Matrix Representations - Changing Bases

1 State Vectors

The main goal is to represent states and operators in different basis. We first use brute force methods for relating basis vectors in one representation in terms of another one. Then we will show the equivalent transformations using matrix operations.

1.1 Inserting the Identity Operator

We begin by using the identity operator in the S_z-basis,

$$I = |{+z}\rangle\langle{+z}| + |{-z}\rangle\langle{-z}|, \tag{1}$$

to derive matrix representations.

We first note that in the S_z-basis the basis states in the S_x and S_y-bases are given by In order to relate

$$\begin{aligned}
|{+x}\rangle &= \tfrac{1}{\sqrt{2}}\left(|{+z}\rangle + |{-z}\rangle\rangle\right), & \langle{+x}| &= \tfrac{1}{\sqrt{2}}\left(\langle{+z}| + \langle{-z}|\rangle\right), \\
|{-x}\rangle &= \tfrac{1}{\sqrt{2}}\left(|{+z}\rangle - |{-z}\rangle\rangle\right), & \langle{-x}| &= \tfrac{1}{\sqrt{2}}\left(\langle{+z}| - \langle{-z}|\rangle\right), \\
|{+y}\rangle &= \tfrac{1}{\sqrt{2}}\left(|{+z}\rangle + i|{-z}\rangle\rangle\right), & \langle{+y}| &= \tfrac{1}{\sqrt{2}}\left(\langle{+z}| - i\langle{-z}|\rangle\right), \\
|{-y}\rangle &= \tfrac{1}{\sqrt{2}}\left(|{+z}\rangle - i|{-z}\rangle\rangle\right), & \langle{-y}| &= \tfrac{1}{\sqrt{2}}\left(\langle{+z}| + i\langle{-z}|\rangle\right).
\end{aligned}$$

$|{+x}\rangle$ in the S_z and S_y bases, we seek a relation between the column representations

$$\begin{pmatrix} \langle{+z}|{+x}\rangle \\ \langle{-z}|{+x}\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_z \quad \text{and} \quad \begin{pmatrix} \langle{+y}|{+x}\rangle \\ \langle{-y}|{+x}\rangle \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix}_x.$$

We first consider the amplitude $\langle{+y}|{+x}\rangle$. Inserting the S_z-basis identity operator (1), we obtain

$$\begin{aligned}
\langle{+y}|{+x}\rangle &= \langle{+y}|\, I\, |{+x}\rangle \\
&= \langle{+y}|\left(|{+z}\rangle\langle{+z}| + |{-z}\rangle\langle{-z}|\right)|{+x}\rangle \\
&= \langle{+y}|{+z}\rangle\langle{+z}|{+x}\rangle + \langle{+y}|{-z}\rangle\langle{-z}|{+x}\rangle.
\end{aligned} \tag{2}$$

Similarly, we have

$$\langle{-y}|{+x}\rangle = \langle{-y}|{+z}\rangle\langle{+z}|{+x}\rangle + \langle{-y}|{-z}\rangle\langle{-z}|{+x}\rangle. \tag{3}$$

Recalling that the goal is to relate $|+x\rangle$ in the S_y and S_z bases, we compute

$$\begin{array}{rr} \langle +z|+x\rangle = \frac{1}{\sqrt{2}}, & \langle +y|+z\rangle = \frac{1}{\sqrt{2}} \\ \langle +z|-x\rangle = \frac{1}{\sqrt{2}}, & \langle +y|-z\rangle = -\frac{i}{\sqrt{2}} \\ \langle -z|+x\rangle = \frac{1}{\sqrt{2}}, & \langle -y|+z\rangle = \frac{1}{\sqrt{2}}, \\ \langle -z|-x\rangle = -\frac{1}{\sqrt{2}}, & \langle -y|-z\rangle = \frac{i}{\sqrt{2}}. \end{array}$$

Therefore, we have

$$\begin{aligned} \langle +y|+x\rangle &= \langle +y|+z\rangle \langle +z|+x\rangle + \langle +y|-z\rangle \langle -z|+x\rangle \\ &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\frac{1}{\sqrt{2}} \\ &= \frac{1-i}{2}, \qquad (4) \\ \langle -y|+x\rangle &= \langle -y|+z\rangle \langle +z|+x\rangle + \langle -y|-z\rangle \langle -z|+x\rangle \\ &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\frac{1}{\sqrt{2}} \\ &= \frac{1+i}{2}. \qquad (5) \end{aligned}$$

So, the matrix representation of $|+x\rangle$ in the S_y-basis is given by

$$|+x\rangle \underset{S_y}{\longrightarrow} \begin{pmatrix} \langle +y|+x\rangle \\ \langle -y|+x\rangle \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1-i \\ 1+i \end{pmatrix}.$$

1.2 A Matrix Approach

A more elegant approach is to write the equations

$$\begin{aligned} \langle +y|+x\rangle &= \langle +y|+z\rangle \langle +z|+x\rangle + \langle +y|-z\rangle \langle -z|+x\rangle . \\ \langle -y|+x\rangle &= \langle -y|+z\rangle \langle +z|+x\rangle + \langle -y|-z\rangle \langle -z|+x\rangle . \end{aligned}$$

in matrix form.

$$\begin{pmatrix} \langle +y|+x\rangle \\ \langle -y|+x\rangle \end{pmatrix} = \begin{pmatrix} \langle +y|+z\rangle & \langle +y|-z\rangle \\ \langle -y|+z\rangle & \langle -y|-z\rangle \end{pmatrix} \begin{pmatrix} \langle +z|+x\rangle \\ \langle -z|+x\rangle \end{pmatrix} \qquad (6)$$

