Review

Differentiation

I. Basic functions.

1. $\frac{d}{dx} x^n = n x^{n-1}$
2. $\frac{d}{dx} \cos x = -\sin x$
3. $\frac{d}{dx} \sin x = \cos x$
4. $\frac{d}{dx} \tan x = \sec^2 x$
5. $\frac{d}{dx} \sec x = \sec x \tan x$
6. $\frac{d}{dx} \ln x = \frac{1}{x}$
7. $\frac{d}{dx} e^x = e^x$

FTC

Ver 1 $\quad \frac{d}{dx} \int_a^x f(x)\,dx = f(x)$

Ver 2 $\quad \int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$

where $F'(x) = f(x)$

Theorems

$u = u(x)$, $v(x)$

1) $\frac{d}{dx} c\,u(x) = c \frac{du}{dx}$
2) $\frac{d}{dx} (u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$
3) $\frac{d}{dx} (u \cdot v) = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$
4) $\frac{d}{dx} \left[\frac{u}{v}\right] = \dfrac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$

Chain rule!

~~$u = u(x$~~)

$u = f(x) \qquad v = g(x)$

$f(g(x)) = (f \circ g)(x)$

1. $\frac{d}{dx} f(v) = \frac{df}{dv} \cdot \frac{dv}{dx}$
2. $\frac{d}{dx} (f \circ g)(x) = f'(g(x)) \cdot g'(x)$

Example

1. $\frac{d}{dx} \sin x^2 = 2x \cos x^2$
2. $\int \sin x^2 \, dx = \ ? \quad -\frac{\cos x^3}{3}$

$? \ -\frac{\cos x^2}{2x} \qquad ? \ -\frac{1}{3} x^3 \cos x^2$

$? \ \left[\frac{\cos x^2}{2}\right]' = -\left(\frac{\sin x^2}{2}\right) 2x$

Ex

$$\int \underbrace{x^2}_{u}\, \underbrace{e^x dx}_{dv} = I$$

Let $u = x^2$ $\quad du = 2x\,dx$

$dv = e^x dx$ $\quad v = e^x$

$$I = x^2e^x - \int 2xe^x dx$$
$$= x^2e^x - 2[xe^x - e^x] + c$$
$$= e^x[x^2 - 2x + 2] + c$$

Note

$$\int \underbrace{x^n}_{u} \underbrace{\begin{cases} e^x \\ \cos x \\ \sin x \end{cases} dx}_{dv}$$

Integrate I.B.P n times.

Trick

$$I = \int x^2 e^x dx$$

+	x^2	e^x
-	$2x$	e^x
+	2	e^x
-	0	e^x

$$I = x^2e^x - 2xe^x + 2e^x$$
$$= e^x[x^2 - 2x + 2] + c$$

5) $\int x^2 \cos 3x\, dx$

+	x^2	$\cos 3x$
-	$2x$	$\frac{1}{3}\sin 3x$
+	2	$-\frac{1}{9}\cos 3x$
-	0	$-\frac{1}{27}\sin 3x$

$$I = \frac{1}{3}x^2 \sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x$$

Note

$$I = \int e^x \begin{cases} \cos x \\ \sin x \end{cases} dx$$

I.B.P *twice* — *flip back*

$$I = \int \underbrace{e^x}_{u}\, \underbrace{\cos x\, dx}_{dv}$$

Let $u = e^x$ $\quad du = e^x dx$

$dv = \cos x\, dx$ $\quad v = \sin x$

$$I = e^x \sin x - \int \underbrace{e^x}_{v}\, \underbrace{\sin x\, dx}_{dv}$$

Let $u = e^x$ $\quad du = e^x dx$

$dv = \sin x\, dx$ $\quad v = -\cos x$

$$I = e^x \sin x - [-e^x \cos x + \int e^x \cos x\, dx]$$

$$I = e^x \sin x + e^x \cos x - I$$
$$2I = e^x \sin x + e^x \cos x$$
$$I = \frac{1}{2}e^x[\sin x + \cos x] + c$$

Ex

~~Ilm~~ $\int \underbrace{\ln x}_{u}\, \underbrace{dx}_{dv} = I$

Let $u = \ln x$ $\quad du = \frac{1}{x}dx$

$dv = dx$ $\quad v = x$

$$I = x\ln x - \int 1\, dx$$
$$= x \ln x - x + c$$

1) $\int \tan^{-1} x\, dx$

2) $\int \sin^{-1} x\, dx$

3) $\int \ln(1 + x^2)\, dx$

1. $\frac{d}{dx} e^u = e^u \cdot \frac{du}{dx}$

2. $\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}$

1. $\int e^u du = e^u + c$

2. $\int \frac{1}{u} du = \ln u + c$

41. $\int \frac{1}{x \ln x} dx = I$

Let $u = \ln x$

$du = \frac{1}{x} dx$

$I = \int \frac{1}{\ln x}\left(\frac{1}{x} dx\right) = \int \frac{1}{u} du$

$= \ln|u| + c$

$= \ln(\ln x) + c$

44.) $\int \frac{e^x}{e^x+1} dx = I$

Let $u = e^x + 1$

$du = e^x dx$

$I = \int \frac{1}{e^x+1} (e^x dx) = \int \frac{1}{u} du$

$= \ln(u) + c$

$= \ln(e^x+1) + c$

Ex $I = \int \frac{1}{e^x} dx = \int e^{-x} dx = -e^{-x} + c$

31) $\int (\sin 3\alpha - \sin 3x) dx$

$= (\sin 3\alpha) x + \frac{1}{3} \cos 3x + c$

47) $\int \frac{1+x}{1+x^2} dx = \int \left(\frac{1}{1+x^2} + \frac{x}{1+x^2}\right) dx$

$= \tan^{-1} x + \frac{1}{2} \ln(1+x^2) + C$

Facts

1. $\int \frac{1}{1+u^2} du = \tan^{-1} u + c$

2. $\int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1} u + c$

3. $\int \frac{1}{u\sqrt{1-u^2}} du = \sec^{-1} u + c$

Integration Techniques

1 Substitution ✓

2. IBP. ✓

Diff	Int
~~Us Stebs~~	
1. Chain Rule	1. Subst
2. Prod. Rule	2. I.B.P
3. Quot. Rule	3. xxxx

Pb 1-29 all

I.B.P.

$\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$

$\int d(u \cdot v) = \int u \cdot dv + \int v \cdot du$

$u \cdot v = \int u\, dv + \int v\, du$

$$\boxed{\int u\, dv = uv - \int v\, du}$$

Ex $\int \underbrace{x}_{u} \underbrace{e^x dx}_{dv} = I$

Let $u = x$ $\quad du = dx$

$dv = e^x dx$ $\quad v = e^x$

$I = xe^x - \int e^x dx$

$= xe^x - e^x + c$

7 $\int (\ln x)^2\,dx = I$

Let $u = (\ln x)^2 \qquad du = \dfrac{2\ln x}{\cancel{x}}\,dx$

$dv = dx \qquad \boxed{v = x}$

$$I = x(\ln x)^2 - \int 2\ln x\,dx$$
$$= x(\ln x)^2 - 2\,[x\ln x - x] + C$$
$$= x(\ln x)^2 - 2x\ln x + 2x + C$$

19) $\int_0^{\frac{\pi}{2}} x\cos 2x\,dx$

$I = \int x\cos 2x\,dx$

+	x	$\cos 2x$
−	1	$\frac{1}{2}\sin 2x$
+	0	$-\frac{1}{4}\cos 2x$

$I = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x$

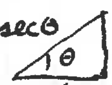
$$I\Big|_0^{\pi/2} = \left[\frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right]_0^{\pi/2}$$
$$= \frac{\pi}{4}\cdot 0 + \left(-\frac{1}{4}\right) - \left[\frac{1}{4}\right] = -\frac{1}{2}$$

Trigonometric Integrals

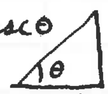
I. Pythagoras Theorem

a) 

$\cos^2\theta + \sin^2\theta = 1$

b) 

$1 + \tan^2\theta = \sec^2\theta$

c) 

$1 + \cot^2\theta = \csc^2\theta$

Pb $\int \sin^n x\cos^m x\,dx$

Case 1 Either n or m is odd
Use Pythagoras Theorem

Case 2 Both, n & m are even
Use $\frac{1}{2}\measuredangle$ (Double angles)

Ex

$$\int \sin^2 x\cos^3 x\,dx$$
$$= \int \sin^2 x\cdot\cos^2 x\,(\cos x\,dx)$$
$$= \int \sin^2 x\,(1 - \sin^2 x)\,(\cos x\,dx)$$
$$= \int u^2(1 - u^2)\,du \qquad (u = \sin x)$$
$$= \int (u^2 - u^4)\,du$$
$$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C$$
$$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

Half / Double angle formulas

II $\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

III $\sin\dfrac{\theta}{2} = \sqrt{\dfrac{1 - \cos\theta}{2}}$

$\cos\dfrac{\theta}{2} = \sqrt{\dfrac{1 + \cos\theta}{2}}$

Other forms

$\sin\theta = \sqrt{\dfrac{1 - \cos 2\theta}{2}}$

$\sin^2\theta = \dfrac{1 - \cos 2\theta}{2}$!

Example

$$\int \sin^3 x\,dx = \int \sin^2 x\,(\sin x\,dx)$$
$$= -\int (1 - \cos^2 x)\,(-\sin x\,dx)$$
$$= -\int (1 - u^2)\,du \qquad (u = \cos x)$$
$$= -\left[u - \frac{u^3}{3}\right] + C$$
$$= -\cos x + \frac{1}{3}\cos^3 x + C$$

$$\int \sin^5 x\,dx = \int \sin^4 x\,(\sin x\,dx)$$
$$= \int (\sin^2 x)^2(\sin x\,dx)$$
$$= -\int (1 - \cos^2 x)^2\,(-\sin x\,dx)$$
$$= -\int (1 - u^2)^2\,du \qquad (u = \cos x)$$
$$= -\int (1 - 2u^2 + u^4)\,du$$
$$= -\left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right] + C$$
$$= -\left[\cos x - \frac{2}{3}\cos^3 x + \frac{1}{5}\cos^5 x\right] + C$$

Ex

$$\int \sin^4 x\,dx = \int (\sin^2 x)^2\,dx$$
$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$
$$= \frac{1}{4}\int (1-\cos 2x)^2\,dx$$
$$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx$$
$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx$$
$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$
$$= \frac{1}{4}\left[\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right] + C$$

* $\int \sin^4 x \cos^2 x\,dx = I$

$$\int \left[\frac{1-\cos 2x}{2}\right]^2 \cdot \left[\frac{1+\cos 2x}{2}\right] dx =$$
$$I = \int \frac{1}{8}\left[1 - 2\cos 2x + \cos^2 2x\right]\left[1+\cos 2x\right] dx$$
$$= \frac{1}{8}\int 1 + \cos 2x - 2\cos 2x - 2\cos^2 2x\,dx$$
$$+ \frac{1}{8}\int (\cos^2 2x + \cos^3 2x)\,dx$$
$$I = \frac{1}{8}\int (1 - \cos 2x - \cos^2 2x + \cos^3 2x)\,dx$$
$$= \frac{1}{8}\int \left(1 - \cos 2x - \left(\frac{1+\cos 4x}{2}\right)\right) dx + \frac{1}{8}\int \cos^2 2x \cdot \cos 2x\,dx$$
$$I = \frac{1}{8}\int \left(\frac{1}{2} - \cos 2x - \frac{1}{2}\cos 4x\right) dx + \frac{1}{8}\int (1-\sin^2 2x)\cos 2x\,dx$$
$$I = \frac{1}{8}\left[\frac{x}{2} - \frac{1}{2}\sin 2x - \frac{1}{8}\sin 4x + \frac{1}{2}\sin 2x - \cancel{\frac{1}{8}}\left(\frac{1}{6}\sin^3 2x\right)\right] + C$$

Do § 7.2 # 1, 3, 5, 7, 9, 13, 17

$$\int 4\sin^2 2x \cos^2 2x\,dx =$$
$$= \int \sin^2 4x\,dx$$
$$= \int \frac{1}{2}(1 - \cos 8x)\,dx$$
$$= \frac{x}{2} - \frac{1}{16}\sin 8x + C$$

$$\int \sec^n x \tan^m x\,dx$$

1) Factor $(\sec x \tan x\,dx)$
   Use $\tan^2 x = \sec^2 x - 1$

2) Factor $(\sec^2 x\,dx) = d(\tan x)$
   Use $\sec^2 x = \tan^2 x + 1$

Ex)

2) $\int \sec^4 x\,dx = \int \sec^2 x \cdot (\sec^2 x\,dx)$
$$= \int (\tan^2 x + 1)(\sec^2 x\,dx)$$
$$= \frac{1}{3}\tan^3 x + \tan x + C$$

Ex $\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}\,dx$
$$= \int \frac{\sec^2 x + \sec x \tan x}{(\sec x + \tan x)}\,dx$$
$$= \int \frac{1}{u}\,du \qquad (u = \sec x + \tan x)$$
$$= \ln|\sec x + \tan x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$
$$\int \tan x\,dx = \ln(\sec x) + C$$

11)

$\int \sec^3 x\, dx$
$\int \tan^2 x \sec x\, dx$ $\Big\}$ I.B.P

$$\int \sec^3 x\, dx = \int \underbrace{\sec x}_{u}\ \underbrace{\sec^2 x\, dx}_{dv}$$

Let $u = \sec x$ $\quad du = \sec x \tan x\, dx$
$dv = \sec^2 x\, dx$ $\quad v = \tan x$

$$I = \sec x \tan x - \int \sec x \tan^2 x\, dx$$

$$\int \tan^5 x\, dx = \int \tan^3 x\, (\tan^2 x\, dx)$$
$$= \int \tan^3 x\, (\sec^2 x - 1)\, dx$$
$$= \int \tan^3 x \sec^2 x\, dx - \int \tan^3 x\, dx$$
$$= \frac{1}{4}\tan^4 x - \int \tan x\, (\sec^2 x - 1)\, dx$$
$$= \frac{1}{4}\tan^4 x - \int \tan x \sec^2 x\, dx - \int \tan x\, dx \qquad + C$$
$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\sec x|$$

Integrals involving $a^2 \pm x^2$

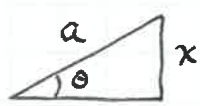
1) $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\frac{x}{a} + c$

2) $\displaystyle\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

3) $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\, dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

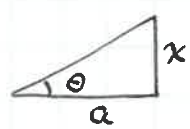
Trig subs

1. $(a^2 - x^2)$



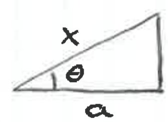
Let $x = a \sin\theta$

2. $(a^2 + x^2)$



Let $x = a \tan\theta$.
$x = a \sinh\theta$

3. $(x^2 - a^2)$



Let $x = a \sec\theta$

$$\int \sin^5 x \cos^5 x\, dx$$
$$= \int \sin^5 x\, (\cos^4 x)(\cos x\, dx)$$
$$= \int \sin^5 x\, (1 - \sin^2 x)^2\, (\cos x\, dx)$$
$$= \int u^5 (1 - u^2)^2\, du \qquad (u = \sin x)$$
$$= \int u^5 (1 - 2u^2 + u^4)\, du$$
$$= \int (u^5 - 2u^7 + u^9)\, du$$
$$= \frac{1}{6}\sin^6 x - \frac{1}{4}\sin^8 x + \frac{1}{10}\sin^{10} x + C$$

Trigonometric Subst.

1. $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1 - x^2}}$

2. $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1 + x^2}$

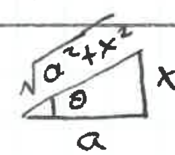
3. $\dfrac{d}{dx}\sec^{-1}x = \dfrac{1}{x\sqrt{x^2 - 1}}$

—o—

1′ $\dfrac{d}{dx}\sinh^{-1}x = \dfrac{1}{\sqrt{1 + x^2}}$

2′ $\dfrac{d}{dx}\tanh^{-1}x = \dfrac{1}{1 - x^2}$

$$\int \frac{1}{a^2 + x^2}\, dx = I$$



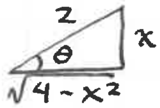
Let $\dfrac{x}{a} = \tan\theta$
$x = a\tan\theta$
$dx = a\sec^2\theta\, d\theta$
$\sqrt{a^2 + x^2} = a\sec\theta$

$$I = \int \frac{1}{a^2\sec^2\theta} \cdot a\sec^2\theta\, d\theta$$
$$= \int \frac{1}{a}\, d\theta = \frac{1}{a}\theta + C$$
$$= \frac{1}{a}\tan^{-1}\frac{x}{a}.$$

434 # 1,3,5,7

Ex

$I = \int \sqrt{4 - x^2}\, dx$



Let $x = 2\sin\theta$

$\sqrt{4-x^2} = 2\cos\theta$

$dx = 2\cos\theta \cdot d\theta$

$I = \int 2\cos\theta \cdot 2\cos\theta\, d\theta$

$= \int 4\cos^2\theta\, d\theta$

$= \int 4\, \frac{1+\cos 2\theta}{2} \cdot d\theta$

$I = 2\int (1 + \cos 2\theta)\, d\theta$

$= 2\left[\theta + \frac{1}{2}\sin 2\theta\right] + c$

$= 2\theta + \sin 2\theta + c$

$= 2\theta + 2\cos\theta \sin\theta + c$

$= 2\sin^{-1}\frac{x}{2} + 2\,\frac{\sqrt{4-x^2}}{2} \cdot \frac{x}{2} + c$

$= 2\sin^{-1}\frac{x}{2} + \frac{1}{2}x\sqrt{4-x^2} + c$

7-3 # 1, 3, 5, 7, 10, 15, 21, 24, 29, 34.

