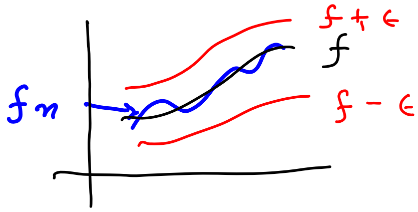
Tuesday, January 15, 2008
11:15 AM

Uniform Convergence

Def Let $f_n(x)$ be sequence of functions

$f_n \to f$ uniformly if for every $\epsilon > 0$
$\exists$ an $N > 0$ such that
$|f_n(x) - f(x)| < \epsilon$ whenever $n > N$, for all x



Theorem (Cauchy Criterium)

$f_n(x) \to f(x)$ uniformly if for every
$\epsilon > 0$, $\exists$ $N > 0$ such that

$|f_n(x) - f_m(x)| < \epsilon$ whenever $n, m > N$
for all x

Proof

Suppose $f_n \to f$ uniformly. Then $\forall \epsilon > 0$
$\exists$ $N_1, N_2 > 0$ such that

$\Rightarrow$ $|f_n(x) - f(x)| < \epsilon/2$ if $n > N_1$
$|f_m(x) - f(x)| < \epsilon/2$ if $m > N_2$

Let $N = \max\{N_1, N_2\}$

$$|f_n(x) - f_m(x)| \le |f_n(x) - f(x)| + |f(x) - f_m(x)|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$< \epsilon$$

$\Leftarrow$) If the Cauchy Criterium holds
then $\forall x$ $f_n(x)$ is a Cauchy seq
$f_n(x) \to f(x)$ uniformly

Def $f(x) = \sum_{n=1}^{\infty} f_n(x)$ we say that

$f(x)$ converges uniformly if the sequence
of partial sums
$s_k = \sum_{n=1}^{k} f_n(x)$ converges uniformly

__Thm__ Suppose $f_n \to f$, and $f_n \leq M_n$ and $\sum^{\infty} M_n$ converges, then $\sum f_n$ converges uniformly.

__Proof__ Given $\epsilon > 0$ pick $N > 0$ such that

$$\left| \sum_{k=0}^{n} M_k - \sum_{k=0}^{m} M_k \right| < \epsilon \quad \text{when } n, m > N$$

We can do this because $\mathbb{R}$ is complete and any convergent sequence is a C.S.

Then

$$\left| \sum_{k=0}^{n} f_k - \sum_{k=0}^{m} f_k \right| < \epsilon \quad \forall x$$

f_k satisfies the Cauchy Criterium

$\therefore f_k \to f$ uniformly

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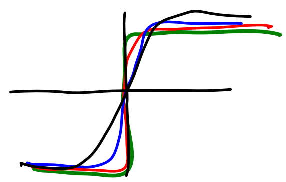
Continuity.

<u>Theorem</u>: If $f_n(x)$ are continuous and $f_n(x) \to f(x)$ uniformily, then $f(x)$ is also continuous

Counter Example

$f_n(x) = \tan^{-1} nx$



$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

$\lim_{n \to \infty} f_n(x) = H(x) \cdot \frac{\pi}{2}$

$f_n \to H$, but H not continuous.

<u>Theorem</u> $\{\mathcal{C}(\mathbb{R}), \sup \|\cdot\|\}$ is complete

The set of continuous functions in $\mathbb{R}$ with the sup norm is complete.

Integration

<u>Theorem</u>: If $f_n \in \mathcal{R}[a,b]$, $\alpha(x)$ is monotonically increasing and $f_n \xrightarrow{u} f$, then

a) $f \in \mathcal{R}[a,b]$

b) $\lim_{n \to \infty} \int_a^b f_n(x)\, d\alpha = \int_a^b f(x)\, d\alpha$

$= \int_a^b \lim_{n \to \infty} f_n(x)\, d\alpha$

or $\int_a^b f_n(x)\, d\alpha \to \int_a^b f(x)\, d\alpha$

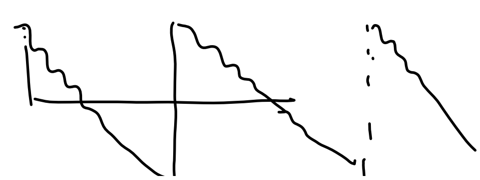
<u>Corollary</u>

If $f(x) = \sum_{n=0}^{\infty} f_n(x)$ which converges Unif.

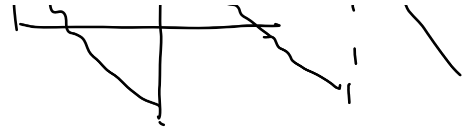
then $\int_a^b f(x)\, dx = \sum_{n=0}^{\infty} \int_a^b f_n(x)\, dx$

Counter Example

Let $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$



Let $f(x) = \sum_{n=1} \frac{\sin nx}{n}$



Differentiation

Theorem: Suppose that f_n is a sequence of differentiable functions which are also cont, and f_n' converge uniformly. Also suppose that $f_n(x_0)$ converges at x_0, Then

a) $f_n(x) \to f(x)$ for all x

b) $f(x)$ is differentiable

c) $\lim_{n\to\infty} f_n' = f'$

Proof Since f_n' are continuous, they are integrable, and by the FTC

$$\int_{x_0}^{x} f_n'(t)\,dt = f_n(x) - f_n(x_0)$$

Suppose $f_n' \xrightarrow{u} g$

$$\lim_{n\to\infty} \int_{x_0}^{x} f'_n(t)\,dt = \int_{x_0}^{x} g(t)\,dt =$$

$$\underbrace{\int_{x_0}^{x} g(t)\,dt}_{\text{Conv}} = \lim_{n\to\infty} [f_n(x) - \underbrace{f_n(x_0)}_{\text{Conv}}]$$

$$\therefore \lim_{n\to\infty} f_n(x) = f(x)$$

$$\int_{x_0}^{x} g(t)\,dt = f(x) - f(x_0)$$

$$g(t) = f'(x)$$

$$\therefore f_n' \to f'$$

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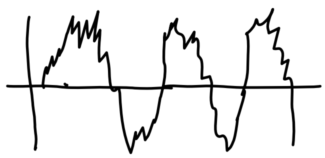
Weierstrak

Example

$$f(x) = \sum_{n=1}^{\infty} a^n \cos(b^n \pi x) \qquad 0 < a < 1 \qquad b\text{-odd}$$

Choose $a = \frac{3}{4}$, $b = 3$



Recall Weierstrak M-Test

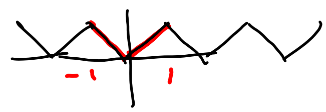
Thm If f_n cont, $f_n < M_n$ and $\sum M_n$ C

Then $\sum f_n$ converges to a cont f.

Here $f_n < \left(\frac{3}{4}\right)^n$

Example:

$\varphi(x) = |x| \quad -1 \le x \le 1$



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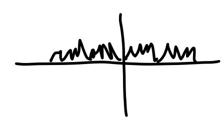

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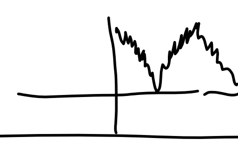
$\varphi(x+2) = \varphi(x)$

Let $f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$ $\rightarrow$



$$|\varphi(x) - \varphi(y)| = \big|\,|x| - |y|\,\big| \le |x-y|$$

$|x+y| \le |x| + |y|$

$x \mapsto x-y$

$|x| \le |x-y| + |y|$

$|x| - |y| \le |x-y|$

$y \mapsto y-x$

If $\epsilon > 0$ is given, choose $\delta = \epsilon$ then

$|x-y| < \delta \Rightarrow |\varphi(x) - \varphi(y)| < \epsilon$

φ is cont and $|\varphi| \le 1$

f_n cont, $|f_n| \le M_n \quad M_n = \frac{3}{4}$.

By M-Test $f_n \to f$ cont.

Claim f is nowhere differentiable

Consider $f(x+\delta_m) - f(x)$ ... $\delta_m = \pm \frac{1}{2} 4^{-m}$ ...

Claim f is nowhere differentiable

Consider $\dfrac{f(x+\delta_m)-f(x)}{\delta_m} = g(x)$ $\quad \delta_m = \pm \frac{1}{2} 4^{-m} \to 0$

$$g(x) = \frac{1}{\pm \frac{1}{2} 4^{-m}} \left[\sum \left(\tfrac{3}{4}\right)^n \varphi(4^n(x+\delta_m) - \sum \left(\tfrac{3}{4}\right)^n \varphi(4^n x) \right]$$

$$= \pm 2 \cdot 4^m \left[\sum_{n=0}^{\infty} \left(\tfrac{3}{4}\right)^n \left[\varphi(4^n(x+\delta_m)) - \varphi(4^n x) \right] \right]$$

$n \begin{cases} > m \\ = m \\ < m \end{cases}$ $\quad$ If $n > m$ $\quad \varphi(4^n(x \pm \frac{1}{2} 4^{-m}) - \varphi(4^n x)$

$$= \varphi(4^n x \pm \tfrac{1}{2} 4^{n-m}) - \varphi(4^n x)$$

$\therefore$
$$= \varphi(4^n x \pm 2K) - \varphi(4^n x)$$

$$= 0 \quad \text{because } \varphi(x+2) = \varphi(x)$$

If $n = m$ $\quad$ Def $\gamma_n = \dfrac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m}$

$$\to \gamma_m = \frac{\varphi(4^n x \pm \frac{1}{2}) - \varphi(4^n x)}{\pm \frac{1}{2} 4^{-m}} = 4^m$$

If $n > m \quad \gamma_n = 0$

$n = m \quad \gamma_m = 4^m$

$n < m \quad |\gamma_n| \le 4^n$

$$|\gamma_n| = \left| \frac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m} \right| \qquad \delta_m = \pm \tfrac{1}{2} 4^{-m}$$

$$= \left| 2 \cdot 4^m \left[\varphi(4^n(x \pm \tfrac{1}{2} 4^{-m}) - \varphi(4^n x) \right] \right|$$

$$\le 2 \cdot 4^m \left| 4^n(x) \pm \tfrac{1}{2} 4^{-m} 4^n - 4^n x \right|$$

$$\le 4^n \; ?$$

$$\Delta = \left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| = \left| \sum_{n=0}^{\infty} \left(\tfrac{3}{4}\right)^n \gamma_n \right|$$

$$= \left| \sum_{n=0}^{m-1} \left(\tfrac{3}{4}\right)^n \gamma_n + \left(\tfrac{3}{4}\right)^m \gamma_m + \cancel{\sum_{n=m+1}^{\infty} \left(\tfrac{3}{4}\right)^n \gamma_n} \right| \quad 0$$

$$= \left| 3^m + \sum_{n=0}^{m-1} \left(\tfrac{3}{4}\right)^n \gamma_n \right|$$

$$\ge 3^m - \sum_{n=0}^{m-1} 3^n$$

$$\ge 3^n - \left(\frac{1-3^m}{1-3} \right) = 3^n - \frac{(1-3^m)}{-2}$$

$$\ge \frac{1}{2} + \frac{3^m}{2}$$

As $m \to \infty$ $\Delta \to \infty$ f not Diff
$\delta_m \to 0$

Thursday, January 24, 2008
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Equicontinuous Families of Functions

<u>Def</u> $\{f_n\}$ is a sequence of functions with with $|f_n(x)| < M$ $\forall x, n$, Uniformly Bd.

<u>Question</u> If $\{f_n\}$ is Uniformly Bd, does it have a subsequence f_{n_k} that converges uniformly

<u>No!</u> Counter-Example

$$f_n(x) = \frac{x^2}{x^2 + (1-nx)^2} \qquad |f_n| \le 1$$

$$f_n \to 0 \qquad f\left(\tfrac{1}{n}\right) = 1$$

<u>Def</u> A family of functions $\mathcal{F}$ is called equicontinuous if $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \quad \text{whenever } d(x,y) < \delta$$

for all x, y, f

<u>Thm 1</u> $\{f_n\}$ - Cont.
$f_n : K \to \mathbb{R}$ K Compact
$f_n \to$ uniformly

$\Rightarrow$ $\{f_n\}$ is equicontinuous

<u>Proof</u> i) f_n c.u means that $\forall \epsilon > 0$, $\exists N > 0$ such that $n > N$
$|f_n(x) - f_N(x)| < \epsilon/3$

ii) f_N are cont on K-cpt $\Rightarrow$ f_n are uniformly cont.
$\exists \delta_n > 0$ s.t $|f_N(x) - f_N(y)| < \epsilon/3$ if $d(x,y) < \delta$

<u>Then</u>

$$\begin{aligned} |f_n(x) - f_n(y)| \le\ & |f_n(x) - f_N(x)| + \\ & |f_N(x) - f_N(y)| + \\ & |f_N(y) - f_n(y)| < \epsilon + \epsilon + \epsilon \end{aligned}$$

$$|f_n(x) - f_n(y)| \le |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$

$\therefore \{f_n\}$ is Equicont.

Theorem If K is compact
f_n cont on K
f_n pt wise bounded and EquiCont.

Then
a) $|f_n| < M$ Unif Bd
b) $\exists$ a sequence f_{n_k} that converges uniformly

—o—

Stone Weierstrass Theorem

Thm: f cont on K $\Rightarrow$ $\exists$ P_n's - Polynomials such that $P_n \to f$ uniformly

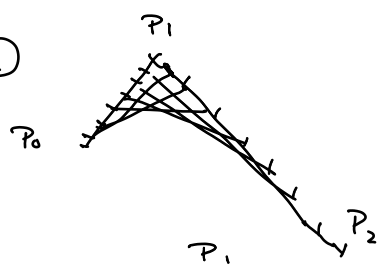
Constructive Proof $K = [0,1]$

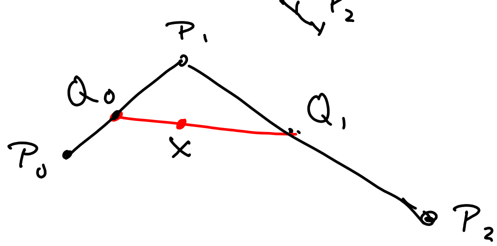
Detour

① Lines

$$\vec{r}(t) = \vec{P}_0 + t\vec{V} \qquad \vec{V} = \vec{P}_1 - \vec{P}_0$$
$$= P_0 + t(P_1 - P_0)$$
$$\vec{r}(t) = (1-t)P_0 + tP_1$$

② Bezier





$$(1-t)P_0 + tP_1 = Q_0$$
$$(1-t)P_1 + tP_2 = Q_1$$

$$\vec{x} = (1-t)Q_0 + tQ_1$$
$$x = (1-t)[(1-t)P_0 + tP_1] + t[(1-t)P_1 + tP_2]$$
$$= (1-t)^2 P_0 + t(1-t)P_1 + t(1-t)P_1 + t^2 P_2$$
$$x = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2$$

Cubic

$$\vec{X} = (1-t)^3 P_0 + 3t^2(1-t)P_1 + 3t(1-t)^2 P_2 + t^3 P_3$$

$$\vec{X} = \sum_{k=0}^{n} \binom{n}{k} P_n t^k (1-t)^{n-k}$$

Def $B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ Bernstein Polynomials

We'll Prove that f cont on $[0,1]$

$$f(x) = \sum_{n=0}^{\infty} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

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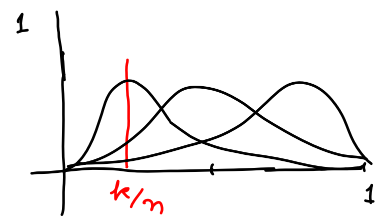
Stone - Weierstrass



Bernstein Polynomials

$$B_{k,n} = \binom{n}{k} x^k (1-x)^{n-k}$$

① Critical Pt, Max

$$f(x) = x^k (1-x)^{n-k}$$
$$f'(x) = k x^{k-1} (1-x)^{n-k} + x^k (n-k)(1-x)^{n-k-1}(-1) = 0$$
$$= x^{k-1}(1-x)^{n-k-1}\left[k(1-x) + (-1)x(n-k)\right] = 0$$
$$= x^{k-1}(1-x)^{n-k-1}\left[k - \cancel{kx} - nx + \cancel{kx}\right] = 0$$

$nx = k \qquad x = \frac{k}{n} \qquad f(x) \geq 0 \qquad f(0) = f(1) = 0$

Max @ $x = \frac{k}{n}$

② $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \qquad a = x \quad b = (1-x)$

$1 = \sum_{n=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$ Partition of Unity

Thm. $f(x)$ cont on $[0,1]$ then

$f_n(x) \doteq \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \qquad f_n \to f$ unif.

