Notes on Elementary Differential Geometry

I. VECTORS.

1.1 Definition. Euclidean n-space $\mathbb{E}^n$ is defined as the set of ordered $n$tuples $\mathbf{p} = (p^1, \ldots, p^n)$ where $p^i \in \mathbb{R}$ for each $i = 1, \ldots, n$

In advanced calculus it is common to identify points in Euclidean space with ordinary vectors because the set of n-tuples as defined above, together the two operations

$$\mathbf{p} + \mathbf{q} = (p^1+q^1, \ldots, p^n+q^n) \qquad \mathbf{p}, \mathbf{q} \in \mathbb{E}^n$$

$$a\,\mathbf{p} = (ap^1, \ldots, ap^n) \qquad \mathbf{p} \in \mathbb{E}^n \quad a \in \mathbb{R}$$

has the structure of a vector space.

1.2 Definition. A real valued function in $\mathbb{E}^n$ is of class C^r if all the partial derivatives of the function up to (including) order r exist and are continuous. The space of infinitely differentiable functions will be denoted by $C^\infty(\mathbb{E}^n)$ or simply $\mathcal{C}^\infty$.

1.3 Definition. Let $\vec{X}_p$ be an ordinary (advanced calculus) vector at the point p and let f be a C^∞ function in an open neighborhood of p. We define the <u>directional</u> derivative $\mathbf{X}_p(f)$ by the equation

$$\mathbf{X}_p(f) = (\vec{\nabla} f)_p \cdot \vec{X}_p. \tag{1}$$

where "∇" is the gradient and "$\cdot$" is the usual dot product.

1.4 Exercise. Show that if $f, g \in C^\infty(\mathbb{E}^n)$ and $a, b \in \mathbb{R}$ then

i) $\mathbf{X}_p(af + bg) = a\mathbf{X}_p(f) + b\mathbf{X}_p(g)$

ii) $\mathbf{X}_p(f \cdot g) = f \cdot \mathbf{X}_p(g) + g \cdot \mathbf{X}_p(f)$

1.5) Definition. A <u>tangent vector</u> (or contravariant vector) at a point $p \in \mathbb{E}^n$ is a real valued function $\mathbf{X}$ on the space of C^∞ functions at p, which satisfies the following properties:

i) $X(af+bg) = aX(f) + bX(g)$

ii) $X(f \cdot g) = f \cdot X(g) + g \cdot X(f)$ (2)

The set of all tangent vectors at a point p in $\mathbb{E}^n$ has the structure of a vector space called the tangent space $T_p(\mathbb{E}^n)$ of $\mathbb{E}^n$ at p.

The definition of a tangent vector given in (1.5) must not be taken lightly; it is a big departure from the usual concept of a "calculus vector". In calculus, a vector is viewed as a set of ordered n-tuples. To a differential geometer, a vector is a linear functional, that is, a function whose inputs are functions and whose outputs are real number. More specifically, if one applies a vector X to a function f, the output is a real number which represents the directional derivative of f in the direction of X.

1.6 Definition. Let $\mathbf{p} \in \mathbb{E}^n$, $\mathbf{p} = (p^1, \ldots, p^n)$. We define the natural coordinate "slot" functions $x^i : \mathbb{E}^n \to \mathbb{R}$ by the requirement that

$$x^i(p^1, \ldots, p^n) = p^i \quad \text{for all } \mathbf{p} \in \mathbb{E}^n \tag{3}$$

1.7. Proposition. Let $X_p = \sum_{i=1}^{n} a^i \left(\frac{\partial}{\partial x^i}\right)_p$ where $a^i \in \mathbb{R}$ and $\left(\frac{\partial}{\partial x^i}\right)_p$ is the partial derivative operator at point p.

Define $X_p(f) = \sum_{i=1}^{n} a^i \left(\frac{\partial f}{\partial x^i}\right)_p \quad \forall f \in C^\infty(\mathbb{E}^n)$. Then X_p is a tangent vector.

Proof:

The proof is elementary. It suffices to show that X_p satisfies conditions (i) and (ii) of definition (1.5) but the conditions hold because of the linearity of partial derivatives and Leibnitz rule for the derivative of a product.

Notation. A. Einstein made a great discovery; he suppressed the sumation symbol $\sum_i$ whenever an expression contained a repeated index. From now on we will use Einstein's sumation convention whenever possible. Thus, for example, the vector defined in proposition 1.7 will be written

$$X_p = a^i \left(\frac{\partial}{\partial x^i}\right)_p = a^1 \frac{\partial}{\partial x^1} + \cdots + a^n \frac{\partial}{\partial x^n} \tag{4}$$

The fact that equation (4) defines a vector may make the novice in differential geometry a little uncomfortable, nonetheless, this is the correct way to define vectors. The quantities a^i in equation (4) are called the components of the vector and the "unit vectors" $X_i = (\partial/\partial x^i)_p$ $(i = 1, \dots n)$ form a <u>basis</u> for the tangent space $T_p(\mathbb{E}^n)$. In other words, the quantities $(\partial/\partial x^i)$ play a role in $\mathbb{E}^n$ analogous to the role of the unit vectors $\vec{i}, \vec{j},$ and $\vec{k}$ in $\mathbb{E}^3$. Thus, for example, the Euclidean vector

$$\vec{X}_p = (3\vec{i} + 4\vec{j} - 3\vec{k})_p = (3, 4, -3)_p$$

will be written in differential geometric notation as

$$X_p = \left(3\frac{\partial}{\partial x} + 4\frac{\partial}{\partial y} - 3\frac{\partial}{\partial z}\right)_p$$

and
$$X_p(f) = 3\left(\frac{\partial f}{\partial x}\right)_p + 4\left(\frac{\partial f}{\partial y}\right)_p - 3\left(\frac{\partial f}{\partial z}\right)_p$$
$$= \vec{X}_p \cdot (\vec{\nabla} f)_p.$$

1.8. Definition. A vector field X in $\mathbb{E}^n$ is a smooth choice of a tangent vector at each point $p \in \mathbb{E}^n$. A general vector field in $\mathbb{E}^n$ may always be written in the form

$$X = a^i(x)\frac{\partial}{\partial x^i} \tag{5}$$

where the a^i's are smooth (C^∞) functions in $\mathbb{E}^n$. The a^i's are called the <u>contravariant</u> <u>components</u> of X

Exercises.

1. Let $\vec{v} = (1, -2, 0)$ and $\vec{w} = (0, 3, -1)$. Write $\vec{v}$, $\vec{w}$ and $2\vec{v} - \vec{w}$ as linear combinations of $X_i(p) = (\partial/\partial x^i)_p$ for an arbitrary point $p \in \mathbb{E}^3$.

2. Let $X_p = 3\left(\frac{\partial}{\partial x}\right)_p + 2\left(\frac{\partial}{\partial y}\right)_p - \left(\frac{\partial}{\partial z}\right)_p$ and $f(x, y, z) = x e^{2yz}$. Compute $X_p(f)$ at the point $p = (-1, -1, 0)$

3. Consider the vector field in $\mathbb{E}^3$ given by $X = 3y\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$.

   a) What are the components of X_p at $p = (-5, 0, 2)$?
   b) Compute $X(f)$ of $f(x, y, z) = x^2 y - y^2 z$

4. Show that the vectors $V_p = x\left(\frac{\partial}{\partial x}\right)_p - \left(\frac{\partial}{\partial y}\right)_p$ and $W_p = \left(\frac{\partial}{\partial x}\right)_p + x\left(\frac{\partial}{\partial y}\right)_p$ are linearly independent at any point $p \in \mathbb{E}^2$.

II. FORMS.

One of the most puzzling ideas in calculus is the notion of the differential. Most calculus textbooks give meaningless definitions such as " $dx \equiv \Delta x$ as $\Delta x \to 0$." In fact, most physicists and many mathematicians go through life not really knowing what is meant by dx. In this section we introduce the machinery necessary to make this concept mathematically rigorous.

2.1 Definition. Let $p \in \mathbb{E}^n$ and let $T_p\mathbb{E}^n$ be the tangent space at p. A <u>one-form at p</u> is a linear map ϕ from $T_p\mathbb{E}^n$ into $\mathbb{R}$. In other words, ϕ is a map satisfying the following properties

a) $\phi(X_p) \in \mathbb{R} \quad \forall\, X_p \in T_p\mathbb{E}^n$

b) $\phi(aX_p + bY_p) = a\,\phi(X_p) + b\,\phi(Y_p) \qquad \forall\, a, b \in \mathbb{R},\ X_p, Y_p \in T_p\mathbb{E}^n$

Algebraically, the space of all one forms at p, is ~~is~~ simply the dual $T^*_p\mathbb{E}$ of the vector space $T_p\mathbb{E}$.(+) A <u>one-form</u> is a smooth choice of a map ϕ as above for each point $p \in \mathbb{E}^n$.

In standard books in calculus, the differential of a real valued function $f : \mathbb{E}^3 \to \mathbb{R}$ is defined by

$$df = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i}\, dx^i = \frac{\partial f}{\partial x^1}\, dx^1 + \frac{\partial f}{\partial x^2}\, dx^2 + \frac{\partial f}{\partial x^3}\, dx^3$$

$$= \frac{\partial f}{\partial x^i}\, dx^i \qquad (\text{Using Einstein's convention})$$

We have now the tools to make the definition above rigorous.

2.2 Definition. Let $f : \mathbb{E}^n \to \mathbb{R}$ be a real valued (C^∞) function. We define the differential df as the one form such that

$$df(X) = X(f) = \nabla f \cdot X \qquad 6)$$

2.3. Exercises

i) Show that $dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j$ (7a)

ii) Show that $df = \frac{\partial f}{\partial x^i}\, dx^i$ (7b)

iii) Show that $d(fg) = f\,dg + g\,df$ (7c)

(+) Consult a standard text in linear algebra for more details ~~about~~ the dual of a vector space.

Before going any further, the reader should make sure that Exercises 2.3 are well understood. In particular, equation (7a) states that the quantities dx^i form a basis for $T^*\mathbb{E}^n$ and that this basis is exactly the dual of the basis $X_i = \partial/\partial x^i$ of $T\mathbb{E}^n$. Thus, the differentials dx^i should be understood mathematically as linear maps whose inputs are vectors and whose outputs are real numbers. Any one-form in $\mathbb{E}^n$ can be written as a linear combination of the dx^i's. That is, any one form ϕ in $\mathbb{E}^n$ looks like this:

$$\phi = A_i(x)\, dx^i \tag{8}$$

One forms are sometimes also called <u>covariant tensors of rank one</u> or simply <u>covariant vectors</u> or just <u>covectors</u>

As we have already mentioned, the space of one forms at a point p has a natural vector space structure. The two operations of addition and scalar multiplication are defined in an obvious way, i.e

i) $(\alpha+\beta)(X) \equiv \alpha(X) + \beta(X)$

ii) $(f\alpha)(X) \equiv f\alpha(X) \qquad \forall\, \alpha,\beta \in T_p^*\mathbb{E}^n, \quad X \in T_p\mathbb{E}^n$ (9)

We now define a new operation between one forms, an operation which is closely related to the notion of the cross product of two Euclidean 3-vectors. The operation will be called the "<u>wedge product</u>." Roughly speaking, what we want is a type of multiplication which is antisymmetric and associative. The multiplication which will be denoted by the symbol "$\wedge$" will be defined in such a way that (for example)

$$dx^i \wedge dx^j = -\, dx^j \wedge dx^i \tag{10a}$$

and

$$dx^i \wedge (dx^j \wedge dx^k) = (dx^i \wedge dx^j) \wedge dx^k. \tag{10b}$$

Notice that (10a) implies that

$$dx^i \wedge dx^i = 0 \tag{11}$$

The formal definition is now given

2.4 Definition. Let α and β be one forms in $\mathbb{E}^n$. By the wedge product $\alpha \wedge \beta$ one means the bilinear map (linear on each slot) from $T\mathbb{E}^n \times T\mathbb{E}^n$ to $\mathbb{R}$ such that

$$(\alpha\wedge\beta)(X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X) \qquad \forall\, X,Y \in T\mathbb{E}^n.$$

2.5 Prop. Let α and β be one forms in $\mathbb{E}^n$. Then

$$\alpha \wedge \beta = -\beta \wedge \alpha \qquad (12)$$

Proof: Let X and Y be arbitrary vectors. Then

$$\begin{aligned}(\alpha\wedge\beta)(X,Y) &= \alpha(X)\beta(Y) - \alpha(Y)\beta(X)\\ &= -[\beta(X)\alpha(Y) - \beta(Y)\alpha(X)]\\ &= -[\beta\wedge\alpha](X,Y)\\ &= [-\beta\wedge\alpha](X,Y)\end{aligned}$$

Notice that (10a) indeed follows from the definition of the wedge product by taking $\alpha = dx^i$ and $\beta = dx^j$ in equation (12)

2.6 Prop. Let $\alpha = A_i dx^i$ and $\beta = B_i dx^i$ be arbitrary one forms in $\mathbb{E}^n$. Then

$$\alpha\wedge\beta = (A_i B_j)\, dx^i \wedge dx^j \qquad (13)$$

Proof. Let X and Y be arbitrary vectors. Then

$$\begin{aligned}(\alpha\wedge\beta)(X,Y) &= A_i dx^i(X)\, B_j dx^j(Y) - A_i dx^i(Y)\, B_j dx^j(X)\\ &= (A_i B_j)[dx^i(X)\,dx^j(Y) - dx^i(Y)\,dx^j(X)]\\ &= (A_i B_j)(dx^i\wedge dx^j)(X,Y)\end{aligned}$$

It is not necessary to memorize equation (13) to compute wedge products. One may just simply compute them using (10) and (11).

2.7 Examples

a) Let $\alpha = x^2 dx - y^2 dy$ and $\beta = dx + dy - 2xy\,dz$. Then

$$\begin{aligned}\alpha\wedge\beta &= (x^2dx - y^2dy)\wedge(dx + dy - 2xy\,dz)\\ &= x^2 dx\wedge dx + x^2 dx\wedge dy - 2x^3y\, dx\wedge dz - y^2 dy\wedge dx - y^2 dy\wedge dy + 2xy^3 dy\wedge dz\\ &= x^2\, dx\wedge dy - 2x^3y\, dx\wedge dz - y^2 dy\wedge dx + 2xy^3\, dy\wedge dz\\ &= (x^2+y^2)(dx\wedge dy) - 2x^3y\, dx\wedge dz + 2xy^3\, dy\wedge dz.\end{aligned}$$

b) Let

$$\begin{aligned}\alpha &= d(r\cos\theta) = -r\sin\theta\, d\theta + \cos\theta\, dr\\ \beta &= d(r\sin\theta) = r\cos\theta\, d\theta + \sin\theta\, dr\end{aligned}$$

then

$$\begin{aligned}\alpha\wedge\beta &= (-r\sin\theta\, d\theta + \cos\theta\, dr)\wedge(r\cos\theta\, d\theta + \sin\theta\, dr)\\ &= -r\sin^2\theta\, d\theta\wedge dr + r\cos^2\theta\, dr\wedge d\theta\\ &= (r\cos^2\theta + r\sin^2\theta)(dr\wedge d\theta)\\ &= r(dr\wedge d\theta).\end{aligned}$$

2.8 Definition. A two-form ϕ is a map from $T\mathbb{E}^n \times T\mathbb{E}^n$ which is:

a) bilinear (ie: linear in each slot)

b) $\phi(X,Y) = -\phi(Y,X) \quad \forall\, X,Y \in T\mathbb{E}^n$ (ie: The map is anti-symmetric)

2.9. Prop. If α and β are one-forms then $\alpha \wedge \beta$ is a two form.

Proof. The proof is elementary. First we show that $\alpha\wedge\beta$ is bilinear. Let $a_1, a_2 \in \mathbb{R}$ and $X_1, X_2, Y \in T\mathbb{E}^n$. Then

$$
\begin{aligned}
(\alpha\wedge\beta)(a_1X_1 + a_2X_2, Y) &= \alpha(a_1X_1 + a_2X_2)\beta(Y) - \alpha(Y)\beta(a_1X_1 + a_2X_2) \\
&= [a_1\alpha(X_1) + a_2\alpha(X_2)]\beta(Y) - \alpha(Y)[a_1\beta(X_1) + a_2\beta(X_2)] \\
&= a_1[\alpha(X_1)\beta(Y) - \alpha(Y)\beta(X_1)] + a_2[\alpha(X_2)\beta(Y) - \alpha(Y)\beta(X_2)] \\
&= a_1(\alpha\wedge\beta)(X_1, Y) + a_2(\alpha\wedge\beta)(X_2, Y).
\end{aligned}
$$

Linearity on the second slot is proved in essentially the same manner. From the definition (2.4) it follows that $(\alpha\wedge\beta)(X,Y) = -(\alpha\wedge\beta)(Y,X)$ by simply interchanging X and Y.

2.10. Remarks.

a) To get our feet back on the ground, we make think of two forms as quantities which resemble elements of area. Example 2.7b is a good illustration of this.

b) Wedge products and two forms are closely related to Jacobians of transformations in $\mathbb{E}^2$. Once again, example 2.7-b is a good illustration. The reader will recognize $\alpha\wedge\beta = dx\wedge dy = r\,dr\wedge d\theta$ as the area element in the plane.

c) Quantities such as $dx\,dy$ and $dy\,dz$ which often appear in calculus text books do not make any sense. In most cases, what it is meant by these gadgets is a wedge product of one forms.

d) We state (without proof) that all two forms ϕ in $\mathbb{E}^n$ are quantities which look like

$$\phi = \frac{1}{2!}F_{ij}\,dx^i\wedge dx^j. \tag{14}$$

In fact, in a more elementary (i.e: sloppier) treatment of this subject, one could define 2-forms as simply objects ϕ which look like the quantity in equation (14). This is what we will do in the next definition.

2.11 Definition: A three form ϕ in $\mathbb{E}^n$ is an object of the following type

$$\phi = \frac{1}{3!}A_{ijk}\,dx^i\wedge dx^j\wedge dx^k. \tag{15}$$

We challenge the reader to come up with a rigorous definition of 3-forms (and n-forms for that matter) in the spirit of (2.8). There is nothing wrong - mind you - with definition 2.11. It is just that the definition is coordinate dependent and mathematicians in general (specially differential geometers) prefer coordinate free definitions, theorems and proofs.

