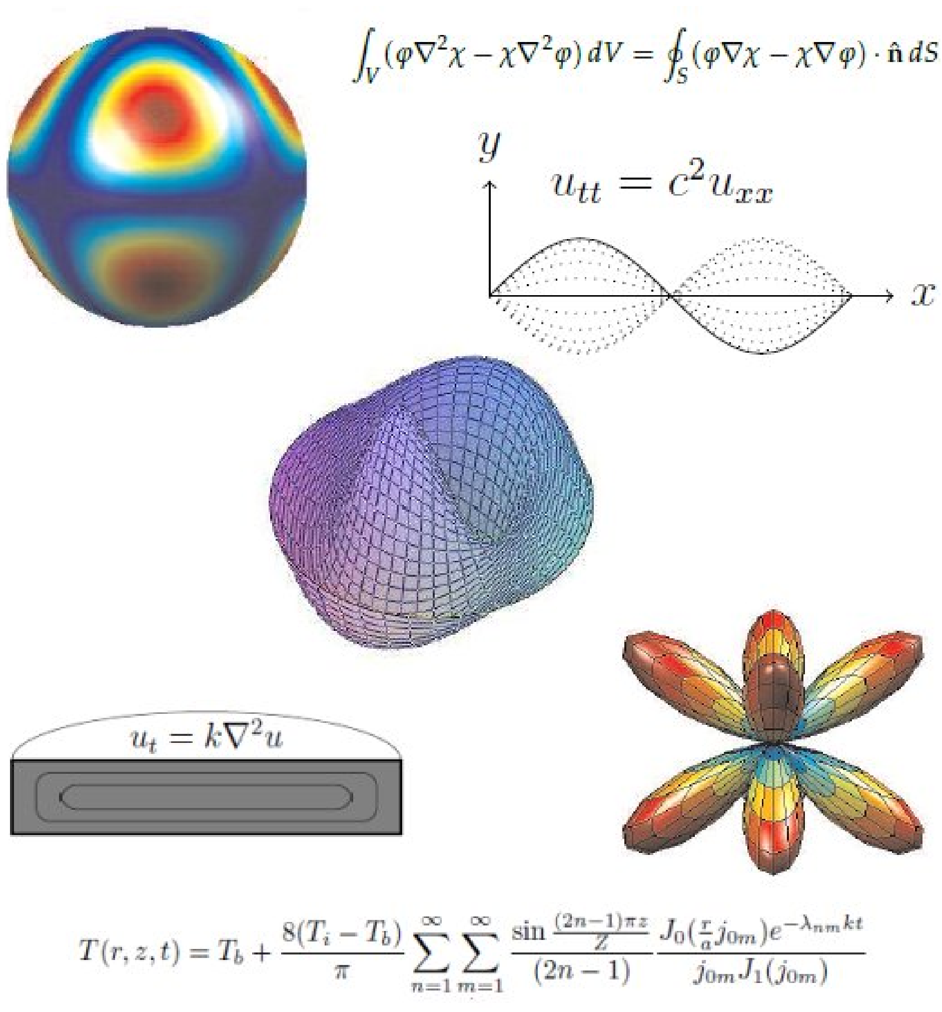
RUSSELL L. HERMAN

A FIRST COURSE IN PARTIAL DIFFERENTIAL EQUATIONS



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INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS by Russell Herman. Versions of these notes have been posted online since Spring 2005.
HTTP://PEOPLE.UNCW.EDU/HERMANR/PDE1/PDEBOOK

August 2024

Contents

Dedicated to those students who have endured previous versions of my notes.

Prologue

"How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality?." - Albert Einstein (1879-1955)

Introduction

THIS SET OF NOTES WAS COMPILED for use in a one semester course on mathematical methods for the solution of partial differential equations typically taken by majors in mathematics, the physical sciences, and engineering. Partial differential equations often arise in the study of problems in applied mathematics, mathematical physics, physical oceanography, meteorology, engineering, and biology, economics, and just about everything else. However, many of the key methods for studying such equations extend back to problems in physics and geometry. In this course we will investigate analytical, graphical, and approximate solutions of some standard partial differential equations. We will study the theory, methods of solution and applications of partial differential equations.

We will first introduce partial differential equations and a few models. A PDE, for short, is an equation involving the derivatives of some unknown multivariable function. It is a natural extension of ordinary differential equations (ODEs), which are differential equations for an unknown function of one variable. We will begin by classifying some of these equations.Our first examples will include the one dimensional heat and wave equations and the two-dimensional Laplace equation on a rectangle. As we progress through the course, we will introduce standard numerical methods since knowing how to numerically solve differential equations can be useful in research. We will also look into the standard solutions, including separation of variables, starting in one dimension and then proceeding to higher dimensions. This naturally leads to finding solutions as Fourier series and special functions, such as Legendre polynomials and Bessel functions. We will end this first course with a short study of first order evolution equations.

The specific topics to be studied in a one semester course and approximate number of lectures will include

- Introduction (1)
- Derivation of Generic Equations (1)
- Separation of Variables (Heat and Wave Equations) (2)
- 1D Wave Equation - d'Alembert Solution (2)
- Classification of Second Order Equations (1)
- Nonhomogeneous Heat Equation and Duhamel's Principle (2)
- Separation of Variables (2D Laplace Equation) (1)
- Fourier Series (3)
- Finite Difference Method, Stability and Computer Implementation (2)
- Sturm-Liouville Theory (3)
- Function Spaces and Special Functions (3)
- Equations in 2D - Laplace's Equation, Vibrating Membranes (2.5)
- 3D Problems and Spherical Harmonics (1.5)
- First Order PDEs (2)
- Conservation Laws and Shocks (1)

A second course in partial differential equations might begin with nonhomogeneous problems and Green's functions, complex analysis, Laplace and Fourier transforms, and more numerical methods. These topics can be found in an expanded version of this text.

Acknowledgments

Most, if not all, of the ideas and examples are not my own. These notes are a compendium of topics and examples that I have used in teaching not only differential equations, but also in teaching numerous courses in physics and applied mathematics. Some of the notions even extend back to when I first learned them in courses I had taken.

I would also like to express my gratitude to the many students who have found typos, or suggested sections needing more clarity in the core set of notes upon which this book was based. This applies to the set of notes used in my mathematical physics course, applied mathematics course, and previous differential equations courses.

1
Second Order Partial Differential Equations

"Either mathematics is too big for the human mind or the human mind is more than a machine." - Kurt Gödel (1906-1978)

1.1 Introduction

In this chapter we will introduce several generic second order linear partial differential equations and see how such equations lead naturally to the study of boundary value problems for ordinary differential equations. These generic differential equations occur in one to three spatial dimensions and are all linear differential equations. A list of these generic equations is provided in Table 1.1. Here we have introduced the Laplacian operator,[1] $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$. Depending on the types of boundary conditions imposed and on the geometry of the system (rectangular, cylindrical, spherical, etc.), one encounters many interesting boundary value problems.

