APPLIED HARMONIC ANALYSIS OUTLINE

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1. Classic Hilbert Spaces and Frequency Representation

Important Hilbert spaces and associated inner products.

$$\mathbb{R}^n,\ \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i,\ \ \mathbb{C}^n,\ \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$

$$\ell^2(\mathbb{Z}),\ \text{Real case } \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=\infty}^{\infty} x_i y_i,\ \text{Complex case } \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=\infty}^{\infty} x_i \overline{y_i}$$

$$L^2[0,1],\ \ \text{Real case } \langle f(x), g(x) \rangle = \int_0^1 f(x) g(x)\, dx$$

$$L^2[0,1],\ \ \text{Complex case } \langle f(x), g(x) \rangle = \int_0^1 f(x) \overline{g(x)}\, dx$$

$$L^2(\mathbb{R}),\ \ \text{Real case } \langle f(x), g(x) \rangle = \int_{-\infty}^{\infty} f(x) g(x)\, dx$$

$$L^2(\mathbb{R}),\ \ \text{Complex case } \langle f(x), g(x) \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)}\, dx$$

Norm $\|f\| = (\langle f, f \rangle)^{1/2}$, $\|\mathbf{x}\| = (\langle \mathbf{x}, \mathbf{x} \rangle)^{1/2}$ Caucy-Schwarz: $|\langle f, g \rangle| \leq \|f\| \cdot \|g\|$

Definitions of periodic functions, Fourier coefficients/ Fourier Series.
Complex case

$$c_n = \int_0^1 f(x) e^{-2\pi i n x} dx \text{ for } n \in \mathbb{Z},\ \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$$

Real case

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx,\ \ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx\ n = 1, 2, 3 \cdots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx\ n = 1, 2, 3 \ldots,\ \ a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

1.1. **Continuous.** .

The Fourier transform and inverse Fourier Transform for functions

$$\hat{f}(w) = \mathcal{F}(f(x))(w) = \int_{-\infty}^{\infty} f(x)e^{-2\pi iwx}dx \text{ and } \mathcal{F}^{-1}(\hat{f}(w))(x) = \int_{-\infty}^{\infty} \hat{f}(w)e^{2\pi iwx}dw$$

Definition of convolution $f * g(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt$ for functions on defined the real line.

Basic properties of the Fourier transform **Exercise** derive 1,3,5,7

Theorem 1.1.

$$1)\mathcal{F}(f(x)+g(x))(w) = \hat{f}(w)+\hat{g}(w) \qquad 2)\mathcal{F}(cf(x))(w) = c\hat{f}(w)$$

$$3)\mathcal{F}(e^{2\pi ikx}f(x))(w) = \hat{f}(w-k) \qquad 4)\mathcal{F}(f(x-k))(w) = e^{-2\pi ikw}\hat{f}(w)$$

$$5)\mathcal{F}(f'(x))(w) = 2\pi iw\hat{f}(w) \qquad 6)\mathcal{F}(xf(x))(w) = \frac{i}{2\pi}\frac{d}{dw}\hat{f}(w)$$

$$7)\mathcal{F}(f*g)(w) = \hat{f}(w)\cdot\hat{g}(w) \qquad 8)\mathcal{F}(\mathcal{F}(f))(x) = f(-x)$$

$$9)\mathcal{F}(\delta_\alpha)(w) = e^{-i\alpha w} \qquad 10)\mathcal{F}(\mathcal{U}_\alpha)(w) = \frac{-ie^{-i\alpha w}}{w}$$

Parseval's Identity, ($\|\mathbf{f}\|^2 = \sum_n |\langle \mathbf{f}, \mathbf{x}_n\rangle|^2$ for any orthonormal basis $\{\mathbf{x}_n\}$)

Euler's Formula $e^{ix} = \cos(x) + i\sin(x)$.

Exercise I) Compute $\mathcal{F}(\mathbf{1}_{[-1/2,1/2)}(x))(w)$ using the definition. Show $\mathcal{F}(f(ax))(w) = \frac{1}{a}\hat{f}(\frac{w}{a})$ for $a > 0$ and use these two to find $\mathcal{F}(\mathbf{1}_{[-1,1)}(x))(w)$.

II) Compute $\mathcal{F}(\cos(2\pi 7x)\mathbf{1}_{[-1/2,1/2)}(x))(w)$ using 1) and 3).

Theorem 1.2. *The sequence $\{f(x)e^{2\pi imx}\}_{m\in\mathbb{Z}}$ is an orthonormal basis for $L^2[0,1]$ if and only if $|f(x)| = 1$ for "almost all" $x \in [0,1]$.(Note: Really true almost everywhere in measure .)*

Proof. We know from above $\{e^{2\pi imx}\}_{m\in\mathbb{Z}}$ is an orthonormal basis for $L^2[0,1]$, so consider

$$\langle f(x)e^{2\pi imx}, f(x)e^{2\pi inx}\rangle = \int_0^1 |f(x)|^2 e^{2\pi i(m-n)x}dx$$

If $|f(x)| = 1$ then

$$\langle f(x)e^{2\pi imx}, f(x)e^{2\pi inx}\rangle = \langle e^{2\pi imx}, e^{2\pi inx}\rangle$$

and hence $\{f(x)e^{2\pi imx}\}_{m\in\mathbb{Z}}$ is an orthonormal basis.

Let $\{f(x)e^{2\pi imx}\}_{m\in\mathbb{Z}}$ be an orthonormal basis then

$$c_n = \langle f(x), f(x)e^{2\pi inx}\rangle = \langle |f(x)|^2, e^{2\pi inx}\rangle = \delta_{0,n}$$

which implies the Fourier series expansion of $|f|^2 = 1$

□

1.2. **Discrete.** The DFT and the IDFT for sequences (also called vectors or signals) of length N.

$$X(k) = \mathcal{F}_N(x)(k) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x(j)e^{-2\pi ijk/N} \text{ and}$$

$$x(k) = \mathcal{F}_N^{-1}(X)(k) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} X(j)e^{2\pi ijk/N}$$

$$x * y(k) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x(j)y(k-j)$$

Exercise Compute DFT and IDFT for $x = (1, 1, i, i)$

If we let T be the translation matrix, M be the modulation matrix defined by

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, M = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & e^{2\pi i/d^2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & e^{4\pi i/d^2} & 0 & \cdots & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & e^{2\pi i(d^2-2)/d^2} & 0 \\ 0 & 0 & \cdots & 0 & 0 & e^{2\pi i(d^2-1)/d^2} \end{bmatrix}$$

and F the discrete Fourier transform matrix, i.e., $F_{j,k} = \dfrac{1}{\sqrt{(N)}} e^{-2\pi i(jk)/N}$ for $j, k \in \mathbb{Z}_N$ then the expected relationships below follow as analogues of the continuous case.

Exercise Write out the Fourier Transform Matrix for $N = 3$ and check it on Matlab fft(eye(N))/sqrt(N) .

Theorem 1.3.

$$1) T = FMF^*, \qquad 2) M^* = FTF^*$$

$$3) F(x * y)(k) = X(k)Y(K) \qquad 4) F(\delta_0) = \frac{1}{\sqrt{N}}\mathbf{1}$$

Theorem 1.4. *For* $\mathbf{v} \in \mathbb{C}^n$ *then* $\{T^j \mathbf{v}\}_{j=0}^{N-1}$ *is an orthonormal basis iff* $|(F\mathbf{v})(k)| = \dfrac{1}{\sqrt{N}}$ *for all* $k \in \mathbb{Z}_n$

Proof. By properties of the DFT (Theorem 1.3 (2))it is enough to show $\{M^j \mathbf{V}\}_{j=0}^{N-1}$ is an orthonormal basis iff $|(\mathbf{V})(k)| = \dfrac{1}{\sqrt{N}}$.

If $|\mathbf{V}(k)| = \dfrac{1}{\sqrt{N}}$, then $\langle M^j \mathbf{V}, M^k \mathbf{V}\rangle = \dfrac{1}{N}\langle M^j \mathbf{1}, M^k \mathbf{1}\rangle = \begin{cases} 0 \text{ if } j \neq k \\ 1 \text{ if } j = k \end{cases}$.

Now assume $\{M^j \mathbf{V}\}_{j=0}^{N-1}$ is an orthonormal basis. Let $\mathbf{f}$ be any norm 1 vector so by Parseval's Theorem we know $\sum_k |\langle f, M^k \mathbf{V}\rangle|^2 = 1$ but

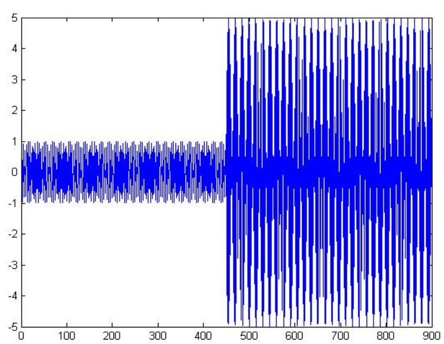
$$1 = \sum_k |\langle \mathbf{f}, M^k \mathbf{V}\rangle|^2 = N \sum_k |\langle \mathbf{f}\mathbf{V}, M^k \frac{\mathbf{1}}{\sqrt{N}}\rangle|^2 = N \sum_k |(\mathbf{f}\mathbf{V})(k)|^2 = N \sum_k |\mathbf{f}(k)|^2 |\mathbf{V}(k)|^2$$

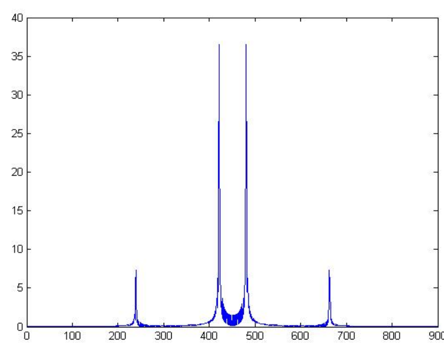
Now consider $\mathbf{f} = T^k \delta_0$ for each k implies $N|\mathbf{V}(k)|^2 = 1$. □

Exercise Derive 3

We will introduce the matrix that acts as a derivative later and recover discrete versions of $\mathcal{F}(f'(x))(w) = iw\hat{f}(w)$ and $\mathcal{F}(xf(x))(w) = i\dfrac{d}{dw}\hat{f}(w)$.