Ex

$\int \frac{1}{x^2+2x+5}\, dx = \int \frac{dx}{(x^2+2x\quad)+5}$

$= \int \frac{dx}{(x^2+2x+1)+4} = \int \frac{1}{4+(x+1)^2}\, dx$

$= \int \frac{1}{4+z^2}\, dz \qquad (z = x+1)$

$= \frac{1}{2}\tan^{-1}\frac{z}{2} + c$

$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$

$\int \frac{1}{(x^2+2x+2)^2}\, dx = \int \frac{1}{[(x^2+2x+1)+1]^2}\, dx$

$= \int \frac{1}{[(x+1)^2+1]^2}\, dx = \int \frac{1}{(z^2+1)^2}\, dz = I$

Let [redacted] $z = \tan\theta$

$\sqrt{z^2+1} = \sec\theta$

$dz = \sec^2\theta\, d\theta$

$I = \int \frac{1}{\sec^4\theta}\sec^2\theta\, d\theta = \int \frac{1}{\sec^2\theta}\, d\theta$

$= \int \cos^2\theta\, d\theta = \int \frac{1+\cos 2\theta}{2}\, d\theta$

$= \int \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$

$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + c$

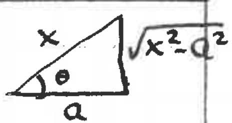
$= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$

$= \frac{1}{2}\tan^{-1} z + \frac{1}{2}\,\frac{z}{\sqrt{1+z^2}}\,\frac{1}{\sqrt{1+z^2}} + c$

$= \frac{1}{2}\tan^{-1} z + \frac{z}{2(1+z^2)} + c$

$= \frac{1}{2}\tan^{-1}(x+1) + \frac{x+1}{2(x^2+2x+2)}$

(10) $\int \frac{\sqrt{x^2-a^2}}{x^4}\, dx = I$



Let $x = a\sec\theta$

$\sqrt{x^2-a^2} = a\tan\theta$

$dx = a\sec\theta\tan\theta\, d\theta$

$I = \int \frac{a\tan\theta}{a^4\sec^4\theta} \cdot a\sec\theta\tan\theta\, d\theta$

$= \frac{1}{a^2}\int \frac{\tan^2\theta}{\sec^3\theta}\, d\theta$

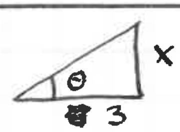
$= \frac{1}{a^2}\int \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^3\theta\, d\theta$

$= \frac{1}{a^2}\int \sin^2\theta\cos\theta\, d\theta$

$= \frac{1}{a^2} \cdot \frac{1}{3}\sin^3\theta + c$

$= \frac{1}{3a^2}\left[\frac{\sqrt{x^2-a^2}}{x}\right]^3 + c$

7 $\int_0^3 \frac{1}{\sqrt{9+x^2}}\, dx$



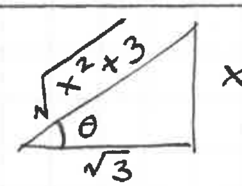
Let $x = 3\tan\theta$

$\frac{x}{3} = \tan\theta$

x	θ
0	0
3	$\frac{\pi}{4}$

etc.

(15) $\int \frac{1}{x\sqrt{x^2+3}}\, dx = I$



Let $x = \sqrt{3}\tan\theta$

$\sqrt{x^2+3} = \sqrt{3}\sec\theta$

$dx = \sqrt{3}\sec^2\theta\, d\theta$

$I = \int \frac{\sqrt{3}\sec^2\theta}{\sqrt{3}\tan\theta \cdot \sqrt{3}\sec\theta}\, d\theta$

$= \frac{1}{\sqrt{3}}\int \frac{\sec\theta}{\tan\theta}\, d\theta = \frac{1}{\sqrt{3}}\int \frac{1}{\cos\theta}\Big/\frac{\sin\theta}{\cos\theta}\, d\theta$

$= \frac{1}{\sqrt{3}}\int \frac{\sin\theta}{\cos^2\theta}\, d\theta$

7 $I = \frac{1}{\sqrt{3}} \int \frac{1}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta}\, d\theta$

$= \frac{1}{\sqrt{3}} \int \frac{1}{\sin\theta}\, d\theta = \frac{1}{\sqrt{3}} \int \csc\theta\, d\theta$

$= \frac{1}{\sqrt{3}} \ln|\csc\theta - \cot\theta| + C$

$= \frac{1}{\sqrt{3}} \ln\left| \frac{\sqrt{x^3+3}}{x} - \frac{\sqrt{3}}{x} \right| + C$

Partial Fractions

Rational Function

$f(x) = \frac{P_n(x)}{Q_m(x)} \qquad n < m$

Ex

$\int \frac{1}{4-x^2}\, dx$

$\frac{1}{4-x^2} = \frac{1}{(2-x)(2+x)}$

$\frac{1}{4-x^2} = \frac{A}{2-x} + \frac{B}{2+x}$

$\frac{1}{4-x^2} = \frac{A(2+x) + B(2-x)}{4-x^2}$

$1 = A(2+x) + B(2-x)$

$1 = A(2+x) + B(2-x)$

I) Let $x = 2$

$1 = A \cdot 4 \Rightarrow A = \frac{1}{4}$

$x = -2 \quad 1 = B \cdot 4 \Rightarrow B = \frac{1}{4}$

$\int \frac{1}{4-x^2}\, dx = \int \frac{\frac{1}{4}}{2-x}\, dx + \int \frac{\frac{1}{4}}{2+x}\, dx$

$= -\frac{1}{4} \ln|2-x| + \frac{1}{4} \ln|2+x|$

$= \frac{1}{4} [\ln|2+x| - \ln|2-x|$

$= \frac{1}{4} \ln\left| \frac{2+x}{2-x} \right| + C$

$= \frac{1}{2} \tanh^{-1} \frac{x}{2} + C$

$1 = x(A-B) + (2A+2B)$

$A - B = 0 \qquad A = B$

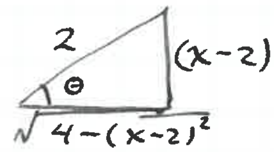
$2A + 2B = 1 \qquad 4A = 1$

$A = \frac{1}{4} = B$

24) $\int \frac{x^2}{\sqrt{4x-x^2}}\, dx = \int \frac{x^2\, dx}{\sqrt{4-(x^2-4x+4)}}$

$= \int \frac{x^2}{\sqrt{4-(x-2)^2}}\, dx$

Let $x - 2 = 2\sin\theta$



Ex

$\int \frac{1}{x^3+x}\, dx = \int \frac{1}{x(x^2+1)}\, dx$

$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$1 = A(x^2+1) + x(Bx+C)$

if $x=0 \quad \boxed{1 = A}$

$x=1 \quad 1 = 2A + B + C$

$\boxed{-1 = B + C}$

$x=-1 \quad 1 = 2A - (-B+C)$

$\boxed{-1 = B - C} \qquad B = -1$

$C = 0$

$\int \frac{1}{x(x^2+1)}\, dx = \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx$

$=$

Quiz

$\int \sqrt{1-4x^2}\, dx$

(19) $\int \frac{4x-1}{(x-1)(x+2)}\,dx$

$$\frac{4x-1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$4x-1 = A(x+2) + B(x-1)$$

$x=1$ $\quad 3 = A(3) \quad A = 1$

$x=-2$ $\quad -9 = -3B \quad B = 3$

$$\int \frac{4x-1}{(x-1)(x+2)} = \int \left(\frac{1}{x-1} + \frac{3}{x+2}\right) dx$$

$$= \ln|x-1| + 3\ln|x+2| + C$$

$$= \ln[(x-1)(x+2)^3] + C$$

Ex

$$\int \frac{x^3-2}{(x-2)^2(x^2+1)^2}\,dx$$

$$\frac{x^3-2}{(x-2)^2(x^2+1)^2} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{Cx+D}{(x^2+1)} + \frac{Ex+F}{(x^2+1)^2}$$

Find the Partial Fraction dec.

49

$$\frac{3x^3-x^2+6x-4}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$

$$I = \frac{A}{2}\ln|x^2+1| + B\tan^{-1}x + \frac{C}{2}\ln|x^2+2| + \frac{D}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

Ex

$$\int \frac{x}{(x+1)(x-2)^2}\,dx$$

$$\frac{x}{(x+1)(x-2)^2} = \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

Try 7.4

\# 9, 11, 15, 19, 23, 27, 37

43, 58, 67*

1

$$\int \frac{2x+5}{x-3}\,dx = \int\left(2 + \frac{11}{x-3}\right)dx$$
$$= 2x + 11\ln|x-3| + C$$

3

$$\int \sin^2 x \cos^3 x \, dx$$
$$= \int \sin^2 x\,(1-\sin^2 x)\cdot \cos x\,dx$$
$$= \int (\sin^2 x - \sin^4 x)\cdot \cos x\,dx$$
$$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

9.

$$\int \underbrace{\ln(1+x^2)}_{u}\,\underbrace{dx}_{dv} = I$$

Let $u = \ln(1+x^2)$ $\qquad u = \dfrac{2x\,dx}{1+x^2}$

$dv = dx$ $\qquad v = x$

$$I = x\ln(1+x^2) - 2\int \frac{x^2}{1+x^2}\,dx$$
$$I = x\ln(1+x^2) - 2\int\left(1 - \frac{1}{1+x^2}\right)dx$$
$$I = x\ln(1+x^2) - 2\left[x - \tan^{-1}x\right] + C$$
$$I = x\ln(1+x^2) - 2x + 2\tan^{-1}x + C$$

7.6 # 1, 3, 6, 9 . . .

$$\int x^5 e^{-x^3}\,dx = U$$

Let $z = -x^3$

$dz = -3x^2\,dx$

$$U = \int -\frac{x^3}{3}\, e^{-x^3}\,(-3x^2dx)$$
$$U = +\frac{1}{3}\int z e^z\,dz$$
$$= +\frac{1}{3}\left[ze^z - e^z\right] + C$$
$$= \frac{1}{3}e^{-x^3}(-x^3-1) + C$$

6)

$$\int_1^2 x^3 \ln x\,dx = I$$

Let $u = \ln x$ $\qquad du = \frac{1}{x}\,dx$

$dv = x^3dx$ $\qquad v = \frac{1}{4}x^4$

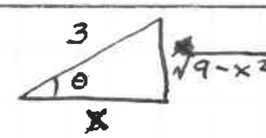
$$I = \frac{1}{4}x^4\ln x\Big]_1^2 - \int_1^2 \frac{1}{4}x^3\,dx$$
$$= \frac{1}{4}x^4\ln x - \frac{1}{16}x^4\Big]_1^2$$
$$I = 4\ln 2 - 1 - \left(0 - \frac{1}{16}\right) = 4\ln 2 - \frac{15}{16}$$

12)

$$\int \tan^3 x \sec^4 x\,dx$$
$$= \int \tan^3 x \sec^2 x\,(\sec^2 x\,dx)$$
$$= \int \tan^3 x\,(\tan^2 x + 1)\,(\sec^2 x\,dx)$$
$$= \int (\tan^5 x + \tan^3 x)(\sec^2 x\,dx)$$
$$= \frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + C$$

15)

$$\int \frac{\sqrt{9-x^2}}{x}\,dx = U$$



Let $x = 3\cos\theta$

$dx = -3\sin\theta\,d\theta$

$\sqrt{9-x^2} = 3\sin\theta$

$$U = -\int \frac{3\sin\theta}{3\cos\theta}\,(3\sin\theta\,d\theta)$$
$$= -3\int \frac{\sin^2\theta}{\cos\theta}\,d\theta = -3\int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta$$
$$= -3\int(\sec\theta - \cos\theta)\,d\theta$$
$$= -3\left[\ln|\sec\theta + \tan\theta| - \sin\theta\right] + C$$
$$= -3\left[\ln\left|\frac{3}{x} + \frac{\sqrt{9-x^2}}{x}\right| - \frac{\sqrt{9-x^2}}{3}\right] + C$$

18)

$$\int \frac{x^3+x+1}{x^4+2x^2+4x}\,dx = I$$

$\boxed{x^3+x+1}$ $\qquad$ Let $u = x^4+2x^2+4x$

$du = (4x^3+4x+4)dx$

$$I = \frac{1}{4}\int \frac{1}{u}\,du \qquad du = 4(x^3+x+1)\,dx$$
$$= \frac{1}{4}\ln|x^4+2x^2+4x| + C$$

27 $\displaystyle\int \frac{6x^2+5x-3}{x^3+2x^2-3x}\,dx = I$

$$\frac{6x^2+5x-3}{x(x+3)(x-1)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1}$$

$$6x^2+5x-3 = A(x+3)(x-1) + Bx(x-1) + Cx(x+3)$$

$x=0 \qquad -3 = -3A \quad \Rightarrow A = 1$

$x=1 \qquad 8 = 4C \quad \Rightarrow C = 2$

$x=-3 \qquad \Rightarrow B = 3$

$$I = \ln|x| + 3\ln|x+3| + 2\ln|x-1|$$

7.5 *Int by intuition.*

\#2) $\displaystyle\int \frac{1}{1+\sqrt[3]{x}}\,dx = J$

Let $\boxed{u = 1+\sqrt[3]{x}}$

$u-1 = \sqrt[3]{x}$

$(u-1)^3 = x$

$\boxed{3(u-1)^2\,du = dx}$

$$J = \int \frac{1}{u}\cdot 3(u-1)^2\,du =$$

$$= 3\int \frac{u^2-2u+1}{u}\,du = 3\int\left(u-2+\frac{1}{u}\right)du$$

$$J = 3\left[\tfrac{1}{2}(1+\sqrt[3]{x})^2 - 2(1+\sqrt[3]{x}) + \ln|1+\sqrt[3]{x}| + C\right.$$

5 $\displaystyle\int \frac{\sqrt{x}}{x+1}\,dx$

Let $u = x+1$

$x = u-1 \qquad\qquad I = \displaystyle\int \frac{\sqrt{u-1}}{u}\,du$

$\sqrt{x} = \sqrt{u-1} \qquad\qquad$ Oops

Let $\begin{cases} u = \sqrt{x} \\ u^2 = x \\ 2u\,du = dx \end{cases}$

$$I = \int \frac{u}{u^2+1}\cdot 2u\,du = 2\int\frac{u^2}{u^2+1}\,du$$

$$= 2\int\frac{(u^2+1)-1}{(u^2+1)}\,du$$

$$= 2\int\left[1-\frac{1}{u^2+1}\right]du$$

$$= 2\left[u - \tan^{-1}u\right] + c$$

$$= 2\left[\sqrt{x}+0 - \tan^{-1}(\sqrt{x}+0)\right] +$$

7.5 |

1, 7, 9, 11, 15, 19, 23*, 25*

7) $\displaystyle\int_5^{10} \frac{x^2}{\sqrt{x-1}}\,dx = I$

Let $u = \sqrt{x-1} \qquad x = u^2+1$

$u^2 = x-1$

$2u\,du = dx$

x	u
5	2
10	3

$$I = \int_2^3 \frac{(u^2+1)^2}{u}\cdot 2u\,du = 2\int_2^3 (u^2+1)^2\,du$$

$$= 2\int_2^3 (u^4+2u^2+1)\,du$$

$$= 2\left[\tfrac{1}{5}u^5 + \tfrac{2}{3}u^3 + u\right]_2^3$$

$$= \frac{1676}{15}$$

19) $\displaystyle\int \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx = I$

Let $y = e^x$

$dy = e^x\,dx$

$$I = \int \frac{y}{y^2+3y+2}\,dy$$

$$\frac{y}{y^2+3y+2} = \frac{y}{(y+2)(y+1)} = \frac{A}{y+2} + \frac{B}{y+1}$$

$y = A(y+1) + B(y+2)$

$-1 = B \qquad A = 2$

$-2 = -A \qquad B = -1$

$$I = 2\ln|y+2| - \ln|y+1| + c$$

$$= 2\ln|e^x+2| - \ln|e^x+1| + c$$

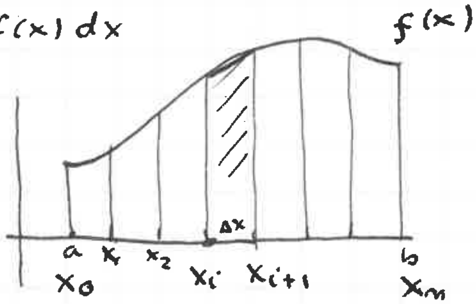
Quiz.