Proof: (See Bartle)

Note ① $\binom{n-1}{k-1} = \dfrac{(n-1)!}{(k-1)!\,(n-k)!} = \dfrac{k}{n}\,\dfrac{n!}{k!\,(n-k)!} = \dfrac{k}{n}\binom{n}{k}$

② $\binom{n-2}{k-2} = \dfrac{(n-2)!}{(k-2)!\,(n-k)!} = \dfrac{k(k-1)}{n(n-1)}\,\dfrac{n!}{k!\,(n-k)!} =$

$= \dfrac{k(k-1)}{n(n-1)}\binom{n}{k}$

① $\boxed{1 = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}}$ Ⓐ *Replace* $n \mapsto n-1$

$1 = \sum_{k=0}^{n-1} \binom{n-1}{k} x^k (1-x)^{n-k-1}$ *Replace* $k \mapsto k-1$

$1 = \sum_{k=1}^{n} \binom{n-1}{k-1} x^{k-1} (1-x)^{n-k}$

$\boxed{x = \sum_{k=0}^{n} \left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}}$ Ⓑ

② *Replace* $n \mapsto n-2$

$1 = \sum_{k=0}^{n-2} \binom{n-2}{k} x^k (1-x)^{n-k-2}$ $k \mapsto k-2$

$1 = \sum_{k=2}^{n} \binom{n-2}{k-2} x^{k-2} (1-x)^{n-k}$

$x^2 = \sum_{k=0}^{n} \frac{k(k-1)}{n(n-1)} \binom{n}{k} x^k (1-x)^{n-k}$

$(n^2-n)\,x^2 = \sum_{k=0}^{n} (k^2-k) \binom{n}{k} x^k (1-x)^{n-k}$

$\left(1-\frac{1}{n}\right)x^2 = \sum \left(\frac{k^2}{n^2} - \frac{k}{n^2}\right)\binom{n}{k} x^k (1-x)^{n-k}$

$\boxed{\left(1-\frac{1}{n}\right)x^2 + \frac{x}{n} = \sum \left(\frac{k^2}{n^2}\right)\binom{n}{k} x^k (1-x)^{n-k}}$ Ⓒ

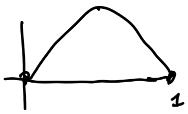
Note $\left(x - \frac{k}{n}\right)^2 = 1x^2 - 2\frac{k}{n}x + \frac{k^2}{n^2}$

Compute

x^2 Ⓐ $- 2x$ Ⓑ $+$ Ⓒ

$$x^2 - 2x^2 + \left(1 - \frac{1}{n}\right)x^2 + \frac{1}{n}x = \sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k}$$

$$-x^2 + x^2 - \frac{1}{n}x^2 + \frac{1}{n}x =$$



$$\boxed{\frac{1}{n}(x)(1-x) = \sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k}}$$

$|x(1-x)| \leq 1/4$

$$|f(x) - B_{n,k}(f)| \leq \sum_{k=0}^{n} \left|f(x) - f\left(\frac{k}{n}\right)\right| \binom{n}{k} x^k (1-x)^{n-k} \qquad |f| < M$$

Given $\epsilon > 0$, $\exists\ \delta > 0$ s.t $|f(x) - f(y)| < \epsilon$ if $|x - y| < \delta$

Uniform cont of f on the cpt space $[0,1]$

Let $n = \sup\left\{\frac{1}{\delta^4}, \frac{M^2}{\epsilon^2}\right\}$ $\qquad n > \frac{1}{\delta^4} \quad \frac{1}{n} < \delta^4$

If $\left|x - \frac{k}{n}\right| < \frac{1}{n^{1/4}}$ $\qquad |f - B| \leq \sum \epsilon \binom{n}{k} (1-x)^{n-k} x^k \quad \frac{1}{n^{1/4}} < \delta$

$< \epsilon/2$

If $\left|x - \frac{k}{n}\right| > \frac{1}{n^{1/4}}$

$\left|x - \frac{k}{n}\right|^2 > \frac{1}{n^{1/2}}$ $\qquad |f - B| < 2M \sum_{n=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$

$$< 2M \sum_{k=0}^{n} \frac{(x - \frac{k}{n})^2}{(x - k/n)^2} x^k (1-x)^{n-k} \binom{n}{k}$$

$$< 2M \frac{1}{n} x(1-x)\sqrt{n}$$

$$< \frac{2M}{\sqrt{n}} \frac{1}{4} \qquad \sqrt{n} > \frac{M}{\epsilon}$$

$$< \frac{1}{2} M \frac{\epsilon}{M} \qquad \frac{1}{\sqrt{n}} < \frac{\epsilon}{M}$$

$$< \epsilon/2$$

$$|f - B| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$< \epsilon.$$

Tuesday, February 05, 2008
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Some Special Functions.

H/w pg 165 # 1, 3, 4, 6, 7, 9, 15, 16, 20, 23

<u>Def</u> $f(x)$ is analytic at $x=0$ if

$* \quad f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad x \in \mathbb{R}$ - Real Analytic

$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad z \in \mathbb{C}$ - Complex Analytic

<u>Def</u> $f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n$ - Analytic at $z=c$

<u>Thm</u>: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$

If $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$, then

- $f(x)$ C.U in $(-R, R)$
- $f(x)$ is cont and diff
- $f'(x) = \sum_{n=0}^{\infty} a_n n x^{n-1}$

<u>Proof</u>
If $|x| < R$ then $|a_n x^n| < |a_n R^n|$

$\sum a_n R^n$ Converges Absolutely

By Weierstrak M-test $\sum a_n x^n$ C.U.

Then we can differentiate term by term

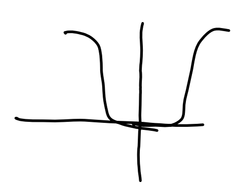
<u>Corollary</u>: $f(x)$ analytic $\Rightarrow$ $f(x)$ is ∞'ly differentiable

Taylor Coeff.

$f(x) = a_0 + a_1 x + a_2 x^2$
$f'(x) = a_1 + 2a_2 x + \cdot \quad \Rightarrow a_n = f^n(0)/n!$
$f''(x) = 2!\, a_2 + \cdot$

$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$

<u>Note</u> $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$



$f^{(n)}(0) = 0 \quad \forall n$

$f^{(n)}(0) = 0 \quad \forall\, n$

$\therefore \quad f(x) \neq \sum \frac{f^{n}(0)}{n!} x^n$

Thm Let $f(x) = \sum a_n x^n$. Then $\lim\limits_{x \to 1} f(x) = \sum\limits_{n=0}^{\infty} a_n$

Proof $|x| < 1$

Def $s_n = a_0 + a_1 + a_2 + \cdots + a_n$
$s_{n-1} = a_0 + a_1 + \cdots + a_{n-1}$

$$a_n = s_n - s_{n-1}$$

$$\sum_{n=0}^{m} a_n x^n = \sum_{n=0}^{m} (s_n - s_{n-1}) x^n \qquad s_{-1} = 0$$

$$= \sum_{n=0}^{m} s_n x^n - \sum_{n=0}^{m} s_{n-1} x^n \qquad n-1 \mapsto n$$

$$= \sum_{n=0}^{m} s_n x^n - \sum_{n=-1}^{m-1} s_n x^{n+1} \qquad n \mapsto n+1$$

$$= \sum_{n=0}^{m-1} s_n x^n + s_m x^m - \left[s_{-1} + x \sum_{n=0}^{m-1} s_n x^n\right]$$

$$= (1-x) \sum_{n=0}^{m-1} s_n x^n + s_m x^m$$

Take lim as $m \to \infty$, Recall $|x| < 1$
$|x|^m \to 0$

$$f(x) = (1-x) \sum_{n=0}^{\infty} s_n x^n$$

$$s = \lim_{n \to \infty} s_n$$

$\forall\, \epsilon > 0 \;\exists\, N > 0$ s.t $|s - s_n| < \frac{\epsilon}{2}$ if $n > N$

If $|x| < 1 \quad \sum\limits_{n=0}^{\infty} x^n = \frac{1}{1-x}$ Geometric Series

$$(1-x) \sum_{n=0}^{\infty} x^n = 1$$

$|1 - x| < \delta$

$$|f(x) - s| = \left| (1-x) \sum_{n=0}^{\infty} (s_n - s) x^n \right|$$

$$\leq \left| (1-x) \sum_{n=0}^{N} (s_n - s) x^n \right| + \left| (1-x) \sum_{n=N+1}^{\infty} \underbrace{(s_n - s)}_{\epsilon/2} x^n \right|$$

$$\leq \underbrace{|1-x| \left| \sum_{n=0}^{N} (s_n - s) x^n \right|}_{\epsilon/2} + \underbrace{\frac{\epsilon}{2} |1-x| \sum x^n}_{\epsilon/2}$$

$$< \epsilon$$

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Multiplying Series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n$$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$$

$$f(x) \cdot g(x) = (a_0 b_0) + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2$$

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{k=0}^{\infty} b_k x^k\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (a_k b_{n-k}) x^n$$
$$= \sum_{n=0}^{\infty} c_n x^n \qquad c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

As $x \to 1$

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{k=0}^{\infty} b_k\right) = \sum_{n=0}^{\infty} c_n \qquad c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Exponentials

Def $\mathrm{Exp}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ Ratio test $\rho = \infty$ Radius of C

Prop $\mathrm{Exp}(x) \cdot \mathrm{Exp}(y) = \mathrm{Exp}(x+y)$

Proof $\mathrm{Exp}(x) \cdot \mathrm{Exp}(y) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{k=0}^{\infty} \frac{y^k}{k!}\right)$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{1}{k!} \frac{n!}{(n-k)!} x^k y^{n-k}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (x+y)^n \qquad \text{binomial}$$

$$= \mathrm{Exp}(x+y)$$

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<u>Def</u> $\mathrm{Exp}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\quad z \in \mathbb{C}$ $\quad \boxed{z = x + iy}$

1) $\mathrm{Exp}(z_1) \cdot \mathrm{Exp}(z_2) = \mathrm{Exp}(z_1 + z_2)$ $\quad \mathrm{Exp}(z) = 1 + z + \frac{z^2}{2!}$

2) Recall $e = \sum_{n=0}^{\infty} \frac{1}{n!} = \mathrm{Exp}(1)$ Def

3) $\mathrm{Exp}(x+iy) = \mathrm{Exp}(x) \cdot \mathrm{Exp}(iy)$

4) $\mathrm{Exp}(x)\,\mathrm{Exp}(-x) = \mathrm{Exp}(x-x) = \mathrm{Exp}(0) = 1$

5) $\mathrm{Exp}'(z) = \sum_{n=1}^{\infty} \frac{n z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!}$ $\quad n-1 \to n$, $\; n \mapsto n+1$

$$= \sum_{n=0} \frac{z^n}{n!}$$

$$= \mathrm{Exp}(z)$$

<u>Prop</u> $x \in \mathbb{R} \Rightarrow \mathrm{Exp}(x) = e^x$

<u>Proof</u>

a) x int $x = n > 0$

$$\mathrm{Exp}(n) = \mathrm{Exp}(\overbrace{1 + 1 + \cdots + 1}^{n})$$

$$= \mathrm{Exp}(1) \cdot \mathrm{Exp}(1) \cdots \mathrm{Exp}(1) \quad (n\text{-times}$$

$$= e \cdot e \cdots e \quad (n\text{-times})$$

$$= e^n$$

b) x - Rational $x = P = \frac{m}{n}$

$m = np$

$$\mathrm{Exp}(m) = e^m = \mathrm{Exp}(np)$$
$$= \mathrm{Exp}(p + \cdots + p) \quad n\text{-times}$$
$$= \mathrm{Exp}(p) \cdot \cdots \mathrm{Exp}(p) \quad n\text{-times}$$

$$e^m = [\mathrm{Exp}(p)]^n$$

$$e^{m/n} = \mathrm{Exp}(p) = e^p$$

c) x real

Def $\quad e^x = \sup_p e^p \qquad p < x \quad p$ rational

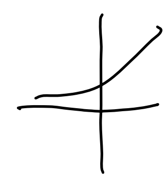
Completeness Property of $\mathbb{R}$

<u>We have:</u>

1) $\mathrm{Exp}'(x) = \mathrm{Exp}(x)$
2) $\mathrm{Exp}(1) = e$
3) $\mathrm{Exp}(0) = 1$
4) $\mathrm{Exp}(x) \cdot \mathrm{Exp}(y) = \mathrm{Exp}(x+y)$
5) $\mathrm{Exp}(xy) = [\mathrm{Exp}(x)]^y$
6) $\mathrm{Exp}(-x) = \dfrac{1}{\mathrm{Exp}(x)}$
7) $\mathrm{Exp}(x) \uparrow$

$x > 0$

$\mathrm{Exp}(x) > 1 + x$



$\mathrm{Exp}(x)$ is invertible.

Def $\quad \operatorname{Ln}(\operatorname{Exp}(x)) = x \qquad \operatorname{Ln}(x) = \operatorname{Log}(x)$

$\operatorname{Exp}(\operatorname{Ln}(x)) = x$

Last Semester

$$(f \circ g)(x) = f(g(x))$$

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$$\Rightarrow \quad [\operatorname{Ln}(\operatorname{Exp}(x))]' = \operatorname{Ln}'(\operatorname{Exp}(x)) \cdot \operatorname{Exp}'(x) = 1$$

$$\operatorname{Ln}'(\operatorname{Exp}(x)) = \frac{1}{\operatorname{Exp}(x)} \qquad \text{Let } y = \operatorname{Exp}(x)$$

$$\operatorname{Ln}'(y) = \frac{1}{y}$$

① $\frac{d}{dx} \operatorname{Log}(x) = \frac{1}{x}$

② $\operatorname{Log}(1) = 0$

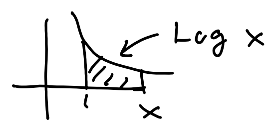
③ $\operatorname{Log}(x) \uparrow$ for $x > 0$

$\operatorname{Log}(\operatorname{Exp}(x)) = x \qquad x = 0$

$\operatorname{Log}(\operatorname{Exp}(0)) = 0$

$\operatorname{Log}(1) = 0$

④ F.T.C : $\operatorname{Log}(x) = \int_1^x \frac{1}{t}\, dt$



⑤ $\operatorname{Log}(xy) = \operatorname{Log} x + \operatorname{Log} y$

⑥ $\operatorname{Log}(e) = 1$

⑦

$\operatorname{Exp}(\operatorname{Log}(x) + \operatorname{Log}(y)) =$

$\operatorname{Exp}(\operatorname{Log}(x)) \cdot \operatorname{Exp}(\operatorname{Log} y)$

$\operatorname{Exp}(\operatorname{Log}(x) + \operatorname{Log}(y)) = x \cdot y$

$\operatorname{Log}(\operatorname{Exp}(\operatorname{Log}(x) + \operatorname{Log}(y))) = \operatorname{Log}(xy)$

$$\operatorname{Log}(x) = \int_1^x \frac{1}{t}\, dt \qquad \text{Let } t = T + 1$$

$$= \int_0^{x-1} \frac{1}{1+T}\, dT \qquad \frac{1}{1+T} = 1 - T + T^2 - \cdots \qquad |T| < 1$$

$x \mapsto x + 1$

$$\operatorname{Log}(1+x) = \int_0^x \frac{1}{1+T}\, dT$$

$$= T - \frac{T^2}{2} + \frac{T^3}{3} \cdots \Big]_0^x$$

$$\operatorname{Log}(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} \qquad |x| < 1$$

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Trig Functions   Trig = Trigonometry