[1] The symbol ∇ is the gradient operator. W. R. Smith suggested in 1870 to P. G. Tait to use nabla, Greek for a harp. Tait and Maxwell referred to ∇ in their letters. William Thomson (Lord Kelvin) introduced it in America in 1884. However, E. B. Wilson and Josiah Gibbs called it "del" in the classic work on vector analysis. The gradient, divergence, curl and Laplacian operators were first introduced by William Rowan Hamilton and at a meeting on July 20, 1846, later published October 1847, Hamilton introduced a rotated version for the gradient operator, $\lhd$.

Name	2 Vars	3 D
Heat Equation	$u_t = k u_{xx}$	$u_t = k\nabla^2 u$
Wave Equation	$u_{tt} = c^2 u_{xx}$	$u_{tt} = c^2\nabla^2 u$
Laplace's Equation	$u_{xx} + u_{yy} = 0$	$\nabla^2 u = 0$
Poisson's Equation	$u_{xx} + u_{yy} = F(x, y)$	$\nabla^2 u = F(x, y, z)$
Schrödinger's Equation	$iu_t = u_{xx} + F(x, t)u$	$iu_t = \nabla^2 u + F(x, y, z, t)u$

Table 1.1: List of generic partial differential equations.

Let's look at the heat equation in one dimension. This could describe the heat conduction in a thin insulated rod of length L. It could also describe the diffusion of pollutant in a long narrow stream, or the flow of traffic down a road. In problems involving diffusion processes, one instead calls this equation the diffusion equation. [We provide a derivation in Section 1.3.2.]

A typical initial-boundary value problem for the heat equation would be that initially one has a temperature distribution $u(x, 0) = f(x)$. Placing the bar in an ice bath and assuming the heat flow is only through the ends of the bar, one has the boundary conditions $u(0, t) = 0$ and $u(L, t) = 0$. Of course, we are dealing with Celsius temperatures and we assume there is

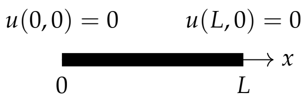


Figure 1.1: One dimensional heated rod of length L.

plenty of ice to keep that temperature fixed at each end for all time as seen in Figure 1.1. So, the problem one would need to solve is given as [IC = initial condition and BC = boundary condition.]

1D Heat Equation

$$\begin{array}{llll} \text{PDE} & u_t = k u_{xx}, & 0 < t, \quad 0 \leq x \leq L, & \\ \text{IC} & u(x,0) = f(x), & 0 < x < L, & \\ \text{BC} & u(0,t) = 0, & t > 0, & \\ & u(L,t) = 0, & t > 0, & \end{array} \qquad (1.1)$$

Here, k is the heat conduction constant and is determined using properties of the bar.

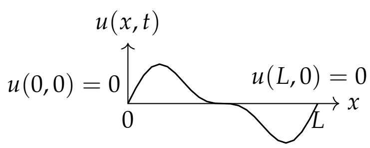


Figure 1.2: One dimensional string of length L.

Another problem that will come up in later discussions is that of the vibrating string. A string of length L is stretched out horizontally with both ends fixed such as a violin string as shown in Figure 1.2. Let $u(x,t)$ be the vertical displacement of the string at position x and time t. The motion of the string is governed by the one dimensional wave equation. [See the derivation in Section 1.3.1.] The string might be plucked, giving the string an initial profile, $u(x,0) = f(x)$, and possibly each point on the string has an initial velocity $u_t(x,0) = g(x)$. The initial-boundary value problem for this problem is given below.

1D Wave Equation

$$\begin{array}{llll} \text{PDE} & u_{tt} = c^2 u_{xx} & 0 < t, \quad 0 \leq x \leq L & \\ \text{IC} & u(x,0) = f(x) & 0 < x < L & \\ & u_t(x,0) = g(x) & 0 < x < L & \\ \text{BC} & u(0,t) = 0 & t > 0 & \\ & u(L,t) = 0 & t > 0 & \end{array} \qquad (1.2)$$

In this problem c is the wave speed in the string. It depends on the mass per unit length of the string, μ, and the tension, τ, placed on the string.

There is a rich history on the study of these and other partial differential equations and much of this involves trying to solve problems in physics.[2] Consider the one dimensional wave motion in the string. Physically, the speed of these waves depends on the tension in the string and its mass density. The frequencies we hear are then related to the string shape, or the allowed wavelengths across the string. We will be interested the harmonics, or pure sinusoidal waves, of the vibrating string and how a general wave on the string can be represented as a sum over such harmonics. This will take us into the field of spectral, or Fourier, analysis. The solution of the heat equation also involves the use of Fourier analysis. However, in this case there are no oscillations in time.

[2] A controversy arose between Leonhard Euler (1707-1783), Jean-Baptiste le Rond d'Alembert (1717-1783), and Daniel Bernoulli (1700-1782). See the story below.

There are many applications that are studied using spectral analysis. At the root of these studies is the belief that continuous waveforms are comprised of a number of harmonics. Such ideas stretch back to the Pythagoreans' study of the vibrations of strings, which led to their program of a world of harmony. This idea was carried further by Johannes Kepler (1571-1630) in his harmony of the spheres approach to planetary orbits. In the 1700's others worked on the superposition theory for vibrating waves on a stretched spring, starting with the wave equation and leading to the superposition of right and left traveling waves. This work was carried out by people such as John Wallis (1616-1703), Brook Taylor (1685-1731) and Jean-Baptiste le Rond d'Alembert (1717-1783).

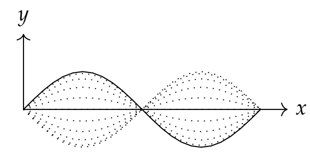


Figure 1.3: Plot of the second harmonic of a vibrating string at different times.

In 1742 d'Alembert solved the wave equation

$$c^2\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0,$$

where y is the string height and c is the wave speed. However, this solution led him and others, like Leonhard Euler (1707-1783) and Daniel Bernoulli (1700-1782), to investigate what "functions" could be the solutions of this equation. In fact, this led to a more rigorous approach to the study of analysis by first coming to grips with the concept of a function. For example, in 1749 Euler sought the solution for a plucked string in which case the initial condition $y(x,0) = h(x)$ has a discontinuous derivative! (We will see how this led to important questions in analysis.)

In 1753 Daniel Bernoulli viewed the solutions as a superposition of simple vibrations, or harmonics. Such superpositions amounted to looking at solutions of the form

$$y(x,t) = \sum_k a_k \sin\frac{k\pi x}{L}\cos\frac{k\pi ct}{L},$$

where the string extends over the interval $[0, L]$ with fixed ends at $x = 0$ and $x = L$.

