}$$

$$V = \iiint dV$$

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}\rho^3\Big]_0^a \sin\varphi\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}a^3 \sin\varphi\, d\varphi\, d\theta$$

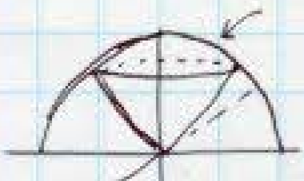
$$= \int_0^{2\pi} -\frac{1}{3}a^3\cos\varphi\Big]_0^{\pi}\, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{3}(a^3)(-1-1)\, d\theta$$

$$= \frac{2}{3}a^3 \cdot 2\pi = \frac{4}{3}\pi a^3$$

Example 2 Find the volume bounded above by the sphere $x^2+y^2+z^2=a^2$ and below by the cone $z^2=x^2+y^2, \quad z>0$

Sol



Sphere: $\rho = a$

Cone $\phi = \frac{\pi}{4}$

$$\begin{aligned}
V &= \iiint dV \qquad \boxed{\text{Jacobian}} \\
&= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{a} \underbrace{\rho^2 \sin\varphi}\, d\rho\, d\varphi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/4} \tfrac{1}{3}a^3 \sin\varphi \, d\varphi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/4} \tfrac{1}{3}a^3 \sin\varphi \, d\varphi\, d\theta \\
&= \int_0^{2\pi} -\tfrac{1}{3}a^3 \cos\varphi \Big]_0^{\pi/4} d\theta \\
&= \int_0^{2\pi} -\tfrac{1}{3}a^3 \left(\tfrac{\sqrt{2}}{2}-1\right) d\theta \\
&= \tfrac{1}{3}a^3\left(1-\tfrac{\sqrt{2}}{2}\right)\cdot 2\pi \\
&= \tfrac{1}{3}\pi a^3 (2-\sqrt{2})
\end{aligned}$$

Oct 22 **Change of variable Theorem**

1) Wedge multiplication

a) $dx \wedge dy = -dy \wedge dx$

b) $dx \wedge dx = 0$

c) $f\,dx \wedge g\,dy = f\cdot g\, dx \wedge dy$

Cartan

Application

Polar coord

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} dx = \cos\theta \cdot dr + r(-\sin\theta)\,d\theta \\ dy = \sin\theta\, dr + r\cos\theta\, d\theta \end{cases}$$

$$\begin{aligned}
dx\wedge dy &= r\cos^2\theta\, dr\wedge d\theta - r\sin^2\theta\, d\theta\wedge dr \\
&= r\cos^2\theta\, dr\wedge d\theta + r\sin^2\theta\, dr\wedge d\theta \\
&= r\, dr\wedge d\theta
\end{aligned}$$

$$\boxed{dA = dx\,dy = r\,dr\,d\theta}$$

In general

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

$$\begin{cases} dx = \frac{\partial x}{\partial u}\cdot du + \frac{\partial x}{\partial v}\cdot dv \\ dy = \frac{\partial y}{\partial u}\, du + \frac{\partial y}{\partial v}\, dv \end{cases}$$

$$\begin{aligned}
dx\wedge dy &= \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v}\, du\wedge dv + \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\, dv\wedge du \\
&= \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) du\wedge dv
\end{aligned}$$

$$dx\wedge dy = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}\, du\wedge dv$$

Jacobian $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$

Thu

$x = \rho\sin\varphi\cos\theta$

$y = \rho\sin\varphi\sin\theta$

$z = \rho\cos\varphi$

Compute $dx\wedge dy\wedge dz$

Change of Variables

Wedge Product (Cartan)

1) $dx \wedge dy = -dy \wedge dx$
2) $dx \wedge dx = -dx \wedge dx = 0$
3) $f\, dx \wedge g\, dy = g \cdot f\, dx \wedge dy$

Polar Coord - Differential of Area

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$dx = \cos\theta\, dr + r(-\sin\theta\, d\theta)$$
$$dy = \sin\theta\, dr + r\cos\theta\, d\theta$$
$$dx \wedge dy = r\cos^2\theta\, dr \wedge d\theta - r\sin^2\theta\, d\theta \wedge dr$$
$$= r\cos^2\theta\, dr \wedge d\theta + r\sin^2\theta\, dr\, d\theta$$
$$dx \wedge dy = r\, dr \wedge d\theta$$

$$\iint dx\, dy = \iint r\, dr\, d\theta$$

General Coord (2-dim)

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

$$\begin{cases} dx = \dfrac{\partial x}{\partial u} du + \dfrac{\partial x}{\partial v} dv \\ \\ dy = \dfrac{\partial y}{\partial u} du + \dfrac{\partial y}{\partial v} dv \end{cases}$$

$$dx \wedge dy = \left(\frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v}\right) du \wedge dv + \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\, dv \wedge du$$

$$dx \wedge dy = \left[\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right] du \wedge dv$$

$$\boxed{dx \wedge dx = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\, du \wedge dv}$$

Hw

Given: $\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$

Compute - simplify

$$dx \wedge dy \wedge dz = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

Review:

1) Limits and continuity
Show how to determine if a function is ~~cont~~ discont.
Ex pg 928 # 9

2) Partial Derivatives
Compute $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$
Ex pg 940 # 21, 49

3) Special equations. pg 941
Heat Eq : 74, 65
Wave Eq : 68
Laplaces eq: 66

4) Differentials (15.4)
$z = f(x,y)$ $dz = f_x dx + f_y dy$
Implicit diff - Ex 940 # 42
958 # 31

5) Chain Rule
$f = f(u,v)$ $u(x,y)$
$v(x,y)$

$f_x = f_u u_x + f_v v_x$
$f_y = f_u u_y + f_v v_y$.
Ex pg 958 # 9

6) Directional Derivative
a) $D_u f(\vec{x}_0) = \vec{\nabla} f(\vec{x}_0) \cdot \vec{u}$ $\|u\| = 1$
b) Max rate of change is in the direction of ∇f
Ex 7 pg 967
16 pg 971

7) Second Derivative Test
$z = f(x,y)$ Pb10, Pg 981
a) $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$ Solve, find the C.P (x_c, y_c)

b) $H(x_c, y_c) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

$$H(x_c, y_c) \begin{cases} > 0, & f_{xx} \begin{cases} > 0 & \ominus \\ < 0 & \oplus \end{cases} \\ < 0 & \text{Saddle} \\ = 0 & \text{Fails} \end{cases}$$

8) Lagrange Mult : # 7, Pg 991
Extrema of $f(x,y)$: $g(x,y) = 0$
Let $F = f - \lambda g$ $\vec{\nabla} F = 0$ Solve

Multiple Integration

Differential of Area

$dA = dx\,dy$ Cartesian

$= r\,dr\,d\theta$ Polar Coord

Differential of Volume

$dV = dx\,dy\,dz$ Cartesian

$= r\,dz\,dr\,d\theta$ Cylindrical

$= \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$ Spherical

$A = \iint dA$ Area

$V = \iiint dV$ Volume

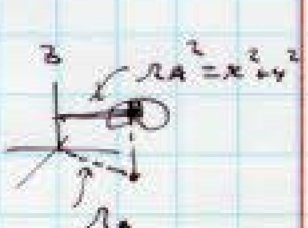
Center of mass

$dm = \rho \begin{cases} dA \\ dV \end{cases}$

$$\bar{x}^i = \frac{\int \bar{x}^i\, dm}{\int dm}$$

Moment of Inertia

$$I_A = \int r_A^2\, dm$$



Ex $I_z = \int (x^2+y^2)\, dm$

Surface Area

$$dS = \iint \sqrt{1+f_x^2+f_y^2}\, dx\,dy$$

Review (Cont)

928 # 9

$$f(x,y) = \frac{8x^2y^2}{x^4+y^4}$$

1) Along $y = 0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,0)\to(0,0)} \frac{0}{x^4+0} = 0$$

2) Along $y = 2x$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,2x)\to(0,0)} \frac{8x^2\cdot 4x^2}{x^4+16x^4}$$

$$= \frac{32}{17} \neq 0.$$

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

940 # 21

$$z = \ln[x+\sqrt{x^2+y^2}]$$

$$\frac{\partial z}{\partial x} = \frac{1}{x+\sqrt{x^2+y^2}} \cdot \frac{\partial}{\partial x}[x+\sqrt{x^2+y^2}]$$

$$= \frac{1}{x+\sqrt{x^2+y^2}}\left[1 + \frac{2x}{2\sqrt{x^2+y^2}}\right]$$

$$\frac{\partial z}{\partial y} = \frac{1}{x+\sqrt{x^2+y^2}} \cdot \frac{\partial}{\partial y}[x+\sqrt{x^2+y^2}]$$

$$= \frac{1}{x+\sqrt{x^2+y^2}}\left[\frac{y}{\sqrt{x^2+y^2}}\right]$$

49) $u = e^{-s}\sin t$

$\begin{cases} u_s = -e^{-s}\sin t \\ u_t = e^{-s}\cos t \end{cases}$

$\begin{cases} u_{ss} = e^{-s}\sin t \\ u_{ts} = -e^{-s}\cos t \end{cases}$ $\quad u_{st} = -e^{-s}\cos t$, $\quad u_{tt} = -e^{-s}\sin t$

Pg 941 # 65

$u = e^{-\alpha^2k^2t}\sin kx$

$u_t = \alpha^2 u_{xx}$

$u_t = -\alpha^2k^2e^{-\alpha^2k^2t}\cdot \sin kx$

$\alpha^2 u_{xx} = -\alpha^2\cdot k^2 e^{-\alpha^2k^2t}\sin kx = u_t$

74 $T(x,y) = 60/(1+x^2+y^2)$

Find $T_x(2,1)$, $T_y(2,1)$ $\quad P=(2,1)$

$T(x,y) = 60[1+x^2+y^2]^{-1}$

$T_x = -60[1+x^2+y^2]^{-2}[2x]$

$T_x = \dfrac{-120x}{(1+x^2+y^2)^2}$ $\quad T_x(2,1) = -\dfrac{20}{3}$

$T_y = \dfrac{-120y}{(1+x^2+y^2)^2}$ $\quad T_x(2,1) = -\dfrac{10}{3}$

$\nabla T_P = \langle -\frac{20}{3}, -\frac{10}{3}\rangle$

68 c) $u = (x-at)^6 + (x+at)^6$

$u_{tt} = a^2 u_{xx}$

$u_t = 6(x-at)^5(-a) + 6(x+at)^5(a)$

$u_{tt} = 30(x-at)^4a^2 + 30(x+at)^4a^2$

$u_{xx} = 6\cdot 5(x-at)^4 + 6\cdot 5(x+at)^4$

$\boxed{u_{tt} = a^2 u_{xx}}$

Tate!