Def
$$\begin{cases} C(x) = \dfrac{Exp(ix) + Exp(-ix)}{2} \\ S(x) = \dfrac{Exp(ix) - Exp(-ix)}{2i} \end{cases}$$
Mult by i, Add

$$\Rightarrow \quad Exp(ix) = C(x) + iS(x)$$
$$Exp(-ix) = C(x) - iS(x)$$
$$Exp(x) = e^x$$

$$\Rightarrow \quad Exp(ix) = \overline{Exp(-ix)}$$

1) $$C'(x) = \frac{i\,Exp(ix) - i\,Exp(-ix)}{2} \qquad \times(-i)$$
$$= -\frac{Exp(ix) - i\,Exp(-ix)}{2i}$$
$$= -S(x)$$
Henri Cartan

$$C'(x) = -S(x)$$
$$S'(x) = C(x)$$

$$e^{ix} = C(x) + iS(x) = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!}$$
$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) + i\left(x - \frac{x^3}{3!} \dots\right)$$
$$C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$
$$S(x) = x - \frac{x^3}{3!} + - = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
$$S'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = C(x)$$

$$C(0) = 1 \qquad C'(x) = -S(x)$$
$$S(0) = 0 \qquad S'(x) = C(x)$$

$$[S^2 + C^2]' = 2SS' + 2CC'$$
$$= 2[SC - CS] = 0$$
$$\Rightarrow S^2 + C^2 = 1$$

Sum Formulas

$$e^{ix} \cdot e^{iy} = e^{ix+iy}$$

$$e^{i(x+y)} = e^{ix} \cdot e^{iy}$$
$$C(x+y) + iS(x+y) = [C(x) + iS(x)][C(y) + iS(y)]$$
$$= [C(x)C(y) - S(x)S(y)] + i[S(x)C(y) + S(y)C(x)]$$

$$\begin{cases} C(x+y) = C(x)C(y) - S(x)S(y) \\ S(x+y) = S(x)C(y) + S(y)C(x) \end{cases}$$
$$x = y \Rightarrow \quad C(2x) = C^2(x) - S^2(x)$$
$$S(2x) = 2S(x)C(x)$$

$$\begin{cases} C(x+y) = C(x)C(y) - S(x)S(y) \\ S(x+y) = S(x)C(y) + S(y)C(x) \end{cases} \qquad x=y \Rightarrow \begin{matrix} C(2x) = C^2(x) - S^2(x) \\ S(2x) = 2S(x)C(x) \end{matrix}$$

Group Structure

$S^1 = \{ e^{ix} \mid x \in \mathbb{R} \}$ *Circle Group*

$x = 0 \quad e^{ix} = \underline{1} \quad \text{id}$

$e^{ix} \cdot e^{iy} = e^{ix} \cdot e^{iy} = e^{i(x+y)} \in G$

$e^{ix} \cdot e^{-ix} = 1$

Matrix Representation

$\varphi : \mathbb{C} \longrightarrow M_{2\times 2} \qquad \varphi : 1=1, \text{ Onto}$

$x + iy \longmapsto \begin{bmatrix} x & y \\ -y & x \end{bmatrix} = x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$x1 + iy \qquad = x(I) + yJ$

$i^2 = -1 \qquad J^2 = -I$

$e^{ix} = c(x) + i s(x) \longmapsto \begin{bmatrix} C(x) & S(x) \\ -S(x) & C(x) \end{bmatrix}$ *Rotation Matrix*

$e^{ix} \cdot e^{iy} = e^{i(x+y)} \qquad \begin{bmatrix} C(x) & S(x) \\ -S(x) & C(x) \end{bmatrix} \cdot \begin{bmatrix} C(y) & S(y) \\ -S(y) & C(y) \end{bmatrix} = \begin{bmatrix} C(x+y) & S(x+y) \\ -S(x+y) & C(x+y) \end{bmatrix}$

$\| e^{ix} \| = (e^{ix} \cdot e^{-ix}) = 1$

$S^2 + C^2 = 1$

$x^2 + y$

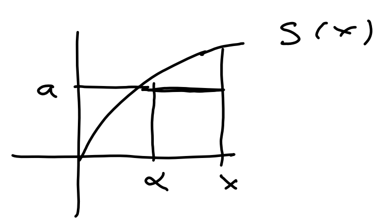
<u>*Theorem*</u>: *There is a value* $x_0 > 0$ *where* $C(x_0) = 0$

<u>*Proof*</u>

Suppose $C(x) > 0 \quad \forall x$

$C(0) = 1 \qquad S'(x) = C(x)$

$S(0) = 0 \qquad C'(x) = -S(x)$



$\Rightarrow S'(x) > 0 \Rightarrow S(x) \uparrow$

on $[\alpha, x]$

Let $S(\alpha) = a$

$S^2 + C^2 = 1$

$a(x - \alpha) < \int_\alpha^x S(x)\, dx$

$< -\int_\alpha^x C'(x)\, dx \quad \Rightarrow \quad a(x-\alpha) \leq C(\alpha) - C(x)$

$a(x-\alpha) < 2$

Can't be true $\forall x$

$x - \alpha < \frac{1}{a} [C(\alpha)]$

$\exists$ *a zero between* $[\alpha, \alpha + \frac{1}{a} C(\alpha)]$

Call the first zero $\frac{\pi}{2}$.

Write $C(x) = \cos x$
$S(x) = \sin x$

Fourier Series

Consider $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ Complex Series

Partial sums
$$f_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

$$f_n(x) = \sum_{k=-n}^{-1} c_k e^{ikx} + c_0 + \sum_{k=1}^{n} e^{ikx} \cdot c_k$$

$$= c_0 + \sum_{k=1}^{n} a_k e^{-ikx} + b_k e^{ikx} \qquad a_k = c_{-k}$$

$$= c_0 + \sum_{k=1}^{n} a_k (\cos kx - i \sin kx) + b_k (\cos kx + i \sin kx)$$

$$= c_0 + \sum_{k=1}^{n} A_k \cos kx + B_k \sin kx \qquad A_k = a_k + b_k$$
$$B_k = i(b_k - a_k)$$

$\longrightarrow$ Trig. Fourier Series

Fourier Coeff

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \qquad x \in [-\pi, \pi]$$

Let $\psi_k = e^{ikx}$ Def $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) \cdot \bar{g}(x)\, dx$

Eval $\langle \psi_k, \psi_\ell \rangle = \int_{-\pi}^{\pi} e^{ikx} e^{-i\ell x}\, dx$

$$= \int_{-\pi}^{\pi} e^{i(k-\ell)x}\, dx$$

$$= \left. \frac{e^{i(k-\ell)x}}{i(k-\ell)} \right]_{-\pi}^{\pi}$$

$$= \frac{2\left[e^{i(k-\ell)\pi} - e^{-i(k-\ell)\pi}\right]}{2i(k-\ell)}$$

$$= \frac{2 \sin(k-\ell)\pi}{(k-\ell)} = 0 \quad \text{if } k \neq \ell$$

If $k = \ell$ $\langle \psi_k, \psi_k \rangle = \int_{-\pi}^{\pi} 1\, dx$
$= 2\pi$

$$\int_{-\pi}^{\pi} \psi_k \bar{\psi}_k\, dx = \begin{cases} 0 & k \neq \ell \\ 2\pi & k = \ell \end{cases}$$

$\checkmark$ Basis

Def $\varphi_k = \frac{1}{\sqrt{2\pi}} \psi_k$ $\Rightarrow \langle \varphi_k, \varphi_\ell \rangle = \delta_{k\ell}$

$\varphi_k = \frac{1}{\sqrt{2\pi}} e^{inx}$

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Fourier Series

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx} \qquad f(x) = f(x+2n\pi)$$

$-\pi \le x \le \pi$ Main Window

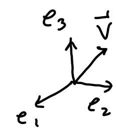
$$\varphi_n = \frac{1}{\sqrt{2\pi}} e^{inx} \qquad \int_{-\pi}^{\pi} \varphi_n(x)\,\bar{\varphi}_m(x)\,dx = \begin{cases} 0 & m \ne n \\ 1 & m = n \end{cases}$$

Def $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\,\bar{g}(x)\,dx$

Then $\langle \varphi_n, \varphi_m \rangle = \delta_{nm}$

$\{\varphi_n\}$ is an orthonormal set

Analogy of
$e_1 = \langle 1, 0, 0 \rangle$
$e_2 = \langle 0, 1, 0 \rangle$ $\qquad \langle e_i, e_j \rangle = \delta_{ij}$
$e_3 = \langle 0, 0, 1 \rangle$



Want $f(x) = \sum_{-\infty}^{\infty} a_n \varphi_n$ $\qquad$ like $\vec{V} = \sum_{n=1}^{3} v^i e_i$

Fourier Coeff

$$\begin{aligned}
\langle f, \varphi_m \rangle &= \langle \sum_{-\infty}^{\infty} a_n \varphi_n, \varphi_m \rangle \\
&= \sum_{-\infty}^{\infty} \langle a_n \varphi_n, \varphi_m \rangle \\
&= \sum_{-\infty}^{\infty} a_n \langle \varphi_n, \varphi_m \rangle \\
&= \sum_{-\infty}^{\infty} a_n \delta_{nm} \\
&= a_m
\end{aligned}$$

$a_m = \langle f, \varphi_m \rangle$

$$a_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)\, e^{-imx}\, dx$$

$$f = \sum_{-\infty}^{\infty} \langle f, \varphi_n \rangle \varphi_n$$

$\vec{V} = v^1 e_1 + v^2 e_2 + v^3 e_3$
$\vec{V} = \langle v^1, v^2, v^3 \rangle$

$\langle e_1, \vec{V} \rangle = v^1 \langle e_1, e_1 \rangle = v^1$
$\langle e_2, \vec{V} \rangle = v^2 \langle e_2, e_2 \rangle = v^2$

$$\boxed{\langle e_i, \vec{V} \rangle = v^i}$$

$$\vec{V} = \sum_{n=1}^{3} \langle e_n, \vec{V} \rangle e_n$$

$$|V|^2 = \vec{V} \cdot \vec{V} = \langle \vec{V}, \vec{V} \rangle$$

$$\Rightarrow \quad \sum (v^i)^2 = \|V\|^2$$

Convergence - Use Mean Square Deviation

Def $\|f\|^2 = \langle f, f \rangle = \int_{-\pi}^{\pi} |f|^2\,dx$

$$\begin{aligned}
\|f - \sum_{-N}^{N} a_n \varphi_n\|^2 &= \langle f - \sum a_n \varphi_n, f - \sum_m a_m \varphi_m \rangle \\
&= \langle f, f \rangle - \langle f, \sum_m a_m \varphi_m \rangle - \langle \sum_n a_n \varphi_n, f \rangle + \langle \sum_n a_n \varphi_n, \sum_m a_m \varphi_m \rangle \\
&= \|f\|^2 - \sum_m \langle f, a_m \varphi_m \rangle - \sum_n \langle a_n \varphi_n, f \rangle + \sum_n \sum_m \langle a_n \varphi_n, a_m \varphi_m \rangle \\
&= \|f\|^2 - \sum_m \bar{a}_m \langle f, \varphi_m \rangle - \sum_m a_m \langle \varphi_m, f \rangle + \sum_n \sum_m a_n \bar{a}_m \langle \varphi_n, \varphi_m \rangle \\
&= \|f\|^2 + \sum_n |a_n|^2 - \sum_n \bar{a}_n \langle f, \varphi_n \rangle - \sum_n a_n \langle \varphi_n, f \rangle \\
&= \|f\|^2 + \sum |a_n - \langle f, \varphi_n \rangle|^2 - \sum |\langle f, \varphi_n \rangle|^2 \ge 0
\end{aligned}$$

$$= \|f\|^2 + \sum |a_n - \langle f,\varphi_n\rangle|^2 - \sum |\langle f,\varphi_n\rangle|^2 \geq 0$$

Min Error if $\boxed{a_n = \langle f, \varphi_n \rangle}$ Fourier Coeff

Then

$$\|f\|^2 - \sum_n |a_n|^2 \geq 0 \qquad \boxed{\sum_n |a_n|^2 \leq \|f\|^2}$$

Bessel's Ineq

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Let $s_N = \sum_{-N}^{N} \langle f, \varphi_n \rangle \varphi_n$

$$= \sum_{-N}^{N} \left[\int_{-\pi}^{\pi} f(t) \frac{e^{-int}}{\sqrt{2\pi}} \, dt \right] \frac{e^{inx}}{\sqrt{2\pi}}$$

$$= \int_{-\pi}^{\pi} \sum_{-N}^{N} \frac{f(t)\, e^{in(x-t)}}{2\pi} \, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{-N}^{N} e^{in(x-t)} \, dt \qquad \text{Let } J = x - t$$

Consider: $\displaystyle\sum_{-N}^{N} e^{inJ} = \sum_{-N}^{-1} e^{inJ} + \sum_{1}^{N} e^{inJ} + 1$

$$\boxed{\begin{aligned} &e^{inJ} = (e^{iJ})^n \\ &\sum_{n=0}^{N} ar^n = \frac{a(1-r^{N+1})}{1-r} \end{aligned}}$$

$$= \sum_{0}^{N} e^{-inJ} + \sum_{0}^{N} e^{inJ} + 1 - 2$$

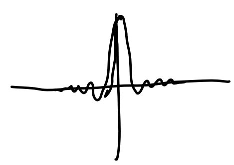
$$= \frac{1 - e^{-i(N+1)J}}{1 - e^{-iJ}} + \frac{1 - e^{i(N+1)J}}{1 - e^{iJ}} - 1$$

$$= \frac{e^{iJ/2} - e^{-i(N+1/2)J}}{e^{iJ/2} - e^{-iJ/2}} + \frac{e^{-iJ/2} - e^{i(N+\frac{1}{2})J}}{e^{-iJ/2} - e^{iJ/2}} - 1$$

$$= \frac{e^{i(N+1/2)J} - e^{-i(N+1/2)J}}{e^{iJ/2} - e^{-iJ/2}}$$

$$= \frac{\sin(N+1/2)J}{\sin J/2} \qquad \text{Dirichlet Kernel}$$

Let $D_N = \dfrac{\sin(N+1/2)J}{\sin J/2}$



$$\int_{-\pi}^{\pi} D_N(J)\, dJ = 2\pi$$

$$f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_N(x-t)\, dt \qquad J = x - t$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+J) D_N(J)\, dJ \qquad \text{because } f(t) = f(t \pm 2n\pi)$$

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Riemann Lebesgue Lemma

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Gamma Function

<u>Def</u> $\Gamma(n+1) = \int_0^\infty x^n e^{-x}\,dx$  $x = e^{\ln x}$

$= \int_0^\infty e^{n\ln x - x}\,dx$

<u>Prop</u> 1 $\Gamma(n+1) = n\Gamma(n)$

<u>Proof</u> $\int_0^\infty \underbrace{x^n}_{u}\, \underbrace{e^{-x}\,dx}_{dv} = -x^n \cancel{e^{-x}}\Big]_0^\infty \to 0 + \int_0^\infty n x^{n-1} e^{-x}\,dx$

Let $u = x^n$ $du = n x^{n-1}$

$dv = e^{-x}\,dx$ $v = -e^{-x}$

$\Gamma(n+1) = n\,\Gamma(n)$

<u>Prop</u> $n \in Z^+$ (n positive Integer) $\Gamma(n+1) = n!$

<u>Proof</u> Induction

1 ① $\Gamma(1) = 1$

$\Gamma(2) = \Gamma(1+1) = 1\,\Gamma(1) = 1$

$\Gamma(3) = \Gamma(2+1) = 2\,\Gamma(2) = 2\cdot 1$ etc

② Assume true for $n = k+1$

$\Gamma(k+1) = k!$

Then if $n = k+2$

$\Gamma(k+2) = (k+1)\,\Gamma(k+1) = (k+1)\,k! = (k+1)!$

<u>Properties</u> :

1) <u>Taylor series</u> Let $f(z) = \Gamma(1+z)$ about $z = 0$

$f(z) = \sum_{n=0}^\infty \frac{f^n(0)}{n!} z^n$ $f(z) = \int_0^\infty t^z e^{-t}\,dt$

$f(z) = \sum_{n=0}^\infty \frac{1}{n!}\left(\int_0^\infty \ln^n t\; e^{-t}\,dt\right) z^n$ $= \int_0^\infty e^{z\ln t - t}\,dt$