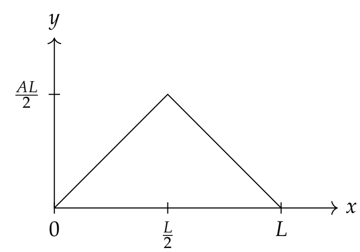


Figure 1.4: Plot of an initial condition for a plucked string.

However, the initial profile for such superpositions is given by

$$y(x,0) = \sum_k a_k \sin\frac{k\pi x}{L}.$$

It was determined that many functions could not be represented by a finite number of harmonics, even for the simply plucked string in Figure 1.4 given by an initial condition of the form

$$y(x,0) = \begin{cases} Ax, & 0 \le x \le L/2 \\ A(L-x), & L/2 \le x \le L \end{cases}$$

Thus, the solution consists generally of an infinite series of trigonometric functions. But it left the question as to how such a non-smooth function could be described by trigonometric functions, which are continuous. Also, people were still trying to understand infinite series and convergence. So, historically, people were not comfortable with these infinite series of trigonometric functions describing the vibrating string. In 1761 Joseph-Louis Lagrange (1736-1813) arrived at a similar solution. Eventually, the problems were settled based on Joseph Fourier's (1768-1830) work around

1820 on heat conduction and trigonometric series and Gustav Lejeune Dirichlet's (1805-1859) work on the convergence of what we now call Fourier series.

Such series expansions were also of importance in Joseph Fourier's (1768-1830) solution of the heat equation. The use of Fourier expansions has become an important tool in the solution of linear partial differential equations, such as the wave equation and the heat equation. More generally, using a technique called the Method of Separation of Variables, allowed higher dimensional problems to be reduced to one dimensional boundary value problems. However, these studies led to very important questions, which in turn opened the doors to whole fields of analysis. Some of the problems raised were

The one dimensional version of the heat equation is a partial differential equation for $u(x,t)$ of the form

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$

Solutions satisfying boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$, are of the form

$$u(x,t) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-n^2\pi^2 t/L^2}.$$

In this case, setting $u(x,0) = f(x)$, one has to satisfy the condition

$$f(x) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

This is another example leading to an infinite series of trigonometric functions.

1. What functions can be represented as the sum of trigonometric functions?
2. How can a function with discontinuous derivatives be represented by a sum of smooth functions, such as the above sums of trigonometric functions?
3. Do such infinite sums of trigonometric functions actually converge to the functions they represent?

There are many other systems for which it makes sense to interpret the solutions as sums of sinusoids of particular frequencies. For example, consider ocean waves. Ocean waves are affected by the gravitational pull of the moon and the sun and other numerous forces. These lead to the tides, which in turn have their own periods of motion. In an analysis of wave heights, one can separate out the tidal components by making use of Fourier analysis. These tidal constituents are due to things like the Earth's rotation, the positions of the Moon and Sun relative to the Earth, and the Moon's elevation above the Equator. there are several hundred constituents of which just a few are dominant.

In the Section 1.4 we describe how to go about solving these linear partial differential equations using the method of separation of variables. We will find that in order to accommodate the initial conditions, we will need to introduce Fourier series before we can complete the problems, which will be the subject of the following chapter. However, we first derive the one-dimensional wave and heat equations.

1.2 *Boundary Value Problems*

You might have only solved initial value problems in your undergraduate differential equations class. For an initial value problem one has to solve a differential equation subject to conditions on the unknown function and its derivatives at one value of the independent variable. For example, for $x = x(t)$ we could have the initial value problem

$$x'' + x = 2, \quad x(0) = 1, \quad x'(0) = 0. \tag{1.3}$$

Typically, initial value problems involve time dependent functions and boundary value problems are spatial. So, with an initial value problem one knows how a system evolves in terms of the differential equation and the state of the system at some fixed time. Then one seeks to determine the state of the system at a later time.

Example 1.1. Solve the initial value problem, $x'' + 4x = \cos t, \quad x(0) = 1, \quad x'(0) = 0$.

Note that the conditions are provided at one time, $t = 0$. Thus, this is an initial value problem. Recall from your course on differential equations that we first find the general solution and then apply the initial conditions. Furthermore, this equation is a nonhomogeneous differential equation, so the solution is a sum of a solution of the homogeneous equation and a particular solution of the nonhomogeneous equation, $x(t) = x_h(t) + x_p(t)$. [See the ordinary differential equations review in the Appendix.]

The solution of $x'' + 4x = 0$ is easily found as

$$x_h(t) = c_1 \cos 2t + c_2 \sin 2t.$$

The particular solution is found using the Method of Undetermined Coefficients. We guess a solution of the form

Method of Undetermined Coefficients.

$$x_p(t) = A \cos t + B \sin t.$$

Differentiating twice, we have

$$x_p''(t) = -(A \cos t + B \sin t).$$

So,

$$x_p'' + 4x_p = -(A \cos t + B \sin t) + 4(A \cos t + B \sin t).$$

Comparing the right hand side of this equation with $\cos t$ in the original problem, we are led to setting $B = 0$ and $A = \frac{1}{3} \cos t$. Thus, the general solution is

$$x(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{3} \cos t.$$

We now apply the initial conditions to find the particular solution. The first condition, $x(0) = 1$, gives

$$1 = c_1 + \frac{1}{3}.$$

Thus, $c_1 = \frac{2}{3}$. Using this value for c_1, the second condition, $x'(0) = 0$, gives $c_2 = 0$. Therefore,

$$x(t) = \frac{1}{3}(2 \cos 2t + \cos t).$$

For boundary values problems, one knows how each point responds to its neighbors, but there are conditions that have to be satisfied at the endpoints. An example would be a horizontal beam supported at the ends, like a bridge. The shape of the beam under the influence of gravity, or other forces, would lead to a differential equation and the boundary conditions at the beam ends would affect the solution of the problem. There are also a variety of other types of boundary conditions. In the case of a beam, one end could be fixed and the other end could be free to move. We will explore the effects of different boundary conditions in our discussions and exercises. But, we will first solve a simple boundary value problem which is a slight modification of the above problem.