Oct 27

15.5 9) $z = \tan^{-1}(2x+y)$

$x = s^2 t, \quad y = s\ln t$

$$\begin{cases} z_s = z_x x_s + z_y y_s \\ z_t = z_x x_t + z_y y_t \end{cases}$$

$$\begin{cases} z_s = \dfrac{1(2)}{1+(2x+y)^2} \cdot 2st + \dfrac{1(1)}{1+(2x+y)^2} \ln t \\[2ex] z_t = \dfrac{1(2)}{1+(2x+y)^2} \cdot s^2 + \dfrac{1}{1+(2x+y)^2} \dfrac{s}{t} \end{cases}$$

31) $xe^y + yz + ze^x = 0$

$e^y dx + xe^y dy + y\,dz + z\,dy + e^x dz + ze^x dx \overset{0}{=}$

$[y+e^x]\,dz = [-e^y - ze^x]\,dx - [xe^y + z]\,dy$

$dz = -\dfrac{[e^y + ze^x]}{[y+e^x]}\,dx - \dfrac{[xe^y+z]}{[y+e^x]}\,dy$

$\dfrac{\partial z}{\partial x} = -\dfrac{e^y + ze^x}{y+e^x} \qquad \dfrac{\partial z}{\partial y} = -\dfrac{xe^y+z}{y+e^x}$

---

971 #16 $f(x,y,z) = \dfrac{x}{y+z} \qquad P(4,1,1) \quad v = \langle 1,2,3\rangle$

$\vec{u} = \dfrac{\langle 1,2,3\rangle}{\sqrt{14}}$

$D_{\vec{u}} f(P) = \vec{\nabla} f(P) \cdot \vec{u}$

$\nabla f = \left\langle \dfrac{1}{y+z}, \dfrac{-x}{(y+z)^2}, \dfrac{-x}{(y+z)^2} \right\rangle$

$\nabla f(P) = \left\langle \dfrac{1}{2}, \dfrac{-4}{4}, \dfrac{-4}{4} \right\rangle = \left\langle \dfrac{1}{2}, -1, -1 \right\rangle$

$D_{\vec{u}} f(P) = \dfrac{1}{\sqrt{14}}\left(\dfrac{1}{2} - 2 - 3\right) = -\dfrac{9}{2\sqrt{14}}$

---

981 #10 $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$

$$\begin{cases} f_x = 6x^2 + y^2 + 10x = 0 \\ f_y = \cancel{y^2/x} = 2xy + 2y = 0 \end{cases}$$

$2y(x+1) = 0$

$y = 0, \quad x = -1$

$6x^2 + 10x = 0 \qquad 6 + y^2 - 10 = 0$

$2x(3x+5) = 0 \qquad y^2 = 4$

$x = 0, -\frac{5}{3} \qquad y = \pm 2$

$(0,0), (-\frac{5}{3}, 0), (-1, 2)\ (-1,-2)$

$f_{xx} = 12x + 10 \qquad f_{xy} = 2y$

$f_{yx} = 2y \qquad f_{yy} = 2x + 2$

$H = (12x+10)(2x+2) - 4y^2$

$= 4(6x+5)(x+1) - 4y^2$

$H(0,0) > 0 \quad f_{xx}(0,0) > 0$ *Min*

$H(-\frac{5}{3}, 0) > 0 \quad f_{xx}(-\frac{5}{3}, 0) < 0$ *Max*

$f_{xx} = 12x + 10$

$H(x,y) = 4(6x+5)(x+1) - 4y^2$

$H(-1,2) < 0$ *Saddle*

$H(-1,-2) < 0$ *Saddle*

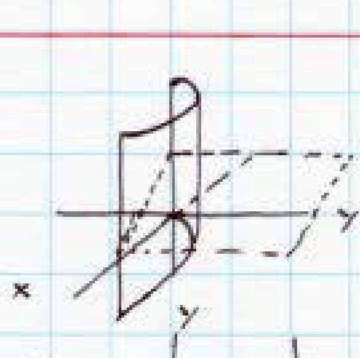
16.7

19 $x = y^2$

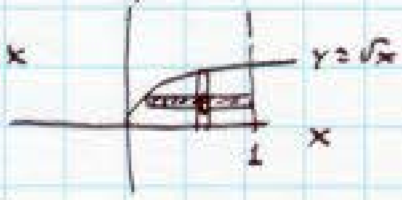
$z = 0$

$x + z = 1$

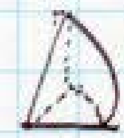
$z = 1 - x$



$$V = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1-x} dz\,dy\,dx$$



$$V = \int_0^1 \int_{y^2}^1 \int_0^{1-x} dz\,dx\,dy$$



Vector Fields:

A vector field is a smooth choice of a vector at each point in space

$$\vec{F}(x,y,z) = P\vec{i} + Q\vec{j} + R\vec{k}$$

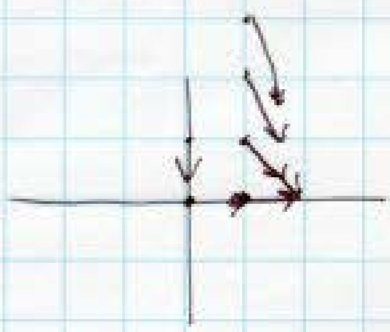
$P = P(x,y,z)$

$Q = Q(x,y,z)$

$R = R(x,y,z)$

Ex $\vec{F} = x\vec{i} - y\vec{j}$

x	y	F
0	0	$0\vec{i} + 0\vec{j}$
0	1	$0\vec{i} - 1\vec{j}$
1	0	$1\vec{i} - 0\vec{j}$
1	1	$1\vec{i} - 1\vec{j}$



Gradient

$\vec{\nabla}$ - "Del" operator

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$f : \mathbb{R}^3 \to \mathbb{R}$

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

Ex $f = xy$

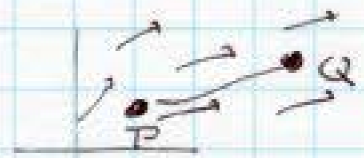
$\nabla f = y\vec{i} + x\vec{j}$

$\nabla f(1,1)$

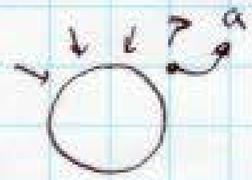
A vector field $\vec{F}$ is called conservative if there exists a function φ such that

$$\vec{F} = \vec{\nabla}\varphi$$

The function φ is called a potential



Conservative $\Rightarrow$ Work done is path independent



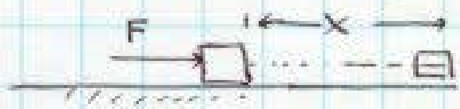
Work - Line Integral.

Do 17.1 # 1, 15, 17, 21, 22, 23, 24

Line Integrals

Work

Case 1

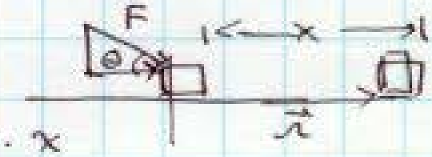


F - Newtons

x - Meters $\quad W = F \cdot x$

W - Joules $\quad 1\text{J} = 1\text{ N}\cdot\text{m}$

Case 2



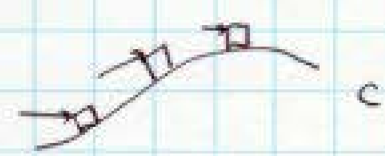
$W = F\cos\theta \cdot x$

$W = F \cdot x \cos\theta$

$W = |\vec{F}| \cdot |\vec{r}| \cdot \cos\theta$

$W = \vec{F} \cdot \vec{r}$ $\quad$ F const, linear motion

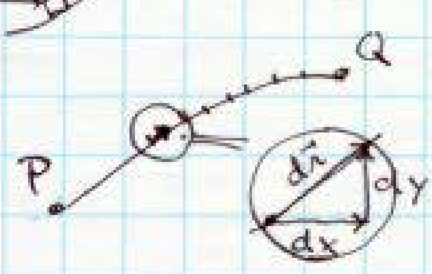
Case 3



$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

$dw = \vec{F} \cdot d\vec{r}$

$$\boxed{W = \int_C \vec{F} \cdot d\vec{r}}$$



$F = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

$W = \int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy$

$C:\quad x = x(t) \qquad dx = x'(t)\,dt$

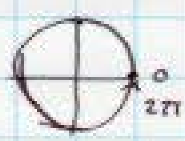
$\qquad y = y(t) \qquad dy = y'(t)\,dt$

$$W = \int_a^b P(x(t),y(t))\cdot x'(t)\,dt + \int_a^b Q(x(t),y(t))\cdot y'(t)\,dt$$

Example

$F = x\,\vec{i} + y\,\vec{j}$

$C: \begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases}$



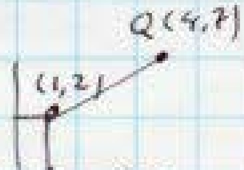
$$W = \int_C \vec{F}\cdot d\vec{r} = \int_C x\,dx + y\,dy$$

$$= \int_0^{2\pi} \cos t\,(-\sin t)\,dt + \sin t\cdot\cos t\,dt$$

$$= \int_0^{2\pi} [(-\cos t\sin t) + (\sin t\cos t)]\,dt$$

$$= \int_0^{2\pi} 0\,dt = 0$$

Ex 2 $\quad \int_C y e^x\,ds$



$\vec{D} = \langle 3, 5\rangle$

Find Eq line

$\begin{cases} x = 1 + 3t \\ y = 2 + 5t \end{cases} \qquad ds = |d\vec{r}|$

$ds = \sqrt{x'^2 + y'^2}\,dt$

$W = \int \vec{F}\cdot d\vec{r} \qquad \vec{F} = F_x(x,y)\,\vec{i} + F_y(x,y)\,\vec{j}$

$\qquad d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

$C:\ x = x(t) \qquad \vec{r} = \vec{r}(t)$

$\quad y = y(t)$

$d\vec{r} = \frac{d\vec{r}}{dt}\cdot dt$

$= \vec{V}\cdot dt \qquad \frac{\vec{V}}{v} = \vec{T} \qquad v = \frac{ds}{dt}$

$= T\cdot v\,dt$

$= T\cdot ds$

$\vec{F}\cdot d\vec{r} = \vec{F}\cdot\vec{T}\,ds$ *Arc Length*

$W = \int \vec{F}\cdot d\vec{r} = \int_C (\vec{F}\cdot\vec{T})\,ds$

Line Integrals

Work:

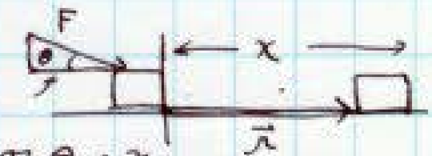
Case 1

F - *Newtons*   *Const*

x - *Meters* $W = F\cdot x$

W - *Joules* $1\,J = 1\,N\text{-}m$

Case 2



$W = F\cos\theta\cdot x$

$\quad = F\cdot x\cos\theta \qquad F = |\vec{F}|$

$\qquad x = |\vec{r}|$

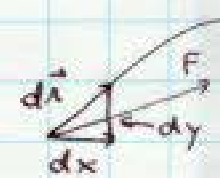
$W = |\vec{F}|\cdot|\vec{r}|\cdot\cos\theta$

$W = \vec{F}\bullet\vec{r}$ <u>F = *Const*</u> !