$f^n(z) = \int_0^\infty (\ln t)^n t^z e^{-t}\,dt$

$f^n(0) = \int_0^\infty (\ln t)^n e^{-t}\,dt$

2) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

<u>Proof</u> $\Gamma(\frac{1}{2}) =$ $= \int_0^\infty t^{-1/2} e^{-t}\,dt$

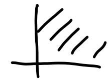
Let $x^2 = t$ $dx = \frac{1}{2} t^{-1/2}\,dt$

$x = t^{1/2}$

Let $x^2 = t$ $\qquad dx = \frac{1}{2} t^{-1/2}\, dt$

$x = t^{1/2}$

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-x^2}\, dx$$



$$[\Gamma(\tfrac{1}{2})]^2 = \left[2\int_0^\infty e^{-x^2}\, dx\right]\left[2\int_0^\infty e^{-y^2}\, dy\right]$$

$$= 4\int_0^\infty\int_0^\infty e^{-x^2-y^2}\, dx\, dy$$

Let $x = r\cos\theta$, $y = r\sin\theta$

$$= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\, dr\, d\theta$$

$$= 4\int_0^{\pi/2} \lim_{R\to\infty} -\frac{1}{2} e^{-r^2}\Big]_0^R\, d\theta$$

$$= 4\cdot\frac{1}{2}\int_0^{\pi/2} d\theta = \pi \qquad \Rightarrow \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

3) $\Gamma(\frac{3}{2}) = \Gamma(1+\frac{1}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$

$\Gamma(\frac{5}{2}) = \Gamma(2+\frac{1}{2}) = \Gamma(1+\frac{3}{2}) =$

$$\Gamma(n+\tfrac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} = \frac{(2n-1)(2n-3)\cdots 3\cdot 1\,\sqrt{\pi}}{2^n}$$

$$= \frac{(2n-1)(2n-2)(2n-3)\cdots 3\cdot 2\cdot 1}{2^n\,(2n-2)(2n-4)\cdots 2}\sqrt{\pi}$$

$$= \frac{(2n-1)!}{2^n\, 2^n\, (n-1)!}\sqrt{\pi}$$

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Beta Function

Def $\beta(x,y) = \dfrac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$

Prop $\beta(x,y) = \int_0^1 z^{x-1}(1-z)^{y-1}\,dz$

Proof:

$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\,dt$ Let $t = x^2$, $dt = 2x\,dx$

$= \int_0^\infty (x^2)^{n-1} e^{-x^2}\, 2x\,dx$

$$\begin{cases} \Gamma(n) = 2\int_0^\infty x^{2n-1} e^{-x^2}\,dx \\ \Gamma(m) = 2\int_0^\infty y^{2m-1} e^{-y^2}\,dy \end{cases}$$

$\Gamma(n)\,\Gamma(m) = 4\int_0^\infty\int_0^\infty x^{2n-1} y^{2m-1} e^{-x^2-y^2}\,dx\,dy$

$\Gamma(n)\,\Gamma(m) = 4\int_0^{\pi/2}\int_0^\infty r^{2n-1} r^{2m-1} (\cos\theta)^{2n-1}(\sin\theta)^{2m-1} e^{-r^2}\, r\,dr\,d\theta$

$= 4\int_0^{\pi/2}\left[(\cos\theta)^{2n-1}(\sin\theta)^{2m-1}\int_0^\infty r^{2n+2m-2}\, r e^{-r^2}\,dr\right]d\theta$

Let $R = r^2$, $dR = 2r\,dr$

$= 2\int_0^{\pi/2}\left[(\cos\theta)^{2n-1}(\sin\theta)^{2m-1}\int_0^\infty R^{n+m-1} e^{-R}\,dR\right]d\theta$

$\Gamma(n)\,\Gamma(m) = \Gamma(n+m)\; 2\int_0^{\pi/2}\left[(\cos\theta)^{2n-1}(\sin\theta)^{2m-1}\right]d\theta$

Let $z = \cos^2\theta$ $\qquad (1-z) = \sin^2\theta$

$dz = -2\cos\theta\sin\theta\,d\theta$

$\dfrac{\Gamma(n)\,\Gamma(m)}{\Gamma(n+m)} = \int_0^{\pi/2} (\cos\theta)^{2n-2}(\sin\theta)^{2m-2}\,(2\cos\theta\sin\theta)\,d\theta$

$= \int_1^0 z^{n-1}(1-z)^{m-1}(-dz)$

$$\boxed{\beta(n,m) = \int_0^1 z^{n-1}(1-z)^{m-1}\,dz = \frac{\Gamma(n)\,\Gamma(m)}{\Gamma(n+m)}}$$

① $\beta(\tfrac12,\tfrac12) = \dfrac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1)} = [\Gamma(\tfrac12)]^2 = \pi \iff \Gamma(\tfrac12) = \sqrt{\pi}$

② $\beta(x,1-x) = \int_0^1 z^{x-1}(1-z)^{-x}\,dz$ $\qquad$ Let $w = \dfrac{z}{1-z}$

$= \int_0^1 \dfrac{z^{x-1}}{(1-z)^x}\,dz$

$w - wz = z$

$w = z + wz$

$z = \dfrac{w}{1+w}$

$dz = (1+w) - w\;\; dw$

$$(1-z)^x$$

$$w = z + wz$$

$$z = \frac{w}{1+w}$$

$$dz = \frac{(1+w) - w}{(1+w)^2}\, dw$$

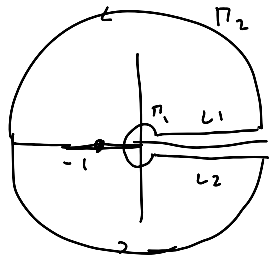
$$1 - z = 1 - \frac{w}{1+w} = \frac{1}{1+w}$$

$$dz = \frac{1}{(1+w)^2}\, dw$$

$$\beta(x, 1-x) = \int \frac{\left(\frac{w}{1+w}\right)^{x-1} (1+w)^x}{(1+w)^2}\, dw = \int_0^\infty \frac{w^{x-1}}{1+w}\, dw \qquad 0 < x-1 < 1$$

Use Residue

$$I = \int_0^\infty \frac{x^\alpha}{1+x}\, dx \qquad 0 < \alpha < 1$$



$$z = re^{i\theta}$$

Consider

$$L_1: \quad z^\alpha = r^\alpha$$

$$J = \int_C \frac{z^\alpha}{1+z}\, dz$$

$$L_2: \quad z^\alpha = r^\alpha e^{2\pi\alpha i}$$

$$= \int_{L_1} + \int_{\Gamma_2} + \int_{L_2} + \int_{\Gamma} f(z)\, dz$$

$$= \int_0^\infty \frac{r^\alpha}{1+r}\, dr + \int_{-\infty}^0 \frac{r^\alpha e^{2\pi\alpha i}}{1+r}\, dr = 2\pi i \operatorname{Res}(z = -1)$$

$$(1 - e^{2\pi\alpha i}) \int_0^\infty \frac{r^\alpha}{1+r}\, dr = e^{\pi i\alpha} \cdot 2\pi i$$

$$\int_0^\infty \frac{r^\alpha}{1+r}\, dr = \frac{(e^{\pi\alpha i})\, 2\pi i}{1 - e^{2\pi\alpha i}} = \frac{2\pi i}{e^{-\pi\alpha i} - e^{\pi\alpha i}}$$

$$= \frac{\pi}{\sin \pi\alpha}$$

$$\int_0^\infty \frac{x^\alpha}{1+x}\, dx = \frac{\pi}{\sin \pi\alpha} \qquad \alpha \mapsto p-1$$

$$\int_0^\infty \frac{x^{p-1}}{1+x}\, dx = \frac{\pi}{\sin \pi p}$$

$$\beta(x, 1-x) = \int_0^\infty \frac{w^{x-1}}{1+w}\, dw = \frac{\pi}{\sin \pi x}$$

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Stirling Approximation

Recall

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx \qquad x = e^{\ln x}$$

$$= \int_0^\infty e^{n\ln x} \cdot e^{-x}\, dx$$

$$= \int_0^\infty e^{n\ln x - x}\, dx$$

Let $x = n + y\sqrt{n}$ $\quad x = 0 \quad \sqrt{n}\, y = -n$

$dx = \sqrt{n}\, dy \qquad y = -\sqrt{n}$

$$\Gamma(n+1) = \int_{-\sqrt{n}}^{\infty} e^{n\ln[n + y\sqrt{n}) - n - y\sqrt{n}} \sqrt{n}\, dy$$

$$\ln(n + y\sqrt{n}) = \ln[n(1 + \tfrac{y}{\sqrt{n}})] \qquad \ln(1+x) = x - \frac{x^2}{2} + \cdots$$

$$= \ln n + \ln(1 + \tfrac{y}{\sqrt{n}})$$

$$= \ln n + \frac{y}{\sqrt{n}} - \frac{y^2}{n\,2}$$

$$n\ln(n + y\sqrt{n}) = n\ln n + \sqrt{n}\, y - \frac{y^2}{2}$$

$$\Gamma(n+1) \doteq \int_{-\sqrt{n}}^{\infty} e^{n\ln n + \cancel{\sqrt{n}y} - \frac{y^2}{2} - n - \cancel{y\sqrt{n}}} \sqrt{n}\, dy$$

$$\doteq \sqrt{n} \int_{-\sqrt{n}}^{\infty} e^{n\ln n - n - y^2/2}\, dy$$

$e^{-y^2/2}$

$$\doteq \sqrt{n}\, e^{n\ln n - n} \int_{-\sqrt{n}}^{\infty} e^{-y^2/2}\, dy$$

$$\doteq \sqrt{n}\, n^n e^{-n} \left[\int_{-\infty}^{\infty} e^{-y^2/2}\, dy - \int_{-\infty}^{-\sqrt{n}} e^{-y^2/2}\, dy \right.$$

0

$$\doteq \sqrt{n}\, n^n e^{-n} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy$$

$$\Gamma(n+1) \simeq \sqrt{n}\, n^n e^{-n}\, 2\sqrt{\pi/2}$$

$$\boxed{\Gamma(n+1) \simeq \sqrt{2\pi n}\; n^n e^{-n}}$$

$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

$\int_0^\infty x^{-1/2} e^{-x}\, dx = \sqrt{\pi}$

Let $x = y^2/2$

$2x = y^2$

$\sqrt{2}\, x^{1/2} = y$

$\frac{1}{2}\sqrt{2}\, x^{-1/2} dx = dy$

$\sqrt{2} \int_0^\infty e^{-\frac{y^2}{2}}\, dy = \sqrt{\pi}$

$\int_0^\infty e^{-\frac{y^2}{2}}\, dy = \sqrt{\frac{\pi}{2}}$

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Analysis in $\mathbb{R}^n$

I. Linear Algebra Review.

1. **Def** A vector space over a field $\mathcal{F}$ $(\mathbb{R}, \mathbb{C})$ is a collection of objects $\mathcal{V}$ called vectors, together with two operations

a) $v, w \in \mathcal{V}$ $\quad v + w \in \mathcal{V}$ $\quad$ Sum

b) $v \in \mathcal{V}$
$k \in \mathcal{F}$ $\quad kv \in \mathcal{V}$ $\quad$ Mult. by a scalar
$\uparrow$ scalars

These operations satisfy 8 ppt's

$$\begin{cases} VS1) & v + w = w + v \\ VS2) & v + (w+z) = (v+w) + z \\ VS3) & \exists\ 0 \in \mathcal{V} \ \text{s.t.}\ v + 0 = v \quad \forall v \\ VS4) & \exists\ -v \in \mathcal{V} \ \text{s.t.}\ v + (-v) = 0 \end{cases}$$

$$\begin{cases} VS5) & (k_1 + k_2) v = k_1 v + k_2 v \\ VS6) & k_1 (k_2 v) = (k_1 k_2) v \\ (VS7) & \exists\ 1 \in \mathcal{F} \ \text{s.t.} \\ & 1 v = v \quad \forall\ v \in \mathcal{V} \\ (VS8) & (-1) v = -v \end{cases}$$

<u>Examples</u>

1) $\mathbb{E}^n = \{ (v^1, \dots, v^n) \} = \mathcal{V}$ $\qquad \mathcal{F} = \mathbb{R} \quad (\mathbb{C}, \mathbb{Q})$

$v = (v^1, \dots v^n)$
$w = (w^1, \dots, w^n)$ $\qquad$ a) $v + w = (v^1 + w^1, \dots, v^n + w^n)$
$\qquad$ b) $kv = \langle kv^1, \dots kv^n)$

2) $\mathcal{P}^3 = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \}$

$f = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ $\qquad$ a) $f + g = \sum_{i=0}^{3} (a_i + b_i) x^i$
$g = b_0 + b_1 x + b_2 x^2 + b_3 x^3$ $\qquad$ b) $kf = \sum_{i=0}^{3} (k a_i) x^i$

3) $\mathcal{M}_{n \times m} = \left\{ \begin{bmatrix} A^1{}_1 & A^1{}_2 & \cdots & A^1{}_m \\ A^2{}_1 & A^2{}_2 & & \\ \vdots & & & \\ A^n{}_1 & & \cdots & A^n{}_m \end{bmatrix} \right\}$

$A = (A^n{}_m)$ $\qquad$ a) $A + B = (A^n{}_m + B^n{}_m)$
$B = (B^n{}_m)$ $\qquad$ b) $kA = (k A^n{}_m)$

4) $\mathcal{V} = \{ a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \}$

5) $\mathcal{V} = \{$ Solutions of $a y'' + b y' + c y = 0 \}$

$a \to a(x), \quad b(x) \quad c(x)$ ✓

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Definitions:

1) $\{v^1, v^2, \dots v^n\}$ is called linearly independent ($l.i$) if $k_1 v^1 + \dots + k_n v^n = 0 \Rightarrow k_1 = k_2 \dots = k_n = 0$

Else, $\{v^1, \dots, v^n\}$ is called $l.d$

2) Given $\{w^1, \dots w^k\}$

$$\text{Span}\{w^1, \dots, w^k\} = \{a_1 w^1 + \dots + a_k w^k\} \quad a\text{'s} \in \mathcal{F}$$

3) $\dim \mathcal{V} = n$ if $\mathcal{V}$ contains n linearly ind. vectors but no set of $n+1$ vectors is $l.i$.

4) A basis for $\mathcal{V}$ is a collection of $l.i$ vectors that spans $\mathcal{V}$.