And now, a little combinatorics! (Yes, I know, I started a sentence with a preposition). Let us count the number of* n-forms in Euclidean space. More specifically, we want to count the dimension of the space of n-forms in the sense of vector spaces. (Think of 0-forms as being ordinary functions)

$\mathbb{E}^2$

	Forms	Dimension
0-forms	f	1
1-forms	$f\,dx,\ g\,dy$	2
2-forms	$f\,dx\wedge dy$	1
3-forms	none	0

$\mathbb{E}^3$

	Forms	Dimension
0-forms	f	1
1-forms	$f\,dx,\ g\,dy,\ h\,dz$	3
2-forms	$f\,dx\wedge dy,\ g\,dx\wedge dz,\ h\,dy\wedge dz$	3
3-forms	$f\,dx\wedge dy\wedge dz$	1
4-forms	none	0

Did you get the pattern?

Exercises

1. Let
$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi.$$
Compute $dx\wedge dy\wedge dz$

2. Let f and g be functions and X and Y be vectors in $\mathbb{E}^n$. Show that
$$(df\wedge dg)(X,Y) = X(f)\,Y(g) - X(g)\,Y(f)$$

* Clearly we mean linearly independent n-forms

(Cont)

3. Let α and β be one-forms. Show that

a) $(\alpha\wedge\beta)(aX+bY, cX+dY) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} (\alpha\wedge\beta)(X,Y) \quad \forall\, a,b \in \mathbb{R}$

b) Show that if ϕ is any two form then

$$\phi(aX+bY, cX+dY) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \phi(X,Y)$$

4. Let $\alpha^i = A^i_j\, dx^j \quad (1 \le i \le 3)$. Show that

a) $\alpha^1\wedge\alpha^2\wedge\alpha^3 = \det|A^i_j|\, dx^1\wedge dx^2\wedge dx^3$.

b) Let $x^i = f^i(y^j) \quad (1 \le i,j \le 3)$. Show that

$$dx^1\wedge dx^2\wedge dx^3 = \det|J|\, dy^1\wedge dy^2\wedge dy^3$$

where J is the Jacobian of the transformation.

5. Give a coordinate free definition of a 3-form in $\mathbb{E}^n$

6. Show that the dimension of the vector space of m-forms in $\mathbb{E}^n$ $(m \le n)$ is $\binom{n}{m}$.

7. Let $\phi = F_{ij}\, dx^i\wedge dx^j$ be any two form. Define $F_{[ij]} = \frac{1}{2}(F_{ij} - F_{ji})$
Show that

$$\phi = F_{[ij]}\, dx^i\wedge dx^j.$$

In other words, one may assume that the components F_{ij} of an arbitrary 2-form are antisymmetric.

* 8) Generalize the statement and proofs of the results in problems 4 and 7 to n-dimensions

$\alpha_1\wedge\alpha_2\wedge\alpha_3 = \sum (-1)^{\operatorname{sgn}\pi}\, \alpha_{\pi_1}\otimes\alpha_{\pi_2}\otimes\alpha_{\pi_3}$ ([illegible]) Abraham Marsden

$\alpha \in \Lambda^p, \quad \beta\in\Lambda^q$

$(\alpha\wedge\beta)(X_1, \dots X_p, X_{p+1}, \dots X_{p+q}) = \sum_{\text{shuffle}} (-1)^\pi \alpha(X_{\pi_1}, \dots X_{\pi_p})\, \beta(X_{\pi_{p+1}}, \dots X_{\pi_{p+q}})$

$\alpha\wedge\beta = \frac{(p+q)!}{p!\,q!} A(\alpha\otimes\beta)$

III Exterior Derivatives.

In this section we introduce a differential operator which generalizes the classical gradient, curl and divergence operators.

Denote by $\Lambda^m_p(\mathbb{E}^n)$ the vector space of m-forms at $p \in \mathbb{E}^n$. This vector space has dimension $\binom{n}{m}$ for $m \le n$ and dimension zero for $m > n$. We shall identify $\Lambda^0_p(\mathbb{E}^n)$ with the space of C^∞ functions at p. Also, we will call $\Lambda^m(\mathbb{E}^n)$ the union of all $\Lambda^m_p(\mathbb{E}^n)$ as p ranges through all the points in $\mathbb{E}^n$. In other words, we have $\Lambda^m = \bigcup_p \Lambda^m_p$.

If $\alpha \in \Lambda^m(\mathbb{E}^n)$ the α can be written as

$$\alpha = A_{i_1 \cdots\cdots i_m}(x)\, dx^{i_1} \wedge \cdots\cdots \wedge dx^{i_m}. \qquad (i_1 < i_2 < \cdots < i_m) \qquad (16)$$

3.1. **Definition.** Let α be an m-form (written in coordinates as in equation 16). The <u>exterior derivative</u> of α is the $(m+1)$-form $d\alpha$ given by

$$d\alpha = \frac{\partial A_{i_1 \cdots\cdots i_m}(x)}{\partial x_{i_0}}\, dx^{i_0} \wedge dx^{i_1} \wedge \cdots\cdots \wedge dx^{i_m} \qquad (17)$$

In the special case when α is a 0-form (ie: a function f) we write

$$df = \frac{\partial f}{\partial x^i}\, dx^i \qquad (18)$$

3.2 **Prop:**

a) $d : \Lambda^m \longrightarrow \Lambda^{m+1}$ (19a)

b) $d^2 = d \circ d = 0$ (19b)

c) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \qquad \forall \alpha \in \Lambda^p,\ \beta \in \Lambda^q.$ (19c)

Proof.

a) Obvious from equation (17)

b) First we prove the proposition for $\alpha = f \in \Lambda^0$. Then

$$d(d\alpha) = d\left(\frac{\partial f}{\partial x^i}\, dx^i\right) = \frac{\partial^2 f}{\partial x^j \partial x^i}\, dx^j \wedge dx^i$$

$$= \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j}\right] dx^j \wedge dx^i \qquad \text{(see Ex. 7 in the previous section)}$$

$$= 0$$

Now suppose that α is of the form in eq (16). Observe that (17) can then be written as

$$d\alpha = dA_{i_1 \cdots\cdots i_m} \wedge dx^{i_1} \wedge \cdots\cdots \wedge dx^{i_m}$$

So,

$$d(d\alpha) = d(dA_{i_1 \cdots\cdots i_m}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_m} = 0$$

(Cont)

c) Let $\alpha \in \Lambda^p$ and $\beta \in \Lambda^q$. Then we can write

$$\alpha = A_{i_1 \cdots\cdots i_p}\, dx^{i_1} \wedge \cdots\cdots dx^{i_p} \quad , \quad \beta = B_{j_1 \cdots\cdots j_q}\, dx^{j_1} \wedge \cdots\cdots dx^{j_q}$$

By definition

$$\alpha \wedge \beta = A_{i_1 \cdots i_p} B_{j_1 \cdots\cdots j_q}\, (dx^{i_1} \wedge \cdots\cdots \wedge dx^{i_p}) \wedge (dx^{j_1} \wedge \cdots\cdots \wedge dx^{j_q})$$

Now we take $\mathbf{d}$ of the last equation and use the fact that for functions $d(fg) = f\,dg + g\,df$. We get:

$$d(\alpha\wedge\beta) = \left[(dA_{i_1 \cdots i_p}) B_{j_1 \cdots j_q} + A_{i_1 \cdots i_p}(dB_{j_1 \cdots j_q})\right] \wedge (dx^{i_1} \wedge \cdots \wedge dx^{i_p}) \wedge (dx^{j_1} \wedge \cdots \wedge dx^{j_q})$$

$$= \left[dA_{i_1 \cdots i_p} \wedge (dx^{i_1} \wedge \cdots\cdots \wedge dx^{i_p})\right] \wedge \left[B_{j_1 \cdots j_q}\, dx^{j_1} \wedge \cdots\cdots \wedge dx^{j_q}\right] +$$

$$\left[A_{i_1 \cdots i_p}\, dx^{i_1} \wedge \cdots\cdots \wedge dx^{i_p}\right] \wedge (-1)^p \left[dB_{j_1 \cdots j_q} \wedge dx^{j_1} \wedge \cdots\cdots \wedge dx^{j_q}\right]$$

The $(-1)^p$ factor comes in because to pass the term $dB_{j_1 \cdots j_q}$ through p one forms of the type dx^i we have to alternate signs p-times. It remains only to observe that the last equation is just what we want.

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p\, \alpha \wedge d\beta .$$

3.3. Examples.

a) Let $\alpha = P(x,y)\,dx + Q(x,y)\,dy$. Then

$$d\alpha = \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy\right) \wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy\right) \wedge dy$$

$$= \frac{\partial P}{\partial y}\, dy \wedge dx + \frac{\partial Q}{\partial x}\, dx \wedge dy$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy .$$

Does this ring a bell in connection with Green's theorem in the plane?

b) Let $\alpha = M(x,y)\,dx + N(x,y)\,dy$, and suppose that $d\alpha = 0$. What can we conclude about M, N and α?

Sol. Since $d\alpha = 0$ then, by the previous example we have that:

$d\alpha = (N_x - M_y)\, dx \wedge dy = 0 \Rightarrow N_x = M_y$. So, it follows that $N = f_y$ and $M = f_x$ for some C^2 function f.

Hence,

$\alpha = f_x\,dx + f_y\,dy = df$.

Does this ring a bell in connection with exact differential equations of first order?

Example 3.3b is a special case of a more general result known as Poincaré's Lemma. To state the Lemma we need a need a short definition

3.4. Definition. A differential form α is called <u>exact</u> if $d\alpha = 0$. A differential form α is called <u>closed</u> if $\alpha = d\beta$ for some β. Clearly if α is closed then it is exact. The converse is not at all obvious

3.5 Theorem. (Poincaré's Lemma). If a differential form is exact, then it is closed.

The proof is too involved to be included in these elementary notes. It must be stated for the sake of rigour that Poincaré's Lemma assumes that the space is simply connected. The theorem is closely related to the weak version of Cauchy's theorem for analytic functions in the plane if one thinks of $f(z)\,dz$ as a complex one form.

<u>The Hodge "*" Operator</u>.

One of the important lessons that students of linear algebra should always learn (but often miss) is that all vector spaces of the same (finite) dimension are isomorphic to each other. That is, they all, in some sense, look like each other and like a Euclidean vector space of the same dimension. As the reader may have already observed, we have encountered this situation a couple of times in these notes. For instance, it is the case that $T_p\mathbb{E}^3$ is three dimensional and hence it is isomorphic to Euclidean 3-space. The isomorphism is given by the command: "Replace $\frac{\partial}{\partial x}$ by $\vec{i}$, $\frac{\partial}{\partial y}$ by $\vec{j}$ and $\frac{\partial}{\partial z}$ by $\vec{k}$.

We have also observed that $T_p\mathbb{E}^n$ is isomorphic to its dual $T_p^*\mathbb{E}^n$ (a fact which is true for any vector space). The isomorphism in this case is given by the map $\partial/\partial x^i \longmapsto dx^i$. This map transforms a contravariant vector into a covariant one.

A much more subtle isomorphism must exist amongst spaces of m-forms in $\mathbb{E}^n$. It has already been pointed out, for example, that $\Lambda^1_p(\mathbb{E}^3)$ and $\Lambda^2_p(\mathbb{E}^3)$ have the same dimensionality (see end of section II) and hence they must be isomorphic. In fact, since the dimension of $\Lambda^m_p(\mathbb{E}^n)$ is given by the binomial coefficient $\binom{n}{m}$ and since it is ~~true~~ that $\binom{n}{m} = \binom{n}{n-m}$, then it must be the case that

$$\Lambda^m_p(\mathbb{E}^n) \text{ is isomorphic to } \Lambda^{n-m}_p(\mathbb{E}^n).$$

To describe the isomorphism, we will need to introduce the totally antisymmetric Levi-Civita permutation symbol which is defined as follows:

$$\epsilon_{i_1\cdots\cdots i_n} = \begin{cases} 1 & \text{if } (i_1,\cdots,i_n) \text{ is an even permutation of } (1,\ldots,n) \\ -1 & \text{if } (i_1,\cdots,i_n) \text{ is an odd permutation of } (1,\ldots\ldots,n) \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Example. In $\mathbb{E}^3$ the only non vanishing components of ϵ_{ijk} are

$$\begin{aligned} \epsilon_{123} &= \epsilon_{231} = \epsilon_{312} = 1 \\ \epsilon_{132} &= \epsilon_{213} = \epsilon_{321} = -1 \end{aligned} \tag{21}$$

The permutation symbols are extremely useful quantities. For example, they may be used to define determinants of matrices. In fact, if $A = (A^i{}_j)$ is a 3×3 matrix, then the equation

$$\det A = |A| = \epsilon_{ijk} A^i{}_1 A^j{}_2 A^k{}_3 \tag{22}$$

is true in view of equations (21). A more thorough discussion of the Levi-Civita symbols will be given later

3.6 Definition. The Hodge $*$-operator is the map $* : \Lambda^m_p(\mathbb{E}^n) \longrightarrow \Lambda^{n-m}_p(\mathbb{E}^n)$ defined in coordinates as follows

$$*(dx^{i_1}\wedge\ldots\ldots\wedge dx^{i_m}) = \frac{1}{(n-m)!}\,\epsilon^{i_1\cdots\cdots i_m}{}_{i_{m+1}\cdots\cdots i_n}\, dx^{i_{m+1}}\wedge\cdots\cdots\wedge dx^{i_n} \tag{23}$$

(Note: This definition will be slightly revised when we introduce a metric)

In the equation above, the ϵ-symbol with upper indices has the same components as the ϵ with the indices down. The $*$-operator is a linear operator, so equation (23) is all one needs to completely specify the map, in view of the fact that the forms $dx^{i_1}\wedge\ldots\wedge dx^{i_m}$ are a basis of the vector space $\Lambda^m_p(\mathbb{E}^n)$.

3.7 Examples:

a) Take $n=3$. (Euclidean 3-space $\mathbb{E}^3$)

$$\begin{aligned} *\,dx^1 &= \frac{1}{2!}\,\epsilon^1{}_{jk}\,dx^j\wedge dx^k = \frac{1}{2!}\left[\epsilon^1{}_{23}\,dx^2\wedge dx^3 + \epsilon^1{}_{32}\,dx^3\wedge dx^2\right] \\ &= \frac{1}{2!}\left[dx^2\wedge dx^3 - dx^3\wedge dx^2\right] \\ &= \frac{1}{2!}\left[dx^2\wedge dx^3 + dx^2\wedge dx^3\right] \end{aligned}$$

Hence

$$\begin{aligned} *\,dx^1 &= dx^2\wedge dx^3 \\ \text{Also}\quad *\,dx^2 &= -dx^1\wedge dx^3 \\ *\,dx^3 &= dx^1\wedge dx^2 \end{aligned} \tag{24}$$

(Cont)

b) Let $f : \mathbb{E}^n \to \mathbb{R}$ be any 0-form. Then

$$
\begin{aligned}
*f &= f\, dx^1 \wedge \cdots\cdots \wedge dx^n \\
&= f\, dV
\end{aligned}
\tag{25}
$$

where the form dV defined in the obvious way, is called the volume form.

c) Let $\alpha = A_1 dx^1 + A_2 dx^2 + A_3 dx^3$ and $\beta = B_1 dx^1 + B_2 dx^2 + B_3 dx^3$.
Then

$$
\begin{aligned}
(\alpha \wedge \beta) &= (A_2B_3 - A_3B_2)(dx^2 \wedge dx^3) + (A_1B_3 - B_3A_1)*(dx^1 \wedge dx^3) + \\
&\quad (A_1B_2 - A_2B_1)*(dx^1 \wedge dx^2) \\
&= (A_2B_3 - A_3B_2)dx^1 - (A_1B_3 - B_3A_1)dx^2 + (A_1B_2 - A_2B_1)dx^3 \\
&= (\vec{A} \times \vec{B})_i\, dx^i .
\end{aligned}
\tag{26}
$$

The previous examples should give the reader a feeling of what goes on. If one thinks of the quantities dx^1, dx^2 and dx^3 as playing the role of $\vec{i}$, $\vec{j}$ and $\vec{k}$ then we see that equations (24) are just the differential geometer's version of the well know equations $\vec{i} = \vec{j} \times \vec{k}$, $\vec{j} = -\vec{i} \times \vec{k}$ and $\vec{k} = \vec{i} \times \vec{j}$. This makes more sense after a close inspection of equation (26) which relates the $*$-map and the wedge product to the cartesian notion of the cross product.

Gradient, Curl and Divergence.

Classical vector analysis is based on the properties of the $*$-operator and the wedge in Euclidean 3-space. We will now show precisely how this works.

1. Let $f : \mathbb{E}^3 \to \mathbb{R}$ be a C^∞ function. Then

$$
df = \frac{\partial f}{\partial x^i} dx^i = \vec{\nabla} f \cdot \vec{dx}
\tag{27}
$$

2. Let $\alpha = A_i dx^i$ be a one form in $\mathbb{E}^3$. Then

$$
\begin{aligned}
(*d)\alpha &= \tfrac{1}{2}\left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x^i}\right) *(dx^i \wedge dx^j) \\
&= (\vec{\nabla} \times \vec{A}) \cdot \vec{dx}
\end{aligned}
\tag{28}
$$

3. Let $\alpha = F_1\, dx^2 \wedge dx^3 + F_2\, dx^3 \wedge dx^1 + F_3\, dx^1 \wedge dx^2$. Then

$$
\begin{aligned}
d\alpha &= \left(\frac{\partial F_1}{\partial x^1} + \frac{\partial F_2}{\partial x^2} + \frac{\partial F_3}{\partial x^3}\right) dx^1 \wedge dx^2 \wedge dx^3 \\
&= (\vec{\nabla} \cdot \vec{F})\, \overline{dV}
\end{aligned}
\tag{29}
$$

3'. Let $\alpha = F_i \, dx^i$, Then

$$*d*\alpha = \vec{\nabla} \cdot \vec{F} \tag{29'}$$

We shall think of the three examples above as th giving the definitions of **grad**, **curl** and **div**.