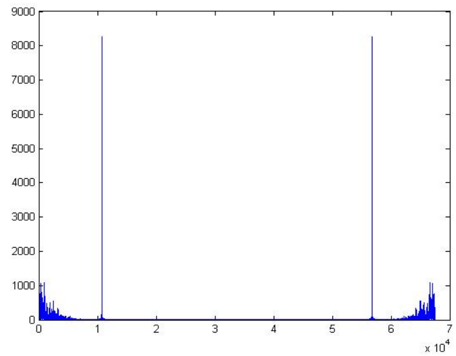
1.3. **Application 1.**

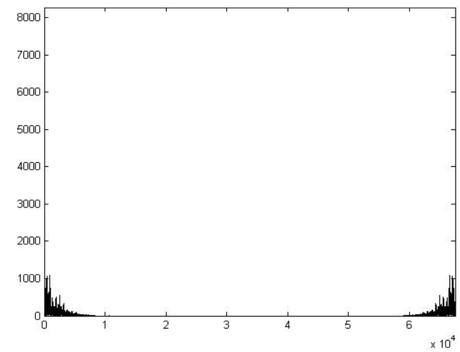




We graph the a signal of two distinct tones and the frequency representation. See Appendix 9.1 for code.

1.4. **Application 2.**





We use an elementary filter and convolution to remove high frequencies. See Appendix 9.3 for code.

2. Linear Algebra and Spectral Theory

Definitions : Linear operator, Upper-lower triangular matrix, similar Matrices, Invertible Matrix, diagonal matrix, diagonalizeable matrix ($A = PDP^{-1}$), eigenvalues, eigenvectors, characteristic equation, singular values, orthogonal or unitary matrix ($UU^* = I$), self-adjoint matrix($A = A^*$), orthogonally diagonalizeable matrix ($A = UDU^*$ with $UU^* = I$), Singular Value Decomposition, least squares solution, linear regression.

Properties of Determinant

$$1) det(A) = det(A^T) \qquad 2) det(A) \neq 0 \text{ iff A is invertible}$$

$$3) det(A^{-1}) = \frac{1}{det(A)} \qquad 4) det(AB) = det(A)det(B)$$

Characteristic polynomial of a matrix A is $p_A(\lambda) = det(A - \lambda I)$.
Theorems/Lemmas you should know the proofs to.

Theorem 2.1. *If two matrices are similar they have the same eigenvalues. (What about eigenvectors? Counterexample?)*

Theorem 2.2. *Real square matrices A and A^T have the same eigenvalues. (What about eigenvectors? Counterexample?)*

Theorem 2.3. *Every transition matrix(a matrix whose columns sum to 1) has 1 as an eigenvalue.*

Theorem 2.4. *If $A = B^*B$ then its eigenvalues are greater then or equal to zero.*

Theorem 2.5. *If A is symmetric it: i) has real eigenvalues(Is converse true? Counterexample?), ii) is orthogonally diagonalizeable. iii) Eigenvectors from distinct eigenvalues of a selfadjoint matrix A are orthogonal. (Counterexample when A is not symmetric)*

Theorem 2.6. *The least squares solution(s) of* $A\mathbf{x} = \mathbf{b}$ *are the non empty solutions of* $A^TA\mathbf{x} = A^T\mathbf{b}$.

Theorem 2.7. *Every matrix* A *has a singular value decomposition of the form* $A = U\Sigma V^T$.

Theorem 2.8. *Show that if* $\sigma = \sqrt{\lambda} \neq 0$ *is a singular value of* A *then it is a singular value of* A^*.

Proof. Recall that singular values of A are the square roots of the eigenvalues of A^TA which we know are all greater then or equal to zero by Theorem2.4. By definition there exists $\mathbf{x} \neq \mathbf{0}$ so that $A^TA\mathbf{x} = \lambda\mathbf{x}$ and so $\lambda\mathbf{x} \neq \mathbf{0}$ and $AA^T(A\mathbf{x}) = A(A^TA\mathbf{x}) = A\lambda\mathbf{x} = \lambda A\mathbf{x}$. Because $A^TA\mathbf{x} \neq \mathbf{0}$ we know $A\mathbf{x} \neq \mathbf{0}$ and therefore $A\mathbf{x}$ is an eigenvector of AA^T corresponding to the eigenvalue λ.

□

Theorem 2.9. *(Spectral Decomposition Theorem) If* A *is symmetric then* $A = \sum \lambda_i \mathbf{u_i}\mathbf{u_i^T}$ *where* $\mathbf{u_i}$ *are orthonormal eigenvectors of* A *in particular* $A\mathbf{x} = \sum \lambda_i \langle \mathbf{x}, \mathbf{u_i} \rangle \mathbf{u_i}$

Exercise: What is the relationship between singularvalues and eigenvalues for a $A = BB^T$?

Definition 2.10. *For a matrix* A *we define* $\|A\| = \sup_{\|x\|=1} \|Ax\|$

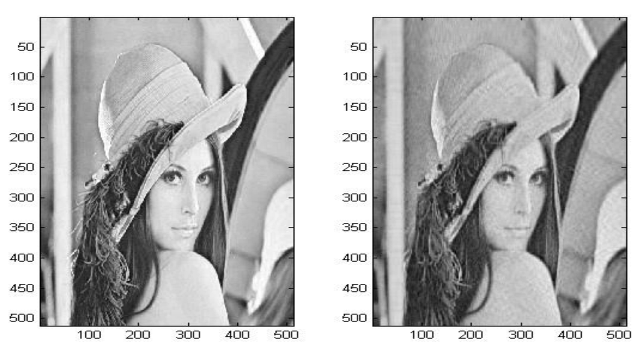
Theorem 2.11. *Show that* $\|A\| = \sigma_N$ *where* σ_N *is the largest singular value of* A

Be able to compute (using Matlab) and use the singular value decomposition.

Be able to use the least squares solution to do a polynomial regression

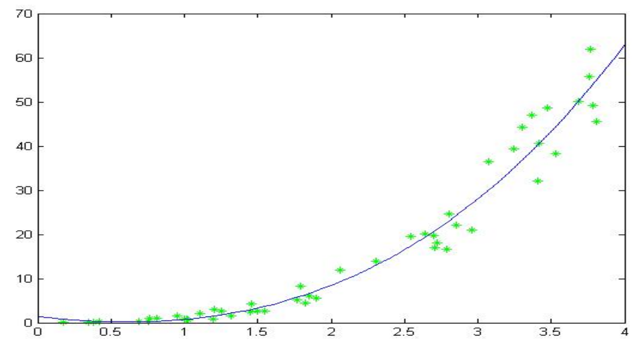
We explored many of these concepts with respect to the Fourier matrix so be familiar with them in that context. In addition, we applied the DFT matrix to numerous signals to find dominant frequencies.

2.1. **Application 1.**



The image on the left is the original image and the one on the left is a reconstruction from a singular value decomposition using 19.16% of the data. See Appendix 9.4 for Matlab code.

2.2. **Application 2.**



We solve the least squares problem to find a polynomial fit to the data points. See Appendix 9.5 for Matlab code.

3. Fourier Transform and the Time-frequency localization

Heisenberg sum. Let $h \in L^2(\mathbb{R})$ and $\|h\| = 1$. If h and $\hat{h}$ are differentiable

$$\|xh(x)\|^2 + \|w\hat{h}(w)\|^2 = \|\frac{d}{dx}\mathcal{F}h(x)\|^2 + \|\frac{d}{dx}h(x)\|^2 = \sigma_0 h \cdot + \sigma_0 \hat{h}$$

Consider the differential operator, $\mathcal{H}(f) = \dfrac{d^2 f}{dx^2} - x^2(f)$, sometimes referred to as the GaussHermite (G-H) differential operator. The eigenfunctions of this differential operator are the Hermite functions with the one corresponding to the smallest eigenvalue being the Gaussian $g(x) = e^{-x^2/2}$ (Note different dimension and Fourier Transforms require different dilations and normalizations). We also point out that since the Fourier transform commutes with this differential operator the eigenvectors are also eigenvectors of the Fourier Transform.