Ex 1) $\mathcal{V} = \mathbb{E}^n$ $\quad e_1 = \langle 1, 0, 0 0 0 \dots \rangle$
$e_2 = \langle 0, 1, 0, 0 \dots \rangle$ $\quad \mathcal{B} = \{e_1, e_2, \dots, e_n\}$

Ex 2) $\mathcal{V} = \mathcal{P}^3$ $\quad e_1 = 1$
$e_2 = x$ $\quad \mathcal{B} = \{e_1, e_2, e_3, e_4\}$
$e_3 = x^2$
$e_4 = x^3$

Notation:

$$A = (A^m{}_n) = \begin{bmatrix} A^1{}_1 & \cdot & A^1{}_n \\ \vdots & & \\ A^m{}_1 & \dots & A^m{}_n \end{bmatrix}$$

$$\begin{cases} ax + by = p \\ cx + dy = q \end{cases} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

<u>*Mult of Matrices*</u>

$$X = \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix} \qquad (AX)^m = \sum_{k=1}^{n} A^m{}_n X^n = \begin{bmatrix} A^1{}_1 x^1 + A^1{}_2 x^2 + \cdots \\ A^2{}_1 x^1 + A^2{}_2 x^2 + \cdots \\ \vdots \\ A^m{}_1 x^1 + \cdots \end{bmatrix}$$

$$A = (A^m{}_n) \qquad B = (B^n{}_k)$$

$$(AB)^m{}_k = \sum_{l=1}^{n} A^m{}_l B^l{}_k = (C)^m{}_k \qquad \boxed{AB = C}$$

<u>Def</u> $I = \begin{bmatrix} 1 & 0 & 0 & \cdot \\ 0 & 1 & 0 & \vdots \\ 0 & 0 & 1 & \\ \vdots & & \cdot & \end{bmatrix} = \mathrm{diag}(1, 1, \cdots)$ *Identity* $\boxed{n \times n}$

Then $AI = IA = A$ $\qquad A = n \times n$

$$(I^j{}_k) = \delta^j{}_k = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

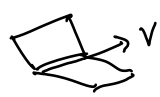
$$\boxed{AX = 0}^T \qquad \boxed{X^T A^T = 0}$$

<u>Ex</u> $\begin{cases} 3x^1 + 2x^2 - x^3 = 0 \\ x^1 - x^2 + 5x^3 = 0 \end{cases} \Rightarrow \begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 5 \end{bmatrix}\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} = 0$

If the system has more variables than equations, then $\exists$ non trivial solutions

$$A \sim \begin{bmatrix} 1 & -1 & 5 \\ 3 & 2 & -1 \end{bmatrix} \; -3 \; \sim \begin{bmatrix} 1 & -1 & 5 \\ 0 & 5 & -16 \end{bmatrix} \; 1/5 \; \sim \begin{bmatrix} 1 & -1 & 5 \\ 0 & 1 & -\frac{16}{5} \end{bmatrix} \; 1$$

$$A \sim \begin{bmatrix} 1 & 0 & 9/5 \\ 0 & 1 & -16/5 \end{bmatrix} \qquad \begin{aligned} x^1 &= -9/5 \; x^3 \\ x^2 &= \left(\tfrac{16}{5}\right) x^3 \end{aligned} \qquad \text{Sol}: \begin{bmatrix} (-9/5)a \\ 16a/5 \\ a \end{bmatrix}$$



$$\text{Sp}\begin{bmatrix} -9/5 \\ 16/5 \\ 1 \end{bmatrix} = V$$

<u>Prop</u> $A \sim R$

Sol of $Ax = 0$

same as $Rx = 0$

A *is* $m \times n$

$m < n$, $\exists$ *non trivial sol.*

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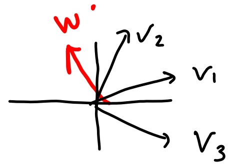
Theorem

If $\{v^1, \dots v^r\}$ spans V then $\dim V \le r$

Ex Pic $V = \mathbb{R}^2$



Proof Suppose $\dim V = n$
Need to show that $n \le r$

Suppose $n > r$, then we can find n l.i. vectors $w^1, \dots, w^n$ with $n > r$

$$\begin{matrix} a_1 \\ a_2 \\ \\ \end{matrix} \begin{cases} w^1 = A^1_1 v^1 + A^1_2 v^2 + \dots + A^1_r v^r \\ w^2 = A^2_1 v^1 + \dots + A^2_r v^r \\ \vdots \\ w^n = A^n_1 v^1 + \dots\dots + A^n_r v^r \end{cases} \Rightarrow w^n = A^n_r v^r$$

$$w^k = A^k_l v^l = \sum_{l=1}^{r} A^k_l v^l \qquad \begin{cases} k = 1 \dots n \\ l = 1, \dots r \end{cases}$$

Consider constants $a_1, \dots, a_n$ s.t

$$a_1 w^1 + a_2 w^2 + \dots + a_n w^n = 0$$

$$a_k w^k = a_k A^k_l v^l = 0$$

$$\sum_{k=1}^{n} a_k w^k = \sum_{k=1}^{n} \sum_{l=1}^{r} a_k A^k_l v^l$$

$$a_1 w^1 + a_2 w^2 + \dots + a_n w^n = (a_1 A^1_1 + a_2 A^2_1 \dots + a_n A^n_1) v^1 + \dots + (a_1 A^1_r + a_2 A^2_r + \dots a_n A^n_r) v^r$$

$$a^T A^T = 0 \qquad \boxed{A^l_k a^k = 0}$$

r equations
n unknowns $r < n$

Hence there are non trivial solutions. Some of a's are not zero, $\Rightarrow$ Contradiction

$\therefore \dim V \le r$.

Theorem: $\dim \mathbb{R}^n = n$

Proof Let $e_1 = \langle 1, 0, 0, 0 \dots \rangle$
$e_2 = \langle 0, 1, 0, 0 \dots \rangle$
$\vdots$
$e_n = \langle 0, 0, \dots, 1 \rangle$

e_k's l i and they span $\mathbb{R}^n$ because $x = \langle a^1, \dots, a^n \rangle = \sum a^k e_k$

$\{e_k\}$ - Standard Basis.

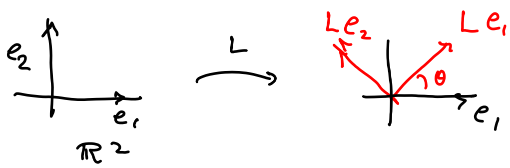
Linear Transformations.

Let V and W be vector spaces
$A : V \to W$ is called linear if
a) $A(x+y) = Ax + Ay$ $\qquad Ax = A(x)$
b) $A(cx) = cAx$ $\qquad \forall\, x, y \in V$, $c \in \mathbb{R}$

Theorem: In any vector space a linear transformation can be represented by a matrix (which depends on the basis)

Ex Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad L =$ rotation by $45°$ c.c.w

$\{e_1, e_2\} =$ Basis



$Le_1 = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

$Le_1 = \frac{\sqrt{2}}{2} e_1 + \frac{\sqrt{2}}{2} e_2$

$Le_2 = -\frac{\sqrt{2}}{2} e_1 + \frac{\sqrt{2}}{2} e_2$

$$L \sim A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

② $V = \mathcal{P}_3 = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3\}$

$D : \mathcal{P}_3 \to \mathcal{P}_3$ $\qquad f \in \mathcal{P}_3$, $Df = f'$ $\qquad \begin{cases} D(f+g) = Df + Dg \\ D(cf) = c\,Df \end{cases}$

$\{e_0, e_1, e_2, e_3\}$

$e_0 = 1$ $\qquad De_0 = 0 = 0e_0 + 0e_1 + 0e_2 + 0e_3$
$e_1 = x$ $\qquad De_1 = 1 = 1e_0 + 0e_1 + 0e_2 + 0e_3$
$e_2 = x^2$ $\qquad De_2 = 2x = 0e_0 + 2e_1 + 0e_2 + 0e_3$
$e_3 = x^3$ $\qquad De_3 = 3x^2 = 0e_0 + 0e_1 + 3e_2 + 0e_3$

$$D \sim M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D \sim M = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In general, Let $\{e_i\}$ = Basis for V
$\{f_k\}$ = Basis for W

$$L : V \to W$$

$$L e_i = \sum A_i{}^k f_k = \sum_k f_k A^k{}_i \qquad L \sim A^T$$

<u>Thm</u> $L : \mathbb{R}^n \to \mathbb{R}^m, \quad M : \mathbb{R}^m \to \mathbb{R}^k$

If $L \sim A$
$M \sim B$ ← Matrix Multiplication
$M \circ L \sim B \cdot A$

<u>Proof</u>

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "L"] \arrow[rr, bend left, "M \circ L"] & \mathbb{R}^m \arrow[r, "M"] & \mathbb{R}^k \\ e_i & f_j & g_k \end{tikzcd}$$

$$L e_i = f_j A^j{}_i \qquad L \sim A$$

$$M f_j = g_k B^k{}_j \qquad M \sim B$$

$$\begin{aligned} (M \circ L)(e_i) &= M(L e_i) \\ &= M(f_j A^j{}_i) \\ &= (M f_j) A^j{}_i = g_k B^k{}_j A^j{}_i \\ &= g_k (BA)^k{}_i \quad \Rightarrow M \circ L \sim BA \end{aligned}$$

<u>Def</u> Let $L : \mathbb{R}^n \to \mathbb{R}^m$

Consider the set $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ of all linear transf

If $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$

$$\boxed{\|L\| = \sup_x |Lx| \quad \text{where } |x| \le 1}$$

$\|L\|\,|x| = |x| \sup |Lx|$ <u>Show</u> $|Lx| \le \|L\|\,|x|$

Tuesday, March 25, 2008
12:36 PM

Differentiation in $\mathbb{R}^n$

Def $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) = \{ L : \mathbb{R}^n \to \mathbb{R}^m,\ L \text{ linear} \}$

Theorem : $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is a metric space
(Norm space)

Def $\|A\| = \sup |Ax| \quad |x| \leq 1$

Equiv:
$$\|A\| = \sup_x \frac{|Ax|}{|x|} \quad |x| \neq 0$$

Fact : $|Ax| \leq \|A\| \, |x|$

Fact : If $|Ax| \leq \lambda |x| \quad \forall x$ then $\|A\| \leq \lambda$ $\Big\{$ Bounded Operator

Prop $\|A + B\| \leq \|A\| + \|B\|$

Pf $|Ax + Bx| \leq |Ax| + |Bx|$

$|(A+B)(x)| \leq \|A\| |x| + \|B\| |x|$ "Drink Sup"

$\|A + B\| \leq \|A\| + \|B\|$

Prop $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is bounded

Proof $A e_i = A_i^j \tilde{e}_j = \sum_{j=1}^m A_i^j \tilde{e}_j$ $\quad$ e_i = Basis for $\mathbb{R}^n$, $\tilde{e}_j$ = Basis for $\mathbb{R}^m$

Use Schwartz Inequality

$$|Ae_i|^2 \leq \sum_{j=1}^m |A_i^j|^2 \, |\tilde{e}_j|^2$$

$$|Ae_i| \leq \left[\sum |A_i^j|^2\right]^{1/2} \Rightarrow \|A\| \leq \left[\sum |A_i^j|^2\right]^{1/2}$$

— o —

Let $f : \mathbb{R}^n \to \mathbb{R}^m$

Def f is called differentiable if $\exists$ a linear map $Df(x) = A$ such that

$$\lim_{|h| \to 0} \frac{|f(x+h) - f(x) - A \cdot h|}{|h|} = 0 \qquad x \in \mathbb{R}^n,\ h \in \mathbb{R}^n$$

Notation $Df(x) = f'(x)$ is called $\Big\{$ differential / Frechet Deriv.

Prop If the differential exists, then it is unique.

Proof Suppose A_1 and A_2 are differentials unique.

Proof Suppose A_1 and A_2 are differentials

$$\Rightarrow \lim_{|h|\to 0} \frac{|f(x+h) - f(x) - A_1 h|}{|h|} = 0$$

$$\lim_{|h|\to 0} \frac{|f(x+h) - f(x) - A_2 h|}{|h|} = 0 \qquad \text{Subtract}$$

$$\lim_{|h|\to 0} \frac{|A_2 h - A_1 h|}{|h|} = 0 \qquad \text{Let } h = tx$$

$$\lim_{|tx|\to 0} \frac{|A_2(tx) - A_1(tx)|}{|tx|} = 0$$

$$\lim_{|x|\to 0} \frac{|A_2(x) - A_1(x)|}{|x|} = 0 \qquad A_2(x) = A_1(x)$$

$$\begin{array}{ccc} T\mathbb{R}^n & \xrightarrow{Df} & T\mathbb{R}^m \\ \downarrow & & \downarrow \\ \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^m \end{array}$$

Notation: Equivalent Definitions of $Df(x)$

① $|f(x_0+h) - f(x_0) - Ah| < \epsilon(h)|h|$ where $\epsilon \to 0$ as $h \to 0$

$\downarrow$ $h = x - x_0$

② f is diff. at x_0 if $\forall \epsilon > 0$, there exists a $\delta > 0$ and a linear map $A = Df$ such that

$$\frac{|f(x) - f(x_0) - A(x_0)\cdot h|}{|x - x_0|} < \epsilon \quad \text{when} \quad |x - x_0| < \delta$$

Tuesday, March 25, 2008
1:25 PM

Chain Rule

$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^k$

Given $f: \mathbb{R}^n \to \mathbb{R}^m$ Differentiable
$g: \mathbb{R}^m \to \mathbb{R}^k$

Let $F = g \circ f$

Then $DF(x_0) = Dg(f(x_0)) \cdot Df(x_0)$

<u>Pf</u> Let $u(x) = f(x) - f(x_0) - A(x - x_0)$ $\quad A = Df \quad y = f(x)$

$v(y) = g(y) - g(y_0) - B(y - y_0) \quad B = Dg$

Then $\forall \epsilon > 0 \ \exists \delta > 0$ s.t

$|u(x)| < \epsilon(x) \cdot (x - x_0)$

if $|x - x_0| < \delta$

$$\frac{|u(x)|}{|x - x_0|} \to 0$$

$$\frac{v(y)}{|y - y_0|} \to 0$$

<u>Then</u>

$$F(x) - F(x_0) - B(A(x - x_0)) = g(f(x)) - g(f(x_0)) - B(A(x - x_0))$$

$$= g(y) - g(y_0) - B(f(x) - f(x_0) - u(x))$$

$$= v(y) + B(y - y_0) - B(y - y_0) + B(u(x))$$

$$= v(y) + B(u(x))$$

want

a) $\dfrac{B(u(x))}{|x - x_0|} \to 0 \qquad \dfrac{|B(u(x))|}{|x - x_0|} \le \|B\| \dfrac{|u(x)|}{|x - x_0|} \to 0$

b) $\dfrac{v(f(x))}{|x - x_0|} \to 0$

$\forall \epsilon > 0 \ \exists \delta > 0$ s.t

$|v(f(x))| < \epsilon |f(x) - f(x_0)|$ if $|f(x) - f(x_0)| < \delta$

$\forall \delta > 0 \ \exists \epsilon_1 > 0$

s.t $|f(x) - f(x_0)| < \delta$ if

$|x - x_0| < \epsilon_1$

$$|v(f(x))| < \epsilon |u(x) + A(x - x_0)|$$
$$< \epsilon |u(x)| + \epsilon \|A\| |x - x_0|$$

$$\frac{|v(f(x))|}{|x - x_0|} < \epsilon \frac{|u(x)|}{|x - x_0|} + \epsilon \|A\| \to 0$$

Thursday, March 27, 2008
12:43 PM

Def Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\quad \{e_i\} \quad \{\hat{e}_j\}$

$$(x^1, \dots, x^n) \overset{f}{\longmapsto} (y^1, \dots, y^m)$$

$$y^1 = f^1(x^1, \dots x^n)$$

$$\vdots$$

$$y^m = f^m(x^1, \dots x^n)$$

$$D_{e_i} f^j(x) = \lim_{t \to 0} \frac{f^j(x + t e_i) - f^j(x)}{t}$$

$$= \frac{\partial f^j}{\partial x^i}$$

Thm: ① $f(x) = k$ (konst) $\qquad Df(x) = 0$

② f linear $\qquad Df(x) = f$

Proof

① $$\lim_{|h| \to 0} \frac{|f(x+h) - f(x) - Df(x) \cdot h|}{|h|} = \lim_{|h| \to 0} \left| \frac{Df(x)|h|}{|h|} \right| = 0$$

$$Df(x) = 0$$

② $$\lim_{|h| \to 0} \frac{|f(x+h) - f(x) - Df(x) \cdot h|}{|h|}$$

$$= \lim_{|h| \to 0} \frac{|\cancel{f(x)} + f(h) - \cancel{f(x)} - Df(x)h|}{|h|} = 0 \qquad Df(x) = f$$

Thm: $$Df(x)\, e_i = \sum_j D_{e_i} f^j(x)\, \hat{e}_j$$

$$\begin{array}{ccc} T\mathbb{R}^n & \xrightarrow{Df} & T\mathbb{R}^m \\ | & & | \\ \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^m \end{array}$$