Example 1.2. Solve the boundary value problem, $x'' + x = 2, \quad x(0) = 1, \quad x(1) = 0.$

Note that the conditions at $t = 0$ and $t = 1$ make this a boundary value problem since the conditions are given at two different points. As with initial value problems, we need to find the general solution and then apply any conditions that we may have. This is a nonhomogeneous differential equation, so the solution is a sum of a solution of the homogeneous equation and a particular solution of the nonhomogeneous equation, $x(t) = x_h(t) + x_p(t)$. The solution of $x'' + x = 0$ is easily found as

$$x_h(t) = c_1 \cos t + c_2 \sin t.$$

The particular solution is again found using the Method of Undetermined Coefficients,

$$x_p(t) = 2.$$

Thus, the general solution is

$$x(t) = 2 + c_1 \cos t + c_2 \sin t.$$

We now apply the boundary conditions and see if there are values of c_1 and c_2 that yield a solution to this boundary value problem. The first condition, $x(0) = 0$, gives

$$0 = 2 + c_1.$$

Thus, $c_1 = -2$. Using this value for c_1, the second condition, $x(1) = 1$, gives

$$0 = 2 - 2\cos 1 + c_2 \sin 1.$$

This yields

$$c_2 = \frac{2(\cos 1 - 1)}{\sin 1}.$$

We have found that there is a solution to the boundary value problem and it is given by[3]

$$x(t) = 2\left(1 - \cos t + \frac{(\cos 1 - 1)}{\sin 1} \sin t\right).$$

[3] This solution can be rewritten as

$$x(t) = 2\frac{\sin 1 - \sin t + \sin(t-1)}{\sin 1}.$$

You can verify that it satisfies the boundary conditions in your head.

Boundary value problems arise in many physical systems, just as the initial value problems we have seen earlier. We will see in the next sections that boundary value problems for ordinary differential equations often appear in the solutions of partial differential equations. However, there is no guarantee that we will have unique solutions of our boundary value problems as we had found in the example above.

Now that we understand simple boundary value problems for ordinary differential equations, we can turn to initial-boundary value problems for partial differential equations. We will see that a common method for studying these problems is to use the method of separation of variables. In this method the problem of solving partial differential equations is to separate the partial differential equation into several ordinary differential equations of which some are boundary value problems of the sort seen in this section.

1.3 *Derivation of Generic 1D Equations*

1.3.1 *Derivation of Wave Equation for String*

THE WAVE EQUATION FOR A ONE DIMENSIONAL STRING is derived based upon simply looking at Newton's Second Law of Motion for a piece of the string plus a few simple assumptions, such as small amplitude oscillations and constant density.

We begin with Newton's Second Law of Motion, $\mathbf{F} = m\mathbf{a}$. The mass of a piece of string of length Δs is $m = \rho(x)\,\Delta s$. From Figure (1.5) an incremental length of the string is given by

$$\Delta s^2 = \Delta x^2 + \Delta u^2.$$

The piece of string undergoes an acceleration of $a = \dfrac{\partial^2 u}{\partial t^2}$.

We will assume that the main force acting on the string is that of tension. Let $T(x,t)$ be the magnitude of the tension acting on the left end of the piece of string. Then, on the right end the tension is $T(x+\Delta x, t)$. At these points the tension makes an angle to the horizontal of $\theta(x,t)$ and $\theta(x+\Delta x, t)$, respectively.

Assuming that there is no horizontal acceleration, the x-component of Newton's Second Law of Motion for the string element is given by

The wave equation is derived from $\mathbf{F} = m\mathbf{a}$.

$$0 = T(x+\Delta x, t)\cos\theta(x+\Delta x, t) - T(x,t)\cos\theta(x,t).$$

The vertical component is given by

$$\rho(x)\Delta s \frac{\partial^2 u}{\partial t^2} = T(x+\Delta x, t)\sin\theta(x+\Delta x, t) - T(x,t)\sin\theta(x,t)$$

The length of the piece of string can be written in terms of Δx,

$$\Delta s = \sqrt{\Delta x^2 + \Delta u^2} = \sqrt{1 + \left(\frac{\Delta u}{\Delta x}\right)^2}\,\Delta x.$$

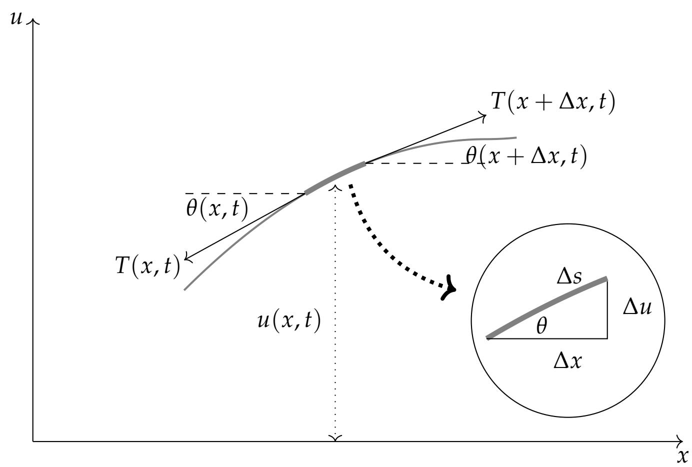


Figure 1.5: A small piece of string is under tension.

and the right hand sides of the component equation can be expanded about $\Delta x = 0$, to obtain

$$\begin{aligned} T(x+\Delta x, t)\cos\theta(x+\Delta x, t) - T(x,t)\cos\theta(x,t) &\approx \frac{\partial (T\cos\theta)}{\partial x}(x,t)\Delta x \\ T(x+\Delta x, t)\sin\theta(x+\Delta x, t) - T(x,t)\sin\theta(x,t) &\approx \frac{\partial (T\sin\theta)}{\partial x}(x,t)\Delta x. \end{aligned}$$

Furthermore, we note that

$$\tan\theta = \lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x}.$$

Now we can divide these component equations by Δx and let $\Delta x \to 0$. This gives the approximations

$$\begin{aligned} 0 &= \frac{T(x+\Delta x, t)\cos\theta(x+\Delta x, t) - T(x,t)\cos\theta(x,t)}{\Delta x} \\ &\approx \frac{\partial (T\cos\theta)}{\partial x}(x,t) \\ \rho(x)\frac{\partial^2 u}{\partial t^2}\frac{\Delta s}{\Delta x} &= \frac{T(x+\Delta x, t)\sin\theta(x+\Delta x, t) - T(x,t)\sin\theta(x,t)}{\Delta x} \\ \rho(x)\frac{\partial^2 u}{\partial t^2}\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2} &\approx \frac{\partial (T\sin\theta)}{\partial x}(x,t). \end{aligned} \tag{1.4}$$

We will assume a small angle approximation, giving

$$\sin\theta \approx \tan\theta = \frac{\partial u}{\partial x},$$

$\cos\theta \approx 1$, and

$$\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2} \approx 1.$$

Then, the horizontal component becomes

$$\frac{\partial T(x,t)}{\partial x} = 0.$$

Therefore, the magnitude of the tension $T(x,t) = T(t)$ is at most time dependent. The vertical component equation is now

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T(t)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = T(t)\frac{\partial^2 u}{\partial x^2}.$$

Assuming that ρ and T are constant and defining

$$c^2 = \frac{T}{\rho},$$

we obtain the one dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$

Note the tension is a force and has units of mass times acceleration. The (linear) density is mass per length. So, the units of $\frac{T}{\rho}$ are acceleration time length, or $(length/time)^2$. Therefore, c has units of speed. Therefore, c is the wave speed.

