Calculus

With

Lecture Notes

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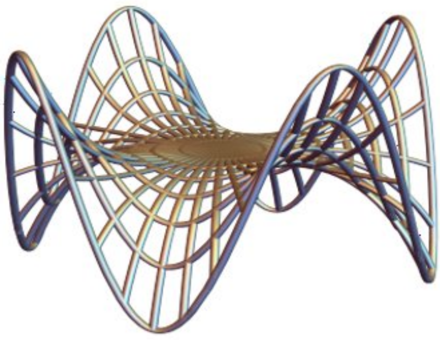


Contents

Chapter 1

Introduction

1.1 Brief History of Calculus

- Greeks

  ◦ Zeno's Paradox ($\tilde{5}00$ BC). One can essentially trace the origins of calculus back to the ancient Greeks. The central concept behind calculus is the idea of limits. The notion is manifested in a number of paradoxes proposed by Zeno of Elea. The most famous is the paradox of Achilles and the tortoise. Achilles chases a tortoise but by the time the man reaches the point where the tortoise was, the tortoise has moved by smaller by finite amount. This process repeats itself over and over again, so how can Achilles catch the turtle? The dichotomy paradox is similar but easier to illustrate. A person moves from a point A to a point B, say, 1 km away. Before the person gets to the point B he must travel $1/2$ the distance, and from there, $1/4$ of the distance, and then $1/8$, and so on. Since the sum $1/2 + 1/4 + 1/8 + \ldots$ never ends, how is it possible for the man to get to point B?

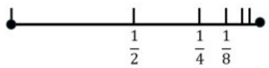
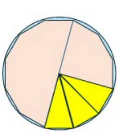
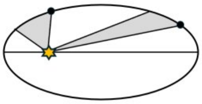


Fig. 1.1: Dichotomy paradox



  ◦ Archimedes ($\tilde{2}50$ BC). with Archimedes' method of exhaustion. The illustration shows the foundation of the method of exhaustion to compute the area of a circle. The idea is to divide the area into central isosceles triangles subtended by a regular polygon, and approximate the area of the circle by the sum of the simpler areas of the isosceles triangles. One continues this process by summing the areas of central triangles of regular polygons of larger and larger number of sides, until the area of the circle not counted is "exhausted". It is an incredible achievement, that by the method of exhaustion, Archimedes found formulas for the area and circumference of a circle, the volumes and surface areas of spheres, cylinders and cones. It will take us the entire semester to derive the latter by the techniques of calculus.

- Kepler

  ○ Kepler ($\tilde{1}$610). Modified the heliocentric theory of Copernicus and introduced new empirical laws for the orbits of planets.
  First law. Planets move on elliptical orbits with the sun at one of the foci.
  Second law. Planets sweep equal areas in equal time.
  Third law. The square if the periods is proportional to the cube of the distances.

- Galileo ($\tilde{1}$620). Galileo Galilei
  - Was the first person to build a telescope.
  - Discovered the first moons in our solar system beyond our own.
  - Introduced the notion of velocity and acceleration.
  - Introduced the notion than in vacuum, al objects fall at the same rate.
  - In his study of kinematics, he discovered the principle of relativity for inertial frames of reference: "It is impossible to conduct a mechanical experiment that will determine whether we are moving (at a constant velocity) or at rest."

- Descartes ($\tilde{1}$630). The main contribution of René Descartes in the context of the history of calculus, was the invention of the eponymously called Cartesian coordinate system. The calculus course should really be called calculus and analytical geometry. The analytic geometry aspect is built on the foundations developed by Descartes. It was in Descartes to whom Newton was referring when said "If I have been further, it is because I was standing on the shoulders of giants."

- Newton ($\tilde{1}$667)
  - Isaac Newton published Principia Mathematica in 1667.
  - Compiled exiting knowledge and established the subject of classical mechanics.
  - Co-invented calculus with the goal of providing a mathematical foundation for Kepler's laws.
  - In the process, introduced the universal law of gravitation.

  - Did pioneering work on optics

- Leibniz (~1675)

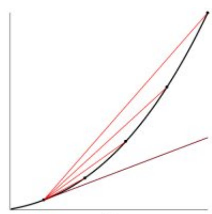
  - Gottfried Leibniz independently co-invented calculus with Newton about 1675 but did not publish any results until 1684.
  - Introduced intuitive notation for derivatives and inntegration,
  - Used differentials in the development of calculus.
  - The product rule for differentiation is attributed to Leibniz.

1.2 Brief overview of Calculus

This course is divided into two major topics: Differential calculus and integral calculus. For each of these topics, we study some applications in the physical sciences and engineering.