Case 3

$\vec{F}$ = *Vector Field*

C : *Curve*



$dw = \vec{F}\cdot d\vec{r}$

$$\boxed{W = \int_C \vec{F}\cdot d\vec{r}}$$

$\vec{F} = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

$|d\vec{r}|^2 = ds^2 = dx^2 + dy^2$

$C:\quad x = x(t), \quad y = y(t)$

Example:

$\vec{F} = x\,\vec{i} - y\,\vec{j}$

$C: \begin{cases} x(t) = \cos t \\ y(t) = +\sin t \end{cases}$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

$$W = \int_C \cancel{y}\,dx - y\,dy = \int_C x\,dx - y\,dy$$

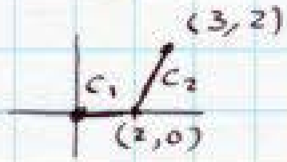
$$= \int_0^{2\pi} (\cos t)(-\sin t)\,dt - \sin t\cdot\cos t\,dt$$

$$= \int_0^{2\pi} -2\sin t\cos t\,dt$$

$$= \int_0^{2\pi} -\sin 2t\,dt$$

$$= \tfrac{1}{2}\cos 2t\Big]_0^{2\pi} = \tfrac{1}{2}[1-1] = 0$$

7 $\int xy\,dx + (x-y)\,dy = W$



$$\begin{cases} C_1: \vec{x} = (0,0) + t(2,0) \\ C_2: \vec{x} = (2,0) + t(1,2) \end{cases}$$

C_1: $x = 2t$, $y = 0$ $\qquad$ C_2: $x = 2 + t$, $y = 2t$

$$W = \int_{C_1} + \int_{C_2} = \int_{C_2}$$

$$= \int_0^1 (2+t)2t\,dt + (2-t)2dt =$$

Conservative Vector Fields.

Definition. $\vec{F}$ is conservative if there exists a function φ such that

$$\boxed{\vec{F} = \vec{\nabla}\varphi} = \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \right\rangle$$

$$\begin{array}{ccc} & \varphi & \\ x & y & z \end{array}$$

Let $d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz$$

$$= \int_C d\varphi = \varphi\Big]_P^Q = \varphi(Q) - \varphi(P)$$

Work is path independent!

Test for path independence

If $\vec{\nabla}\times F = 0$ then $\vec{F} = \nabla\varphi$

Ex: $\boxed{13}$

$$F = x^3y^4\,\vec{i} + x^4y^3\,\vec{j}$$

$$\vec{r}(t) = \sqrt{t}\,\vec{i} + (1+t^3)\vec{j} \qquad 0 \le t \le 1$$

$P(0,1), Q(1,2)$

a)
$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3y^4 & x^4y^3 & 0 \end{vmatrix} = [4x^3y^3 - 4x^3y^3]\,\vec{k}$$

$$F = \nabla\varphi = \frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j} \qquad \frac{\partial\varphi}{\partial x} = x^3y^4$$

$$\varphi = \frac{1}{4}x^4y^4 \qquad \frac{\partial\varphi}{\partial y} = x^4y^3$$

$$\int\vec{F}\cdot d\vec{r} = \frac{1}{4}(x^4y^4)\Big]_{(0,1)}^{(1,2)} = \frac{1}{4}[16]$$

$$= \frac{1}{4}(16) = 4$$

HW: 17.3 # 3-9, 13-17 odd

Conservative Fields

The following are equivalent:

1) $\vec{\nabla}\times\vec{F} = 0$
2) $\vec{F} = \nabla\varphi$
3) $\int_C \vec{F}\cdot d\vec{r}$ - Path independent
4) $\int_C \vec{F}\cdot d\vec{r} = \int_C d\varphi = \varphi(Q) - \varphi(P)$
5) $\oint_C \vec{F}\cdot d\vec{r} = 0$

Exact differentials. ($\mathbb{R}^2$)

Def A differential 1-form is an expression of the type

$$\alpha = P(x,y)\,dx + Q(x,y)\,dy$$

$$\beta = T\,dS + dQ + P\,dV \qquad \text{(1st Law of Th)}$$

Def. An exact differential form is one that can be written as $d\varphi$ for some function φ

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy$$

$\alpha = P\,dx + Q\,dy$ is exact if $P = \frac{\partial\varphi}{\partial x}$ $\quad Q = \frac{\partial\varphi}{\partial y}$

Example $\qquad \boxed{\text{Exact D.E}}$

$$\alpha = 2xy^3dx + 3x^2y^2dy \quad (=0)$$

is exact because

$$P = 2xy^3 = \frac{\partial}{\partial x}(x^2y^3)$$

$$Q = 3x^2y^2 = \frac{\partial}{\partial y}(x^2y^3)$$

$$\alpha = d(x^2y^3)$$

Suppose α is exact

$$\alpha = P\,dx + Q\,dy = \vec{F}\cdot d\vec{r}$$

$$P = \frac{\partial\varphi}{\partial x},\quad Q = \frac{\partial\varphi}{\partial y}$$

$$F = P\,\vec{i} + Q\,\vec{j}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & 0 \end{vmatrix} = (\varphi_{yx} - \varphi_{xy})\,\vec{k}$$

$$= 0$$

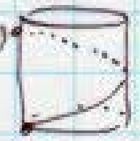
11/09 17.3

(17) $\vec{F} = (2xz + \sin y)\vec{i} + (x\cos y)\vec{j} + x^2\vec{k}$

$\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$

$0 \le t \le 2\pi$

$Q(1,0,2\pi)$



$P(1,0,0)$

Find $\int_C \vec{F}\cdot d\vec{r}$.

Step 1. Find $\vec{\nabla}\times\vec{F}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ 2xz+\sin y & x\cos y & x^2 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial z}(x\cos y) - \frac{\partial}{\partial y}(x^2)\right] + j\left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial z}(2xz+\sin y)\right] + k\left[\frac{\partial}{\partial y}(2xz+\sin y) - \frac{\partial}{\partial x}(x\cos y)\right] = 0$$

Hence

$\vec{F} = \nabla\varphi$

$\frac{\partial\varphi}{\partial x} = 2xz + \sin y \qquad \varphi = x^2 z + x\sin y$

$\frac{\partial\varphi}{\partial y} = x\cos y$

$\frac{\partial\varphi}{\partial z} = x^2$

$$\int_C F\cdot dr = \varphi\Big]_P^Q = x^2 z + x\sin y\Big]_{(1,0,0)}^{(1,0,2\pi)}$$

$= 2\pi$

Do 17.3 # 19, 21, 27, 28

Conservative Force Fields

1) $\vec{\nabla}\times F = 0$
2) $\vec{F} = \vec{\nabla}\varphi$
3) $\int_C \vec{F}\cdot d\vec{r} = \int_C d\varphi$ Path ind.
4) $\oint_C \vec{F}\cdot d\vec{r} = 0$

Ex 7 $\vec{F} = (2x\cos y - y\cos x)\vec{i} + (-x^2\sin y - \sin x)\vec{j}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x\cos y - y\cos x & -x^2\sin y - \sin x & 0 \end{vmatrix}$$

$$= [-2x\sin y - \cos x - (-2x\sin y - \cos x)]\cdot\vec{k} = 0$$

$F = \nabla\varphi = \frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}j$

$\frac{\partial\varphi}{\partial x} = 2x\cos y - y\cos x \qquad \varphi = x^2\cos y - y\sin x$

$\frac{\partial\varphi}{\partial y} = -x^2\sin y - \sin x \qquad \varphi = x^2\cos y - y\sin x$

If $\varphi: \mathbb{R}^2 \to \mathbb{R} \qquad \varphi = \varphi(x,y)$

$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy \leftarrow$ Exact Diff

If $\vec{F} = \vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j}$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

$$\int\vec{F}\cdot d\vec{r} = \int\frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy$$

$$= \int_C d\varphi = \varphi\Big]_P^Q$$

Ex 18

$\vec{F} = 4xe^z\vec{i} + \cos y\,\vec{j} + 2x^2e^z\vec{k}$

$\vec{r} = t\vec{i} + t^2\vec{j} + t^4 k \qquad 0 \le t \le 1$

To Compute $W = \int_C \vec{F}\cdot d\vec{r} \qquad P(0,0,0) \quad Q(1,1,1)$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4xe^z & \cos y & 2x^2e^z \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(4xe^z - 4xe^z) + \vec{k}(0) = 0$$

$\frac{\partial\varphi}{\partial x} = 4xe^z \qquad \varphi = 2x^2e^z + \sin y$

$\frac{\partial\varphi}{\partial y} = \cos y \qquad \varphi = \sin y + 2x^2e^z$

$\frac{\partial\varphi}{\partial z} = 2x^2e^z$

$$W = \int_C F\cdot dr = \sin y + 2x^2e^z\Big]_{P(0,0,0)}^{Q(1,1,1)}$$

$= \sin 1 + 2e$

Green's Theorem.

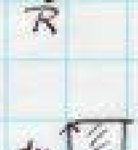
(Stoke's Theorem in 2-dim)

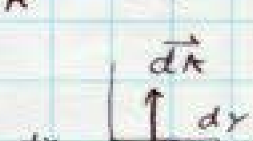
Let $\vec{F}$ be a V.F. and C: be a (simple) close curve

Then:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (\vec{\nabla}\times\vec{F})\cdot d\vec{A}$$

In $\mathbb{R}^2$.