We say that a function has good time-frequency localization if the Heisenberg sum is $\|xh(x)\|^2 + \|w\hat{h}(w)\|^2$ small. The minimizer of the Heisenberg sum above over normalized functions is exactly this Gaussian. In other words in this sense, the Gaussian has "the best" time-frequency localization. Some functions are so poorly time-frequency localized that the sum is infinite. Can you come up with any?

4. Discrete Fourier Transform and Time-frequency localization

We develop a discrete version of the Heisenberg sum. Let $h \in L^2(\mathbb{R})$ and $\|h\| = 1$. If h and $\hat{h}$ are differentiable

$$\|xh(x)\|^2 + \|w\hat{h}(w)\|^2 = \|\frac{d}{dx}\mathcal{F}h(x)\|^2 + \|\frac{d}{dx}h(x)\|^2 = \sigma_0 h \cdot + \sigma_0 \hat{h}$$

We begin by replacing the derivative with the matrix D, the finite difference operator, and denote $\Delta = D^* D$

$$D = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & 1 & -1 \\ -1 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}, \Delta = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix},$$

We observe that $D = I - T$, $T^{-1} = T^*$ and $M^{-1} = M^*$. So if one defines $X^2 = F^*\Delta F$ we can apply $TF = FM$ to get

$$F^*\Delta F = F^*(I-T)^*(I-T)F = F^*(-T^* + 2I - T)F = -M + 2I - M^*$$

Since I and M are both diagonal matrices, so is X^2 with $(j+1)^{th}$ diagonal element

$$-e^{2\pi ij/d^2} + 2 - e^{-2\pi ij/d^2} = -2\cos(2\pi j/d^2) + 2 = 4\sin^2(\pi j/d^2).$$

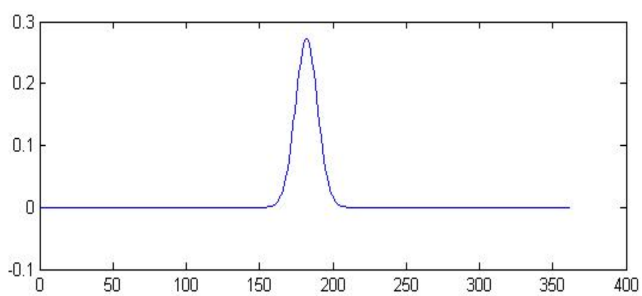
Finally we observe that since $\Delta F = FX^2$ the columns of the Fourier matrix are the eigenvectors of Δ with the diagonal elements of X^2 as the corresponding eigenvalues. Now that we have established a "derivative" let us give an equivalent derivative that helps us fill in the gaps in generalizing form the continuous case to the discrete case. Since X^2 is a diagonal matrix with all positive entries we will let X be the natural diagonal matrix obtained by simply taking the positive square root of these entries. Hence, we define $D1 = FXF^*$ as our symmetric derivative and we recover the discrete form of the continuous properties 5)$\mathcal{F}(f'(x))(w) = iw\hat{f}(w)$ and 6)$\mathcal{F}(xf(x))(w) = i\frac{d}{dw}\hat{f}(w)$ form above. Namely:

$$1) FX = D1F, \qquad\qquad 2) D1F = FX$$

We propose the following norm as the replacement:$d^2(\|Dg\|^2 + \|DFg\|^2)$. To see why, let g be a function in $\ell^2_{\mathbb{Z}_{d^2}}$ we let h be a function on $\mathbb{R}$ to $\mathbb{C}$ defined by $h(n/d) = \sqrt{d}g(n)$ for $n = -\lfloor d^2/2 \rfloor, \cdots \lfloor (d^2-1)/2 \rfloor$. It is easy to show $\|g\|_{\ell^2_{\mathbb{Z}_{d^2}}} \approx \|h\|_{L^2(\mathbb{R})}$. Now using the fact that $\sin(x) \approx x$ for small x we get :

$$
\begin{aligned}
d^2(\|Dg\|^2+\|DFg\|^2) &= 4d^2(\sum_{j=0}^{d^2-1}\sin^2(\pi j/d^2)|(Fg)(j)|^2+\sum_{j=0}^{d^2-1}\sin^2(\pi j/d^2)|g(j)|^2) \\
&\approx 4d^2(\sum_{j=-\lfloor d^2/2\rfloor}^{\lfloor (d^2-1)/2\rfloor}(\pi j/d^2)^2|\frac{\hat{h}(j/d)}{\sqrt{d}}|^2+\sum_{j-\lfloor d^2/2\rfloor}^{\lfloor (d^2-1)/2\rfloor}(\pi j/d^2)^2|\frac{h(j/d)}{\sqrt{d}}|^2) \\
&= 4d^2\frac{\pi^2}{d^2}(\sum_{j=-\lfloor d^2/2\rfloor}^{\lfloor (d^2-1)/2\rfloor}(j/d)^2|\hat{h}(j/d)|^2\frac{1}{d}+\sum_{j-\lfloor d^2/2\rfloor}^{\lfloor (d^2-1)/2\rfloor}(\pi j/d)^2|h(j/d)|^2\frac{1}{d}) \\
&\approx (4\pi^2)\int_{\mathbb{R}} w^2|\hat{h}(w)|^2\,dw+\int_{\mathbb{R}} x^2|h(x)|^2\,dx \\
&= (4\pi^2)(\|w\hat{h}(w)\|^2+\|xh(x)\|^2)
\end{aligned}
$$

We have produced a finite version of the G-H differential operator that asymptotically seems to agree with the Heisenberg Sum when we apply norms. Hence we can consider the eigenvectors of this positive definite linear operator $(\Delta + F^*\Delta F)$ and refer to them as the finite Hermite functions. As usual, we label the eigenvector with the smallest eigenvalue as the Gaussian and provide an image below when $d = 19$.



5. Time-Frequency Representations: Continuous and Discrete

5.1. **Continuous.** It is easy to show that $e^{2\pi imx}$ is an orthonormal basis for $L^2[0,1]$. One may translate this basis around to get an orthonormal basis for $L^2(\mathbb{R})$, i.e. $e_{m,n} = e^{2\pi imx}\mathbf{1}_{[n,n+1)}$ is an orthonormal basis of $L^2(\mathbb{R})$. In other words one may create an orthonormal basis in $L^2(R)$ by applying time shifts (from $\mathbf{1}_{[0,1)}$ to $\mathbf{1}_{[n,n+1)}$) and frequency shifts (from $\mathbf{1}_{[0,1)}$ to $e^{2\pi imx}\mathbf{1}_{[0,1)}$) to the single function $\mathbf{1}_{[0,1)}$. A fundamental tool used in analyzing sequences created by time-frequency shifts of a single function is the Zak transform:

$$\mathbf{Z}(f(x))\,(t,v) = \sum_{n\in\mathbb{Z}} f(t-n)e^{-2\pi inv}.$$

This is a **linear** map from $L^2(\mathbb{R})$ to $L^2([0,1]\times[0,1])$.

a) Show another representation of the Zak transform is

$$\mathbf{Z}(f(x))\,(t,v) = \sum_{m,n\in\mathbb{Z}} \langle f, e_{m,n}\rangle e^{2\pi imt} e^{2\pi inv}.$$

Why is Z a unitary map?

b) For $k\in\mathbb{Z}$ show

$$\mathbf{Z}(f(x-k))(t,v) = e^{2\pi ikv}\mathbf{Z}(f(x))(t,v)$$

and

$$\mathbf{Z}(e^{2\pi ikx}f(x))(t,v) = e^{2\pi ikt}\mathbf{Z}(f(x))(t,v).$$

c) For fixed f,g in $L^2(\mathbb{R})$ define time-frequency convolution

$$f\odot g(x) = \sum_{m,n}\langle f, e_{m,n}\rangle e^{2\pi imx} g(x-n).$$

Use **a), b)** to show

$$\mathbf{Z}(f\odot g(x))(t,v) = \mathbf{Z}(f(x)(t,v))\mathbf{Z}(g(x))(t,v)$$

d) What can you say about $\mathbf{Z}(g)(t,v)$ if $\{T^nM^mg\}_{m,n\in\mathbb{Z}} = \{g(x-n)e^{2\pi imx}\}_{m,n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{R})$?