1.3.2 Derivation of 1D Heat Equation

CONSIDER A ONE DIMENSIONAL ROD of length L as shown in Figure 1.6. It is heated and allowed to sit. The heat equation is the governing equation which allows us to determine the temperature of the rod at a later time.

In some applications the heat equation is called the diffusion equation. The derivation in Section 7.4.1 leads to the conservation law 7.28,

$$u_t(x,t) + \phi_x(x,t) = f(x,t).$$

Defining the flux as $\phi(x,t) = -ku_x(x,t)$, leads to a diffusion equation with a source,

$$u_t = ku_{xx} + f.$$

We begin with some simple thermodynamics. Recall that to raise the temperature of a mass m by ΔT takes thermal energy given by

$$Q = mc\Delta T,$$

assuming the mass does not go through a phase transition (like melting). Here c is the specific heat capacity of the substance. So, we will begin with the heat content of the rod at position x and time t as

$$Q = mcT(x,t)$$

and assume that m and c are constant.

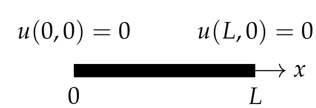


Figure 1.6: One dimensional heated rod of length L.

We will also need Fourier's law of heat transfer or heat conduction . This law simply states that heat energy flows from warmer to cooler regions and is written in terms of the heat energy flux, $\phi(x,t)$. The heat energy flux, or flux density, gives the rate of energy flow per area. Thus, the amount of heat energy flowing over the left end of the region of cross section A in time Δt is given $\phi(x,t)\Delta t A$. The units of $\phi(x,t)$ are then $\mathrm{J/s/m^2} = \mathrm{W/m^2}$.

Fourier's law of heat conduction states that the flux density is proportional to the gradient of the temperature,

$$\phi = -K\frac{\partial T}{\partial x}.$$

Here K is the thermal conductivity and the negative sign takes into account the direction of flow from higher to lower temperatures.

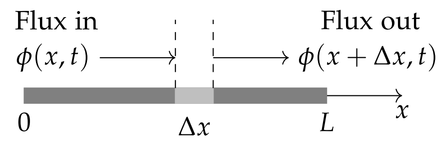


Figure 1.7: A one dimensional rod of length L. Heat can flow through increment Δx.

Now, we make use of the conservation of energy. Consider a small section of the rod of width Δx as shown in Figure 1.7. The rate of change of the energy through this section is due to energy flow through the ends. Namely,

$$\text{Rate of change of heat energy} = \text{Heat in} - \text{Heat out}.$$

The energy content of the small segment of the rod is given by

$$\Delta Q = (\rho A \Delta x) c T(x, t + \Delta t) - (\rho A \Delta x) c T(x, t).$$

The flow rates across the boundaries are given by the flux.

$$(\rho A \Delta x) c T(x, t + \Delta t) - (\rho A \Delta x) c T(x, t) = [\phi(x, t) - \phi(x + \Delta x, t)] \Delta t A.$$

Dividing by Δx and Δt and letting $\Delta x, \Delta t \to 0$, we obtain

$$\frac{\partial T}{\partial t} = -\frac{1}{c\rho} \frac{\partial \phi}{\partial x}.$$

Using Fourier's law of heat conduction,

$$\frac{\partial T}{\partial t} = \frac{1}{c\rho} \frac{\partial}{\partial x} \left(K \frac{\partial T}{\partial x} \right).$$

Assuming K, c, and ρ are constant, we have the one dimensional heat equation as used in the text:

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2},$$

where $k = \frac{k}{c\rho}$.

1.4 *Separation of Variables*

SOLVING MANY OF THE LINEAR PARTIAL DIFFERENTIAL EQUATIONS presented in the first section can be reduced to solving ordinary differential equations. We will demonstrate this by solving the initial-boundary value problem for the heat equation as given in (1.1). We will employ a method typically used in studying linear partial differential equations, called the Method of Separation of Variables. In the next subsections we describe how this method works for the one-dimensional heat equation, one-dimensional wave equation, and the two-dimensional Laplace equation.

1.4.1 The 1D Heat Equation

WE WANT TO SOLVE THE HEAT EQUATION,

$$u_t = k u_{xx}, \quad 0 < t, \quad 0 \leq x \leq L.$$

subject to the boundary conditions

$$u(0,t) = 0, u(L,t) = 0, \quad t > 0,$$

and the initial condition

$$u(x,0) = f(x), \quad 0 < x < L.$$

We begin by assuming that u can be written as a product of single variable functions of each independent variable,

$$u(x,t) = X(x)T(t).$$

Substituting this guess into the heat equation, we find that

$$XT' = kX''T.$$

The prime denotes differentiation with respect to the independent variable and we will suppress the independent variable in the following unless needed for emphasis.

Solution of the 1D heat equation using the Method of Separation of Variables.

Dividing both sides of this result by k and $u = XT$, yields

$$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X}.$$

We have separated the functions of time on one side and space on the other side. The constant k could be on either side of this expression, but we moved it to make later computations simpler.

The only way that a function of t equals a function of x is if the functions are constant functions. Therefore, we set each function equal to a constant, λ : [For example, if $Ae^{ct} = ax^2 + b$ is possible for any x or t, then this is only possible if $a = 0$, $c = 0$ and $b = A$.]

$$\underbrace{\frac{1}{k}\frac{T'}{T}}_{\text{function of } t} = \underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{\lambda.}_{\text{constant}}$$

This leads to two equations:

$$T' = k\lambda T, \tag{1.5}$$

$$X'' = \lambda X. \tag{1.6}$$

These are ordinary differential equations. The general solutions to these constant coefficient equations are readily found as

$$T(t) = Ae^{k\lambda t}, \tag{1.7}$$

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}. \tag{1.8}$$

We need to be a little careful at this point. The aim is to force the final solutions to satisfy both the boundary conditions and initial conditions. Also, we should note that λ is arbitrary and may be positive, zero, or negative. We first look at how the boundary conditions on $u(x,t)$ lead to conditions on $X(x)$.

The first boundary condition is $u(0,t) = 0$. This implies that

$$X(0)T(t) = 0, \quad \text{for all } t.$$

The only way that this is true is if $X(0) = 0$. Similarly, $u(L,t) = 0$ for all t implies that $X(L) = 0$. So, we have to solve the boundary value problem

$$X'' - \lambda X = 0, \quad X(0) = 0, X(L) = 0. \tag{1.9}$$

An obvious solution is $X \equiv 0$. However, this implies that $u(x,t) = 0$, which is not an interesting solution. We call such solutions, $X \equiv 0$, trivial solutions and will seek nontrivial solution for these problems.