$d\vec{A} = (dx)\vec{i} \times (dy)\vec{j} = (dx\cdot dy)(\vec{i}\times\vec{j})$

$d\vec{A} = (dx\,dy)\vec{k}$

Green's (Stokes 2-d) Theorem

Let $F = P(x,y)\vec{i} + Q(x,y)\vec{j}$

$d\vec{A} = dx\,dy\,\vec{k}$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

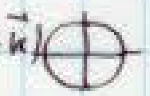
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (\vec{\nabla}\times\vec{F})\cdot d\vec{A}$$

$$\boxed{\oint_C P\,dx + Q\,dy = \iint (Q_x - P_y)\,dx\,dy}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

Ex 1

$\boxed{\text{Pb2}}$ $\oint_C y\,dx - x\,dy = \iint 2\vec{k}\cdot(dx\,dy\,\vec{k})$



$= -\iint 2\,dx\,dy$

$\vec{F} = y\,\vec{i} - x\,\vec{j}$ $\quad = \boxed{-2\pi}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & -x & 0 \end{vmatrix} = (-1-1)k = -2\vec{k}$$

Do 1, 3, 5, 7, 9 $\boxed{17.4}$

Green's Theorem: (Stoke's Thm 2D)

$\vec{F}$ _ a Vector Field

C _ a simple closed curve

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (\vec{\nabla}\times\vec{F})\cdot d\vec{A}$$

$\mathbb{R}^2$

$d\vec{A} = (dx\,\vec{i})\times(dy\,\vec{j})$

$dA = (dx\,dy)\,k$

In particular; if $\vec{F} = P\vec{i} + Q\vec{j}$

$$\oint_C \vec{F}\cdot d\vec{r} = \int P\,dx + Q\,dy$$

$$\left\{\begin{array}{l} \vec{\nabla}\times\vec{P} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix} = (Q_x - P_y)\vec{k} \\ d\vec{A} = (dx\,dy)\,k \end{array}\right.$$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Example Pb#1 $\quad$ 17.4

$\oint_C y^3\,dx - x^3\,dy \qquad x^2 + y^2 = 4$

$\vec{F} = y^3\,\vec{i} - x^3\,\vec{j}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y^3 & -x^3 & 0 \end{vmatrix} = (-3x^2 - 3y^2)\,k$$

$dA = dx\,dy\,\vec{k}$

$$W = \iint_R (-3x^2 - 3y^2)\,dx\,dy$$

$$= -3\iint_R (x^2+y^2)\,dx\,dy$$

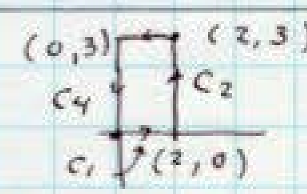
$$= -3\int_0^{2\pi}\int_0^2 r^2\cdot r\,dr\,d\theta$$

$$= -3\int_0^{2\pi} \tfrac{1}{4}r^4\Big]_0^2\,d\theta = -3(4)\,2\pi$$

$$= -24\pi$$

17.4 #1:

$\oint_C xy^2\,dx + x^3\,dy = W$



C_1: $x = 0 + 2t$, $y = 0 + 0t$ $\qquad$ C_3: $x = 2 - 2t$, $y = 3$

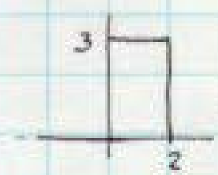
C_2: $x = 2 + 0t$, $y = 0 + 3t$ $\qquad$ C_4: $x = 0 + 0t$, $y = 3 - 3t$

$$W = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

$$= \int_0^1 0 + \int_0^1 8(3\,dt) + \int_0^1 (2-2t)\cdot 9(-2\,dt) +$$

$$= \int_0^1 [24 + (-36) + 36t]\,dt$$

$$= -12 + 18 = 6$$



b) $\vec{F} = xy^2\,\vec{i} + x^3\,\vec{j}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xy^2 & x^3 & 0 \end{vmatrix} = (3x^2 - 2xy)\,\vec{k}$$

$$W = \int_0^2\int_0^3 (3x^2 - 2xy)\,dy\,dx$$

$$= \int_0^2 3x^2y - xy^2\Big]_0^3\,dx$$

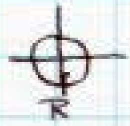
$$= \int_0^2 (9x^2 - 9x)\,dx$$

$$= 3x^3 - \tfrac{9}{2}x^2\Big]_0^2 = 24 - 18 = 6$$

11/11

Green's Theorem

17.4 **2)** $\oint_c y\,dx - x\,dy = W$



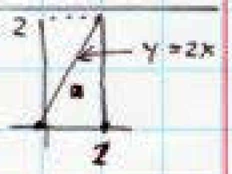
$\vec{F} = y\vec{i} - x\vec{j}$

$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & -x & 0 \end{vmatrix} = (-1-1)\vec{k} = -2\vec{k}$

$W = \iint_R 2\,dA = -2\pi$

---

3) $\oint (xy\,dx + x^2y^3dy) = W$

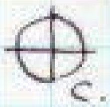


$F = xy\vec{i} + x^2y^3\vec{j}$

$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xy & x^2y^3 & 0 \end{vmatrix} = (2xy^3 - x)k$

$W = \iint_R (2xy^3 - x)\,dA = \int_0^1\int_0^{2x}(2xy^3 - x)\,dy\,dx$

5) $P = x^4y^5 \quad Q = -x^7y^6$



$\vec{F} = x^4y^5\vec{i} - x^7y^6\vec{j}$

a) $C: \; y = \sin t \qquad 0 \le t \le 2\pi$

$x = \cos t$

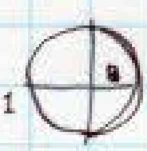
$W = \oint_C F\cdot dr = \oint_C x^4y^5dx - x^7y^6dy$

$= \int_0^{2\pi}\cos^4 t\,\sin^5 t(-\sin t)\,dt + \int_0^{2\pi} -\cos^7 t\,\sin^6 t\,\cos t\,dt$

$W = -\int_0^{2\pi}(\cos^4 t\,\sin^6 t + \cos^8 t\,\sin^6 t)\,dt$

b) $\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^4y^5 & -x^7y^6 & 0 \end{vmatrix} = (-7x^6y^6 - 5x^4y^4)\vec{k}$

$W = \iint_R -7x^6y^6 - 5x^4y^4\,dA$

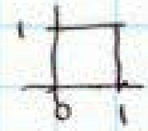


$= -\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(7x^6y^6 + 5x^4y^4)\,dy\,dx$

Do # 11, 13, 14, 17, 19.

---

7) $\oint e^y dx + 2xe^y dy = W$



$F = e^y\vec{i} + 2xe^y\vec{j}$

$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ e^y & 2xe^y & 0 \end{vmatrix} = (2e^y - e^y)\vec{k}$

$W = \int_0^1\int_0^1 e^y\,dy\,dx = \int_0^1(e-1)\,dx = e-1$

17.3 # 13.

$\vec{F} = x^3y^4\vec{i} + x^4y^3\vec{j}$

$C: \; \vec{r}(t) = \sqrt{t}\,\vec{i} + (1+t^3)\vec{j} \qquad 0 \le t \le 1$

$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^3y^4 & x^4y^3 & 0 \end{vmatrix} = (4x^3y^3 - 4x^3y^3)\vec{k} = 0$

$F = \nabla\varphi \qquad \frac{\partial\varphi}{\partial x} = x^3y^4 \qquad \varphi = \frac{1}{4}x^4y^4$

$\frac{\partial\varphi}{\partial y} = x^4y^3$

$\int \vec{F}\cdot d\vec{r} = \varphi\Big]_P^Q = \frac{1}{4}x^4y^4\Big]_{(0,1)}^{(1,2)}$

$= \frac{1}{4}(16) = 4$

17.4 # 6

$P = y^2\sin x \quad Q = x^2\sin y$

$y = x^2 \qquad (0,0) \to (1,1)$

$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix}$

$\begin{cases} x = t \\ y = t^2 \end{cases}$

$W = \int y^2\sin x\,dx + x^2\sin y\,dy$

$W = \int_0^1 t^4\sin t\,dt + t^2\sin t^2(2t\,dt)$

$W = \int_0^1\int_0^{x^2}(2x\sin y - 2y\sin x)\,dy\,dx$

Divergence

$\partial_x = \frac{\partial}{\partial x}$

Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$= F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$

Def $\quad \vec{\nabla} = \partial_x\vec{i} + \partial_y\vec{j} + \partial_z\vec{k}$

$\text{Div}(\vec{F}) = \vec{\nabla}\cdot\vec{F}$

$= \frac{\partial}{\partial x}P + \frac{\partial Q}{\partial y} + \frac{\partial}{\partial z}R$

Ex $\quad F = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$

$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$

$= 2x + 2y + 2z$

$\boxed{17.5}$ $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

1) $\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$

2) $\vec{\nabla}\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

3) Theorems:

$\begin{cases} a) & \vec{\nabla}\times\vec{F} = 0 \quad \text{iff} \quad \vec{F} = \nabla\varphi \\ b) & \vec{\nabla}\cdot\vec{F} = 0 \quad \text{iff} \quad \vec{F} = \vec{\nabla}\times\vec{A} \end{cases}$ ← new

$\begin{cases} a') & \vec{\nabla}\times(\nabla\varphi) = 0 \\ b') & \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0 \end{cases}$

Another interpretation of Green's Thm

Dim 2

$\vec{F} = P\vec{i} + Q\vec{j}$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

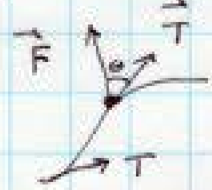
$ds^2 = dx^2 + dy^2$.

