1

Second Order Partial Differential Equations

"Either mathematics is too big for the human mind or the human mind is more than a machine." - Kurt Gödel (1906-1978)

1.1 Introduction

In this chapter we will introduce several generic second order linear partial differential equations and see how such equations lead naturally to the study of boundary value problems for ordinary differential equations. These generic differential equations occur in one to three spatial dimensions and are all linear differential equations. A list of these generic equations is provided in Table 1.1. Here we have introduced the Laplacian operator,[1] $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$. Depending on the types of boundary conditions imposed and on the geometry of the system (rectangular, cylindrical, spherical, etc.), one encounters many interesting boundary value problems.

[1] The symbol ∇ is the gradient operator. W. R. Smith suggested in 1870 to P. G. Tait to use nabla, Greek for a harp. Tait and Maxwell referred to ∇ in their letters. William Thomson (Lord Kelvin) introduced it in America in 1884. However, E. B. Wilson and Josiah Gibbs called it "del" in the classic work on vector analysis.

Name	2 Vars	3 D
Heat Equation	$u_t = k u_{xx}$	$u_t = k\nabla^2 u$
Wave Equation	$u_{tt} = c^2 u_{xx}$	$u_{tt} = c^2\nabla^2 u$
Laplace's Equation	$u_{xx} + u_{yy} = 0$	$\nabla^2 u = 0$
Poisson's Equation	$u_{xx} + u_{yy} = F(x,y)$	$\nabla^2 u = F(x,y,z)$
Schrödinger's Equation	$iu_t = u_{xx} + F(x,t)u$	$iu_t = \nabla^2 u + F(x,y,z,t)u$

Table 1.1: List of generic partial differential equations.

Let's look at the heat equation in one dimension. This could describe the heat conduction in a thin insulated rod of length L. It could also describe the diffusion of pollutant in a long narrow stream, or the flow of traffic down a road. In problems involving diffusion processes, one instead calls this equation the diffusion equation. [We provide a derivation in Section 1.3.2.]

A typical initial-boundary value problem for the heat equation would be that initially one has a temperature distribution $u(x,0) = f(x)$. Placing the bar in an ice bath and assuming the heat flow is only through the ends of the bar, one has the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$. Of course, we are dealing with Celsius temperatures and we assume there is

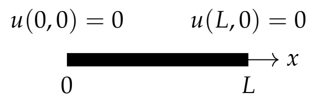


Figure 1.1: One dimensional heated rod of length L.

plenty of ice to keep that temperature fixed at each end for all time as seen in Figure 1.1. So, the problem one would need to solve is given as [IC = initial condition and BC = boundary condition.]

1D Heat Equation

$$\begin{array}{lcc} \text{PDE} & u_t = k u_{xx}, & 0 < t, \quad 0 \le x \le L, \\ \text{IC} & u(x,0) = f(x), & 0 < x < L, \\ \text{BC} & u(0,t) = 0, & t > 0, \\ & u(L,t) = 0, & t > 0, \end{array} \tag{1.1}$$

Here, k is the heat conduction constant and is determined using properties of the bar.

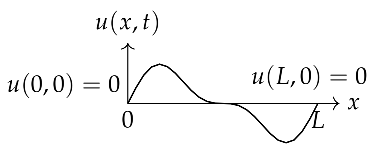


Figure 1.2: One dimensional string of length L.

Another problem that will come up in later discussions is that of the vibrating string. A string of length L is stretched out horizontally with both ends fixed such as a violin string as shown in Figure 1.2. Let $u(x,t)$ be the vertical displacement of the string at position x and time t. The motion of the string is governed by the one dimensional wave equation. [See the derivation in Section 1.3.1.] The string might be plucked, giving the string an initial profile, $u(x,0) = f(x)$, and possibly each point on the string has an initial velocity $u_t(x,0) = g(x)$. The initial-boundary value problem for this problem is given below.

1D Wave Equation

$$\begin{array}{lcc} \text{PDE} & u_{tt} = c^2 u_{xx} & 0 < t, \quad 0 \le x \le L \\ \text{IC} & u(x,0) = f(x) & 0 < x < L \\ & u_t(x,0) = g(x) & 0 < x < L \\ \text{BC} & u(0,t) = 0 & t > 0 \\ & u(L,t) = 0 & t > 0 \end{array} \tag{1.2}$$

In this problem c is the wave speed in the string. It depends on the mass per unit length of the string, μ, and the tension, τ, placed on the string.

There is a rich history on the study of these and other partial differential equations and much of this involves trying to solve problems in physics.[2] Consider the one dimensional wave motion in the string. Physically, the speed of these waves depends on the tension in the string and its mass density. The frequencies we hear are then related to the string shape, or the allowed wavelengths across the string. We will be interested the harmonics, or pure sinusoidal waves, of the vibrating string and how a general wave on the string can be represented as a sum over such harmonics. This will take us into the field of spectral, or Fourier, analysis. The solution of the heat equation also involves the use of Fourier analysis. However, in this case there are no oscillations in time.

[2] A controversy arose between Leonhard Euler (1707-1783), Jean-Baptiste le Rond d'Alembert (1717-1783), and Daniel Bernoulli (1700-1782). See the story below.

There are many applications that are studied using spectral analysis. At the root of these studies is the belief that continuous waveforms are comprised of a number of harmonics. Such ideas stretch back to the Pythagoreans' study of the vibrations of strings, which led to their program of a world of harmony. This idea was carried further by Johannes Kepler (1571-1630) in his harmony of the spheres approach to planetary orbits. In the 1700's others worked on the superposition theory for vibrating waves on a stretched spring, starting with the wave equation and leading to the superposition of right and left traveling waves. This work was carried out by people such as John Wallis (1616-1703), Brook Taylor (1685-1731) and Jean-Baptiste le Rond d'Alembert (1717-1783).

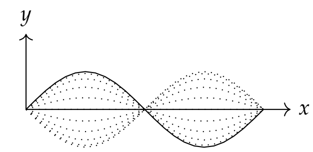


Figure 1.3: Plot of the second harmonic of a vibrating string at different times.

In 1742 d'Alembert solved the wave equation

$$c^2 \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0,$$

where y is the string height and c is the wave speed. However, this solution led him and others, like Leonhard Euler (1707-1783) and Daniel Bernoulli (1700-1782), to investigate what "functions" could be the solutions of this equation. In fact, this led to a more rigorous approach to the study of analysis by first coming to grips with the concept of a function. For example, in 1749 Euler sought the solution for a plucked string in which case the initial condition $y(x,0) = h(x)$ has a discontinuous derivative! (We will see how this led to important questions in analysis.)

In 1753 Daniel Bernoulli viewed the solutions as a superposition of simple vibrations, or harmonics. Such superpositions amounted to looking at solutions of the form

$$y(x,t) = \sum_k a_k \sin \frac{k\pi x}{L} \cos \frac{k\pi ct}{L},$$

where the string extends over the interval $[0, L]$ with fixed ends at $x = 0$ and $x = L$.

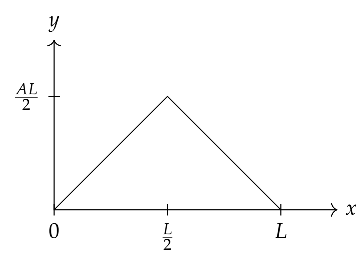


Figure 1.4: Plot of an initial condition for a plucked string.

However, the initial profile for such superpositions is given by

$$y(x,0) = \sum_k a_k \sin \frac{k\pi x}{L}.$$

It was determined that many functions could not be represented by a finite number of harmonics, even for the simply plucked string in Figure 1.4 given by an initial condition of the form

$$y(x,0) = \begin{cases} Ax, & 0 \le x \le L/2 \\ A(L-x), & L/2 \le x \le L \end{cases}$$

Thus, the solution consists generally of an infinite series of trigonometric functions. But it left the question as to how such a non-smooth function could be described by trigonometric functions, which are continuous. Also, people were still trying to understand infinite series and convergence. So, historically, people were not comfortable with these infinite series of trigonometric functions describing the vibrating string. In 1761 Joseph-Louis Lagrange (1736-1813) eventually arrived at a similar solution. Eventually, the problems were settled based on Joseph Fourier's (1768-1830) work around 1820 on heat conduction and trigonometric series and Lejeune Dirichlet's (1805-1859) work on the convergence of what we now call Fourier series.

Such series expansions were also of importance in Joseph Fourier's (1768-1830) solution of the heat equation. The use of Fourier expansions has become an important tool in the solution of linear partial differential equations, such as the wave equation and the heat equation. More generally, using a technique called the Method of Separation of Variables, allowed higher dimensional problems to be reduced to one dimensional boundary value problems. However, these studies led to very important questions, which in turn opened the doors to whole fields of analysis. Some of the problems raised were

The one dimensional version of the heat equation is a partial differential equation for $u(x,t)$ of the form

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$

Solutions satisfying boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$, are of the form

$$u(x,t) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-n^2\pi^2 t/L^2}.$$

In this case, setting $u(x,0) = f(x)$, one has to satisfy the condition

$$f(x) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

This is another example leading to an infinite series of trigonometric functions.

1. What functions can be represented as the sum of trigonometric functions?
2. How can a function with discontinuous derivatives be represented by a sum of smooth functions, such as the above sums of trigonometric functions?
3. Do such infinite sums of trigonometric functions actually converge to the functions they represent?

There are many other systems for which it makes sense to interpret the solutions as sums of sinusoids of particular frequencies. For example, consider ocean waves. Ocean waves are affected by the gravitational pull of the moon and the sun and other numerous forces. These lead to the tides, which in turn have their own periods of motion. In an analysis of wave heights, one can separate out the tidal components by making use of Fourier analysis. These tidal constituents are due to things like the Earth's rotation, the positions of the Moon and Sun relative to the Earth, and the Moon's elevation above the Equator. there are several hundred constituents of which just a few are dominant.

In the Section 1.4 we describe how to go about solving these linear partial differential equations using the method of separation of variables. We will find that in order to accommodate the initial conditions, we will need to introduce Fourier series before we can complete the problems, which will be the subject of the following chapter. However, we first derive the one-dimensional wave and heat equations.

1.2 *Boundary Value Problems*

You might have only solved initial value problems in your undergraduate differential equations class. For an initial value problem one has to solve a differential equation subject to conditions on the unknown function and its derivatives at one value of the independent variable. For example, for $x = x(t)$ we could have the initial value problem

$$x'' + x = 2, \quad x(0) = 1, \quad x'(0) = 0. \tag{1.3}$$

Typically, initial value problems involve time dependent functions and boundary value problems are spatial. So, with an initial value problem one knows how a system evolves in terms of the differential equation and the state of the system at some fixed time. Then one seeks to determine the state of the system at a later time.

Example 1.1. Solve the initial value problem, $x'' + 4x = \cos t, \quad x(0) = 1, \quad x'(0) = 0.$

Note that the conditions are provided at one time, $t = 0$. Thus, this an initial value problem. Recall from your course on differential equations that we first find the general solution and then apply the initial conditions. Furthermore, this equation is a nonhomogeneous differential equation, so the solution is a sum of a solution of the homogeneous equation and a particular solution of the nonhomogeneous equation, $x(t) = x_h(t) + x_p(t)$. [See the ordinary differential equations review in the Appendix.]

The solution of $x'' + 4x = 0$ is easily found as

$$x_h(t) = c_1 \cos 2t + c_2 \sin 2t.$$

The particular solution is found using the Method of Undetermined Coefficients. We guess a solution of the form

$$x_p(t) = A \cos t + B \sin t.$$

Differentiating twice, we have

$$x_p''(t) = -(A \cos t + B \sin t).$$

So,

$$x_p'' + 4x_p = -(A \cos t + B \sin t) + 4(A \cos t + B \sin t).$$

Comparing the right hand side of this equation with $\cos t$ in the original problem, we are led to setting $B = 0$ and $A = \frac{1}{3} \cos t$. Thus, the general solution is

$$x(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{3} \cos t.$$

We now apply the initial conditions to find the particular solution. The first condition, $x(0) = 1$, gives

$$1 = c_1 + \frac{1}{3}.$$

Thus, $c_1 = \frac{2}{3}$. Using this value for c_1, the second condition, $x'(0) = 0$, gives $c_2 = 0$. Therefore,

$$x(t) = \frac{1}{3}(2 \cos 2t + \cos t).$$

For boundary values problems, one knows how each point responds to its neighbors, but there are conditions that have to be satisfied at the endpoints. An example would be a horizontal beam supported at the ends, like a bridge. The shape of the beam under the influence of gravity, or other forces, would lead to a differential equation and the boundary conditions at the beam ends would affect the solution of the problem. There are also a variety of other types of boundary conditions. In the case of a beam, one end could be fixed and the other end could be free to move. We will explore the effects of different boundary conditions in our discussions and exercises. But, we will first solve a simple boundary value problem which is a slight modification of the above problem.

Example 1.2. Solve the boundary value problem, $x'' + x = 2, \quad x(0) = 1, \quad x(1) = 0.$

Note that the conditions at $t = 0$ and $t = 1$ make this a boundary value problem since the conditions are given at two different points. As with initial value problems, we need to find the general solution and then apply any conditions that we may have. This is a nonhomogeneous differential equation, so the solution is a sum of a solution of the homogeneous equation and a particular solution of the nonhomogeneous equation, $x(t) = x_h(t) + x_p(t)$. The solution of $x'' + x = 0$ is easily found as

$$x_h(t) = c_1 \cos t + c_2 \sin t.$$

The particular solution is again found using the Method of Undetermined Coefficients,

$$x_p(t) = 2.$$

Thus, the general solution is

$$x(t) = 2 + c_1 \cos t + c_2 \sin t.$$

We now apply the boundary conditions and see if there are values of c_1 and c_2 that yield a solution to this boundary value problem. The first condition, $x(0) = 0$, gives

$$0 = 2 + c_1.$$

Thus, $c_1 = -2$. Using this value for c_1, the second condition, $x(1) = 1$, gives

$$0 = 2 - 2\cos 1 + c_2 \sin 1.$$

This yields

$$c_2 = \frac{2(\cos 1 - 1)}{\sin 1}.$$

We have found that there is a solution to the boundary value problem and it is given by

$$x(t) = 2\left(1 - \cos t \frac{(\cos 1 - 1)}{\sin 1} \sin t\right).$$

Boundary value problems arise in many physical systems, just as the initial value problems we have seen earlier. We will see in the next sections that boundary value problems for ordinary differential equations often appear in the solutions of partial differential equations. However, there is no guarantee that we will have unique solutions of our boundary value problems as we had found in the example above.

Now that we understand simple boundary value problems for ordinary differential equations, we can turn to initial-boundary value problems for partial differential equations. We will see that a common method for studying these problems is to use the method of separation of variables. In this method the problem of solving partial differential equations is to separate the partial differential equation into several ordinary differential equations of which some are boundary value problems of the sort seen in this section.

1.3 Derivation of Generic 1D Equations

1.3.1 Derivation of Wave Equation for String

THE WAVE EQUATION FOR A ONE DIMENSIONAL STRING is derived based upon simply looking at Newton's Second Law of Motion for a piece of the string plus a few simple assumptions, such as small amplitude oscillations and constant density.

We begin with $\mathbf{F} = m\mathbf{a}$. The mass of a piece of string of length Δs is $m = \rho(x)\,\Delta s$. From Figure (1.5) an incremental length of the string is given by

$$\Delta s^2 = \Delta x^2 + \Delta u^2.$$

The piece of string undergoes an acceleration of $a = \dfrac{\partial^2 u}{\partial t^2}$.