There are three cases to consider, depending on the sign of λ. [Note, eventually you need not consider all of the cases as you will inherently know which values of λ lead to nontrivial solutions.]

Case I. $\underline{\lambda > 0}$

In this case we have the exponential solutions

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}. \tag{1.10}$$

For $X(0) = 0$, we have

$$0 = c_1 + c_2.$$

We will take $c_2 = -c_1$. Then,

$$X(x) = c_1(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}) = 2c_1 \sinh \sqrt{\lambda}x.$$

Applying the second condition, $X(L) = 0$ yields

$$c_1 \sinh \sqrt{\lambda}L = 0.$$

This will be true only if $c_1 = 0$, since $\lambda > 0$ and $L > 0$. Thus, the only solution in this case is the trivial solution, $X(x) = 0$.

Case II. $\underline{\lambda = 0}$

For this case it is easier to set λ to zero in the differential equation. So, $X'' = 0$. Integrating twice, one finds

$$X(x) = c_1 x + c_2.$$

Setting $x = 0$, we have $c_2 = 0$, leaving $X(x) = c_1 x$. Setting $x = L$, we find $c_1 L = 0$. So, $c_1 = 0$ and we are once again left with a trivial solution.

Case III. $\underline{\lambda < 0}$

In this case is would be simpler to write $\lambda = -\mu^2$. Then the differential equation is

$$X'' + \mu^2 X = 0.$$

The general solution is

$$X(x) = c_1 \cos \mu x + c_2 \sin \mu x.$$

At $x = 0$ we get $0 = c_1$. This leaves $X(x) = c_2 \sin \mu x$.

At $x = L$, we find

$$0 = c_2 \sin \mu L.$$

So, either $c_2 = 0$ or $\sin \mu L = 0$. $c_2 = 0$ leads to a trivial solution again. But, there are cases when the sine is zero. Namely,

$$\mu L = n\pi, \quad n = 1, 2, \ldots.$$

Note that $n = 0$ is not included since this leads to a trivial solution. Also, negative values of n are redundant, since the sine function is an odd function.

In summary, we can find solutions to the boundary value problem (1.9) for particular values of λ. The solutions are

$$X_n(x) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

for

$$\lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$

We should note that the boundary value problem in Equation (1.9) is an eigenvalue problem. We can recast the differential equation as

Eigenvalue Problem

$$LX = \lambda X,$$

where

$$L = D^2 = \frac{d^2}{dx^2}$$

is a linear differential operator. The solutions, $X_n(x)$, are called eigenfunctions and the λ_n's are the eigenvalues. We will elaborate more on this characterization later in the next chapter.

We have found the product solutions of the heat equation (1.1) satisfying the boundary conditions. These are

Product solutions.

$$u_n(x,t) = e^{k\lambda_n t} \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots. \tag{1.11}$$

However, these do not necessarily satisfy the initial condition $u(x,0) = f(x)$. What we do get is

$$u_n(x,0) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots.$$

So, if the initial condition is in one of these forms, we can pick out the right value for n and we are done.

For other initial conditions, we have to do more work. Note, since the heat equation is linear, the linear combination of the product solutions is also a solution of the heat equation. The general solution satisfying the given boundary conditions is given as

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \sin \frac{n\pi x}{L}. \tag{1.12}$$

General solution.

The coefficients in the general solution are determined using the initial condition. Namely, setting $t = 0$ in the general solution, we have

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

So, if we know $f(x)$, can we find the coefficients, b_n? If we can, then we will have the solution to the full initial-boundary value problem.

The expression for $f(x)$ is a Fourier sine series. We will need to digress into the study of Fourier series in order to see how one can find the Fourier series coefficients given $f(x)$. The solution is provided in Section 2.5. Before proceeding, we will show that this process is not uncommon by applying the Method of Separation of Variables to the wave equation in the next section.

1.4.2 *The 1D Wave Equation*

In this section we will apply the Method of Separation of Variables to the one dimensional wave equation, given by

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial^2 x}, \quad t > 0, \quad 0 \le x \le L, \tag{1.13}$$

subject to the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0,$$

and the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L.$$

This problem applies to the propagation of waves on a string of length L with both ends fixed so that they do not move. $u(x,t)$ represents the vertical displacement of the string over time. The derivation of the wave equation in Section 1.3.1 assumed that a small vertical displacement and a uniform string. The constant c is the wave speed, given by

$$c = \sqrt{\frac{\tau}{\mu}},$$

where τ is the tension in the string and μ is the mass per unit length. We can understand this in terms of string instruments. The tension can be adjusted to produce different tones and the makeup of the string (nylon or steel, thick or thin) also has an effect. In some cases the mass density is changed simply by using thicker strings. Thus, the thicker strings in a piano produce lower frequency notes.

The u_{tt} term gives the acceleration of a piece of the string. The u_{xx} is the concavity of the string. Thus, for a positive concavity the string is curved upward near the point of interest. Thus, neighboring points tend to pull the string upward towards the equilibrium position. If the concavity is negative, it would cause a negative acceleration.

The solution of this problem is easily found using separation of variables. We let $u(x,t) = X(x)T(t)$. Then we find

$$XT'' = c^2 X'' T,$$

Solution of the 1D wave equation using the Method of Separation of Variables.

which can be rewritten as

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X}.$$

Again, we have separated the functions of time on one side and space on the other side. Therefore, we set each function equal to a constant, λ.

$$\underbrace{\frac{1}{c^2}\frac{T''}{T}}_{\text{function of } t} = \underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{\lambda.}_{\text{constant}}$$

This leads to two equations:

$$T'' = c^2 \lambda T, \tag{1.14}$$

$$X'' = \lambda X. \tag{1.15}$$

As before, we have the boundary conditions on $X(x)$:

$$X(0) = 0, \quad \text{and} \quad X(L) = 0,$$

giving the solutions, as shown in Figure 1.8,

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = -\left(\frac{n\pi}{L}\right)^2.$$

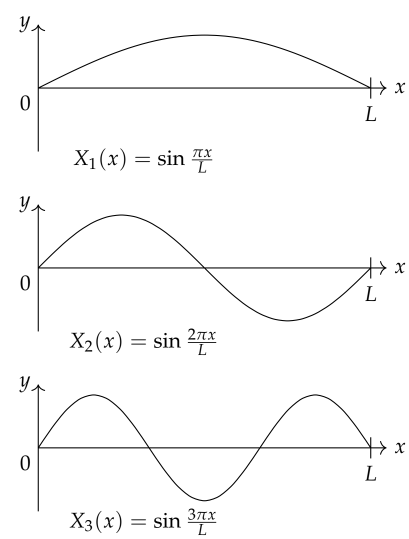


Figure 1.8: The first three harmonics, eigenfunctions with $n = 1, 2, 3$, of the vibrating string.