$W = \oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy = \iint(\vec{\nabla}\times\vec{F})\cdot d\vec{A}$

$= \iint(Q_x - P_y)\,dA$

$\oint_{C=\partial R} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$W = \oint_C \left(P\frac{dx}{ds} + Q\frac{dy}{ds}\right) ds$



$= \oint \langle P, Q\rangle\cdot\left\langle \frac{dx}{ds}, \frac{dy}{ds}\right\rangle ds$

$= \oint(\vec{F}\cdot\vec{T})\,ds \qquad T = \frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j}$

$\boxed{\oint_C(\vec{F}\cdot\vec{T})\,ds = \iint_R(\vec{\nabla}\times\vec{F})\cdot d\vec{A}}$ $\qquad \vec{F}\cdot T = |\vec{F}|\cos\theta$

$\vec{T} = \frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j}$

$\vec{N}\cdot\vec{T} = 0 \quad |\vec{N}| = 1$

$\vec{N} = \frac{dy}{ds}\vec{i} - \frac{dx}{ds}\vec{j}$

$\oint_C(\vec{F}\cdot\vec{N})\,ds = \oint\langle P, Q\rangle\cdot\left\langle\frac{dy}{ds}, -\frac{dx}{ds}\right\rangle ds$

$= \int\left(P\frac{dy}{ds} - Q\frac{dx}{ds}\right) ds$

$= \oint(P\,dy - Q\,dx)$

$\oint_{C=\partial R}(\vec{F}\cdot\vec{N})\,ds = \oint -Q\,dx + P\,dy$

$= \iint_R\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA \qquad \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -Q & P & 0 \end{vmatrix}$

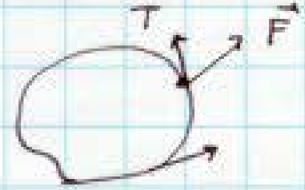
$= \iint_R(\vec{\nabla}\cdot\vec{F})\cdot dA$

Summary

1) $\oint_{C=\partial R}(\vec{F}\cdot\vec{T})\,ds = \oint_C \vec{F}\cdot d\vec{r} = \iint_R(\vec{\nabla}\times\vec{F})\cdot d\vec{A}$

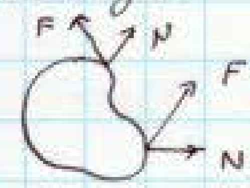
2) $\oint_{C=\partial R}(\vec{F}\cdot\vec{N})\,ds = \iint_R(\vec{\nabla}\cdot\vec{F})\,dA$

Physics



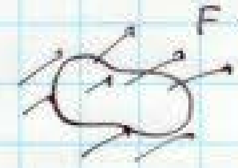
$\oint(\vec{F}\cdot\vec{T})\,ds$ = Work

$\vec{F}\cdot\vec{T}$ = Component of $\vec{F}$ tang. to C



$\oint(\vec{F}\cdot\vec{N})\,ds$

$\vec{F}\cdot\vec{N}$ = Component of $\vec{F}$ normal to C



$\oint(\vec{F}\cdot\vec{N})\,ds = \iint(\vec{\nabla}\cdot\vec{F})\,dA$

if $\vec{\nabla}\cdot\vec{F} = 0 \Rightarrow \oint(\vec{F}\cdot\vec{N})\,ds = 0$

No net flow!

$\vec{\nabla}\cdot\vec{F}\begin{cases} > 0 & \text{source} \\ < 0 & \text{sink} \\ = 0 & \text{Incompressible} \end{cases}$

DO: 17.5 # 3, 7, 15, 13, 30a, 31c, 33*, 34*, 36*

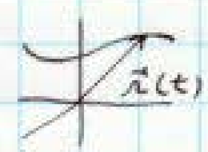
$\boxed{17.6}$ Surfaces - Surface Area

Curve:

$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

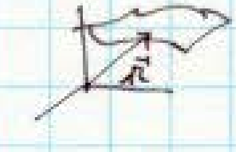
1-parameter

t - "time"



Surfaces

$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$



Example

$\begin{cases} x(u,v) = a\sin u\cos v \\ y(u,v) = a\sin u\sin v \\ z(u,v) = a\cos u \end{cases}$ $\qquad x^2 + y^2 + z^2 = a^2$

Sphere

Surface (Parametric Form)

$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

Ex 44, (17.6)

$$\begin{cases} x = a\,u\cos v & 0 \le u \le 2 \\ y = b\,u\sin v & 0 \le v \le 2\pi \\ z = u^2 \end{cases}$$

Recall: $\begin{cases} \cos^2\theta + \sin^2\theta = 1 \\ \cosh^2\theta - \sinh^2\theta = 1 \end{cases}$

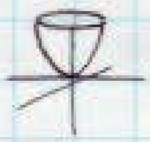
$(\frac{x}{a})^2 = u^2\cos^2 v$

$(\frac{y}{b})^2 = u^2\sin^2 v$

$(\frac{x}{a})^2 + (\frac{y}{b})^2 = u^2(\cos^2 v + \sin^2 v)$

$= z$

$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$



Elliptic Paraboloid

Ex $x^2 + y^2 - 4z^2 = 1$

Parametric form

a) Stupid way $4z^2 = x^2 + y^2 - 1$

$z = \frac{1}{2}\sqrt{x^2+y^2-1}$

$\vec{r}(u,v) = (u, v, \frac{1}{2}\sqrt{u^2+v^2-1})$

$x = u$

$y = v$

$z = \frac{1}{2}\sqrt{u^2+v^2-1}$

b) $x = \cos u \cosh v$

$y = \sin u \cosh v$

$\frac{z}{2} = \sinh v$

~~Verify~~ Verify

$x^2+y^2 = (\cos^2 u + \sin^2 u)\cosh^2 v$

$x^2+y^2 = \cosh^2 v$

$(\frac{z}{2})^2 = \sinh^2 v$

$x^2+y^2 - \frac{z^2}{4} = 1$

Surface Area:

$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$

$\vec{r} = \vec{X}$

$$\boxed{d\vec{S} = |\vec{X}_u \times \vec{X}_v|\,du\,dv}$$

$S = \iint |\vec{X}_u \times \vec{X}_v|\,du\,dv$

$|A| = \sqrt{A\cdot A}$

$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = \begin{vmatrix} \vec{A}\cdot\vec{C} & \vec{A}\cdot\vec{D} \\ \vec{B}\cdot\vec{C} & \vec{B}\cdot\vec{D} \end{vmatrix}$

$(\vec{X}_u\times\vec{X}_v)\cdot(X_u\times X_v) = \begin{vmatrix} \vec{X}_u\cdot\vec{X}_u & \vec{X}_u\cdot\vec{X}_v \\ \vec{X}_v\cdot\vec{X}_u & \vec{X}_v\cdot\vec{X}_v \end{vmatrix}$

$= \begin{vmatrix} E & F \\ F & G \end{vmatrix} = EG - F^2$

$|\vec{X}_u\times\vec{X}_v| = \sqrt{EG-F^2}$

$$\boxed{\begin{aligned} S &= \iint \sqrt{EG-F^2}\,du\,dv \\ E &= \vec{X}_u\cdot\vec{X}_u \\ F &= \vec{X}_u\cdot\vec{X}_v \\ G &= \vec{X}_v\cdot\vec{X}_v \end{aligned}}$$

Example:

$$\begin{cases} x = a\sin v\cos u & 0 \le v \le \pi \\ y = a\sin v\sin u & 0 \le u \le 2\pi \\ z = a\cos v \end{cases}$$

$\vec{X}(u,v) = \langle a\sin v\cos u, a\sin v\sin u, a\cos v\rangle$

$\vec{X}_u = \langle -a\sin v\sin u, a\sin v\cos u, 0\rangle$

$\vec{X}_v = \langle a\cos v\cos u, a\cos v\sin u, -a\sin v\rangle$

$E = a^2\sin^2 v = a^2\sin^2 v$

$F = 0$

$G = a^2\cos^2 v + a^2\sin^2 v = a^2$

$S = \int_0^{\pi}\int_0^{2\pi} a^2\sin v\,du\,dv$

$= 2\pi\int_0^{\pi} a^2\sin v\,dv$

$= 2\pi a^2(-\cos v)\Big]_0^{\pi}$

$= 2\pi a^2(+1+1) = 4\pi a^2$ ✓

Surfaces

$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$

$x = x(u,v)$

$y = y(u,v)$

$z = z(u,v)$

Example 45 p 1126 17.6

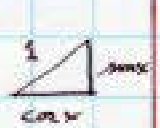
$x = a \sin u \cos v \qquad 0 \le u \le \pi$

$y = b \sin u \sin v \qquad 0 \le v \le 2\pi$

$z = c \cos u$

Recall: $\cos^2 x + \sin^2 x = 1$

$\cosh^2 x - \sinh^2 x = 1$



$\frac{x}{a} = \sin u \cos v$

$\frac{y}{b} = \sin u \sin v$

$\frac{z}{c} = \cos u$

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = \sin^2 u + \cos^2 u = 1$

Ellipsoid

46 $x = a \cosh u \cos v$

$y = b \cosh u \sin v$

$z = c \sinh u$

$\left(\frac{x}{a}\right)^2 = \cosh^2 u \cos^2 v$

$\left(\frac{y}{b}\right)^2 = \cosh^2 u \sin^2 v$

$\left(\frac{z}{c}\right)^2 = \sinh^2 u$

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 1$ *Hyperboloid of 1 sheet*

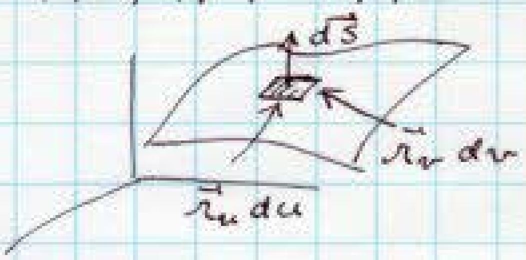
Surface Area:

$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$

$x = x(u,v)$

$y = y(u,v)$

$z = z(u,v)$



$$\boxed{dS = |\vec{r}_u \times \vec{r}_v|\,du\,dv}$$

Fact: $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = \begin{vmatrix} \vec{A}\cdot\vec{C} & \vec{A}\cdot\vec{D} \\ \vec{B}\cdot\vec{C} & \vec{B}\cdot\vec{D} \end{vmatrix}$

$A\cdot A = |A|^2$

$$(\vec{r}_u \times \vec{r}_v)\cdot(\vec{r}_u \times \vec{r}_v) = \begin{vmatrix} \vec{r}_u\cdot\vec{r}_u & \vec{r}_u\cdot\vec{r}_v \\ \vec{r}_v\cdot\vec{r}_u & \vec{r}_v\cdot\vec{r}_v \end{vmatrix} = \begin{vmatrix} E & F \\ F & G \end{vmatrix} = EG - F^2$$

$$\boxed{d\vec{S} = \sqrt{EG-F^2}\,du\,dv}$$

$$\boxed{E = \vec{r}_u\cdot\vec{r}_u \qquad F = \vec{r}_u\cdot\vec{r}_v \qquad G = \vec{r}_v\cdot\vec{r}_v}$$

Example:

