Math 418/518 Homework 4

Directions: NEATLY write all solutions on your own paper. Use may use Maple or another computer program to find the solutions. Include any print-outs if you use a CAS.

1)Apply the Laplace Transform to Bessel's equation of order 0,

$$ty'' + y' + ty = 0 \quad t > 0,$$

and show $(s^2+1)Y' = -sY$. Solve this differential equation in Y. Since J_0 is a solution of Bessel's equation of order 0 what can you conclude about it's Laplace transform? The other solution Y_0 doe not have a Laplace transform.

2) Use properties of the Laplace transform to solve

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial u^2}{\partial^2 x} + e^{-t}, \text{ for } \; u(0,t) = 0, t > 0 \quad \text{and } u(x,0) = 0, \; \frac{\partial u}{\partial t}(x,0) = 0, x > 0$$

3) A function $f(x) \in L^2(\mathbb{R})$ is said to **1-band limited** if $|\hat{f}(w)| = 0$ for $|w| > 1$. Since $\hat{f}(w)$ is defined on a finite interval it has a Fourier series representation $\hat{f}(w) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi w}$. Show $c_n = \sqrt{\frac{\pi}{2}} f(-n\pi)$.

4) Show that if $f(x)$ is 1-bandlimited

$$f(x) = \sum_{n=-\infty}^{\infty} f(n\pi) \frac{\sin(x - n\pi)}{(x - n\pi)}$$

Extra credit: Use 4 to show

$$\hat{f}(w) = \sqrt{\frac{\pi}{2}} (\mathcal{U}_{-1}(x) - \mathcal{U}_1(x)) \sum_{n=-\infty}^{\infty} f(n\pi) e^{-in\pi w},$$

then use this to solve the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ for } (-\infty < x < \infty, \; t > 0) \text{ and } u(x,0) = f(x),$$

in the case where $f(x)$ is 1-bandlimited.