<u>Gauss-Maxwell Equations.</u>

The classical equations of Maxwell describing electromagnetic phenomena as

$$\begin{aligned} \vec{\nabla}\cdot\vec{E} &= 4\pi\rho & \qquad \vec{\nabla}\times\vec{B} &= 4\pi\vec{J} + \partial\vec{E}/\partial t \\ \vec{\nabla}\cdot\vec{B} &= 0 & \qquad \vec{\nabla}\times\vec{E} &= -\partial\vec{B}/\partial t \end{aligned} \tag{30}$$

Our task is to cast these equations into the language of forms.
Define the Maxwell 2-form F by the equation

$$F = \tfrac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu \qquad (\mu, \nu = 0, 1, 2, 3)$$

where

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \qquad \begin{aligned} \vec{E} &= (E_x, E_y, E_z) \\ \vec{B} &= (B_x, B_y, B_z) \end{aligned} \tag{31}$$

and $dx^0 = dt$, $dx^1 = dx$, $dx^2 = dy$, $dx^3 = dz$.

Written out in complete detail, the Maxwell 2-form is given by

$$\begin{aligned} F = {} & E_x \, dt \wedge dx^1 + E_y \, dt \wedge dx^2 + E_z \, dt \wedge dx^3 + \\ & B_z \, dx^1 \wedge dx^2 - B_y \, dx^1 \wedge dx^3 + B_x \, dx^2 \wedge dx^3 \end{aligned} \tag{32}$$

We also define the current source one form

$$J = J_\mu \, dx^\mu \equiv \rho \, dt + J_1 \, dx^1 + J_2 \, dx^2 + J_3 \, dx^3 \tag{33}$$

3.8. Prop. Maxwell equations (30) are equivalent to the equations

$$\begin{aligned} dF &= 0 \\ d*F &= 4\pi * J. \end{aligned} \tag{34}$$

<u>Proof</u>. We could save some time by using eqs (27-29) but we won't. Instead, we will use the definitions of "d" and "*" directly.

$$
\begin{aligned}
dF = {} & \frac{\partial E_x}{\partial x^2}\, dx^2 \wedge dt \wedge dx^1 + \frac{\partial E_x}{\partial x^3}\, dx^3 \wedge dt \wedge dx^1 + \\
& \frac{\partial E_y}{\partial x^1}\, dx^1 \wedge dt \wedge dx^2 + \frac{\partial E_y}{\partial x^3}\, dx^3 \wedge dt \wedge dx^2 + \\
& \frac{\partial E_z}{\partial x^1}\, dx^1 \wedge dt \wedge dx^3 + \frac{\partial E_z}{\partial x^2}\, dx^2 \wedge dt \wedge dx^3 + \\
& \frac{\partial B_z}{\partial t}\, dt \wedge dx^1 \wedge dx^2 + \frac{\partial B_z}{\partial x^3}\, dx^3 \wedge dx^1 \wedge dx^2 + \\
& -\left(\frac{\partial B_y}{\partial t}\, dt \wedge dx^1 \wedge dx^3 + \frac{\partial B_y}{\partial x^2}\, dx^2 \wedge dx^1 \wedge dx^3\right) + \\
& \frac{\partial B_x}{\partial t}\, dt \wedge dx^2 \wedge dx^3 + \frac{\partial B_x}{\partial x^1}\, dx^1 \wedge dx^2 \wedge dx^3 .
\end{aligned}
$$

Collecting similar terms and using the antisymmetry of "$\wedge$" we get

$$
\begin{aligned}
dF = {} & -\left(\frac{\partial E_z}{\partial x^2} - \frac{\partial E_y}{\partial x^3} + \frac{\partial B_x}{\partial t}\right) dt \wedge dx^2 \wedge dx^3 + \\
& \left(\frac{\partial E_x}{\partial x^3} - \frac{\partial E_z}{\partial x^1} - \frac{\partial B_y}{\partial t}\right) dt \wedge dx^1 \wedge dx^3 + \\
& \left(\frac{\partial E_x}{\partial x^2} - \frac{\partial E_y}{\partial x^1} + \frac{\partial B_z}{\partial t}\right) dt \wedge dx^1 \wedge dx^2 + \\
& \left(\frac{\partial B_x}{\partial x^1} + \frac{\partial B_y}{\partial x^2} + \frac{\partial B_z}{\partial x^3}\right) dx^1 \wedge dx^2 \wedge dx^3 = 0
\end{aligned}
$$

Hence:

$$
\left.
\begin{aligned}
\frac{\partial E_z}{\partial x^2} - \frac{\partial E_y}{\partial x^3} &= -\frac{\partial B_x}{\partial t} \\
-\left(\frac{\partial E_x}{\partial x^3} - \frac{\partial E_z}{\partial x^1}\right) &= -\frac{\partial B_y}{\partial t} \\
\frac{\partial E_x}{\partial x^2} - \frac{\partial E_y}{\partial x^1} &= -\frac{\partial B_z}{\partial t}
\end{aligned}
\right\} \Rightarrow \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}
$$

and:

$$
\left.\frac{\partial B_x}{\partial x^1} + \frac{\partial B_y}{\partial x^2} + \frac{\partial B_z}{\partial x^3} = 0 \right\} \Rightarrow \quad \vec{\nabla} \cdot \vec{B} = 0 .
$$

So, the equation $dF = 0$ yields the first two Maxwell equations. We leave it to the reader to verify that $d{*}F = 4\pi {*}J$ yields the other two. (Ex. 5)

Note that since F is a closed form, then by Poincaré's lemma, in a simply connected region we have

$$
F = dA \quad \text{for some one form } A . \tag{35}
$$

The one form A is called the potential. If we write the potential as

$$
A = A_\mu\, dx^\mu
$$

we have

$$
F_{\mu\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \tag{36}
$$

The treatment of the various concepts of vector calculus given so far can also be given in a more classical fashion with the help of the permutation symbols.

For example, let $\vec{A} = \langle A_1, A_2, A_3 \rangle$ and $\vec{B} = \langle B_1, B_2, B_3 \rangle$ be any two Euclidean vectors. It follows from equations (23) and (26) that

$$(\vec{A} \times \vec{B})_k = \epsilon^{ij}{}_k A_i B_j$$

and

$$(\vec{\nabla} \times \vec{A})_k = \epsilon^{ij}{}_k \frac{\partial A_j}{\partial x^i}$$

. The classical divergence may also be written in a simple way

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_i}{\partial x^i} .$$

In fact, using the equations above, together with the easy to verify relation (see Ex. 3)

$$\epsilon^{ijm} \epsilon_{klm} = \delta^i_k \delta^j_l - \delta^i_l \delta^j_k , \tag{37}$$

one can derive most of the classical vector relations.

3.9 Example.

$$[A \times (B \times C)]_l = \epsilon^{mn}{}_l A_m (B \times C)_n$$

$$= \epsilon^{mn}{}_l A_m (\epsilon^{jk}{}_n B_j C_k)$$

$$= \epsilon^{mn}{}_l \epsilon^{jk}{}_n A_m B_j C_k$$

$$= \epsilon_{mnl} \epsilon^{jkn} A^m B_j C_k$$

$$= (\delta^j_l \delta^k_m - \delta^k_l \delta^j_m) A^m B_j C_k$$

$$= B_l A^m C_m - C_l A^m B_m$$

or

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

Exercises.

1. Compute the exterior derivative of the forms α where
   a) $\alpha = yz\,dx + x^2 dy$
   b) $\alpha = x^2 y\, dx\wedge dy + z^2 dx\wedge dz + (x+y)\, dy\wedge dz$

2. Let $\alpha = \dfrac{x\,dx + y\,dy}{x^2+y^2}$. a) Show that in a simply connected region not containing the origin α is exact
   b) Find f such that $\alpha = df$.

3. Show that $\epsilon^{ijk}\epsilon_{ijk} = 3!$ and $\epsilon^{ijm}\epsilon_{klm} = \delta^i_k\delta^j_l - \delta^i_l\delta^j_k$.

4. Using the fact that $d^2 = 0$ show that in Euclidean 3-space
   a) $\vec{\nabla}\times\vec{\nabla}f = 0$
   b) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$

5. Show that $d{*}F = 4\pi{*}J$ is equivalent to the other two Maxwell equations

6. Let $\vec{A} = (A_1, A_2, A_3)$, $\vec{B} = (B_1, B_2, B_3)$, $\vec{C} = (C_1, C_2, C_3)$. Show that

$$\epsilon^{ijk} A_i B_j C_k = \vec{A}\cdot(\vec{B}\times\vec{C})$$

7. Using the methods of this chapter prove the following well known vector identities
   a) $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A}$
   b) $\vec{\nabla}\cdot(f\vec{A}) = f\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}f$
   c) $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$
   d) $\vec{\nabla}\times(f\vec{A}) = f\vec{\nabla}\times\vec{A} - \vec{A}\times\vec{\nabla}f$.

8. Show that $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = \begin{vmatrix} \vec{A}\cdot\vec{C} & \vec{B}\cdot\vec{C} \\ \vec{A}\cdot D & \vec{B}\cdot\vec{D} \end{vmatrix}$

9. Given F as in (31) compute ${*}(F\wedge{*}F)$ and ${*}(F\wedge F)$

10. Show that ${*}{*}\alpha = (-1)^{p(n-p)}\alpha$ for any $\alpha\in\Lambda^p(\mathbb{E}^n)$

IV TENSORS.

Tensors are physical quantities which undergo certain special transformations under changes of coordinates. Most physicists define tensors as quantities which transform like tensors. Although this kind of definition is somewhat ackward for the pure mathematician, it is nontheless the easiest and most intuitive way to introduce tensors; this is, therefore, the approach that we take in these notes.

Contravariant Tensors.

Consider a tangent vector X whose components in a coordinate system $(x^1, \ldots x^n)$ in $\mathbb{E}^n$ at a point P are given by

$$X = V^i(x) \frac{\partial}{\partial x^i}\Big|_P .$$

We would like to investigate what happens to the components $V^i(x)$ of the vector if the coordindates x^i undergo a typical coordinate transformation of the form(1)

$$\bar{x}^i = f^i(x^j) , \qquad x^i = g^i(\bar{x}^j). \tag{38}$$

The answer is simple. Using the chain rule of partial differentiation we have

$$\frac{\partial}{\partial x^i}\Big|_P = \frac{\partial \bar{x}^k}{\partial x^i} \cdot \frac{\partial}{\partial \bar{x}^k}\Big|_P \tag{39}$$

Hence, in the new coordinate system the vector X has components

$$\begin{aligned} X &= V^i(x(\bar{x})) \cdot \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial}{\partial \bar{x}^k}\Big|_P \\ &= \bar{V}^k(\bar{x}) \frac{\partial}{\partial \bar{x}^k}\Big|_P \end{aligned}$$

where

$$\bar{V}^k = V^i \frac{\partial \bar{x}^k}{\partial x^i} . \tag{40.}$$

In short, the components of the vector transform with the Jacobian matrix. A set of quantities which transforms as in equation (40) is called a contravariant vector of rank one. More generally, we have

4.1 Definition. A set of quantities $T^{i_1 \ldots i_m}$ is called a contravariant tensor of rank m if under coordi-

(1) Assume $\det|J| \neq 0$.

nate changes as in (38), the quantities transform according to

$$\bar{T}^{k_1 \cdots k_n} = T^{i_1 \cdots i_n} \frac{\partial \bar{x}^{k_1}}{\partial x^{i_1}} \cdots\cdots \frac{\partial x^{k_n}}{\partial x^{i_n}} \tag{41}$$

*Note. Mathematically inclined readers should be aware that what a physicist calls a contravariant tensor is really the quantity consisting of the components of a tensor. In more mathematical terms a rank p tensor T is a multilinear map $T: T^*(\mathbb{E}^n) \times \overset{(p\text{-times})}{\cdots\cdots} \times T^*(\mathbb{E}^n) \longrightarrow \mathbb{R}$. If T_1 is a rank-p tensor and T_2 a rank-q tensor, then the tensor product is a rank-p+q tensor defined by

$(T_1 \otimes T_2)(\alpha^1, \ldots \alpha^p, \beta^1, \ldots, \beta^q) = T_1(\alpha^1, \ldots, \alpha^p)\, T_2(\beta^1, \ldots, \beta^q)$, $\alpha, \beta \in T^*(\mathbb{E}^n)$.

Any rank-p tensor T may be expressed in local coordinates as

$T = T^{i_1 \cdots i_p}\, \partial/\partial x^{i_1} \otimes \cdots\cdots \otimes \partial/\partial x^{i_p}$. The $T^{i_1 \cdots i_p}$'s are called the contravariant components of the tensor. Covariant tensors can be defined in a similar way. A covariant tensor of rank-p locally looks like this:

$T = T_{i_1 \cdots i_p}\, dx^{i_1} \otimes \cdots\cdots \otimes dx^{i_p}$.

We point out that equation (40) represents a linear transformation. In other words, a change of coordinates in Euclidean space induces a linear transformation on the tangent space. The linear transformation is represented by the Jacobian matrix. In a similar way, we may think of equation (41) as representing a multilinear transformation in the sense that it is linear in each index. For each index, the matrix representing the transformation is once again the Jacobian.

The collection of tensors of rank p at a point in $\mathbb{E}^n$ has the structure of a vector space. A tensor field is a smooth choice of a tensor at each point of $\mathbb{E}^n$.

Covariant Tensors.

Consider now a one form $\alpha = A_i dx^i$ and once again we perform a coordinate transformation as in (38). Then

$$\begin{aligned}\alpha &= A_i dx^i \\ &= A_i \frac{\partial x^i}{\partial \bar{x}^k} d\bar{x}^k \\ &= \bar{A}_k d\bar{x}^k\end{aligned}$$

where

$$\bar{A}_k = A_i \frac{\partial x^i}{\partial \bar{x}^k} \tag{42}$$

A quantity which transforms according to (42) is called a covariant tensor of rank 1.

Generalizing, a covariant tensor of rank p is a set of quantities $A_{i_1 \cdots i_p}$ which transforms according to

$$\bar{A}_{k_1 \ldots k_p} = A_{i_1 \ldots i_p} \frac{\partial x^{i_1}}{\partial \bar{x}^{k_1}} \cdots \frac{\partial x^{i_p}}{\partial \bar{x}^{k_p}}. \quad (43)$$

The most elementary example of a rank-2 covariant tensor is the metric tensor which represents the square of the differential of arc length. We will discuss this important tensor in some detail.

We recall that in Euclidean space the square of the arc length differential in Cartesian coordinates is given by

$$ds^2 = \sum_{i=1}^{m} dx^i \cdot dx^i = \delta_{ij}\, dx^i dx^j \quad (44a)$$

In cylindrical coordinates the formula is

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2 = g_{ij}\, dx^i dx^j \quad (44b)$$

where $x^i = (r, \theta, z)$ and $g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

In spherical coordinates the formula is

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin\theta\, d\varphi^2 = g_{ij}\, dx^i dx^j \quad (44c)$$

where

$x^i = (\rho, \theta, \varphi)$ and $g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2 \sin\theta \end{bmatrix}$.

In general, quantities of the form[1]

$$ds^2 = g_{ij}\, dx^i dx^j \quad (45)$$

where the g_{ij}'s are given by symmetric matrices, such as those in equations 44, are called metrics. The quantities g_{ij} so defined represent the components of a (symmetric) covariant rank-2 tensor as we now demonstrate.

Let $x^i = x^i(\bar{x}^k)$, then

$$ds^2 = g_{ij} \left(\frac{\partial x^i}{\partial \bar{x}^k} d\bar{x}^k\right)\left(\frac{\partial x^j}{\partial \bar{x}^l} d\bar{x}^l\right)$$

$$= g_{ij} \frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^l} d\bar{x}^k d\bar{x}^l$$

$$= \bar{g}_{kl}\, d\bar{x}^k d\bar{x}^l$$

where

$$\bar{g}_{kl} = g_{ij} \frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^l} \quad (46)$$

Thus, g_{ij} transforms like a tensor and hence it is a tensor.

It is standard notation to denote by g^{ij} the components of the inverse matrix of g_{ij}, that is

$$g^{ik} g_{jk} = \delta^i_j. \quad (47)$$

1) Technically eq. (45) should be written $ds^2 = g_{ij}\, dx^i \otimes dx^j$

In the special case when $g_{ij} = \delta_{ij}$ we may think of the metric as the geometric object which represents the standard inner product of Euclidean vectors in the following sense

$$\langle \vec{x}, \vec{y} \rangle = g_{ij} x^i y^j = \delta_{ij} x^i x^j$$

In the more general case when g_{ij} is an arbitrary symmetric matrix, the quantity $g_{ij} x^i y^j$ is called a quadratic form.

One may also think of the metric as a map which takes covariant vectors into contravariant ones and vice-versa. The mechanism to get such map is obtained by using the g_{ij}'s to raise and lower indices according to the following formulas

$$g^{ij} A_j = A^i$$
$$A_i \, g^{ij} = A^j \qquad (48)$$

Notice that the covariant and contravariant components of a vector in Euclidean space with the standard metric $g_{ij} = \delta_{ij}$ are identical (i.e $A^i = \delta^{ij} A_j = A_i$). In this case, the tensors are called cartesian tensors and it makes no difference whether the indices are up or down.

Exercises

a) Show that if g_{ij} is diagonal then $g^{11} = 1/g_{11}$, $g^{22} = 1/g_{22} \dots$ etc

b) Let $ds^2 = dx^2 + dy^2 + dz^2$ and let $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. Derive equation (44 b)

c) Find the length of the helix $x = a\cos\theta$, $y = a\sin\theta$, $z = b\theta$ for values of θ between $\theta = 0$ and $\theta = 2\pi$.

d) Define the length of a Eucledean vector $\vec{x}$ by $|\vec{x}|^2 = \delta_{ij} x^i x^j$. Show that the length is preserved under orthogonal transformations.

Before give more examples of tensors we point out that it is possible to define tensors of mixed types. A tensor is of type (p, q) if it transforms as a contravariant tensor on p indices and as covariant one on q-indices. The transformation law is

$$\bar{T}^{ij\cdots}{}_{kl\cdots} = T^{mn\cdots}{}_{rs\cdots} \frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial \bar{x}^j}{\partial x^n} \cdots \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^s}{\partial \bar{x}^l} \cdots \qquad (49)$$

The permutation symbols.

