Math 418/518 Review Test II

Be able to define the Fourier transform and inverse Fourier Transform for functions

$$\hat{f}(w) = \mathcal{F}(f(x))(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx \text{ and } \mathcal{F}^{-1}(\hat{f}(w))(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}dw$$

Be able to compute a fourier transform (sinc, gaussian,$e^{-|x|}$). Know the **definition of convolution** $f * g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-t)g(t)\,dt$ for functions on defined the real line. Know these operational properties of the Fourier Transform(next page) and be able to derive and use them. (Section 7.2)

Examples: Compute $\mathcal{F}(1_{[-1/2,1/2)}(x))(w)$ using the definition. Show $\mathcal{F}(f(ax))(w) = \frac{1}{a}\hat{f}(\frac{w}{a})$ for $a > 0$ and use these two to find $\mathcal{F}(1_{[-1,1)}(x))(w)$.

Convert a Partial Differential equation into an ordinary one using the Fourier Transform and solve.

Example: Convert and solve the PDE $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $u(x,0) = f(x)$ by transforming it into an ODE (Also examples 1-5 sec 7.3) .

Be able to derive(and apply) the the solution to the heat equation using Gauss's kernel $g_t(x) = \frac{1}{\sqrt{2t}}e^{-x^2/4t}$ (7.4) pg 421

Be able to derive(and apply) the the solution to the Dirichlet Problem on the upper half-plane using the Poisson kernel $P_y(x) = \sqrt{\frac{2}{\pi}}\frac{y}{x^2+y^2}$ (7.5) pg 430

Be able to take/**DEFINE** derivatives and Fourier Transforms of generalized functions and be able to apply the operational properties. Example 6 pg453

Proof of the Sampling Theorem pg 552-553

Know the definitions of the DFT and the IDFT for sequences (also called vectors or signals) of length N. Be able to compute a simple DFT and IDFT. (say for $x = (1, 1, i, i)$)

$$X(k) = \mathcal{F}_N(x)(k) = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} x(j)e^{2\pi ijk/N} \text{ and}$$

$$x(k) = \mathcal{F}_N^{-1}(X)(k) = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} X(j)e^{-2\pi ijk/N}$$

Proof of Theorrem 1 10.3 pg 565

Proof of Parseval's Theorem pg591

Fourier Operational Properties

1)$\mathcal{F}(f(x)+g(x))(w)=\hat{f}(w)+\hat{g}(w)$ 2)$\mathcal{F}(cf(x))(w)=c\hat{f}(w)$

3)$\mathcal{F}(e^{ikx}f(x))(w)=\hat{f}(w-k)$ 4)$\mathcal{F}(f(x-k))(w)=e^{-ikw}\hat{f}(w)$

5)$\mathcal{F}(f'(x))(w)=iw\hat{f}(w)$ 6)$\mathcal{F}(xf(x))(w)=i\frac{d}{dw}\hat{f}(w)$

7)$\mathcal{F}(f*g)(w)=\hat{f}(w)\cdot\hat{g}(w)$ 8)$\mathcal{F}(\mathcal{F}(f))(x)=f(-x)$

9)$\mathcal{F}(\mathcal{U}_a(x))(w)=\frac{-ie^{-iaw}}{\sqrt{2\pi}w}$ 10)$\mathcal{F}(\delta_a(x))(w)=\frac{e^{-iaw}}{\sqrt{2\pi}}$

11)$\mathcal{F}(f(ax))(w)=\frac{1}{a}\hat{f}(\frac{w}{a})$ for $a>0$ 12)$\mathcal{F}^{-1}(\hat{f}(\frac{w}{a}))(w)=af(ax)$

Euler's Formula $e^{ix}=\cos(x)+i\sin(x)$

Elementary Ordinary Differential Equations

First Order Linear constant Coef: $y'=\alpha y$, $y=A_0e^{\alpha x}$

Second order Linear constant Coef: $\alpha y''+\beta y'+\gamma y=0$ let $\lambda_{1,2}=\frac{-\beta\pm\sqrt{\beta^2-4\alpha\gamma}}{2\alpha}$

$$y=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x} \text{ for } \beta^2-4\alpha\gamma\neq 0,\ y=c_1e^{\lambda_1 x}+xc_2e^{\lambda_1 x} \text{ for } \beta^2-4\alpha\gamma=0$$

Special Case 1 $y''+\gamma^2y=0$ $y=c_1e^{i\gamma x}+c_2e^{-i\gamma x}$ or $y=d_1\cos(\gamma x)+d_2\sin(\gamma x)$

Special Case2 $y''-\gamma^2y=0$ $y=c_1e^{\gamma x}+c_2e^{-\gamma x}$ or $y=d_1\cosh(\gamma x)+d_2\sinh(\gamma x)$

Bernoulli zero $xy''+y'+\lambda^2xy=0$, $y=d_1J_0(\alpha x)+d_2Y_0(\alpha x)$. J_0 and Y_0 are the **Bessel functions of order 0 of the first and second kind.** We denote the zeros of J_0 by α_n

Euler's Equation $x^2y''+xy'-n^2y=0$

$$y=c_1x^n+c_2x^{-n},\ \text{for } n=1,2,3\ldots$$

$$y=c_1+c_2\ln(r),\ \text{for } n=0$$