$\vec{r} = \sin u \cos v\,\vec{i} + \sin u \sin v\,\vec{j} + \cos u\,\vec{k}$

$0 \le u \le \pi, \quad 0 \le v \le 2\pi$

$\vec{r}_u = \langle \cos u \cos v, \cos u \sin v, -\sin u \rangle$

$\vec{r}_v = \langle -\sin u \sin v, \sin u \cos v, 0 \rangle$

$E = \cos^2 u + \sin^2 u = 1 \qquad F = 0$

$G = \sin^2 u$

$S = \int_0^{2\pi}\int_0^{\pi} \sqrt{\sin^2 u}\,du\,dv$

$= \int_0^{2\pi}\int_0^{\pi} \sin u\,du\,dv$

$= \int_0^{2\pi} -\cos u \Big]_0^{\pi} dv = \int_0^{2\pi} 2\,dv$

$= 4\pi$ $\boxed{S = 4\pi a^2}$

Do: 7, 9, 10, 17, 19, 21, 23
33, 35, 37, 39, 45, 46

31c (17.5)

$\nabla\left(\frac{1}{r}\right) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\left(\frac{1}{r}\right)$

$\frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}$

$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{2x}{2(x^2+y^2+z^2)^{3/2}} = -\frac{x}{r^3}$

$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = \frac{-2y}{2(x^2+y^2+z^2)^{3/2}} = -\frac{y}{r^3}$

$\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = \frac{-2z}{2(x^2+y^2+z^2)^{3/2}} = \frac{-z}{r^3}$

$\vec{\nabla}\left(\frac{1}{r}\right) = -\left(\frac{x\vec{i} + y\vec{j} + z\vec{k}}{r^3}\right) = -\frac{\vec{r}}{r^3}$

Coulomb Potential

Surfaces & Surface Area

33, Section 17.6

$$\begin{cases} x+2y+z=4 \quad \text{Area} \\ x^2+y^2=4 \end{cases}$$

Note: $z = 4 - x - 2y$

$\vec{r}(u,v) = u\vec{i} + v\vec{j} + (4-u-2v)\vec{k}$

$$\begin{cases} x = u \\ y = v \\ z = 4-u-2v \end{cases} \qquad S = \iint \sqrt{EG-F^2}\,du\,dv \qquad E = \vec{r}_u\cdot\vec{r}_u \quad F = \vec{r}_u\cdot\vec{r}_v \quad G = \vec{r}_v\cdot\vec{r}_v$$

$\vec{r}_u = \langle 1, 0, -1\rangle$

$\vec{r}_v = \langle 0, 1, -2\rangle$

$E = 2 \qquad G = 5 \qquad F = 2$

$S = \iint \sqrt{10-4}\,du\,dv$

$= \sqrt{6}\iint_D du\,dv = \sqrt{6}\,(\pi\cdot 2^2) = 4\sqrt{6}\,\pi$

$= \sqrt{6}\int_0^{2\pi}\int_0^2 r\,dr\,d\theta = 4\sqrt{6}\,\pi.$

37) $y = 4x + z^2$

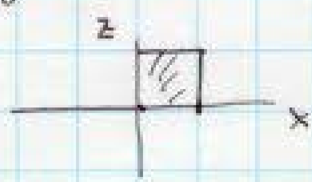
$x=0,\ x=1,\ z=0,\ z=1$

$\vec{r}(u,v) = u\vec{i} + (4u+v^2)\vec{j} + v\,k$

$x = u$

$z = v$

$y = 4u+v^2$



$\vec{r}_u = \langle 1, 4, 0\rangle \qquad E = 17 \qquad F =$

$\vec{r}_v = \langle 0, 2v, 1\rangle \qquad G = 4v^2+1$

$F = 8v$

$S = \int_0^1\int_0^1 \sqrt{17(4v^2+1) - 64v^2}\,dv\,du$

$= \int_0^1\int_0^1 \sqrt{4v^2+17}\,dv\,du = 4.28$

19 $\quad y = 6 - 3x^2 - 2z^2$

Find parametric equations.

a) Stupid way.

$\vec{r}(u,v) = u\vec{i} + (6-3u^2-2v^2)\vec{j} + v\,k$

$x = u \qquad y = 6 - 3u^2 - 2v^2$

$z = v$

b) Fancy way $\quad (\cos^2\theta + \sin^2\theta = 1)$

$x = (\sqrt{3})^{-1}\cos\theta \qquad 3x^2 = \cos^2\theta$

$z = (\sqrt{2})^{-1}\sin\theta \qquad 2z^2 = \sin^2\theta$

$3x^2 + 2z^2 = 1$

$3x^2 = (6-u)\cosh^2 v$

$2z^2 = (6-u)\sinh^2 v$

$y = 6-u$

$3x^2 - 2z^2 = 6 - u = y$

$3x^2 = (6-u)^2\cosh^2 v$

$2z^2 = (6-u)^2\sinh^2 v \qquad \sqrt{\cdot}$

$y^2 = (6-u)^2$

$$\begin{cases} x = \frac{1}{\sqrt{3}}(6-u)\cosh v \\ z = \frac{1}{\sqrt{2}}(6-u)\sinh v \\ y = 6-u \end{cases}$$

39) $x^2+y^2+z^2 = a^2$

$x^2 + y^2 = ax$

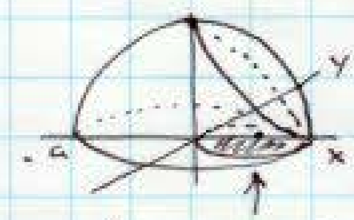
$(x^2 - ax + \frac{a^2}{4}) + y^2 = 0 + \frac{a^2}{4}$

$(x - \frac{a}{2})^2 + y^2 = \frac{a^2}{4}$

$C(\frac{a}{2}, 0)$

$r = a/2$

$z = \sqrt{a^2-x^2-y^2}$



$r^2 = a\,r\cos\theta$

$r = a\cos\theta$

$$\begin{cases} x = u \\ y = v \\ z = \sqrt{a^2-u^2-v^2} \end{cases}$$

$\vec{X}_u = \langle 1, 0, \frac{-u}{\sqrt{a^2-u^2-v^2}}\rangle$

$\vec{X}_v = \langle 0, 1, \frac{-v}{\sqrt{a^2-u^2-v^2}}\rangle$

$E = 1 + \frac{u^2}{(a^2-u^2-v^2)} = \frac{a^2-v^2}{(a^2-u^2-v^2)}$

$F = \frac{uv}{(a^2-u^2-v^2)}$

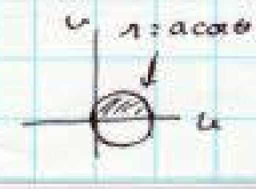
$G = 1 + \frac{v^2}{a^2-u^2-v^2} = \frac{a^2-u^2}{(a^2-u^2-v^2)}$

$EG - F^2 = \frac{(a^2-v^2)(a^2-u^2) - u^2v^2}{(a^2-u^2-v^2)^2}$

$= \frac{a^4 - a^2v^2 - a^2u^2 + \cancel{u^2v^2} - \cancel{u^2v^2}}{(a^2-u^2-v^2)^2}$

$= \frac{a^2(a^2-v^2-u^2)}{(a^2-u^2-v^2)^2} = \frac{a^2}{a^2-u^2-v^2}$

$dS = \iint \frac{a}{\sqrt{a^2-u^2-v^2}}\,du\,dv$



$$S = 4\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta} \frac{a}{\sqrt{a^2-r^2}}\, r\,dr\,d\theta$$

$$S = 4a\int_0^{\frac{\pi}{2}} -\sqrt{a^2-r^2}\,\Big]_0^{a\cos\theta} d\theta$$

$$= -4a\int_0^{\pi/2} (a\sin\theta - a)\,d\theta$$

$$= -4a^2\,[-\cos\theta - \theta]_0^{\pi/2}$$

$$= +4a^2\,[(0-1) + \tfrac{\pi}{2}] = 2a^2(\pi - 2)$$

Differential of Surface

Curves:

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

$$\vec{F} = F_1\,\vec{i} + F_2\,\vec{j} + F_3\,\vec{k}$$

$$\vec{F}\cdot d\vec{r} = F_1\,dx + F_2\,dy + F_3\,dz$$

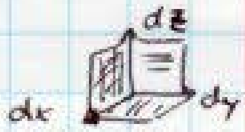
$\int \vec{F}\cdot d\vec{r}$ = Line integral

Surfaces

$$dx\,\vec{i} \times dy\,\vec{j} = dx\,dy\,\vec{k}$$

$$dz\,\vec{k} \times dx\,\vec{i} = dz\,dx\,\vec{j}$$

$$dy\,\vec{j} \times dz\,\vec{k} = dy\,dz\,\vec{i}$$



$$d\vec{S} = dy\,dz\,\vec{i} + dz\,dx\,\vec{j} + dx\,dy\,\vec{k}$$

$$dS = dy\wedge dz + dz\wedge dx + dx\wedge dy$$

$\boxed{\int \vec{F}\cdot d\vec{S}}$ Flux.

$$\int P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$$

Surface Integrals / Gauss' Thm

$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$\Phi = \iint \vec{F}\cdot d\vec{S}$

$= \iint P\, dy\wedge dz + Q\, dz\wedge dx + R\, dx\wedge dy$

$S: \ r(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

Need different computational tool.

Picture

$(u,v) \xrightarrow{\vec{r}(u,v)}$

$d\vec{S} = (\vec{r}_u \times \vec{r}_v)\, du\, dv$

$dS = |\vec{r}_u \times \vec{r}_v|\, du\, dv = \sqrt{EG - F^2}\, du\, dv$

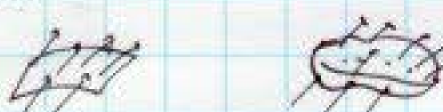
$$\boxed{\Phi = \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\, du\, dv}$$

Use for computation.

What does this mean?