Pf

By def $$\lim_{|h| \to 0} \frac{|f(x+h) - f(x) - Df(x)h|}{|h|} = 0$$

let $h = t e_i$

$$\lim_{t\to 0} \frac{|f(x+te_i) - f(x) - Df(x)(te_i)|}{t} = 0$$

$$\lim_{t\to 0} \frac{|f(x+te_i) - f(x) - t\,Df(x)e_i|}{t} = 0$$

$$\lim_{t\to 0} \frac{f(x+te_i) - f(x)}{t} = Df(x)e_i$$

$$D_{e_i} f^j(x) = [Df(x)]^j e_i$$

$$\sum_{j=1}^{m} D_{e_i} f^j(x)\, \hat{e}_j = Df(x)\, e_i$$

$$[Df(x)] = [D_{e_i} f^j(x)]$$

$$= \left[\frac{\partial f^i}{\partial x^j}\right] \quad \text{Jacobian}$$

In coord

$$\begin{array}{ccccc} T\mathbb{R}^n & \xrightarrow{Df} & T\mathbb{R}^m & \xrightarrow{Dg} & T\mathbb{R}^k \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^m & \xrightarrow{g} & \mathbb{R}^k \\ \{x^i\} & & \{y^m\} & & \{z^k\} \end{array}$$

$$D(g\circ f)(x) = Dg(f(x))\cdot Df(x)$$

$$z^k = g^k(y^m)$$
$$y^m = f^m(x^i)$$

$$\boxed{\frac{\partial z^k}{\partial x^i} = \frac{\partial z^k}{\partial y^m}\frac{\partial y^m}{\partial x^i}}$$

$$D(g\circ f) = Dg\cdot Df$$

Special Case

$$F = f\circ\gamma$$

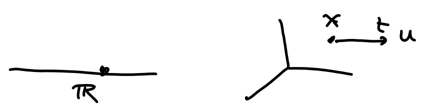
$$\begin{array}{ccccc} \mathbb{R} & \xrightarrow{\gamma} & \mathbb{R}^n & \xrightarrow{f} & \mathbb{R} \\ t & & x & & \text{Linda} \end{array}$$

$$DF(t) = Df(\gamma(t))\cdot D\gamma(t)$$

$$\frac{d(\text{Linda})}{dt} = \frac{\partial(\text{Linda})}{\partial x^i}\cdot\frac{dx^i}{dt}$$

Let $\gamma(t) = x + tu$

$$df = \frac{\partial f}{\partial x^i}\,dx^i$$



$$\frac{df}{dt} = \frac{\partial f}{\partial x^i}\cdot u^i = \nabla f(x)\cdot\vec{u}$$

Last time
Chain Rule

$$\begin{array}{ccccc} T\mathbb{R}^n & \xrightarrow{Df} & T\mathbb{R}^m & \xrightarrow{Dg} & T\mathbb{R}^k \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^m & \xrightarrow{g} & \mathbb{R}^k \end{array}$$

$$\begin{array}{ccccc} \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^m & \xrightarrow{g} & \mathbb{R}^k \\ x^i & & y^j & & z^k \end{array}$$

$$D(g \circ f)(x) = Dg(f(x)) \cdot Df(x)$$

In Matrix form $\quad \dfrac{\partial z^k}{\partial x^i} = \dfrac{\partial z^k}{\partial y^j}\dfrac{\partial y^j}{\partial x^i}$

Second Order Chain Rule

$$\frac{\partial^2 z^k}{\partial x^j \partial x^i} = \frac{\partial}{\partial x^j}\left[\frac{\partial z^k}{\partial x^i}\right] = \frac{\partial}{\partial x^j}\left[\frac{\partial z^k}{\partial y^n}\frac{\partial y^n}{\partial x^i}\right]$$

$$= \frac{\partial}{\partial x^j}\left[\frac{\partial z^k}{\partial y^n}\right]\frac{\partial y^n}{\partial x^i} + \frac{\partial z^k}{\partial y^n}\frac{\partial^2 y^n}{\partial x^j \partial x^i}$$

$$= \frac{\partial}{\partial y^m}\left[\frac{\partial z^k}{\partial y^n}\right]\frac{\partial y^m}{\partial x^j}\frac{\partial y^n}{\partial x^i} + \frac{\partial z^k}{\partial y^n}\frac{\partial^2 y^n}{\partial x^j \partial x^i}$$

$$= \frac{\partial^2 z^k}{\partial y^m \partial y^n}\frac{\partial y^m}{\partial x^j}\frac{\partial y^n}{\partial x^i} + \frac{\partial z^k}{\partial y^n}\frac{\partial^2 y^n}{\partial x^j \partial x^i}$$

Ex $\quad z = f(x,y) \qquad \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad x^i = (x,y)$

$\quad x = x(u,v) \qquad (u,v) \longmapsto (x,y) \longrightarrow z \qquad u^j = (u,v)$

$\quad y = y(u,v)$

$$\frac{\partial f}{\partial u^i} = \frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial u^i}$$

$$f_u = f_x x_u + f_y y_u$$

$$f_{uu} = [f_x]_u x_u + f_x x_{uu} + [f_y]_u y_u + f_y y_{uu}$$

$$= [f_{xx} x_u + f_{xy} y_u]\, x_u + f_x x_{uu} +$$
$$[f_{yx} x_u + f_{yy} y_u]\, y_u + f_y y_{uu}$$

$$\begin{cases} f_{uu} = f_{xx} x_u^2 + 2 f_{xy} x_u y_u + f_{yy} y_u^2 + f_x x_{uu} + f_y y_{uu} \\ f_{vv} = f_{xx} x_v^2 + 2 f_{xy} x_v y_v + f_{yy} y_v^2 + f_x x_{vv} + f_y y_{vv} \end{cases}$$

Tuesday, April 01, 2008
12:48 PM

Inverse Function Theorem

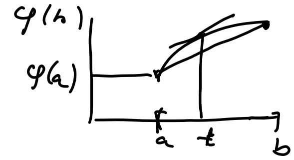
① Rev $\mathbb{R} \xrightarrow{\varphi} \mathbb{R}$ $\boxed{MVT}$

φ is diff on (a,b)

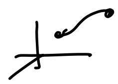
φ is cont on $[a,b]$

then $\exists$ $t \in (a,b)$ s.t

$\varphi(b) - \varphi(a) = \varphi'(t)(b-a)$



Thm $\mathbb{R} \xrightarrow{\varphi} \mathbb{R}^n$



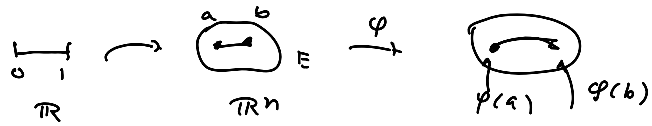
$\exists$ t s.t $|\varphi(b) - \varphi(a)| \leq \|D\varphi(t)\| \, |b-a|$

② Let $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ and $E \subset \mathbb{R}^n$ is a convex set and $\|D\varphi(x)\| \leq M$ for all $x \in E$ then

$$|\varphi(b) - \varphi(a)| \leq \|D\varphi(x)\| \, |b-a|$$
$$\leq M |b-a| \qquad b, a \in E$$

Proof

$\mathbb{R} \xrightarrow{\gamma} \mathbb{R}^n \xrightarrow{\varphi} \mathbb{R}^m$ Let $\gamma(t) = ta + (1-t)b$



$D\gamma(t) = a - b$

Let $g(t) = (\varphi \circ \gamma)(t)$

$$Dg(t) = D\varphi(\gamma(t)) \cdot D\gamma(t)$$
$$|Dg(t)| = |D\varphi(\gamma(t))| \cdot |D\gamma(t)|$$
$$\left|\frac{g(1) - g(0)}{1 - 0}\right| \leq M |b-a|$$
$$|\varphi(b) - \varphi(a)| \leq M |b-a|$$

Contractions If $X \xrightarrow{\varphi} Y$ is a function from Metric space X to metric space Y and

$$d(\varphi(b), \varphi(a)) \leq c \, d(b,a) \qquad 0 < c < 1 \text{ then}$$

φ is called a contraction $\qquad a, b \in \mathcal{U} \subset X$

Theorem If X is complete then $\exists$ a unique fixed point x such $\varphi(x) = x$

Proof

Thursday, April 03, 2008
12:27 PM

Inverse Function Theorem

<u>Contractions</u> Let X be a metric space

$\varphi : X \to X$ is called a contraction if
$d(\varphi(x), \varphi(y)) < c\, d(x,y)$ $\quad 0 < c < 1$
$\quad x, y \in X$

<u>Thm</u>: (Contraction Mapping Theorem)
If $\varphi : \overset{\boxed{\text{cont}}}{X} \to X$ is a contraction and X is <u>complete</u> then $\exists!$ x such that $\varphi(x) = x$

<u>Proof</u>:

<u>Def</u> $x_{n+1} = \varphi(x_n)$ $\quad n = 0, 1, 2 \ldots$ $\quad$ Then

$$d(x_2, x_1) = d(\varphi(x_1), \varphi(x_0)) < c\, d(x_1, x_0)$$
$$d(x_3, x_2) = d(\varphi(x_2), \varphi(x_1)) < c\, d(x_2, x_1) < c^2 d(x_1, x_0)$$
$$\vdots$$
$$d(x_{n+1}, x_n) < c^n d(x, x_0)$$

Suppose $m > n > N$. Then

$$d(x_m, x_n) \le d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n)$$
$$\le (c^{m-1} + c^{m-2} + \cdots + c^n)\, d(x_1, x_0)$$
$$< \frac{c^n}{1-c} d(x_1, x_0) < \epsilon \quad \text{if } N \text{ large enough.}$$

$\therefore$ x_n is a C.S $\quad x_n \to x$

$$\varphi(x) = \lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} x_{n+1} = x$$

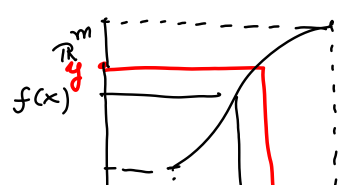
<u>I.F.T</u>: Let $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $Df(x)$ is cont. in an open set $a \in E$ and $Df(a)$ is invertible, then

a) $\exists$ open sets $a \in U$ and $f(a) = b \in V$ with f 1-1 and $f(U) = V$

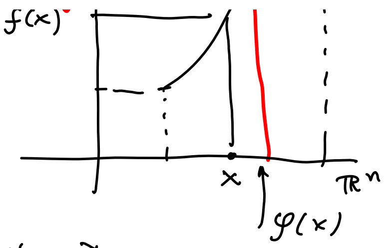
b) If $g = f^{-1}$ then g is continuously differentiable

<u>Proof</u> (a) Let $A = Df(a)$
$x \in E$
Let $\varphi(x) = x + A^{-1}(y - f(x))$

<u>Pic</u>



$x \in E$



Let $\varphi(x) = x + A^{-1}(y - f(x))$

Then $D\varphi(x) = 1 + A^{-1}(0 - Df(x))$
$= 1 - A^{-1}(Df(x))$
$= A^{-1}(A - Df(x))$
$= A^{-1}(Df(a) - Df(x))$

Pick λ
$|Df(a) - Df(x)| < \lambda$
$\|A^{-1}\| < \frac{1}{2\lambda}$

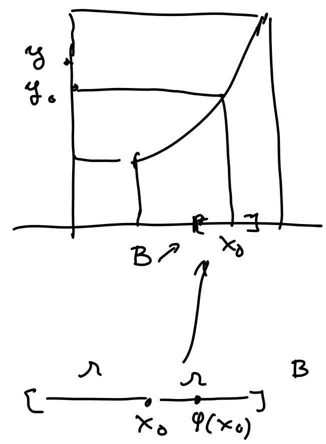
$|D\varphi(x)| < \|A^{-1}\| \, |Df(a) - Df(x)|$
$< \frac{1}{2}$

$\therefore \quad |\varphi(x_1) - \varphi(x_2)| < \frac{1}{2}|x_1 - x_2| \qquad x_1, x_2 \in E$

φ is a contraction map
$\exists$ unique x such $\varphi(x) = x \Rightarrow f(x) = y$

On some nbhd $\mathcal{U}$ of a and V of b, f is 1-1

Need to show V is open



Pick $y_0 \in V$

Let $x_0 \in B_r(x_0) \subset \overline{B}_r \subset \mathcal{U}$

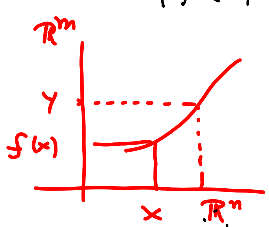
If $|y - y_0| < \lambda r$ then $y \in V$

$\varphi(x_0) = x_0 + A^{-1}(y - f(x_0))$
$= x_0 + A^{-1}(y - y_0)$

$|\varphi(x_0) - x_0| = \|A^{-1}\| \, |y - y_0|$
$< \frac{1}{2\lambda} \lambda r$

If $y = f(x)$ $\qquad < \frac{r}{2}$

$|\varphi(x) - x_0| \leq |\varphi(x) - \varphi(x_0)| + |\varphi(x_0) - x_0|$
$\leq \frac{1}{2}|x - x_0| + \frac{r}{2}$



$< \frac{1}{2}r + \frac{1}{2}r =$

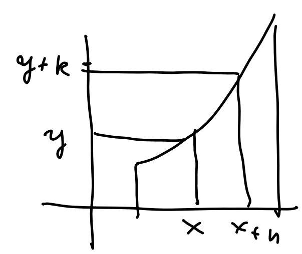
$< r \qquad x \in B$

$f(x) = y \in V \qquad V$ open!

$\varphi(x) = x - A^{-1}(y - f(x))$

f is 1-1
on $\mathcal{U}$ onto V
$\exists \; g = f^{-1}$

b) Let $y, y+k \in V$



$y = f(x), \quad y + k = f(x+h)$

$\varphi(x+h) - \varphi(x) = (x+h) + A^{-1}[y - f(x+h)]$
$- [x + A^{-1}(\cancel{x}y - f(x))]$

$|\varphi(x+h) - \varphi(x)| = |h + A^{-1}(y - (y+k))|$
$= |h + A^{-1}(-k)|$
$= |h - A^{-1}k| < \frac{1}{2}|x + h - x|$

$Df(x)^{-1} \; \exists$

$\Rightarrow |h - A^{-1}(k)| < \frac{1}{2}|h| \Rightarrow |A^{-1}k| > \frac{1}{2}|h|$

Let $g(f(x)) = x$ $\qquad T = Df^{-1}$ $\qquad \|A^{-1}\|\,|k| > \frac{1}{2}|h|$

$$g(y+k) - g(y) - T(k) = x+h-x - T(f(x+h)-f(x))$$
$$= h - T(f(x+h) - f(x))$$
$$= T(Df(h) - f(x+h) + f(x))$$

$$\frac{|g(y+k) - g(y) - T(k)|}{|k|} = \frac{|T(f(x+h) - f(x) - Df(h)|}{|k|}$$

$$\leq \frac{\|T\|\,|f(x+h) - f(x) - Df(h)|}{\frac{1}{2}|h|}\|A^{-1}\| \to 0$$

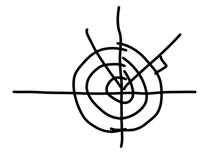
$\therefore\ T = Dg$.

Tuesday, April 08, 2008
12:56 PM

Example:

(1) $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\qquad x^i = f^i(u^\beta)$

$(x,y) \longmapsto (r,\theta)$ $\qquad dx^i = \dfrac{\partial f^i}{\partial u^\beta} du^\beta$



$x = r\cos\theta \qquad dx = \cos\theta\, dr - r\sin\theta\, d\theta$

$y = r\sin\theta \qquad dy = \sin\theta\, dr + r\cos\theta\, d\theta$

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}\begin{bmatrix} dr \\ d\theta \end{bmatrix} \qquad \left|\frac{\partial f^i}{\partial u^\beta}\right| = r \neq 0 \quad \text{if } r \neq 0$$

$k = A h$

$A^{-1} k = h$

$$\begin{bmatrix} dr \\ d\theta \end{bmatrix} = \frac{1}{r}\begin{bmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$\frac{\partial r}{\partial x} = \cos\theta \qquad \frac{\partial r}{\partial y} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = -\frac{1}{r}\sin\theta \qquad \frac{\partial \theta}{\partial y} = \frac{1}{r}\cos\theta$$

Ex 2 $\qquad f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$u = \frac{1}{2}\ln(x^2+y^2) = \ln\sqrt{x^2+y^2}$

$v = \tan^{-1}\dfrac{y}{x}$

$w = u + iv \qquad u = u(x,y)$

$z = x + iy \qquad v = v(x,y)$

$w = f(z)$ Diff

(1) $\begin{cases} u_{xx} + u_{yy} = 0 \\ v_{xx} + v_{yy} = 0 \end{cases}$

Conjugate Harmonic.

$$\begin{cases} du = \dfrac{x}{x^2+y^2}\,dx + \dfrac{y}{x^2+y^2}\,dy \\ dv = \dfrac{-y}{x^2+y^2}\,dx + \dfrac{x}{x^2+y^2}\,dy = d\theta \end{cases}$$

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \frac{1}{x^2+y^2}\begin{bmatrix} x & y \\ -y & x \end{bmatrix}\begin{bmatrix} dx \\ dy \end{bmatrix} \qquad x^2+y^2 \neq 0 \checkmark$$

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}\begin{bmatrix} du \\ dv \end{bmatrix} \qquad \begin{cases} \dfrac{\partial x}{\partial u} = x & \dfrac{\partial x}{\partial v} = -y \\ \dfrac{\partial y}{\partial u} = y & \dfrac{\partial y}{\partial v} = x \end{cases}$$

Q. What is going on?

$w = u + iv$

$z = x + iy = re^{i\theta} \qquad r^2 = x^2+y^2$

$\theta = \tan^{-1}\frac{y}{x}$

$$w = \ln z = \ln re^{i\theta} \qquad 0 \le \theta < 2\pi$$

$$= \ln r + i\theta$$

$$u + iv = \ln\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x} \qquad \begin{cases} u = \ln\sqrt{x^2+y^2} \\ v = \tan^{-1} y/x \end{cases}$$

Inverse:

$e^w = z$

$e^{u+iv} = x + iy$

$e^u e^{iv} = x + iy$

$x + iy = e^u[\cos v + i\sin v]$

$\begin{cases} x = e^u\cos v \\ y = e^u\sin v \end{cases}$

$\dfrac{\partial x}{\partial u} = x \qquad \dfrac{\partial x}{\partial v} = -y$

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Implicit Functions.

Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^n$

$$(x^1, \dots, x^n, y^1, \dots, y^m) \xrightarrow{F} (z^1, \dots, z^n)$$

$$z^1 = F^1(x^1, \dots x^n, y^1, \dots y^m) = 0 \qquad F(x, y) = 0$$

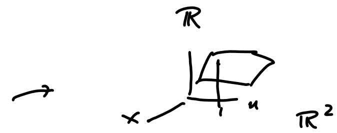
$$\vdots$$

$$z^n = F^n(x^1, \dots, x^n, y^1, \dots y^m) = 0$$

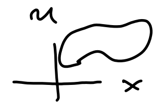
Special Cases

$F: \mathbb{R}^2 \to \mathbb{R}$

$(x,u) \mapsto f(x,u) = z$



At (x_0, u_0) $\quad f(x_0, u_0) = 0$



$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial u}du = 0$$

$$\frac{\partial f}{\partial x}dx = -\frac{\partial f}{\partial u}du$$

$$\boxed{\frac{dx}{du} = \frac{-f_u}{f_x}} \qquad f_x(x_0,u_0) \neq 0$$

$x^3 + y^3 = 1$

$3x^2\,dx = -3y^2\,dz$

$$\begin{cases} \dfrac{dx}{dy} = -\dfrac{y^2}{x^2} \\ \dfrac{dy}{dx} = -\dfrac{x^2}{y^2} \end{cases}$$

$a\,dx + b\,dy = 0$ Linear

$a\,dx = b\,dy$

$dx = a^{-1}b\,dy$

$f: \mathbb{R}^{n+m} \to \mathbb{R}^n$

$(x^1, \dots x^n, u^1, \dots u^m) \to z^n = f^n(x^1, \dots x^n, u^1, \dots u^m] = 0$

$f^n(\vec{x}_0, \vec{u}_0) = 0$

$$\begin{cases} df^1 = \dfrac{\partial f^1}{\partial x^1}dx^1 + \dots + \dfrac{\partial f^n}{\partial x^n}dx^n + \dfrac{\partial f^1}{\partial u^1}du^1 + \dots + \dfrac{\partial f^1}{\partial u^m}du^m = 0 \\ \vdots \\ df^n = \dfrac{\partial f^n}{\partial x^1}dx^1 + \dots + \dfrac{\partial f^n}{\partial x^n}dx^n + \dfrac{\partial f^n}{\partial u^1}du^1 + \dots + \dfrac{\partial f^n}{\partial u^m}du^m = 0 \end{cases}$$

(1)
$$\underbrace{\begin{bmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \frac{\partial f^n}{\partial x^1} & \cdots & \frac{\partial f^n}{\partial x^n} \end{bmatrix}}_{A_x} \begin{bmatrix} dx^1 \\ \vdots \\ dx^n \end{bmatrix} + \begin{bmatrix} \frac{\partial f^1}{\partial u^1} & \cdots & \frac{\partial f^1}{\partial u^m} \\ \vdots & & \\ \frac{\partial f^n}{\partial u^1} & \cdots & \frac{\partial f^n}{\partial u^m} \end{bmatrix} \begin{bmatrix} du^1 \\ \vdots \\ du^m \end{bmatrix} = 0$$

$A_x \cdot d\vec{x} + B_u \cdot du = 0$

"In principle" $\quad x^n = g^n(u^m)$

(2)
$$\boxed{dx^n = \frac{\partial x^n}{\partial u^m}du^m} = \frac{\partial x^n}{\partial u^1}du^1 + \dots + \frac{\partial x^n}{\partial u^m}du^m$$

$$①+② \Rightarrow \quad \frac{\partial x^n}{\partial u^m} = -\begin{bmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \vdots & & \\ \frac{\partial f^n}{\partial x^1} & \cdots & \frac{\partial f^n}{\partial x^n} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial f^1}{\partial u^1} & \cdots & \frac{\partial f^1}{\partial u^m} \\ \vdots & & \\ \frac{\partial f^n}{\partial u^1} & \cdots & \frac{\partial f^n}{\partial u^m} \end{bmatrix} \quad ✱$$

$$= -\left[\frac{\partial(f^1,\dots,f^n)}{\partial(x^1,\dots,x^n)}\right]^{-1}\left[\frac{\partial(f^1,\dots,f^n)}{\partial(u^1,\dots,u^m)}\right]$$

$$f^n(x^k, u^\alpha) = 0 \qquad n,k = 1,\dots,n \qquad \alpha,\beta = 1,\dots,m$$

$$df^n = \frac{\partial f^n}{\partial x^k}dx^k + \frac{\partial f^n}{\partial u^\alpha}du^\alpha = 0$$

$$\frac{\partial f^n}{\partial x^k}dx^k = -\frac{\partial f^n}{\partial u^\alpha}du^\alpha$$

$$\frac{\partial x^k}{\partial u^\alpha} = -\left(\underbrace{\frac{\partial f^n}{\partial x^k}}_{\neq 0}\right)^{-1}\left(\frac{\partial f^n}{\partial u^\alpha}\right) \quad ✱$$

Ex
$$\begin{cases} x^2+2y^2+u^2-v^2+w^2 = 21 = f^1 \\ -x^2+y^2+u^2+v^2+w^2 = 1 = f^2 \end{cases} \qquad P(3,2,1,1,2)$$

$$f: \mathbb{R}^{2+3} \to \mathbb{R}^2$$

$$x = x^1 = g^1(u,v,w) \qquad \begin{Bmatrix} x^1_u, & x^1_v, & x^1_w \\ x^2_u, & x^2_v, & x^2_w \end{Bmatrix} \qquad \frac{\partial(x^1,x^2)}{\partial(u,v,w)}$$
$$y = x^2 = g^2(u,v,w)$$

Sol
$$\begin{cases} 2x\,dx + 4y\,dy + 2u\,du - 2v\,dv + 2w\,dw = 0 \\ -2x\,dx + 2y\,dy + 2u\,du + 2v\,dv + 2w\,dw = 0 \end{cases}$$

$$\begin{bmatrix} 2x & 4y \\ -2x & 2y \end{bmatrix}\begin{bmatrix} dx \\ dy \end{bmatrix} + \begin{bmatrix} 2u & -2v & 2w \\ 2u & 2v & 2w \end{bmatrix}\begin{bmatrix} du \\ dv \\ dw \end{bmatrix} = 0$$

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = -\begin{bmatrix} x & 2y \\ -x & y \end{bmatrix}^{-1}\begin{bmatrix} u & -v & w \\ u & v & w \end{bmatrix}\begin{bmatrix} du \\ dv \\ dw \end{bmatrix}$$

$$= -\frac{1}{3xy}\begin{bmatrix} y & -2y \\ x & x \end{bmatrix}\begin{bmatrix} u & -v & w \\ u & v & w \end{bmatrix}\begin{bmatrix} du \\ dv \\ dw \end{bmatrix}$$

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = -\frac{1}{3xy}\begin{bmatrix} -yu & -3yv & -yw \\ 2xu & 0 & 2xw \end{bmatrix}\begin{bmatrix} du \\ dv \\ dw \end{bmatrix}$$

$$\left.\frac{\partial x}{\partial u}\right|_P = -\frac{1}{3xy}(-yu)\Big|_P = \qquad P(3,2,1,1,2)$$

$$= -\frac{1}{3(6)}(-2\cdot 1) = \frac{1}{9}$$

Thursday, April 10, 2008
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Differential Forms - Stoke's Thm

$V = \mathbb{R}^3 \; (x, y, z)$

1-form $\quad \alpha = f_1 dx^1 + f_2 dx^2 + f_3 dx^3$

Ex $\quad \alpha = 3dx + x^2 y \, dy - (3z + x) dz$

$f: \mathbb{R}^3 \to \mathbb{R} \quad \Rightarrow f = f(x, y, z)$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$ Exact diff

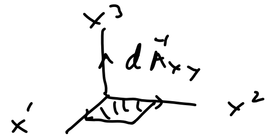
$$df = \vec{\nabla} f \cdot d\vec{x}$$

Def $\quad \boxed{dx^i \wedge dx^j = -dx^j \wedge dx^i} \quad i, j = 1, 2, 3$

Ex $\quad dx^1 \wedge dx^2 = -dx^2 \wedge dx^1 \qquad \wedge$ Wedge

$dx^1 \wedge dx^1 = -dx^1 \wedge dx^1 = 0$

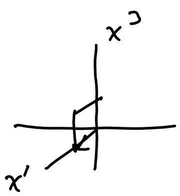
(1)



Classically

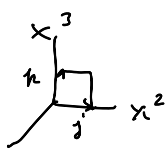
$d\vec{A}_{xy} = \vec{i} dx \times \vec{j} dy = dx\,dy\,\vec{k} \qquad \boxed{dA_{xy} = dx \wedge dy}$

(2)



$dA_{xz} \; - \vec{i}\, dx^1 \times \vec{k}\, dx^3 = -dx^1 dx^3 \vec{j} \qquad \boxed{dA_{xz} = -dx^1 \wedge dx^3}$

(3)



$dA_{yz} = \vec{j}\, dx^2 \times \vec{k}\, dx^3 = dx^2 dx^3 \vec{i} \qquad \boxed{dA_{yz} = dx^2 \wedge dx^3}$

2- form

$$\omega = f_1\, dx^2 \wedge dx^3 - f_2\, dx^1 \wedge dx^3 + f_3\, dx^1 \wedge dx^2$$

Given $\omega = f_1\, dx^1 + f_2\, dx^2 + f_3\, dx^3 = f_i\, dx^i$

$$\boxed{d\omega = df_i \wedge dx^i}$$

$$= df_1 \wedge dx^1 + df_2 \wedge dx^2 + df_3 \wedge dx^3$$

$\vec{F} = \langle f_1, f_2, f_3 \rangle$

$$= \left(\cancel{\frac{\partial f_1}{\partial x^1} dx^1} + \frac{\partial f_1}{\partial x^2} dx^2 + \frac{\partial f_1}{\partial x^3} dx^3\right) \wedge dx^1 +$$

$$\left(\frac{\partial f_2}{\partial x^1} dx^1 + \cancel{\frac{\partial f_2}{\partial x^2} dx^2} + \frac{\partial f_2}{\partial x^3} dx^3\right) \wedge dx^2 +$$

$$\left(\frac{\partial f_3}{\partial x^1} dx^1 + \frac{\partial f_3}{\partial x^2} dx^2 + \cancel{\frac{\partial f_3}{\partial x^3} dx^3}\right) \wedge dx^3$$

$$= \left(\frac{\partial f_3}{\partial x^2} - \frac{\partial f_2}{\partial x^3}\right) dx^2 \wedge dx^3 - \left(\frac{\partial f_3}{\partial x^1} - \frac{\partial f_1}{\partial x^3}\right) dx^1 \wedge dx^3 +$$

$$\left(\frac{\partial f_2}{\partial x^1} - \frac{\partial x^1}{\partial x^2}\right) dx^1 \wedge dx^2$$

$\omega = F \cdot dr$ $\qquad$ $d\omega = (\vec{\nabla} \times \vec{F})\, d\vec{S}$

$$\int_{c = \partial s} \omega = \int_s d\omega$$

Tuesday, April 15, 2008
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Measure Theory

Let $\mathcal{A}$ be a collection of open sets in a space X $(\mathbb{R}, \mathbb{R}^n)$.

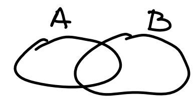
Def $\mathcal{A}$ is called Boolean Algebra if

1) $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$.

2) $A \in \mathcal{A}$ then $\tilde{A} \in \mathcal{A}$.

a) De-Morgan's Laws $\widetilde{(A \cup B)} = A^c \cap B^c$

Hence $A \cap B \in \mathcal{A}$



b) $A_1, \dots A_k \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{k} A_i \in \mathcal{A}$

$\Rightarrow \bigcap_{i=1}^{k} A_i \in \mathcal{A}$.

Prop
If $\mathcal{C}$ is a collection of sets, there is a smallest algebra $\mathcal{A}$ which contains $\mathcal{C}$

Pf $\mathcal{A} = \bigcap_{\mathcal{B} \subset \mathcal{F}} \mathcal{B}$ $\quad \mathcal{F} = \{$ set of all algebras containing $\mathcal{C}\}$

Def An algebra $\mathcal{A}$ is called a σ-algebra if $\{A_n\}$ is sequence of sets in $\mathcal{A}$, then

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$$

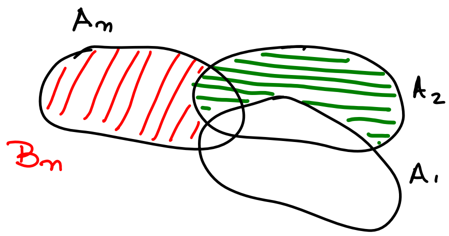
Note $\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$.

Prop: If $\{A_n\}$ is a sequence in $\mathcal{A}$, There exists a sequence $\{B_n\}$ such that

$$B_n \cap B_m = \phi \qquad \forall\, n \neq m$$

and $\displaystyle\bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A_k$

Proof



Let $B_n = A_n \sim [A_1 \cup A_2 \cup \cdots \cup A_{n-1}]$

$$B_n = A_n \cap \tilde{A}_1 \cap \tilde{A}_2 \cap \cdots \cap \tilde{A}_{n-1}$$

Suppose $m < n$ $\qquad B_m \subset A_m$

$$\begin{aligned} B_m \cap B_n &\subset A_m \cap B_n \\ &\subset A_m \cap [A_n \cap \tilde{A}_{n-1} \cdots \cap \tilde{A}_m \cap \cdots \cap \tilde{A}_2 \cap \tilde{A}_1] \\ &\subset (A_m \cap \tilde{A}_m) \cap (A_n \ ---- \ ) \\ &\subset \phi \cap (\cdots) \\ &= \phi \end{aligned}$$

Def Let X be a topological space $(X = \mathbb{R}, \mathbb{R}^n)$

The Borel algebra of X is the smallest algebra which contains the open sets.