Theorem 5.1. *(Balian-Low) If $\{T^nM^mg\}_{m,n\in\mathbb{Z}} = \{g(x-n)e^{2\pi imx}\}_{m,n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{R})$ then $\|xg(x)\|^2 + \|w\hat{g}(w)\|^2 = \infty$.*

Example 5.2. *By above we know $e_{m,n} = e^{2\pi imx}\mathbf{1}_{[n,n+1)}$ is an orthonormal basis for $L^2(\mathbb{R})$, so in this case $g(x) = \mathbf{1}_{[0,1)}$. Compute $\|xg(x)\|^2 + \|w\hat{g}(w)\|^2$.*

Definition 5.3. *For given parameters a,b we define the* **Gabor System** *as $\{T^{na}M^{mb}g\}_{m,n\in\mathbb{Z}} = \{g(x-na)e^{2\pi imbx}\}_{m,n\in\mathbb{Z}}$*

Remark 5.4. *Let us put two things together. If we want good time frequency-localization and something close to an orthonormal basis perhaps for $g(x) = e^{-\pi x^2}$ and some a and b the sequence $\{T^{na}M^{mb}g\}_{m,n\in\mathbb{Z}} = \{g(x-na)e^{2\pi imbx}\}_{m,n\in\mathbb{Z}}$ is a basis?*

5.2. **Discrete.** We examine creating o.n. bases with time-frequency shifts of a vector in $\mathbb{C}^{d^2}$. We chose a square dimension so that we can apply a finite Zak transform to the window. Let $f \in \mathbb{C}^{d^2}$ be periodic in that $f(d^2+j) = f(j)$ so that $\mathbb{C}^{d^2} \cong \ell^2(\mathbb{Z}_{d^2})$. We define the Zak transform as the unitary operator $Z : \ell^2(\mathbb{Z}_{d^2}) \to \ell^2(\mathbb{Z}_{d^2})$,

$$(Zf)(md+n) = \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}_d} f(m+kd) e^{\frac{-2\pi ikn}{d}}$$

and its inverse Z^*

$$(Z^*F)(md+n) = \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}_d} F(nd+k) e^{\frac{2\pi ikm}{d}}$$

Remark 5.5. *Matlab code for Zak matrix Z*

```
F2=fft(eye(d))/sqrt(d);
for j = 0:d^2-1;
Z(j+1,n+1+floor(j/d)) = F2(mod(j,d)+1,:); %Zak transform
end;
```

Remark 5.6. *Alternatively, the finite Zak transform can be described as the synthesis operator for the canonical Gabor system, i.e., the Gabor system with "block" window* $\mathbf{b}$ *so that*

$$\mathbf{b}(i) = \begin{cases} \frac{1}{\sqrt{d}} & \text{if } 0 \le i < d \\ 0 & \text{else} \end{cases}$$

Let M, N, T be operators defined on $\ell^2(\mathbb{Z}_{d^2})$ by :

$(T^m f)(n) = f(n-m)$ and $(M^m f)(n) = e^{2\pi imn/d^2} f(n)$ for $m, n \in \mathbb{Z}_{d^2}$.

$(N^{jd} f)(md+n) = e^{2\pi ijm/d} f(md+n)$ for $m, n \in \mathbb{Z}_d$

We consider the special case of the d^{th} power of M and N. Here $(M^d f)(n) = e^{2\pi in/d} f(n)$ for $n \in \mathbb{Z}_{d^2}$. and $(N^d f)(n) = e^{2\pi i\lfloor n/d \rfloor/d} f(n)$ for $n \in \mathbb{Z}_{d^2}$.

Now we are ready to show the finite Zak transform changes Translation to Modulation and sends Modulation to a Modulation as in the continuous case.

Proposition 5.7. *a)* $ZT^{jd} = M^{jd} Z$

b) $ZM^{jd} = N^{jd} Z$

Proof. a)

$$\begin{aligned}\sqrt{d}(ZT^{jd}f)(md+n) &= \sum_{k\in\mathbb{Z}_d} f(m+jd+kd)e^{\frac{-2\pi ikn}{d}} \\ &= \sum_{k\in\mathbb{Z}_d} f(m+(k+j)d)e^{\frac{-2\pi ikn}{d}} \\ &= \sum_{\ell\in\mathbb{Z}_d} f(m+\ell d)e^{\frac{-2\pi i(\ell-j)n}{d}} \\ &= e^{\frac{2\pi ijn}{d}} \sum_{\ell\in\mathbb{Z}_d} f(m+\ell d)e^{\frac{-2\pi i\ell n}{d}} = \sqrt{d}(M^{jd}Zf)(md+n)\end{aligned}$$

b) First we note that $(M^{jd}f)(n) = e^{2\pi ijn/d}f(n)$

$$\begin{aligned}\sqrt{d}(ZM^{jd}f)(md+n) &= \sum_{k\in\mathbb{Z}_d} e^{2\pi ij(m+kd)/d}f(m+kd)e^{\frac{-2\pi ikn}{d}} \\ &= \sum_{k\in\mathbb{Z}_d} e^{2\pi ijm/d}f(m+kd)e^{\frac{-2\pi ikn}{d}} \\ &= e^{\frac{2\pi ijm}{d}} \sum_{k\in\mathbb{Z}_d} f(m+kd)e^{\frac{-2\pi ikn}{d}} = \sqrt{d}(N^{jd}Zf)(md+n)\end{aligned}$$

Alternatively we write $ZT^dZ^* = M^d$ and $ZM^dZ^* = N^d$.

□

Now let us consider the **Finite Gabor System** generated with d translations with d modulations, that is, given $g \in \ell(\mathbb{Z}_{d^2})$ we have the $d^2 \times d^2$ system/matrix $\{T^{nd}M^{md}g\}_{m,n\in\mathbb{Z}_d}$. If we consider this in matrix form we get

$$G = [g\ (T^dg)\dots(T^{N-d}g)\ (M^dg)\,(T^dM^dg)\dots(M^{N-d}g)\,(T^dM^{N-d}g)\cdots(T^{N-d}M^{N-d}g)]$$

In some sense we have seen this before but either only Translates or only Modulations. For example the Identity matrix can be described as a translates of the delta function δ_0 and the Fourier matrix can be described as Modulations of the normalized constant function $\frac{1}{\sqrt{N}}\mathbf{1}$. We now give the Time-Frequency version of Theorem 1.4.

Theorem 5.8. *Let $g \in \ell(\mathbb{Z}_{d^2})$, $\|g\| = 1$ then $G = \{T^{nd}M^{md}g\}_{m,n\in\mathbb{Z}_d}$, is an orthonormal system iff $|(Zg)(n)| = \frac{1}{d}$ for all n*

Proof. Applying the Zak Transform to $\{T^{nd}M^{md}g\}_{m,n\in\mathbb{Z}_d}$ gives $\{M^nN^mZ(g)\}_{m,n\in\mathbb{Z}_d}$. M and N are both diagonal matrices, i.e. multiplication operators, and is easy to show that for

$|H(n)| = \frac{1}{d}$ the system $\{M^n N^m H)\}_{m,n \in \mathbb{Z}_d}$ is an orthonormal basis. The rest follows from the fact that the Zak Transform is a Unitary operator.

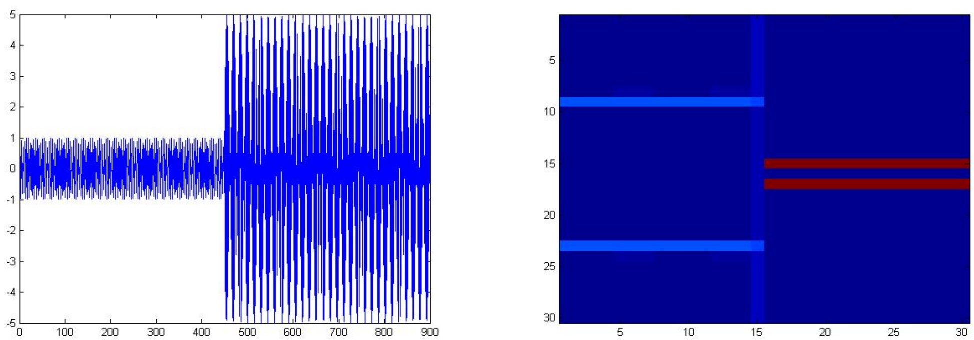
□

One last observation we include here without proof regarding the interplay between the finite Zak transform and Gabor systems. Is we define the Gabor A_g matrix for a window g to be the matrix whose rows are $\{T^{nd} M^{md} g\}_{m,n \in \mathbb{Z}_d}$ then for any vectors g, h, f we have

$$Z(A_g A_h^*(f)) = d^2 Z(g) \cdot \overline{Z(h)} \cdot Z(f)$$

where the multiplication on the right hand side of the equation is point wise in each component of the vector.

5.3. **Application.**



We give a time frequency representation of a signal of two distinct tones using the inverse Zak transform. See Appendix 9.6 for Matlab code.

6. Frames

Formal definition. For any of the vector spaces $\mathbb{X}$ defined in Section 1, i.e. $\mathbb{C}^n$, ℓ_2, we can define a **frame** as a sequence of vectors or functions $\{\mathbf{g_n}\} \subset \mathbb{X}$ so there exists $0 < A \leq B$ satisfying the inequality

$$A\|\mathbf{f}\|^2 \leq \sum_n |\langle \mathbf{f}, \mathbf{g}_n \rangle|^2 \leq B\|\mathbf{f}\|^2 \text{ for all } \mathbf{f} \in \mathbb{X} \tag{1}$$

We have seen a version of this before and we know tons of examples. Unlike in previous sections we will start with the finite cases and extend to the infinite ones.

Example 6.1. *If A=B=1 the inequality above looks like Parsevals identity. Therefore, any orthonormal basis is a frame.*

Example 6.2. *Any basis in $\mathbb{C}^d$ is also a frame. Why? What are the eigenvalues of GG^*? What does this have to do with the inequality above. Hint if $G = [\mathbf{g}_1\ \mathbf{g}_2 \cdots \mathbf{g}_d]$ then $\|G^*\mathbf{f}\|^2 =$*

$\sum_n |\langle \mathbf{f}, \mathbf{g}_n \rangle|^2$. *But we also know* $\|G^*\mathbf{f}\|^2 = \langle G^*\mathbf{f}, G^*\mathbf{f}\rangle = \langle GG^*\mathbf{f}, \mathbf{f}\rangle$ *and since the matrix* G *is invertible we know the eigenvalues of* GG^* *are* > 0 *and hence*

$$\lambda_1\|\mathbf{f}\|^2 \leq \|G^*\mathbf{f}\|^2 \leq \lambda_d\|f\|^2$$

Another way to describe the frame inequality (2) is to say that A, B *are the smallest and largest eigenvalues of* GG^*

In finite dimensions if a frame has a finite number of elements it is simply a spanning set. In matrix form this turns out to be a non square matrix that is full rank.