$\Phi = \iint \vec{F}\cdot d\vec{S} = \iint \vec{F}\cdot\frac{(\vec{r}_u\times\vec{r}_v)}{|\vec{r}_u\times\vec{r}_v|}\, |\vec{r}_u\times\vec{r}_v|\, du\, dv$

$\Phi = \iint \vec{F}\cdot d\vec{S} = \iint (\vec{F}\cdot\vec{N})\, dS$



Flux

$F(x,y,z) = c \qquad \vec{N} = \frac{\vec{\nabla} F}{|\vec{\nabla} F|}$

Theorem

$$\boxed{\Phi = \oiint_{S=\partial V} \vec{F}\cdot d\vec{S} = \iiint_V (\vec{\nabla}\cdot\vec{F})\, dV}$$

Example. Page 1134

Find $\Phi = \oiint F\cdot dS \quad \vec{F} = \langle y, x, z\rangle$

$S_1: \ z = 1 - x^2 - y^2$

$S_2 = \ z = 0$

$\Phi_1 = \iint_{S_1} \vec{F}\cdot d\vec{S} \qquad \vec{r} = \langle u, v, 1 - u^2 - v^2\rangle$

$\vec{r}_u = \langle 1, 0, -2u\rangle \qquad x = u = r\cos\theta$

$\vec{r}_v = \langle 0, 1, -2v\rangle \qquad y = v = r\sin\theta$

$\vec{r}_u \times \vec{r}_v = \langle 2u, 2v, 1\rangle$

$\vec{F} = \langle v, u, 1 - u^2 - v^2\rangle$

$\Phi_1 = \iint (4uv + 1 - u^2 - v^2)\, du\, dv$

$= \int_0^{2\pi}\int_0^1 (4r^2\cos\theta\sin\theta + 1 - r^2)\, r\, dr\, d\theta$

$= \frac{\pi}{2}$

$\Phi_2: \quad z = 0 \qquad \vec{F} = \langle x, y, 0\rangle$

$\vec{N} = \langle 0, 0, 1\rangle$

$\Phi_2 = \iint (\vec{F}\cdot\vec{N})\, dS = \iint 0\, dx\, dy = 0$

$\Phi = \Phi_1 + \Phi_2 = \frac{\pi}{2}$

Monday $\iint \vec{F}\cdot d\vec{S} = \iiint \nabla\cdot F\, dV$

$= \iiint 1\, dV$

$=$ [redacted]

Do: 17.7 # 5, 7, 9, 11, 13

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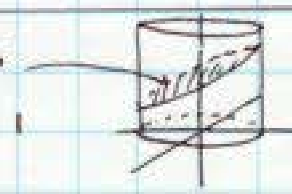
11) $\iint_S yz\, dS \qquad z = y + 3$

$x^2 + y^2 = 1$



$\vec{r} = \langle u, v, v+3\rangle$

$\vec{r}_u = \langle 1, 0, 0\rangle \qquad E = 1 \qquad F = 0$

$\vec{r}_v = \langle 0, 1, 1\rangle \qquad G = 2$

$\Phi = \iint_S v(v+3)\sqrt{2}\, du\, dv$

$= \sqrt{2}\int_0^{2\pi}\int_0^1 r\sin\theta(r\sin\theta + 3)\, r\, dr\, d\theta$

$= \sqrt{2}\int_0^{2\pi} (r^3\sin^2\theta + 3r^2\sin\theta)\, dr\, d\theta$

$= \sqrt{2}\int_0^{2\pi} (\frac{1}{4}\sin^2\theta + \sin\theta)\, d\theta$

$= \sqrt{2}\int_0^{2\pi} [\frac{1}{4}(\frac{1}{2} - \frac{1}{2}\cos 2\theta) + \sin\theta]\, d\theta$

$= \sqrt{2}\,[\frac{1}{8}\theta - \frac{1}{16}\sin 2\theta - \cos\theta]_0^{2\pi}$

$= \sqrt{2}\,[\frac{\pi}{4}]$

Ex 5 Page 1135

$F = z\vec{i} + y\vec{j} + x\vec{k}$

Find flux across $x^2 + y^2 + z^2 = 1$

Sol Need to compute $\Phi = \int \vec{F}\cdot d\vec{s}$

Use $\iint_S \vec{F}\cdot d\vec{S} = \iiint_V (\vec{\nabla}\cdot\vec{F})\, dV$

a) $\vec{\nabla}\cdot F = 0 + 1 + 0 \qquad V = \frac{4}{3}\pi r^3$

$\Phi = \iiint_V 1.\, dV = \frac{4}{3}\pi$

$= \int_0^{\pi}\int_0^{2\pi}\int_0^1 \rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi$

---

23 $\quad F = \langle x, y, z\rangle \qquad S: x^2 + y^2 + z^2 = 9$

$\vec{\nabla}\cdot F = 3$

$\Phi = \iint \vec{F}\cdot d\vec{S} = \iiint_V 3\, dV = 3(\frac{4}{3}\pi\cdot 3^3)$

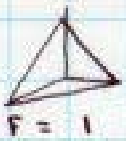
$= 36\pi\cdot 3 = 108\pi$

DO # 17.7, 19, 23, 27, 25

17.7 7) Find: $\iint yz\,dS$

$S: x+y+z=1$ First Octant

$z = 1-x-y$



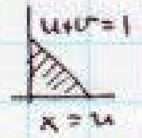
$\vec{r} = \langle u, v, 1-u-v \rangle$

$\vec{r}_u = \langle 1, 0, -1 \rangle \quad E = 2 \quad F = 1$

$\vec{r}_v = \langle 0, 1, -1 \rangle \quad G = 2$

$I = \iint_R v(1-u-v)\sqrt{2(2)-1}\,du\,dv$

$= \sqrt{3}\int_0^1\int_0^{1-u}(v-uv-v^2)\,dv\,du$



$= \frac{1}{24}$

Example 5 Page 1135

Flux $\vec{F} = z\vec{i} + y\vec{j} + x\vec{k}$

$S: x^2+y^2+z^2=1$

Sol: $\vec{\nabla}\cdot\vec{F} = \frac{\partial z}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial x}{\partial z} = 1$ $\boxed{V = \frac{4}{3}\pi r^3}$

$\Phi = \iint_{S=\partial V}\vec{F}\cdot d\vec{S} = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV = \iiint_V dV$

$= \frac{4}{3}\pi$

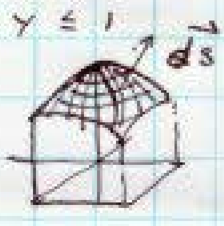
17.9 Do # 3, 5, 8, 13, 15, 19

Ex 19 Page 1138

$\vec{F} = \langle xy, yz, zx \rangle \qquad 0 \le x \le 1$

$S: z = 4-x^2-y^2 \qquad 0 \le y \le 1$

Find $\int \vec{F}\cdot d\vec{S} = \phi$



$\vec{r} = \langle u, v, 4-u^2-v^2 \rangle$

$\vec{r}_u = \langle 1, 0, -2u \rangle$

$\vec{r}_v = \langle 0, 1, -2v \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 2u, 2v, 1 \rangle$

$\vec{F} = \langle uv, v(4-u^2-v^2), u(4-u^2-v^2) \rangle$

$\vec{F}\cdot d\vec{S} = 2u^2v + 2v^2(4-u^2-v^2) + u(4-u^2-v^2)\,dA$

$\Phi = \int_0^1\int_0^1 [2u^2v + (4-u^2-v^2)(2v^2+u)]\,du\,dv$

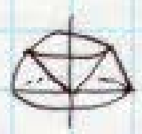
$= \frac{713}{180}$

Ex 21 Page 1126 Section 17.6

$x^2+y^2+z^2=4$ above $z=\sqrt{x^2+y^2}$

$x = 2\sin\varphi\cos\theta \qquad y=0 \quad z=\pm x$

$y = 2\sin\varphi\sin\theta$



$z = 2\cos\varphi \qquad 0 \le \varphi \le \pi/4$

$0 \le \theta \le 2\pi$

More Gauss Theorem Examples

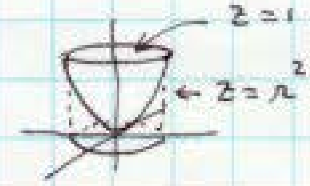
Fact $\iint_{S=\partial V}\vec{F}\cdot d\vec{S} = \iiint_V(\vec{\nabla}\cdot\vec{F})\,dV$

Pb 4 17.9

$\vec{F} = \langle xz, yz, 3z^2 \rangle$

$S: z = x^2+y^2$

$z = 1$



$\Phi = \iint \vec{F}\cdot d\vec{S}$

$\vec{\nabla}\cdot\vec{F} = z + z + 6z = 8z$

$\Phi = \iiint 8z\,dV$

$= \int_0^{2\pi}\int_0^1\int_{r^2}^1 8z\,r\,dz\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 [4z^2]_{r^2}^1\,r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 (4-4r^4)r\,dr\,d\theta$

$= (2-\frac{4}{6})2\pi = \frac{8}{3}\pi$

19) $F = \langle xz^2, \frac{1}{3}y^3+\tan z, x^2z+y^2 \rangle$

$S_1: x^2+y^2+z^2=1$

$\Phi = \iint \vec{F}\cdot d\vec{S} = \iiint \vec{\nabla}\cdot\vec{F}\,dV$ S closed

$= \underbrace{\iint_{S_1} + \iint_{S_2} \vec{F}\cdot d\vec{S}}_{} = \underbrace{\iiint \vec{\nabla}\cdot\vec{F}\,dV}_{}$

$\iint_{S_1}\vec{F}\cdot d\vec{S} = \iiint(\vec{\nabla}\cdot\vec{F})\,dV - \iint_{S_2}\vec{F}\cdot d\vec{S}$

$\vec{\nabla}\cdot\vec{F} = z^2 + y^2 + x^2 = x^2+y^2+z^2$

$\iiint \rho^2\,dV = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^1 \rho^2\cdot\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$

$= \frac{1}{5}\int_0^{\pi/2}\int_0^{2\pi}\sin\varphi\,d\theta\,d\varphi$

$= \frac{2\pi}{5} \qquad S_0 = 4\pi r^2$

$\iint_{S_2}\vec{F}\cdot d\vec{S} = -\iint \langle 0, \frac{1}{3}y^3, y^2 \rangle\cdot\langle 0,0,1 \rangle\,dy\,dx$

$= -\iint_R y^2\,dy\,dx =$

$= -\int_0^{2\pi}\int_0^1 r^2\sin^2\theta\,r\,dr\,d\theta$

$= -\frac{1}{4}\int_0^{2\pi}\sin^2\theta\,d\theta$

$= -\frac{1}{4}\int_0^{2\pi}(\frac{1}{2}-\frac{1}{2}\cos 2\theta)\,d\theta$

$= -\frac{1}{4}[\frac{1}{2}\cdot 2\pi] = \frac{\pi}{4}$

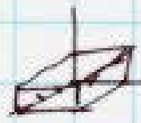
$\iint_{S_1}\vec{F}\cdot d\vec{S} = \frac{2\pi}{5} + \frac{\pi}{4} = \pi\left[\frac{8+5}{20}\right] = \frac{13\pi}{20}$