1. $\displaystyle\int \frac{x^2+1}{x^2-x}\,dx$

2. $\displaystyle\int \frac{x^2}{\sqrt{x-1}}\,dx$

Trapezoidal rule

$\int_a^b f(x)\,dx$



$$\Delta A_i = \frac{1}{2}\left[f(x_i) + f(x_{i+1})\right]\cdot \Delta x$$

$$\boxed{A \doteq \sum_{i=0}^{n-1}\left[\frac{f(x_i)+f(x_{i+1})}{2}\right]\cdot \Delta x}$$

$x_1 = a + \Delta x$

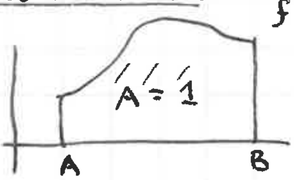
$x_2 = a + 2\cdot\Delta x$ $\qquad \boxed{\Delta x = \frac{b-a}{n}}$

$\boxed{x_i = a + i\cdot\Delta x}$

$$A = \sum_{i=0}^{n-1}\left[\frac{y_i + y_{i+1}}{2}\cdot \Delta x\right]$$

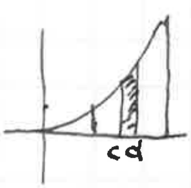
Probabilities

$f(x) > 0$



Ex

$f(x) = x^2 \qquad 1 \le x \le 4$



$f(x) \mapsto \frac{1}{21}x^2$

$f(x) = k e^{-5x} \qquad 0 \le x \le \infty$

Def $P(x)$ is a Probability

$\int_a^b P(x)\,dx = 1$

1) $E(x) = \int_a^b x\cdot P(x)\,dx$

2) $P(c \le x \le d) = \int_c^d P(x)\,dx$

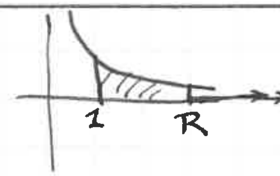
Improper Integrals

Type 1 $\quad \int_1^\infty \frac{1}{x^3}\,dx$

Type 2 $\quad \int_0^1 \frac{1}{\sqrt{x}}\,dx$

Ex 1

$$\int_1^\infty \frac{1}{x^3}\,dx = I$$



$$I = \lim_{R\to\infty}\int_1^R \frac{1}{x^3}\,dx$$

$$= \lim_{R\to\infty}\left[\frac{-1}{2x^2}\right]_1^R \qquad \text{Converges}$$

$$= \lim_{R\to\infty}\left[-\frac{1}{2R^2} + \frac{1}{2}\right] = \frac{1}{2}$$

P-test. $\quad P > 0$

$$\int_1^\infty \frac{1}{x^P}\,dx = \lim_{R\to\infty}\int_1^R x^{-P}\,dP$$

$$= \lim_{R\to\infty}\left[\frac{x^{-P+1}}{-P+1}\right]_1^R = \lim_{R\to\infty}\left[\frac{x^{1-P}}{1-P}\right]_1^R$$

$$= \lim_{R\to\infty}\left[\frac{R^{1-P}}{1-P} - \frac{1}{1-P}\right] = -\frac{1}{1-P} \qquad P > 1$$

$$= \infty \qquad P \le 1$$

$$\boxed{\int_1^\infty \frac{1}{x^P}\,dx = \begin{cases}\frac{1}{P-1} & P > 1\\ \infty & P \le 1\end{cases}}$$

$\int_1^\infty \frac{1}{x^{1.00001}}\,dx$ Converges

$\int_1^\infty \frac{1}{x^{0.99999}}\,dx$ diverges

Ex 2

$$\int_0^1 \frac{1}{x^{1/2}}\,dx = \lim_{\epsilon\to 0}\int_\epsilon^1 \frac{1}{x^{1/2}}\,dx$$

$$\lim_{\epsilon\to 0}\int_\epsilon^1 x^{-\frac{1}{2}}\,dx = \lim_{\epsilon\to 0}\left[2x^{\frac{1}{2}}\right]_\epsilon^1$$

$$= \lim_{\epsilon\to 0}\left[2 - 2\sqrt{\epsilon}\right] = 2.$$

$$\int_0^1 \frac{1}{x^{1/2}}\,dx = 2x^{\frac{1}{2}}\Big]_0^1 = 2$$

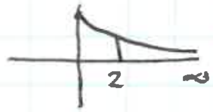
$\boxed{\text{SCORE} = 0}$

Pb For what values of P does the following int. converge

$$\int_0^1 \frac{1}{x^P}\,dx = \begin{cases} \frac{1}{1-P} & 0<P<1 \\ \infty & P \ge 1 \end{cases}$$

Piece of that's amore Pizza Pie Integral.

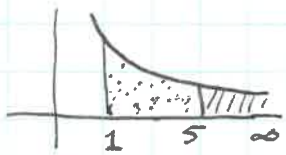
4 $\int_2^\infty \frac{1}{(x+3)^{3/2}}\,dx = I$



$I = \int_2^\infty \frac{1}{(x+3)^{3/2}}\,dx$

Let $z = x+3$

$dz = dx$

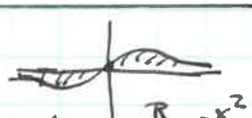


$I = \int_5^\infty \frac{1}{z^{3/2}}\,dz$

$= \int_1^\infty \frac{1}{z^{3/2}}\,dz - \int_1^5 \frac{1}{z^{3/2}}\,dz$

$= \frac{1}{\frac{3}{2}-1} - \int_1^5 \frac{1}{z^{3/2}}\,dz$ etc

11 $\int_{-\infty}^\infty x e^{-x^2}\,dx = 0$



$I = \lim_{R\to\infty} \int_{-R}^0 x e^{-x^2}\,dx + \lim_{R\to\infty}\int_0^R x e^{-x^2}\,dx$

$= \lim_{R\to\infty}\left[\frac{e^{-x^2}}{-2}\right]_{-R}^0 + \lim_{R\to\infty}\left[\frac{e^{-x^2}}{-2}\right]_0^R$

$= -\frac{1}{2} + [0 - (-\frac{1}{2})] = 0$

66) $\int \frac{x\ln x}{\sqrt{x^2-1}}\,dx = I$

Let $dv = \frac{x}{\sqrt{x^2-1}}\,dx \quad v = \sqrt{x^2-1}$

$u = \ln x \quad du = \frac{1}{x}\,dx$

$I = \ln x \cdot \sqrt{x^2-1} - \int \frac{\sqrt{x^2-1}}{x}\,dx$

Let $x = \sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$

$\sqrt{x^2-1} = \tan\theta$

$I = \ln x\sqrt{x^2-1} - \int \frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta$

$= \ln x\sqrt{x^2-1} - \int[-1+\sec^2\theta]\,d\theta$

$= \ln x\sqrt{x^2-1} + \theta - \tan\theta + c$

27) $\int_0^3 \frac{1}{\sqrt{x}}\,dx = \int_0^1 \frac{1}{x^{1/2}}\,dx + \int_1^3 \frac{1}{x^{1/2}}\,dx$

$= \frac{1}{1-\frac{1}{2}} + \int_1^3 \frac{1}{x^{1/2}}\,dx$

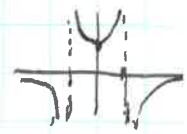
$= 2 - \frac{2}{3}x^{1/2}\Big]_1^3 = 2 - \frac{2}{3}[3^{1/2}-1]$

$= 2\sqrt{3}.$

Do # 7.9 # 3, 9, 11, 21, 23, 29, 30, 33, 39, 58

39) $\int_{-2}^2 \frac{1}{x^2-1}\,dx = 2\int_0^2 \frac{1}{x^2-1}\,dx$

$= -2\int_0^2 \frac{1}{1-x^2}\,dx$



$I = -2\left[\int_0^1 \frac{1}{1-x^2}\,dx + \int_1^2 \frac{1}{1-x^2}\,dx\right]$

$= -2\left[\lim_{a\to 1}\int_0^a \frac{1}{1-x^2}\,dx + \lim_{a\to 1^+}\int_a^2 \frac{dx}{1-x^2}\right]$

$= -2\left\{\lim_{a\to +1}\left[\tanh^{-1}x\right]_0^a + \lim_{a\to 1^+}\tanh^{-1}x\right]_a^2$

$= -2[\infty - -] = \text{Div}.$

$\int \underbrace{\ln(4+x^2)}_{u}\underbrace{dx}_{dv} = I$

Let $u = \ln(4+x^2) \quad dv = dx$

$du = \frac{2x}{4+x^2}\,dx \quad v = x$

$I = x\ln(4+x^2) - \int \frac{2x^2}{4+x^2}\,dx$

$= x\ln(4+x^2) - 2\int[1 - \frac{4}{4+x^2}]\,dx$

$I = x\ln(4+x^2) - 2[x - \frac{4}{2}\tan^{-1}\frac{x}{2}] + c$

$= x\ln(4+x^2) - 2x + 4\tan^{-1}\frac{x}{2} + c$

Differential Equations.

Ex1. $F = ma \Rightarrow m\frac{d^2x}{dt^2} = F$

Linear, 2nd Order

↳ Coeff do not depend on t!

Ex2 $\frac{dx}{dt} = e^t\cos(x^2+t)$

First order, non linear

Ex $3y'' + 2y' + y = 0$

2nd order, linear

Ex $4x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = x$

2nd order, linear

Ex $3y\frac{d^2y}{dx^2} + y^2\frac{dy}{dx} + x^3 = 0$

General linear 2nd d.e.

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = f(x)$$

Separable equations 1st order

$$\boxed{\begin{gathered}\frac{dy}{dx} = f(x)\cdot g(y) \\ \int \frac{1}{g(y)}\,dy = \int f(x)\,dx\end{gathered}}$$

Ex $\frac{dy}{dx} = \frac{y}{4+x^2}$

$\int \frac{1}{y}\,dy = \int \frac{1}{4+x^2}\,dx$

$\ln y = \frac{1}{2}\tan^{-1}\frac{x}{2} + c$

$y = e^{\frac{1}{2}\tan^{-1}\frac{x}{2} + c}$

Ex $\frac{dy}{dx} = 2x$

$y = x^2 + c$

Initial Conditions

Ex $\frac{dy}{dx} = 2x, \quad y(2) = 4$

$y = x^2 + c$

$4 = 4 + c \qquad c = 0$

$y = x^2$

Ex 10 $\frac{dy}{dx} = \frac{1+x}{xy} \qquad x > 0, \quad y(1) = -4$

$\int y\,dy = \int \frac{1+x}{x}\,dx = \int \left(\frac{1}{x} + 1\right)dx$

$\frac{1}{2}y^2 = \ln|x| + x + c$

$y^2 = 2\ln x + 2x + C$

$16 = 2\ln 1 + 2 + C \qquad C = 14$

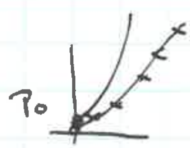
$y^2 = 2\ln x + 2x + 14$

Page 491 # 1, 5, 9, 11, 14, 15,

Logistic Model.

a) Malthus.

$\frac{dP}{dt} = kP$



$\int \frac{1}{P}\,dP = \int k\,dt$

$\ln P = kt + c$

$P = e^{kt+c} = e^c \cdot e^{kt}$

$P = P_0 e^{kt} \qquad (t=0, P=P_0)$

$\frac{dP}{dt} = kP(M-P) \qquad t=0, P=P_0$

k, M fixed

$\int \frac{1}{P(M-P)}\,dP = \int k\,dt$

$\frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$

$\frac{1}{P(M-P)} = \frac{A(M-P) + BP}{P(M-P)}$

Let $P = 0 \qquad A\cdot M = 1 \qquad A = 1/M$

$P = M \qquad BM = 1 \qquad B = 1/M$

$\int \frac{dP}{P(M-P)} = \int\left[\frac{1/M}{P} + \frac{1/M}{M-P}\right]dP$

$= \frac{1}{M}\ln P + \left(-\frac{1}{M}\right)\ln(M-P)$ ✓

$= \frac{1}{M}\left[\ln P - \ln(M-P)\right] = kt + C$

$\therefore \ln P - \ln(M-P) = (Mk)t + C$

$\ln\left(\frac{P}{M-P}\right) = (Mk)t + c$

$\frac{P}{M-P} = e^{(Mk)t+c} = e^{Mkt}\cdot e^c$

$\frac{P}{M-P} = c.e^{Mkt}$

$P = ce^{Mkt}(M-P)$

$= Mce^{Mkt} - P\cdot ce^{Mkt}$

$P + P\cdot ce^{Mkt} = Mce^{Mkt}$

$P[1 + ce^{Mkt}] = Mce^{Mkt}$

$P = \frac{Mce^{Mkt}}{1 + ce^{Mkt}}$

$\boxed{\text{Hw. 1}}$

$P = \frac{Mce^{Mkt}}{1 + ce^{Mkt}} \qquad t=0, P=P_0$

find C.

Rewrite sol.

8.1 # 1, 5, 9, 11, 14, 15
17, 31, 33, 35, 36

Linear First Order Eq'n

$a(x)y' + b(x)y = c(x)$

$$\boxed{y' + p(x)y = q(x)}$$

To solve:

Let $\mu = \int p(x)\,dx$

Multiply by e^{μ}

$\therefore$ Left hand side

$\frac{d}{dx}[e^{\mu}y] = e^{\mu}q(x)$

$e^{\mu}y = \int e^{\mu}q(x)\,dx$

$y = e^{-\mu}\cdot\int e^{\mu}q(x)\,dx$

Ex: $xy' + y = 7x^2$

$*\quad y' + \frac{1}{x}y = 7x$

$P(x) = \frac{1}{x}\qquad \mu = \int P(x)\,dx$

$= \int \frac{1}{x}\,dx$

$= \ln x\qquad$ Int. Fact.

$e^{\mu} = x$

$xy' + y = 7x^2$

$\frac{d}{dx}(xy) = 7x^2$

$xy = \frac{7x^3}{3} + c\qquad !$

$y = \frac{7}{3}x^2 + \frac{c}{x}$

$y' + P(x)y = q(x)$

F.T.C $\qquad \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

$e^{\int p\,dx}y' + p\,e^{\int p\,dx}y = e^{\int p\,dx}\cdot q(x)$

Claim

$\frac{d}{dx}\left[e^{\int p\,dx}\cdot y\right] = e^{\int p\,dx}\cdot q(x)$

$e^{\int p\,dx}y = \int e^{\int p\,dx}\cdot q(x)\,dx$

$y = e^{-\int p\,dx}\cdot\int e^{\int p\,dx}q(x)\,dx$

7) $y' - 2xy = x\qquad P(x) = -2x$

$\int p\,dx = -x^2$

$\mu = e^{\int p\,dx} = e^{-x^2}$

$e^{-x^2}y' - 2xe^{-x^2}y = xe^{-x^2}$

$\frac{d}{dx}[e^{-x^2}\cdot y] = xe^{-x^2}$

$e^{-x^2}y = \int xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2} + c$

$y = -\frac{1}{2} + ce^{+x^2}$

9) $y'\cos x = y\sin x + \sin 2x$

$-\frac{\pi}{2} < x < \frac{\pi}{2}$

$y'\cos x - y\sin x = \sin 2x$

$\frac{d}{dx}(\cos x\cdot y) = \sin 2x$

$(\cos x)y = -\frac{1}{2}\cos 2x + C$

$y = -\frac{1}{2}\frac{\cos 2x}{\cos x} + c\cdot\sec x$

(4) $y' - (\tan x)y = \frac{\sin 2x}{\cos x}$

$P = -\tan x\qquad \int p\,dx = -\ln[\sec x]$

$\mu = e^{-\ln(\sec x)} = \frac{1}{e^{\ln(\sec x)}} = \frac{1}{\sec x}$

$\mu = \cos x$

TOPS FORM 3314

Homogeneous Equations

General 1st-order O.D.E.

$$y' = f(x,y)$$

The eq is called homog. if $f(x,y) = F\left(\frac{y}{x}\right)$

Sol

* $y' = F\left(\frac{y}{x}\right)$. Let $u = \frac{y}{x}$

$\therefore y = x \cdot u$

$$y' = \boxed{x \cdot \frac{du}{dx} + u = F(u)}$$

$$x\frac{du}{dx} = F(u) - u$$

$$\int \frac{1}{F(u)-u}\,du = \int \frac{1}{x}\,dx$$

Ex 35

$$\frac{dy}{dx} = \frac{x^2+y^2}{xy} = \frac{1+\left(\frac{y}{x}\right)^2}{(y/x)}$$

Let $u = \frac{y}{x} \Rightarrow y = x \cdot u$

$$x\frac{du}{dx} + u = \frac{1+u^2}{u}$$

$$x\frac{du}{dx} = \frac{1+u^2}{u} - u = \frac{1}{u}$$

$$\int u\,du = \int \frac{1}{x}\,dx \qquad \text{Let } x>0$$

$$\frac{u^2}{2} = \ln|x| + c$$

$$u^2 = 2\ln x + c$$

$$\left(\frac{y}{x}\right)^2 = 2\ln x + c$$

$$y^2 = 2x^2\ln x + cx^2$$

$$y = \pm\sqrt{2x^2\ln x + cx^2}$$

Quiz:

Solve:

1. $\frac{dy}{dt} = \frac{ty+3t}{t^2+1} \qquad y(0)=0$

2. $y' - 2xy = 2x e^{x^2}$

$$\int \frac{1-2v-v^2}{(1+v)(1+v^2)}\,dv = \int \frac{1}{x}\,dx$$

$$\frac{1-2v-v^2}{(1+v)(1+v^2)} = \frac{A}{1+v} + \frac{Bv+c}{1+v^2}$$

$$A\ln(1+v) + \tfrac{1}{2}B\ln(1+v^2) + c\tan^{-1}v$$

$$A\ln(1+v) + \frac{B}{2}\ln(1+v^2) + c\tan^{-1}v = \ln x + c$$

$$A\ln\left(1+\frac{y}{x}\right) + \frac{B}{2}\ln\left(1+\frac{y^2}{x^2}\right) + c\tan^{-1}\frac{y}{x} = \ln x + c$$

Ex 34

$$y' = \frac{x+y}{x-y} = \frac{1+\frac{y}{x}}{1-y/x}$$

Do 15.1 # 29–37

Let $u = \frac{y}{x} \Rightarrow y = x \cdot u$

$$x \cdot \frac{du}{dx} + u = \frac{1+u}{1-u}$$

$$x\frac{du}{dx} = \frac{1+u}{1-u} - u = \frac{1+u-(u-u^2)}{1-u}$$

$$x\frac{du}{dx} = \frac{1+u^2}{1-u} \Rightarrow \int \frac{1-u}{1+u^2}\,du = \int \frac{1}{x}\,dx$$

$$\tan^{-1}u - \frac{1}{2}\ln(1+u^2) = \ln x + c$$

$$\tan^{-1}\frac{y}{x} - \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) = \ln x + c$$

37) $xy' = y + x e^{y/x}$

$$y' = \frac{y}{x} + e^{y/x}$$

Let $u = y/x \qquad y = xu$

$$y' = x\frac{du}{dx} + u = u + e^u$$

$$x\frac{du}{dx} = e^u$$

$$\int e^{-u}\,du = \int \frac{1}{x}\,dx$$

$$-e^{-u} = \ln x + c \qquad \text{if } x \ge 0$$

$$-e^{-y/x} = \ln x + c$$

$$1 = -(\ln x + c)e^{y/x}$$

31) $y' = \ln y - \ln x$

$y' = \ln\left(\frac{y}{x}\right)$ Homog!