Example 6.3. *For* $\mathbb{R}^2$ $\{\mathbf{g_n}\}$ $=\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}\frac{-1}{2}\\\frac{\sqrt{3}}{2}\end{bmatrix}, \begin{bmatrix}\frac{-1}{2}\\\frac{-\sqrt{3}}{2}\end{bmatrix}\}$ *or in matrix form* $G = \begin{bmatrix}1 & \frac{-1}{2} & \frac{-1}{2}\\ 0 & \frac{\sqrt{3}}{2} & \frac{-\sqrt{3}}{2}\end{bmatrix}$

We compte the eigenvalues of $GG^* = \begin{bmatrix}\frac{3}{2} & 0\\ 0 & \frac{3}{2}\end{bmatrix}$ *and it is easy to see that* $A = B = \dfrac{3}{2}$

What is an example of a frame in finite dimensions with an infinite number of elements?

Example 6.4. *Consider the set in* $\mathbb{R}^2$ $\{\begin{bmatrix}1\\0\end{bmatrix}\} \bigcup \{\begin{bmatrix}0\\\frac{1}{n}\end{bmatrix}\}_{n=1}^{\infty}$ *Claim* $A = 1$ *and* $B = \dfrac{\pi^2}{6}$ *will work in inequality (2)*

Definition 6.5. *A* ***dual frame*** *to the frame* $\{\mathbf{g_n}\}\mathbf{n} = \mathbf{1^N} \subset \mathbb{X}$ *is any set of vectors* $\{\mathbf{h_n}\}_{\mathbf{n=1}}^{\mathbf{N}} \subset \mathbb{X}$ *so that for all* $x \in \mathbb{X}$ *we have*

$$\mathbf{x} = \sum_{n=1}^{N}\langle \mathbf{x}, \mathbf{h_n}\rangle \mathbf{g_n} = \sum_{n=1}^{N}\langle \mathbf{x}, \mathbf{g_n}\rangle \mathbf{h_n}.$$

In matrix form a frame G *can be represented as the columns of a matrix*

$$G = [\mathbf{g_1}\ \mathbf{g_2}\ \cdots \mathbf{g_N}]$$

and a dual is a any matrix

$$H = [\mathbf{h_1}\ \mathbf{h_2}\ \cdots \mathbf{h_N}]$$

so that $GH^* = HG^* = I$. *If* $dim(\mathbb{X}) = d$ *what can you say about* N *compared to* d*?* $\geq, =\leq \ldots$*?*

IMPORTANT Difference between frames and bases is that the duals of a frame do not have to be unique.

Exercise: Find three distinct duals to the frame $\{\mathbf{g_n}\}$ $=\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}\frac{-1}{2}\\\frac{\sqrt{3}}{2}\end{bmatrix}, \begin{bmatrix}\frac{-1}{2}\\\frac{-\sqrt{3}}{2}\end{bmatrix}\}$.

Definition 6.6. *Let the Frobenius norm of a matrix, even non square, be defined to be* $\|A\|_{fr} = \sqrt{\sum\sum |a_{i,j}|^2}$

Definition 6.7. *We define the canonical dual of a frame n*

$$G = [\mathbf{g_1}\ \mathbf{g_2}\ \cdots \mathbf{g_N}]$$

to be the columns of the matrix $G_0 = (GG^*)^{-1}G$. Can you show this is a dual frame?

Definition 6.8. *The trace of a square matrix A is the the sum of the diagonal element, or* $Tr(A) = \sum_{i=1}^{N} a_{ii}$

Proposition 6.9. *If A and and B are both square $N \times N$ matrices*

1) $Tr(A) = Tr(A^t)$

2) $Tr(AB) = Tr(BA)$

3) $Tr(BAB^{-1}) = Tr(A)$

4) $Tr(A + B) = Tr(A) + Tr(B)$

If A and and B are both non square $M \times N$ matrices

5) $\langle A, B\rangle = Tr(B^*A) = Tr(A^*B)$ *is an inner product for non square matrices.*

6) $0 \le \langle A, A\rangle = \|A\|_{fr}^2$ *is a norm for the set of non square matrices.*

Theorem 6.10. *The canonical dual of a finite frame G of $\mathbb{C}^n$ is the dual that minimizes the Frobenius norm.*

Proof. Recall $G_0 = (GG^*)^{-1}G$ and the fact that if H is a dual of G then so is any $H + K$ where $GK^* = KG^* = 0$ (Why?). Consider any dual of the form $G_0 + K$. First show that $G_0K^* = 0$ and compute the $\langle H + K, H + K\rangle$ using the properties above. □

Definition 6.11. *A frame $\{\mathbf{g_n}\} \subset \mathbb{X}$ is called a* ***tight frame***

$$\sum_n |\langle \mathbf{f}, \mathbf{g}_n\rangle|^2 = B\|\mathbf{f}\|^2 \text{ for all } \mathbf{f} \in \mathbb{X} \tag{2}$$

that is $A = B \neq 0$ and if $A = B = 1$ it is called a **Parseval frame**.

Example 6.12. *What do we need to do to turn this into a Parseval frame?* $\{\mathbf{g_n}\} = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}\frac{-1}{2}\\ \frac{\sqrt{3}}{2}\end{bmatrix}, \begin{bmatrix}\frac{-1}{2}\\ \frac{-\sqrt{3}}{2}\end{bmatrix}\}$.

Theorem 6.13. *If the columns of a $d \times N$ matrix G form a frame then the columns of $(GG^*)^{-1/2}G$ form a Parseval frame.*

Proof. See homework 3. □

It is common to use the derivative to create a new norm in for function on the real line namely for $f \in L^2(R)$ consider the space of functions so that $\|\frac{d^k f}{dx^k}\| < \infty$. There are a number of variations on the norm such as $\sum_{n=0}^{k} \|f^n\|$ and $\sqrt{\sum_{n=0}^{\infty}(1+n^2)|\hat{f}(n)|^2}$. Since not all functions in $L^2(\mathbb{R})$ have derivatives (Example?) one needs to massage these definitions by saying the the derivatives exist in some weak sense. We already have a discreet version of the derivative but we are going to tweak it a bit to make it invertible(Why do we want it invertible?), hence we introduce

$$D = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

With this new "Discrete Derivative" we define a norm on non square $d \times N$ matrices $A_{d\times N}$ by

$$\|A\|_k = \|D^k_{N\times N} A^*_{N\times d}\|_{fr}$$

Definition 6.14. *For a given frame matrix E for $\mathbb{R}^d$ we define the k^{th} Sobolev dual to be*

$$F = (E\nabla^{-2k}E^*)^{-1}E\nabla^{-2k}$$

where $\nabla^{2k} = D^{*k}D^k$.

Theorem 6.15. *Given a frame E the k^{th} Sobolev dual F is the minimizer of both and $\|F\|_k$ over all dual frames of E.*

Proof. The argument for the kth case is nearly identical to k=1 so for simplicity we prove the result for $k = 1$. Observe that F is a dual frame to E iff FD^* is a dual frame to ED^{-1}, that is $FD^*D^{*-1}E^* = FE^* = I$. By Lemma 6.10, if FD^* is the canonical dual of ED^{-1} then it is the minimizer over the Frobenius norm. So let

$$FD^* = (ED^{-1}D^{*-1}E^*)^{-1}ED^{-1}.$$

Multiplying both sides of the equation on the right by the inverse of D^* and simplifying the notation gives us that F is the Sobolev dual

□

7. Critical Sampling v.s. Over Sampling: Bases v.s. Frames

Definition 7.1. *A function $f \in L^2(\mathbb{R})$ is called* **band-limited** *if $\hat{f}(w)$ vanishes outside a certain interval (i.e. band). Alternatively we say $\hat{f}(w) = 0$ for $|w| > W$ and refer to W as the* **band width**.

Exercise: Give an example of examples of a function with bandwidth 3?