The main difference from the solution of the heat equation is the form of the time function. Namely, from Equation (1.14) we have to solve

$$T'' + \left(\frac{n\pi c}{L}\right)^2 T = 0. \tag{1.16}$$

This equation takes a familiar form. We let

$$\omega_n = \frac{n\pi c}{L},$$

then we have

$$T'' + \omega_n^2 T = 0.$$

This is the differential equation for simple harmonic motion and ω_n is the angular frequency. The solutions are easily found as

$$T(t) = A_n \cos\omega_n t + B_n \sin\omega_n t. \tag{1.17}$$

Therefore, we have found that the product solutions of the wave equation take the forms $\sin \frac{n\pi x}{L} \cos \omega_n t$ and $\sin \frac{n\pi x}{L} \sin \omega_n t$. The general solution, a superposition of all product solutions, is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right] \sin \frac{n\pi x}{L}. \tag{1.18}$$

General solution.

This solution satisfies the wave equation and the boundary conditions. We still need to satisfy the initial conditions. Note that there are two initial conditions, since the wave equation is second order in time.

First, we have $u(x,0) = f(x)$. Thus,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}. \tag{1.19}$$

In order to obtain the condition on the initial velocity, $u_t(x,0) = g(x)$, we need to differentiate the general solution with respect to t:

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} \left[-A_n \sin \frac{n\pi ct}{L} + B_n \cos \frac{n\pi ct}{L} \right] \sin \frac{n\pi x}{L}. \tag{1.20}$$

Then, we have from the initial velocity

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin \frac{n\pi x}{L}. \tag{1.21}$$

So, applying the two initial conditions, we have found that $f(x)$ and $g(x)$, are represented as Fourier sine series. In order to complete the problem we need to determine the coefficients A_n and B_n for $n = 1, 2, 3, \ldots$. Once we have these, we have the complete solution to the wave equation. This is done in Section 2.6.

We had seen similar results for the heat equation. In the next chapter we will find out how to determine these Fourier coefficients for such series of sinusoidal functions. However, we can use other techniques to solve the wave equation which we will explore next.

1.5 *d'Alembert's Solution of the Wave Equation*

A GENERAL SOLUTION OF THE ONE-DIMENSIONAL WAVE EQUATION can be found. This solution was first found by Jean-Baptiste le Rond d'Alembert (1717-1783) and is referred to as d'Alembert's formula. In this section we will derive d'Alembert's formula and then use it to arrive at solutions to the wave equation on infinite, semi-infinite, and finite intervals.

We consider the wave equation in the form $u_{tt} = c^2 u_{xx}$ and introduce the transformation

$$u(x,t) = U(\xi, \eta), \quad \text{where} \quad \xi = x + ct \text{ and } \quad \eta = x - ct.$$

We will see that ξ and η are the characteristics of the wave equation.

In order to transform the wave equation into an equation in the new variables, we need to see how the derivatives transform. For example, we apply a multivariable chain rule,

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial U(\xi,\eta)}{\partial x} \\ &= \frac{\partial U(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial U(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial x} \\ &= \frac{\partial U(\xi,\eta)}{\partial \xi} + \frac{\partial U(\xi,\eta)}{\partial \eta}. \end{aligned} \tag{1.22}$$

Therefore, as an operator, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}.$$

Similarly, one can show that

$$\frac{\partial}{\partial t} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}.$$

Using these results, the wave equation becomes

$$\begin{aligned} 0 &= u_{tt} - c^2 u_{xx} \\ &= \left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) u \\ &= \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u \\ &= \left(c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta} + c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right)\left(c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta} - c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}\right) U \\ &= -4c^2\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} U. \end{aligned} \tag{1.23}$$

Therefore, the wave equation has transformed into the simpler equation,

$$U_{\eta\xi} = 0.$$

Not only is this simpler, but as we will see in the next section, the wave equation is a hyperbolic equation. Of course, it is also easy to integrate. Since

$$\frac{\partial}{\partial \eta}\left(\frac{\partial U}{\partial \xi}\right) = 0,$$

we find that

$$\frac{\partial U}{\partial \xi} = \text{ constant with respect to } \eta = \Gamma(\xi).$$

A further integration gives

$$U(\xi,\eta) = \int^{\xi} \Gamma(\xi')\, d\xi' + G(\eta) \equiv F(\xi) + G(\eta).$$

Therefore, we have as the general solution of the wave equation,

$u(x,t)$ = sum of left and right traveling waves.

$$u(x,t) = F(x+ct) + G(x-ct), \tag{1.24}$$

where F and G are two arbitrary, twice differentiable functions. As t is increased, we see that $F(x+ct)$ gets horizontally shifted to the left and $G(x-ct)$ gets horizontally shifted to the right. As a result, we conclude that the solution of the wave equation can be seen as the sum of left and right traveling waves.

Let's use initial conditions to solve for the unknown functions. We let

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad |x| < \infty.$$

Applying this to the general solution, we have

$$f(x) = F(x) + G(x) \tag{1.25}$$
$$g(x) = c[F'(x) - G'(x)]. \tag{1.26}$$

We need to solve for $F(x)$ and $G(x)$ in terms of $f(x)$ and $g(x)$. Integrating Equation (1.26), we have

$$\frac{1}{c}\int_0^x g(s)\,ds = F(x) - G(x) - F(0) + G(0).$$

Adding this result to Equation (1.25), and solving for $F(x)$ gives

$$F(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\,ds + \frac{1}{2}[F(0) - G(0)].$$

Subtracting from Equation (1.25) solving for $G(x)$, gives

$$G(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds - \frac{1}{2}[F(0) - G(0)].$$

Now we can write out the solution $u(x,t) = F(x+ct) + G(x-ct)$, yielding d'Alembert's solution

d'Alembert's solution

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds. \tag{1.27}$$

When $f(x)$ and $g(x)$ are defined for all $x \in R$, the solution is well-defined. However, there are problems on more restricted domains. In the next examples we will consider the semi-infinite and finite length string problems.

For each example we will need to consider the domain of dependence and the domain of influence of specific points. These concepts are shown in Figure 1.9. The domain of dependence of point P is red region. The point P depends on the values of u and u_t at points inside this domain. The domain of influence of P is the blue region. The points in this region are influenced by the values of u and u_t at P.

