Math 418/518 Homework 2

Directions: NEATLY write all solutions on your own paper. Solutions should include details like integration by parts and reasons for convergence or divergence. 418, 1-4 518, 1-5

1) Use Fourier transform techniques to solve

$$u_t = 3t^2 u_{xx} \text{ for } \infty < x < \infty,\ t > 0 \text{ and } u(x,0) = f(x).$$

You may leave your answer as the convolution of two functions of x and t.

2. Solve the Dirichlet Problem in the upper half plane

$$\nabla^2 u = 0 \text{ for } \infty < x < \infty,\ y > 0 \text{ and } u(x,0) = \mathcal{U}_{-1}(x) - \mathcal{U}_1(x).$$

3) Using properties of the Fourier transform show $P_{y_1} * P_{y_2}(x) = P_{y_1+y_2}(x)$.

4) Define the DFT by $\mathcal{F}_N(\mathbf{x})(k) = \dfrac{1}{\sqrt{N}} \displaystyle\sum_{n=0}^{N-1} \mathbf{x}(n) e^{-2\pi i n k/N}$. Define $T(\mathbf{x})(k) = \mathbf{x}(k+1)$ and Define $M(\mathbf{x})(k) = e^{2\pi i k/N}\mathbf{x}(k)$ Show $\mathcal{F}_N T \mathcal{F}_N^* = M$. That is, the Fourier Transform diagonalizes T

5) Using the notation from 4) define $D = I - T$ to be the circulant difference operator and show $\mathcal{F}_N$ diagonalizes the finite Laplacian $L = D^*D$, i.e. $X = \mathcal{F}_N L \mathcal{F}_N^*$, is diagonal. Finally, show that $\mathcal{F}_N$ commutes with $L + X$

Matlab code that illustrates the properties in 4) and 5)

```
d=4
T =diag(ones(1,d-1),1);
T(d,1)=1
x=rand(d,1)
T*x
D=eye(d)-T
F=fft(eye(d))/sqrt(d)
F*T*F'
L=D'*D;
X=F'*L*F
X+L
% eig(X+L)
F'*(X+L)*F
```