We have already introduced the permutation symbols in equations (20). In order to be consistent with our summation convention it will convenient to also introduce ϵ-symbols with upper indices $\epsilon^{i_1,\dots,i_m}$ whose numerical components are identical to those of $\epsilon_{i_1,\dots,i_m}$.

A closely related, and equally as useful quantity is the so-called generalized Kronecker delta defined as follows

$$\delta^{k_1,\dots\dots k_m}_{l_1,\dots\dots l_m} \equiv \begin{vmatrix} \delta^{k_1}_{l_1} & \cdots & \delta^{k_1}_{l_m} \\ \vdots & & \vdots \\ \delta^{k_m}_{l_1} & \cdots & \delta^{k_m}_{l_m} \end{vmatrix} = \begin{cases} 1 & \text{if } (k_1,\dots,k_m) \text{ is an even permutation of } (l_1,\dots,l_m) \text{ and the } k_i\text{'s are all different} \\ -1 & \text{if } (k_1,\dots,k_m) \text{ is an odd permutation of } (l_1,\dots,l_m) \text{ and the } k_i\text{'s are all different.} \\ 0 & \text{otherwise.} \end{cases} \qquad (50)$$

Thus, for example:

$$\delta^{11}_{ij} = \delta^{22}_{ij} = \delta^{33}_{ij} = \delta^{ij}_{11} = \delta^{ij}_{22} = \delta^{ij}_{33} = 0$$

$$\delta^{12}_{12} = \delta^{13}_{13} = \delta^{23}_{23} = \delta^{21}_{21} = \delta^{31}_{31} = \delta^{32}_{32} = 1$$

$$\delta^{12}_{21} = \delta^{13}_{31} = \delta^{23}_{32} = \delta^{21}_{12} = \delta^{31}_{13} = \delta^{32}_{23} = -1.$$

It follows almost immediately by looking at the definitions of the ϵ's and the δ's that

$$\epsilon_{i_1\dots\dots i_m} = \delta^{1\dots\dots m}_{i_1\dots\dots i_m} \quad \text{and} \quad \epsilon^{i_1\dots\dots i_m} = \delta^{i_1\dots\dots i_m}_{1\dots\dots n}. \qquad (51)$$

We also have the slightly less obvious relation

$$\epsilon_{i_1\dots\dots i_m}\epsilon^{k_1\dots\dots k_m} = \delta^{k_1\dots\dots k_m}_{i_1\dots\dots i_m}, \qquad (52)$$

which can be verified as follows: The left hand side is zero unless all the i's (as well as the k's) are different. If the k's are an even permutation of the i's and the i's are all different, then both factors in the left in equation (52) have the same signature and hence the product is equal to 1. On the other hand, if the k's are all different, and they are and odd permutation of the i's, then the two factors have opposite signs, and thus their product is -1. This is exactly the definition of the δ's.

With a little effort, it is possible to establish a few other identities which are useful in manipulating and simplifying quantities involving ϵ's and δ's. We will mention some of these.

Let $A_{i_1 \ldots\ldots i_m}$ be a quantity which is completely antisymmetric with respect to all of its indices. The components of an n-form is an example of one set of such quantities. We then have

$$\delta^{i_1 \ldots\ldots i_m}_{k_1 \ldots\ldots k_m} A_{i_1 \ldots\ldots i_m} = m!\, A_{k_1 \ldots\ldots k_m}. \tag{53}$$

To corroborate the veracity of the last equation, one may reason as follows: The left hand side of (53) is the sum of $m!$ terms of which one is exactly $A_{k_1 \ldots\ldots k_m}$. Each of the other non-zero terms involves a permutation of $(k_1 \ldots\ldots k_m)$. If the permutation is even then the term is equal to $A_{k_1 \ldots\ldots k_m}$. If the permutation is odd then the antisymmetry of the A's gives you a minus sign, but the δ gives you another, hence the term is also equal to $A_{k_1 \ldots\ldots k_m}$.

A special example of equation (53) is

$$\delta^{i_1 \ldots\ldots i_n}_{k_1 \ldots\ldots k_n} \epsilon_{i_1 \ldots\ldots i_n} = n!\, \epsilon_{k_1 \ldots\ldots k_n}. \tag{54}$$

Another special example is obtained if we take $(k_1 \ldots\ldots k_n)$ to be the set $\{1, \ldots\ldots n\}$. Then (51) together with (54) yield

$$\epsilon^{i_1 \ldots\ldots i_n} \epsilon_{i_1 \ldots\ldots i_n} = n! \tag{55}$$

We leave it as an exercise to the reader to prove that

$$\epsilon^{i_1 \ldots\ldots i_m j_1 \ldots\ldots j_{n-m}} \epsilon_{k_1 \ldots\ldots k_m j_1 \ldots\ldots j_{n-m}} = (n-m)!\, \delta^{i_1 \ldots\ldots i_m}_{k_1 \ldots\ldots k_m} \tag{56}$$

A natural question to ask, at this point, is whether the ϵ's or the δ's transform as the components of a tensor. To answer this question we need the generalization of equation (22). If $A = (A^i{}_j)$ is an $n \times n$ matrix then

$$\det A = |A| = \epsilon_{i_1 \ldots\ldots i_n} A^{i_1}_1 \ldots\ldots A^{i_n}_n. \tag{57}$$

or equivalently

$$\epsilon_{i_1 \ldots\ldots i_n} A^{i_1}_{k_1} \ldots\ldots A^{i_n}_{k_n} = |A|\, \epsilon_{k_1 \ldots\ldots k_n} \tag{58a}$$

$$\epsilon^{k_1 \ldots\ldots k_n} A^{i_1}_{k_1} \ldots\ldots A^{i_n}_{k_n} = |A|\, \epsilon^{i_1 \ldots\ldots i_n} \tag{58b}$$

Consider now a coordinate transformation

$$x^i = f^i(\bar{x}^j)$$

and let

$$A^i{}_j = \frac{\partial x^i}{\partial \bar{x}^j} = (J)^i{}_j$$

Then, using (58a) and (58b) we find immediately that

$$\epsilon_{k_1 \cdots\cdots k_m} = \frac{1}{|J|} \epsilon_{i_1 \cdots\cdots i_m} \frac{\partial x^{i_1}}{\partial \bar{x}^{k_1}} \cdots\cdots \frac{\partial x^{i_m}}{\partial \bar{x}^{k_m}} \tag{59a}$$

$$\epsilon^{k_1 \cdots\cdots k_m} = |J| \, \epsilon^{i_1 \cdots\cdots i_m} \frac{\partial \bar{x}^{k_1}}{\partial x^{i_1}} \cdots\cdots \frac{\partial \bar{x}^{k_m}}{\partial x^{i_m}} \tag{59b}$$

Equations (59) show that the ϵ's are almost tensors. Quantities which transform as above are called relative tensors of weights -1 and $+1$ respectively. On the other hand, the generalized Kronecker delta, which is obtained (as in (52)) by multiplying two relative tensors of weights $+1$ and -1, transforms as the components of a tensor of type (m,m).

We also point out that in view of equation (46) and (44a) we have

$$|J|^2 = \left|\frac{\partial x^i}{\partial \bar{x}^j}\right| \cdot \left|\frac{\partial x^i}{\partial \bar{x}^j}\right| = \left|\frac{\partial x^i}{\partial \bar{x}^j} \cdot \frac{\partial x^i}{\partial \bar{x}^j}\right| = |g_{ij}| \equiv g$$

Hence

$$|J| = \sqrt{g} \tag{60}$$

Because of this fact, one often encounters in the literature the following quantities

$$e_{k_1 \cdots\cdots k_m} = \sqrt{g} \, \epsilon_{k_1 \cdots\cdots k_m} \tag{61a}$$

$$e^{k_1 \cdots\cdots k_m} = \frac{1}{\sqrt{g}} \epsilon^{k_1 \cdots\cdots k_m} \tag{61b}$$

The e's are, then, the components of a tensor. We must be careful at this point. The indices of the tensorial objects e are raised and lowered with the metric g_{ij}. The indices of the relative tensors ϵ are manipulated with the δ_{ij}. This is the reason why the ϵ's with upper indices are numerically equivalent to the ϵ's with lower indices.

In view of the insight we have gained into the nature of the permutation symbols, it becomes necessary to revise the definition of the Hodge $*$ operator. The new, metric-compatible definition is

$$*\,(dx^{i_1} \wedge \cdots\cdots \wedge dx^{i_m}) = \frac{\sqrt{g}}{(n-m)!} \epsilon^{i_1 \cdots\cdots i_m}{}_{i_{m+1} \cdots\cdots i_n} dx^{i_{m+1}} \wedge \cdots\cdots \wedge dx^{i_n} \tag{62}$$

Curvilinear Coordinates.

In this section we apply all the previous formalism to treat vector calculus in curvilinear coordinates, and we give a new derivation of the formulas for the gradient, the curl and the divergence in orthogonal coordinates. We will specialize the entire discussion to three dimensions, although the formalism is valid in any number of dimensions.

In the previous sections we have discussed several coordinate transformations. The coordinate systems we encountered have a common feature, namely, the coefficients g_{ij} of the metric tensor are diagonal. (See eqs 44) Coordinate systems of this type will be called orthogonal curvilinear coordinate systems or simply orthogonal coordinates. To be more specific, let $\{x^i\}$ be the standard Euclidean coordinates and consider the transformation

$$x^i = x^i(u^k) \tag{63}$$

The coordinate system $\{u^k\}$ is orthogonal if

$$g_{kl} = \delta_{ij} \frac{\partial x^i}{\partial u^k} \frac{\partial x^j}{\partial u^l} \tag{64}$$

is a diagonal matrix. In other words, the line metric is of the form

$$ds^2 = g_{11}(du^1)^2 + g_{22}(du^2)^2 + g_{33}(du^3)^2 \tag{65}$$

To connect with classical notation we introduce the <u>frame forms</u> θ^i

$$\begin{aligned} \theta^1 &= \sqrt{g_{11}}\, du^1 \equiv h_1 du^1 \\ \theta^2 &= \sqrt{g_{22}}\, du^2 \equiv h_2 du^2 \\ \theta^3 &= \sqrt{g_{33}}\, du^3 = h_3 du^3 \end{aligned} \tag{66}$$

"Classical notation"
$\vec{u}_1 = h_1 \vec{\nabla} u_1$
$\vec{u}_2 = h_2 \vec{\nabla} u_2$
$\vec{u}_3 = h_3 \vec{\nabla} u_3$.

We may also define the <u>frame vectors</u> e_i by

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{g_{11}}} \frac{\partial}{\partial u^1} = \frac{1}{h_1} \frac{\partial}{\partial u^1} \\ e_2 &= \frac{1}{\sqrt{g_{22}}} \frac{\partial}{\partial u^2} = \frac{1}{h_2} \frac{\partial}{\partial u^2} \\ e_3 &= \frac{1}{\sqrt{g_{33}}} \frac{\partial}{\partial u^3} = \frac{1}{h_3} \frac{\partial}{\partial u^3} . \end{aligned} \tag{67}$$

<u>Example</u>. If $ds^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta \, d\varphi^2$, then

$$\theta^1 = d\rho, \quad \theta^2 = \rho\, d\theta, \quad \theta^3 = \rho \sin\theta \, d\varphi \; ; \quad h_1 = 1, \; h_2 = \rho, \; h_3 = \rho \sin\theta .$$

These definitions insure that the "duality" of the basis forms and basis vectors is preserved. That is

$$\theta^i(e_j) = \delta^i_j .$$

Gradient.

Let $f = f(x^i)$. Then, as we have seen

$$df = \frac{\partial f}{\partial x^i} dx^i = \vec{\nabla} f \cdot \vec{dx}$$

If we now subject this last equation to the coordinate transformation $x^i = x^i(u^k)$ we get

$$df = \frac{\partial f}{\partial u^i} \frac{\partial u^i}{\partial x^k} dx^k$$

$$= \frac{\partial f}{\partial u^i} du^i = \frac{1}{h_i} \frac{\partial f}{\partial u^i} \theta^i = e_i(f) \theta^i \qquad (68)$$

In other words, the components of the gradient in the coframe $\{\theta^i\}$ are

$$\left(\frac{1}{h_1} \frac{\partial f}{\partial u^1}, \frac{1}{h_2} \frac{\partial f}{\partial u^2}, \frac{1}{h_3} \frac{\partial f}{\partial u^3} \right) \qquad (69)$$

Curl.

Let $F = F_1 \theta^1 + F_2 \theta^2 + F_3 \theta^3$. That is, F is a vector field whose components in the $\{\theta^i\}$ coframe are (F_1, F_2, F_3). We want to compute the curl of F in this coframe. For this, we recall that curl $= *d$ (eq. 28). We have:

$$F = F_1 \theta^1 + F_2 \theta^2 + F_3 \theta^3$$
$$= (h_1 F_1) du^1 + (h_2 F_2) du^2 + (h_3 F_3) du^3$$
$$= (hF)_i \, du^i . \qquad \text{where } (hF)_i \equiv h_i F_i$$

$$dF = \frac{1}{2} \left[\frac{\partial (hF)_i}{\partial u^j} - \frac{\partial (hF)_j}{\partial u^i} \right] du^j \wedge du^i$$

$$= \frac{1}{2 h_i h_j} \left[\frac{\partial (hF)_i}{\partial u^j} - \frac{\partial (hF)_j}{\partial u^i} \right] \theta^j \wedge \theta^i$$

$$*dF = \epsilon^{ij}{}_k \left[\frac{1}{h_i h_j} \left(\frac{\partial (hF)_i}{\partial u^j} - \frac{\partial (hF)_j}{\partial u^i} \right) \right] \theta^k = (\nabla \times F)_k \theta^k$$

Thus, the components of the curl in the θ-coframe are

$$\left(\frac{1}{h_2 h_3} \left[\frac{\partial (h_3 F_3)}{\partial u^2} - \frac{\partial (h_2 F_2)}{\partial u^3} \right], \frac{1}{h_3 h_1} \left[\frac{\partial (h_1 F_1)}{\partial u^3} - \frac{\partial (h_3 F_3)}{\partial u^1} \right], \frac{1}{h_1 h_2} \left[\frac{\partial (h_2 F_2)}{\partial u^3} - \frac{\partial (h_3 F_3)}{\partial u^2} \right] \right) \qquad (70)$$

Divergence.

As before, let $F = F_1\theta^1 + F_2\theta^2 + F_3\theta^3$. We compute the divergence recalling that $\mathrm{Div} = *F*$.

$$F = F_1\theta^1 + F_2\theta^2 + F_3\theta^3$$

$$*F = F_1\,\theta^2\wedge\theta^3 + F_2\,\theta^3\wedge\theta^1 + F_3\,\theta^1\wedge\theta^2$$

$$= (h_2h_3F_1)\,du^2\wedge du^3 + (h_1h_3F_2)\,du^3\wedge du^1 + (h_1h_2F_3)\,du^1\wedge du^2$$

$$d*F = \left[\frac{\partial(h_2h_3F_1)}{\partial u^1} + \frac{\partial(h_1h_3F_2)}{\partial u^2} + \frac{\partial(h_1h_2F_3)}{\partial u^3}\right] du^1\wedge du^2\wedge du^3$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3F_1)}{\partial u^1} + \frac{\partial(h_1h_3F_2)}{\partial u^2} + \frac{\partial(h_1h_2F_3)}{\partial u^3}\right] \theta^1\wedge\theta^2\wedge\theta^3$$

$$*d*F = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3F_1)}{\partial u^1} + \frac{\partial(h_1h_3F_2)}{\partial u^2} + \frac{\partial(h_1h_2F_3)}{\partial u^3}\right] \tag{71}$$

Laplacian.

Substituting for the components F_i in equation (71) the components of the gradient (as in 68)) we get the formula for the Laplacian

$$\Delta f = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u^1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{h_1h_3}{h_2}\frac{\partial f}{\partial u^2}\right) + \frac{\partial}{\partial u^3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u^3}\right)\right. \tag{72}$$

Exercises

1. Compute the metric tensors in cylindrical and in spherical coordinates (eq 44) using eq. (46)

2. The contravariant components of a cartesian vector are $\vec{A} = (A^1, A^2, A^3)$. Find the covariant components in spherical coordinates

3. Show that if $\vec{A} = (A^1, A^2, A^3)$ and $\vec{B} = (B_1, B_2, B_3)$ then
$$\cos(\vec{A},\vec{B}) = \frac{g_{ij} A^i B^j}{(g_{k\ell} A^k A^\ell)^{1/2} (g_{mn} B^m B^n)^{1/2}} .$$

4. Show that a) $\delta^{ijk}_{ijk} = 3!$ $(i,j,k = 1,2,3)$. b) $\delta^{ijk\ell}_{ijk\ell} = 4!$ $(i,j,k,\ell = 1, \dots 4)$

5) Show that if A_{ij} is an antisymmetric tensor $(i,j = 1, \dots, n)$ then the number of distinct terms is $n(n-1)/2$

6) Show that if A_{ijk} is symmetric tensor $(i,j,k = 1, \dots, n)$ then the number of distinct terms in the set $\{A_{ijk}\}$ is N where
$$N = n + n(n-1) + \frac{n(n-1)(n-2)}{3!}$$

7) Show that $e_{i_1 \dots i_m}$ and $e^{i_1 \dots i_m}$ as defined in equations (61) are the components of a tensor

8) Verify that $\delta^{ij}_{k\ell} A^{k\ell} = A^{ij} - A^{ji}$

9) Verify that $\delta^{ijk}_{\ell mn} A^{\ell mn} = A^{ijk} - A^{ikj} + A^{jki} - A^{jik} + A^{kij} - A^{kji}$

10) Compute a) δ^i_i, b) δ^{ij}_{ji} c) $\delta^i_j \delta^j_k \delta^k_i$ $(i,j,k = 1, \dots, n)$

11) Prove that the generalized Kronecker deltas $\delta^{i_1 \dots i_m}_{k_1 \dots k_m}$ transform as the components of a tensor.