Theorem 7.2. Sampling Theorem *Suppose f is a bandlimited function with bandwidth with W then*

$$f(x) = \sum_{n\in\mathbb{Z}} f(\frac{n}{2W})\frac{\sin(\pi(2Wx-n))}{\pi(2Wx-n)}$$

Proof. Since $\hat{f}(w) = 0$ if $w > |W|$, $f(x) = \int_{-W}^{W} \hat{f}(w)e^{2\pi iwx}dw$ and $\hat{f}$ is compactly supported so $\hat{f}(w) = \sum c_n e^{\pi inw/W} = \sum c_{-n}e^{-\pi inw/W}$ with $c_{-n} = \frac{1}{2W}\int_{-W}^{W}\hat{f}(w)e^{\pi iwn/W}dw = \frac{1}{2W}f(\frac{n}{2W})$. Putting these together yields

$$\begin{aligned}
f(x) &= \int_{-W}^{W}\sum_{n\in\mathbb{Z}}\frac{1}{2W}f(\frac{n}{2W})e^{-\pi inw/W}e^{2\pi iwx}dw \\
&= \sum_{n\in\mathbb{Z}}\frac{1}{2W}f(\frac{n}{2W})\int_{-W}^{W}e^{\pi iw(2Wx-n)/W}dw \\
&= \sum_{n\in\mathbb{Z}} f(\frac{n}{2W})\frac{1}{2\not{W}}\frac{e^{\pi iw(2Wx-n)/W}}{(\pi i(2Wx-n)/\not{W})}\Bigg|_{-W}^{W} \\
&= \sum_{n\in\mathbb{Z}} f(\frac{n}{2W})\frac{1}{\pi(2Wx-n)}\frac{e^{\pi i(2Wx-n)} - e^{-\pi i(2Wx-n)}}{(2i)} \\
&= \sum_{n\in\mathbb{Z}} f(\frac{n}{2W})\frac{\sin(\pi(2Wx-n))}{\pi(2Wx-n)}
\end{aligned}$$

Note if you define bandlimited with the Fourier transform given in Maple $\hat{f}1(w) = \int_{-infty}^{\infty} f(x)e^{-iwx}dx$ instead of the one used for these notes the sampling theorem becomes

$$f(x) = \sum_{n\in\mathbb{Z}} f(\frac{n\pi}{W})\frac{\sin(Wx-n\pi)}{(Wx-n\pi)}$$

□

7.1. **Taking advantage of Over Sampling: Quantization.** Given an expansion of the form $f(x) = \sum_n x_n e_n$, find an expression $\tilde{f}(x) = \sum_n q_n e_n$ so $\|f-\tilde{f}\|$ is *small* **and** q_n come from **a finite alphabet**. For example, $q_n \in \{-1, 1\}$.

Example 7.3. *For* **bandlimited functions** $f(x)$ *with* $|\hat{f}(w)|, |f(x)| < 1$ *and representation* $f(x) = \frac{1}{\lambda}\sum_n f(\frac{n}{\lambda})g(x - \frac{n}{\lambda})$,

we would like representations *of the form*

$$\tilde{f}(x) = \frac{1}{\lambda}\sum_n q_n g(x - \frac{n}{\lambda}) \text{ with } q_n \in \{-1, 1\}$$

$$\textit{\textbf{AND}}\ |f(t) - \tilde{f}(t)| \textit{ is } \text{small.}$$

Sigma Delta($\Sigma\Delta$ For Finite Frames) scheme.

First order $\Sigma\Delta$We introduce a state variable u_n and let $0 = u_0$ then we find inductively $u_n = x_n - q_n + u_{n-1}$ and $q_n = Q_\delta(x_n + u_{n-1})$ or

$$\Delta u_n = u_n - u_{n-1} = x_n - q_n$$

and

$$\widetilde{x} = \sum_{n=1}^{N} x_n \mathbf{f}_n$$

Higher order $\Sigma\Delta$ For an k^{th} order scheme $\Delta^k u_n = x_n - q_n$

$$\|x - \tilde{x}\|_2 = \|\sum_{n=1}^{N-k} u_n \Delta^k f_n + \sum_{j=1}^{k} u_{N-j+1}\Delta^{k-1} f_{N-j+1}||$$

Our next example will be for a vector in $\mathbb{R}^2$ jus to make sure the notation is clear.

Example 7.4. *(From Anna Maser's thesis)*
We will begin with an original signal, $\vec{x}$ *and a sampling frame,* E. *We will show how to obtain an error magnitude between the original signal,* $\vec{x}$, *and the reconstructed signal,* $\widetilde{x}$, *through the use of a first-order quantization scheme.*

Let $\vec{x} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{4} \end{bmatrix}$ *be an original signal and* E *be the sampling frame matrix such that the rows of* E *correspond to* e_n *and*

$$E = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}. \tag{3}$$

We apply E *to* $\vec{x}$ *to achieve an oversampled representation of* $\vec{x}$*, a vector originally in* $\mathbb{R}^2$ *and now in* $\mathbb{R}^6$*. The result follows:*

$$E\vec{x} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}. \tag{4}$$

We then apply the first order quantization scheme defined above to send each of our x_n*'s to a member of our quantization alphabet,* $\{-1, 1\}$*, while keeping track of the error between each*

x_n and q_n as we proceed.

$$
\begin{aligned}
u_0 &= 0 \\
q_1 &= Q(u_0 + x_1) = Q(0 + \frac{1}{3}) = 1 \\
u_1 &= u_0 + x_1 - q_1 = 0 + \frac{1}{3} - 1 = -\frac{2}{3} \\
q_2 &= Q(u_1 + x_2) = Q(-\frac{1}{3}) = -1 \\
u_2 &= u_1 + x_2 - q_2 = -\frac{1}{3} + 1 = \frac{2}{3} \\
q_3 &= Q(u_2 + x_3) = 1 \\
u_3 &= u_2 + x_3 - q_3 = 1 - 1 = 0 \\
q_4 &= Q(u_3 + x_4) = Q(-\frac{1}{4}) = -1 \\
u_4 &= u_3 + x_4 - q_4 = \frac{3}{4} \\
q_5 &= Q(u_4 + x_5) = Q(\frac{1}{2}) = 1 \\
u_5 &= u_4 + x_5 - q_5 = -\frac{1}{2} \\
q_6 &= Q(u_5 + x_6) = Q(-\frac{3}{4}) = -1 \\
u_6 &= u_5 + x_6 - q_6 = \frac{1}{4}.
\end{aligned} \tag{5}
$$

Now, the quantized vector consisting of the q_n's, or quantized coefficients, from above is

$$\vec{q} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

and the vector of state variables is

$$\vec{\mathbf{u}} = \begin{bmatrix} 0 \\ -\frac{2}{3} \\ \frac{2}{3} \\ 0 \\ \frac{3}{4} \\ -\frac{1}{2} \\ \frac{1}{3} \end{bmatrix}.$$

Stability is the task of keeping the state variables bounded. Notice that stability conditions are met in this example because the state vector, $\vec{\mathbf{u}}$*, is bounded above by* 1 *and below by* -1*. We will now reconstruct the signal. This can be done using any dual frame matrix. In this case, we will use the canonical dual .*

To find the canonical dual, we use the property $E_c^*E = I_d$*.*

(6) $$E_c = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{3} & 0 \\ \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \\ 0 & \frac{1}{3} \\ 0 & \frac{1}{3} \end{bmatrix}.$$

We then calculate E_c^*E *using* E_c *from (6) and our sampling frame matrix,* E*, from (3).*

(7) $$E_c^*E = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Applying E_c^* *to* $\vec{\mathbf{q}}$*, we get*

(8) $$E_c^*\vec{\mathbf{q}} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

(9) $$= \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} = \widetilde{\mathbf{x}}.$$

Then, we calculate the magnitude of the error by

$$\begin{aligned} \|\mathbf{x} - \widetilde{\mathbf{x}}\| &= \left\| \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{4} \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} \right\|. \\ &= \left\| \begin{bmatrix} 0 \\ \frac{1}{12} \end{bmatrix} \right\| \\ &= \sqrt{0^2 + \left(\frac{1}{12}\right)^2} = \frac{1}{12}. \end{aligned} \tag{10}$$

Let's revisit the error between $\mathbf{x}$ and $\tilde{\mathbf{x}}$. Given any dual frame $F = \{\mathbf{f_n}\}$ we get

$$\begin{aligned} \mathbf{x} - \widetilde{\mathbf{x}} &= \sum_{n=1}^{N} x_n \mathbf{f}_n - \sum_{n=1}^{N} q_n \mathbf{f}_n \\ &= \sum_{n=1}^{N} (x_n - q_n)\mathbf{f}_n \\ &= \sum_{n=1}^{N} (u_n - u_{n-1})\mathbf{f}_n \\ &= \sum_{n=1}^{N} (\Delta u_n)\mathbf{f}_n \\ &= \sum_{n=1}^{N} u_n(\Delta \mathbf{f}_n) + u_N \underline{f}_N - u_0 \mathbf{f}_1 \text{ summation by parts} \\ &= \sum_{n=1}^{N} u_n(\Delta \underline{f}_n) + u_N \mathbf{f}_N \text{ since we get to choose } u_0 = 0 \end{aligned}$$

This should look familiar. We can stat this as the matrix equation

$$\mathbf{x} - \tilde{\mathbf{x}} = FD^*\mathbf{u}$$

where F is any dual frame to E, D is the invertible finite difference operator defined in 6 and $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}$ Now the beauty of frames is that we get to choose any dual frame that suits us. So instead of choosing the canonical dual to reconstruct, how about we chose the Sobolev dual $F = (E\nabla^{-2}E^*)^{-1}E\nabla^{-2}$

$$F = \left(\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0.0560 & -0.0720 \\ -0.0720 & 0.1640 \end{bmatrix} \begin{bmatrix} 15 & 14 & 12 & 9 & 6 & 3 \\ 6 & 6 & 6 & 6 & 5 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 0.4080 & 0.3520 & 0.2400 & 0.0720 & -0.0240 & -0.0480 \\ -0.0960 & -0.0240 & 0.1200 & 0.3360 & 0.3880 & 0.2760 \end{bmatrix}$$

Why are we guessing this may be a better choice? Well

$$\|\mathbf{x} - \tilde{\mathbf{x}}\| = \|FD^*\mathbf{u}\| \leq \|FD^*\|\|\mathbf{u}\|$$

here $\|FD^*\|$ represents the operator norm of the matrix, (see Definition 2.10 and we have already seen in Theorem 6.15 that this dual minimizes the Frobenious norm of this matrix. It turns out that the Sobolev dual also minimizes the operator norm. Let see what happens in the example above.