We will assume that the main force acting on the string is that of tension. Let $T(x,t)$ be the magnitude of the tension acting on the left end of the piece of string. Then, on the right end the tension is $T(x+\Delta x, t)$. At these points the tension makes an angle to the horizontal of $\theta(x,t)$ and $\theta(x+\Delta x, t)$, respectively.

The wave equation is derived from $\mathbf{F} = m\mathbf{a}$.

Assuming that there is no horizontal acceleration, the x-component in the second law, $m\mathbf{a} = \mathbf{F}$, for the string element is given by

$$0 = T(x+\Delta x, t)\cos\theta(x+\Delta x, t) - T(x,t)\cos\theta(x,t).$$

The vertical component is given by

$$\rho(x)\Delta s\frac{\partial^2 u}{\partial t^2} = T(x+\Delta x, t)\sin\theta(x+\Delta x, t) - T(x,t)\sin\theta(x,t)$$

The length of the piece of string can be written in terms of Δx,

$$\Delta s = \sqrt{\Delta x^2 + \Delta u^2} = \sqrt{1 + \left(\frac{\Delta u}{\Delta x}\right)^2}\,\Delta x.$$

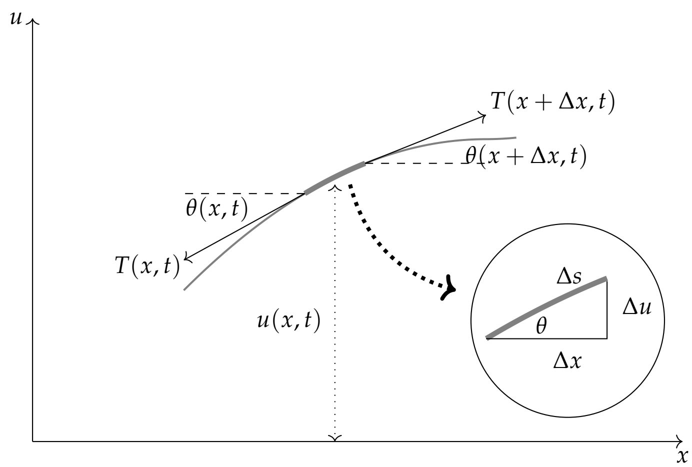


Figure 1.5: A small piece of string is under tension.

and the right hand sides of the component equation can be expanded about $\Delta x = 0$, to obtain

$$\begin{aligned} T(x+\Delta x, t)\cos\theta(x+\Delta x, t) - T(x,t)\cos\theta(x,t) &\approx \frac{\partial (T\cos\theta)}{\partial x}(x,t)\Delta x \\ T(x+\Delta x, t)\sin\theta(x+\Delta x, t) - T(x,t)\sin\theta(x,t) &\approx \frac{\partial (T\sin\theta)}{\partial x}(x,t)\Delta x. \end{aligned}$$

Furthermore, we note that

$$\tan\theta = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x}.$$

Now we can divide these component equations by Δx and let $\Delta x \to 0$. This gives the approximations

$$\begin{aligned} 0 &= \frac{T(x+\Delta x, t)\cos\theta(x+\Delta x, t) - T(x,t)\cos\theta(x,t)}{\Delta x} \\ &\approx \frac{\partial (T\cos\theta)}{\partial x}(x,t) \\ \rho(x)\frac{\partial^2 u}{\partial t^2}\frac{\Delta s}{\Delta x} &= \frac{T(x+\Delta x, t)\sin\theta(x+\Delta x, t) - T(x,t)\sin\theta(x,t)}{\Delta x} \\ \rho(x)\frac{\partial^2 u}{\partial t^2}\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2} &\approx \frac{\partial (T\sin\theta)}{\partial x}(x,t). \end{aligned} \tag{1.4}$$

We will assume a small angle approximation, giving

$$\sin\theta \approx \tan\theta = \frac{\partial u}{\partial x},$$

$\cos\theta \approx 1$, and

$$\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2} \approx 1.$$

Then, the horizontal component becomes

$$\frac{\partial T(x,t)}{\partial x} = 0.$$

Therefore, the magnitude of the tension $T(x,t) = T(t)$ is at most time dependent.

The vertical component equation is now

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T(t)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = T(t)\frac{\partial^2 u}{\partial x^2}.$$

Assuming that ρ and T are constant and defining

$$c^2 = \frac{T}{\rho},$$

we obtain the one dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$

Note the tension is a force and has units of mass times acceleration. The (linear) density is mass per length. So, the units of $\frac{T}{\rho}$ are acceleration time length, or $(length/time)^2$. Therefore, c has units of speed. Therefore, c is the wave speed.

1.3.2 Derivation of 1D Heat Equation

CONSIDER A ONE DIMENSIONAL ROD of length L as shown in Figure 1.6. It is heated and allowed to sit. The heat equation is the governing equation which allows us to determine the temperature of the rod at a later time.

We begin with some simple thermodynamics. Recall that to raise the temperature of a mass m by ΔT takes thermal energy given by

$$Q = mc\Delta T,$$

assuming the mass does not go through a phase transition (like melting). Here c is the specific heat capacity of the substance. So, we will begin with the heat content of the rod at position x and time t as

$$Q = mcT(x,t)$$

and assume that m and c are constant.

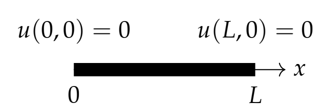


Figure 1.6: One dimensional heated rod of length L.

We will also need Fourier's law of heat transfer or heat conduction . This law simply states that heat energy flows from warmer to cooler regions and is written in terms of the heat energy flux, $\phi(x,t)$. The heat energy flux, or flux density, gives the rate of energy flow per area. Thus, the amount of heat energy flowing over the left end of the region of cross section A in time Δt is given $\phi(x,t)\Delta tA$. The units of $\phi(x,t)$ are then $\mathrm{J/s/m^2} = \mathrm{W/m^2}$.

Fourier's law of heat conduction states that the flux density is proportional to the gradient of the temperature,

$$\phi = -K\frac{\partial T}{\partial x}.$$

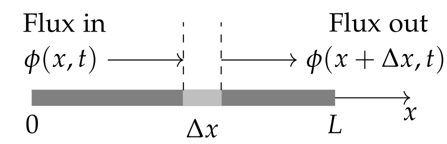


Figure 1.7: A one dimensional rod of length L. Heat can flow through increment Δx.

Here K is the thermal conductivity and the negative sign takes into account the direction of flow from higher to lower temperatures.

Now, we make use of the conservation of energy. Consider a small section of the rod of width Δx as shown in Figure 1.7. The rate of change of the energy through this section is due to energy flow through the ends. Namely,

$$\text{Rate of change of heat energy} = \text{Heat in} - \text{Heat out}.$$

The energy content of the small segment of the rod is given by

$$\Delta Q = (\rho A \Delta x) c T(x, t+\Delta t) - (\rho A \Delta x) c T(x,t).$$

The flow rates across the boundaries are given by the flux.

$$(\rho A \Delta x) c T(x, t+\Delta t) - (\rho A \Delta x) c T(x,t) = [\phi(x,t) - \phi(x+\Delta x, t)]\Delta t A.$$

Dividing by Δx and Δt and letting $\Delta x, \Delta t \to 0$, we obtain

$$\frac{\partial T}{\partial t} = -\frac{1}{c\rho}\frac{\partial \phi}{\partial x}.$$

Using Fourier's law of heat conduction,

$$\frac{\partial T}{\partial t} = \frac{1}{c\rho}\frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right).$$

Assuming K, c, and ρ are constant, we have the one dimensional heat equation as used in the text:

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2},$$

where $k = \dfrac{k}{c\rho}$.

1.4 *Separation of Variables*

SOLVING MANY OF THE LINEAR PARTIAL DIFFERENTIAL EQUATIONS presented in the first section can be reduced to solving ordinary differential equations. We will demonstrate this by solving the initial-boundary value problem for the heat equation as given in (1.1). We will employ a method typically used in studying linear partial differential equations, called the Method of Separation of Variables. In the next subsections we describe how this method works for the one-dimensional heat equation, one-dimensional wave equation, and the two-dimensional Laplace equation.

1.4.1 The 1D Heat Equation

We want to solve the heat equation,

$$u_t = ku_{xx}, \quad 0 < t, \quad 0 \leq x \leq L.$$

subject to the boundary conditions

$$u(0,t) = 0, u(L,t) = 0, \quad t > 0,$$

and the initial condition

$$u(x,0) = f(x), \quad 0 < x < L.$$

Solution of the 1D heat equation using the method of separation of variables.

We begin by assuming that u can be written as a product of single variable functions of each independent variable,

$$u(x,t) = X(x)T(t).$$

Substituting this guess into the heat equation, we find that

$$XT' = kX''T.$$

The prime denotes differentiation with respect to the independent variable and we will suppress the independent variable in the following unless needed for emphasis.

Dividing both sides of this result by k and $u = XT$, yields

$$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X}.$$

We have separated the functions of time on one side and space on the other side. The constant k could be on either side of this expression, but we moved it to make later computations simpler.

The only way that a function of t equals a function of x is if the functions are constant functions. Therefore, we set each function equal to a constant, λ : [For example, if $Ae^{ct} = ax^2 + b$ is possible for any x or t, then this is only possible if $a = 0$, $c = 0$ and $b = A$.]

$$\underbrace{\frac{1}{k}\frac{T'}{T}}_{\text{function of } t} = \underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{\lambda.}_{\text{constant}}$$

This leads to two equations:

$$T' = k\lambda T, \tag{1.5}$$

$$X'' = \lambda X. \tag{1.6}$$

These are ordinary differential equations. The general solutions to these constant coefficient equations are readily found as

$$T(t) = Ae^{k\lambda t}, \tag{1.7}$$

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}. \tag{1.8}$$

We need to be a little careful at this point. The aim is to force the final solutions to satisfy both the boundary conditions and initial conditions. Also, we should note that λ is arbitrary and may be positive, zero, or negative. We first look at how the boundary conditions on $u(x,t)$ lead to conditions on $X(x)$.

The first boundary condition is $u(0,t) = 0$. This implies that

$$X(0)T(t) = 0, \quad \text{for all } t.$$

The only way that this is true is if $X(0) = 0$. Similarly, $u(L,t) = 0$ for all t implies that $X(L) = 0$. So, we have to solve the boundary value problem

$$X'' - \lambda X = 0, \quad X(0) = 0, X(L) = 0. \tag{1.9}$$

An obvious solution is $X \equiv 0$. However, this implies that $u(x,t) = 0$, which is not an interesting solution. We call such solutions, $X \equiv 0$, trivial solutions and will seek nontrivial solution for these problems.

There are three cases to consider, depending on the sign of λ. [Note, eventually you need not consider all of the cases as you will inherently know which values of λ lead to nontrivial solutions.]

Case I. $\underline{\lambda > 0}$

In this case we have the exponential solutions

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}. \tag{1.10}$$

For $X(0) = 0$, we have

$$0 = c_1 + c_2.$$

We will take $c_2 = -c_1$. Then,

$$X(x) = c_1(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}) = 2c_1 \sinh \sqrt{\lambda}x.$$

Applying the second condition, $X(L) = 0$ yields

$$c_1 \sinh \sqrt{\lambda}L = 0.$$

This will be true only if $c_1 = 0$, since $\lambda > 0$ and $L > 0$. Thus, the only solution in this case is the trivial solution, $X(x) = 0$.

Case II. $\underline{\lambda = 0}$

For this case it is easier to set λ to zero in the differential equation. So, $X'' = 0$. Integrating twice, one finds

$$X(x) = c_1 x + c_2.$$

Setting $x = 0$, we have $c_2 = 0$, leaving $X(x) = c_1 x$. Setting $x = L$, we find $c_1 L = 0$. So, $c_1 = 0$ and we are once again left with a trivial solution.

Case III. $\underline{\lambda < 0}$
In this case is would be simpler to write $\lambda = -\mu^2$. Then the differential equation is

$$X'' + \mu^2 X = 0.$$

The general solution is

$$X(x) = c_1 \cos \mu x + c_2 \sin \mu x.$$

At $x = 0$ we get $0 = c_1$. This leaves $X(x) = c_2 \sin \mu x$.
At $x = L$, we find

$$0 = c_2 \sin \mu L.$$

So, either $c_2 = 0$ or $\sin \mu L = 0$. $c_2 = 0$ leads to a trivial solution again. But, there are cases when the sine is zero. Namely,

$$\mu L = n\pi, \quad n = 1, 2, \ldots.$$

Note that $n = 0$ is not included since this leads to a trivial solution. Also, negative values of n are redundant, since the sine function is an odd function.

In summary, we can find solutions to the boundary value problem (1.9) for particular values of λ. The solutions are

$$X_n(x) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

for

$$\lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$

We should note that the boundary value problem in Equation (1.9) is an eigenvalue problem. We can recast the differential equation as

Eigenvalue Problem

$$LX = \lambda X,$$

where

$$L = D^2 = \frac{d^2}{dx^2}$$

is a linear differential operator. The solutions, $X_n(x)$, are called eigenfunctions and the λ_n's are the eigenvalues. We will elaborate more on this characterization later in the next chapter.

We have found the product solutions of the heat equation (1.1) satisfying the boundary conditions. These are

Product solutions.

$$u_n(x, t) = e^{k\lambda_n t} \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots. \tag{1.11}$$

However, these do not necessarily satisfy the initial condition $u(x, 0) = f(x)$. What we do get is

$$u_n(x, 0) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots.$$

So, if the initial condition is in one of these forms, we can pick out the right value for n and we are done.

For other initial conditions, we have to do more work. Note, since the heat equation is linear, the linear combination of the product solutions is also a solution of the heat equation. The general solution satisfying the given boundary conditions is given as

General solution.

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \sin \frac{n\pi x}{L}. \tag{1.12}$$

The coefficients in the general solution are determined using the initial condition. Namely, setting $t = 0$ in the general solution, we have

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

So, if we know $f(x)$, can we find the coefficients, b_n? If we can, then we will have the solution to the full initial-boundary value problem.

The expression for $f(x)$ is a Fourier sine series. We will need to digress into the study of Fourier series in order to see how one can find the Fourier series coefficients given $f(x)$. The solution is provided in Section 2.5. Before proceeding, we will show that this process is not uncommon by applying the Method of Separation of Variables to the wave equation in the next section.

1.4.2 *The 1D Wave Equation*

In this section we will apply the Method of Separation of Variables to the one dimensional wave equation, given by

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial^2 x}, \quad t > 0, \quad 0 \le x \le L, \tag{1.13}$$

subject to the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0,$$

and the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L.$$

This problem applies to the propagation of waves on a string of length L with both ends fixed so that they do not move. $u(x,t)$ represents the vertical displacement of the string over time. The derivation of the wave equation in Section 1.3.1 assumed that a small vertical displacement and a uniform string. The constant c is the wave speed, given by

$$c = \sqrt{\frac{\tau}{\mu}},$$

where τ is the tension in the string and μ is the mass per unit length. We can understand this in terms of string instruments. The tension can be adjusted to produce different tones and the makeup of the string (nylon or steel, thick or thin) also has an effect. In some cases the mass density is changed simply by using thicker strings. Thus, the thicker strings in a piano produce lower frequency notes.

The u_{tt} term gives the acceleration of a piece of the string. The u_{xx} is the concavity of the string. Thus, for a positive concavity the string is curved upward near the point of interest. Thus, neighboring points tend to pull the string upward towards the equilibrium position. If the concavity is negative, it would cause a negative acceleration.

Solution of the 1D wave equation using the Method of Separation of Variables.