9) $(x^2-2xy-y^2)y' = x^2+2xy-y^2$

$$y' = \frac{x^2+2xy-y^2}{x^2-2xy-y^2} = \frac{1+2(y/x)-(y/x)^2}{1-2(y/x)-(y/x)^2}$$

Let $v = \frac{y}{x} \qquad y = x \cdot v$

$$x \cdot \frac{dv}{dx} + v = \frac{1+2v-v^2}{1-2v-v^2}$$

$$x \cdot \frac{dv}{dx} = \frac{1+2v-v^2}{1-2v-v^2} - v$$

$$x\frac{dv}{dx} = \frac{1+v+v^2+v^3}{1-2v-v^2}$$

$$\frac{1-2v-v^2}{1+v+v^2+v^3}\,dv = \frac{1}{x}\,dx$$

10) $\frac{dy}{dx} = \frac{e^x - y}{x} = \frac{e^x}{x} - \frac{1}{x}y$

$\frac{dy}{dx} + \frac{1}{x}y = \frac{e^x}{x}$

$x\frac{dy}{dx} + y = e^x$

$(x \cdot y)' = e^x$

$xy = e^x + c$

$y = \frac{1}{x}(e^x + c)$

Fact!

$$\boxed{e^{ix} = \cos x + i \sin x}$$

$$e^{i\pi} = -1$$

$$e^{2i\pi} = 1$$

Superposition Principle.

Let $L = \left(a\frac{d^2}{dx^2} + b\frac{d}{dx} + c\right)$

so that

$$Ly = ay'' + by' + cy = 0$$

Prop

a) $Ly_1 = 0 \Rightarrow L(c_1 y_1) = 0$

b) $Ly_1 = 0$, $Ly_2 = 0 \Rightarrow L(y_1 + y_2) = 0$

Pf: a) $L(c_1 y_1) = c_1 L(y_1) = 0$

b) $L(y_1 + y_2) = L(y_1) + L(y_2) = 0$

Linear

Corollary

If y_1, y_2 are solutions to $L(y) = 0$, then $c_1 y_1 + c_2 y_2$ is also a solution!

Ex

$y'' + 36y = 0$ $\quad m^2 = -36$

Let $y = e^{mx}$ $\quad m = \pm 6i$

$m^2 + 36 = 0$ $\quad y = c_1 \cos 6x + c_2 \sin 6x$

$y'' + y' + y = 0$ $\quad$ Let $y = e^{mx}$

$m^2 + m + 1 = 0$

$$m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$$m = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$y = c_1 e^{-\frac{1}{2}x} \cos \frac{\sqrt{3}}{2}x + c_2 e^{-\frac{1}{2}x} \sin \frac{\sqrt{3}}{2}x$$

Linear 2nd order O.D.E.

$a(x)y'' + b(x)y' + c(x)y = f(x)$

Const. coeff, $f(x) = 0$

$$\boxed{ay'' + by' + cy = 0}$$

Sol

$y'' + 3y' + 2y = 0$ $\quad$ Let $y = e^{mx}$

$m^2 e^{mx} + 3 \cdot m e^{mx} + 2e^{mx} = 0$

$(m^2 + 3m + 2)e^{mx} = 0$

$(m^2 + 3m + 2) = 0$

$(m+2)(m+1) = 0$ $\quad m = -2, -1$

$y_1 = e^{-2x}$, $y_2 = e^{-x}$

Ans $y = c_1 e^{-2x} + c_2 e^{-x}$

Problem $ay'' + by' + cy = 0$

Let $y = e^{mx}$

$e^{mx}(am^2 + bm + c) = 0$

Solve $am^2 + bm + c = 0$

Case 1

$m = m_1, m_2$ real

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

Case 2

$m = m_1, m_1$

$$\boxed{y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}}$$

Case 3

$$\boxed{m = a \pm ib}$$

$y = e^{(a+ib)x} = e^{ax} \cdot e^{ibx}$

$= \cancel{e} = e^{ax}[\cos bx + i \sin bx]$

$$\boxed{y = c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx}$$

8.2 # 3, 5, 6, 7, 8, 11, 15, 32

Arc Length

$$\boxed{ds^2 = dx^2 + dy^2} \leftarrow$$

Case 1

$$y = f(x)$$
$$\frac{dy}{dx} = f'(x)$$
$$dy = f'(x)\,dx$$

$$ds^2 = dx^2 + [f'(x)]^2 dx^2$$
$$= [1 + f'(x)^2]\,dx^2$$
$$ds = \sqrt{1 + [f'(x)]^2}\,dx$$

$$s = \int \sqrt{1 + [f'(x)]^2}\,dx$$

6) $y = \dfrac{x^3}{6} + \dfrac{1}{2x} \qquad 1 \le x \le 2$

$$ds^2 = dx^2 + dy^2$$
$$dy = \left(\tfrac{1}{2}x^2 - \frac{1}{2x^2}\right)dx$$

$$ds^2 = dx^2 + \left(\tfrac{1}{2}x^2 - \frac{1}{2x^2}\right)^2 dx^2$$
$$= dx^2 + \left(\tfrac{1}{4}x^4 - \tfrac{1}{2} + \frac{1}{4x^4}\right)dx^2$$
$$= \left[1 + \left(\frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}\right)\right]dx^2$$
$$= \left[\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}\right]dx^2$$
$$= \left[\frac{x^2}{2} + \frac{1}{2x^2}\right]^2 dx^2$$

$$\int_1^2 ds = \int \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)dx$$
$$= \frac{x^3}{6} - \frac{1}{2x}\Big]_1^2 = \frac{17}{12}$$

11) $y = \ln(1 - x^2) \qquad 0 \le x \le \frac{1}{2}$

$$dy = \left(\frac{-2x}{1 - x^2}\right)dx$$
$$ds^2 = dx^2 + dy^2$$
$$= dx^2 + \frac{4x^2}{(1-x^2)^2}dx^2$$
$$= \left[1 + \frac{4x^2}{(1-x^2)^2}\right]dx^2$$
$$= \left[\frac{(1-x^2)^2 + 4x^2}{(1-x^2)^2}\right]dx^2$$

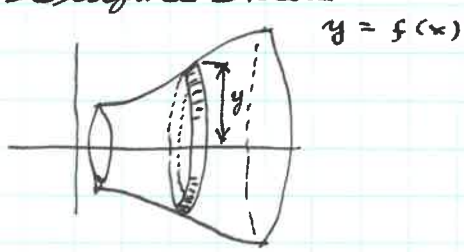
$$ds^2 = \frac{(1 - 2x^2 + x^4) + 4x^2}{(1-x^2)^2}dx^2$$
$$= \frac{1 + 2x^2 + x^4}{(1-x^2)^2}dx^2$$
$$= \frac{(1+x^2)^2}{(1-x^2)^2}dx^2$$

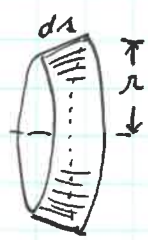
$$ds = \frac{1+x^2}{1-x^2}\,dx$$

$1-x^2 \,\overline{)\,1+x^2}$
$\quad -1$
$-x^2+1 \,\overline{)\,x^2+1}$
$\quad x^2-1$
$\quad 2$

$$ds = \left(-1 + \frac{2}{1-x^2}\right)dx$$
$$s = \int_0^{1/2}\left(-1 + \frac{2}{1-x^2}\right)dx$$
$$= -x + 2\tanh^{-1}x\Big]_0^{1/2}$$
$$= -x + 2\cdot\tfrac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\Big]_0^{1/2}$$
$$= \ln 3 - \tfrac{1}{2}$$

Surface Area





$$dS = 2\pi r\,ds$$
$$= 2\pi y\,ds$$
$$\boxed{S = 2\pi \int y\,ds} \qquad \boxed{1-5}$$

Parametric Equations

1) $y = f(x)$ *Explicit.*

2) $f(x,y) = 0$ *Implicit*

$f(x,y) = C$ $\quad x^2 + y^2 - 1 = 0$

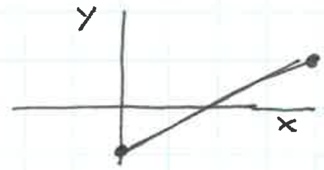
3) $\left.\begin{cases} x = f(t) \\ y = g(t) \end{cases}\right\}$ *Parametric*

$\uparrow$ *Parameter*

Ex

$\begin{cases} x = 3t + 1 \\ y = 2t - 1 \end{cases}$

t	x	Y
0	1	-1
1	4	1
2	7	3



Eliminate the Parameter.

$x = 3t + 1 \qquad \frac{x-1}{3} = t$

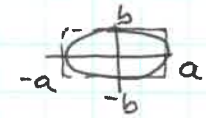
$y = 2t - 1 \qquad \frac{y+1}{2} = t$

$\frac{y+1}{2} = \frac{x-1}{3} \qquad \left\{ \frac{y+1}{x-1} = \frac{2}{3} \right.$

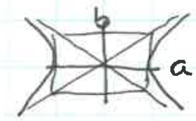
Line $m = \frac{2}{3}$ $P(1,-1)$

1) *Lines* $\begin{cases} x = at + x_0 \\ y = bt + y_0 \end{cases}$ $\quad m = b/a$ $\quad P(x_0, y_0)$

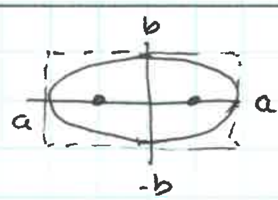
2) *Ellipse* $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$



3) *Hypos* $\Big\{$



Ellipses



① $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

② $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$ $\qquad \frac{x}{a} = \cos t$, $\quad \frac{y}{b} = \sin t$

$\begin{cases} \frac{x^2}{a^2} = \cos^2 t \\ \frac{y^2}{b^2} = \sin^2 t \end{cases} \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Hypo's

$\cosh t = \frac{e^t + e^{-t}}{2}$

$\sinh t = \frac{e^t - e^{-t}}{2}$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$x = a\cosh t$

$y = b\sinh t$

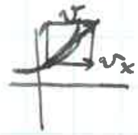
$\cosh^2 t - \sinh^2 t = 1$

$\left.\begin{array}{l} \frac{x^2}{a^2} = \cosh^2 t \\ \frac{y}{b} = \sinh^2 t \end{array}\right\} \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

\# 9.1 : 2, 7, 8, 9, 18, 21, 25, 26

Derivatives in Parametric Form

$$x = x(t) \qquad \frac{dx}{dt} = v_x$$
$$y = y(t) \qquad \frac{dy}{dt} = v_y$$



$$\frac{dy}{dx} = ?$$

$$\frac{dy}{dx} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}} \qquad v = \sqrt{v_x^2 + v_y^2}$$
$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$v = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$ds^2 = dx^2 + dy^2$$

Ex

$$\begin{cases} x = r\cos\omega t \\ y = r\sin\omega t \end{cases}$$

$$\begin{cases} \frac{dx}{dt} = -r\omega\sin\omega t \\ \frac{dy}{dt} = r\omega\cos\omega t \end{cases}$$

$$\frac{d^2x}{dt^2} = -r\omega^2\cos\omega t$$
$$\frac{d^2y}{dt^2} = -r\omega^2\sin\omega t$$

$$v = \sqrt{r^2\omega^2\sin^2\omega t + r^2\omega^2\cos^2\omega t}$$
$$= \sqrt{r^2\omega^2(\sin^2\omega t + \cos^2\omega t)}$$
$$v = r\omega$$
$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{r^2\omega^4(1)}$$

8. $y = \frac{x^2}{2} - \frac{\ln x}{4}$

$$dy = \left(x - \frac{1}{4x}\right)dx \qquad 2 \le x \le 4$$

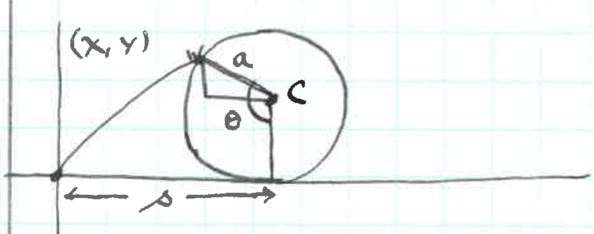
$$ds^2 = dx^2 + \left(x - \frac{1}{4x}\right)^2 dx^2$$
$$= \left[1 + \left(x - \frac{1}{4x}\right)^2\right] dx^2$$
$$= \left[1 + \left(x^2 - \frac{1}{2} + \frac{1}{16x^2}\right)\right] dx^2$$
$$= \left[x^2 + \frac{1}{2} + \frac{1}{16x^2}\right] dx^2$$
$$= \left[x + \frac{1}{4x}\right]^2 dx^2$$

$$ds = \left(x + \frac{1}{4x}\right)dx$$
$$s = \int_2^4 \left(x + \frac{1}{4x}\right)dx =$$
$$= (8-2) + \frac{1}{4}\ln(2) = 6 + \frac{\ln 2}{4}$$

$$v = r\omega \qquad \omega = \frac{v}{r}$$
$$a = r\omega^2$$
$$a = r\left(\frac{v^2}{r^2}\right) \Rightarrow \boxed{a = \frac{v^2}{r}}$$

Recall

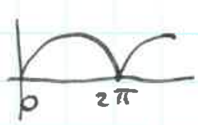
$$\cos\left(\theta - \frac{\pi}{2}\right) = \cos\theta\cos\frac{\pi}{2} + \sin\theta\sin\frac{\pi}{2}$$
$$\cos\left(\theta - \frac{\pi}{2}\right) = +\sin\theta$$
$$\sin\left(\theta - \frac{\pi}{2}\right) = \sin\theta\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\cos\theta$$
$$\sin\left(\theta - \frac{\pi}{2}\right) = -\cos\theta$$



$$C\,(a\theta, a)$$

$$\begin{cases} x = a\theta - a\cos\left[\theta - \frac{\pi}{2}\right] \\ y = a + a\sin\left[\theta - \frac{\pi}{2}\right] \end{cases}$$

$$\begin{cases} x = a\theta - a\sin\theta \\ y = a - a\cos\theta \end{cases}$$



$$x(t) = at - a\sin t$$
$$y(t) = a - a\cos t$$

$$\begin{cases} \frac{dx}{dt} = a - a\cos t = v_x \\ \frac{dy}{dt} = a\sin t = v_y \end{cases}$$

$$v = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \qquad \boxed{ds^2 = dx^2 + dy^2}$$

$$\frac{ds}{dt} = \sqrt{(a - a\cos t)^2 + (a\sin t)^2}$$
$$= \sqrt{a^2 - 2a^2\cos t + a^2\cos^2 t + a^2\sin^2 t}$$
$$= \sqrt{a^2 - 2a^2\cos t + a^2}$$
$$= \sqrt{2a^2 - 2a^2\cos t}$$
$$= \sqrt{2}\,a\sqrt{1 - \cos t}$$

$$\int ds = \sqrt{2}\,a\int\sqrt{1 - \cos t}\,dt$$
$$= \sqrt{2}\,a\int\sqrt{2}\sqrt{\frac{1 - \cos t}{2}}\,dt$$
$$= 2a\int_0^{2\pi}\sin\frac{t}{2}\,dt$$
$$= -2a\cdot 2\cos\frac{t}{2}\Big]_0^{2\pi} = -4a[-1 - 1]$$
$$= 8a$$

Graphing curves.

Parametric Eqs.:

1. Lines
2. Circles, Ellipses
3. Hyperbolas
4. Parabolas
5. Cycloid.

---

6. Hypocycloid
7. Epicycloid
8. Witch of Agnesi 
   1719 - 1799

Polar Coordinates

1 Circles $r = a$,
$r = a\cos\theta$, $r = a\sin\theta$

2. Cardioids $r = a(1 \pm \cos\theta)$ 
$r = a(1 \pm \sin\theta)$

---

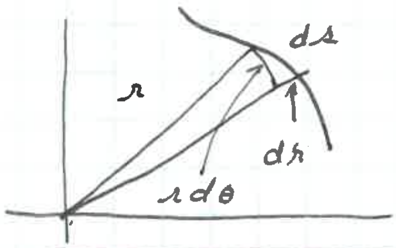
3. Limaçons
4. Roses $r = a\cos n\theta$
$r = a\sin n\theta$
5. Lemniscates.

Other.