12) Show that $\delta^{12}_{ij} \delta^{ik}_{12} = \delta_j{}^k$ $(i,j,k = 1, 2)$

13) (Derivative of a determinant). Let $A = (A^i{}_j)$ be an $n \times n$ matrix and let $\Delta^i{}_j$ be the cofactor matrix.

a) Show that $\Delta^1{}_i = \epsilon_{i,i_2,i_3,\dots,i_m} A^{i_2}_2 A^{i_3}_3 \dots A^{i_m}_m$.

b) Show that $\det A = |A| = A^{i_1}_1 \epsilon_{i_1 i_2 \dots i_m} A^{i_2}_2 \dots A^{i_m}_m$.

c) By differentiating $|A| = \epsilon_{i_1 \dots i_m} A^{i_1}_1 \dots A^{i_m}_m$ show that
$$\frac{\partial |A|}{\partial x^j} = \frac{\partial A^{i_1}_1}{\partial x^j} \Delta^1_{i_1} + \frac{\partial A^{i_2}_2}{\partial x^j} \Delta^2_{i_2} + \dots \frac{\partial A^{i_m}_m}{\partial x^j} \Delta^m_{i_m}$$
$$= \frac{\partial A^i_k}{\partial x^j} \Delta^k_i$$

(Cont)

d) Using the results of part c show that

$$\frac{\partial}{\partial x_j}(\ln|A|) = \frac{\partial A^i{}_k}{\partial x_j} \cdot A^k{}_i$$

14) Show that if $A^i{}_j$ transforms as a tensor then $\frac{d}{\partial x_k} A^i{}_j$ does not transform as a tensor.

15) Show that in spherical coordinates

a) $\Delta f = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}$

b) $\vec{\nabla} \times \vec{F} = \frac{1}{\rho \sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta F_3) - \frac{\partial F_2}{\partial \varphi}\right]\theta^1 + \frac{1}{\rho}\left[\frac{1}{\sin\theta}\frac{\partial F_1}{\partial \varphi} - \frac{\partial}{\partial \rho}(\rho F_3)\right]\theta^2 + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho F_2) - \frac{\partial F_1}{\partial \theta}\right]\theta^3$

16) Elliptic cylindrical coordinates. Let $x = c\xi\eta$, $y = c\sqrt{(\xi^2-1)(1-\eta^2)}$ $z = z$
Show that

$$ds^2 = c^2\left(\frac{\xi^2-\eta^2}{\xi^2-1}\right)d\xi^2 + c^2\left(\frac{\xi^2-\eta^2}{1-\eta^2}\right)d\eta^2 + dz^2$$

17) Parabolic coordinates. Let $x = \frac{1}{2}(\eta^2 - \xi^2)$, $y = \xi\eta$, $z = z$. Then

$$ds^2 = (\xi^2+\eta^2)\,d\xi^2 + (\xi^2+\eta^2)\,d\eta^2 + dz^2$$

18) Show that in spherical coordinates $\nabla\left(\frac{1}{\rho}\right) = \nabla \times (\cos\theta\, \nabla\varphi)$

V. CURVES IN $\mathbb{E}^3$

5.1. Definition. A curve $\alpha(t)$ is a C^∞ map from an open subset of $\mathbb{R}$ into $\mathbb{E}^3$.
The curve assigns to each value of a parameter t in $\mathbb{R}$ a point $(\alpha^1(t), \alpha^2(t), \alpha^3(t))$ in $\mathbb{E}^3$

$$U \subset \mathbb{R} \xrightarrow{\alpha} \mathbb{E}^3$$

$$t \longmapsto \alpha(t) = (\alpha^1(t), \alpha^2(t), \alpha^3(t))$$

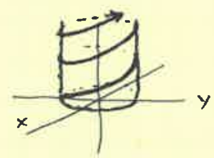
One may think of the parameter t as being "time" and the curve α as the trajectory of a point moving in time.

Examples:

1) $\alpha(t) = (a_1 t + b_1,\ a_2 t + b_2,\ a_3 t + b_3)$.

This curve is a straight line passing through the point $\mathbf{p} = (b_1, b_2, b_3)$ in the direction of $\vec{q} = (a_1, a_2, a_3)$

2) $\alpha(t) = (a\cos\omega t,\ a\sin\omega t,\ bt)$.



This curve is called a circular helix. It maybe thought of as the curve described by the hypothenuse of a right triangle wrapped around a cylinder of radius a.

Remark. In classical books in elementary differential geometry one finds that curves are often defined in terms of a position Euclidean vector

$$\vec{\mathbf{x}}(t) = (x(t), y(t), z(t)). \qquad (73)$$

or

$$x^i = x^i(t) \qquad (i = 1,2,3)$$

This is an abuse of notation but it is convenient and we will often refer to this notation in this section. The modern version of equation (73) is

$$\mathbf{X}(t) = (x^i \circ \alpha)(t)\frac{\partial}{\partial x^i} \qquad (74)$$

since the x^i's are coordinate functions which map from $\mathbb{E}^3$ into $\mathbb{R}$.

5.2
Definition. The velocity $V(t)$ of a curve $\alpha(t)$ is defined as $V(t) = \alpha'(t)$. The acceleration $A(t)$ is given by $A(t) = \alpha''(t)$. What is meant by $\alpha'(t)$ in this definition is the following tangent vector (see diagram). If $V(t) \neq 0$ then $\alpha(t)$ is called regular

$$V(t) = \alpha'(t) = \frac{d}{dt}(x^i \circ \alpha)(t) \cdot \frac{\partial}{\partial x^i}\Big|_{\alpha(t)} \tag{75}$$

$$\begin{tikzcd}
\frac{d}{dt} \in T\mathbb{R} \arrow[r, "\alpha_*"] \arrow[d] & TE^3 \ni \alpha'(t) \arrow[d] & \\
\mathbb{R} \arrow[r, "\alpha"] & E^3 \arrow[r, "x^i"] & \mathbb{R}
\end{tikzcd}$$

The vector field $\alpha'(t)$ is sometimes referred to as the push-forward $\alpha_*(\frac{\partial}{\partial t})$ of the tangent vector $d/dt \in T\mathbb{R}$. The magnitude of the vector $\alpha'(t)$ is called the <u>speed</u> of the curve. The classical version of (75) is

$$\mathbf{V}(t) = \mathbf{X}'(t) = (x'(t), y'(t), z'(t)) \tag{75}$$

Notation: $\dot{\mathbf{X}} \equiv \mathbf{X}'(t)$, $\ddot{\mathbf{X}} \equiv \mathbf{X}''(t)$ etc.

5.3. **Definition.** If $t = t(s)$ is a smooth real valued function and if $\alpha(t)$ is a curve in E^3, we say that the curve $\beta(s) = \alpha(t(s))$ is a reparametrization of the curve $\alpha(t)$.

A common and convenient reparametrization a curve is obtained by using the arclength s as the parameter. The arc length is computed as a function of t using the formula

$$s(t) = \int^t \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt = \int_a^t \|\alpha'(u)\|\, du \tag{76}$$

A curve parametrized by arc length is called a <u>unit speed</u> curve because in this case the velocity vector $\mathbf{T}(s) = \beta'(s)$ has unit length.

$$\mathbf{T}(s) = \beta'(s) = \alpha'(t(s)) \cdot t'(s) = \frac{\alpha'(t(s))}{\|\alpha'(t(s))\|}$$

(classically)

$$\mathbf{T}(s) = \frac{d\mathbf{X}}{ds} = \frac{d\mathbf{X}}{dt} \Big/ \frac{ds}{dt} = \dot{\mathbf{X}} / \|\dot{\mathbf{X}}\|$$

Example: Let $\alpha(t) = (a\cos\omega t, a\sin\omega t, bt)$

$$\mathbf{V}(t) = (-a\omega\sin\omega t, a\omega\cos\omega t, b)$$

$$s(t) = \int \sqrt{a^2\omega^2(\sin^2\omega t + \cos^2\omega t) + b^2}\, dt$$

$$= ct \quad \text{where} \quad c = \sqrt{a^2\omega^2 + b^2}$$

The helix of unit speed is then given by

$$\beta(s) = \left(a\cos\frac{\omega s}{c},\ a\sin\frac{\omega s}{c},\ \frac{bs}{c}\right)$$

Frenet Frame.

Let $\beta(s)$ be a curve parametrized by arc length and let $T(s)$ be the unit tangent vector. We have

$$T \cdot T = 1$$

Differentiating this equation with respect to s we get

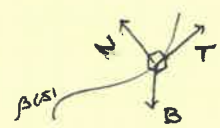
$$2T \cdot T' = 0$$

hence $T'(s)$ is orthogonal to T. Let N be a unit vector orthogonal to T and define the curvature κ of the curve by

$$T'(s) = \kappa N(s) \tag{77}$$

We also define the binormal vector $B = T \times N$. The three vectors T, N and B form an orthonormal set. That is

$$\begin{aligned} T\cdot T = N \cdot N = B \cdot B = 1 \\ T \cdot N = T \cdot B = N \cdot B = 0 \end{aligned} \tag{78}$$



If we differentiate the relation $B \cdot B = 1$ we find that $B \cdot B' = 0$; hence B' is orthogonal to B. Furthermore, if we differentiate the equation $T \cdot B = 0$ we find that $B' \cdot T = -T' \cdot B = -\kappa N \cdot B = 0$ hence B' is orthogonal to T. We conclude that

$$B'(s) = -\tau N(s) \tag{79}$$

for some τ. Equation (79) defines the so called <u>torsion</u> τ of the curve.

5.4 **Theorem.** (Frenet equations). Let $\beta(s)$ be a unit speed curve with curvature κ and torsion τ. Then

$$\begin{aligned} T' &= \quad \kappa N \\ N' &= -\kappa T \quad\quad\quad + \tau B \\ B' &= \quad -\tau N \end{aligned} \qquad \text{or} \qquad \begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \tag{80}$$

Proof: We only need to establish the equation for N'. Differentiating $N \cdot N = 1$ we find that N' is orthogonal to N, hence

$$N' = aT + bB$$

$$N' \cdot T = aT \cdot T = -N \cdot T' = -N \cdot (\kappa N) = -\kappa \quad \therefore \quad a = -\kappa$$

$$N' \cdot B = bB \cdot B = -B' \cdot N = \tau N \cdot N = \tau \quad \therefore \quad b = \tau$$

Notation. If $\vec{A}, \vec{B}$ and $\vec{C}$ are Euclidean vectors, we define the scalar triple product $(\vec{A}\vec{B}\vec{C}) = \vec{A}\cdot(\vec{B}\times\vec{C})$.

5.5 **Proposition.** Let $\beta(s)$ be a unit speed curve with curvature $\kappa > 0$ and torsion τ. Then

	(Classically)	
a) $\kappa^2 = \|\beta''(s)\|^2 = \beta''(s)\cdot\beta''(s)$	$\kappa = X''\cdot X''$	(81a)
b) $\tau = \dfrac{\beta'(s)\cdot[\beta''(s)\times\beta'''(s)]}{\beta''(s)\cdot\beta''(s)}$	$\tau = \dfrac{(X'X''X''')}{X''\cdot X''}$	(81b)

Proof.
If $\beta(s)$ is a unit speed curve then $T = \beta'(s)$. Hence

a) $T' = \beta''(s) = \kappa N$

$$\therefore \; \beta''\cdot\beta'' = (\kappa N)\cdot(\kappa N)$$
$$= \kappa^2 \qquad \therefore \; \kappa^2 = \|\beta''\|^2$$

b)
$$T'' = \beta'''(s) = \kappa' N + \kappa N'$$
$$= \kappa' N + \kappa(-\kappa T + \tau B)$$
$$= -\kappa^2 T + \kappa' N + \kappa\tau B$$

$\therefore$

$$\beta'\cdot[\beta''\times\beta'''] = T\cdot[\kappa N\times(-\kappa^2 T + \kappa' N + \kappa\tau B)]$$
$$= T\cdot[+\kappa^3 B + \kappa^2\tau T]$$
$$= \kappa^2\tau$$

$$\therefore \qquad \tau = \frac{(\beta'\beta''\beta''')}{\kappa^2} = \frac{(\beta'\beta''\beta''')}{\beta''\cdot\beta''}$$

5.6. **Proposition.** Let $\alpha(t)$ be a curve of speed v and curvature κ. Assume that $\alpha(t)$ is regular; Then

$$V(t) = vT \qquad (82)$$
$$A(t) = \frac{dv}{dt}T + \kappa v^2 N.$$

Proof. Let $t = t(s)$ where s is arclength. Since $V(t)\neq 0$ we can write $s = s(t)$. Using the chain rule for derivatives on the curve $\beta(s(t)) = \alpha(t)$ we get

$$\alpha'(t) = \beta'(s(t))\cdot s'(t)$$
$$\therefore \quad V = T\cdot v \quad\Rightarrow\quad V(t) = vT(s(t))$$

$$\alpha''(t) = \frac{dv}{dt}T + v\,T'(s(t))\cdot s'(t)$$
$$= \frac{dv}{dt}T + v\cdot(\kappa N)\cdot v$$
$$= \frac{dv}{dt}T + v^2\kappa N.$$

If the curve $\alpha(t)$ is not parametrized by arc length, we may use the following formulas to compute κ and τ.

5.7 Proposition. *If $\alpha(t)$ is a regular curve in $\mathbb{E}^3$ then*

$$\kappa^2 = \frac{(\dot\alpha \times \ddot\alpha)\cdot(\dot\alpha \times \ddot\alpha)}{(\dot\alpha\cdot\dot\alpha)^3} = \frac{\|\dot\alpha \times \ddot\alpha\|^2}{\|\dot\alpha\|^6} \tag{83}$$

$$\tau = \frac{(\dot\alpha\,\ddot\alpha\,\dddot\alpha)}{(\dot\alpha\times\ddot\alpha)\cdot(\dot\alpha\times\ddot\alpha)} = \frac{(\dot\alpha\,\ddot\alpha\,\dddot\alpha)}{\|\dot\alpha\times\ddot\alpha\|^2}$$

Proof:

$$\dot\alpha = v T \qquad \Rightarrow \quad v = \|\dot\alpha\|$$

$$\ddot\alpha = \dot v T + v^2 \kappa N$$

$$\dddot\alpha = (v^2\kappa)\dot N + \cdots$$

$$= v^3\kappa N' + \cdots \qquad (\text{since } \dot N = \tfrac{dN}{ds}\cdot\tfrac{ds}{dt}).$$

$$= v^3 \kappa \tau B + \cdots$$

We only need the term containing B since $\dot\alpha \times \ddot\alpha$ is proportional to B

$$\dot\alpha \times \ddot\alpha = v^3 \kappa B \qquad (\dot\alpha\times\ddot\alpha)\cdot\dddot\alpha = v^6\kappa^2\tau$$

$$\|\dot\alpha\times\ddot\alpha\| = v^3\kappa \qquad \tau = \frac{(\dot\alpha\,\ddot\alpha\,\dddot\alpha)}{v^6\kappa^2}$$

$$\kappa = \frac{\|\dot\alpha\times\ddot\alpha\|}{v^3}. \qquad \tau = \frac{(\dot\alpha\,\ddot\alpha\,\dddot\alpha)}{\|\dot\alpha\times\ddot\alpha\|^2}.$$

Example (*Helix*)

$$\beta(s) = \left(a\cos\frac{\omega s}{c},\ a\sin\frac{\omega s}{c},\ \frac{bs}{c}\right) \qquad c = \sqrt{a^2\omega^2+b^2}$$

$$\beta'(s) = \left(-\frac{a\omega}{c}\sin\frac{\omega s}{c},\ \frac{a\omega}{c}\cos\frac{\omega s}{c},\ \frac{b}{c}\right)$$

$$\beta''(s) = \left(-\frac{a\omega^2}{c^2}\cos\frac{\omega s}{c},\ -\frac{a\omega^2}{c^2}\sin\frac{\omega s}{c},\ 0\right)$$

$$\beta'''(s) = \left(\frac{a\omega^3}{c^3}\sin\frac{\omega s}{c},\ -\frac{a\omega^3}{c^3}\cos\frac{\omega s}{c},\ 0\right)$$

$$\kappa^2 = \beta''\cdot\beta'' = a^2\omega^4/c^4 \Rightarrow \kappa = \pm\frac{a\omega^2}{c^2}$$

$$\tau = \frac{(\beta'\beta''\beta''')}{\beta''\cdot\beta''} = \frac{b}{c}\begin{vmatrix} -\frac{a\omega^2}{c^2}\cos\frac{\omega s}{c} & -\frac{a\omega^2}{c^2}\sin\frac{\omega s}{c} \\ \frac{a\omega^3}{c^3}\sin\frac{\omega s}{c} & -\frac{a\omega^3}{c^3}\cos\frac{\omega s}{c}\end{vmatrix}\frac{c^4}{a^2\omega^4}$$

$$= \frac{b}{c}\cdot\frac{a^2\omega^5}{c^5}\,\frac{c^4}{a^2\omega^4}$$

$$\therefore \quad \kappa = \frac{\pm a\omega^2}{a^2\omega^2+b^2} \qquad \tau = \frac{b\omega}{a^2\omega^2+b^2}$$

Notice that if $b=0$ the curve $\beta(s)$ becomes a circle. In this case $\tau = 0$ and $\kappa = 1/a$.

Example. (*Plane curve*)

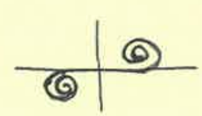
Take $\alpha(t) = (x(t), y(t), 0)$

$$\dot\alpha = (\dot x, \dot y, 0) \qquad \kappa = \frac{\|\dot\alpha\times\ddot\alpha\|}{\|\dot\alpha\|^3} = \frac{|\dot x\ddot y - \dot y\ddot x|}{(\dot x^2+\dot y^2)^{3/2}}$$

$$\ddot\alpha = (\ddot x, \ddot y, 0)$$

$$\dddot\alpha = (\dddot x, \dddot y, 0) \qquad \tau = 0$$

Example. (*Cornu spiral*)



Take $\beta(s) = (x(s), y(s), 0) \qquad x(s) = \int_0^s \cos\frac{t^2}{2c^2}\,dt, \quad y(s) = \int_0^s \sin\frac{t^2}{2c^2}\,dt$

$$\beta'(s) = \left(\cos\frac{s^2}{2c^2},\ \sin\frac{s^2}{2c^2},\ 0\right) \qquad \|\beta'\| = v = 1 \quad \therefore\ s = \text{arc length}$$

$$\kappa = |x'y'' - x'y''| = (\beta''\cdot\beta'')^{1/2} = \left\|\left(-\frac{s}{c^2}\sin\frac{s^2}{2c^2},\ \frac{s}{c^2}\cos\frac{s^2}{2c^2},\ 0\right)\right\| = s/c^2$$

The integrals defining x and y are called Fresnel Integrals; they are important in the theory of diffraction.