The solution of this problem is easily found using separation of variables. We let $u(x,t) = X(x)T(t)$. Then we find

$$XT'' = c^2 X'' T,$$

which can be rewritten as

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X}.$$

Again, we have separated the functions of time on one side and space on the other side. Therefore, we set each function equal to a constant, λ.

$$\underbrace{\frac{1}{c^2}\frac{T''}{T}}_{\text{function of } t} = \underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{\lambda.}_{\text{constant}}$$

This leads to two equations:

$$T'' = c^2 \lambda T, \tag{1.14}$$

$$X'' = \lambda X. \tag{1.15}$$

As before, we have the boundary conditions on $X(x)$:

$$X(0) = 0, \quad \text{and} \quad X(L) = 0,$$

giving the solutions, as shown in Figure 1.8,

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = -\left(\frac{n\pi}{L}\right)^2.$$

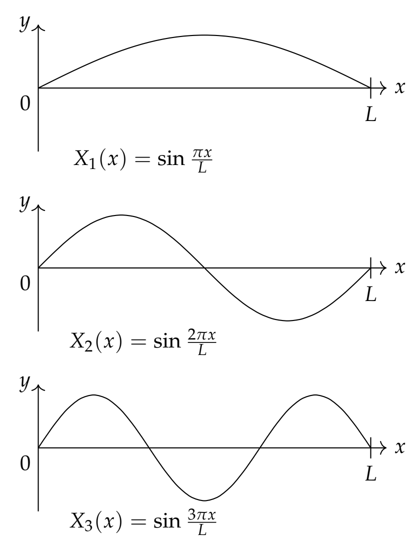


Figure 1.8: The first three harmonics, eigenfunctions with $n = 1, 2, 3$, of the vibrating string.

The main difference from the solution of the heat equation is the form of the time function. Namely, from Equation (1.14) we have to solve

$$T'' + \left(\frac{n\pi c}{L}\right)^2 T = 0. \tag{1.16}$$

This equation takes a familiar form. We let

$$\omega_n = \frac{n\pi c}{L},$$

then we have

$$T'' + \omega_n^2 T = 0.$$

This is the differential equation for simple harmonic motion and ω_n is the angular frequency. The solutions are easily found as

$$T(t) = A_n \cos\omega_n t + B_n \sin\omega_n t. \tag{1.17}$$

Therefore, we have found that the product solutions of the wave equation take the forms $\sin \frac{n\pi x}{L} \cos \omega_n t$ and $\sin \frac{n\pi x}{L} \sin \omega_n t$. The general solution, a superposition of all product solutions, is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right] \sin \frac{n\pi x}{L}. \tag{1.18}$$

General solution.

This solution satisfies the wave equation and the boundary conditions. We still need to satisfy the initial conditions. Note that there are two initial conditions, since the wave equation is second order in time.

First, we have $u(x,0) = f(x)$. Thus,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}. \tag{1.19}$$

In order to obtain the condition on the initial velocity, $u_t(x,0) = g(x)$, we need to differentiate the general solution with respect to t:

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} \left[-A_n \sin \frac{n\pi ct}{L} + B_n \cos \frac{n\pi ct}{L} \right] \sin \frac{n\pi x}{L}. \tag{1.20}$$

Then, we have from the initial velocity

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin \frac{n\pi x}{L}. \tag{1.21}$$

So, applying the two initial conditions, we have found that $f(x)$ and $g(x)$, are represented as Fourier sine series. In order to complete the problem we need to determine the coefficients A_n and B_n for $n = 1, 2, 3, \ldots$. Once we have these, we have the complete solution to the wave equation. This is done in Section 2.6.

We had seen similar results for the heat equation. In the next chapter we will find out how to determine these Fourier coefficients for such series of sinusoidal functions. However, we can use other techniques to solve the wave equation which we will explore next.

1.5 d'Alembert's Solution of the Wave Equation

A GENERAL SOLUTION OF THE ONE-DIMENSIONAL WAVE EQUATION can be found. This solution was first found by Jean-Baptiste le Rond d'Alembert (1717-1783) and is referred to as d'Alembert's formula. In this section we will derive d'Alembert's formula and then use it to arrive at solutions to the wave equation on infinite, semi-infinite, and finite intervals.

We consider the wave equation in the form $u_{tt} = c^2 u_{xx}$ and introduce the transformation

$$u(x,t) = U(\xi, \eta), \quad \text{where} \quad \xi = x + ct \text{ and } \quad \eta = x - ct.$$

We will see that ξ and η are the characteristics of the wave equation.

In order to transform the wave equation into an equation in the new variables, we need to see how the derivatives transform. For example, we apply a multivariable chain rule,

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial U(\xi,\eta)}{\partial x} \\ &= \frac{\partial U(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial U(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial x} \\ &= \frac{\partial U(\xi,\eta)}{\partial \xi} + \frac{\partial U(\xi,\eta)}{\partial \eta}. \end{aligned} \tag{1.22}$$

Therefore, as an operator, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}.$$

Similarly, one can show that

$$\frac{\partial}{\partial t} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}.$$

Using these results, the wave equation becomes

$$\begin{aligned} 0 &= u_{tt} - c^2 u_{xx} \\ &= \left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) u \\ &= \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u \\ &= \left(c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta} + c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right)\left(c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta} - c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}\right) U \\ &= -4c^2\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} U. \end{aligned} \tag{1.23}$$

Therefore, the wave equation has transformed into the simpler equation,

$$U_{\eta\xi} = 0.$$

Not only is this simpler, but as we will see in the next section, the wave equation is a hyperbolic equation. Of course, it is also easy to integrate. Since

$$\frac{\partial}{\partial \eta}\left(\frac{\partial U}{\partial \xi}\right) = 0,$$

we find that

$$\frac{\partial U}{\partial \xi} = \text{ constant with respect to } \eta = \Gamma(\xi).$$

A further integration gives

$$U(\xi,\eta) = \int^{\eta} \Gamma(\xi')\, d\xi' + G(\eta) \equiv F(\xi) + G(\eta).$$

Therefore, we have as the general solution of the wave equation,

$$u(x,t) = F(x+ct) + G(x-ct), \tag{1.24}$$

where F and G are two arbitrary, twice differentiable functions. As t is increased, we see that $F(x+ct)$ gets horizontally shifted to the left and $G(x-ct)$ gets horizontally shifted to the right. As a result, we conclude that the solution of the wave equation can be seen as the sum of left and right traveling waves.

$u(x,t)$ = sum of left and right traveling waves.

Let's use initial conditions to solve for the unknown functions. We let

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad |x| < \infty.$$

Applying this to the general solution, we have

$$f(x) = F(x) + G(x) \tag{1.25}$$
$$g(x) = c[F'(x) - G'(x)]. \tag{1.26}$$

We need to solve for $F(x)$ and $G(x)$ in terms of $f(x)$ and $g(x)$. Integrating Equation (1.26), we have

$$\frac{1}{c}\int_0^x g(s)\,ds = F(x) - G(x) - F(0) + G(0).$$

Adding this result to Equation (1.25), and solving for $F(x)$ gives

$$F(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\,ds + \frac{1}{2}[F(0) - G(0)].$$

Subtracting from Equation (1.25) solving for $G(x)$, gives

$$G(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds - \frac{1}{2}[F(0) - G(0)].$$

Now we can write out the solution $u(x,t) = F(x+ct) + G(x-ct)$, yielding d'Alembert's solution

d'Alembert's solution

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds. \tag{1.27}$$

When $f(x)$ and $g(x)$ are defined for all $x \in R$, the solution is well-defined. However, there are problems on more restricted domains. In the next examples we will consider the semi-infinite and finite length string problems.

For each example we will need to consider the domain of dependence and the domain of influence of specific points. These concepts are shown in Figure 1.9. The domain of dependence of point P is red region. The point P depends on the values of u and u_t at points inside the domain. The domain of influence of P is the blue region. The points in the region are influenced by the values of u and u_t at P.

Example 1.3. Use d'Alembert's solution to solve

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \leq x < \infty.$$

The d'Alembert solution is not well-defined for this problem because $f(x-ct)$ is not defined for $x - ct < 0$ for $c, t > 0$. There are

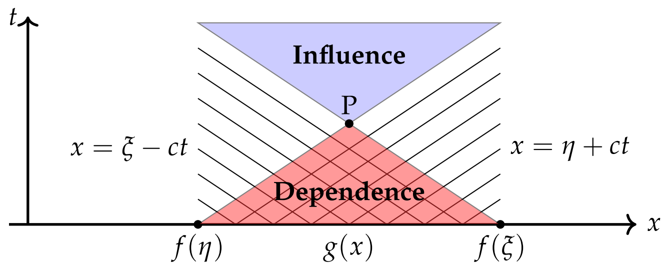


Figure 1.9: The domain of dependence of point P is red region. The point P depends on the values of u and u_t at points inside the domain. The domain of influence of P is the blue region. The points in the region are influenced by the values of u and u_t at P.

similar problems for $g(x)$. This can be seen by looking at the characteristics in the xt-plane. In Figure 1.10 there are characteristics emanating from the points marked by η_0 and ξ_0 that intersect in the domain $x > 0$. The point of intersection of the blue lines have a domain of dependence entirely in the region $x, t > 0$, however the domain of dependence of point P reaches outside this region. Only characteristics $\xi = x + ct$ reach point P, but characteristics $\eta = x - ct$ do not. But, we need $f(\eta)$ and $g(x)$ for $x < ct$ to form a solution.

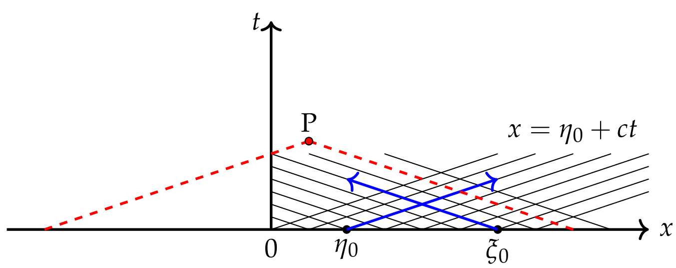


Figure 1.10: The characteristics for the semi-infinite string indicating no characteristics of the form $\eta = x - ct$ reach points like P.

This can be remedied if we specified boundary conditions at $x = 0$. For example, we will assume the end $x = 0$ is fixed,

Fixed end boundary condition

$$u(0, t) = 0, \quad t \geq 0.$$

Imagine an infinite string with one end (at $x = 0$) tied to a pole. In Figure 1.11 we see how this added information along the time axis provides information that can be propagated to points in the first quadrant of the xt-plane.

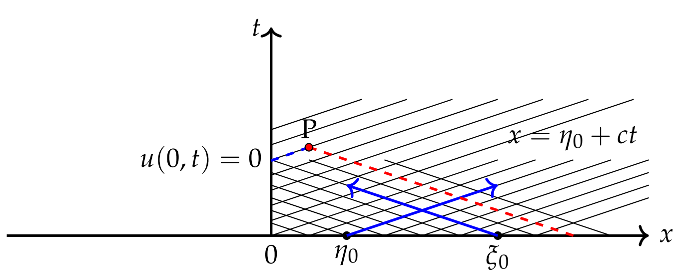


Figure 1.11: The characteristics for the semi-infinite string indicating the boundary condition $u(0, t) = 0$ and the new characteristics starting with $x = 0$.

Let's see how this can be used to construct the solution analytically. Since $u(x, t) = F(x + ct) + G(x - ct)$, we have

$$u(0, t) = F(ct) + G(-ct) = 0.$$

Letting $\zeta = -ct$, this gives $G(\zeta) = -F(-\zeta)$, $\zeta \leq 0$.

Note that

$$\begin{aligned} G(\zeta) &= \frac{1}{2}f(\zeta) - \frac{1}{2c}\int_0^{\zeta} g(s)\,ds \\ -F(-\zeta) &= -\frac{1}{2}f(-\zeta) - \frac{1}{2c}\int_0^{-\zeta} g(s)\,ds \\ &= -\frac{1}{2}f(-\zeta) + \frac{1}{2c}\int_0^{\zeta} g(\sigma)\,d\sigma. \end{aligned} \tag{1.28}$$

Comparing the expressions for $G(\zeta)$ and $-F(-\zeta)$, we see that

$$f(\zeta) = -f(-\zeta), \quad g(\zeta) = -g(-\zeta).$$

These relations imply that we can extend the functions into the region $x < 0$ if we make them odd functions, or what are called odd extensions. An example is shown in Figure 1.12.

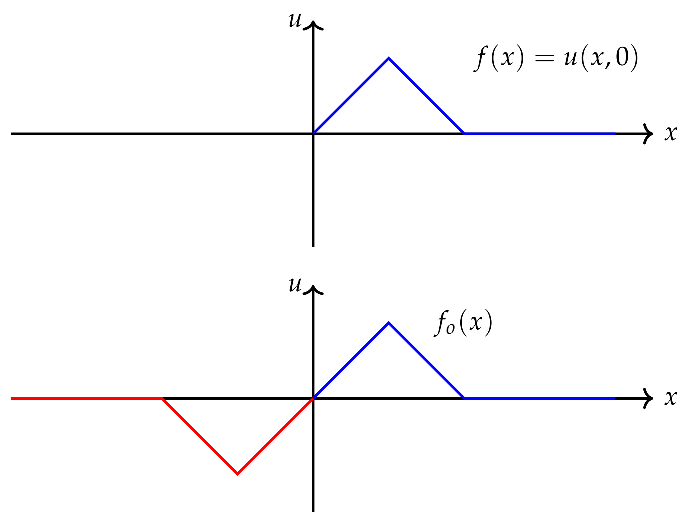


Figure 1.12: The initial condition and its odd extension. The odd extension is obtained through reflection of $f(x)$ about the origin.

Free end boundary condition

Another type of boundary condition is if the end $x = 0$ is free,

$$u_x(0,t) = 0, \quad t \geq 0.$$

In this case we could have an infinite string tied to a ring and that ring is allowed to slide freely up and down a pole.

As a homeowrk exercise you can prove that this leads to

$$f(-\zeta) = f(\zeta), \quad g(-\zeta) = g(\zeta).$$

Thus, we can use an even extension of these function to produce solutions.

Example 1.4. Solve the initial-boundary value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 \leq x < \infty, t > 0. \\ u(x,0) &= \begin{cases} x, & 0 \leq x \leq 1, \\ 2-x, & 1 \leq x \leq 2, \\ 0, & x > 2, \end{cases} \quad 0 \leq x < \infty \end{aligned}$$

$$\begin{aligned} u_t(x,0) &= 0, \quad 0 \le x < \infty. \\ u(0,t) &= 0, \quad t > 0. \end{aligned} \tag{1.29}$$

This is a semi-infinite string with a fixed end. Initially it is plucked to produce a nonzero triangular profile for $0 \le x \le 2$. Since the initial velocity is zero, the general solution is found from d'Alembert's solution,

$$u(x,t) = \frac{1}{2}[f_o(x+ct) + f_o(x-ct)],$$

where $f_o(x)$ is the odd extension of $f(x) = u(x,0)$. In Figure 1.12 we show the initial condition and its odd extension. The odd extension is obtained through reflection of $f(x)$ about the origin.

The next step is to look at the horizontal shifts of $f_o(x)$. Several examples are shown in Figure 1.13.These show the left and right traveling waves.