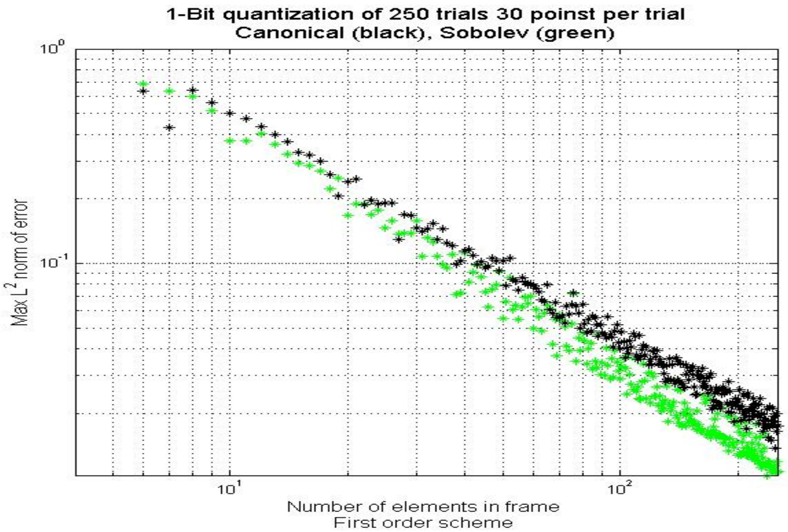
$$\widetilde{\mathbf{x_2}} = \begin{bmatrix} 0.4080 & 0.3520 & 0.2400 & 0.0720 & -0.0240 & -0.0480 \\ -0.0960 & -0.0240 & 0.1200 & 0.3360 & 0.3880 & 0.2760 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 0.2480 \\ -0.1760 \end{bmatrix}.$$

Computing the error gives.

$$\|\mathbf{x} - \widetilde{\mathbf{x_2}}\| = \left\| \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{4} \end{bmatrix} - \begin{bmatrix} 0.2480 \\ -0.1760 \end{bmatrix} \right\| = 0.1130 > \frac{1}{12}.$$

It seems to be worse instead of better. However lets try a few more and let Matlab do the heavy lifting.

7.2. **Applications.**



Math 592/475 Homework 1

Directions: NEATLY write all solutions on your own paper. Solutions should include details.

1) Show that every symmetric matrix $A = A^T$ can be written as $A = UDU^T$ where D is a diagonal matrix and $UU^T = I$. We shall proceed by induction.

i) Show for n=1 the statement is true. (Yes it is so easy it seems hard)

ii) Assume true for $n-1$. Now let $\mathbf{x}$ be any normalized eigenvector of an $n \times n$ symmetric matrix A with eigenvalue λ , i.e., $A\mathbf{x} = \lambda\mathbf{x}$ and $\|\mathbf{x}\| = 1$. By the Gram-Schmidt process there exist $\{\mathbf{y_1}, \mathbf{y_2} \cdots \mathbf{y_{n-1}}\}$ so that $\{\mathbf{y_1}, \mathbf{y_2} \cdots \mathbf{y_{n-1}}, \mathbf{x}\}$ is an orthonormal basis for $\mathbb{R}^n$.

Use the $n \times (n-1)$ matrix $Y = [\mathbf{y_1}\, \mathbf{y_2} \cdots \mathbf{y_{n-1}}]$ and A to create an $(n-1) \times (n-1)$ symmetric matric (be sure to show it is symmetric) and apply the induction for $(n-1)$ to get a $(n-1) \times (n-1)$ diagonal matrix D and an orthogonal matrix $U = [\mathbf{u_1}\, \mathbf{u_2} \cdots \mathbf{u_{n-1}}]$

iii) Show BAB^T is diagonal and $BB^T = I$ if $B^T = [\mathbf{x}\, Y\mathbf{u}_1\, Y\mathbf{u}_2 .... Y\mathbf{u}_{n-1}]$

2) Let $\{\mathbf{u_1}, \mathbf{u_2} \cdots \mathbf{u_p}\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$. Define the $n \times p$ matrix

$$U = [\mathbf{u_1}\, \mathbf{u_2} \cdots \mathbf{u_p}]$$

and for any $\mathbf{y} \in \mathbb{R}^n$ define $\hat{\mathbf{y}} = UU^T\mathbf{y}$.

i) Show $\hat{\mathbf{y}} - \mathbf{y} \perp W$

ii) Show that $\|\hat{\mathbf{y}} - \mathbf{y}\| \leq \|\mathbf{v} - \mathbf{y}\|$ for all $\mathbf{v} \in \mathbb{W}$

Math 592 Homework 1-2014

Directions: NEATLY write all solutions on your own paper. Solutions should include details.

1) Load the file I emailed into Matlab with
[NUM,Text,Raw]=xlsread('roundhill-Barnes.xls'); clf plot(NUM(:,3))
then perform a regression of your choosing taht fits the data well visually. See appendix 9.5.

2) If a matrix A is diagonalizeable, i.e. $A = PDP^{-1}$ then we define $A^{\alpha} = PD^{\alpha}P^{-1}$. Use this to find a Translation operator that corresponds to translating a vector by 1/2 a unit. Illustrate with Matlab.

3) For a matrix A we define $\|A\| = \sup_{\|x\|=1} \|Ax\|$. Show that $\|A\| = \sigma_N$ where σ_N is the largest singular value of A

Math 475/592 Homework 2

Directions: NEATLY write all solutions on your own paper.

1) Recall that $e^{2\pi imx}$ is an orthonormal basis for $L^2[0,1]$. One may translate this basis around to get an orthonormal basis for $L^2(\mathbb{R})$, i.e. $e_{m,n} = e^{2\pi imx}\mathbf{1}_{[n,n+1)}$ is an orthonormal basis of $L^2(\mathbb{R})$. In other words one may create an orthonormal basis in $L^2(R)$ by applying time shifts (from $\mathbf{1}_{[0,1)}$ to $\mathbf{1}_{[n,n+1)}$) and frequency shifts (from $\mathbf{1}_{[0,1)}$ to $e^{2\pi imx}\mathbf{1}_{[0,1)}$) to the single function $\mathbf{1}_{[0,1)}$. A fundamental tool used in analyzing sequences created by time-frequency shifts of a single function is the Zak transform:

$$\mathbf{Z}(f(x))\,(t,v) = \sum_{n\in\mathbb{Z}} f(t-n)e^{-2\pi inv}.$$

This is a **linear** map from $L^2(\mathbb{R})$ to $L^2([0,1]\times[0,1])$.

a) Show another representation of the Zak transform is

$$\mathbf{Z}(f(x))\,(t,v) = \sum_{m,n\in\mathbb{Z}} \langle f, e_{m,n}\rangle e^{2\pi imt}e^{2\pi inv}.$$

Why is Z a unitary map?

b) For $k \in \mathbb{Z}$ show

$$\mathbf{Z}(f(x-k))(t,v) = e^{2\pi ikv}\mathbf{Z}(f(x))(t,v)$$

and

$$\mathbf{Z}(e^{2\pi ikx}f(x))(t,v) = e^{2\pi ikt}\mathbf{Z}(f(x))(t,v).$$

c) For fixed f, g in $L^2(\mathbb{R})$ define time-frequency convolution

$$f \odot g(x) = \sum_{m,n}\langle f, e_{m,n}\rangle e^{2\pi imx}g(x-n).$$

Use **a), b)** to show

$$\mathbf{Z}(f \odot g(x))(t,v) = \mathbf{Z}(f(x)(t,v))\mathbf{Z}(g(x))(t,v)$$

d) What can you say about $\mathbf{Z}(g)(t,v)$ if $\{T^nM^mg\}_{m,n\in\mathbb{Z}} = \{g(x-n)e^{2\pi imx}\}_{m,n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{R})$?

Math 475/592 Homework 3

Directions: NEATLY write all solutions on your own paper.

1) Suppose $\mathbf{f}$ is an eigenvector of $\Delta + F^*\Delta F$ on $\mathbb{C}^N$ where F is the the discrete Fourier transform and

$$\Delta = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}.$$

a) Why is the associated eigenvalue $\lambda > 0$

b) What are the eigenvalues of F? Why?

c) If the associated eigenvalue λ is unique show that f is also an eigenvector of F.

2) Suppose the columns of a $d \times N$ matrix X form a frame for $\mathbb{R}^d$. Let $S = XX^*$ with bounds $0 < A \leq B$.

a) Show that $BS^{-1} = \sum_{n=0}^{\infty}(I - \frac{S}{B})^n$. Hint from Calculus II you know the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ when $|x| < 1$.

b) Show that $(S)^{-1/2}X$ is a Parseval frame, where $S^{-1/2}$ is defined to be the symmetric matrix so that $S^{-1/2}S^{-1/2} = S^{-1}$. How do you know it exists?