$= \frac{13\pi}{20}$

8) $\vec{F} = \langle x^2y, -x^2z, z^2y \rangle$ ~~$V = (1,1,1)$~~

$x=0 \quad y=0 \quad z=0$

$x=3 \quad y=2 \quad z=1$



$$\iint_S F\cdot dS = \iiint_V \vec{\nabla}\cdot\vec{F}\, dV = \Phi$$

$$\vec{\nabla}\cdot\vec{F} = 2xy - 0 + 2zy$$

$$\Phi = \int_0^3\int_0^2\int_0^1 (2xy + 2zy)\, dz\, dy\, dx$$

$$= \int_0^3\int_0^2 [2xyz + z^2y]_0^1\, dy\, dx$$

$$= \int_0^3\int_0^2 [2xy + y]\, dy\, dx$$

$$= \int_0^3 [xy^2 + \tfrac{1}{2}y^2]_0^2\, dx$$

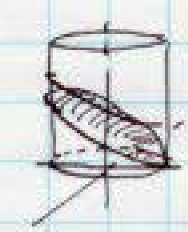
$$= \int_0^3 (4x + 2)\, dx$$

$$= 2x^2 + 2x\Big]_0^3 = 18 + 6 = 24$$

15) $F = \langle y e^{z^2}, y^2, e^{xy} \rangle$

S: $x^2 + y^2 = 9$

$z = 0, \quad z = y - 3$



$$\vec{\nabla}\cdot\vec{F} = 0 + 2y + 0$$

$$\Phi = \iint \vec{F}\cdot d\vec{S} = \iiint 2y\, dV$$

dz

$$= \int_0^{2\pi}\int_0^3\int_0^{r\sin\theta - 3} 2r\sin\theta\; r\, dz\, dr\, d\theta$$

11/30 Last Calculus Homework!

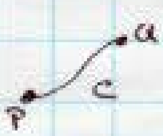
17.8 # 3, 5, 7, 9, 13, 17

Review

1) Line Integrals

a) Open Curves

$$W = \int_C \vec{F}\cdot d\vec{r}$$



i) If $\vec{\nabla}\times\vec{F} = 0$ Then $F = \nabla\varphi$

$$\int_C \vec{F}\cdot d\vec{r} = \varphi\Big]_P^Q$$

ii) If $\vec{\nabla}\times\vec{F} \neq 0$.

Parametrize curve

Compute $W = \int P\,dx + Q\,dy + R\,dz$

b) Closed Curves

$$W = \oint_{C=\partial S} \vec{F}\cdot d\vec{r} = \iint_{S_1} (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$$

$$= \iint_{S_2} (\vec{\nabla}\times\vec{F})\, d\vec{S}$$

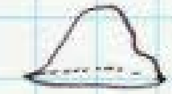
Replace S by simplest surface with the same boundary!

2) Surface Integrals

a) Open Surfaces

$$\Phi = \iint \vec{F}\cdot d\vec{S}$$

$$= \iint_S \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\, du\, dv$$



b) Closed Surfaces

$$\Phi = \iint_{S=\partial V} \vec{F}\cdot d\vec{S} = \iiint_V (\vec{\nabla}\cdot\vec{F})\, dV$$

Physics

1) $\int_C \vec{F}\cdot d\vec{r} = \int (\vec{F}\cdot\vec{T})\, ds$

$ds = \sqrt{dx^2 + dy^2 + dz^2}$

= Work!

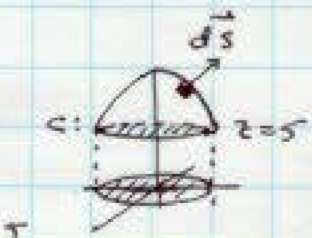
2) $\iint \vec{F}\cdot d\vec{S} = \iint (\vec{F}\cdot\vec{N})\, dS$

$dS = \sqrt{EG - F^2}\, du\, dv$

= Flux

Line Integrals and Stoke's Thm in Dim 3.

Example # 2 $\boxed{17.8}$



~~$\vec{F} = \langle y,$~~

$\vec{F} = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$

S: $z = 9 - x^2 - y^2 \quad z = 5$

Find $\iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = W$

a) Line Integral method

$$W = \iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r} \qquad 5 = 9 - x^2 - y^2$$

$$= \int_C yz\,dx + xz\,dy + xy\,dz \qquad \begin{cases} x^2 + y^2 = 4 \\ z = 5 \end{cases}$$

C: $x = 2\cos t, \; y = 2\sin t, \; z = 5$

$$W = \int_0^{2\pi} [(2\sin t)5(-2\sin t) + (2\cos t)5(2\cos t)]\, dt$$

$$= 20\int_0^{2\pi} (\cos^2 t - \sin^2 t)\, dt$$

$$= 20\int_0^{2\pi} \cos 2t\, dt$$

$$= 20\,\tfrac{1}{2}\sin 2t\Big]_0^{2\pi} = 0$$

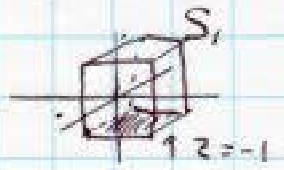
b) Replace S: by S_2: $z = 5$

$d\vec{S} = (dx\,dy)\,\vec{k}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ yz & xz & xy \end{vmatrix} = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$\Phi = \iint_{S_2} 0\cdot dx\,dy = 0$$

17.8
5. *Find* $\iint (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$



$\vec{F}\ \langle xyz,\ xy,\ x^2yz\rangle$

$S_1 \mapsto S_2:\quad z=-1$

$-1\le x, y, \le 1$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ xyz & xy & x^2yz \end{vmatrix}$$

$\vec{\nabla}\times\vec{F} = x^2z\,\vec{i} + (xy - 2xyz)\,\vec{j} + (y - xz)\,\vec{k}$

$d\vec{S} = (dx\,dy)\,\vec{k}$

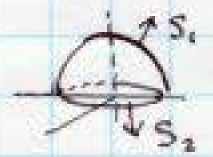
$I = \iint_{S_2} (y - xz)\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1}(y+x)\,dy\,dx$

$= \int_{-1}^{1}(\frac{1}{2}y^2 + xy)\Big|_{-1}^{1}dx = \int_{-1}^{1} 2x\,dx = 0$

17.9
19 $\iint \vec{F}\cdot d\vec{S} = \Phi$

$\vec{F} = \langle z^2x,\ (\frac{1}{3}y^3 + \tan z),\ x^2z + y^2\rangle$

$S_1:\ x^2+y^2+z^2 = 1$



$\vec{\nabla}\cdot\vec{F} = (z^2+y^2+x^2)$

$S = S_1 \cup S_2$

$\Phi = \iint_S F\cdot ds - \iint_{S_2} F\cdot ds$

$S_2:\ z=0$

$= \iiint_{\Delta}(\nabla\cdot F)\,ds - \iint_{S_2}\vec{F}\cdot d\vec{S}$ $\quad d\vec{S}_2 = dx\,dy\,(-\vec{k})$

$= -dx\,dy\,\vec{k}$

$= \iiint_{\Delta}\rho^2(\rho^2\sin\varphi)\,d\rho\,d\varphi\,d\theta - \iint_{\Theta} y^2\,dx\,dy$

$= \frac{13}{20}\pi$ (*See page 46*)

12/1 *Practice Problems*

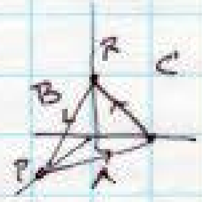
P 1154 # 3, 5, 13, 15, 27, 30, 31, 35

17.8 # 7

$\vec{F} = \langle x+y^2,\ y+z^2,\ z+x^2\rangle$

C.

Find $\oint_C \vec{F}\cdot d\vec{r} = \iint_{\Delta}(\vec{\nabla}\times\vec{F})\cdot d\vec{S}$



$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x+y^2 & y+z^2 & z+x^2 \end{vmatrix}$$

$= \langle -2z, -2x, -2y\rangle$

$A = \langle -1, 1, 0\rangle$

$B = \langle -1, 0, 1\rangle$

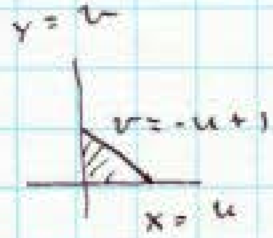
$\vec{N} = \langle 1, 1, 1\rangle$

$N = \langle 1, 1, 1\rangle$

$P = (1, 0, 0)$

$(x-1) + y + z = 0$

$z = 1 - x - y$



$\vec{r} = \langle u, v, 1-u-v\rangle$

$\vec{r}_u = \langle 1, 0, -1\rangle$

$\vec{r}_v = \langle 0, 1, -1\rangle$

$\vec{r}_u\times\vec{r}_v = \langle 1, 1, 1\rangle$

$\iint(\vec{\nabla}\times\vec{F})\,d\vec{S} = \iint(\vec{\nabla}\times\vec{F})\cdot(\vec{r}_u\times\vec{r}_v)\,du\,dv$

$= \iint(-2z - 2x - 2y)\,du\,dv$

$= \iint[-2(1-u-v) - 2u - 2v]\,du\,dv$

$= -2\int_0^1\int_0^{-u+1}[(1-u-v) + u + v]\,du\,dv$

$= -2\int_0^1\int_0^{-u+1} 1\,dv\,du$

$= -2\cdot\frac{1}{2} = -1$

1) $\oint \vec{F}\cdot d\vec{r} = \iint(\vec{\nabla}\times\vec{F})\cdot d\vec{S}$

2) $d\vec{S} = (\vec{r}_u\times\vec{r}_v)\,du\,dv$

Use Stokes Theorem to compute

$W = \int_C y\,dx + (2x - z)\,dy + (z - x)\,dz$

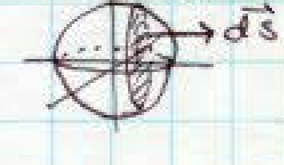
C: *Intersection* $\quad x^2+y^2+z^2 = 10$

$y = 1$

Solution $\quad x^2+y^2 = 9$

$F = \langle y,\ 2x - z,\ z - x\rangle$

$\oint F\cdot dr = \iint(\vec{\nabla}\times\vec{F})\cdot d\vec{S}$



$\vec{\nabla}\times\vec{F} = \langle 1, 1, 1\rangle$

$d\vec{S} = (dz\,dx)\,\vec{j}$

$W = \iint_{O} dz\,dx = \pi\cdot 3^2 = 9\pi$

$d\vec{S} = dx\,dy\,\vec{k}$

$d\vec{S} = dz\,dx\,\vec{j}$

$d\vec{S} = dy\,dz\,\vec{i}$