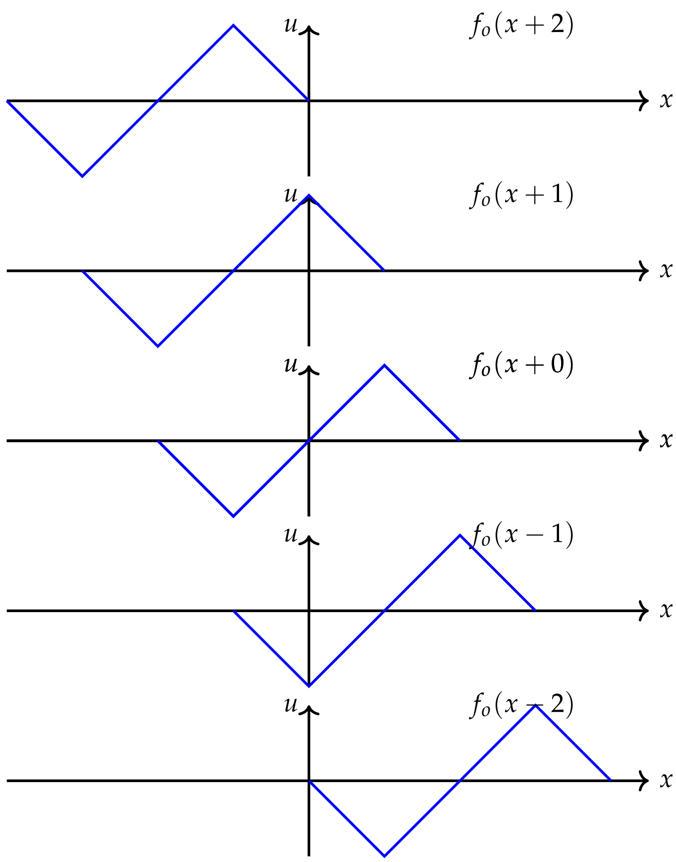
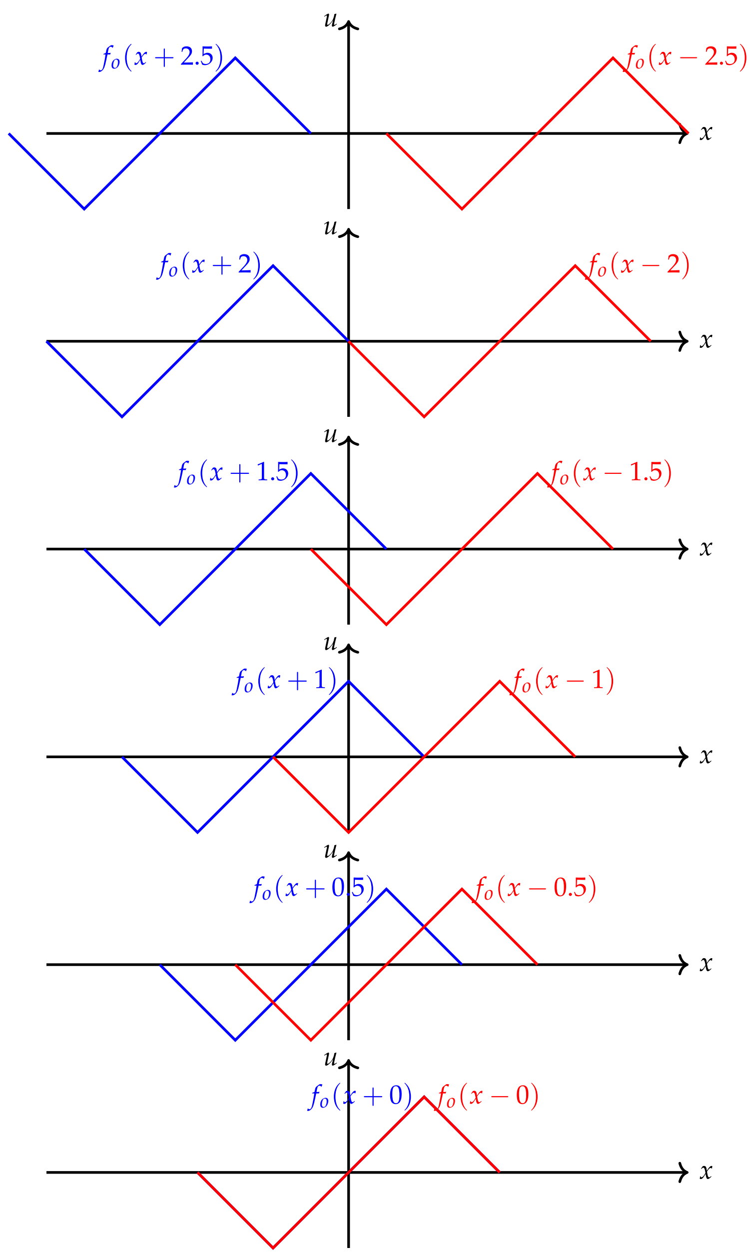


Figure 1.13: Examples of $f_o(x+ct)$ and $f_o(x-ct)$. We see that $f(x-1)$ and $f(x-2)$ have moved to the right, while $f(x+1)$ and $f(x+2)$ have moved to the left.

In Figure 1.14 we show superimposed plots of $f_o(x+ct)$ and $f_o(x-ct)$ for given times. The initial profile is at the bottom. By the time we have $ct = 2$, the full traveling wave has emerged. The solution to the problem emerges on the right side of Figure 1.23 by averaging each plot.

Figure 1.14: Superimposed plots of $f_o(x+ct)$ and $f_o(x-ct)$ for given times. The initial profile in at the bottom. By the time $ct = 2$ the full traveling wave has emerged.



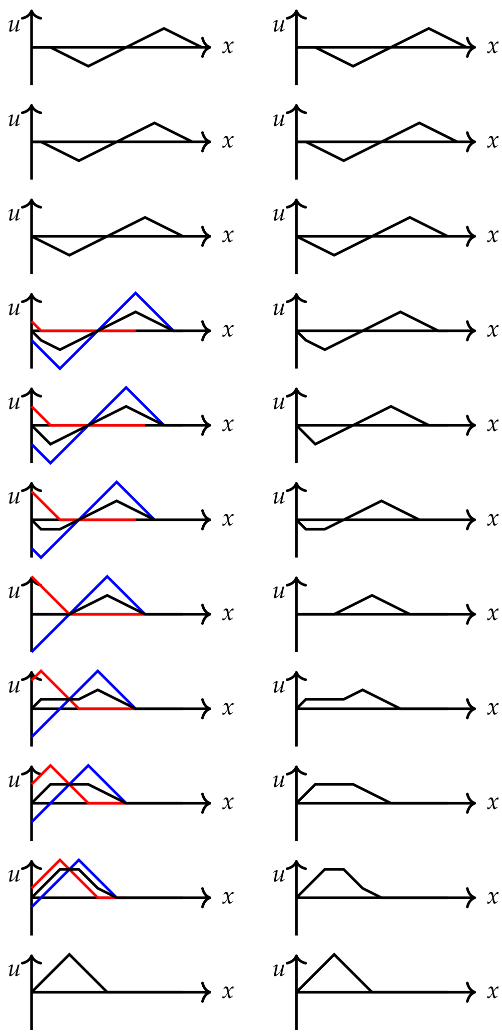


Figure 1.15: On the left is a plot of $f(x+ct)$, $f(x-ct)$ from Figure 1.14 and the average, $u(x,t)$. On the right the solution alone is shown for $ct = 0$ at bottom to $ct = 1$ at top for the semi-infinite string problem

Example 1.5. Use d'Alembert's solution to solve

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le \ell.$$

The general solution of the wave equation was found in the form

$$u(x,t) = F(x+ct) + G(x-ct).$$

However, for this problem we can only obtain information for values of x and t such that $0 \le x+ct \le \ell$ and $0 \le x-ct \le \ell$. In Figure 1.16 the characteristics $x = \xi + ct$ and $x = \eta - ct$ for $0 \le \xi, \eta \le \ell$ are shown. The main (gray) triangle, which is the domain of dependence of the point $(\ell, 2, \ell/2c)$, is the only region in which the solution can be found based solely on the initial conditions. As with the previous problem, boundary conditions will need to be given in order to extend the domain of the solution.

In the last example we saw that a fixed boundary at $x = 0$ could be satisfied when $f(x)$ and $g(x)$ are extended as odd functions. In Figure 1.17 we indicate how the characteristics are affected by drawing in the new ones as red dashed lines. This allows us to now construct solutions based on the initial conditions under the line $x = \ell - ct$ for $0 \le x \le \ell$. The new region for which we can construct solutions from the initial conditions is indicated in gray in Figure 1.17.

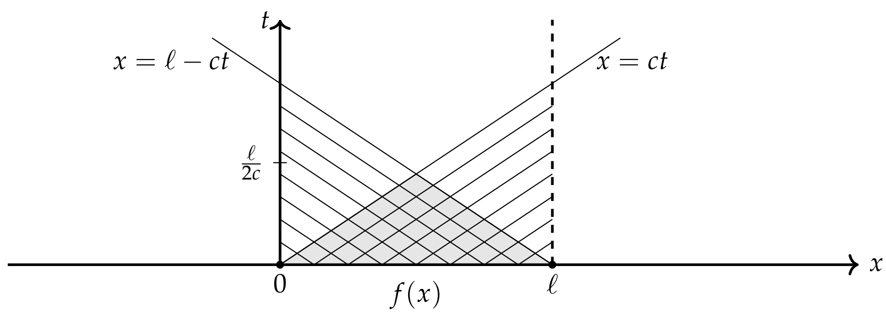


Figure 1.16: The characteristics emanating from the interval $0 \le x \le \ell$ for the finite string problem.

We can add characteristics on the right by adding a boundary condition at $x = \ell$. Again, we could use fixed $u(\ell, t) = 0$, or free, $u_x(\ell, t) = 0$, boundary conditions. This allows us to now construct solutions based on the initial conditions for $\ell \le x \le 2\ell$.

Let's consider a fixed boundary condition at $x = \ell$. Then, the solution must satisfy

$$u(\ell, t) = F(\ell + ct) + G(\ell - ct) = 0.$$

To see what this means, let $\zeta = \ell + ct$. Then, this condition becomes (since $ct = \zeta - \ell$)

$$F(\zeta) = -G(2\ell - \zeta), \quad \ell \le \zeta \le 2\ell.$$

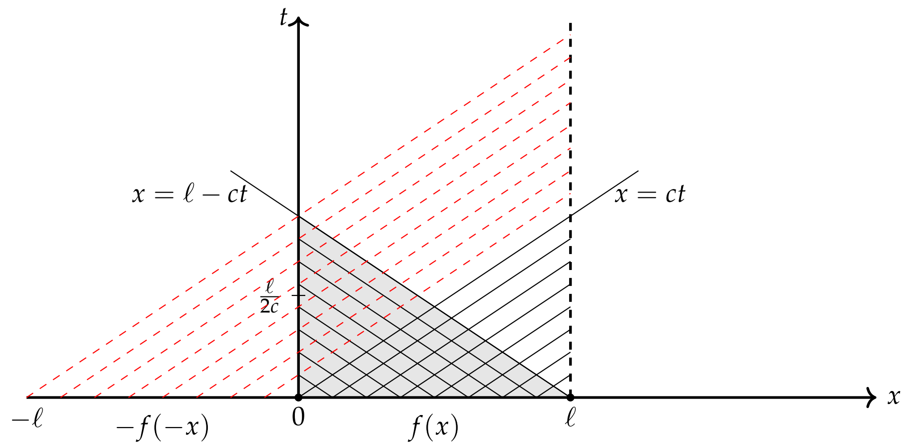


Figure 1.17: The red dashed lines are the characteristics from the interval $[-\ell, 0]$ from using the odd extension about $x = 0$.

Observe that $G(2\ell - \zeta)$ is defined for $0 \leq 2\ell - \zeta \leq \ell$. Therefore, this is a well-defined extension of the domain of $F(x)$.

Using the expressions for $F(x)$ and $G(x)$ in terms of $f(x)$ and $g(x)$, we have

$$\begin{aligned} F(\zeta) &= \frac{1}{2}f(\zeta) + \frac{1}{2c}\int_0^\ell g(s)\,ds. \\ -G(2\ell - \zeta) &= -\frac{1}{2}f(2\ell - \zeta) + \frac{1}{2c}\int_0^{2\ell-\zeta} g(s)\,ds \\ &= -\frac{1}{2}f(2\ell - \zeta) - \frac{1}{2c}\int_0^{\zeta} g(2\ell - \sigma)\,d\sigma \end{aligned} \tag{1.30}$$

Comparing the expressions for $G(\zeta)$ and $-G(2\ell - \zeta)$, we see that

$$f(\zeta) = -f(2\ell - \zeta), \quad g(\zeta) = -g(2\ell - \zeta).$$

These relations imply that we can extend the functions into the region $x > \ell$ if we consider an odd extension of $f(x)$ and $g(x)$ about $x = \ell$. This will give the blue dashed characteristics in Figure 1.18 and a larger gray region to construct the solution.

So far we have extended $f(x)$ and $g(x)$ to the interval $-\ell \leq x \leq 2\ell$ in order to determine the solution over a larger xt-domain. For example, the function $f(x)$ has been extended to

$$f_{ext}(x) = \begin{cases} -f(-x), & -\ell < x < 0, \\ f(x), & 0 < x < \ell, \\ -f(2\ell - x), & \ell < x < 2\ell. \end{cases}$$

A similar extension is needed for $g(x)$. Inserting these extended functions into d'Alembert's solution, we can determine $u(x,t)$ in the region indicated in Figure 1.18.

Even though the original region has been expanded, we have not determined how to find the solution throughout the entire strip, $[0, \ell] \times [0, \infty)$.

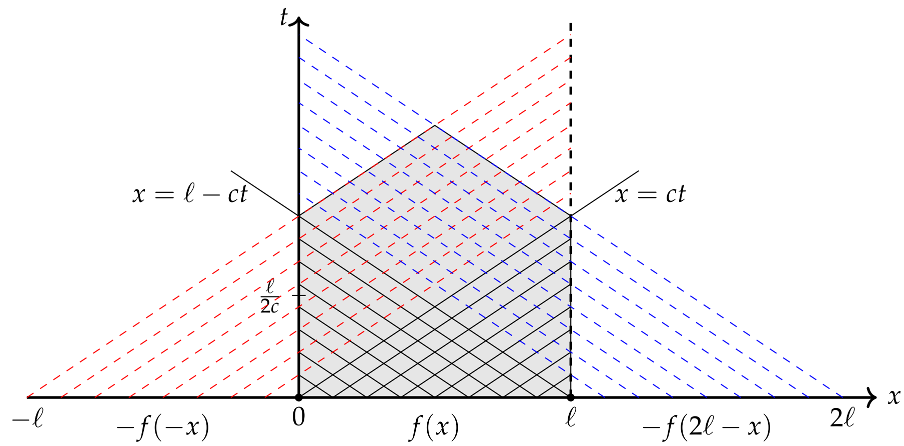


Figure 1.18: The red dashed lines are the characteristics from the interval $[-\ell, 0]$ from using the odd extension about $x = 0$ and the blue dashed lines are the characteristics from the interval $[\ell, 2\ell]$ from using the odd extension about $x = \ell$.

This is accomplished by periodically repeating these extended functions with period 2ℓ. This can be shown from the two conditions

$$\begin{aligned} f(x) &= -f(-x), \quad -\ell \leq x \leq 0, \\ f(x) &= -f(2\ell - x), \quad \ell \leq x \leq 2\ell. \end{aligned} \tag{1.31}$$

Now, consider

$$\begin{aligned} f(x+2\ell) &= -f(2\ell - (x-2\ell)) \\ &= -f(-x) \\ &= f(x). \end{aligned} \tag{1.32}$$

This shows that $f(x)$ is periodic with period 2ℓ. Since $g(x)$ satisfies the same conditions, then it is as well.

In Figure 1.19 we show how the characteristics are extended throughout the domain strip using the periodicity of the extended initial conditions. The characteristics from the interval endpoints zig zag throughout the domain, filling it up. In the next example we show how to construct the odd periodic extension of a specific function.