1.2.1 Differential Calculus

One of the first things we learn in coordinate geometry is that a straight line has equation $y = mx + b$, where m is the slope and b the y-intersect. The question is how to generalize the notion of a slope of a line to slope of curve. By slope of a curve at a particular point $x = a$, we must mean the slope of the tangent line to the curve at that point. The problem is that to find the slope of a line, we need two points.

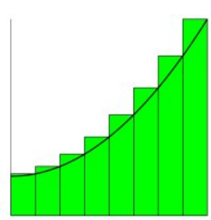


The process envisioned by Newton and Leibniz is this. Pick another point $x = b$ on the curve near the point with $x = a$. Then it makes sense to find the slope of the secant line passing through those to points. The closer b is to a, tho closer the secants lines are to the slope of the tangent line.

The slope of the tangent line is obtained by a limit process of the slopes of the secants, as b gets arbitrarily close to a, as illustrated in the picture. The slope of the tangent of a function $y = f(x)$ at a point $x = a$ is called the derivative $f'(a)$ at that point. The slope at any point x is another function $f'(x)$.

1.2.2 Integral Calculus

The second major topic is integral calculus. The foundations of the theory are very easy to explain.



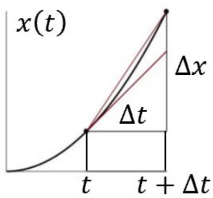
Let $y = f(x)$ which for now we assume to be positive. The problem is to find the area under the curve between $x = a$ and $x = b$. In general, the area formulas we learn in geometry will not do. Instead, the divide the interval $[a, b]$ into n pieces of width Δx and approximate the area under the curve by the sums of the areas of the rectangles with base Δx and height given by the value of the function at some point in each subinterval. In the figure here, we have chosen to evaluate the heights at the right endpoints of the subintervals. Of course, the sum of the areas of the rectangles gives only an approximation of the area under the curve, but the approximation gets better and better as we increase the number of rectangles. The area we seek is called the integral of $f(x)$ over $[a, b]$. The integral is obtained by a limit process in which we let n be arbitrarily large, or equivalently, we let Δx approach zero.

Chapter 2

Limits and Derivatives

2.1 Tangents and Velocity

The most fundamental concept in kinematics is that if a particle is moving on a straight line at a constant velocity v, then, the position $x(t)$ along that line is given by the equation $x(t) = vt + b$. The number b is the position coordinate at time $t = 0$.



In coordinate geometry, we have a linear equation in which the slope $v = \Delta x/\Delta t$ is the velocity and b is the x-intersect in an x-t coordinate plane. We would like to consider the case where the trajectory of the particle is given by a more general curve $x(t)$. It makes sense to identify the slope of the curve at time t with the slope of the tangent to the curve at that point. Our goal then, is to find a process to determine the slope of the line tangent to a curve at a particular point with coordinate $(t, x(t)$. The problem is that find the slope of line one needs two points, and here we only have the point of tangency. The idea in calculus is to pick a point nearby with coordinate $t + \Delta t$. We denote the corresponding change of the x-coordinate by Δx. the line passing through the points $(t, x(t))$ and $(t + \Delta t, x(t + \Delta t))$ is called a secant line. The slope of the secant line is given by

$$m_{sec} = \frac{\Delta x}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

The slope of the secant line is called the average velocity of the curve between the two points. If Δt is small, the slope of the secant line gives an approximation to the slope of the tangent line. As the value of the increment Δt becomes smaller and smaller so that the second point gets closer and closer to the given point, the slope of the secant line gets closer to the slope of the tangent line we seek. We define the slope of the tangent line $x'(t)$ as

$$x'(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}. \tag{2.1}$$

It is common to denote independent coordinate increment by $h = \Delta t$. The quantity $x'(t)$ is called the instantaneous velocity at time t. The expression $\lim_{\Delta t \to 0}$ is read as the limit as Δt approaches 0, meaning that Δt becomes arbitrarily close to 0, but it is not equal to 0; else we would only have one point and the notion of slope would make no sense. We defer to a later section to write a precise definition of limits. One of the goals in this chapter is to understand how to estimate or compute a derivative

a) Graphically
b) Numerically
c) Analytically

2.1.1 Example Estimate graphically and numerically the value of the derivative of $x(t) = t^2$ at $t = 1$.
Solution: First step is to graph the function. In Maple, the simple command to plot(t^2) displays the function on an automatically generated window range.