Cissoid of Diocles

Folium of Descartes

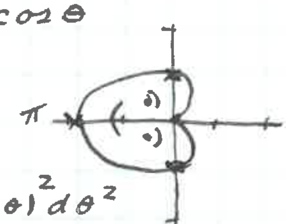
Areas and Arc Length in Polar Coordinates



$$\boxed{\begin{aligned} ds^2 &= dr^2 + r^2 d\theta^2 \\ dA &= \tfrac{1}{2} r^2 d\theta \end{aligned}}$$

9.5) 3, 5, 9, 15, 19 Area
43, 49, 48, 50 Length

Ex Find the length of the cardioid $r = 1 - \cos\theta$
$\theta \in [0, \pi]$



$dr = \sin\theta \, d\theta$

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\theta^2 \\ &= (\sin\theta \, d\theta)^2 + (1-\cos\theta)^2 d\theta^2 \\ &= [\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta] \, d\theta^2 \\ &= (2 - 2\cos\theta) \, d\theta^2 \\ &= 2(1-\cos\theta) \, d\theta^2 \end{aligned}$$

$$\int ds = \sqrt{2} \int \sqrt{1-\cos\theta} \, d\theta$$

$$s = \sqrt{2} \cdot 2 \int_0^{\pi} \sqrt{2} \sqrt{\frac{1-\cos\theta}{2}} \, d\theta.$$

$$= 4 \int_0^{\pi} \sin\frac{\theta}{2} \, d\theta = -8 \cos\frac{\theta}{2} \Big]_0^{\pi}$$

$$= -8[0 - 1] = 8$$

49) $r = \cos^4\left(\frac{\theta}{4}\right) \qquad \theta \in [0, 2\pi)$

$ds^2 = dr^2 + r^2 d\theta^2$

$dr = 4 \cdot \cos^3\left(\frac{\theta}{4}\right)\left(-\sin\frac{\theta}{4}\right)\left(\frac{1}{4}\right) \cdot d\theta$

$dr = -\cos^3\left(\frac{\theta}{4}\right)\sin\left(\frac{\theta}{4}\right) d\theta$

$ds^2 = \cos^6\left(\frac{\theta}{4}\right)\sin^2\left(\frac{\theta}{4}\right) d\theta^2 + \cos^8\left(\frac{\theta}{4}\right) d\theta^2$

$ds^2 = \cos^6\left(\frac{\theta}{4}\right)\left[\sin^2\left(\frac{\theta}{4}\right) + \cos^2\frac{\theta}{4}\right] d\theta^2$

$ds = \cos^3\frac{\theta}{4}\, d\theta$

$s = \int_0^{2\pi} \cos^3\frac{\theta}{4}\, d\theta$

$= \int_0^{2\pi} \cos^2\frac{\theta}{4} \cdot \cos\frac{\theta}{4}\, d\theta$

$= \int_0^{2\pi} \left(1 - \sin^2\frac{\theta}{4}\right) \cos\frac{\theta}{4}\, d\theta$

$= \cancel{4\left[\theta + \frac{1}{3}\cos^3\frac{\theta}{4}\right]_0^{2\pi}}$ Let $u = \sin\frac{\theta}{4}$

$du = \frac{1}{4}\cos\frac{\theta}{4}\, d\theta$

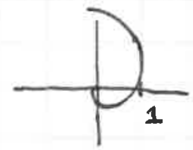
$s = 4\int_0^1 (1 - u^2)\, du$

$= 4\left[1 - \frac{1}{3}\right] = \frac{8}{3}$

Graph $\theta = \frac{\pi}{3}$ 

$r = e^{\theta}$ $\cancel{0 \le \theta \le \pi}$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

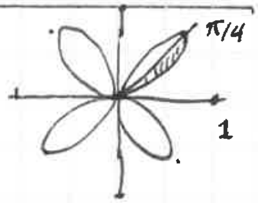


$dA = \frac{1}{2} e^{2\theta} d\theta$

$A = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\pi/2} e^{2\theta} d\theta = \frac{1}{2}\,\frac{1}{2}\, e^{2\theta}\Big]_{-\pi/2}^{\pi/2}$

$A = \frac{1}{4}\left[e^{\pi} - e^{-\pi}\right]$

5) $r = \sin 2\theta$.



$dA = \frac{1}{2} r^2 d\theta$

$A = \cancel{2}\int_0^{\pi/6} \frac{1}{2}\sin^2 2\theta\, d\theta$

$= \frac{1}{2}\int_0^{\pi/6} \left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right) d\theta$

$= \frac{1}{2}\left[\frac{\theta}{2} - \frac{1}{8}\sin 4\theta\right]_0^{\pi/6}$

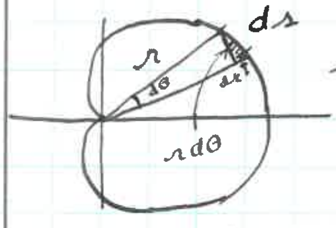
$= \frac{\pi}{24} - \frac{1}{16}\sin\frac{2\pi}{3}$

$= \frac{\pi}{24} - \frac{\sqrt{3}}{32}$

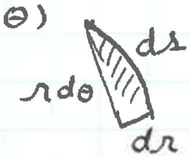
<u>Rev</u> 7-9:15.

<u>Do</u> 9.2 # 40, 1, 3, 9, 15, 17, 26, 27, 29, 32.

Pumpkin Pie.



$r = f(\theta)$



Arc Length

$$\boxed{ds^2 = dr^2 + r^2 d\theta^2}$$

Area

$$\boxed{dA = \frac{1}{2} r^2 d\theta}$$

$ds^2 = dx^2 + dy^2$

$x = r\cdot\cos\theta$

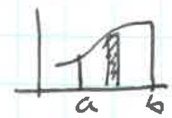
$y = r\sin\theta$

$dx = -r\sin\theta\, d\theta + \cos\theta\, dr$

$dy = r\cos\theta\, d\theta + \sin\theta\, dr$

$ds^2 = [-r\sin\theta\, d\theta + \cos\theta\, dr]^2 + [r\cos\theta\, d\theta + \sin\theta\, d$

$=$



$A = \int_a^b f(x)\,dx$

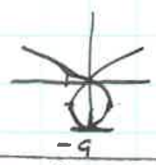
15 $x = t(t^2 - 3)$ $\quad y = 3(t^2 - 3)$

$dy = 6t\,dt$

$dx = (3t^2 - 3)\,dt$

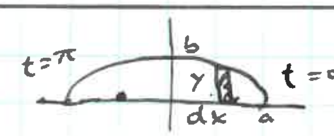
$\frac{dy}{dx} = \frac{6t}{3t^2 - 3}$ $\qquad \frac{dy}{dx} = 0$ if $t = 0$

$\frac{dy}{dx} = \text{"}\infty\text{"}$ $\quad t = \pm 1$



29) $x = a\cos\theta$

$y = b\sin\theta$



$dA = y \cdot dx$

$dA = (b\sin\theta)(-a\sin\theta)\,d\theta$

$-A = -4ab\int_0^{\pi/2} \sin^2\theta\, d\theta$

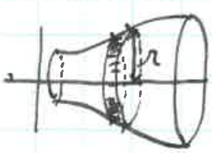
$= -4ab\int_0^{\pi/2} [\frac{1}{2} - \frac{1}{2}\cos 2\theta]\,d\theta$

$= -4ab\,[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta]_0^{\pi/2}$

$= -4ab\,[\frac{\pi}{4}] =$

$A = \pi ab$

Surface Area

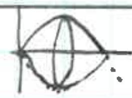


$dS = 2\pi r\,ds$

$\boxed{S = 2\pi\int r\,ds}$ ✓

P. 504

5) $y = \sin x \quad 0 \le x \le \pi$



$ds^2 = dx^2 + dy^2$

$dy = \cos x\,dx$

$ds^2 = dx^2 + \cos^2 x\,dx^2$

$= (1 + \cos^2 x)\,dx^2$

$ds = \sqrt{1 + \cos^2 x}\,dx$

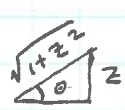
$dS = d\Sigma = 2\pi\sin x\sqrt{1 + \cos^2 x}\,dx$

$S = 2\pi\int_0^{\pi} \sin x\sqrt{1 + \cos^2 x}\,dx$

Let $z = \cos x \quad dz = -\sin x\,dx$

$S = -2\pi\int_1^{-1}\sqrt{1 + z^2}\,dz$

$= 2\pi\int_{-1}^{1}\sqrt{1 + z^2}\,dz$



Let $z = \tan\theta$

$dz = \sec^2\theta\,d\theta$

$S = 2\pi\int_{-\pi/4}^{\pi/4}\sec^3\theta\,d\theta$

$I = \int\sec^3\theta\,d\theta = \int\underbrace{\sec\theta}_{u}\,\underbrace{\sec^2\theta\,d\theta}_{dv}$

$I = uv - \int v\,du$

$= \sec\theta\tan\theta - \int\tan^2\theta\sec\theta\,d\theta$

$= \sec\theta\tan\theta - \int(\sec^2\theta - 1)\sec\theta\,d\theta$

$I = \sec\theta\tan\theta - \int\sec^3\theta\,d\theta + \int\sec\theta\,d\theta$

$2I = \sec\theta\tan\theta + \int\sec\theta\,d\theta$

$I = \frac{1}{2}\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|$

$S = 2\pi\{\frac{1}{2}[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|]_{-\pi/4}^{\pi/4}$

Surface Area

23) $x = a\cos^3 t \qquad 0 \le \theta \le \frac{\pi}{2}$

$y = a\sin^3 t$

$\boxed{dS = 2\pi r\, ds}$

$\begin{cases} dx = -3a\cos^2 t \sin t\, dt \\ dy = 3a\sin^2 t \cos t\, dt \end{cases}$

$\begin{cases} dx^2 = 9a^2\cos^4 t \sin^2 t\, dt^2 \\ dy^2 = 9a^2 \sin^4 t \cos^2 t\, dt^2 \end{cases}$

$ds^2 = 9a^2 \sin^2 t \cos^2 t \; dt^2$

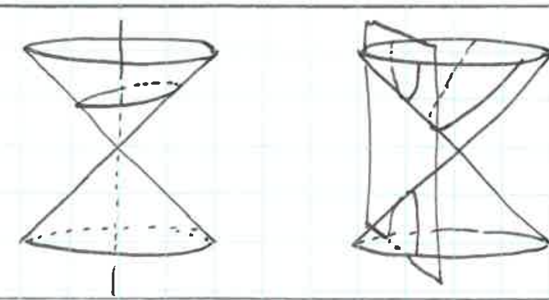
$ds = 3a \sin t \cos t\, dt$

$dS = 2\pi a \sin^3 t\, (3a \sin t \cos t\, dt)$

$dS = 6\pi a^2 \sin^4 t \cos t\, dt$

$S = 6\pi a^2 \int_0^{\pi/2} \sin^4 t \cos t\, dt$

$= 6\pi a^2 \frac{1}{5} \sin^5 t \Big]_0^{\pi/2} = \frac{6}{5}\pi a^2$

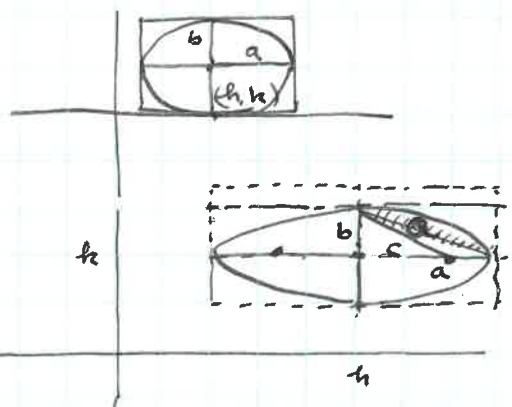


Conic Sections

Ellipses

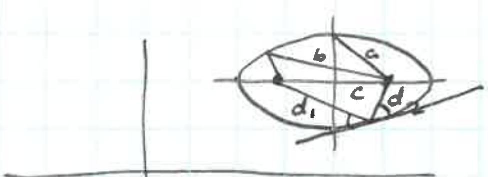
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$\begin{cases} x = a\cos\theta + h \\ y = b\sin\theta + k \end{cases}$

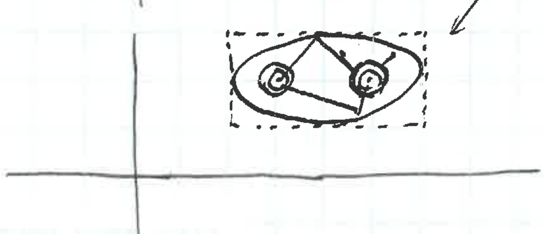


$c^2 = a^2 - b^2$

$f_\pm \;(h \pm c, k)$



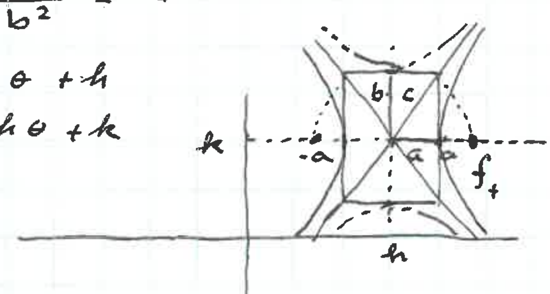
$d_1 + d_2 = \text{const}$



Hyperbolas

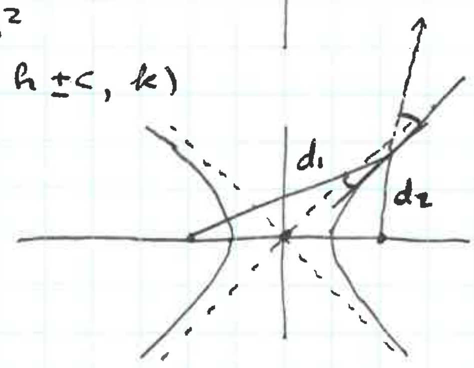
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = \pm 1$$

$\begin{cases} x = a\cosh\theta + h \\ y = b\sinh\theta + k \end{cases}$



$c^2 = a^2 + b^2$

$f_\pm = (h \pm c, k)$



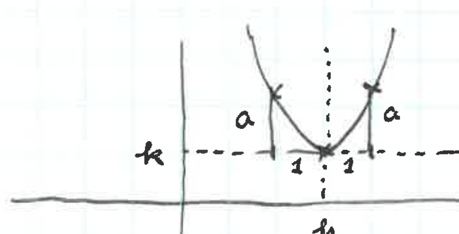
$d_1 - d_2 = \text{const.}$

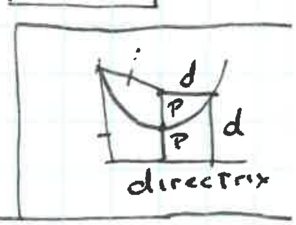
Parabolas

$y = ax^2 + bx + c$

① $4P(y-k) = (x-h)^2$

② $(y-k) = a(x-h)^2 \qquad \boxed{a = \frac{1}{4P}}$

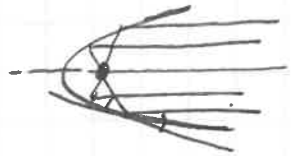




$y = ax^2$

x	Y
1	a

Geometry



Example 9.6

#19)

$2y^2 - 3x^2 - 4y + 12x + 8 = 0$

$2y^2 - 4y \quad -3x^2 + 12x = -8$

$2(y^2 - 2y + 1) - 3(x^2 - 4x + 4) = -8 + 2 - 12$

$2(y-1)^2 - 3(x-2)^2 = -18$

$3(x-2)^2 - 2(y-1)^2 = 18$

$\cancel{x^3}$

$\frac{(x-2)^2}{6} - \frac{(y-1)^2}{9} = 1$

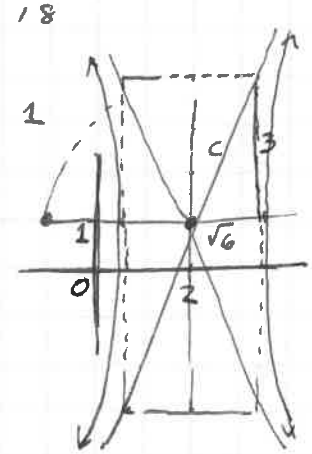
$(h, k) = (2, 1)$

$a = \sqrt{6}$

$b = 3$

$c = \sqrt{6+9} = \sqrt{15}$

$f_{\pm} = (2 \pm \sqrt{15}, \cancel{\sqrt{8}}\ 1)$



Review for test 2.

1 $(x^2+y^2)\,dx + xy\,dy = 0$

$xy\,dy = -(x^2+y^2)\,dx$

$\dfrac{dy}{dx} = -\dfrac{(x^2+y^2)}{(xy)}\,\dfrac{(1/x^2)}{(1/x^2)}$

$\dfrac{dy}{dx} = \dfrac{-\left(1+\frac{y^2}{x^2}\right)}{(y/x)}$

Let $u = \dfrac{y}{x}$ $\quad y = xu$

$x\cdot\dfrac{du}{dx} + u = -\dfrac{(1+u^2)}{u}$

$x\dfrac{du}{dx} = -\dfrac{(1+u^2)}{u} - u$

$x\dfrac{du}{dx} = \dfrac{-1-u^2-u^2}{u}$

$x\dfrac{du}{dx} = \dfrac{-1-2u^2}{u}$

$\displaystyle\int \frac{u}{1+2u^2}\,du = -\int \frac{1}{x}\,dx$

$\frac{1}{4}\ln|1+2u^2| = -\ln|x| + c$

$\frac{1}{4}\ln\left|1+2\frac{y^2}{x^2}\right| = -\ln x + c$

2) $2\dfrac{dy}{dx} - y = e^{x/2}$ $\qquad \mu = e^{\int P\,dx}$

(*) $\dfrac{dy}{dx} - \dfrac{1}{2}y = \dfrac{1}{2}e^{x/2}$ $\qquad \boxed{\text{Linear}}$

$P = -\frac{1}{2}$ $\qquad \int P\,dx = -\frac{x}{2}$

$\mu = e^{-x/2}$

$e^{-\frac{x}{2}}\dfrac{dy}{dx} - \frac{1}{2}e^{-\frac{x}{2}}y = \frac{1}{2}$

$\dfrac{d}{dx}\left(e^{-\frac{x}{2}}y\right) = \frac{1}{2}$

$e^{-\frac{x}{2}}y = \frac{x}{2} + c$

$\boxed{y = \left(\frac{x}{2}+c\right)e^{\frac{x}{2}}}$

3) $y'' + y' + 2y = 0$ $\qquad y = e^{mx}$

$m^2 + m + 2 = 0$

$m = \dfrac{-1 \pm \sqrt{1-8}}{2} = -\frac{1}{2} \pm \frac{i\sqrt{7}}{2}$

$y = c_1 e^{-\frac{x}{2}}\cos\frac{\sqrt{7}}{2}x + c_2 e^{-\frac{x}{2}}\sin\frac{\sqrt{7}}{2}x$

$ay'' + by' + cy = 0$

Let $y = e^{mx}$

$m = \alpha \pm i\beta$

$y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x.$

$\quad = e^{\alpha x}[c_1\cos\beta x + c_2\sin\beta x]$

4) $y'' - 4y' + 4y = 0$

$m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$ $\qquad m = 2, 2$

$y = c_1 e^{2x} + c_2 x e^{2x}.$

1 $\begin{cases} x = 2\cos^2 t + 1 & 0 \le t \le \frac{\pi}{2} \\ y = 2\sin^2 t - 2 \end{cases}$

$ds^2 = dx^2 + dy^2$

$ds^2 = (-4\cos t\sin t\,dt)^2 + (4\sin t\cos t\,dt)^2$

$\quad = (16\cos^2 t\sin^2 t + 16\sin^2 t\cos^2 t)\,dt^2$

$ds^2 = (32\cos^2 t\sin^2 t)\,dt^2$

$ds = 4\sqrt{2}\cos t\sin t\,dt$

$s = 4\sqrt{2}\displaystyle\int_0^{\pi/2}\cos t\sin t\,dt$

$\quad = -4\sqrt{2}\left[\frac{1}{2}\cos^2 t\right]_0^{\pi/2}$

$\quad = -4\sqrt{2}\left[-\frac{1}{2}\right] = 2\sqrt{2}$

Area bd by

$\begin{cases} r = \sec\theta \\ r = \csc\theta \end{cases}$ $\qquad r = \dfrac{1}{\cos\theta}$ $\qquad r\cos\theta = 1$

$\qquad r\sin\theta = 1$

$x = 1$

$y = 1$

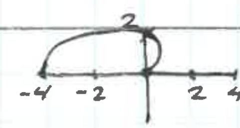
$A = 1$

$x - 1 = 2\cos^2 t$

$y + 2 = 2\sin^2 t$

$(x-1) + (y+2) = 2$

2) $r = 2 - 2\cos\theta$



$r = 2(1-\cos\theta)$

$dr = 2\sin\theta\,d\theta$

$ds^2 = dr^2 + r^2 d\theta^2$

$\quad = 4\sin^2\theta\,d\theta^2 + 2^2(1-\cos\theta)^2 d\theta^2$

$\quad = [4\sin^2\theta + 4(1 - 2\cos\theta + \cos^2\theta)]\,d\theta^2$

$\quad = (4 + 4 - 8\cos\theta)\,d\theta$

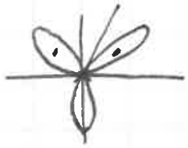
$\quad = 8(1-\cos\theta)\,d\theta^2$

$\quad = 16\left[\dfrac{1-\cos\theta}{2}\right]d\theta^2$

$ds = 4\sqrt{\dfrac{1-\cos\theta}{2}}\,d\theta = 4\sin\frac{\theta}{2}\,d\theta$

$s = 2\displaystyle\int_0^{\pi} 4\sin\frac{\theta}{2}\,d\theta = -16\left[\cos\frac{\theta}{2}\right]_0^{\pi} = +16$

3) $r = 2\sin 3\theta$.