Some geometrical insight into the significance of κ and τ can be gained by considering a curve $\beta(s)$ expanded by a Taylor series at the origin.

$$\beta(s) = \beta(0) + \beta'(0)s + \frac{\beta''(0)}{2}s^2 + \frac{\beta'''(0)}{3!}s^3 + \cdots \tag{84}$$

If s represents the arc length parameter then

$$\beta'(0) = T(0) \equiv T_0$$
$$\beta''(0) = \kappa N|_0 \equiv \kappa_0 N_0$$
$$\beta'''(0) = (-\kappa^2 T + \kappa' N + \kappa\tau B)|_{s=0} \equiv -\kappa_0^2 T_0 + \kappa_0' N_0 + \kappa_0\tau_0 B_0$$

Keeping only the lowest terms in the components of T, N and B we get the "Frenet aproximation" to the curve

$$\beta(s) \cong \beta(0) + s_0 T_0 + \tfrac{1}{2}\kappa_0 N_0 s^2 + \tfrac{1}{6}\kappa_0\tau_0 B_0 s^3 \tag{85}$$

The first two terms represent the linear aproximation to the curve. The first three terms approximate the curve by a parabola which lies in the T-N plane. The parabola is called the osculating parabola, and the T-N plane is called the osculating plane. If $\kappa=0$ then locally the curve is a straight line. If $\tau_0=0$ then locally the curve is a plane curve in the osculating plane. In this sense κ measures the deviation of the curve from being a straight line, and τ measures the deviation of the curve from being a plane curve

5.8. **Fundamental Theorem for curves.** Let $\kappa(s)$ and $\tau(s)$, $s>0$ be any two analytic functions. There exists a unique curve (unique up to its position in space) for which s is the arc length, $\kappa(s)$ the curvature and $\tau(s)$ the torsion.

Proof: Pick a point in E^3. Without loss of generality we may take this point to be the origin. At this point pick any orthogonal frame $\{T, N, B\}$. The curve is then determined uniquely by its Taylor expansion in the Frenet frame as in equation (85).

Note. It is possible to prove the theorem just assuming that $\kappa(s)$ and $\tau(s)$ are continuous. The proof, however becomes much harder. (See O'Neill, Elem. Diff. Geom for example.)

5.9 **Prop.** A curve with $\kappa=0$ is a straight line (Ex. 3)

5.10 **Prop.** A curve with $\tau=0$ is a plane curve

Proof: If $\tau=0$ then $\dot\alpha\ddot\alpha\dddot\alpha=0$. This means that the three vectors $\dot\alpha, \ddot\alpha$, and $\dddot\alpha$ are linearly dependent and hence, 5.10 (cont)

there exists functions $a_1(s)$, $a_2(s)$ and $a_3(s)$ such that

$$a_3 \dddot{\alpha} + a_2 \ddot{\alpha} + a_1 \dot{\alpha} = 0$$

This linear homogeneous differential will have a solution of the form

$$\alpha = \vec{c}_1 \alpha_1 + \vec{c}_2 \alpha_2 + \vec{c}_3 \quad , \quad c_i = \text{constant vectors.}$$

This curve lies on the plane $(\vec{X} - \vec{c}_3) \cdot \vec{m} = 0$ where $\vec{m} = \vec{c}_1 \times \vec{c}_3$

Exercises

1.) Compute κ and τ for the following curves

a) $\alpha(t) = (t, t^2, t^3)$

b) $\alpha(t) = (t - \sin t, 1 - \cos t, t)$

c) $\alpha(t) = (3t - t^3, 3t^2, 3t + t^3)$ (Show : $\kappa^2 = \tau^2$)

2) Show that for the curve $\alpha(t) = (\cosh t, \sinh t, 1)$ the arc length $s(t)$ is given by $s(t) = \sqrt{2}\sinh t$

3) Show that if a curve $\alpha(t)$ has $\kappa = 0$ then α is a straight line

4) Let $\alpha(t)$ be a circular helix with binormal B. Show that the angle between B and the axis of the cylinder in which $\alpha(t)$ lies is constant.

5) (Struik) When a rigid body rotates about a point there exists an axis of instantaneous rotations. Show that this axis for the moving trihedron (moving frame) has the direction of the vector $R = \tau T + \kappa B$. Show that the Frenet frame formulas can be written as

$$T' = R \times T \qquad N' = R \times N \qquad B' = R \times B$$

6) (Spherical Image). Let $\beta(s)$ be a unit speed curve and $T(s)$ the unit tangent. As the parameter s varies the endpoints of the vector $T(s)$ describe a curve on a unit sphere. The curve $\gamma(s) = T(s)$ is called the sperical image of β. Compute the curvature κ_γ and torsion τ_γ of γ given the curvature and torsion of $\beta(s)$ and then show that

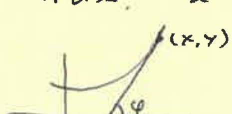
$$\kappa_\gamma(s) = \sqrt{1 + (\tau/\kappa)^2}\,, \qquad \tau_\gamma(s) = \frac{(\tau/\kappa)'}{\kappa\,[1 + (\tau/\kappa)^2]}$$

7) Show that the curve $\alpha(t) = (a\sin^2 t, a\sin t\cos t, a\cos t)$ lies on a sphere

8) Show that the cubic curve $\alpha(t) = (at, bt^2, t^3)$ is a helix if $2b^2 = 3a$ (Show that κ/τ = constant)

9) Consider a plane curve $\beta(s) = (x(s), y(s), 0)$ with curvature κ. Let φ be the angle subtended by the tangent to the curve at a pt and the x axis. Show that if $\kappa(s)$ is given, then

$$\varphi = \int^s \kappa\, ds \qquad x = \int^\varphi R\cos\varphi\, d\varphi, \qquad y = \int^\varphi R\sin\varphi\, d\varphi \qquad R = \frac{1}{\kappa}$$



These are called the <u>natural equations</u>

10) Find the curves $\alpha(s) = (x(s), y(s), 0)$ whose natural equations are

a) $\kappa = \text{const} = a$ (circle)
b) $\kappa = a s$ (Cornu spiral)
c) $\kappa = a s + b$ (logarithmic spiral)
$= s \cot\varphi + p \csc\varphi$ ($\varphi, p = \text{const}$)
d) $R^2 = 2as$
$\varphi = \sqrt{\frac{2}{a}}\sqrt{s}$. (Involute of a circle)

11) (STRUIK). Show that if $\kappa(s)$ and $\tau(s)$ are given, then the equation of a curve having κ and τ as curvature and torsion can be obtained as the solution $\alpha(s)$ of the following differential equation ($\kappa, \tau \neq 0$)

$$\alpha^{(iv)} - \left(2\frac{\kappa'}{\kappa} + \frac{\tau'}{\tau}\right)\alpha''' + \left(\kappa^2 + \tau^2 - \frac{\kappa\kappa'' - 2\kappa'^2}{\kappa^2} + \frac{\kappa'\tau'}{\kappa\tau}\right)\alpha'' + \kappa^2\left(\frac{\kappa'}{\kappa} - \frac{\tau'}{\tau}\right)\alpha' = 0$$

12) (Spherical curves). Show that a curve whose natural coordinates ~~(s,a~~) satisfies the equation

$$R^2 + (TR')^2 = a^2$$

lies entirely on a sphere

VI SURFACES IN $\mathbb{E}^3$.

Standard Connection in $\mathbb{E}^3$.

Following the conventions established so far, given a vector $\vec{X} = a\vec{i} + b\vec{j} + c\vec{k}$ in $\mathbb{E}^3$, we interpret the vector as being a differential operator $\mathbf{X} = a(\partial/\partial x) + b(\partial/\partial y) + c(\partial/\partial z)$, so that X acting on a function f yields the directional derivative of that function in the direction of $\vec{X}$

$$\mathbf{X}(f) = \vec{\nabla} f \cdot \vec{X}$$

We would now like to extend this idea to define the derivative of a vector field Y in the direction of X. Let

$$Y = f^1(x)\,\partial/\partial x + f^2(x)\,\partial/\partial y + f^3(x)\,\partial/\partial z$$
$$= f^i(x)\,\partial/\partial x^i$$

be a C^∞ vector field and let X be as above

6.1 Definition. The covariant derivative of Y in the direction of **X** is the vector field $\overline{\nabla}_{\mathbf{X}} Y$ whose components are

$$\overline{\nabla}_X Y = X(f^1)\frac{\partial}{\partial x} + \mathbf{X}(f^2)\frac{\partial}{\partial y} + \mathbf{X}(f^3)\frac{\partial}{\partial z}$$
$$= \mathbf{X}(f^i)\frac{\partial}{\partial x^i} \qquad (86)$$

In other words, in Euclidean space with the standard metric, the covariant derivative of a vector field Y in the direction of X is the the vector field whose components are the directional derivatives of the components of Y.

Example. Take $\mathbf{X} = x\frac{\partial}{\partial x} + xz\frac{\partial}{\partial y}$. $Y = x^2\frac{\partial}{\partial x} + xy^2\frac{\partial}{\partial z}$.

$$\overline{\nabla}_{\mathbf{X}} Y = \left(x\cdot\frac{\partial(x^2)}{\partial x} + xz\frac{\partial(x^2)}{\partial y}\right)\frac{\partial}{\partial x} + \left(x\,\frac{\partial(xy^2)}{\partial x} + xz\frac{\partial(xy^2)}{\partial y}\right)\frac{\partial}{\partial z}$$
$$= 2x^2\frac{\partial}{\partial x} + (xy^2 + 2x^2yz)\frac{\partial}{\partial z}.$$

6.2 Prop. Let X_1, X_2, Y_1, Y_2 be vector fields and f a C^∞ function. Then, the following properties hold

a) $\overline{\nabla}_{fX_1} Y_1 = f\,\overline{\nabla}_{X_1} Y_1$

b) $\overline{\nabla}_{X_1+X_2} Y_1 = \overline{\nabla}_{X_1} Y_1 + \overline{\nabla}_{X_2} Y_1$ (87)

c) $\overline{\nabla}_{X_1}(fY_1) = X_1(f)\,Y_1 + f\,\overline{\nabla}_{X_1}(Y_1)$

d) $\overline{\nabla}_{X_1}(Y_1+Y_2) = \overline{\nabla}_{X_1} Y_1 + \overline{\nabla}_{X_1} Y_2$.

The proof follows trivially from definition (1.5)

Let $\langle X,Y\rangle$ denote the usual dot product of vectors in $\mathbb{E}^n$. That is, if $X = X^i\,\partial/\partial x^i$, $Y = Y^j\,\partial/\partial x^j$, we take $\langle X,Y\rangle = \delta_{ij} X^i Y^j$. We then have the following proposition

6.3 Prop.

a) $\bar\nabla_X Y - \bar\nabla_Y X = [X,Y]$ (88a)

b) $X\langle Y,Z\rangle = \langle \bar\nabla_X Y, Z\rangle + \langle Y, \bar\nabla_X Z\rangle$. (88b)

where $[X,Y]f \equiv X(Y(f)) - Y(X(f))$.

The proof is elementary and will be left as an exercise.

Any operator $\bar\nabla_X$ which satisfies properties (87) is called a <u>connection</u>. If in addition the operator satisfies properties (88) is called a Riemannian connection. The standard connection in $\mathbb{E}^n$ is the one defined above which has been chosen to be "compatible" with the standard metric $\langle X,Y\rangle$ via equations (88). If we changed the definition of the inner product, say $\langle X,Y\rangle = g_{ij}X^iY^j$, we would of course get a different Riemannian connection.

Surfaces in $\mathbb{E}^3$

6.4 Definition. A C^k coordinate chart is a C^k map X from an open subset of $\mathbb{E}^2$ into $\mathbb{E}^3$. We will always assume that the Jacobian of the map has maximal rank (Rank = 2)

$$X : U \subset \mathbb{E}^2 \longrightarrow \mathbb{E}^3$$
$$(u,v) \xrightarrow{\;X\;} (x(u,v),\, y(u,v),\, z(u,v)) \qquad (89)$$

The notation in local coordinates

$$x^i = x^i(u^\alpha) \qquad \text{where} \quad i=1,2,3,\quad \alpha=1,2 \qquad (90)$$

is also used often in the literature. This notation is useful because it allows us to work the the tensor formalism introduced in earlier chapters.

The assumption that the Jacobian $(J)^i{}_\alpha = \partial x^i/\partial u^\alpha$ is of maximal rank means that, by the implicit function theorem, it is possible (in principle) to (locally) solve for one of the coordinates say x^3 in terms of the other two

$$x^3 = f(x^1, x^2) \qquad (91)$$

The locus of points in $\mathbb{E}^3$ satisfying the equations $x^i = x^i(u^\alpha)$ can also be represented implicitly by an expression of the form

$$F(x^1, x^2, x^3) = 0 \qquad (92)$$

6.5 Definition. A coordinate chart transformation is a map φ from an open subset $U \subset \mathbb{E}^2$ to an open subset $V \subset \mathbb{E}^2$

$$\varphi : (u^1, u^2) \in U \longmapsto (\bar{u}^1, \bar{u}^2) \in V$$

such that the jacobian determinant

$$|D| = \begin{vmatrix} \frac{\partial u^1}{\partial \bar{u}^1} & \frac{\partial u^1}{\partial \bar{u}^2} \\ \frac{\partial u^2}{\partial \bar{u}^1} & \frac{\partial u^2}{\partial \bar{u}^2} \end{vmatrix}$$

is nowhere zero.

If $\mathbf{X}(u^1, u^2)$ is a coordinate chart, then a transformation $u^\alpha = u^\alpha(\bar{u}^\beta)$ induces another coordinate chart $\mathbf{Y}(\bar{u}^1, \bar{u}^2)$. We assume that the new chart satisfies the conditions of the definition (6.4) (See fig. 6.1)

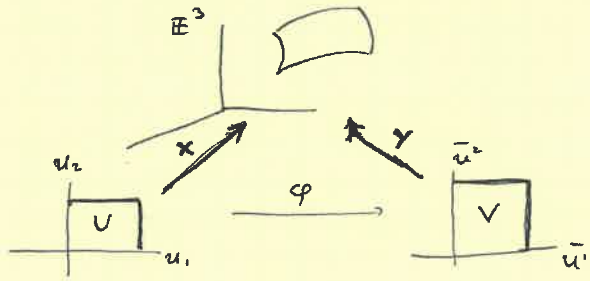


Fig. 6.1.

Two charts are said to be C^k equivalent if whenever a set of points in $\mathbb{E}^3$ is contained in the image of both charts $\mathbf{X}$ and $\mathbf{Y}$ then the map φ and its inverse φ^{-1} are C^k differentiable. In more lucid terms, this just says that the sets $\mathbf{X}(u^1, u^2)$ and $\mathbf{Y}(\bar{u}^1, \bar{u}^2)$ are different but equivalent parametrizations of the same set of points in $\mathbb{E}^3$.

6.6. Definition. A C^k differentiable surface M in $\mathbb{E}^3$ is a set of points in $\mathbb{E}^3$ such that

a) If $p \in M$ then p belongs to some C^k chart $\mathbf{X}$. (That is, locally a surface looks like a piece of $\mathbb{E}^2$)

b) If $p \in M$ belongs to two different charts $\mathbf{X}$ and $\mathbf{Y}$ then the charts $\mathbf{X}$ and $\mathbf{Y}$ are C^k equivalent. (That is, the charts represent the same set of points near p and they "patch up" in a C^k fashion)

Intuitively we may think of a surface locally as a patch and globally, as a quilt consisting of one or more patches sewn together smoothly

Notation. Given a parametrization of a surface in a local chart $x(u,v) = x(u^1,u^2) = x(u^\alpha)$, we write

$$x_u = x_1 = \frac{\partial x}{\partial u} = \frac{\partial x}{\partial u^1} \qquad x_{uu} = x_{11} = \frac{\partial^2 x}{\partial u^2}$$

$$x_v = x_2 = \frac{\partial x}{\partial v} = \frac{\partial x}{\partial u^2} \qquad x_{uv} = x_{12} = \frac{\partial^2 x}{\partial u \partial v} \quad \text{etc}$$

$$x_\alpha = \frac{\partial x}{\partial u^\alpha}, \qquad x_{\alpha\beta} = \frac{\partial^2 x}{\partial u^\alpha \partial u^\beta} \quad \text{etc.}$$

The vectors x_α are tangent to the surface's parametric curves $u = \text{const}$, and $v = \text{const}$.

First Fundamental Form.