Ex If $X = \mathbb{R}$ ① $I = (a,b)$

② $A = \bigcup^{\infty} A_n \qquad A_n$ closed

③ $\displaystyle A = \bigcup_{n=1}^{\infty} \left[a + \tfrac{1}{n},\ b - \tfrac{1}{n}\right] = (a,b)$

Def 1) A closed interval in $\mathbb{R}$ is a set $I = [a, b]$

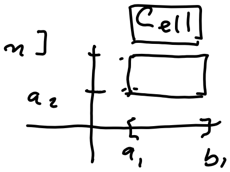
$[a, b] = \{ x \mid a \le x \le b \}$

2) An open interval in $\mathbb{R}$ is a set of the form $I = (a, b) = \{ x \mid a < x < b \}$

3) A closed interval in $\mathbb{R}^n$ is a set of the form

$$I^k = [a_1, b_1] \times [a_2, b_2] \times \dots [a_n, b_n]$$

$$= \prod_{k=1}^{n} [a_k, b_k]$$



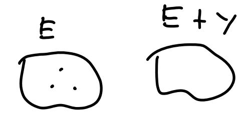
Let $I = [a, b]$

Def $\ell(I) = b - a$

$$\ell(I^k) = \prod_{k=1}^{n} (b_k - a_k) = (b_1 - a_1)(b_2 - a_2) \dots (b_k - a_n)$$

A measure should have the following features

a) $m(I) = b - a \qquad m(I^n) = \prod_{k=1}^{n} (b_k - a_k)$



b) $E_1 \cap E_2 \dots \cap E_n = \phi$

$$m\left(\bigcup^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k)$$ Countably Additive

c) m - Translation invariant

$m(E + y) = m(E)$

$\boxed{\text{d) Any set is measurable.}}$ Too strong.

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Lebesgue Measure in $\mathbb{R}$

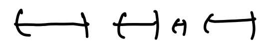
Let A be a set in $\mathbb{R}$ and $\{I_n\}$ be the set of all countable collections which contain A.

$$A \subset \bigcup_n I_n$$

Def the outer measure $m^*(A)$ is defined by

$$m^*(A) = \inf_{A \subset \bigcup I_n} \sum_i l(I_n)$$

P.C



Prop ① $m^*(\phi) = 0$

② $A \subset B$
$m^*(A) = m^*(B)$ H.w.

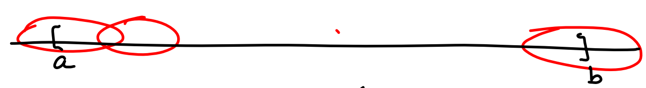
Thm $m^*([a,b]) = b - a$

Pf

(i) $[a,b] \subset (a-\epsilon, b+\epsilon)$

$$m^*([a,b]) \leq b - a + 2\epsilon \implies m^*([a,b]) \leq b - a$$

(ii) Let $\{I_n\}$ be cover of $[a,b]$



By Heine Borel there is a finite subcover.

Thursday, April 17, 2008
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Last Time

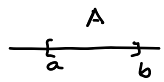
Outer Measure

$$m^*(A) = \inf_{A \subset \cup I_n} \sum \ell(I_n) \qquad I_n \text{ open}$$

Prop $m^*([a,b]) = b-a$

Proof

ⓐ



Let $I = (a-\epsilon, b+\epsilon)$

$m^* A \leq b - a + 2\epsilon \quad \forall \epsilon$

$m^* A \leq b - a$

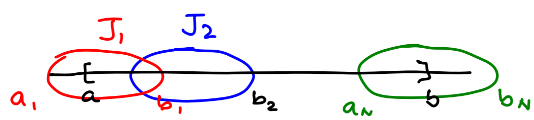
ⓑ Show that $m^* A \geq (b-a)$

Let $A \subset \dot{\cup} I_n$

$A = [a,b]$ closed, bd

Heine Borel

A = compact



A is already covered by a finite subset

$A = \bigcup_{n=1}^{K} I_n \qquad \exists\ J_1 \in \{I_n\}$ such that

$a \in J_1 = (a_1, b_1) \qquad a_1 < a < b_1$

$b_1 \in J_2 = (a_2, b_2) \qquad a_2 < b_1 < b_2$

$\vdots \qquad\qquad\qquad\qquad \vdots$

$b \in J_N = (a_N, b_N) \qquad a_N < b < b_N$

$$\sum_{k=1}^{N} \ell(J_k) = (b_1 - a_1) + (b_2 - a_2) + \cdots + (b_N - a_N)$$

$$\geq b_N - a_1 \geq b - a$$

$\therefore\ m^*[a,b] = b - a$

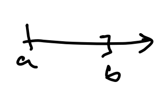
Prop I - any interval

$m^*(I) = \ell(I)$

Ex ① $I = (a, b]$

Let $J = [a,b] = \bar{I}$

② $I = [a, \infty)$



$m^* I = \infty$

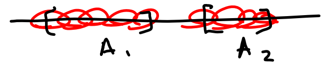
Thm

If A_n is a sequence of sets, then

$$m^*\left(\bigcup^{\infty} A_n\right) \leq \sum^{\infty} m^*(A_n)$$

If A_n is a sequence of sets, then

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n)$$



For each A_n, find a cover by intervals $I_{n,k}$

$$A_1 \subset \bigcup_{k=1}^{\infty} I_{1,k}$$

$$\vdots$$

$$A_2 \subset \bigcup^{\infty} I_{2,k}$$

with $m^*(A_n) < \sum_{k=1}^{\infty} l(I_{n,k}) + \frac{\epsilon}{2^k}$

$I_{n,k}$ is a countable union of countable sets, so $I_{n,k}$ is countable

$$m^*\left(\bigcup_n A_n\right) \leq \sum_k \sum_n l(I_{n,k}) + \frac{\epsilon}{2^k} \to \epsilon$$

$$\leq \sum m^*(A_n)$$

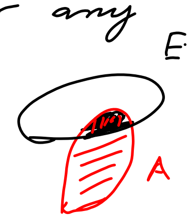
<u>Corollary</u> if A is countable, $m^*(A) = 0$

<u>Corollary</u> $[0,1]$ is uncountable

<u>Lebesgue Measure</u> (Carathéodory)

<u>Def</u> A set E is measurable if for any set A

$$\boxed{m^*(A) = m^*(A \cap E) + m^*(A \cap \tilde{E})}$$



Since

$$A = (A \cap E) \cup (A \cap \tilde{E})$$

It is clear

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap \tilde{E})$$

Thus, to prove that a set E is measurable it suffices to show that

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap \tilde{E})$$

Scheme: Let $\mathcal{M} = \{\text{Measurable sets}\}$

- 1) Show that $\mathcal{M}$ is an algebra
  - a) $E_1, E_2 \in \mathcal{M} \Rightarrow E_1 \cup E_2 \in \mathcal{M}$
  - b) $E_2 \in \mathcal{M} \Rightarrow \tilde{E}_2 \in \mathcal{M}$
- 2) Note: $\mathcal{M}$ contains the open sets in $\mathbb{R}$

3) Show that $\mathcal{M}$ is σ-Algebra

a) $\left(\bigcup^{\infty} E_n\right) \in \mathcal{M}$

(4) Borel Alg is the smallest σ-algebra that contains the open sets
$\Rightarrow$ Borel Sets are measurable
$m = m^*$ restricted to the Borel sets is the Lebesgue Measure

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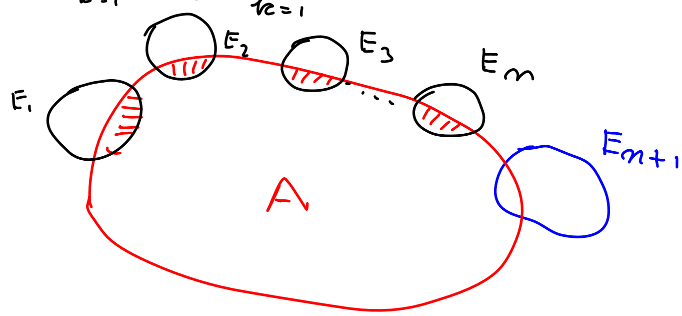
Theorem: The set of measurable sets $\mathcal{M}$ is a σ-algebra.

Proof

Lemma: If $\{E_n\}$ is a finite collection of measurable disjoint sets ($E_n \cap E_k = \phi$) then

$$m^*\left[A \cap \bigcup_{k=1}^{n} E_k\right] = \sum_{k=1}^{n} m^*(A \cap E_k)$$

Pic



Proof: By induction

a) $k=1$ Trivial

b) Assume true for $k=n$. Will show that the assertion is true for $k=n+1$

$$\left(A \cap \bigcup_{k=1}^{n+1} E_k\right) \cap E_{n+1} = A \cap E_{n+1}$$

$$\left(A \cap \bigcup_{k=1}^{n+1} E_k\right) \cap \tilde{E}_{n+1} = A \cap \bigcup_{k=1}^{n} E_k$$

$$m^*\left[\left(A \cap \bigcup^{n+1} E_k\right) \cap E_{n+1}\right] + m^*\left[\left(A \cap \bigcup^{n+1} E_k\right) \cap \tilde{E}_{n+1}\right] =$$

$$= m^*\left(A \cap \bigcup_{k=1}^{n+1} E_k\right) = m^*(A \cap E_{n+1}) + \sum_{k=1}^{n} m^*(A \cap E_k)$$

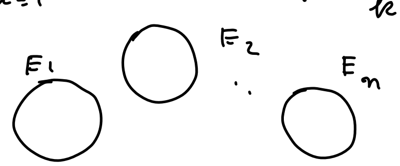
$$= \sum_{k}^{n+1} m^*(A \cap E_k) \quad \checkmark$$

Proof of Theorem

Let $\{E_n\}$ be a sequence of measurable sets. WLG we may assume that the sets are pairwise disjoint $E_n \cap E_m = \phi$

Let $E = \bigcup_{k=1}^{\infty} E_k \qquad K_n = \bigcup_{k=1}^{n} E_k \in \mathcal{M}$

K_n :



$K_n \subset E$

$\tilde{K}_n \supset \tilde{E}$

$A \cap \tilde{K}_n \supset A \cap \tilde{E}$

$$K_n \supset E$$
$$A \cap \tilde{K}_n \supset A \cap \tilde{E}$$

$$m^* A = m^*_n (A \cap K_n) + m^* (A \cap \tilde{K}_n)$$
$$\geq \sum_{k=1}^{n} m^* (A \cap E_k) + m^* (A \cap \tilde{E})$$
$$m^* A \geq \sum_{k=1}^{\infty} m^* (A \cap E_k) + m^* (A \cap \tilde{E})$$
$$\geq m^* (A \cap E) + m^* (A \cap \tilde{E}) \;. \Rightarrow E \in \mathcal{M}$$

$\mathcal{M}$ is a σ-Algebra

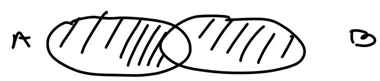
$\mathcal{B}$ = Borel Algebra is the smaller σ-Alg

Contain the open sets in $\mathbb{R}$

∴ Borel sets are measurable

Characterization of Measurable sets.

<u>Def</u> $S(A, B) = A \triangle B = (A \cup B) \sim (A \cap B) = (A \sim B) \cup (B \sim A)$



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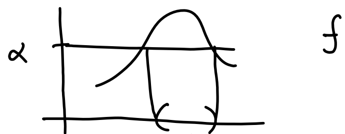
Measurable Functions

Def If Dom(f) is measurable we say that f is measurable if

$$f^{-1}\{(\alpha,\infty)\} = \{x \mid f(x) > \alpha \quad \forall \alpha\} \text{ is measurable}$$



$f^{-1}(\alpha,\infty) \in \mathcal{M}$.

Prop Function f is measurable if any of the following are true

a) $f^{-1}\{(\alpha,\infty)\} \in \mathcal{M}$
b) $f^{-1}\{[\alpha,\infty)\} \in \mathcal{M}$
c) $f^{-1}\{(-\infty,\alpha)\} \in \mathcal{M}$
d) $f^{-1}\{(-\infty,\alpha]\} \in \mathcal{M}$

Idea $\bigcap^{\infty}(\alpha-\frac{1}{n},\infty) = [\alpha,\infty)$

Thm If f is measurable then
$f^{-1}\{(\alpha,\beta)\} \in \mathcal{M}$ Idea $(\alpha,\beta) = (\alpha,\infty)\cap(-\infty,\beta)$

Thm f is measurable iff the inverse image of a measurable set is measurable

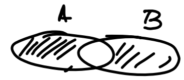
Littlewood Three Principle $m^*(E) < \infty$

1. A measurable set is <u>nearly</u> a finite union of open intervals

2. A measurable function on a sets with finite measure is <u>nearly</u> cont.

3. A convergent sequence of measurable functions is <u>nearly</u> uniformly cont.

P1. Def $A \Delta B = (A \sim B) \cup (B - A)$



a) $m(E) < \infty$, $\exists$ $A = \bigcup^{m} O_m$ such that $(A \Delta E) < \epsilon$ $\forall \epsilon$

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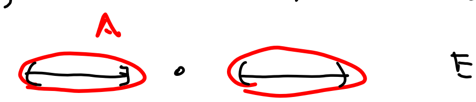
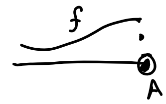


b) $\exists$ $\mathcal{U}$ open s.t $m[\mathcal{U} \sim E] < \epsilon$ $\forall \epsilon$

P.2| If f is measurable on E, then there exists a closed set A, $m(A) < \epsilon$ and $f|_{E \sim A}$ is cont



P.3 Egoroff's Theorem

If $\{f_n\} \to f$ is a seq of measurable functions on E, with $m(E) < \infty$, then $\exists$ A closed with $m(A) < \epsilon$ such that $f_n \to f$ uniformly on $E \sim A$

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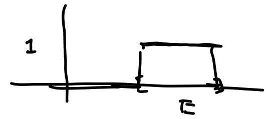
Lebesgue Integration

Def $\chi_E = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$



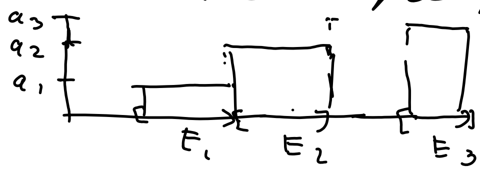
χ_E is measurable iff E is measurable

Def If $\{a_1, \dots a_n\}$ is a finite set of numbers, let

$$\varphi(x) = \sum a_i \chi_{E_i} \qquad m(E_i) < \infty$$

$\varphi(x)$ is called a simple function

Pic



Def $\varphi = \sum a_i \chi_{A_i}$ is in canonical form if

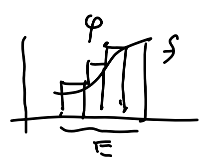
$$A_i = \{x \mid \varphi(x) = a_i\} = \varphi^{-1}(a_i) \qquad A_i \cap A_j = \phi$$

$$\int \varphi = \sum a_i m(A_i)$$

Prop Given f on a measurable set E

$$\inf_{f \le \varphi} \int \varphi = \sup_{f \ge \psi} \int \psi \quad \text{iff } f \text{ is measurable}$$

Def: $\int f = \inf_{f \le \varphi} \int \varphi$



Thm $f \le g$
$\Rightarrow \int f \le \int g$

Thm $\int f + g = \int f + \int g$

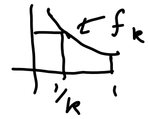
Convergence Theorems

① Bounded Convergence Theorem

If $\{f_n\}$ is a sequence of functions, $|f_n| < M$ and $f_n \to f$, then
$$\int f = \lim \int f_n$$

② Monotone Convergence Theorem

$$f_n \uparrow f \Rightarrow \int f_n \to \int f$$



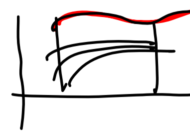
③ Lebesgue Dominated Theorem

③ Lebesgue Dominated Theorem

f_n measurable on E, $m(E) < \infty$

$|f_n| < g$, g integrable and $f_n \to f$

then $\int f = \lim \int f_n$



<u>Def</u> $\mathcal{L}^2[a,b] = \{ f \mid \int_a^b |f|^2 < \infty \}$ $\quad \langle f, g \rangle = \int_a^b f g$

$\|f\|_2 = \left[\int_a^b |f|^2 \right]^{1/2} < \infty$ $\quad \mathcal{L}^2$ Norm

$f_n \to f$ in the mean (norm) if

$\|f_n - f\| \to 0$

<u>Thm</u> $f_n = \sum \langle f, \varphi_n \rangle \varphi_n \qquad \langle \varphi_m, \varphi_n \rangle = \delta_{nk}$

$\|f_n - f\| \to 0$

<u>Thm</u> $\| \; \|$ is complete $\quad$ Riesz - Fischer Thm