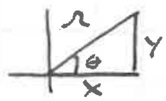


θ	r
0	0
$\frac{\pi}{3}$	0

$$A = \int_0^{\frac{\pi}{3}} \frac{1}{2} r^2 d\theta$$
$$= \frac{1}{2}\int_0^{\pi/3} 4\sin^2 3\theta \, d\theta$$
$$= 2\int_0^{\pi/3}\left(\frac{1}{2} - \frac{1}{2}\cos 6\theta\right) d\theta$$
$$= 2\left[\frac{\theta}{2} - \frac{1}{12}\sin 6\theta\right]_0^{\pi/3}$$
$$= \frac{\pi}{3}$$

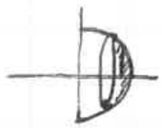
6) $r = \sin\theta \Rightarrow r^2 = r\sin\theta$

$x^2 + y^2 = y$



5) $x = \cos\theta \quad 0 \le \theta \le \frac{\pi}{4}$

$y = \sin\theta$



$dS = 2\pi r \, ds$

$dS = 2\pi(\sin\theta)[\sin^2\theta + \cos^2\theta] d\theta^2$

$dS = 2\pi \sin\theta \, d\theta$

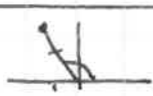
$-S = 2\pi \int_0^{\pi/4} \sin\theta \, d\theta$

$= 2\pi[-\cos\theta]_0^{\pi/4}$

$= 2\pi\left[-\frac{\sqrt{2}}{2} + 1\right]$

$S = 2\pi\left[\frac{\sqrt{2}}{2} - 1\right] = (\pi\sqrt{2} - 2\pi)$

7)



$r = \sqrt{2} \qquad x = \sqrt{2}\cos\frac{2\pi}{3}$

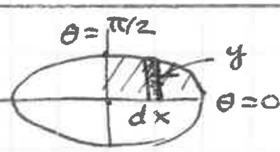
$\theta = 2\pi/3 \qquad y = \sqrt{2}\sin\frac{2\pi}{3}$

$x = \sqrt{2}\cdot\left(-\frac{1}{2}\right) = -\frac{\sqrt{2}}{2}$

$y = \sqrt{2}\cdot\frac{\sqrt{3}}{2} = \sqrt{6}/2$

29) $x = a\cos\theta$

$y = b\sin\theta$

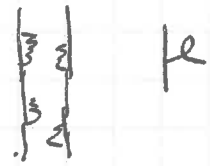


$dA = y \cdot dx$

$\frac{A}{4} = -\int_{\pi/2}^{0} b\sin\theta \, a\sin\theta \, d\theta$

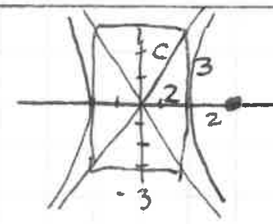
$\frac{A}{4} = -ab\int_{\pi/2}^{0} \sin^2\theta \, d\theta$

$A = \pi ab$



4) $x = 2\cosh t$

$y = 3\sinh t$



$\left(\frac{x}{2}\right)^2 - \left(\frac{y}{3}\right)^2 = 1$

$c = \sqrt{13}$

$f_\pm\ (\pm\sqrt{13}, 0)$

6) $4x^2 - y^2 + 8x + 2y - 1 = 0$

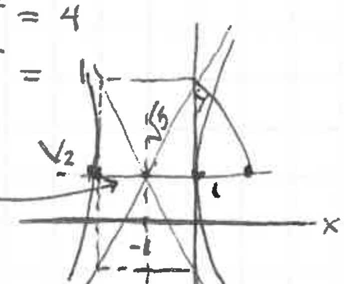
$4(x^2 + 2x \quad) - (y^2 - 2y \quad) = 1$

$4(x^2 + 2x + 1) - (y^2 - 2y + 1) = 1 + 4 - 1$

$4(x^2 + 2x + 1) - (y^2 - 2y + 1) = 4$

$4(x+1)^2 - (y-1)^2 = 4$

$\frac{(x+1)^2}{1} - \frac{(y-1)^2}{4} = 1$



$(h, k) = (-1, 1)$

$a = 1$

$b = 2$

$c = \sqrt{1+4} = \sqrt{5}$

$f_\pm\ (-1 \pm \sqrt{5}, 1)$

$V_1\ (0, 1) \quad V_2\ (-2, 1)$

9.6 #5

$(x+1) = 2(y-3)^2$

$\boxed{4P(x-h) = (y-k)^2}$

$\frac{1}{2}(x+1) = (y-3)^2$

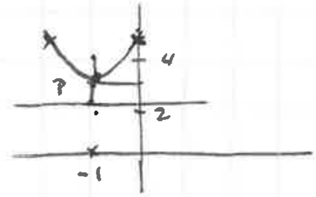
$4P = \frac{1}{2}$

$P = \frac{1}{8}$

$C(-1, 3)$

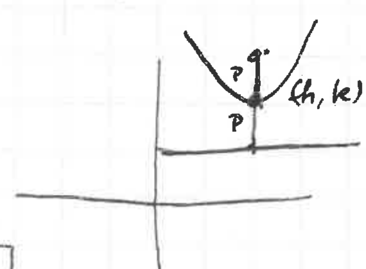
$a = 2$

$P = \frac{1}{8}$



$y = 3 - \frac{1}{8} = 23/8$

$f = (-1, 3 + \frac{1}{8})$



$f(h, k+P)$

$y = k - P$

$\cup \quad \cap \quad \subset \quad \supset$

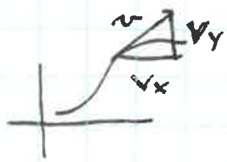
13 $x = e^{-t}$ $\quad v_y = \frac{dy}{dt}, \quad V_x = \frac{dx}{dt}$

$y = te^{2t}$

Find $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $\quad \boxed{v = \sqrt{V_x^2 + V_y^2} \quad ds/dt}$

$$\begin{cases} \frac{dy}{dt} = 2te^{2t} + e^{2t} \\ \frac{dx}{dt} = -e^{-t} \end{cases} \qquad \begin{cases} dy = (2te^{2t} + e^{2t})dt \\ dx = -e^{-t}dt \end{cases}$$

$$\frac{dy}{dx} = \frac{2te^{2t} + e^{2t}}{-e^{-t}}$$



Cardioids

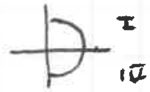
$r = a(1+\cos\theta)$

$r = a(1-\cos\theta)$

$r = a(1+\sin\theta)$

$r = a(1-\sin\theta)$

$\frac{dy}{d\theta} - (\tan\theta)y = 1 \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$



$P = -\tan\theta$

$\int P\, d\theta = -\ln|\sec\theta|$

$\mu = e^{\int P} = e^{-\ln(\sec\theta)} = \frac{1}{\sec\theta} = \cos\theta$

$\cos\theta \frac{dy}{d\theta} - (\sin\theta)y = \cos\theta$

$\frac{d}{d\theta}(\cos\theta \cdot y) = \cos\theta$

$(\cos\theta)\, y = \sin\theta + c$

$y = \tan\theta + c\sec\theta$

$y = ax^2 \quad \cup \qquad x = ay^2 \quad \subset$

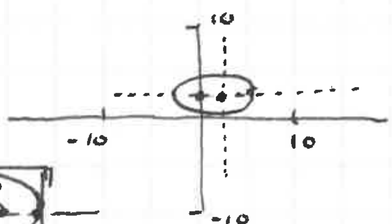
$y = -ax^2 \quad \wedge \qquad x = -ay^2 \quad \supset$

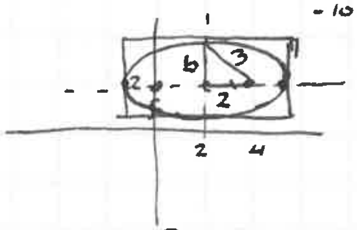
---

31) $(h,k) = (2,2)$

$f\,(0,2)$

$V\,(5,2)$

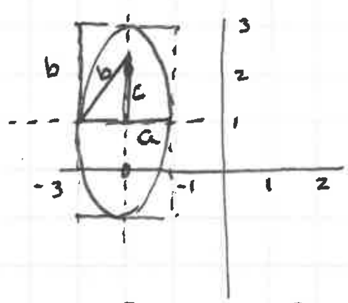




$a = 3$

$b = \sqrt{5}$

$$\frac{(x-2)^2}{9} + \frac{(y-2)^2}{5} = 1$$



$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

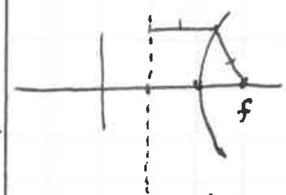
$a = 1,\ b = 2 \quad c = \sqrt{4-1} = \cancel{2\sqrt{2}}$

$c = \sqrt{3} \qquad f\pm(-2,\, 1 \pm \sqrt{3})$

$e^{-\ln x} = \frac{1}{e^{\ln x}} = \frac{1}{x}$

---

23) $f(3,0)$ Directrix $x = 1$

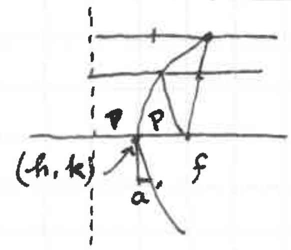


$V\,(2,0) \quad P = 1$

$(h,k) = (2,0) \quad P = 1$

$4p(x-h) = (y-k)^2$

$4(x-2) = y^2$



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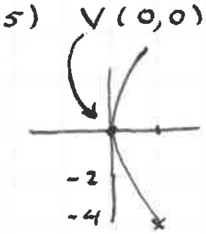
25) $V\,(0,0) \qquad P\,(1,-4)$



$(x-h)^{\cancel{2}} = a(y-k)^2$

$x = ay^2$

$1 = a(-4)^2 = 16a$

$a = \frac{1}{16} \qquad x = \frac{1}{16}y^2$

$\boxed{16x = y^2}$

Sequences – Series.

Examples – Sequences

1) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \}$

2) $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots, \frac{1}{2^n}\}$

$\{a_n\} = \{\frac{1}{2^n}\} \quad n = 1, 2, 3, 4, \dots$

3) $\{\frac{1}{2}, \frac{1}{8}, \frac{1}{4}, \frac{1}{32}, \frac{1}{16}, \dots \}$

Examples Series

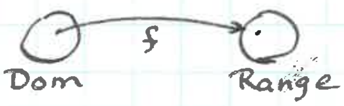
1. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

2. $s = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n} + \dots$

$= \sum_{n=1}^{\infty} \frac{1}{2^n}$

3. $\frac{1}{2} + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \dots$

Def. A sequence is a real valued function whose domain is the set of positive integers.



Ex 1

Domain		Range	
1	$\longrightarrow$	1	$= a_1$
2	$\longrightarrow$	1/2	$= a_2$
3	$\longrightarrow$	1/3	$= a_3$
4	$\longrightarrow$	1/4	$= a_4$
5	$\longrightarrow$	1/5	$= a_5$
⋮			

Notation 1 $\quad f(n) = \frac{1}{n}$

Seq $= \{a_n\}$ where $\boxed{a_n = \frac{1}{n}}$

$\lim_{n\to\infty} \frac{1}{n} = 0$

Def

$\lim_{n\to\infty} a_n = L$ if for any number $\epsilon > 0$, there exists an integer $N > 0$ such that $|a_n - L| < \epsilon$ whenever $n > N$.

$a_n \to L \qquad \boxed{a_n = f(n)}$

Limit Theorem

If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ then

a) $\lim_{n\to\infty} (k a_n) = k \cdot L$

b) $\lim_{n\to\infty} (a_n + b_n) = L + M$

c) $\lim_{n\to\infty} (a_n b_n) = L \cdot M$

d) $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{L}{M}$ if $M \neq 0$!

e) $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$

if f is a cont. function.

Do 1 – 11 in #0 § 1

Growth of functions / seq

Given $f(x) \longrightarrow a_n$

Def $a_n = f(n) \quad n = 1, 2, \ldots$

Ex

$$f(x) = \frac{1}{x^2}$$

Def $a_n = \dfrac{1}{n^2} = f(n)$

$$\ln x < x^p \propto e^x \propto x^x$$

$$\ln n \propto n^p \propto e^n \propto n! \propto n^n$$

$$n! = n \cdot (n-1)(n-2) \cdots 2 \cdot 1$$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

Ex

$$a_n = \frac{n^2}{e^n}$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^2}{e^n}$$

$$= \lim_{n \to \infty} \frac{2n}{e^n} = \lim_{n \to \infty} \frac{2}{e^n} = 0$$

Converges

10.1 # 13, 15, 23, 27, 31, 36, 33

L'Hôpital's Theorem

If $\lim_{x \to \infty} f(x) = \infty \quad (0)$

$\lim_{x \to \infty} g(x) = \infty \quad (0)$

$$\Rightarrow \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

23. $a_n = \frac{\pi^n}{3^n} = \left(\frac{\pi}{3}\right)^n$

$\frac{\pi}{3} > 1 \quad \lim_{n\to\infty} \left(\frac{\pi}{3}\right)^n = \infty$

Fact $\lim_{n\to\infty} a^n \begin{cases} \pm\infty & |a|>1 \\ 1 & |a|=1 \\ 0 & |a|<1 \end{cases}$

33 $a_n = n^{-1/n}$

$\ln a_n = -\frac{1}{n} \ln n$

$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} -\frac{\ln n}{n}$

$= \lim_{n\to\infty} \frac{-\frac{1}{n}}{1} = 0$

$\ln a_n \to 0$

$a_n \to 1$

Infinite Series

$s = a_0 + a_1 + a_2 + \cdots + a_n + \cdots$

Consider the sequence

$\{s_0, s_1, s_2, \cdots s_n, \cdots\}$

$s_0 = a_0$

$s_1 = a_0 + a_1$

$s_2 = a_0 + a_1 + a_2$

$\vdots$

$s_n = a_0 + a_1 + \cdots + a_n.$

$s = \lim_{n\to\infty} s_n \begin{cases} \text{Exists} & s \text{ Conv} \\ \sim\text{Exist} & s \text{ Div} \end{cases}$

Notation

$s_n = \sum_{k=0}^{n} a_k$

$s = \lim_{n\to\infty} \sum_{k=0}^{n} a_k$

$s = \sum_{k=0}^{\infty} a_k$

27) $a_n = \frac{\ln n^2}{n} = \frac{2 \ln n}{n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2 \cdot \frac{1}{n}}{1} = 0$

36) $a_n = \frac{n \cos n}{n^2+1}$

$\left|\frac{n \cos n}{n^2+1}\right| < \frac{n}{n^2+1}$

$\lim_{n\to\infty} \frac{n}{n^2+1} = 0$

$\therefore \lim_{n\to\infty} \left|\frac{n \cos n}{n^2+1}\right| = 0$

$a_n = \{1, \frac{\pi}{2}, \sqrt{3}, 1.72, -7, \ldots\}$

Ex $s = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

$s_0 = 1$

$s_1 = 1 + \frac{1}{2} = \frac{3}{2}$

$s_2 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$

$s_n = \{1, \frac{3}{2}, \frac{11}{6}, \quad\}$

Ex

$s = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$

$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots\cdots + \frac{1}{2^n}.$

$\frac{1}{2} s_n = \quad \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}$

$(1 - \frac{1}{2}) s_n = 1 + \frac{1}{2^{n+1}}$

$s_n = \frac{1 + \frac{1}{2^{n+1}}}{1 - \frac{1}{2}}$

$\lim_{n\to\infty} s_n = \frac{1}{1-\frac{1}{2}} = \frac{1}{1/2} = 2$

TOPS FORM 3314

Geometric Series

$$S = a + ar + ar^2 + \cdots + ar^n + \cdots$$

$$S_n = a + ar + \cdots + ar^n$$

$$rS_n = ar + \cdots + ar^n + ar^{n+1}$$

$$(1-r)S_n = a - ar^{n+1}$$

$$S_n = \frac{a - ar^{n+1}}{1-r}$$

$$\lim_{n\to\infty} S_n = \begin{cases} \frac{a}{1-r} & |r|<1 \\ \pm\infty & |r|\ge 1 \end{cases}$$

Conclusion !

$$\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{if } |r|<1}$$

Ex $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} \cdots$

$$= \frac{1}{1-\frac{1}{2}} = 2$$

Ex $S = 1 + \frac{1}{3} + \frac{1}{9} + \quad = \sum_{n=0} \frac{1}{3^n}$

$$= \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

10.2 # 1, 4, 5, 7, 9, 11, 13, 14, 15, 16, 17, 21, 27, 29, 43, 45, 31, 35, 49

Telescoping Series

$$\sum_{n=4}^{\infty} \frac{3}{n(n-1)}$$

$$\frac{3}{n(n-1)} = \frac{A}{n} + \frac{B}{n-1}$$

$3 = A(n-1) + B\cdot n$

$3 = B$

$3 = -A$

$$\sum_{n=4}^{\infty} \frac{3}{n(n-1)} = \sum_{n=4}^{\infty} \left[\frac{-3}{n} + \frac{3}{n-1}\right]$$

$$= 3\left[\left(-\frac{1}{4} + \frac{1}{3}\right) + \left(-\frac{1}{5} + \frac{1}{4}\right) + \left(-\frac{1}{6} + \frac{1}{5}\right)\right]$$

$$= 3\left(\frac{1}{3}\right)$$

$$= 1$$

$\sum_{n=1}^{\infty} b_n$ where $b_n = (a_n - a_{n+1})$

ie $\sum_{n=1}^{\infty} (a_n - a_{n+1}) = S$

$$S = (a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4)$$

If $\begin{cases} a_n \to 0 \\ S = a_1 \end{cases}$

$S_1 = a_1 - a_2$

$S_2 = a_1 - a_3$

$S_3 = a_1 - a_4$

$S_n = a_1 - a_n$

$\lim_{n\to\infty} S_n = a_1$

if $\boxed{\lim_{n\to\infty} a_n = 0}$!