Example 1.6. Construct the odd periodic extension of the plucked string initial profile given by

$$f(x) = \begin{cases} x, & 0 \leq x \leq \frac{\ell}{2}, \\ \ell - x, & \frac{\ell}{2} \leq x \leq \ell, \end{cases}$$

satisfying fixed boundary conditions at $x = 0$ and $x = \ell$.

We first take the solution and add the odd extension about $x = 0$. Then we add an extension beyond $x = \ell$. This process is shown in Figure 1.20.

We can use the odd periodic function to construct solutions. In this case we use the result from the last example for obtaining the solution of the problem in which the initial velocity is zero, $u(x,t) = \frac{1}{2}[f(x+ct) + f(x -$

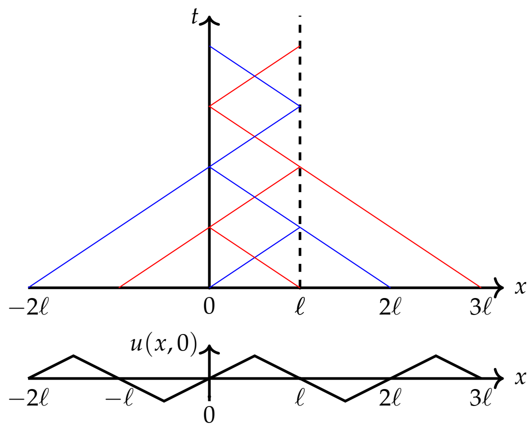


Figure 1.19: Extending the characteristics throughout the domain strip.

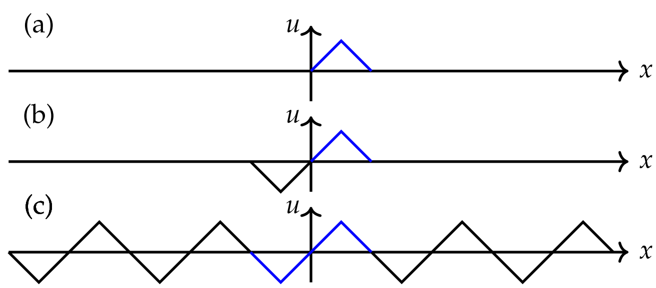


Figure 1.20: Construction of odd periodic extension for (a) The initial profile, $f(x)$. (b) Make $f(x)$ an odd function on $[-\ell, \ell]$. (c) Make the odd function periodic with period 2ℓ.

$ct)]$. Translations of the odd periodic extension are shown in Figure 1.21. In Figure 1.22 we show superimposed plots of $f(x+ct)$ and $f(x-ct)$ for different values of ct. A box is shown inside which the physical wave can be constructed. The solution is an average of these odd periodic extensions within this box. This is displayed in Figure 1.23.

1.6 Classification of Second Order PDEs

We have studied several examples of partial differential equations, the heat equation, the wave equation, and Laplace's equation. These equations are examples of parabolic, hyperbolic, and elliptic equations, respectively. Given a general second order linear partial differential equation, how can we tell what type it is? This is known as the classification of second order PDEs.

Let $u = u(x,y)$. Then, the general form of a linear second order partial differential equation is given by

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y). \tag{1.33}$$

In this section we will show that this equation can be transformed into one of three types of second order partial differential equations.

Let $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$ be an invertible transformation from

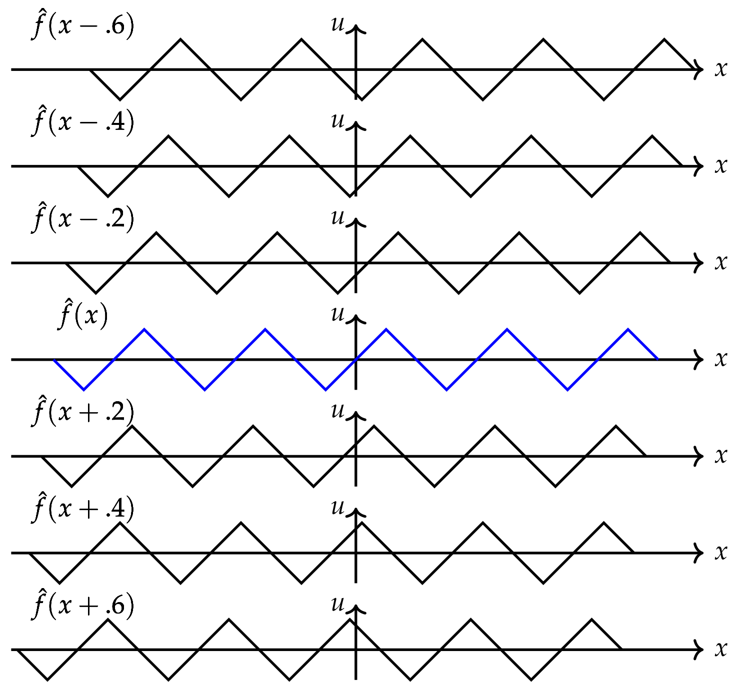


Figure 1.21: Translations of the odd periodic extension.

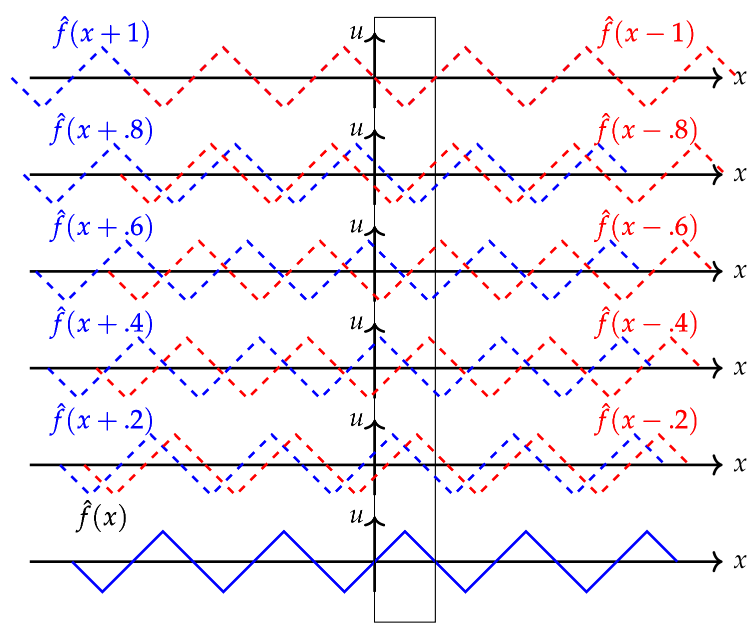


Figure 1.22: Superimposed translations of the odd periodic extension.

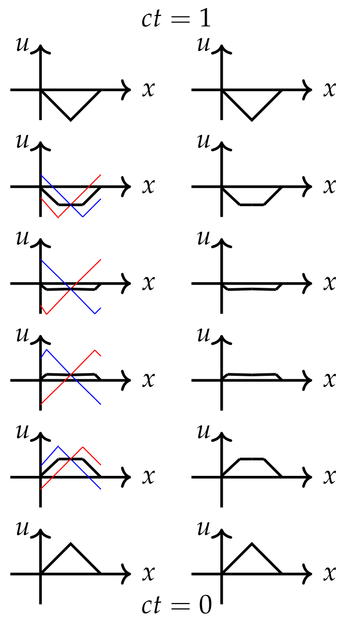


Figure 1.23: On the left is a plot of $f(x+ct)$, $f(x-ct)$ from Figure 1.22 and the average, $u(x,t)$. On the right the solution alone is shown for $ct = 0$ to $ct = 1$.

coordinates (ξ, η) to coordinates (x, y). Furthermore, let

$$u(x(\xi,\eta), y(\xi,\eta)) = U(\xi,\eta).$$

How does the partial differential equation (1.33) transform?

We first need to transform the derivatives of $u(x,t)$. We have

$$\begin{aligned}
u_x &= U_\xi \xi_x + U_\eta \eta_x, \\
u_y &= U_\xi \xi_y + U_\eta \eta_y, \\
u_{xx} &= \frac{\partial}{\partial x}(U_\xi \xi_x + U_\eta \eta_x), \\
&= U_{\xi\xi}\xi_x^2 + 2U_{\xi\eta}\xi_x\eta_x + U_{\eta\eta}\eta_x^2 + U_\xi \xi_{xx} + U_\eta \eta_{xx}, \\
u_{yy} &= \frac{\partial}{\partial y}(U_\xi \xi_y + U_\eta \eta_y), \\
&= U_{\xi\xi}\xi_y^2 + 2U_{\xi\eta}\xi_y\eta_y + U_{\eta\eta}\eta_y^2 + U_\xi \xi_{yy} + U_\eta \eta_{yy}, \\
u_{xy} &= \frac{\partial}{\partial y}(U_\xi \xi_x + U_\eta \eta_x), \\
&= U_{\xi\xi}\xi_x\xi_y + U_{\xi\eta}\xi_x\eta_y + U_{\xi\eta}\xi_y\eta_x + U_{\eta\eta}\eta_x\eta_y + U_\xi \xi_{xy} + U_\eta \eta_{xy}.
\end{aligned} \tag{1.34}$$

Inserting these derivatives into Equation (1.33), we have

$$\begin{aligned}
g - fU &= au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y \\
&= a\left(U_{\xi\xi}\xi_x^2 + 2U_{\xi\eta}\xi_x\eta_x + U_{\eta\eta}\eta_x^2 + U_\xi \xi_{xx} + U_\eta \eta_{xx}\right) \\
&\quad +2b\left(U_{\xi\xi}\xi_x\xi_y + U_{\xi\eta}\xi_x\eta_y + U_{\xi\eta}\xi_y\eta_x \right. \\
&\quad \left. + U_{\eta\eta}\eta_x\eta_y + U_\xi \xi_{xy} + U_\eta \eta_{xy}\right) \\
&\quad +c\left(U_{\xi\xi}\xi_y^2 + 2U_{\xi\eta}\xi_y\eta_y + U_{\eta\eta}\eta_y^2 + U_\xi \xi_{yy} + U_\eta \eta_{yy}\right)
\end{aligned}$$

$$
\begin{aligned}
& +d\left(U_\xi \xi_x + U_\eta \eta_x\right) \\
& +e\left(U_\xi \xi_y + U_\eta \eta_y\right) \\
= \quad & (a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2)U_{\xi\xi} \\
& +(2a\xi_x\eta_x + 2b\xi_x\eta_y + 2b\xi_y\eta_x + 2c\xi_y\eta_y)U_{\xi\eta} \\
& +(a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2)U_{\eta\eta} \\
& +(a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y)U_\xi \\
& +(a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y)U_\eta \\
= \quad & AU_{\xi\xi} + 2BU_{\xi\eta} + CU_{\eta\eta} + DU_\xi + EU_\eta. \qquad (1.35)
\end{aligned}
$$

Picking the right transformation, we can eliminate some of the second order derivative terms depending on the type of differential equation. These choices can lead to one of three types of second order partial differential equations: elliptic, hyperbolic, or parabolic. We'll assume that at least one of the coefficients, A, B, C, is not zero.

For example, if transformations can be found to make $A \equiv 0$ and $C \equiv 0$, then the equation reduces to

$$U_{\xi\eta} = \text{ lower order terms.}$$

Hyperbolic case.

Such an equation is called hyperbolic. A generic example of a hyperbolic equation is the wave equation.

The conditions that $A \equiv 0$ and $C \equiv 0$ give the conditions

$$
\begin{aligned}
a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 &= 0. \\
a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 &= 0. \qquad (1.36)
\end{aligned}
$$

We seek ξ and η satisfying these two equations, which are of the same form. Let's assume that $\xi = \xi(x, y)$ is a constant curve in the xy-plane. Furthermore, if this curve is the graph of a function, $y = y(x)$, then $\xi = \xi(x, y(x))$ and

$$\frac{d\xi}{dx} = \xi_x + \frac{dy}{dx}\xi_y = 0.$$

Then, we have

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}.$$

Inserting this expression in the equation for $A = 0$, we find

$$
\begin{aligned}
A &= a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 \\
&= \xi_y^2\left(a\left(\frac{\xi_x}{\xi_y}\right)^2 + 2b\frac{\xi_x}{\xi_y} + c\right) \\
&= \xi_y^2\left(a\left(\frac{dy}{dx}\right)^2 - 2b\frac{dy}{dx} + c\right) = 0. \qquad (1.37)
\end{aligned}
$$

Characteristic curves.

This equation is satisfied if $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}.$$

So, for $A = 0$, we choose ξ and η to be constant on these characteristic curves.

Example 1.7. Show that $u_{xx} - u_{yy} = 0$ is hyperbolic.

In this case we have $a = 1 = -c$ and $b = 0$. Then,

$$\frac{dy}{dx} = \pm 1.$$

This gives $y(x) = \pm x + c$. So, we choose ξ and η constant on these characteristic curves. Therefore, we let $\xi = x - y$, $\eta = x + y$.

Let's see if this transformation transforms the differential equation into a canonical form. Let $u(x, y) = U(\xi, \eta)$. Then, the needed derivatives become

$$\begin{aligned}
u_x &= U_\xi \xi_x + U_\eta \eta_x = U_\xi + U_\eta. \\
u_y &= U_\xi \xi_y + U_\eta \eta_y = -U_\xi + U_\eta. \\
u_{xx} &= \frac{\partial}{\partial x}(U_\xi + U_\eta) \\
&= U_{\xi\xi}\xi_x + U_{\xi\eta}\eta_x + U_{\eta\xi}\xi_x + U_{\eta\eta}\eta_x \\
&= U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}. \\
u_{yy} &= \frac{\partial}{\partial y}(-U_\xi + U_\eta) \\
&= -U_{\xi\xi}\xi_y - U_{\xi\eta}\eta_y + U_{\eta\xi}\xi_y + U_{\eta\eta}\eta_y \\
&= U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}. \qquad (1.38)
\end{aligned}$$

Inserting these derivatives into the differential equation, we have

$$0 = u_{xx} - u_{yy} = 4U_{\xi\eta}.$$

Thus, the transformed equation is $U_{\xi\eta} = 0$, showing it is a hyperbolic equation. In fact, we have already seen this form of equation when solving the wave equation in Section 1.5.

We have seen that A and C vanish for $\xi(x, y)$ and $\eta(x, y)$ constant along the characteristics

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}$$

for second order hyperbolic equations. This is possible when $b^2 - ac > 0$ since this leads to two characteristics.

In general, if we consider the second order operator

$$L[u] = a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy},$$

then this operator can be transformed to the new form

$$L'[U] = BU_{\xi\eta}$$

if $b^2 - ac > 0$. An example of a hyperbolic equation is the wave equation, $u_{tt} = u_{xx}$.

When $b^2 - ac = 0$, then there is only one characteristic solution, $\frac{dy}{dx} = \frac{b}{a}$. This is the parabolic case. But, $\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}$. So,

$$\frac{b}{a} = -\frac{\xi_x}{\xi_y},$$

which we can write as a first order partial differential equation for ξ,[3]

[3] We will see in Chapter 7 that this first order PDE can be solved using the method of characteristics. The result is that ξ is constant along the characteristics given by $\frac{dy}{dx} = \frac{b}{a}$.

$$a\xi_x + b\xi_y = 0.$$

Also, $b^2 - ac = 0$ implies that $c = b^2/a$.

Parabolic case.