Evaluating the matrices on the right side of the equation, we obtain

$$\begin{pmatrix} \langle +y|+x\rangle \\ \langle -y|+x\rangle \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1-i \\ 1+i \end{pmatrix}.$$

Similarly, the matrix representation of $|{-x}\rangle$ in the S_y basis is given by

$$\begin{aligned}\begin{pmatrix} \langle +y|-x\rangle \\ \langle -y|-x\rangle \end{pmatrix} &= \begin{pmatrix} \langle +y|+z\rangle & \langle +y|-z\rangle \\ \langle -y|+z\rangle & \langle -y|-z\rangle \end{pmatrix}\begin{pmatrix} \langle +z|-x\rangle \\ \langle -z|-x\rangle \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix}.\end{aligned}$$

Therefore, we have found that representations of $|\pm x\rangle$, or any state $|\psi\rangle$, in the S_y and S_z bases are given by the transformation

$$|\psi\rangle_y = \mathbf{R}^\dagger\,|\psi\rangle_z$$

whose matrix representation is

$$\mathbb{S}^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

We represent such a representation in the new basis as

$$|\psi\rangle \xrightarrow[S_y]{} \begin{pmatrix} \langle +y|\psi\rangle \\ \langle -y|\psi\rangle \end{pmatrix} = \begin{pmatrix} \langle +y|+z\rangle & \langle +y|-z\rangle \\ \langle -y|+z\rangle & \langle -y|-z\rangle \end{pmatrix}\begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix}. \tag{7}$$

Now consider the matrix

$$\mathbb{S} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

Note that the columns of $\mathbb{S}$ are the representations of $|\pm y\rangle$ in the S_z basis:

$$\begin{aligned}|+y\rangle &= \frac{1}{\sqrt{2}}\left(|+z\rangle + i|-z\rangle\right)\right) \xrightarrow[S_z]{} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \\ |-y\rangle &= \frac{1}{\sqrt{2}}\left(|+z\rangle - i|-z\rangle\right)\right), \xrightarrow[S_z]{} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix},\end{aligned}$$

Similarly, the rows of $\mathbb{S}^\dagger$ are the representations of $\langle \pm y|$ in the S_z basis.

$$\begin{aligned}\langle +y| &= \frac{1}{\sqrt{2}}\left(\langle +z| - i\langle -z|\rangle\right) \xrightarrow[S_z]{} \frac{1}{\sqrt{2}}(1, -i), \\ \langle -y| &= \frac{1}{\sqrt{2}}\left(\langle +z| + i\langle -z|\rangle\right) \xrightarrow[S_z]{} \frac{1}{\sqrt{2}}(1, i).\end{aligned}$$

Example Change $|+y\rangle$ from the S_z to S_x basis.

We know that

$$\begin{aligned}|+x\rangle &= \frac{1}{\sqrt{2}}\left(|+z\rangle + |-z\rangle\right)\right), \\ |-x\rangle &= \frac{1}{\sqrt{2}}\left(|+z\rangle - |-z\rangle\right)\right),\end{aligned}$$

Then, we write

$$\mathbb{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Therefore,

$$\mathbb{S}^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then,

$$|+y\rangle \underset{S_x}{\longrightarrow} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}.$$

2 Operators in New Bases

We can use rotations to represent operators, $\hat{A}$, in the S_y basis. In general, we compute matrix elements of the matrix representation of the operator, ${}_y\langle\phi|\hat{A}|\psi\rangle_y$, by using the identity

$$I = \hat{R}\hat{R}^\dagger = \hat{R}^\dagger\hat{R}$$

and the change of representations of states. Namely, we have

$$|\psi\rangle_y = \hat{R}^\dagger |\psi\rangle_z, \qquad |\psi\rangle_z = \hat{R} |\psi\rangle_y,$$

to obtain a general expression for matrix elements of $\hat{A}$:

$$\begin{aligned} {}_y\langle\phi|\hat{A}|\psi\rangle_y &= {}_y\langle\phi|\hat{R}\hat{R}^\dagger\hat{A}\hat{R}^\dagger\hat{R}|\psi\rangle_y \\ &= {}_z\langle\phi|R^\dagger\hat{A}\hat{R}|\psi\rangle_z \end{aligned} \tag{8}$$

This means that

$$\hat{A} \underset{S_y}{\longrightarrow} \mathbb{S}^\dagger\mathbb{A}\mathbb{S} \quad \text{where} \quad \hat{A} \underset{S_z}{\longrightarrow} \mathbb{A}. \tag{9}$$

In other words, we compute $\mathbb{S}^\dagger\mathbb{A}\mathbb{S}$ in the S_z-basis to obtain the representation of $\hat{A}$ in the S_y-basis.

Example Represent $\hat{J}_z$ in the S_y-basis.

We know that $\hat{J}_z$ is diagonal in the S_z-basis. So,

$$\hat{J}_z \underset{S_z}{\longrightarrow} \mathbb{J}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

No, we compute the matrix representation of $\hat{J}_z$ in the S_y-basis using the similarity transformation, $\mathbb{S}^\dagger\mathbb{J}_z\mathbb{S}$,

$$\begin{aligned} \hat{J}_z \underset{S_y}{\longrightarrow} \mathbb{S}^\dagger\mathbb{J}_z\mathbb{S} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \\ &= \frac{\hbar}{4} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \\ &= \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$