$S = (1-2) + (2-3) + (3-4) \cdots$

Diverges

Pb

$0.32323232 = 0.\overline{32} = x$

$$x = \frac{32}{100} + \frac{32}{(100)^2} + \frac{32}{(100)^3} + \cdots$$

$$= \frac{\frac{32}{100}}{1 - \frac{1}{100}} = \frac{32}{100-1} = \frac{32}{99}$$

$x = 0.32323232$

$100x = 32.32323232$

$99x = 32$

$x = \frac{32}{99}$

$0.99999999\ldots = 1$

$$1 = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{9}{10^k}$$

$x = 2.166666$

$10x = 21.6666.$

$100x = 216.6666$

$100x - 10x = 216 - 21$

$x = \dfrac{195}{90}$

TOPS FORM

Memory Refreshing -

Seq $\{a_n\} = \{a_0, a_1, a_2, \cdots\}$

Ser $s = \sum_n a_n$

Partial Sums

$$s_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + \cdots + a_n$$

$$s = \lim_{n\to\infty} s_n = \sum_{k=0}^{\infty} a_n$$

Geometric Series

$$s = a_0 + a_0 r + \ldots a_0 r^n \ldots$$

$$s = \sum_{k=0}^{\infty} a_0 r^n \begin{cases} \frac{a_0}{1-r} & |r|<1 \\ \text{Div} & |r| \geq 1 \end{cases}$$

Telescoping

$$s = \sum_{n=0}^{\infty} (a_n - a_{n+1}) = \begin{cases} a_0 & |a_n| \to 0 \\ \text{Div} & \end{cases}$$

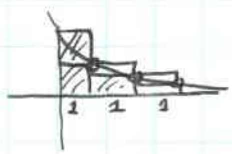
Ex

$\sigma = \{a_n\}, \quad a_n = 3\left(\frac{1}{2^n}\right) \quad n = 0, 1, \cdots$

$\sigma = \left\{3, \frac{3}{2}, \frac{3}{2^2}, \frac{3}{2^3}, \cdots \right\}$

$f(x) = \frac{3}{2^x} = 3\left(\frac{1}{2^x}\right)$

$= 3\left(\frac{1}{2}\right)^x$



$s = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} \ldots$

$= \sin x + \frac{1}{2}\sin 2x + \frac{1}{3} \cdots$

Does $\sum \frac{1}{n}$ converge? $\boxed{\text{Harmonic Series}}$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \frac{1}{8} + \cdots$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \cdots + \frac{1}{8}\right) + \cdots$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

$$> 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{\frac{1}{2}} + \underbrace{\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)}_{\frac{1}{2}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \cdots = \text{Div}$$

Tests for Convergence

1) *If* $\sum_{n=1}^{\infty} a_n$ C *then* $a_n \to 0$

in other words!

If $\lim_{n\to\infty} a_n \neq 0$ *then* $\sum_{n=1}^{\infty} a_n$ D.

Note

$s = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \ldots$ C

$s = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \ldots$ D

$s = \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \cdots \frac{n}{n+1} \cdots$

Diverges because $a_n \not\to 0$

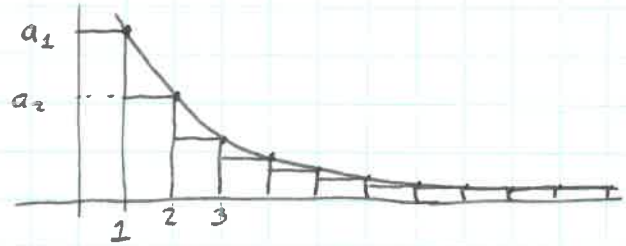
$$\int_0^{\infty} \frac{1}{2^x}\,dx < \infty$$

$$\int_0^{\infty} e^{-x}\,dx < \infty$$

The Integral Test.

Thm. Let $s = \sum_{n=1}^{\infty} a_n$ and let $f(x)$ be a function such that $f(n) = a_n$, then $\int_1^{\infty} f(x)\,dx$ and $\sum_{n=1}^{\infty} a_n$

either both converge or both diverge.



Examples:

1. Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

Ans: Let $f(x) = \frac{1}{x}$ so that $f(n) = a_n = 1/n$

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{R\to\infty} \ln x \Big]_1^R$$

$$= \lim_{R\to\infty} [\ln R - 0] = \infty$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n}$ Div. by the $\int$. test.

Ex 2. The p-test.

Does $\sum_{n=1} \frac{1}{n^p}$ converge?

Ans.

Let $f(x) = \frac{1}{x^p}$ so that $f(n) = \frac{1}{n^p}$.

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & p > 1 \\ \infty & 0 < p < 1 \end{cases}$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^p}$ C. if $p > 1$ by the $\int$. test

D if $p \leq 1$

Ex Does $\sum_{n=1} \frac{\ln n}{n}$ C.

Ans Let $f(x) = \frac{\ln x}{x}$, $f(n) = a_n$

$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \frac{1}{2}\ln^2 x \Big]_1^{\infty}$$

$$= \lim_{R\to\infty} \frac{1}{2}\ln^2 x \Big]_1^R = \infty$$

$\therefore \sum_{n=1}^{\infty} \frac{\ln n}{n}$ Div. by $\int$. test

Pg 603. # 1, 3, 7, 9, 11, 16, 19, 20, 26, 27

Quiz

1. Determine whether the given sequences converge or diverge. If convergent, find the limit

   a) $a_n = \frac{n^2-1}{n^2+1}$

   b) $a_n = \frac{\ln(n^2)}{n}$

2. Sum the series

$$\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n}$$

$a_n \geq 0$!

16 $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$. Let $a_n = \frac{\ln n}{n^2}$,

$f(x) = \frac{\ln x}{x^2}$ so that $f(n) = a_n$

$\int_1^\infty \frac{\ln x}{x^2}\, dx = I$

Let $u = \ln x$ $\quad dv = \frac{1}{x^2}\,dx$

$du = \frac{1}{x}\,dx \quad v = -\frac{1}{x}$

$I = -\frac{1}{x}\ln x \Big]_1^\infty + \int_1^\infty \frac{1}{x^2}\,dx$

$= \lim_{R\to\infty} \left[-\frac{1}{x}\ln x - \frac{1}{x}\right]_1^R = 1$

$\therefore \sum a_n$ Conv. by the $\int$ test

20) $\sum_{n=3}^{\infty} \frac{1}{n \ln n [\ln(\ln n)]^p} = \sum_{n=3}^{\infty} a_n$

Let $f(x) = \frac{1}{x \ln x [\underbrace{\ln(\ln x)}_{y}]^p}$

so that $f(n) = a_n$.

$\int_{x=3}^\infty f(x)\,dx = \int_{x=3}^\infty \frac{1}{y^p}\,dy < \infty$

where

$y = \ln(\ln x) \quad dy = \frac{1}{x \ln x}\,dx$

$\therefore \sum a_n$ C. for $p > 1$ by the $\int$ test.

Examples

Ex 1 $s = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Let $a_n = \frac{1}{n^2+1}$, $b_n = \frac{1}{n^2}$

Now, $n^2 + 1 > n^2$

$\therefore \frac{1}{n^2+1} < \frac{1}{n^2}$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2+1} < \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ Conv. by the p-test $(p=2)$

$\therefore \sum_{n=1}^{\infty} a_n$ Conv by the comparison test.

9) $\sum_{n=1}^{\infty} n e^{-n^2}$. Let $a_n = n e^{-n^2}$ and $f(x) = x e^{-x^2}$

so that $f(n) = a_n$.

$\int_1^\infty x e^{-x^2}\,dx = -\frac{1}{2} \lim_{R\to\infty} e^{-x^2}\Big]_1^R$

$= -\frac{1}{2}[0 - 1] = \frac{1}{2}$.

$\therefore \sum a_n$ Conv by the $\int$ test

Ex

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ Converges by the p-test where $p = \frac{3}{2} > 1$

Comparison Tests

1. If $a_n \leq b_n$ for $n > N$ then $\sum a_n \leq \sum b_n$
$\therefore \sum b_n$ C $\Rightarrow \sum a_n$ C.

2. If $a_n \geq b_n$ for $n > N$ then $\sum a_n \geq \sum b_n$
$\therefore \sum b_n = \infty \Rightarrow \sum a_n = \infty$

Idea: Compare to a series whose behaviour is known

a) – Geometric series

b) → p-series

a) $\sum a_0 r^n$ $\qquad$ b) $\sum \frac{1}{n^p}$

Ex 2

$s = \sum_{n=2}^{\infty} \frac{1}{n^2-1}$

Let $a_n = \frac{1}{n^2-1}$, $b_n = \frac{1}{n^2}$

$n^2 - 1 < n^2$

$\frac{1}{n^2-1} > \frac{1}{n^2}$

$\sum_{n=2}^{\infty} \frac{1}{n^2-1} > \sum_{n=2}^{\infty} \frac{1}{n^2}$

$\sum \frac{1}{n^2}$ Conv by the p-Test

Oh-oh ☹

Limit Comparison test.

Suppose that

$0 < \lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| < \infty$ then the series

$s_1 = \Sigma a_n$ and $s_2 = \Sigma b_n$

either both converge or both diverge.

Ex $\sum_{n=3}^{\infty} \frac{n^2}{\sqrt{n^7+2}}$. Let $a_n = \frac{n^2}{\sqrt{n^7+2}}$

$b_n = \frac{n^2}{n^{7/2}} = \frac{1}{n^{3/2}}$. $\quad b_n = \frac{n^2}{\sqrt{n^7}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{\sqrt{n^7+2}} \cdot \frac{\sqrt{n^7}}{n^2} = 1$

Σb_n Conv by the p test $p = \frac{3}{2} > 1$

$\therefore \Sigma a_n$ Conv by the L.C.T

Ex $\Sigma' \frac{1}{n^2-1} = s$

Let $a_n = \frac{1}{n^2-1}$ and $b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \left|\frac{n^2}{n^2-1}\right| = 1$

Σb_n Conv by p test $\Rightarrow \Sigma a_n$ C. by L.C.T

Note. L.C.T Always! works if a_n is a rational function of n

Ex $a_n = \frac{n^2-n}{n^4+3n-1}$ — $b_n = \frac{1}{n^2}$

$a_n = \frac{\sqrt{n}}{n^3+4}$ — $b_n = \frac{\sqrt{n}}{n^3} = \frac{1}{n^{5/2}}$

pg 608 # 1, 7, 9, 15, 19, 21, 23, 26

* $\boxed{32}$

a) Quiz Sol.

1 a) $\lim_{n\to\infty} \frac{n^2-1}{n^2+1} \overset{L.H}{=} \lim_{n\to\infty} \frac{2n}{2n} = 1$

b) $\lim_{n\to\infty} \frac{\ln n^2}{n} = \lim_{n\to\infty} \frac{2\ln n}{n}$

$\overset{L.H}{=} \lim_{n\to\infty} \frac{2/n}{1} = 0$

Ex $\lim_{n\to\infty} \frac{7n^3+2n-1}{2n^3+n^2+8} = \frac{7}{2}$

2. $\sum_{n=1}^{\infty} \frac{3^n+2^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n$

$= \frac{1/2}{1-1/2} + \frac{1/3}{1-1/3} = 1 + \frac{1}{2} = \frac{3}{2}$

23) $\sum \frac{n^2-n+2}{\sqrt[4]{n^{10}+n^5+3}} = \sum a_n$ $\boxed{\lim_{n\to\infty} \frac{a_n}{b_n} = 1}$

Let $b_n = \frac{n^2}{\sqrt[4]{n^{10}}} = \frac{n^2}{n^{10/4}} = \frac{1}{n^{1/2}}$

$\sum \frac{1}{n^{1/2}}$ Div by the p test, $p = \frac{1}{2}$

$\therefore \sum a_n$ Div by the L.C.T.

$y = n^{\frac{1}{n}}$

$\ln y = \frac{1}{n} \ln n = \frac{\ln n}{n}$

$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0$

$\ln y \to 0$ $\qquad n^{\frac{1}{n}} \to 1$

$y \to e^0 = 1$ $\qquad \frac{1}{n^{1/n}} \to 1$

Ratio test (Most important)

Let $s = \sum_{n=1}^{\infty} a_n$

Define $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$\rho \begin{cases} > 1 & \text{Div} \\ < 1 & \text{Con} \\ = 1 & \text{Fails.} \end{cases}$$

Works for all $\sum a_n$ where a_n involves exponential and factorials

---

Ex 29.

$s = \sum \frac{1}{n!}$. Let $a_n = \frac{1}{n!}$

$\rho = \lim_{n\to\infty} \left| \frac{1}{(n+1)!} \cdot \frac{n!}{1} \right|$

$= \lim_{n\to\infty} \left| \frac{n!}{(n+1)n!} \right| = 0 < 1$

s Converges by the ratio test

$s = \sum_{n=0}^{\infty} \frac{1}{n!} = e$ (!)

---

32) $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ Let ~~$a_n = n$~~

Let $a_n = \frac{1}{n^{1+\frac{1}{n}}}$ $\quad b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^{1+\frac{1}{n}}} = \lim_{n\to\infty} \frac{n}{n \cdot n^{1/n}}$

$= \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$. $\sum b_n$ Div $\Rightarrow \sum a_n$ Div by L.C.T

---

19) $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$. Let $a_n = \frac{1}{1+\sqrt{n}}$, $b_n = \frac{1}{\sqrt{n}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} \overset{L.H}{=} \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{2\sqrt{n}}} = 1$

Ex 25 p. 608

$s = \sum \frac{(n+1)}{n \cdot 2^n}$. Let $a_n = \frac{n+1}{n \cdot 2^n}$

$\rho = \lim_{n\to\infty} \left| \frac{n+2}{(n+1)2^{n+1}} \cdot \frac{n \cdot 2^n}{n+1} \right|$

$= \lim_{n\to\infty} \left| \frac{n(n+2) \cdot 2^n}{(n+1)^2 \cdot 2^n \cdot 2} \right| = \frac{1}{2} < 1$

s Converges by the ratio test

Note $n! = n(n-1) \cdots 2 \cdot 1$

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$6! = 6 \cdot 5! = 6 \cdot 5 \cdot 4!$

$\boxed{n! = n(n-1)!}$

$\boxed{(n+1)! = (n+1)n!}$

10.6)

21) $\sum \cos(n\frac{\pi}{3}) \cdot \frac{1}{n!} = \sum a_n$

$\rho = \lim_{n\to\infty} \left| \frac{\cos((n+1)\pi/3)}{(n+1)!} \cdot \frac{n!}{\cos(n\pi/3)} \right|$

$= \lim_{n\to\infty} \left| \frac{\cos[(n+1)\pi/3]}{\cos n\pi/3} \cdot \frac{n!}{(n+1)n!} \right| = 0$

s converges absolutely by ratio test

TOPS FORM

Test $\sum (-1)^n a_n = \sum b_n$.

$a_n \geq 0$

1) Test for A C

$\sum |b_n| = \sum a_n$

Pink.

2. Alt series test

Only one test

i) $a_n \to 0$?

ii) Alt

iii) $a_{n+1} < a_n$.

Ex $\sum \frac{(-1)^n}{\sqrt{n}}$

D☺!
10.6 # 1 - 32

i) Alt ✓

ii) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ ✓

iii) Is $a_{n+1} < a_n$?