Inserting these expression into coefficient B, we have

$$\begin{aligned} B &= 2a\xi_x\eta_x + 2b\xi_x\eta_y + 2b\xi_y\eta_x + 2c\xi_y\eta_y \\ &= 2(a\xi_x + b\xi_y)\eta_x + 2(b\xi_x + c\xi_y)\eta_y \\ &= 2\frac{b}{a}(a\xi_x + b\xi_y)\eta_y = 0. \end{aligned} \quad (1.39)$$

Therefore, in the parabolic case, $A = 0$ and $B = 0$, and $L[u]$ transforms to

$$L'[U] = CU_{\eta\eta}$$

when $b^2 - ac = 0$. This is the canonical form for a parabolic operator. An example of a parabolic equation is the heat equation, $u_t = u_{xx}$.

Elliptic case.

Finally, when $b^2 - ac < 0$, we have the elliptic case. In this case we cannot force $A = 0$ or $C = 0$. However, we can force $B = 0$. As we just showed, we can write

$$B = 2(a\xi_x + b\xi_y)\eta_x + 2(b\xi_x + c\xi_y)\eta_y.$$

Letting $\eta_x = 0$, we can choose ξ to satisfy $b\xi_x + c\xi_y = 0$. This leads to

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = a\xi_x^2 - c\xi_y^2 = \frac{ac - b^2}{c}\xi_x^2$$

$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = c\eta_y^2$$

Furthermore, setting $\frac{ac - b^2}{c}\xi_x^2 = c\eta_y^2$, we can make $A = C$ and, finally, $L[u]$ transforms to

$$L'[U] = A[U_{\xi\xi} + U_{\eta\eta}]$$

when $b^2 - ac < 0$. This is the canonical form for an elliptic operator. An example of an elliptic equation is Laplace's equation, $u_{xx} + u_{yy} = 0$.

In summary, we have can classify second order partial differential equations by considering the sign of $b^2 - ac$. This is displayed in the box below.

Classification of Second Order PDEs
The second order differential operator

$$L[u] = a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy},$$

can be transformed to one of the following forms:

- $b^2 - ac > 0$. Hyperbolic: $L[u] = B(x,y)u_{xy}$
- $b^2 - ac = 0$. Parabolic: $L[u] = C(x,y)u_{yy}$
- $b^2 - ac < 0$. Elliptic: $L[u] = A(x,y)[u_{xx} + u_{yy}]$

As a final note,[4] the terminology used in this classification is borrowed from the general theory of quadratic equations which are the equations for translated and rotated conics. Recall from your calculus class that the general quadratic equation in two variable takes the form

[4] The rest of this section is about the classification of conics and is not essential to the study of PDEs.

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0. \tag{1.40}$$

One can complete the squares in x and y to obtain the new form

$$a(x-h)^2 + 2bxy + c(y-k)^2 + f' = 0.$$

So, translating points (x,y) using the transformations $x' = x - h$ and $y' = y - k$, we find the simpler form

$$ax^2 + 2bxy + cy^2 + f = 0.$$

Here we dropped all primes.

We can also introduce transformations to simplify the quadratic terms. Consider a rotation of the coordinate axes by θ, as shown in Figure 1.24,

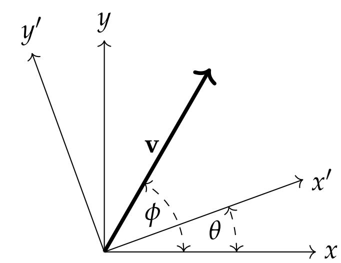


Figure 1.24: Comparison of the coordinate systems. Using the addition formula for trigonometric, functions we have $x' = r\cos\phi\cos\theta + v\sin\phi\sin\theta$ and $y' = r\sin\phi\cos\theta - v\cos\phi\sin\theta$. We can use the polar forms $x = r\cos\phi$ and $y = r\sin\phi$ to obtain system (1.41).

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta, \end{aligned} \tag{1.41}$$

or the inverse form,

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta. \end{aligned} \tag{1.42}$$

The resulting equation takes the form

$$Ax'^2 + 2Bx'y' + Cy'^2 + D = 0,$$

where

$$\begin{aligned} A &= a\cos^2\theta + 2b\sin\theta\cos\theta + c\sin^2\theta. \\ B &= (c-a)\sin\theta\cos\theta + b(\cos^2\theta - \sin^\theta). \\ C &= a\sin^2\theta - 2b\sin\theta\cos\theta + c\cos^2\theta. \end{aligned} \tag{1.43}$$

We can eliminate the $x'y'$ term by forcing $B = 0$. Since $\cos^2\theta - \sin^2\theta = \cos 2\theta$ and $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$, we have

$$B = \frac{(c-a)}{2}\sin 2\theta + b\cos 2\theta = 0.$$

Therefore, the condition for eliminating the $x'y'$ term is

$$\cot(2\theta) = \frac{a-c}{2b}.$$

Furthermore, one can show that $b^2 - ac = B^2 - AC$. From the form $Ax'^2 + 2Bx'y' + Cy'^2 + D = 0$, the resulting quadratic equation takes one of the following forms:

- $b^2 - ac > 0$. Hyperbolic: $Ax^2 - Cy^2 + D = 0$.
- $b^2 - ac = 0$. Parabolic: $Ax^2 + By + D = 0$.
- $b^2 - ac < 0$. Elliptic: $Ax^2 + Cy^2 + D = 0$.

Thus, one can see the connection between the classification of quadratic equations and second order partial differential equations in two independent variables.

1.7 The Nonhomogeneous Heat Equation

In this section we discuss nonhomogeneous initial-boundary value problems in the form of the nonhomogeneous heat equation. Either the partial differential equation is nonhomogeneous, or the boundary conditions are nonhomogeneous. This will lead to the notion of a Green's function.[5] As with the earlier solutions of the heat and wave equations, we will come to a point where will will need to determine some Fourier coefficients, which we study in the next chapter.

[5] Green's functions were first introduced by the British mathematician George Green (1793-1841) in 1828 in a memoir which was hardly known. In 1845 a young William Thomson (1824-1907), later to be known as Lord Kelvin, saw a footnote in Robert Murphy's (1806-1843) 1832 paper, referring to Green's essay. When Thomson found the memoir he made Green known to the world.

1.7.1 Nonhomogeneous Time Independent Boundary Conditions

Consider the nonhomogeneous heat equation with nonhomogeneous boundary conditions:

$$\begin{aligned} u_t - ku_{xx} &= h(x), \qquad 0 \le x \le L, \quad t > 0, \\ u(0,t) &= a, \quad u(L,t) = b, \\ u(x,0) &= f(x). \end{aligned} \tag{1.44}$$

We are interested in finding a particular solution to this initial-boundary value problem. In fact, we can represent the solution to the general nonhomogeneous heat equation as the sum of two solutions that solve different problems.

First, we consider a function, $v(x,t)$, which satisfies the homogeneous problem,

$$\begin{aligned} v_t - kv_{xx} &= 0, \qquad 0 \le x \le L, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \\ v(x,0) &= g(x). \end{aligned} \tag{1.45}$$

Note that the differential equation as well as the boundary conditions are homogeneous. In this problem we will call $v(x,t)$ the transient solution.

We will also need a steady state solution to the original problem. A steady state solution is one that satisfies $u_t = 0$. Let $w(x)$ be the steady state solution. It satisfies the problem

$$\begin{aligned} -kw_{xx} &= h(x), \qquad 0 \le x \le L. \\ w(0) &= a, \quad w(L) = b. \end{aligned} \tag{1.46}$$

The steady state solution, $w(t)$, satisfies a nonhomogeneous differential equation with nonhomogeneous boundary conditions. The transient solution, $v(t)$, satisfies the homogeneous heat equation with homogeneous boundary conditions and satisfies a modified initial condition.

Now consider $u(x,t) = w(x) + v(x,t)$, the sum of the steady state solution, $w(x)$, and the transient solution, $v(x,t)$. We first note that $u(x,t)$ satisfies the nonhomogeneous heat equation,

$$\begin{aligned} u_t - ku_{xx} &= (w+v)_t - (w+v)_{xx} \\ &= v_t - kv_{xx} - kw_{xx} \equiv h(x). \end{aligned} \tag{1.47}$$

The solution to the nonhomogeneous problem will be the sum of the steady state and transient solutions, $u(x,t) = w(x) + v(x,t)$.

The boundary conditions are also satisfied by $u(x,t)$. Evaluating, $u(x,t)$ at $x = 0$ and $x = L$, we have

$$\begin{aligned} u(0,t) &= w(0) + v(0,t) = a, \\ u(L,t) &= w(L) + v(L,t) = b. \end{aligned} \tag{1.48}$$

Finally, the initial condition gives

$$u(x,0) = w(x) + v(x,0) = w(x) + g(x).$$

The transient solution satisfies

$$v(x,0) = f(x) - w(x).$$

Thus, if we set $g(x) = f(x) - w(x)$, then $u(x,t) = w(x) + v(x,t)$ will be the solution of the nonhomogeneous boundary value problem. We all ready know how to solve the homogeneous problem to obtain $v(x,t)$. So, we only need to find the steady state solution, $w(x)$.

There are several methods we could use to solve Equation (1.46) for the steady state solution. One is the Method of Variation of Parameters, which is closely related to the Green's function method for boundary value problems which we describe in the Appendix. However, we will just integrate the differential equation for the steady state solution directly to find the solution. From this solution we will be able to read off the Green's function.

Integrating the steady state equation (1.46) once, yields

$$\frac{dw}{dx} = -\frac{1}{k}\int_0^x h(z)\,dz + A,$$

where we have been careful to include the integration constant, $A = w'(0)$. Integrating again, we obtain

$$w(x) = -\frac{1}{k}\int_0^x \left(\int_0^y h(z)\,dz\right)\,dy + Ax + B, \tag{1.49}$$

where a second integration constant has been introduced, $B = w(0)$. This gives the general solution for Equation (1.46).

The boundary conditions can now be used to determine the constants. It is clear that $B = a$ for the condition at $x = 0$ to be satisfied. The second condition gives

$$b = w(L) = -\frac{1}{k}\int_0^L \left(\int_0^y h(z)\,dz\right)\,dy + AL + a.$$

Solving for A, we have

$$A = \frac{1}{kL}\int_0^L \left(\int_0^y h(z)\,dz\right)\,dy + \frac{b-a}{L}.$$

Inserting the integration constants in Equation (1.49), the solution of the boundary value problem for the steady state solution is then

The steady state solution.

$$w(x) = -\frac{1}{k}\int_0^x \left(\int_0^y h(z)\,dz\right)\,dy + \frac{x}{kL}\int_0^L \left(\int_0^y h(z)\,dz\right)\,dy + \frac{b-a}{L}x + a.$$

This is sufficient for an answer, but it can be written in a more compact form. In fact, we will show that the solution can be written in a way that a Green's function can be identified.

Writing the steady state solution in a compact form and introducing the Green's function.

First, we rewrite the double integrals as single integrals. We can do this using integration by parts. Consider integral in the first term of the solution,

$$I = \int_0^x \left(\int_0^y h(z)\,dz\right)\,dy.$$

Setting $u = \int_0^y h(z)\,dz$ and $dv = dy$ in the standard integration by parts formula, we obtain

$$\begin{aligned} I &= \int_0^x \left(\int_0^y h(z)\,dz\right)\,dy \\ &= y\int_0^y h(z)\,dz\Big|_0^x - \int_0^x yh(y)\,dy \\ &= \int_0^x (x-y)h(y)\,dy. \end{aligned} \qquad (1.50)$$

Thus, the double integral has now collapsed to a single integral. Replacing the integral in the solution, the steady state solution becomes

$$w(x) = -\frac{1}{k}\int_0^x (x-y)h(y)\,dy + \frac{x}{kL}\int_0^L (L-y)h(y)\,dy + \frac{b-a}{L}x + a.$$

We can make a further simplification by combining these integrals. This can be done if the integration range, $[0, L]$, in the second integral is split into two pieces, $[0, x]$ and $[x, L]$. Writing the second integral as two integrals over these subintervals, we obtain

$$\begin{aligned} w(x) &= -\frac{1}{k}\int_0^x (x-y)h(y)\,dy + \frac{x}{kL}\int_0^x (L-y)h(y)\,dy \\ &\quad + \frac{x}{kL}\int_x^L (L-y)h(y)\,dy + \frac{b-a}{L}x + a. \end{aligned} \qquad (1.51)$$

Next, we rewrite the integrands,

$$\begin{aligned} w(x) &= -\frac{1}{k}\int_0^x \frac{L(x-y)}{L}h(y)\,dy + \frac{1}{k}\int_0^x \frac{x(L-y)}{L}h(y)\,dy \\ &\quad +\frac{1}{k}\int_x^L \frac{x(L-y)}{L}h(y)\,dy + \frac{b-a}{L}x + a. \end{aligned} \tag{1.52}$$

It can now be seen how we can combine the first two integrals:

$$w(x) = -\frac{1}{k}\int_0^x \frac{y(L-x)}{L}h(y)\,dy + \frac{1}{k}\int_x^L \frac{x(L-y)}{L}h(y)\,dy + \frac{b-a}{L}x + a.$$

The resulting integrals now take on a similar form and this solution can be written compactly as

$$w(x) = \int_0^L G(x,y)\left(-\frac{1}{k}h(y)\right)\,dy + \frac{b-a}{L}x + a, \tag{1.53}$$

where

$$G(x,y) = \begin{cases} -\dfrac{x(L-y)}{L}, & 0 \le x \le y, \\ -\dfrac{y(L-x)}{L}, & y \le x \le L, \end{cases} \tag{1.54}$$

is the Green's function for this problem.

The Green's function for the steady state problem.

The full solution to the original problem can be found by adding to this steady state solution a solution of the homogeneous problem,

$$\begin{aligned} u_t - ku_{xx} &= 0, \qquad 0 \le x \le L, \quad t > 0, \\ u(0,t) &= 0, \quad u(L,t) = 0, \\ u(x,0) &= f(x) - w(x). \end{aligned} \tag{1.55}$$

Example 1.8. Solve the nonhomogeneous problem,

$$\begin{aligned} u_t - u_{xx} &= 10, \qquad 0 \le x \le 1, \quad t > 0, \\ u(0,t) &= 20, \quad u(1,t) = 0, \\ u(x,0) &= 2x(1-x). \end{aligned} \tag{1.56}$$

In this problem we have a rod initially at a temperature of $u(x,0) = 2x(1-x)$. The ends of the rod are maintained at fixed temperatures and the bar is continually heated at a constant temperature, represented by the source term, 10.

First, we find the steady state temperature, $w(x)$, satisfying

$$\begin{aligned} -w_{xx} &= 10, \qquad 0 \le x \le 1. \\ w(0,t) &= 20, \quad w(1,t) = 0. \end{aligned} \tag{1.57}$$

Using the general solution, we have

$$w(x) = -\int_0^1 10G(x,y)\,dy - 20x + 20,$$

where the Green's function in Equation (1.54) becomes

$$G(x,y) = \begin{cases} -x(1-y), & 0 \le x \le y, \\ -y(1-x), & y \le x \le 1, \end{cases}$$

we compute the solution

$$\begin{aligned} w(x) &= \int_0^x 10y(1-x)\,dy + \int_x^1 10x(1-y)\,dy - 20x + 20 \\ &= 5(x-x^2) - 20x + 20, \\ &= 20 - 15x - 5x^2. \end{aligned} \tag{1.58}$$

Checking this solution, it satisfies both the steady state equation and boundary conditions.