Let $x^i(u^\alpha)$ be a local parametrization of a surface. Then, the Euclidean inner product in $\mathbb{E}^3$ induces an inner product in the space of tangent vectors to the surface. This metric in the surface is obtained as follows

$$dx^i = \frac{\partial x^i}{\partial u^\alpha} du^\alpha \qquad \text{(classically)} \quad dx = x_u du + x_v dv$$

$$\begin{aligned} \therefore \quad ds^2 &= \delta_{ij}\, dx^i dx^k \\ &= \delta_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} du^\alpha du^\beta \\ ds^2 &= g_{\alpha\beta}\, du^\alpha du^\beta \\ &= g_{11}\, du^2 + 2 g_{12}\, du\, dv + g_{22}\, dv^2 \end{aligned} \qquad \begin{aligned} ds^2 &= dx \cdot dx \\ &= (x_u du + x_v dv) \cdot (x_u du + x_v dv) \\ &= (x_u \cdot x_u) du^2 + 2\, x_u x_v\, du\, dv + (x_v \cdot x_v) dv^2 \\ &= E\, du^2 + 2F\, du\, dv + G\, dv^2 \end{aligned}$$

where
$$\begin{aligned} E &= g_{11} = x_u \cdot x_u \\ F &= g_{12} = x_u \cdot x_v \\ G &= g_{22} = x_v \cdot x_v \end{aligned} \qquad \text{or} \quad g_{\alpha\beta} = x_\alpha \cdot x_\beta = \langle x_\alpha, x_\beta \rangle$$

We find then, that any surface M in Euclidean three space comes equipped with a natural metric

$$ds^2 = g_{\alpha\beta}\, du^\alpha du^\beta \qquad 93)$$

induced by the metric in $\mathbb{E}^3$. A pair (M, g) where M is a surface and $g = g_{\alpha\beta}\, du^\alpha \otimes du^\beta$ is a metric on M is called a Riemannian manifold of dimension 2. The study of surfaces in $\mathbb{E}^3$ is the study of two dimensional Riemannian submanifolds of $\mathbb{E}^3$.
The metric g is classically called the first fundamental form of the surface. In modern differential geometry, the first fundamental form is denoted by

$$\begin{aligned} I(X,Y) &= g(X,Y) \\ &= \langle X, Y \rangle \end{aligned} \qquad \text{where } X, Y = \text{tangent vectors to } M \qquad (94)$$

Examples:

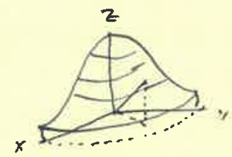
1) *Sphere.*

$$x = a\,\cancel{\sin}\,\theta\cos\varphi$$
$$y = a\sin\theta\sin\varphi$$
$$z = a\cos\theta$$

$$J = \begin{array}{c} \begin{bmatrix} -a\cos\varphi\sin\theta & a\cos\theta\sin\varphi & -a\sin\theta \\ -a\cos\theta\sin\varphi & a\sin\theta\cos\varphi & 0 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow x_u \\ \leftarrow x_v \end{array}$$

$$ds^2 = a^2 d\theta^2 + a^2\,\cancel{\sin}^2\theta\, d\varphi^2 \qquad E = a^2,\ F = 0,\ G = a^2\sin^2\theta$$

2) *Surface of revolution*

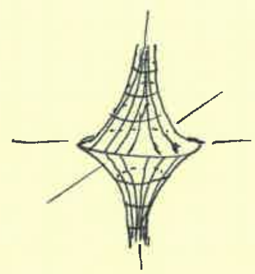


$$x = r\cos\theta,$$
$$y = r\sin\theta$$
$$z = f(r)$$

$$J = \begin{bmatrix} \cos\theta & \sin\theta & f'(r) \\ -r\sin\theta & r\cos\theta & 0 \end{bmatrix}$$

$$ds^2 = [1 + f'^2(r)]\,dr^2 + r^2 d\varphi^2 \qquad E = 1 + f'^2 \quad F = 0 \quad G = r^2$$

3) *Pseudosphere*

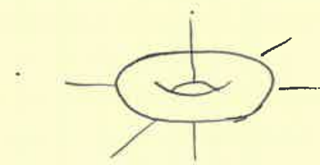


$$x = a\sin u\cos v$$
$$y = a\sin u\sin v$$
$$z = a\left(\cos u + \ln\tan\frac{u}{2}\right)$$

$$ds^2 = a^2\cot^2 u\, du^2 + a^2\sin^2 u\, dv^2$$

$$E = a^2\cot^2 u \quad F = 0 \quad G = a^2\sin^2 u$$

4) *Torus (of revolution)*



$$x = (b + a\cos u)\cos v$$
$$y = (b + a\cos u)\sin v$$
$$z = a\sin u$$

$$ds^2 = a^2 du^2 + (b + a\cos u)^2 dv^2 \qquad E = a^2 \quad F = 0 \quad G = (b + a\cos u)^2$$

5) *Helicoid*

$$x = u\cos v$$
$$y = u\sin v$$
$$z = av$$

$$ds^2 = du^2 + (u^2 + a^2)\,dv^2$$

$$E = 1 \quad F = 0 \quad G = u^2 + a^2$$

6) *Catenoid (of revolution)*

$$x = u\cos v$$
$$y = u\sin v$$
$$z = c\cosh^{-1}\frac{u}{c}$$

$$ds^2 = \frac{u^2}{u^2 - c^2}\,du^2 + u^2 dv^2 \qquad E = \frac{u^2}{u^2 - c^2} \quad F = 0 \quad G = u^2.$$

Orthogonal Coordinates.

All of the examples above have $F=0$. We now explore the geometrical implications of this property. Let A and B be vectors tangent to the surface at a point. Since X_1 and X_2 span the tangent space of the surface, we can write A and B as linear combinations of the X_α's.

$$A = A^\alpha X_\alpha$$
$$B = B^\alpha X_\alpha$$

The A^α's and B^α's are called the curvilinear components of the vectors. Computing the length of this vectors we get

$$\begin{aligned}\|A\|^2 = \langle A, A\rangle &= \langle A^\alpha X_\alpha, A^\beta X_\beta\rangle = A^\alpha A^\beta \langle X_\alpha, X_\beta\rangle \\ &= g_{\alpha\beta} A^\alpha A^\beta \end{aligned} \tag{95}$$

$$\|B\|^2 = g_{\alpha\beta} B^\alpha B^\beta$$

Also

$$\begin{aligned}\langle A, B\rangle &= \langle A^\alpha X_\alpha, B^\beta X_\beta\rangle = A^\alpha B^\beta \langle X_\alpha, X_\beta\rangle \\ &= g_{\alpha\beta} A^\alpha B^\beta .\end{aligned} \tag{96}$$

The angle φ subtended by these two vectors satisfies the equation

$$\cos\varphi = \frac{\langle A, B\rangle}{\|A\|\,\|B\|} = \frac{g_{\alpha\beta} A^\alpha B^\beta}{\sqrt{g_{\alpha\beta}A^\alpha A^\beta}\sqrt{g_{\alpha\beta}B^\alpha B^\beta}} \cdot = \frac{I(A,B)}{\sqrt{I(A,A)}\sqrt{I(B,B)}} \tag{97}$$

Let $u^\alpha = \varphi^\alpha(t)$ and $u^\alpha = \psi^\alpha(t)$ be two curves on the surface. Then the total differentials $du^\alpha = \frac{\partial\varphi^\alpha}{\partial t}dt$ and $\delta u^\alpha = \frac{\partial\psi^\alpha}{\partial t}dt$ are the components of infinitesimal vectors tangent to the curves. The angle between the curves thus satisfies

$$\cos\varphi = \frac{g_{\alpha\beta}\, du^\alpha \delta u^\beta}{\sqrt{g_{\alpha\beta}\, du^\alpha du^\beta}\sqrt{g_{\alpha\beta}\,\delta u^\alpha \delta u^\beta}} . \tag{98}$$

In particular, if the two sets of curves happened to be the parametric curves $u = \text{const}$, and $v = \text{const}$, we have

$du = 0$, dv arbitrary
$\delta u =$ arbitrary, $\delta v = 0$

$$\therefore \cos\varphi = \frac{g_{12}\, du\, dv}{\sqrt{g_{11}\, du^2}\sqrt{g_{22}\, dv^2}} = \frac{g_{12}}{\sqrt{g_{11} g_{22}}} = \frac{F}{\sqrt{EG}} . \tag{99}$$

The parametric lines are orthogonal if

$$F = 0 \tag{100}$$

Second Fundamental Form.

Let $x = x(u^\alpha)$ be a coordinate patch on a surface M and consider a curve on the surface. Such a curve would then be given by taking $u^\alpha = u^\alpha(s)$ where, for convenience, we choose to parametrize the curves by arc length s. Using the chain rule we can then compute the tangent T to the curve

$$T = \frac{dx}{ds} = x_\alpha \frac{du^\alpha}{ds} \qquad (101)$$

The vector T is tangent to the surface but the vector $T' = dT/ds$ does not in general have this property. The vector T' may be decomposed, however, into its normal components and tangential components (see fig)

$$\begin{aligned} T' &= \vec{K}_N + \vec{K}_G \\ &= k_n N + K_G \qquad (\text{where } k_n = \|\vec{K}_n\|) \end{aligned} \qquad (102)$$

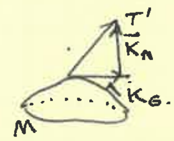


fig.

The quantity k_n is called the normal curvature and $\vec{K}_G$ is called the geodesic curvature vector. To get a formula for k_n we compute

$$T' = \frac{dT}{ds} = \left(\frac{dx_\alpha}{du^\beta}\frac{du^\beta}{ds}\right)\frac{du^\alpha}{ds} + x_\alpha \frac{d^2u^\alpha}{ds^2}$$

$$T' = x_{\alpha\beta}\frac{du^\beta}{ds}\frac{du^\alpha}{ds} + x_\alpha \frac{d^2u^\alpha}{ds^2}$$

Taking the inner product of the last equation with N we get

$$k_n = \langle N, T'\rangle = \langle x_{\alpha\beta}, N\rangle \frac{du^\alpha}{ds}\frac{du^\beta}{ds}$$

$$k_n = \frac{b_{\alpha\beta}\,du^\alpha du^\beta}{g_{\alpha\beta}\,du^\alpha du^\beta} \qquad (103)$$

where

$$b_{\alpha\beta} = \langle x_{\alpha\beta}, N\rangle \qquad (104)$$

$$ds^2 = g_{\alpha\beta}\,du^\alpha du^\beta$$

The quantity $b_{\alpha\beta}\,du^\alpha du^\beta$ is called classically the second fundamental form II. That is

$$k_n = \frac{\mathrm{II}}{\mathrm{I}} = \frac{e\,du^2 + 2f\,du\,dv + g\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2} \qquad (105)$$

where $e = b_{11}$, $f = b_{12}$, $g = b_{22}$

Explicit formulas for $e, f,$ and g are easy to obtain since

$$N = \frac{x_u \times x_v}{\| x_u \times x_v \|} \tag{106}$$

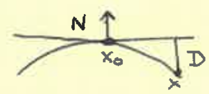
The results are

$$e = \frac{x_{uu} \cdot (x_u \times x_v)}{\sqrt{EG - F^2}} \qquad f = \frac{x_{uv} \cdot (x_u \times x_v)}{\sqrt{EG - F^2}} \qquad g = \frac{x_{vv} \cdot (x_u \times x_v)}{\sqrt{EG - F^2}} \tag{107}$$

Also note that since $\langle x_\alpha, N \rangle = 0$ we have

$$b_{\alpha\beta} = \langle x_{\alpha\beta}, N \rangle = -\langle x_\alpha, N_\beta \rangle \tag{108}$$

The first fundamental form on a surface measures the (square) infinitesimal distance between two ~~(nearby pts)~~ neighboring points. Is there an analogous interpretation of the second fundamental form? The answer is yes. The second fundamental form measures the infinitesimal distance between a point on the surface and the tangent plane of a neighboring point (see fig)



To see this, let $x_{(0)} = x(u^\alpha_0)$ be a point on the surface. Taking the Taylor expansion we get

$$x(u^\alpha_0 + du^\alpha) = x_{(0)} + x_{(0)\alpha}\, du^\alpha + \tfrac{1}{2} x_{(0)\alpha\beta}\, du^\alpha du^\beta + \cdots \tag{109}$$

The distance D is given by

$$\begin{aligned} D &= \langle N, x - x_0 \rangle \\ &= \tfrac{1}{2} \langle x_{\alpha\beta}, N \rangle du^\alpha du^\beta \qquad \text{(dropping the "o's")} \\ &= \tfrac{1}{2}\, \mathrm{II} \end{aligned} \tag{110}$$

The second fundamental form contains information about the shape of a surface at a point. In particular if $b = |b_{\alpha\beta}| > 0$ at a point then, in view of the discussion above, all the neighboring points lie on the same side of the tangent plane and we have what is called an elliptic point. The following terminology is often used

$$b = eg - f^2 \begin{cases} > 0 & \text{elliptic pt.} \\ < 0 & \text{Hyperbolic pt.} \\ = 0 & \text{Parabolic pt.} \end{cases} \tag{111}$$




 etc.

6.7 Definition Let X be a vector tangent to a surface and N be the unit normal to the surface. The map

$$LX = \bar{\nabla}_X N \tag{112}$$

is called the <u>Gauss map</u>.

6.8 **Prop.** L is a linear transformation on the tangent space.
Proof:
Linearity follows from the linearity of $\bar{\nabla}$. That $LX \in TM$ whenever $X \in TM$ follows because

$$\langle N, N\rangle = 1$$
$$\therefore\ X\langle N,N\rangle = 2\langle \bar{\nabla}_X N, N\rangle$$
$$= 2\langle LX, N\rangle = 0 \qquad \therefore\ LX \perp N$$

We define the quantity

$$\mathrm{II}(X,Y) = \langle LX, Y\rangle \qquad X,Y \in TM \tag{113}$$

6.9 Prop. The Gauss map is symmetric (ie $\mathrm{II}(X,Y) = \mathrm{II}(Y,X)$) if $[X,Y] \in TM$ whenever $X, Y \in TM$.
Proof.

$$\mathrm{II}(X,Y) - \mathrm{II}(Y,X) = \langle \bar{\nabla}_X N, Y\rangle - \langle \bar{\nabla}_Y N, X\rangle$$
$$= \langle N, \bar{\nabla}_Y X\rangle - \langle N, \bar{\nabla}_X Y\rangle$$
$$= \langle N, \bar{\nabla}_Y X - \bar{\nabla}_X Y\rangle$$
$$= \langle N, [Y,X]\rangle = \langle [X,Y], N\rangle$$
$$= 0 \quad \text{iff} \quad [Y,X] \in TM.$$

6.10) <u>Definition</u>. The eigenvalues of L are called the principal curvatures of M and the eigenvectors are called the principal directions. That is, if

$$LX_1 = k_1 X_1 \quad \text{and} \quad LX_2 = k_2 X_2 \qquad X_1, X_2 \in TM \tag{114}$$

then k_1 and k_2 are the principal curvatures.

The invariants

$$K = \det L$$
$$H = \tfrac{1}{2}\operatorname{Tr} L \tag{115}$$

are called the Gaussian and the mean curvature of the surface. Clearly

$$K = k_1 k_2$$
$$H = \tfrac{1}{2}(k_1 + k_2) \tag{116}$$

The reader should note that the two uses of the symbol $\mathrm{I\!I}$ are consistent of equation (108) $b_{\alpha\beta} = -\langle X_\alpha, N_\beta \rangle$. That is

$$b_{\alpha\beta} = -\langle LX_\beta, X_\alpha \rangle = \mathrm{I\!I}(X_\alpha, X_\beta) \tag{117}$$

6.11 Theorem. Let X and Y be any two tangent vectors to a surface M. Then

$$\begin{aligned} LX \times LY &= K(X \times Y) \\ LX \times Y + X \times LY &= 2H(X \times Y) \end{aligned} \tag{118}$$

Proof.
Since LX and $LY \in TM$ then we can write

$$\begin{aligned} LX &= aX + bY \\ LY &= cX + dY \end{aligned}$$

Computing cross products we get:

$$\begin{aligned} LX \times LY &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} X \times Y \\ &= (\det L)\, X \times Y \\ &= K(X \times Y) \end{aligned}$$

Similarly

$$\begin{aligned} (LX \times Y) + (X \times LY) &= (a+d)(X \times Y) \\ &= (\operatorname{Tr} L)(X \times Y) \\ &= 2H(X \times Y) \end{aligned}$$

6.12. Corollary.

$$K = \frac{eg - f^2}{EG - F^2} \qquad H = \frac{1}{2}\, \frac{Eg - 2fF + eG}{(EG - F^2)} \tag{119}$$

Proof:
Starting with equations (118) we take the dot product of both sides with $(X \times Y)$ using the vector identity

$$\langle V \times W, X \times Y \rangle = \begin{vmatrix} \langle V, X \rangle & \langle V, Y \rangle \\ \langle W, X \rangle & \langle W, Y \rangle \end{vmatrix}.$$

After solving the equations for H and K the results are

$$K = \frac{\begin{vmatrix} \langle LX, X \rangle & \langle LX, Y \rangle \\ \langle LY, X \rangle & \langle LY, Y \rangle \end{vmatrix}}{\begin{vmatrix} \langle X, X \rangle & \langle X, Y \rangle \\ \langle Y, X \rangle & \langle Y, Y \rangle \end{vmatrix}} \qquad 2H = \frac{\begin{vmatrix} \langle LX, X \rangle & \langle LX, Y \rangle \\ \langle Y, X \rangle & \langle Y, Y \rangle \end{vmatrix} + \begin{vmatrix} \langle X, X \rangle & \langle X, Y \rangle \\ \langle LY, X \rangle & \langle LY, Y \rangle \end{vmatrix}}{\begin{vmatrix} \langle X, X \rangle & \langle X, Y \rangle \\ \langle Y, X \rangle & \langle Y, Y \rangle \end{vmatrix}}$$

The result then follows by taking $X = X_u$, $Y = Y_v$

The formulas just derived are very useful for actual computations

6.13 Euler's Theorem. Let x_1 and x_2 be the eigenvectors of L and let $x = (\cos\theta)x_1 + (\sin\theta)x_2$. Then

$$\mathrm{II}(x,x) = k_1\cos^2\theta + k_2\sin^2\theta \tag{120}$$

Proof: Easy. Just compute $\mathrm{II}(x,x) = \langle Lx, x\rangle$

$$\begin{aligned}\langle Lx, x\rangle &= \langle (\cos\theta)k_1 x_1 + (\sin\theta)k_2 x_2, (\cos\theta)x_1 + (\sin\theta)x_2\rangle \\ &= k_1\cos^2\theta + k_2\sin^2\theta\end{aligned}$$

