Study Guide for Vector Calculus Exam 2
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1. Functions

A. Let $z = f(x,y)$. The Domain of f is the set of all admissible values of x and y. Exclude from the domain those points where the function is not defined:
Example:

a) $f(x,y) = \text{sqrt}(9 - x^2 - y^2)$. $Df = \{(x,y) \in \Re^2 \mid x^2 + y^2 \leq 9\}$. The domain is a disk of radius 3, including the points of the boundary. In fact the graph is a semi-sphere of radius 3
b) $f(x,y) = \ln(9 - x^2 - y^2)$. $Df = \{(x,y) \in \Re^2 \mid x^2 + y^2 \leq 9\}$. The domain is the interior of a disk of radius 3.
c) $f(x,y) = 1/(9 - x^2 - y^2)$. $Df = \{(x,y) \in \Re^2 \mid x^2 + y^2 \neq 9\}$. The domain is the entire plane except for the points in the a circle of radius 3

B. In describing the traces ($x = k$, $y = k$, $y = k$) recall that:

a) $x^2 + y^2 = c^2$ are circles
b) $(x/a)^2 + (y/b)^2 = c^2$ are ellipses
c) $(x/a)^2 - (y/b)^2 = c^2$ are hyperbolas

2. Partial Derivatives:

A. Know the basic differentiation rules (Addition, Subtraction, Product and Quotient, and Chain rules.
B. Know the formulas for the derivatives of all the elementary functions including:
   a) Power rule
   b) All Trig Functions
   c) The inverse functions $\sin^{-1}(x)$, $\tan^{-1}(x)$ and $\sec^{-1}(x)$
   d) The functions $\sinh(x)$ and $\cosh(x)$
   e) The inverse functions $\sinh^{-1}(x)$, $\tanh^{-1}(x)$

3. Limits:

A. To show that limit as $(x,y) \to (0,0)$ of $f(x,y)$ does not exist first try approaching the origin along a line $y = mx$. If the result depends on m, then limit does not exist. This works whenever $f(x,y)$ involves a fraction in which all the terms in the numerator and in the denominator have powers that add up to the same number.
B. Example: $f(x,y) = 5xy^2 / (x^3 + 7y^3)$. All powers add up to 3. Along $y = mx$ the function reduces to $5m^2 / (1 + 7m^3)$. Since the limit cannot depend on m, the limit does not exist.
C. If the exponents in the terms of the fraction do not add up to the same number, try approaching the origin along $y = m x^2$. The idea is the same: to try to get all the terms to add up to the same power.

4. Differentials

A. If $z = f(x,y)$, then $dz = z_x\, dx + z_y\, dy$.

B. Notice that this is the infinitesimal version of the equation of the tangent plane: $(z - z_o) = z_x\,(x - x_o) + z_y\,(y - y_o)$

C. Use differentials to find partial derivatives of implicit functions $F(x,y,z)=0$.

   a. Take the differential to **linearize** the equation.

   b. Solve the linear for dz and read the partial derivatives from the coefficients of dx and dz

   c. Example: $xz^2 - 5y^3 + z^4 = 4$.

   $(dx)\, z^2 + x\,(2z dz) - 5(3y^2\, dy) + 4z^3\, dz = d(4) = 0$

   $[x\,(2z) + 4z^3\,]\, dz = - z^2\, dx - 5(3y^2\,)dy$

   $dz = [- z^2 / (2xz + 4z^3)]\, dx - [15y^2 / (2xz + 4z^3)]dy$

   So: $z_x = - z^2 / (2xz + 4z^3)$

   $z_x = 15y^2 / (2xz + 4z^3)$

4. Directional Derivatives.

A. given a function $z = f(x,y)$, a point $P\,(x_o, y_o)$ and a **unit** vector **u**= < a,b>, the directional derivative of f at P in the direction of u is given by:

   $D_{\mathbf{u}}\, f(P) = \nabla f \bullet \mathbf{u}\,(P) = a\, f_x + b\, f_y$

B. $D_{\mathbf{u}}\, f(P)$ represents the rate of change of the surface ("the slope of the mountain") in the direction **u.**

C. If the direction is given by:

   a. An arbitrary vector **v**, then take $\mathbf{u} = \mathbf{v} / |\,\mathbf{v}\,|$

   b. A direction from P to a point Q, take $\mathbf{u} = PQ / |\,PQ|$

   c. An angle θ, take $u = <\cos\theta, \sin\theta>$

D. The maximum rate of change occurs in the direction of the Gradient. That is: the maximum occurs when $\mathbf{u} = \nabla f / |\nabla f\,|$.

E. The maximum rate of change is $|\nabla f\,|$.

F. If $F(x,y,z) = k$, then ∇F is normal to the implicitly defined surface. Thus, you can use the ∇F to find the equation of the tangent plane at a point.

   a. Example. Let : $x^2 + 5y^2 + z^2 = 7$, then

   $\nabla F = < 3z, 10y, 2z>$ and the normal at (1,5,1) is **N** = <3,50,2>.

   The equation of the tangent plane is $3(x-1) + 50(y-5) + 2(z-1) = 0$

   b. Example: Let $z = f(x,y) = x^2 + 9y^2$. Find the normal to the paraboloid at the point (1,1).

   Solution: Let $F(x,y,z) = f(x,y) - z$. Then $\nabla F = < f_x, f_y, -1>$.

   So, $\nabla F = <2x, 18y, -1>$ is normal to the surface at any point

   At (1,1), $\nabla F = <2, 18, -1>$

   The equation of the tangent plane is $2(x-1)+18(y-1)-(z-10)=0$.

5. Second Derivative Test for Optimization

A. Let $z = f(x,y)$

a. Compute f_x and f_y and set to equal to zero. Solve the equations $f_x = 0$ and $f_y = 0$. The solutions are called critical points. Denote these by (x_c, y_c).

b. Compute the Hessian

$$H(x,y) = \det \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}\, f_{xy} - (f_{xy})^2$$

c. Evaluate the Hessian at the all the critical points:

If $H(x_c, y_c) > 0$ and $f_{xx}(x_c, y_c) > 0$, then there is a loc **min** at (x_c, y_c).
If $H(x_c, y_c) > 0$ and $f_{xx}(x_c, y_c) < 0$, then there is a loc max at (x_c, y_c).

If $H(x_c, y_c) < 0$ and $f_{xx}(x_c, y_c). > 0$, then there is a saddle at (x_c, y_c).
If $H(x_c, y_c) = 0$, the test fails.

B. If $z = f(x,y)$ is quadratic, then the surface is either a paraboloid, a saddle or a cylinder. In the former two cases there is only one critical point.