The transient solution satisfies

$$\begin{aligned} v_t - v_{xx} &= 0, \qquad 0 \le x \le 1, \quad t > 0, \\ v(0,t) &= 0, \quad v(1,t) = 0, \\ v(x,0) &= x(1-x) - 10. \end{aligned} \tag{1.59}$$

Recall, that we have determined the solution of this problem as

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x,$$

where the Fourier coefficients, b_n, are given in terms of the initial temperature distribution. In the next chapter we will see that these are given by

$$b_n = 2\int_0^1 [x(1-x) - 10] \sin n\pi x\,dx, \quad n = 1, 2, \ldots.$$

Therefore, the full solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x + 20 - 15x - 5x^2.$$

Note that for large t, the transient solution tends to zero and we are left with the steady state solution as expected.

1.7.2 *The Green's Function*

While Green's functions are explored in the second semester of a PDE course or an ODE course, it is useful to point them out as we progress. A Green's function is the response to a point source. For

$$w_{xx} = f(x), \quad w(0) = a, \quad w(L) = b,$$

we seek a function $G(x, x')$ which satisfies

$$G_{xx}(x, x') = \delta(x - x'), \quad G(0, x') = 0, \quad G(L, x')) = 0,$$

where $\delta(x - x')$ is the Dirac delta function representing a unit impulse.

The Dirac delta function, which is described in more detail in the two semester text, satisfies two properties: $\delta(x) = 0$, $x \neq 0$, and

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1.$$

From this definition one can show that

$$\int_{-\infty}^{\infty} \delta(x-x')f(x)\,dx = f(x').$$

Using the differential equations satisfied by $w(x)$ and $G(x,x')$, we have

$$\begin{aligned}
\int_0^L \left(w''(x)G(x,x') - w(x)G_{xx}(x,x')\right)\,dx &= \int_0^L \frac{d}{dx}\left(w'(x)G(x,x') - w(x)G_x(x,x')\right)\,dx \\
\int_0^L \left(f(x)G(x,x') - w(x)\delta(x-x')\right)\,dx &= \left[w'(x)G(x,x') - w(x)G_x(x,x')\right]_0^L \\
\int_0^L f(x)G(x,x')\,dx - w(x') &= -w(L)G_x(L,x') + w(0)G_x(0,x') \\
w(x') &= \int_0^L f(x)G(x,x')\,dx + bG_x(L,x') - aG_x(0,x').
\end{aligned} \tag{1.60}$$

This tells us that if we know the Green's function, then we can write down the solution to the nonhomogeneous differential equation. But, first we need to interchange the independent variables.

Noting that when $f(x) = \delta(x-\xi)$, then $w(x) = G(x,\xi)$, and $a = b = 0$. So, for this special problem, the result gives

$$G(x',\xi) = \int_0^L \delta(x-\xi)G(x,x')\,dx = G(\xi,x').$$

This property is call reciprocality for the Green's function.

We can rewrite the solution for $w(x)$ by interchanging x and x' in the solution as

$$w(x) = \int_0^L f(x')G(x,x')\,dx' + bG_{x'}(x,L) - aG_{x'}(x,0).$$

If we use the Green's function in Equation (1.54) for the previous steady state problem, where $f(x) = -\frac{1}{k}h(x)$, then we have

$$w(x) = -\frac{1}{k}\int_0^L h(x')G(x,x')\,dx' + b\frac{x}{L} + a\frac{L-x}{L}).$$

since

$$G_y(x,y) = \begin{cases} \dfrac{x}{L}, & 0 \le x \le y, \\ -\dfrac{L-x}{L}, & y \le x \le L, \end{cases}$$

The solution is the same as Equation (1.53).

1.7.3 *Time Dependent Boundary Conditions*

In the last section we solved problems with time independent boundary conditions using equilibrium solutions satisfying the steady state heat equation sand nonhomogeneous boundary conditions. When the boundary conditions are time dependent, we can also convert the problem to an auxiliary problem with homogeneous boundary conditions.

Consider the problem

$$\begin{aligned} u_t - ku_{xx} &= h(x), \quad 0 \le x \le L, \quad t > 0, \\ u(0,t) &= a(t), \quad u(L,t) = b(t), \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le L. \end{aligned} \tag{1.61}$$

We define $u(x,t) = v(x,t) + w(x,t)$, where $w(x,t)$ is a modified form of the steady state solution from the last section,

$$w(x,t) = a(t) + \frac{b(t) - a(t)}{L} x.$$

Noting that

$$\begin{aligned} u_t &= v_t + \dot{a} + \frac{\dot{b} - \dot{a}}{L} x, \\ u_{xx} &= v_{xx}, \end{aligned} \tag{1.62}$$

we find that $v(x,t)$ is a solution of the problem[6]

$$\begin{aligned} v_t - kv_{xx} &= h(x) - \left[\dot{a}(t) + \frac{\dot{b}(t) - \dot{a}(t)}{L} x\right], \quad 0 \le x \le L, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \quad t > 0, \\ v(x,0) &= f(x) - \left[a(0) + \frac{b(0) - a(0)}{L} x\right], \quad 0 \le x \le L. \end{aligned} \tag{1.63}$$

[6] The bracketed expression look complicated, but we could write instead

$$\begin{aligned} v_t - kv_{xx} &= h(x) - w_t(x,t) \\ v(x,0) &= f(x) - w(x,0). \end{aligned}$$

Thus, we have converted the original problem into a nonhomogeneous heat equation with homogeneous boundary conditions and a new source term and new initial condition.

Example 1.9. Solve the problem

$$\begin{aligned} u_t - u_{xx} &= x, \quad 0 \le x \le 1, \quad t > 0, \\ u(0,t) &= 2, \quad u(L,t) = t, \quad t > 0 \\ u(x,0) &= 3\sin 2\pi x + 2(1-x), \quad 0 \le x \le 1. \end{aligned} \tag{1.64}$$

We first define $w(x,t) = 2 + (t-2)x$. Then, we have

$$u(x,t) = v(x,t) + 2 + (t-2)x,$$

where $v(x,t)$ satisfies the problem

$$\begin{aligned} v_t - v_{xx} &= 0, \quad 0 \le x \le 1, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \quad t > 0, \\ v(x,0) &= 3\sin 2\pi x, \quad 0 \le x \le 1. \end{aligned} \tag{1.65}$$

Note that this problem was rigged so that the initial value problem is easily solved. as we have seen earlier in the chapter, the general solution is given by

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x e^{-n^2\pi^2 t}.$$

Since $v(x,0) = 3\sin 2\pi x$, the $b_n's$ all vanish except for $b_2 = 3$. This gives $v(x,t) = 3\sin 2\pi x e^{-4\pi^2 t}$. Therefore, we have found the solution

$$u(x,t) = 3\sin 2\pi x e^{-4\pi^2 t} + 2 + (t-2)x.$$

1.7.4 Duhamel's Principle

THE IDEA THAT ONE CAN SOLVE A NONHOMOGENEOUS partial differential equation by associating it with a related homogeneous problem with nonhomogeneous initial conditions is know as Duhamel's Principle named after Jean-Marie Constant Duhamel (1797-1872). We apply this principle to the heat equation.

Recall the heated rod in Section 1.3.2 where we derived the heat equation without a heat source. When there is a heat source, the one-dimensional heat equation with homogeneous boundary conditions becomes

$$\begin{aligned} u_t - ku_{xx} &= Q(x,t), \qquad 0 \le x \le L, \quad t > 0, \\ u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le L. \end{aligned} \tag{1.66}$$

Let us consider a rod at initially zero temperature, $f(x) = 0$. We apply a source of heat energy $Q(x,0)$ to every point on the rod at $t = 0$. The temperature is then approximately $Q(x,0)\,\delta s$, assuming the time δs is too short for the energy to diffuse. At a later time, after diffusion and with no additional heat source, the temperature is $\widetilde{v}(x,t)\,\delta s$, where $v(x,t)$ satisfies the (heat energy) diffusion equation, but with initial heat energy $Q(x,0)$,

$$\begin{aligned} \widetilde{v}_t &= k\widetilde{v}_{xx}, \quad 0 \le x \le L, \quad t > 0, \\ \widetilde{v}(0,t) &= 0, \quad \widetilde{v}(L,t) = 0, \quad t > 0, \\ \widetilde{v}(x,0) &= Q(x,0), \quad 0 \le x \le L. \end{aligned} \tag{1.67}$$

Now, we consider what happens when the source is turned on at time $t = s - \delta s$ and turned off at $t = s$. This leads to a new solution, $\overline{v}(x,t;s)$, which satisfies

$$\begin{aligned} \overline{v}_t &= k\overline{v}_{xx}, \quad 0 \le x \le L, \quad t > 0, \\ \overline{v}(0,t;s) &= 0, \quad \overline{v}(L,t;s) = 0, \quad t > 0, \\ \overline{v}(x,s;s) &= Q(x,s), \quad 0 \le x \le L. \end{aligned} \tag{1.68}$$

Here we are given $Q(x,s)$ as the initial condition at $t = s$. Then, adding all of the sources before time t, we have the solution

$$u(x,t) = \int_0^t \overline{v}(x,t;s)\,ds.$$

The initial value for $\overline{v}(x,t;s)$ is given at $t = s$ while that for $\widetilde{v}(x,t)$ is at $t = 0$. So, we need to relate these functions. The relation between the solutions is found as

$$\overline{v}(x,t;s) \approx \widetilde{v}(x,t-s;s).$$

Therefore, $\widetilde{v}(x,0;s) = Q(x,s)$. Now, we can write the total solution over time as

$$u(x,t) = \int_0^t \widetilde{v}(x,t-s;s)\,ds,$$

where we have modified the problem solved by $\widetilde{v}(x,t;s)$ as

$$\begin{aligned} \widetilde{v}_t &= k\widetilde{v}_{xx}, \quad 0 \le x \le L, \quad t > 0, \\ \widetilde{v}(0,t;s) &= 0, \quad \widetilde{v}(L,t;s) = 0, \quad t > 0, \\ \widetilde{v}(x,0;s) &= Q(x,s), \quad 0 \le x \le L. \end{aligned} \tag{1.69}$$

Proof. Differentiating the solution for $u(x,t)$ with respect to t, we have

$$\begin{aligned} u_t(x,t) &= \widetilde{v}(x,0;t) + \int_0^t \widetilde{v}_t(x,t-s;s)\,ds \\ &= Q(x,t) + \int_0^t k\widetilde{v}_{xx}(x,t-s;s)\,ds \\ &= Q(x,t) + ku_{xx}(x,t). \end{aligned} \tag{1.70}$$

□

We see that the solution of a problem with a source can be converted to a related homogeneous differential equation with an initial condition. This is the essence of Duhamel's Principle. We demonstrate this in the following example.

Example 1.10. Use Duhamel's Principle to solve the initial-boundary value problem

$$\begin{aligned} u_t - ku_{xx} &= t\sin x, \qquad 0 \le x \le \pi, \quad t > 0, \\ u(0,t) &= 0, \quad u(\pi,t) = 0, \quad t > 0, \\ u(x,0) &= 0, \quad 0 \le x \le \pi. \end{aligned} \tag{1.71}$$

We first need to solve the related problem,

$$\begin{aligned} \widetilde{v}_t &= k\widetilde{v}_{xx}, \qquad 0 \le x \le \pi, \quad t > 0, \\ \widetilde{v}(0,t) &= 0, \quad \widetilde{v}(\pi,t) = 0, \quad t > 0, \\ \widetilde{v}(x,0) &= s\sin x, \quad 0 \le x \le \pi. \end{aligned} \tag{1.72}$$

One can show that the series solution is given by

$$\widetilde{v}(x,t;s) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx.$$

Setting $t = 0$,

$$\widetilde{v}(x,0;s) = \sum_{n=1}^{\infty} b_n \sin nx = s\sin x.$$

We can see that all of the b_n's vanish except that $n = 1$ case. Therefore, the solution is $\widetilde{v}(x,0) = se(-kt)\sin x$.

The final solution to the given problem is found through integration:

$$\begin{aligned} u(x,t) &= \int_0^t \widetilde{v}(x,t-s;s)\,ds \\ &= \int_0^t se^{-k(t-s)} \sin x\,ds \\ &= \left(\frac{t}{k} - \frac{e^{-kt}-1}{k^2}\right) \sin x. \end{aligned} \tag{1.73}$$

1.8 Laplace's Equation in 2D

Another generic partial differential equation is Laplace's equation, $\nabla^2 u = 0$. Laplace's equation arises in many applications. As an example, consider a thin rectangular plate with boundaries set at fixed temperatures. Assume that any temperature changes of the plate are governed by the heat equation, $u_t = k\nabla^2 u$, subject to these boundary conditions. However, after a long period of time the plate may reach thermal equilibrium. If the boundary temperature is zero, then the plate temperature decays to zero across the plate. However, if the boundaries are maintained at a fixed nonzero temperature, which means energy is being put into the system to maintain the boundary conditions, the internal temperature may reach a nonzero equilibrium temperature. Reaching thermal equilibrium means that asymptotically in time the solution becomes time independent. Thus, the equilibrium state is a solution of the time independent heat equation, $\nabla^2 u = 0$.

Thermodynamic equilibrium, $\nabla^2 u = 0$.

A second example comes from electrostatics. Letting $\phi(\mathbf{r})$ be the electric potential, one has for a static charge distribution, $\rho(\mathbf{r})$, that the electric field, $\mathbf{E} = \nabla\phi$, satisfies one of Maxwell's equations, $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$. In regions devoid of charge, $\rho(\mathbf{r}) = 0$, the electric potential satisfies Laplace's equation, $\nabla^2\phi = 0$.

Incompressible, irrotational fluid flow, $\nabla^2\phi = 0$, for velocity $\mathbf{v} = \nabla\phi$. See more in Section 1.8.

As a final example, Laplace's equation appears in two-dimensional fluid flow described by the velocity field $\mathbf{v}(\mathbf{r})$. For an incompressible flow, $\nabla \cdot \mathbf{v} = 0$. If the flow is irrotational, then $\nabla \times \mathbf{v} = 0$. We can introduce a velocity potential, $\mathbf{v} = \nabla\phi$. Thus, $\nabla \times \mathbf{v}$ vanishes by a vector identity and $\nabla \cdot \mathbf{v} = 0$ implies $\nabla^2\phi = 0$. So, once again we obtain Laplace's equation.

Solutions of Laplace's equation are called *harmonic functions*. These are often encountered in complex analysis where one applies complex variable techniques to solve the two-dimensional Laplace equation. In this section we will apply the Method of Separation of Variables to solve simple examples of Laplace's equation in two dimensions. Three-dimensional problems will be studied in Chapter 6. Parts of this discussion are repeated with more general boundary conditions in Section 6.3.

Example 1.11. Equilibrium Temperature Distribution for a Rectangular Plate

Let's consider Laplace's equation in Cartesian coordinates,

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, \quad 0 < y < H$$

with the boundary conditions

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) = f(x), \quad u(x, H) = 0.$$

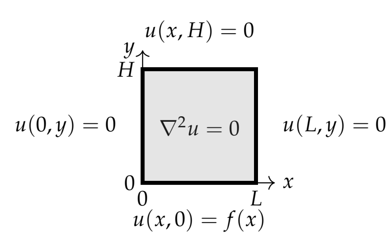


Figure 1.25: In this figure we show the domain and boundary conditions for the example of determining the equilibrium temperature distribution for a rectangular plate.

The boundary conditions are shown in Figure 6.8. Note that there is no time-dependence and, therefore, no need for initial conditions.

As with the heat and wave equations, we can solve this problem using the method of separation of variables. Let $u(x, y) = X(x)Y(y)$.