6.14 Gauss & Weingarten equations. We have already seen that if $x(u^\alpha)$ is a coordinate chart of a surface M then the two vectors x_α are tangential to the surface. On the other hand the vectors $x_{\alpha\beta}$ are, in general, not tangential. The equation of Gauss is simply the decomposition of $x_{\alpha\beta}$ into its tangential and normal components. In a similar fashion, the vector N_α is tangential to M since N_α is orthogonal to the unit normal N. The equation of Weingarten is an expression of N_α as a linear combination of the tangent vectors x_α. The Gauss & Weingarten equations are

$$x_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta}x_\gamma + b_{\alpha\beta}N \tag{121}$$

$$N_\alpha = b^\beta{}_\alpha x_\beta \tag{122}$$

The quantities $\Gamma^\gamma_{\alpha\beta}$ are called the Christoffel symbols. To obtain an expression for them we take the inner product of Gauss's equation with the vector x_δ. The result is

$$\langle x_{\alpha\beta}, x_\delta\rangle = \Gamma^\gamma_{\alpha\beta}\, g_{\gamma\delta} \equiv [\alpha\beta;\delta]$$

$$\therefore \qquad \Gamma^\gamma_{\alpha\beta} = g^{\gamma\delta}\langle x_{\alpha\beta}, x_\delta\rangle \tag{123}$$

The coefficients of N in Gauss's formula are precisely the components of the second fundamental form as can be readily verified by taking the inner product of both sides of the equation with N

$$b_{\alpha\beta} = \langle x_{\alpha\beta}, N\rangle$$

The quantities $b^\beta{}_\alpha$ in Weingarten's equation can be evaluated by taking the inner product of the equation with x_δ. We get

$$\langle N_\alpha, x_\delta\rangle = b^\beta{}_\alpha\, g_{\beta\delta} = -\langle N, x_{\delta\alpha}\rangle = -b_{\delta\alpha}$$

$$\therefore \quad b^\beta{}_\alpha = -g^{\beta\delta}b_{\delta\alpha} \tag{124}$$

6.14-1

Proposition: $\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\mu} (g_{\alpha\mu,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu})$ (124.1)

Proof. Differentiate the equation $g_{\alpha\beta} = \langle x_\alpha, x_\beta \rangle$ with respect to u_μ. The result is

$$g_{\alpha\beta,\mu} = \langle x_{\alpha\mu}, x_\beta \rangle + \langle x_\alpha, x_{\beta\mu} \rangle = [\alpha\mu;\beta] + [\beta\mu;\alpha]$$

Also

$$g_{\alpha\mu,\beta} = \langle x_{\alpha\beta}, x_\mu \rangle + \langle x_\alpha, x_{\mu\beta} \rangle = [\alpha\beta;\mu] + [\mu\beta;\alpha]$$

$$g_{\mu\beta,\alpha} = \langle x_{\mu\alpha}, x_\beta \rangle + \langle x_\mu, x_{\beta\alpha} \rangle = \langle \mu\alpha;\beta \rangle + [\alpha\beta;\mu]$$

Adding the last two equations and subtracting the first, we get

$$2[\alpha\beta;\mu] = g_{\alpha\mu,\beta} + g_{\beta\mu,\alpha} - g_{\alpha\beta,\mu} \qquad (124.2)$$

$$\therefore \quad \Gamma^{\gamma}_{\alpha\beta} = g^{\gamma\mu}[\alpha\beta;\mu] = \frac{1}{2} g^{\gamma\mu}(g_{\alpha\mu,\beta} + g_{\beta\mu,\alpha} - g_{\alpha\beta,\mu})$$

Corollary: If $g_{12} = 0$ (Orthogonal coordinates) then

$$\Gamma^1_{11} = \frac{1}{2g_{11}} \frac{\partial g_{11}}{\partial u^1} \qquad \Gamma^1_{12} = \frac{1}{2g_{11}} \frac{\partial g_{11}}{\partial u^2} \qquad \Gamma^1_{22} = -\frac{1}{2g_{11}} \frac{\partial g_{22}}{\partial u^1}$$

$$\Gamma^2_{22} = \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial u^2} \qquad \Gamma^2_{21} = \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial u^1} \qquad \Gamma^2_{11} = -\frac{1}{2g_{22}} \frac{\partial g_{11}}{\partial u^2} \qquad (124.3)$$

Proof. Compute using (124.1) and the fact that if $g_{12} = 0$ then $g^{11} = 1/g_{11}$ and $g^{22} = 1/g_{22}$.

Corollary: In a general coordinate system ($g_{12} \neq 0$) we have

$$\Gamma^1_{11} = \frac{1}{2g}(GE_u - 2FF_u + FE_v) \qquad \Gamma^2_{22} = \frac{1}{2g}(EG_v - 2FF_v + FG_u)$$

$$\Gamma^1_{12} = \frac{1}{2g}(GE_v - FG_u) \qquad \Gamma^2_{21} = \frac{1}{2g}(EG_u - FE_v) \qquad (124.4)$$

$$\Gamma^1_{22} = \frac{1}{2g}(2GF_v - GG_u - FG_v) \qquad \Gamma^2_{11} = \frac{1}{2g}(2EF_u - EE_v - FE_u)$$

where $g = EG - F^2$.

The computations above show that the Christoffel symbols depend only on the first fundamental form; that is, they only depend on the metric. Quantities such as these are called "intrinsic" to the surface

6.15. Gauss Coddazzi Equations.

The classical Gauss and Coddazzi equations are obtained simply by noticing that $x_{\alpha\beta\nu} = x_{\alpha\nu\beta}$ and $N_{\alpha\beta} = N_{\beta\alpha}$. We compute using the Gauss - Weingarten equations.

Computation:

$$\begin{aligned} x_{\alpha\beta\nu} &= (\Gamma^{\mu}{}_{\alpha\beta} x_{\mu} + b_{\alpha\beta} N)_{,\nu} \\ &= \Gamma^{\mu}{}_{\alpha\beta,\nu} x_{\mu} + \Gamma^{\mu}{}_{\alpha\beta} (\Gamma^{\rho}_{\mu\nu} x_{\rho} + b_{\mu\nu} N) + b_{\alpha\beta,\nu} N + b_{\alpha\beta} N_{\nu} \\ &= \Gamma^{\mu}{}_{\alpha\beta,\nu} x_{\mu} + \Gamma^{\mu}{}_{\alpha\beta} \Gamma^{\rho}_{\mu\nu} x_{\rho} + \Gamma^{\mu}_{\alpha\beta} b_{\mu\nu} N + b_{\alpha\beta,\nu} N + b_{\alpha\beta} (b^{\rho}{}_{\nu} x_{\rho}) \end{aligned}$$

Separating the tangential and normal components and ~~separations~~) relabeling indices we get

$$x_{\alpha\beta\nu} = (\Gamma^{\mu}{}_{\alpha\beta,\nu} + \Gamma^{\rho}_{\alpha\beta} \Gamma^{\mu}_{\rho\nu} + b_{\alpha\beta} b^{\mu}{}_{\nu}) x_{\mu} + (b_{\alpha\beta,\nu} + b_{\mu\nu} \Gamma^{\mu}{}_{\alpha\beta}) N$$
$$x_{\alpha\nu\beta} = (\Gamma^{\mu}{}_{\alpha\nu,\beta} + \Gamma^{\rho}_{\alpha\nu} \Gamma^{\mu}_{\rho\beta} + b_{\alpha\nu} b^{\mu}{}_{\beta}) x_{\mu} + (b_{\alpha\nu,\beta} + b_{\mu\beta} \Gamma^{\mu}{}_{\alpha\nu}) N$$

Subtracting the last two equations and equating the tangential and normal components to zero we get

$$R^{\mu}{}_{\alpha\nu\beta} = b_{\alpha\nu} b^{\mu}{}_{\beta} - b_{\alpha\beta} b^{\mu}{}_{\nu} \tag{125}$$

and

$$b_{\alpha\beta,\nu} - b_{\alpha\nu,\beta} = b_{\mu\nu} \Gamma^{\mu}{}_{\alpha\beta} - b_{\mu\beta} \Gamma^{\mu}{}_{\alpha\nu} \tag{126}$$

where

$$R^{\mu}{}_{\alpha\nu\beta} = \Gamma^{\mu}{}_{\alpha\beta,\nu} - \Gamma^{\mu}{}_{\nu\beta,\alpha} + \Gamma^{\rho}_{\alpha\beta} \Gamma^{\mu}_{\rho\nu} - \Gamma^{\rho}_{\alpha\nu} \Gamma^{\mu}_{\rho\beta} . \tag{127}$$

The quantity $R^{\mu}{}_{\alpha\nu\beta}$ is called the Riemann tensor for the surface. In two dimensions this tensor only has one independent component, namely R_{1212}. All all other components are equal to plus or minus this value

6.16 Theorema Egregium. (Gauss)

$$K = -\frac{R_{1212}}{g} \tag{128}$$

Proof. We rewrite the Gauss and Coddazzi equation (125) by lowering one index with the metric $g_{\alpha\beta}$ of the surface.

$$R_{\mu\alpha\nu\beta} = b_{\alpha\nu} b_{\mu\beta} - b_{\alpha\beta} b_{\mu\nu}.$$

So

$$\begin{aligned} R_{1212} &= b_{21} b_{12} - b_{22} b_{11} \\ &= -f^2 + eg. \end{aligned}$$

Hence $K = \dfrac{eg - f^2}{g} = -\dfrac{R_{1212}}{g}$.

The theorem is remarkable because it states that the Gaussian curvature depends only on the metric of the surface (i.e: the first fundamental form.)

6.17) *Corollary.*

$$K = \frac{1}{g}\left[\frac{\partial^2 g_{12}}{\partial u^1 \partial u^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{(\partial u^2)^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{(\partial u^1)^2} - (\Gamma^\mu{}_{11}\Gamma^\nu{}_{22} - \Gamma^\mu{}_{12}\Gamma^\nu{}_{21})\, g_{\mu\nu}\right] \tag{129}$$

Proof

$$K = -\frac{R_{1212}}{g} = -\frac{g_{1\mu}}{g} R^\mu{}_{212}$$

$$= -\frac{1}{g}\left[\Gamma^\mu{}_{22,1} - \Gamma^\mu{}_{12,2} + \Gamma^\rho{}_{22}\Gamma^\mu{}_{\rho 1} - \Gamma^\rho{}_{21}\Gamma^\mu{}_{\rho 2}\right] g_{1\mu}$$

$$= \frac{1}{g}\left\{ \frac{\partial}{\partial u^2}[12;1] - \frac{\partial}{\partial u^1}[22;1] - (\Gamma^\mu{}_{11}\Gamma^\nu{}_{22} - \Gamma^\mu{}_{12}\Gamma^\nu{}_{21})\, g_{\mu\nu}\right\}$$

An easy computation using (124.2) yields the result.

The formula in this corollary gives an expression for the Gaussian curvature in terms of the metric. There are many other neat expressions for K in the classical literature and we refer the reader to the standard books on the subject for these. One formula that is particularly useful can be obtained using (124.3) when $g_{12} = 0$. The result is

$$K = -\frac{1}{\sqrt{g_{11} g_{12}}}\left[\frac{\partial}{\partial u^1}\left(\frac{1}{\sqrt{g_{11}}}\frac{\partial \sqrt{g_{22}}}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{1}{\sqrt{g_{22}}}\frac{\partial\sqrt{g_{11}}}{\partial u^2}\right)\right] \tag{130}$$

Notice that if $g_{11} = g_{12} = f^2$ this formula is very closely related to the formula for the Laplacian. Can you figure out the significance of this?!

VII STRUCTURE EQUATIONS.

Frames and Coframes in $\mathbb{E}^3$.

7.1) Definition. A frame in $\mathbb{E}^3$ is a triplet $\{e_i\}$ of mutually orthogonal unit vectors in $\mathbb{E}^3$. The vectors e_i of a frame are, of course, expressible ~~as~~ a linear combination of the standard frame vectors $\partial/\partial x^i = \partial_i$. In other words, we can write

$$e_i = h_i^{\ j}\,\partial_j \tag{131}$$

We will denote the dual forms to the vectors e_i by θ^i. Since $\theta^i(e_j) = \delta^i_{\ j}$ we then have

$$\theta^i = h^i_{\ j}\,dx^j \tag{132}$$

where $h^i_{\ j}$ is the matrix inverse to the matrix $h_i^{\ j}$. That is

$$h^i_{\ k}\,h_j^{\ k} = \delta^i_{\ j}\,. \tag{133}$$

7.2. Definition. Let e_i be a frame in $\mathbb{E}^3$ and X be any vector field in $\mathbb{E}^3$. The connection coefficients of the frame are defined by the formula

$$\bar\nabla_X e_i = \omega^j_{\ i}(X)\,e_j\,. \tag{134}$$

This formula makes sense because the covariant derivative of e_i in the direction of X is a vector and as such it is expressible as a linear combination of the elements of the frame. Note that since $\theta^k(e_j) = \delta^k_{\ j}$ we have

$$\omega^k_{\ j}(X) = \theta^k(\bar\nabla_X e_j) = \langle \theta^k, \bar\nabla_X e_j\rangle \tag{135}$$

7.3) Prop. $\omega^k_{\ j}$ is a linear transformation in the tangent space of $\mathbb{E}^3$. i.e. $\omega^k_{\ j}$ is a matrix of one-forms.

Proof. Linearity follows from the linearity of $\bar\nabla$. That $\omega^i_{\ j}$ maps vectors into numbers is obvious from the definition (7.2)

7.4. Prop. $\omega_{ij} = -\omega_{ji}$ where $\omega_{ij} = g_{ik}\,\omega^k_{\ j}$.

Proof.

In the frame e_i we have a metric $\langle e_i, e_j\rangle = g_{ij}$. Assume that g_{ij} has constant components. Then

$$\bar\nabla_X\langle e_i, e_j\rangle = \langle \omega^k_{\ i}(X)e_k, e_j\rangle + \langle e_i, \omega^k_{\ j}(X)e_k\rangle = 0$$

$$\therefore\ g_{kj}\,\omega^k_{\ i} + g_{ik}\,\omega^k_{\ j} = 0$$

$$\omega_{ji} + \omega_{ij} = 0 \tag{136}$$

7.5. Prop. $\omega = (dh)h^{-1}$ (In matrix form) (137)

Proof.

$$\theta^i = h^i_j\, dx^j \quad , \quad e_i = h_i^{\ j}\, \partial_j$$

$$\begin{aligned} \bar\nabla_X e_i = \omega^k_{\ i}(X)\, e_k &= X(h_i^{\ j})\,\partial_j \\ &= dh_i^{\ j}(X)\,\partial_j \\ &= dh_i^{\ j}(X)\, h^k_{\ j}\, e_k . \quad \Rightarrow \quad \omega^k_{\ i}(X) = dh_i^{\ j}(X) \cdot h^k_{\ j} . \end{aligned}$$

7.6. Prop. (Cartan Eq's. for $\mathbb{R}^3$)

$$d\theta^i - \omega^i_{\ k}\wedge\theta^k = 0 \qquad (d\theta - \omega\wedge\theta = 0) \qquad (138)$$
$$d\omega^i_{\ j} - \omega^i_{\ k}\wedge\omega^k_{\ j} = 0 \qquad (d\omega - \omega\wedge\omega = 0)$$

Proof: a) $d\theta^i = d(h^i_{\ j}\, dx^j) = dh^i_{\ j}\wedge dx^j = dh^i_{\ j}\wedge h^j_{\ k}\,\theta^k = dh^i_{\ j}\, h^j_{\ k}\wedge\theta^k$
$= \omega^i_{\ k}\wedge\theta^k$

b) $d(d\theta^i) = 0 = d\omega^i_{\ k}\wedge\theta^k - \omega^i_{\ k}\wedge d\theta^k$
$= d\omega^i_{\ k}\wedge\theta^k - \omega^i_{\ k}\wedge(\omega^k_{\ j}\wedge\theta^j)$
$= (d\omega^i_{\ j} - \omega^i_{\ k}\wedge\omega^k_{\ j})\wedge\theta^j = 0 . \ \therefore\ d\omega^i_{\ j} - \omega^i_{\ k}\wedge\omega^k_{\ j} = 0.$

Cartan Equations for Surfaces.

Let $x = x(u^1, u^2)$ be a coordinate patch for a surface M in $\mathbb{R}^3$. We pick a frame in $\mathbb{R}^3$ as follows. We take e_1 and e_2 to be any two vectors in TM and let $e_3 = N$, the unit normal. We of course require that the vectors be orthonormal. Let $\{\theta^1, \theta^2, \theta^3\}$ be the dual coframe. Then $\theta^3 = 0$! on the surface. The only nonzero forms on the surface are then: $\theta^1, \theta^2, \omega^1_{\ 2}, \omega^1_{\ 3}$ and $\omega^2_{\ 3}$. In view of Cartan's equations (138) we have the following proposition

7.7) Prop. Let $\{\theta^1, \theta^2, \theta^3\}$ be an adapted coframe to a surface $M \subset \mathbb{R}^3$. Then:

a) $d\theta^1 = \omega^1_{\ 2}\wedge\theta^2$
b) $d\theta^2 = \omega^2_{\ 1}\wedge\theta^1$
c) $d\theta^3 = \omega^3_{\ 1}\wedge\theta^1 + \omega^3_{\ 2}\wedge\theta^2 = 0$

} Gauss equations

d) $d\omega^1_{\ 2} = \omega^1_{\ 3}\wedge\omega^3_{\ 2}$
e) $d\omega^1_{\ 3} = \omega^1_{\ 2}\wedge\omega^2_{\ 3}$
f) $d\omega^2_{\ 3} = \omega^2_{\ 1}\wedge\omega^1_{\ 3}$

} Gauss Coddazzi equations (139)

7.8) Theorem.

$$d\omega^1_{\ 2} = -K\ \theta^1\wedge\theta^2$$
$$\omega^1_{\ 3}\wedge\theta^2 + \theta^1\wedge\omega^2_{\ 3} = 2H\ \theta^1\wedge\theta^2$$

Proof. We prove the first part

$$L e_1 = \nabla_{e_1} e_3 = \omega^1_{\ 3}(e_1)\, e_1 + \omega^2_{\ 3}(e_1)\, e_2$$
$$L e_2 = \nabla_{e_2} e_3 = \omega^1_{\ 3}(e_2)\, e_1 + \omega^2_{\ 3}(e_2)\, e_3$$

$$K = \det L = \omega^1_{\ 3}(e_1)\,\omega^2_{\ 3}(e_2) - \omega^1_{\ 3}(e_2)\,\omega^2_{\ 3}(e_1) = -(\omega^1_{\ 3}\wedge\omega^3_{\ 2})(e_1, e_2) = -d\omega^1_{\ 2}(e_1, e_2)$$

The theorem follows