Method 1

$n+1 > n$

$\sqrt{n+1} > \sqrt{n}$

$\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ ✓

Method 2

Let $f(x) = \frac{1}{\sqrt{x}}$

$f'(x) = \frac{-1}{2x^{3/2}} < 0$

seq a_n is decreasing ✓

14) $\sum (-1)^n \frac{\arctan n}{n^3}$

Note that $|\arctan n| < \frac{\pi}{2}$

$\frac{\arctan n}{n^3} < \frac{\pi/2}{n^3}$

$\sum \frac{\pi/2}{n^3}$ C. by p test

$\sum \frac{\tan^{-1} n}{n^3}$ C. by comparison

$\therefore \sum (-1)^n \frac{\arctan n}{n^3}$ C.A.

10) $\sum \frac{n!}{(-10)^n}$ $a_n \to \infty$. D.

24) $\sum \frac{(-1)^n}{n \ln n}$. Let $a_n = \frac{1}{n \ln n}$ $f(x) = \frac{1}{x \ln x}$

$\int_1^\infty \frac{1}{x \ln x}\,dx = \lim_{R\to\infty} [\ln(\ln x)]_1^R = \infty$

$\therefore \sum a_n$ Div. $\Rightarrow \sum (-1)^n a_n$ does not C.A.

Test for C.C.

i) Alt ✓

ii) $\lim_{n\to\infty} \frac{1}{n \ln n} = 0$ ✓

iii) $n+1 > n$

$\ln(n+1) > \ln n$

$\therefore (n+1)\ln(n+1) > n \ln n$

$\frac{1}{(n+1)\ln(n+1)} < \frac{1}{n \ln n}$ ✓

$\therefore \sum (-1)^n \frac{1}{n \ln n}$. C.C.

Let $f(x) = \frac{1}{x \ln x}$

$f'(x) = \frac{x \ln x \cdot 0 - \frac{d}{dx}(x \ln x)}{x \ln x^2}$

$f'(x) = \frac{-[x \cdot \frac{1}{x} + \ln x]}{x \ln x^2}$

$f'(x) = \frac{-[1 + \ln x]}{x \ln x^2} < 0$ if $x > 1$

$\therefore$ ~~f(an+1)~~ $f(n+1) < f(n)$

Power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n.$$

Ex $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$!

x	f(x)
0	0
1	$\sum \frac{1}{n!} =$
2	$\sum \frac{2^n}{n!}$
3	$\sum \frac{3^n}{n!}$

Test for AC

3) $\sum \frac{(-3)^n}{n^3}$ 26) $\sum \frac{(-2)^n n^2}{(n+2)!}$

19) $\sum \frac{n!}{(-10)^n}$ x) $\sum \frac{(n!)^2}{(2n)!}$

Hw 10.6 # 4, 7, 8, 15, 17, 20
26, 31, 29, 36

Test 3 ~~Wed Dec~~
~~Thurs / Dec 5.~~

$\sum \frac{(n!)^2}{(2n)!}$ Let $a_n = \frac{(n!)^2}{(2n)!}$

$$\rho = \lim_{n\to\infty} \left| \frac{[(n+1)!]^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{[(n+1)n!]^2 \cdot (2n)!}{(2n+2)(2n+1)(2n)! \cdot (n!)^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{(2n+2)(2n+1)} \right| = \frac{1}{4} < 1$$

C. by ratio test

Alternating series.

Ex

$s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots$

First $s = \sum (-1)^n a_n$

$s = \sum (-1)^{n+1} a_n$

Not Alt $s = \sum (-1)^{2n+1} a_n$

not alt $s = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8}$

Ex

$s = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} \dots$

$s = \sum (-1)^n \frac{1}{2^n} < \sum \frac{1}{2^n} < \infty$

s converges absol.

Def Let $s = \sum b_n = \sum (-1)^n a_n$
where $a_n \geq 0$
s converges absolutely if
$s' = \sum |b_n| = \sum a_n$ C.

17) $\sum \frac{(n+1)5^n}{n \cdot 3^{2n}}$ Let $a_n = \frac{(n+1)5^n}{n \cdot 3^{2n}}$

$$\rho = \lim_{n\to\infty} \left| \frac{(n+2)5^{n+1}}{(n+1)3^{2n+2}} \cdot \frac{n\, 3^{2n}}{(n+1)5^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n(n+2)}{(n+1)^2} \cdot \frac{5}{9} \right| = \frac{5}{9} < 1$$

Series C.A. by ratio test.

Test #3 Dec 4 - Θ

10) $\underbrace{\sum (-1)^{n-1} \frac{\sqrt{n}}{n+1}}_{b_n}$. Let $a_n = \frac{\sqrt{n}}{n+1}$, $c_n = \frac{\sqrt{n}}{n} = \frac{1}{n^{1/2}}$

$\sum \lim_{n\to\infty} \left| \frac{a_n}{c_n} \right| = \lim_{n\to\infty} \left| \frac{\sqrt{n}}{n+1} \cdot \frac{n}{\sqrt{n}} \right| = 1$

$\sum c_n$ diverges by p-test $p = \frac{1}{2}$

$\sum a_n$ diverges by L.C.T

$\therefore \sum b_n$ does not C.A.

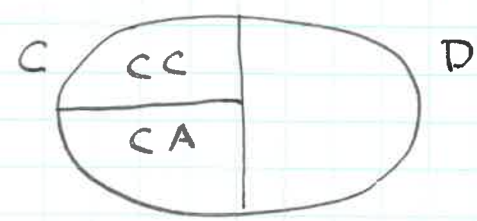
Alt. Harmonic series!

$s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots$

Test

1) Strictly alt
2) $a_n \to 0$
3) $a_{n+1} < a_n$

Conditional Convergence



TOPS FORM 3314

Convergence of Power series.

$$f(x) = \sum_{n=0}^{\infty} c_n (x-c)^n$$

1. Ratio test. Set $\rho < 1$
2. Check end pts

Ex

$$f(x) = \sum_{n=4}^{\infty} \underbrace{\frac{(-1)^n x^n}{(n-3)}}$$

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(n-2)} \cdot \frac{(n-3)}{(-1)^n x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x(n-3)}{(n-2)} \right| = |x| < 1$$

$-1 < x < 1$ C.A.

Check end points.

$x = -1$ $\sum \frac{(-1)^n(-1)^n}{(n-3)} = \sum \frac{1}{n-3}$ D

$x = 1$ $\sum \frac{(-1)^n}{n-3}$.   C. C.

$f(x)$ Conv. for $\underline{-1 < x \le 1}$
Interval of C.

$|x| < 1 \leftarrow$ $1 = \underline{\text{radius of } C}$

Do. 10.8 # 3, 7, 9, 18, 21, 25, 31*

Ex

$$f(x) = \sum \frac{(-1)^{n+2}(x-3)^n}{(n+4)\cdot 2^n}$$

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+3}(x-3)^{n+1}}{(n+5)\, 2^{n+1}} \cdot \frac{(n+4)\, 2^n}{(-1)^{n+2}(x-3)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-3)(n+4)}{(n+5)\cdot 2} \right| = \left| \frac{x-3}{2} \right| < 1$$

$\Rightarrow |x-3| < 2$ 2 - radius of C.

$-2 < x-3 < 2$

$1 < x < 5$

Check $x = 1$

$$\sum \frac{(-1)^{n+2}(-2)^n}{(n+4)\, 2^n} = \sum \frac{(-1)^{n+2}(-1)^n}{n+4}$$

$$= \sum \frac{(-1)^{2n+2}}{n+4} \quad \text{D.}$$

Check $x = +5$

$$\sum \frac{(-1)^{n+2}\, 2^n}{(n+4)\, 2^n} = \sum \frac{(-1)^{n+2}}{n+4} \quad \text{C.C}$$

Series conv for $1 < x \le 5$

TOPS FORM 3314
LITHO IN U.S.A.

21) $\sum \frac{2^n (x-3)^n}{n+3}$

$$\rho = \lim_{n\to\infty} \left| \frac{2^{n+1}(x-3)^{n+1}}{(n+3+1)} \frac{(n+3)}{2^n (x-3)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2(x-3)(n+3)}{(n+4)} \right|$$

$$= |2(x-3)| < 1$$

$$|x-3| < \frac{1}{2} = r$$

Taylor Series (about $x=0$)

$f(x) = \sin x$

Assume

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \cdots$$

$$f(x) = \sum_{n=0} a_n (x-0)^n$$

$$f'(x) = a_1 + 2\cdot a_2 x^1 + 3\cdot a_3 x^2 + \cdots$$

$$f''(x) = (2\cdot 1) a_2 + (3\cdot 2) a_3 x^1 + \cdots\cdots$$

$$f'''(x) = (3\cdot 2\cdot 1) a_3 + (4\cdot 3\cdot 2) a_4 x^1 + \cdots$$

$f(0) = a_0$	$a_0 = f(0)$
$f'(0) = a_1$	$a_1 = f'(0)$
$f''(0) = (2\cdot 1) a_2$	$a_2 = f''(0)/2!$
$f'''(0) = (3\cdot 2\cdot 1) a_3$	$\vdots$
$f^{(n)}(0) = n!\, a_n$	$a_n = \frac{f^{(n)}(0)}{n!}$

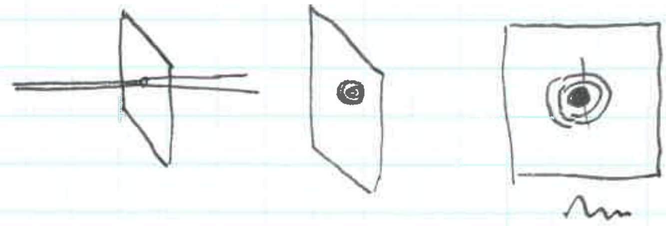
$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (x-0)^n}$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n}$$

Taylor's Formula about $x = c$

Do 10.10 # 1, 5, 6, 7, 11, 17, 19, 21, 25, 26



Ex $f(x) = \sin x$ $(x=0)$

$f(x) = \sin x$	$f(0) = 0$
$f'(x) = \cos x$	$f'(0) = 1$
$f''(x) = -\sin x$	$f''(0) = 0$
$f'''(x) = -\cos x$	$f'''(0) = -1$
$f^{iv}(x) = \sin x$	$f^{iv}(0) = 0$
$f^{(v)}(x) = \cos x$	$f^{(v)}(0) = 1$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Ex $f(x) = \cos x$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Taylor Series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} x^n} \quad !$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(c)}{n!} (x-c)^n$$

Ex $f(x) = \frac{1}{1-x}$ $(x=0)$

$$\begin{array}{r|l} & 1 + x + x^2 \\ \hline 1-x & 1 \\ & \underline{1-x} \\ & x \\ & \underline{x - x^2} \\ & x^2 \\ & \underline{x^2 - x^3} \\ & x^3 \end{array}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 \cdots$$

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$$\frac{1}{1-x} = 1 + x + x^2 + x^3 \dots \qquad |x|<1$$

$$\frac{1}{1-2} = 1 + 2 + 2^2 + 2^3 + \dots \qquad \text{F}$$

$$\frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \qquad \text{T}$$

$f(x) = \ln x$

Consider

$f(x) = \ln(1-x)$

$$\int \frac{1}{1-x}\,dx = \int (1 + x + x^2 \dots)$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} \dots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}$$

$x \mapsto -x$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

$\int \sin x^2\,dx$

Frenel Integral

Ex Expand $f(x) = \frac{1}{1+x^2}$ $(x=0)$

Sol

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 \dots = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 \dots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Expand $f(x) = \tan^{-1} x$ $(x=0)$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} \dots + (C=0)$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots$$

$$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots\right]$$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} \dots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + = \sum_0^{\infty} \frac{x^n}{n!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots \qquad |x|<1$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 \dots$$

Expand $f(x) = \sin x^2$

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} \dots + \mathcal{E}$$

$$\int_0^1 \sin x^2\,dx = \left.\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} \dots\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!}$$

$\mathcal{E} < 0.000013$

Expand $f(x) = \tanh^{-1} x$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} \dots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 \dots$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} \dots$$

643 # 2, 4, 5, 17, 19, 21, 22
23, 25, 33, 34, 37, 40

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23 $f(x) = \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots$

$\cos 2x = 1 - \frac{2^2x^2}{2!} + \frac{2^4x^4}{4!} - \frac{2^6x^6}{6!} \dots$

$-\frac{1}{2}\cos 2x = -\frac{1}{2} + \frac{2x^2}{2!} - \frac{2^3x^4}{4!} + \frac{2^5x^6}{6!}$

$\frac{1}{2} - \frac{1}{2}\cos 2x = \frac{2x^2}{2!} - \frac{2^3x^4}{4!} + \frac{2^5x^6}{6!} \dots$

22 $f(x) = x e^{-x}$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

$xe^{-x} = x - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!} \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!}$

25 $\sum \frac{(-1)^n \ln n}{\sqrt{n}}$ .. Let $a_n = \frac{\ln n}{\sqrt{n}}$, $b_n = \frac{1}{\sqrt{n}}$

$\frac{\ln n}{\sqrt{n}} > \frac{1}{\sqrt{n}}$. $\sum \frac{1}{\sqrt{n}}$ D $\Rightarrow \sum a_n$ C$\cancel{A}$

Int test

$\int \frac{\ln n}{\sqrt{n}}\, dn = I.$ Let $u = \ln n$ $\quad du = \frac{dn}{n}$

$dv = \frac{1}{\sqrt{n}}\, dn \quad \frac{v}{2} = \sqrt{n}$

$I = 2\sqrt{n}\ln n - \int \frac{2\sqrt{n}}{n}\, dn$

$= 2\sqrt{n}\ln n - 4\sqrt{n}$

$\int_1^{\infty} \frac{\ln n}{\sqrt{n}}\, dn = \lim_{R\to\infty} [2\sqrt{n}\ln n - 4\sqrt{n}]_1^R$

$= \infty$

Test for C.C.

i) $\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = 0$ ✓

ii) Alt ✓

iii) $f(x) = \frac{\ln x}{\sqrt{x}}$ $\quad f'(x) = \sqrt{x} \cdot \frac{1}{x} - \dots$

$f'(x) = \frac{\sqrt{x} \cdot \frac{1}{x} - \ln x \cdot (+\frac{1}{2x^{1/2}})}{(\sqrt{x})^2}$

$= \frac{2 - \ln x}{2x\sqrt{x}} < 0$ if $x > 10$

Series C.C. by Alt. test

Final Dec 15, 8:00 AM.

10.2 #24 $\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)}$ s

$\frac{1}{(3n-2)(3n+1)} = \frac{A}{3n-2} + \frac{B}{3n+1}$

$1 = A(3n+1) + B(3n-2)$

$n = \frac{2}{3}$ $\quad 1 = 3A \quad A = 1/3$

$n = -\frac{1}{3}$ $\quad 1 = -3B \quad B = -1/3$

$s = \sum_{n=1}^{\infty} \frac{1}{3}\left[\frac{1}{3n-2} - \frac{1}{3n+1}\right] = \frac{1}{3}$

10) $\sum \frac{(-1)^{n-1}\sqrt{n}}{(n+1)}$ Let $a_n = \frac{\sqrt{n}}{n+1}$, $b_n = \frac{1}{\sqrt{n}}$

$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \left|\frac{\sqrt{n}}{n+1} \cdot \frac{\sqrt{n}}{1}\right| = 1$

$\sum b_n$ D by p-test $\Rightarrow \sum a_n$ C$\cancel{A}$ "~"

Test for C.C.

i) $\lim_{n\to\infty} a_n = 0$

ii) Alt ✓

iii) $f(x) = \frac{\sqrt{x}}{x+1}$

$f'(x) = \frac{(x+1)\frac{1}{2\sqrt{x}} - \sqrt{x}}{(x+1)^2}$

$f'(x) = \frac{(x+1) - 2x}{\sqrt{2x}(x+1)^2} < 0$ if $x > 1$

$\therefore$ CC by Alt. test.

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Euler's Formula.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

$$e^{ix} = 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} \cdots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} + \cdots$$

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right)$$

$$\boxed{e^{ix} = \cos x + i \sin x}$$

Ex

$y'' + y = 0$ Let $y = e^{rx}$

$r^2 + 1 = 0 \quad r^2 = -1 \quad r = \pm i$

$y_1 = e^{ix} = \cos x + i \sin x$

$y_2 = e^{-ix} = \cos x - i \sin x$

$y = c_1 e^{ix} + c_2 e^{-ix}$

$y = d_1 \cos x + d_2 \sin x.$

Note

$$(1+x)^{1/2} = 1\, 1^{1/2} x^0 + \frac{1}{2} 1^{-\frac{1}{2}} x^1 - \frac{1}{4} 1^{-\frac{3}{2}} x^2$$

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 \cdots$$

$$(1+x)^{-1} = 1 \cdot 1^{-1} x^0 - 1\, 1^{-2} x^1 + 1\, 1^{-3} x^2 - x^3$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 \cdots$$

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 \cdots$$

$$\sqrt{101} = (1+100)^{\frac{1}{2}} = 1 + 50 -$$

etc, etc, etc.

Binomial Expansion.

$(x+y)^2 = x^2 + 2x^1y^1 + y^2$

$(x+y)^3 = x^3 + 3x^2y^1 + 3xy^2 + y^3$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Pascal's Δ.

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
```

$(x+y)^5 = 1x^5y^0 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 \cdots$

$(x+y)^6 = 1x^6y^0 + 6x^5y^1 + 15x^4y^2 + 20x^3y^3 \cdots$

Def $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$

Ex

$\binom{4}{0} = \dfrac{4!}{0!\,4!} = 1$

$\binom{4}{1} = \dfrac{4!}{1!\,3!} = 4$

$\binom{4}{2} = \dfrac{4!}{2!\,2!} = 6$

$(x+y)^4 = \binom{4}{0}x^4y^0 + \binom{4}{1}x^3y^1 + \binom{4}{2}x^2y^2$

$(x+y)^n = \binom{n}{0}x^ny^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 \cdots$

10.11 # 1, 3, 5, 7

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