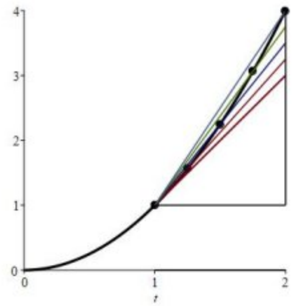


Fig. 2.1: Derivative of t^2 at $t = 1$

However, it it by far more clear to include in the graphic a sequence of secant lines with slope approaching the tangent line, as shown in figure 2.1. Of course, the generic Maple code to produce such a figure is more elaborate. The code used here appears in the appendix 4.1.2. Inspection of graph of the tangent line allow us to estimate the slope as approximately $\Delta x/\Delta t \approx 2$. Of course this is a very crude approximation which ia not to be trusted beyond one or two significant figures.

A numerical estimate of the slope of the tangent is obtained by evaluating the slope of secant lines at $t = 1$ and $t \pm \Delta t$. A good choice is to pick δt to be an integer power of 0.1. This is easily done in a graphing calculator, but it is more efficient to use a Maple code as shown in the appendix 4.1.1

t	m (sec)	t	m (sec)
1.100000	2.100000	0.900000	1.900000
1.010000	2.010000	0.990000	1.990000
1.001000	2.001000	0.999000	1.999000
1.000100	2.000100	0.999900	1.999900
1.000010	2.000010	0.999990	1.999990
1.000001	2.000000	0.999999	1.999999

Although not a proof, evaluation of the slopes of the secant lines as the second point approaches $t = 1$ from the left and from the right, gives a compelling argument that the slope of the tangent line at $t = 1$ is 2. That is, the velocity $x'(t)$ at $t = 1$ is 2.

2.1.2 Example Estimate graphically and numerically the value of the derivative of $x(t) = t^3 - t + 1$ at $t = 1$.
Solution: Using maple code 4.1.2 with input function $x(t) = t^3 - t + 1$, we obtain the graph 2.2. In the graph of the tangent line in red, we have $\Delta t = 1$ and $\Delta x \approx 2$, so geometrically, a rough guess for the derivative is 2. As before,

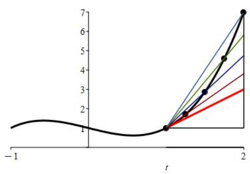


Fig. 2.2: Derivative of $t^3 - t + 1$ at $t = 1$

we evaluate the slopes of the secant lines using the maple code 4.1.1 for the given function. The table of values is given by

t	m (sec)	t	m (sec)
1.100000	2.310000	0.900000	1.710000
1.010000	2.030100	0.990000	1.970100
1.001000	2.003001	0.999000	1.997001
1.000100	2.000300	0.999900	1.999700
1.000010	2.000030	0.999990	1.999970
1.000001	2.000003	0.999999	1.999997

Our educated guess is that the velocity at $t = 1$ is $x'(1) = 2$.

2.1.3 Example Estimate graphically and numerically the value of the derivative of $x(t) = \sin(t)$ at $t = 0$.
Solution: Using maple code 4.1.2 with input function $x(t) = t^3 - t + 1$, we obtain the graph 2.3. In the graph of the tangent line in red, we have $\Delta t = \pi/8$ and $\Delta x \approx 0.40$, so geometrically, a rough guess for the derivative is 1.02. Using

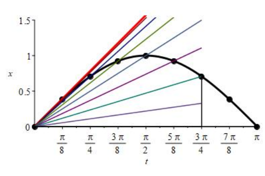


Fig. 2.3: Derivative of $\sin(t)$ at $t = 0$

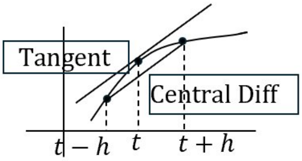
Maple to valuate the slopes of the secant lines we obtain the following table

t	m (sec)	t	m (sec)
0.100000	0.998334	−0.100000	0.998334
0.010000	0.999983	−0.010000	0.999983
0.001000	1.000000	−0.001000	1.000000
0.000100	1.000000	−0.000100	1.000000

It is safe to guess that $x'(0) = 1$. This means that if $|t|$ is small, then sine function $\sin(t)$ is approximately equal to t. The smallest the value of $|t|$, the better the approximation $\sin(t) \approx 1$. Try this on a graphing calculator for a small value of t such as $t = 0.00214$. We get $\sin(0.00214) \approx 0.0021399984$.