9. Application Appendix

9.1. **DFT.** Hearing and "seeing" constant frequencies with the DFT.

Note: Latex and the Matlab editor interpret the symbol ' differently so if you copy and paste this code into a Matlab editor you may have to do some editing

```
N=2^10;
n=[1:N];
freq=2*pi/N;
f=1000;
%f=100*rand(1);
jj=[1:N]; % creating two tone signal
kk=[floor(N/2):N];
x=sin(400*jj/pi);
x(kk)=5*cos(50*kk/pi);
%ra=10*rand(1,N)'; %noise
%x=2*sin(n*f*freq)';
%x=2*sin(n*f*freq)'+ra; %noise+signal
%x=x-mean(x);% subtract mean
clf
figure(1)
plot(x)
figure(2)
y=fft(x);
plot(abs(y))
wavplay(x,44000)
figure(1)
```

9.2. **Vector convolution.** function vconv.m

```
function y = vconv(x,y)
%%%vector conculution
A=fft(x);
B=fft(y);
N=length(x);
y =ifft(A.*B)/sqrt(N)    ;
```

9.3. **Using convolution to filter noise.** .

Requirements:
1)noisy signal. 'noisythink1.wav' in the current code.
2) vconv.m function must be in same folder.

```
clf
clear all
```

```
y = wavread('noisythink1.wav');
wavplay(y,44000)
N=length(y);
f1=ones(1,floor(N/8));
Freqfilt=[f1 zeros(1,N-2*length(f1)) f1 ]';
%Freqfilt=[f1 zeros(1,N-length(f1)) ]';
filt=real(ifft(Freqfilt)/norm(Freqfilt));
filtered=N*real(vconv(y,filt));
filtered=filtered/max(filtered);
wavplay(filtered,44000)
figure(1)
plot(abs(fft(y)))
figure(2)
plot(abs(fft(filtered)),'k')
axis([0 N 0 max(abs(fft(y)))])
```

9.4. **Singular Value Decomposition.** .

Requires 512×512 image

```
L=imread('Lenna512.jpg');
N=length(L);
K=50; %change to get different compression percents
X=L(1:N,1:N,1);%+L(1:N,1:N,2)+L(1:N,1:N,3);
X=double(X);
 [U,S,V]=svd(X);
U1=U;
S1=S;
V1=V;
figure(1)
colormap(gray)
subplot(121)
imagesc(U*S*V')
U(:,K:N)=0;
S(:,K:N)=0;
V(:,K:N)=0;
subplot(122)
imagesc(U*S*V')
percent_original_coeff =100*(2*(K-1)*N+K-1)/N^2% percent of coefficients kept
```

9.5. **Regression.**

```
clf
clear all;
```

```
L=4;
N=50;
%X(1,:)=L*rand(1,N);
X(1,:)=L*rand(1,N);
X(2,:)=(X(1,:)+((1/2)-rand(1,N))/2).^3;
A=ones(2,N);
A(2,:)=X(1,:);
A(3,:)=X(1,:).^2;
A(4,:)=X(1,:).^3;
Y=X(2,:)
C=inv(A*A');
B=C*A*Y'
j=L*(0:N)/N;
plot(X(1,:),X(2,:),'g*')
hold on
l=B(2)*j+B(1);
l=l+B(3)*j.^2+B(4)*j.^3;
plot(j,l)
```

9.6. **Zak transform.** .

Creates two tone signal to illustrate the time frequency representation, i.e. a spectrogram, via the inverse Zak transform

```
clear all
d=30;N=d^2;
F2=fft(eye(d))/sqrt(d); %Small Fourier for creation of Zak
Z = zeros(N,N);
F=fft(eye(N))/d;
n = 0:d-1;
n=d*n;
for j = 0:d^2-1;
   Z(j+1,n+1+floor(j/d)) = F2(mod(j,d)+1,:); %Zak transform
end;
jj=[1:N]; % creating two tone signal
kk=[floor(N/2):N];
f=sin(400*jj/pi);
f(kk)=5*cos(50*kk/pi);
Zf=Z'*f';  %taking inverse Zak to create spectrogram
ZF=zeros(d,d);
for k=1:d    %turning the vector into
figure(1) %original signal a matrix
for m=1:d
```

```
ZF(k,m)=Zf((k-1)*d+m);
end;
end;
clf
plot(f);
figure(2) %Fourier transform
plot(abs(F*f'))
figure(3) % spectrogram-Inverse Zak
imagesc(abs(ZF))
```

9.7. **Sobolev Dual.** .

```
clear all
N=6;
D = diag(ones(1,N),0) + diag(-ones(1,N-1),1);%invertible Difference Matrix
D(N,N) = 1;
E=[ones(1,N/2) zeros(1,N/2) ;zeros(1,N/2) ones(1,N/2) ]
S=inv(E*(inv(D'*D))*E')*E*inv(D'*D)
inv(D'*D)
inv(E*(inv(D'*D))*E')
q=[1 -1 1 -1 1 -1 ]';
S*q-[1/3; -1/4]
norm(ans)
```

9.8. **Sigma Delta.** . Needs Q.M in same folder.

```
clear all
format long
S=3;
pts=30;
K=1;
T=250;
gamma=.2;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Compute optimal sig-delt dual frame for odd Nth roots of unity frame.
% Compute MSE second oreder schemes for canonical dual and
% alternate dual

% Do not treat boundary term as random - use exact value.             %
%                                                                     %
% S = starting size of Nth roots of unity                             %
% pts = number of test points for computing error                       %
% 2*K = number of quantizer levels (large K gives small delta below)
```

```
% T= number of trials, i.e., computes MSE for S upt S+T roots of unity


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for t=1:T
N=S+t;
%%%%%%%%%%%%% Construct Nth roots of unity frame matrix %%%%%%%%%%%%%%%%%
%for j = 1:N
 %    E(:,j) = [cos(2*pi*(j-1)/N); sin(2*pi*(j-1)/N)];
%end
%%%%%%%%%%%%% Construct sampling frame %%%%%%%%%%%%%%%%%
for j = 1:floor(N/2)
    E(:,j) = [1;0];
    E(:,j+floor(N/2)) =[0;1];
end
 E(:,N) =[0;1];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

Edual = ((E*(E.'))^(-1))*E;    %%% canonical dual of E
%%%%%%%%%%%%%% Compute MSE %%%%%%%%%%%%%%%%%%%%%%%%%%%%%% %%%%%%%%%%%%%%%
delt= 1/(K-.5);                %%% quantizer stepsize
%%% random test points  (as column vectors)
m=1;
while m<pts+1 ;
       tt(:,m)=rand(2,1);
                    if abs(max(tt(1,m),tt(2,m))) <1;
                        P(1,m)=tt(1,m)-1/2; P(2,m)=tt(2,m)-1/2;
                        m=m+1;
                    else

                    end
end
%%%%%%%%%%%%% Mtrices for computing Sobolev dual frame for first order %%%%
D = diag(ones(1,N),0) + diag(-ones(1,N-1),1);
D(N,N) = 1; %%%


Dinv=inv(D);

M = Dinv*(Dinv.');
F = ((E*M*(E.'))^(-1))*(E*M); %%% Sobolev dual F*(E.') =Identity
```

```
X2 = (E.')*P;   %%% frame coefficients of test pts with respect to E
for m =1:pts   %%%
     q = zeros(1,N);
     x = X2(:,m);
     u = zeros(1,N);

    q(1) = Q(x(1) ,delt,K);
    u(1) = 0 ;


     for j = 2:N
          q(j) = Q(u(j-1)+ x(j),delt,K);

          u(j) = u(j-1) +x(j)-q(j);
     end

     xtilde_can = Edual*(q');   %%% reconstruct with canonical dual
     xtilde_alt = F*(q');       %%% reconstruct with alternate dual

     error_sq_can(m) = norm((P(:,m) - xtilde_can));
     error_sq_alt(m) = norm((P(:,m) - xtilde_alt));

end

%%% MSE for second order recon w/ canonical dual and alt dual
trial_second_can(1:S)=1;
trial_second_alt(1:S)=1;
trial_second_can(t+S)=max(error_sq_can);
trial_second_alt(t+S)=max(error_sq_alt);
ee(t+S)=300/(t+S)^(2);
ee2(t+S)=10/(t+S)^(1);
end


figure(1)
loglog(trial_second_alt,'g*')
hold on
loglog(trial_second_can,'k*')
%loglog(ee2,'b-','linewidth',3)
%loglog(ee,'r-','linewidth',3)
```

```
axis([S+1 S+T min(trial_second_alt) 1])
xlabel({['Number of elements in frame'];['First order scheme']},'fontsize',12)
ylabel('Max L^2 norm of error','fontsize',12)
title({['1-Bit quantization of ',num2str(T),' trials ' num2str(pts), ' points per trial'];
['Canonical (black), Sobolev (green)']},'fontsize',13)
grid on
hold off
```

Q.m

```
function y = Q(x,delt,K)
%%% 2K level midrise quant w/ stepsize delt

A = linspace((-K+.5)*delt, (K-.5)*delt,2*K);

D = abs(A-x);

[Y,I]= sort(D);

y = A(I(1));
```

References