Then, Laplace's equation becomes

$$X''Y + XY'' = 0$$

and we can separate the x and y dependent functions and introduce a separation constant, λ,

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

Thus, we are led to two differential equations,

$$\begin{aligned} X'' + \lambda X &= 0, \\ Y'' - \lambda Y &= 0. \end{aligned} \tag{1.74}$$

From the pair of boundary conditions $u(0, y) = 0, u(L, y) = 0$, we have $X(0) = 0, X(L) = 0$. So, we have the usual eigenvalue problem for $X(x)$,

$$X'' + \lambda X = 0, \quad X(0) = 0, X(L) = 0.$$

The solutions to this problem are given by

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots.$$

The general solution of the equation for $Y(y)$ is given by

$$Y(y) = c_1 e^{\sqrt{\lambda} y} + c_2 e^{-\sqrt{\lambda} y}.$$

The boundary condition $u(x, H) = 0$ implies $Y(H) = 0$. So, we require that

$$Y(H) = c_1 e^{\sqrt{\lambda} H} + c_2 e^{-\sqrt{\lambda} H} = 0.$$

We solve this for c_2,

$$c_2 = -c_1 e^{2\sqrt{\lambda} H}.$$

Note: Having carried out this computation, we can now see that it would be better to guess this form in the future. So, for $Y(H) = 0$, one would guess a solution $Y(y) = \sinh\sqrt{\lambda}(H - y)$. For $Y(0) = 0$, one would guess a solution $Y(y) = \sinh\sqrt{\lambda} y$. Similarly, if $Y'(H) = 0$, one would guess a solution $Y(y) = \cosh\sqrt{\lambda}(H - y)$.

Inserting this result into the expression for $Y(y)$, we have

$$\begin{aligned} Y(y) &= c_1 e^{\sqrt{\lambda} y} - c_1 e^{2\sqrt{\lambda} H} e^{-\sqrt{\lambda} y} \\ &= c_1 e^{\sqrt{\lambda} H}\left(e^{-\sqrt{\lambda} H} e^{\sqrt{\lambda} y} - e^{\sqrt{\lambda} H} e^{-\sqrt{\lambda} y}\right) \\ &= c_1 e^{\sqrt{\lambda} H}\left(e^{-\sqrt{\lambda}(H-y)} - e^{\sqrt{\lambda}(H-y)}\right) \\ &= -2c_1 e^{\sqrt{\lambda} H} \sinh\sqrt{\lambda}(H - y). \end{aligned} \tag{1.75}$$

Since we already know the values of the eigenvalues λ_n from the eigenvalue problem for $X(x)$, we have that the y-dependence is given by

$$Y_n(y) = \sinh\frac{n\pi(H - y)}{L}.$$

So, the product solutions are given by

$$u_n(x, y) = \sin\frac{n\pi x}{L} \sinh\frac{n\pi(H - y)}{L}, \quad n = 1, 2, \ldots.$$

These solutions satisfy Laplace's equation and the three homogeneous boundary conditions and in the problem. We should note that the constant $-2c_1e^{\sqrt{\lambda}H}$ can be absorbed into the final product solutions and, therefore, can be dropped. Another way to take care of this factor is to redefine $c_1 = -\frac{1}{2}e^{-\sqrt{\lambda}H}$.

The remaining boundary condition, $u(x,0) = f(x)$, still needs to be satisfied. Inserting $y = 0$ in the product solutions does not satisfy the boundary condition unless $f(x)$ is proportional to one of the eigenfunctions $X_n(x)$. So, we first write down the general solution as a linear combination of the product solutions,

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi(H-y)}{L}. \tag{1.76}$$

Now, we apply the boundary condition, $u(x,0) = f(x)$, to find that

$$f(x) = \sum_{n=1}^{\infty} a_n \sinh\frac{n\pi H}{L}\sin\frac{n\pi x}{L}. \tag{1.77}$$

Defining $b_n = a_n \sinh\frac{n\pi H}{L}$, this becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}. \tag{1.78}$$

We see that the determination of the unknown coefficients, b_n, is simply done by recognizing that this is a Fourier sine series. We now move on to the study of Fourier series and provide more complete answers in Section 6.3.

Example 1.12. Consider the boundary value problem depicted in Figure 1.26

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad 0 \le x \le 1, \quad 0 \le y \le 1, \\ u(0,y) &= 0, \quad u(1,y) = \sin 3\pi y, \quad 0 \le y \le 1, \\ u(x,0) &= 2\sin\pi x, \quad u(x,1) = 0, \quad 0 \le x \le 1. \end{aligned} \tag{1.79}$$

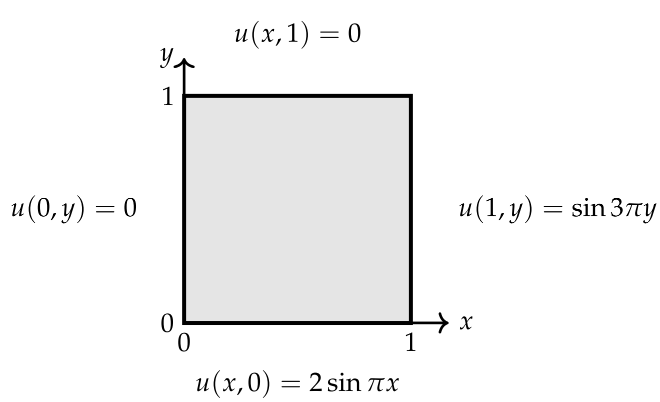


Figure 1.26: A boundary value problem for the 2D Laplace equation given by system (1.79).

In this problem there are two non-vanishing boundary conditions. But we can relate this problem to the problem with one non-vanishing boundary condition because we are dealing with a linear problem. In Figure 1.27 we depict how to split up a more general problem into two simpler problems. Thus, we have to solve Laplace's equation for solutions $u_1(x,y)$ and $u_2(x,y)$ which satisfy the boundary conditions

$$\begin{aligned} u_1(0,y) &= 0, \quad u_1(1,y)=0, \quad 0\le y\le 1,\\ u_1(x,0) &= f(x), \quad u_1(x,1)=0, \quad 0\le x\le 1,\\ u_2(0,y) &= 0, \quad u_2(1,y)=g(y), \quad 0\le y\le 1,\\ u_2(x,0) &= 0, \quad u_2(x,1)=0, \quad 0\le x\le 1. \end{aligned} \tag{1.80}$$

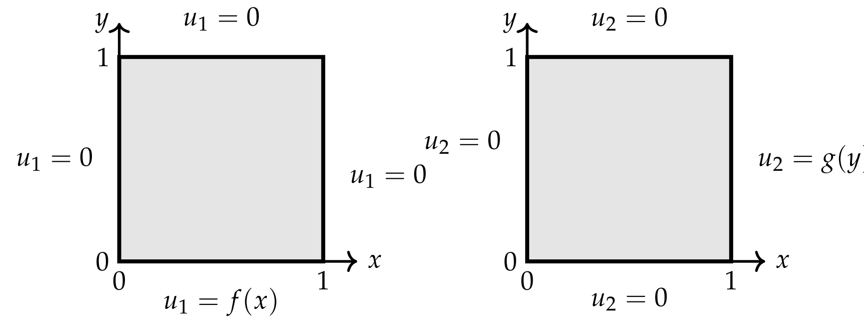


Figure 1.27: Creating two boundary value problems from that shown in Figure (1.26).

From the last example, we have

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi(1-y).$$

Similarly, we can write the general solution for the second problem as

$$u_2(x,y) = \sum_{m=1}^{\infty} b_m \sin m\pi y \sinh m\pi x.$$

Then, the solution to the full problem is given by $u(x,y) = u_1(x,y) + u_2(x,y)$, or

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi(1-y) + \sum_{m=1}^{\infty} b_m \sin m\pi y \sinh m\pi x.$$

Applying this to the original problem, where $f(x) = 2\sin \pi x$ and $g(y) = \sin 3\pi y$, we see that the only nonzero coefficients are a_1 and b_3. In particular, we have

$$\begin{aligned} u(x,0) &= a_1 \sin \pi x \sinh \pi = 2\sin \pi x,\\ u(1,y) &= b_3 \sin 3\pi y \sinh 3\pi = \sin 3\pi y. \end{aligned} \tag{1.81}$$

Therefore, $a_1 = \dfrac{2}{\sinh \pi}$ and $b_3 = \dfrac{1}{\sinh 3\pi}$ and the solution is given as

$$u(x,y) = \frac{2}{\sinh \pi} \sin \pi x \sinh \pi(1-y) + \frac{1}{\sinh 3\pi} \sin 3\pi y \sinh 3\pi x. \tag{1.82}$$

A plot of this solution is shown in Figure 1.28.

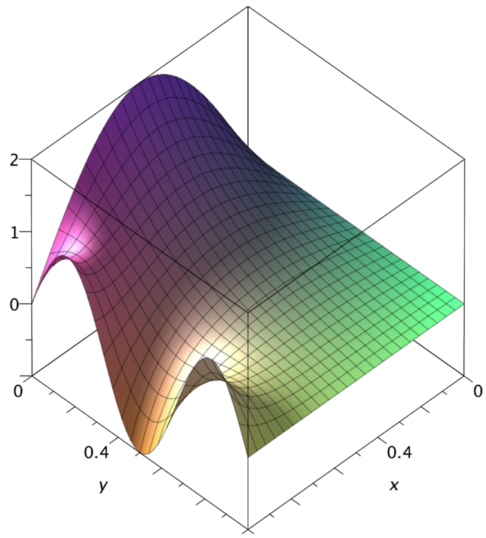


Figure 1.28: Plot of the solution in Equation (1.82) of the two-dimensional Laplace equation with boundary conditions $u(0,y) = 0$, $u(1,y) = \sin 3\pi y$, $u(x,0) = 2\sin \pi x$, $u(x,1) = 0$.

Problems

1. Solve the following initial value problems.

a. $x'' + x = 0, \quad x(0) = 2, \quad x'(0) = 0.$

b. $y'' + 2y' - 8y = 0, \quad y(0) = 1, \quad y'(0) = 2.$

c. $x^2y'' - 2xy' - 4y = 0, \quad y(1) = 1, \quad y'(1) = 0.$

2. Solve the following boundary value problems directly, when possible.

a. $x'' + x = 2, \quad x(0) = 0, \quad x'(1) = 0.$

b. $y'' + 2y' - 8y = 0, \quad y(0) = 1, \quad y(1) = 0.$

c. $y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$

3. Consider the boundary value problem for the deflection of a horizontal beam fixed at one end,

$$\frac{d^4y}{dx^4} = C, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(L) = 0, \quad y'''(L) = 0.$$

Solve this problem assuming that C is a constant.

4. Find the product solutions, $u(x,t) = T(t)X(x)$, to the heat equation, $u_t - u_{xx} = 0$, on $[0,\pi]$ satisfying the boundary conditions $u_x(0,t) = 0$ and $u(\pi,t) = 0$.

5. Find the product solutions, $u(x,t) = T(t)X(x)$, to the wave equation $u_{tt} = 2u_{xx}$, on $[0,2\pi]$ satisfying the boundary conditions $u(0,t) = 0$ and $u_x(2\pi,t) = 0$.

6. Find product solutions, $u(x,y) = X(x)Y(y)$, to Laplace's equation, $u_{xx} + u_{yy} = 0$, on the unit square satisfying the boundary conditions $u(0,y) = 0$, $u(1,y) = g(y)$, $u(x,0) = 0$, and $u(x,1) = 0$.

7. Consider the following boundary value problems. Determine the eigenvalues, λ, and eigenfunctions, $y(x)$ for each problem.

In problem d you will not get exact eigenvalues. Show that you obtain a transcendental equation for the eigenvalues in the form $\tan z = 2z$. Find the first three eigenvalues numerically.

a. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = 0.$

b. $y'' - \lambda y = 0, \quad y(-\pi) = 0, \quad y'(\pi) = 0.$

c. $x^2 y'' + xy' + \lambda y = 0, \quad y(1) = 0, \quad y(2) = 0.$

d. $(x^2 y')' + \lambda y = 0, \quad y(1) = 0, \quad y'(e) = 0.$

8. Classify the following equations as either hyperbolic, parabolic, or elliptic.

a. $u_{yy} + u_{xy} + u_{xx} = 0.$

b. $3u_{xx} + 2u_{xy} + 5u_{yy} = 0.$

c. $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0.$

d. $y^2 u_{xx} + 2xy u_{xy} + (x^2 + 4x^4) u_{yy} = 0.$

9. Use d'Alembert's solution to prove

$$f(-\zeta) = f(\zeta), \quad g(-\zeta) = g(\zeta)$$

for the semi-infinite string satisfying the free end condition $u_x(0,t) = 0$.

10. Derive a solution similar to d'Alembert's solution for the equation $u_{tt} + 2u_{xt} - 3u = 0$.

11. Construct the appropriate periodic extension of the plucked string initial profile given by

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{\ell}{2}, \\ \ell - x, & \frac{\ell}{2} \le x \le \ell, \end{cases}$$

satisfying the boundary conditions at $u(0,t) = 0$ and $u_x(\ell,t) = 0$ for $t > 0$.

12. Find and sketch the solution of the problem

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 \le x \le 1, t > o \\ u(x,0) &= \begin{cases} 0, & 0 \le x < \frac{1}{4}, \\ 1, & \frac{1}{4} \le x \le \frac{3}{4}, \\ 0, & \frac{3}{4} < x \le 1, \end{cases} \\ u_t(x,0) &= 0, \\ u(0,t) &= 0, \quad t > 0, \\ u(1,t) &= 0, \quad t > 0, \end{aligned}$$

13. Find the solution to the heat equation:

PDE: $u_t = 2u_{xx}$, $0 \le x \le 1$, $t > 0$.

BC: $u(0,t) = -1$, $u_x(1,t) = 1$.

IC: $u(x,0) = x + \sin \frac{3\pi x}{2} - 1$.

14. Find the solution to the heat equation:

PDE: $u_t = 5u_{xx}$, $0 \le x \le 10$, $t > 0$.

BC: $u_x(0,t) = 2$, $u_x(10,t) = 3$.

IC: $u(x,0) = \frac{x^2}{20} + 2x + \cos \pi x$.

15. Use Duhamel's Principle to find the solution to the nonhomogeneous heat equation:

PDE: $u_t - u_{xx} = t \sin x$, $0 \le x \le \pi$, $t > 0$.

BC: $u(0,t) = 0$, $u(\pi,t) = 0$.

IC: $u(x,0) = 0$.

16. Find the solution to the nonhomogeneous heat equation:

PDE: $u_t - u_{xx} = t(\sin 2\pi x + 2x)$, $0 \le x \le 1$, $t > 0$.

BC: $u(0,t) = 1$, $u(1,t) = t^2$.

IC: $u(x,0) = 1 + \sin 3\pi x - x$.

17. The nonhomogeneous problem for the wave equation,

PDE: $u_{tt} - c^2 u_{xx} = h(x,t)$, $-\infty < x < \infty$, $t > 0$,

IC: $u(x,0) = 0$, $u_t(x,0) = 0$,

can be solved using Duhamel's Principle for the wave equation. Namely, one solves the problem

PDE: $\tilde{v}_{tt} - c^2 \tilde{v}_{xx} = 0$, $-\infty < x < \infty$, $t > 0$,

IC: $\tilde{v}(x,0;s) = 0$, $\tilde{v}_t(x,0;s) = h(x,s)$,

and the solution to the nonhomogeneous equation is given by

$$u(x,t) = \int_0^t \tilde{v}(x, t-s; s)\, ds.$$

Verify that this is the solution.

18. Solve the initial value problem:

PDE: $u_{tt} - u_{xx} = x - t$, $-\infty < x < \infty$, $t > 0$,

IC: $u(x,0) = x^2$, $u_t(x,0) = \sin x$.

Hint: Split the problem into two terms, one for which you can use Duhamel's Principle from the previous problem and the other which can be solved using d'Alembert's solution.