Solution of the Heat Equation with Nonhomogeneous BCs

MAT 418/518 Fall 2020, by Dr. R. L. Herman

Department of Mathematics and Statistics, UNCW

Homogeneous Boundary Conditions

We first consider the problem[a]

$$u_t = ku_{xx}, \quad 0 < x < L, \quad t > 0$$
$$u(0,t) = 0, \quad u(0,t) = 0, \quad t > 0$$
$$u(x,0) = f(x), \quad 0 \le x \le L. \tag{1}$$

Using the Method of Separation of Variables, we let $u(x,t) = X(x)T(t)$, leading to solving

$$X'' + \kappa^2 X = 0, \quad X(0) = X(L) = 0.$$

The solution of this eigenvalue problem is

$$X(x) = \sin\frac{n\pi x}{L}, \quad \kappa = \frac{n\pi}{L}, n = 1, 2, \ldots.$$

The general solution of (1) is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-n^2\pi^2 kt/L^2}. \tag{2}$$

The Fourier coefficients are found as

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

[a] Note - Other boundary conditions, such as insulating, mixed, or periodic, boundary conditions will lead to other solutions.

Time-Independent BCs

Let's assume fixed conditions at $x = 0, L$:

$$u_t = ku_{xx}, \quad 0 < x < L, \quad t > 0$$
$$u(0,t) = a, \quad u(L,t) = b, \quad t > 0$$
$$u(x,0) = f(x), \quad 0 \le x \le L. \tag{3}$$

We seek particular [**steady-state** ($u_t = 0$)] solutions satisfying these boundary conditions. Then,

$$u_p''(x) = 0, \quad 0 < x < L,$$
$$u_p(0) = a, \quad u_p(L) = b. \tag{4}$$

Assuming the form $u_p(x) = cx + d$, we find

$$u_p(x) = \frac{(b-a)x}{L} + a. \tag{5}$$

If $u(x,t) = u_p(x) + v(x,t)$, then $v(x,t)$ is a solution to

$$v_t = kv_{xx}, \quad 0 < x < L, \quad t > 0$$
$$v(0,t) = 0, \quad v(L,t) = 0, \quad t > 0$$
$$v(x,0) = f(x) - u_p(x), \quad 0 \le x \le L. \tag{6}$$

The general solution of (3) is found as

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-n^2\pi^2 kt/L^2} + \frac{(b-a)x}{L} + a, \tag{7}$$

where b_n is determined using the modified initial condition, $v(x,0) = f(x) - u_p(x)$.

General Problem with Time-Dependent Boundary Conditions

The homogeneous[a] heat equation with time-dependent boundary conditions is given by

$$u_t = ku_{xx}, \quad 0 < x < L, \quad t > 0,$$
$$u(0,t) = a(t), \quad u(L,t) = b(t), \quad t > 0,$$
$$u(x,0) = f(x), \quad 0 \le x \le L. \tag{8}$$

We seek solutions of the form

$$u(x,t) = v(x,t) + w(x,t),$$

where

$$w(x,t) = [b(t) - a(t)]\frac{x}{L} + a(t). \tag{9}$$

[a] The nonhomogeneous equation, $u_t - ku_{xx} = F(x,t)$, can be solved using $u(x,t) = v(x,t) + w(x,t) + u_2(x,t)$ where $u_2(x,t)$ satisfies the nonhomogeneous problem with homogeneous boundary conditions using Duhamel's Principle.

Problem for $v(x,t)$

$v(x,t)$ satisfies the initial-boundary value problem

$$v_t - kv_{xx} = -[b'(t) - a'(t)]\frac{x}{L} - a'(t),$$
$$v(0,t) = 0, \quad v(L,t) = 0,$$
$$v(x,0) = f(x) - [b(0) - a(0)]\frac{x}{L} - a(0).$$

This is a nonhomogeneous heat equation with homogeneous boundary conditions.

Nonhomogeneous Heat Equation with Homogeneous BCs

The equation for $v(x,t)$ can be written in the general form

$$v_t - kv_{xx} = h(x,t), \quad 0 < x < L, \quad t > 0,$$
$$v(0,t) = 0, \quad v(L,t) = 0, \quad t > 0,$$
$$v(x,0) = g(x), \quad 0 \le x \le L. \tag{10}$$

Once again, we split Problem (10) into two problems. Let

$$v(x,t) = u_1(x,t) + u_2(x,t),$$

where u_1 and u_2 satisfy the following two problems.

Problem for $u_1(x,t)$

$$u_{1t} - ku_{1xx} = 0, \quad 0 < x < L, \quad t > 0,$$
$$u_1(0,t) = 0, \quad u_1(L,t) = 0, \quad t > 0,$$
$$u_1(x,0) = g(x), \quad 0 \le x \le L.$$

This is the familiar homogeneous heat equation with homogeneous boundary conditions. The solutions are found using the Method of Separation of Variables.

Problem for $u_2(x,t)$

$$u_{2t} - ku_{2xx} = h(x,t), \quad 0 < x < L, \quad t > 0,$$
$$u_2(0,t) = 0, \quad u_2(L,t) = 0, \quad t > 0,$$
$$u_2(x,0) = 0, \quad 0 \le x \le L.$$

This is a nonhomogeneous heat equation with homogeneous initial conditions. We use **Duhamel's Principle** to convert this problem with a source to an initial value problem.

Duhamel's Principle

The solution of a heat equation with a source and homogeneous boundary conditions may be found by solving a homogeneous heat equation with nonhomogeneous boundary conditions.

ODE Version

Let $\mathbf{X} : \mathbb{R} \to \mathbb{R}$ and $\mathbf{X}(t) = U(t)\mathbf{X}_0$ be the solution of $\dot{\mathbf{X}} = A\mathbf{X}$, $\mathbf{X}(0) = \mathbf{X}_0$.
Consider

$$\mathbf{X}(t) = \int_0^t U(t-s)\mathbf{Y}(s)\,ds.$$

$\mathbf{X}(t)$ satisfies the inhomogeneous problem

$$\left(\frac{d}{dt} - A\right)\mathbf{X} = \mathbf{Y}(s), \quad \mathbf{X}(0) = \mathbf{0}.$$

Solution for $u_2(x,t)$

Solve for $\tilde{v}(x,t;s)$ in the problem

$$\tilde{v}_t - k\tilde{v}_{xx} = 0, \quad 0 < x < L, \quad t > 0,$$
$$\tilde{v}(0,t;s) = 0, \quad \tilde{v}(L,t;s) = 0,$$
$$\tilde{v}(x,0;s) = h(x,s). \tag{11}$$

Then, $v(x,t;s) = \tilde{v}(x,t-s;s)$ satisfies

$$v_t - kv_{xx} = 0, \quad 0 < x < L, \quad t > s,$$
$$v(0,t;s) = 0, \quad v(L,t;s) = 0,$$
$$v(x,0;s) = h(x,s). \tag{12}$$

$v(x,t;s)$ is the solution when the source is turned on at time $t = s - \Delta s$ and turned off at $t = s$. A superposition of these incremental sources gives the solution

$$u_2(x,t) = \int_0^t v(x,t;s)\,ds$$
$$= \int_0^t \tilde{v}(x,t-s;s)\,ds. \tag{13}$$

Green's Function, $G(x,y)$

The **steady state solution**, satisfying

$$-kw_{xx} = h(x), \quad 0 < x < L,$$
$$w(0) = a, \quad w(L) = b, \tag{14}$$

can be found by direct integration as

$$w(x) = -\int_0^L G(x,y)\left(-\frac{1}{k}h(y)\right)dy + (b-a)\frac{x}{L} + a.$$