Numerical Differentiation

In real-life applications in science and engineering, we often encounter functions defined by data. The data might not be modelled by an easily recognizable formula. In these cases it is necessary to calculate derivatives by a numerical scheme called the method of differences. Suppose the data is given by a discrete function $x(t)$, and $t = a$ is a data point that has left and right neighbors. with coordinates $(a-h)$ and $a+h$. As shown in the picture, the best approximation to the slope of the tangent is given be the slope of the secant line between the two adjacent points, that is



$$x'(a) \approx \frac{x(a+h) - x(a-h)}{2h} \tag{2.2}$$

Suppose there are n data points. One can index the n data points by natural numbers $k = 1 \ldots n$. The formula above can be easily generalized for all but the end-points so it can be implemented by computers. The formula is called the method of **central differences.** If needed, for the endpoints, one can use obvious “backward” or “forward” differences.

2.1.4 Example .

The table below shows the position of a particle for the first 8 seconds:

t	1	2	3	4	5	6	7	8
x	1.89	1.98	2.12	2.24	2.45	2.73	2.94	3.12

Estimate the instantaneous velocity at time $t = 5$.

Solution: Simple.

$$x'(5) \approx \frac{2.73 - 2.24}{6 - 4} =$$

2.2 Limit of Function

2.2.1 Definition (Non-Rigorous intuitive definition) Let $y = f(x)$, and c be a real number. We say that

$$\lim_{x \to c} f(x) = L,$$

if $f(x)$ is close to L whenever x is sufficiently close to c. Equivalently,

$$|x - c| \quad \text{small} \quad \Rightarrow \quad |f(x) - L| \quad \text{small.}$$

The concept here is easy to understand intuitively, but it lacks mathematical rigor because it is not clear how "close" is "sufficiently close."

2.2.2 Example Find $\lim_{x\to 2}(2+(x+2)\cos(x+2))$.

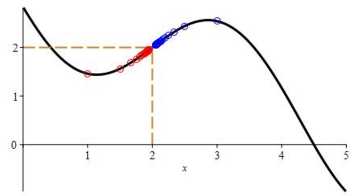


Fig. 2.4: Limit example

Chapter 3

Differentiation Rules

Chapter 4

Appendix

4.1 Maple Codes

4.1.1 Slope of Secants Numerically

The non-obvious part of this code is the syntax for formatting the string output of the printf command. In the argument %9.6f, the digit 9 is the spacing and the digit 6 is the number of decimals. The lines of a do-loop are separated by "Shift-Enter".

```
> restart;
> Digits := 12;
> with(plots);
> a:= 1.0: b:= 2.0: n:= 6:  h:= 0.1:
> f := x -> evalf(x^2);
> printf("\n x        m (sec)      x        m (sec)\n");
  printf("  ------------------------------------------\n");
  for i to n do
    X[i] := a+h^i;
    X2[i] := a-h^i;
    Y[i] := (f(X[i])-f(a))/h^i;
    Y2[i] := (f(X2[i])-f(a))/(-h^i);
  printf("%9.6f  %9.6f  %9.6f%9.6f\n",X[i],Y[i],X2[i],Y2[i]);
  end do;
```

```
 t            m (sec)      t            m (sec)
 ----------------------------------------------
 1.100000     2.100000     0.900000     1.900000
 1.010000     2.010000     0.990000     1.990000
 1.001000     2.001000     0.999000     1.999000
 1.000100     2.000100     0.999900     1.999900
 1.000010     2.000010     0.999990     1.999990
 1.000001     2.000000     0.999999     1.999999
```

4.1.2 Slope of Secants Graphically

Generic worksheet to plot a function on an interval $[0, b]$, the tangent at $t = a$ and n secants. Input function here is $f(t) = t^2$.

```
restart:
with(plottools): with(plots): with(student):
 a := 1:    b := 2:    n := 4:
 Delta x:=(b-a)/n :
 f:=t -> t^2 :
slope:= (a, y1, b, y2) -> (y2 - y1)/(b - a):
msec:= x -> slope(a, f(a), x, f(x)):
plotf:= plot(f(t), t = 0 .. b, color = black, thickness = 3):
tgt:=plot(f(a)+eval(diff(f(x),x),x = a)*(x - a),x = a..b,
          thickness=2):
plotSec:=plot([seq(f(a)+msec(a+i Delta x)(x - a),
               i=1..n)],x=a..b):
Points:= {}:
 for i  to  n  do:
    Points:= Points {\bf union}
    {pointplot ([a+i Delta x,f(a+i Delta x)],
         symbol=solidcircle, symbolsize=20)};
   od:
p0:=plot(0, a..b,color=black):
p2:=plot([t,f(a),t=a..b],color=black):
p3:=plot([b,t,t=f(a)..f(b)],color=black):
display(plotf, Points, tgt, plotSec, p0, p2, p3,
        tickmarks=[3, 6])
```