Example 1.3. Use d'Alembert's solution to solve

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x < \infty.$$

The d'Alembert solution is not well-defined for this problem because $f(x-ct)$ is not defined for $x - ct < 0$ for $c, t > 0$. There are similar problems for $g(x)$. This can be seen by looking at the characteristics

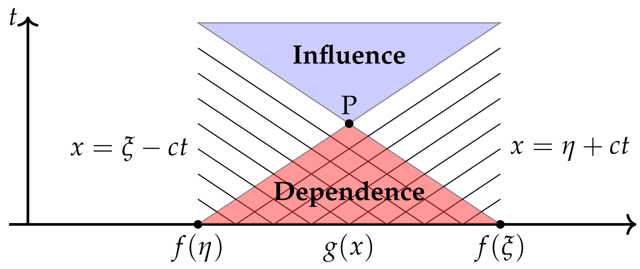


Figure 1.9: The domain of dependence of point P is red region. The point P depends on the values of u and u_t at points inside the domain. The domain of influence of P is the blue region. The points in the region are influenced by the values of u and u_t at P.

For example, at $t = 0$ the value $f(\xi)$ at $x = \xi$ is carried along the characteristic $x = \xi - ct$. Similarly, the value $f(\eta)$ reaches P along the characteristic $x = \eta + ct$. All values of $g(x)$ for $\eta \le x \le \xi$ contribute to the solution, $u(x,t)$ at P.

in the xt-plane. In Figure 1.10 there are characteristics emanating from the points marked by η_0 and ξ_0 that intersect in the domain $x > 0$. The point of intersection of the blue lines have a domain of dependence entirely in the region $x, t > 0$, however the domain of dependence of point P reaches outside this region. Only characteristics $\xi = x + ct$ reach point P, but characteristics $\eta = x - ct$ do not. However, we need $f(\eta)$ and $g(x)$ for $x < ct$ to form a solution.

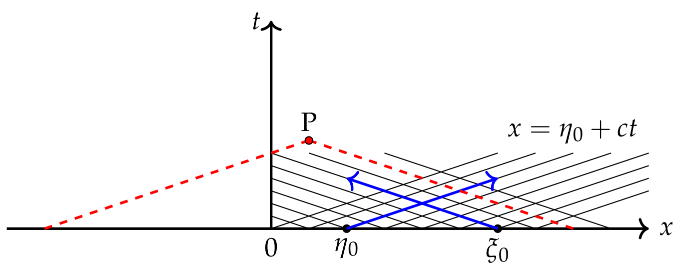


Figure 1.10: The characteristics for the semi-infinite string indicating no characteristics of the form $\eta = x - ct$ reach points like P.

This can be remedied if we specified boundary conditions at $x = 0$. For example, we will assume the end $x = 0$ is fixed,

Fixed end boundary condition

$$u(0,t) = 0, \quad t \ge 0.$$

Imagine an infinite string with one end (at $x = 0$) tied to a pole. In Figure 1.11 we see how this added information along the time axis provides information that can be propagated to points in the first quadrant of the xt-plane.

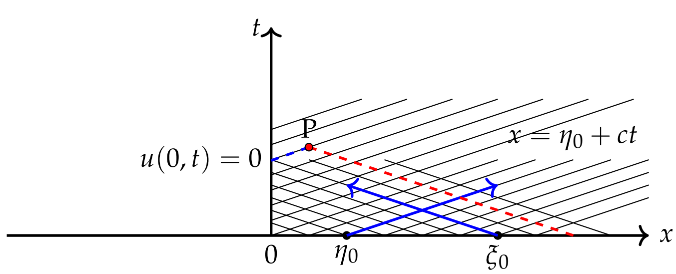


Figure 1.11: The characteristics for the semi-infinite string indicating the boundary condition $u(0,t) = 0$ and the new characteristics starting with $x = 0$.

Let's see how this can be used to construct the solution analytically. Since $u(x,t) = F(x + ct) + G(x - ct)$, we have

$$u(0,t) = F(ct) + G(-ct) = 0.$$

Letting $\zeta = -ct$, this gives $G(\zeta) = -F(-\zeta)$, $\zeta \leq 0$.

Note that

$$\begin{aligned} G(\zeta) &= \frac{1}{2}f(\zeta) - \frac{1}{2c}\int_0^{\zeta} g(s)\,ds \\ -F(-\zeta) &= -\frac{1}{2}f(-\zeta) - \frac{1}{2c}\int_0^{-\zeta} g(s)\,ds \\ &= -\frac{1}{2}f(-\zeta) + \frac{1}{2c}\int_0^{\zeta} g(\sigma)\,d\sigma. \end{aligned} \tag{1.28}$$

Comparing the expressions for $G(\zeta)$ and $-F(-\zeta)$, we see that

$$f(\zeta) = -f(-\zeta), \quad g(\zeta) = -g(-\zeta).$$

These relations imply that we can extend the functions into the region $x < 0$ if we make them odd functions, or what are called odd extensions. An example is shown in Figure 1.12.

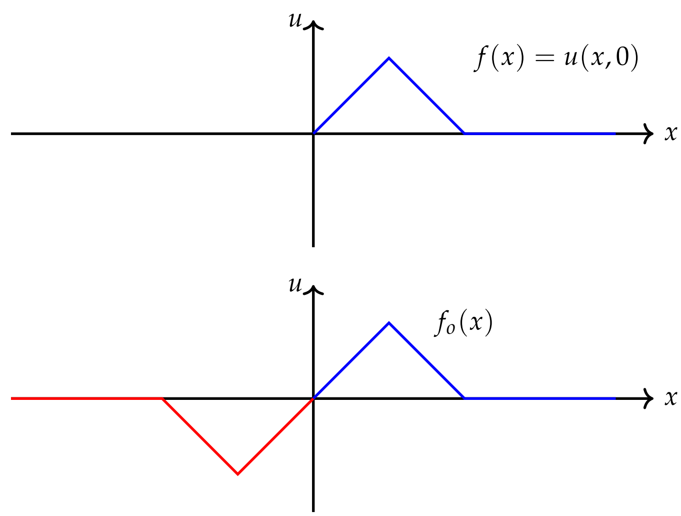


Figure 1.12: The initial condition and its odd extension. The odd extension is obtained through reflection of $f(x)$ about the origin.

Free end boundary condition

Another type of boundary condition is if the end $x = 0$ is free,

$$u_x(0,t) = 0, \quad t \geq 0.$$

In this case we could have an infinite string tied to a ring and that ring is allowed to slide freely up and down a pole.

As a homework exercise you can prove that this leads to

$$f(-\zeta) = f(\zeta), \quad g(-\zeta) = g(\zeta).$$

Thus, we can use an even extension of these function to produce solutions.

Example 1.4. Solve the initial-boundary value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 \leq x < \infty, t > 0. \\ u(x,0) &= \begin{cases} x, & 0 \leq x \leq 1, \\ 2-x, & 1 \leq x \leq 2, \\ 0, & x > 2, \end{cases} \quad 0 \leq x < \infty \end{aligned}$$

$$\begin{aligned} u_t(x,0) &= 0, \quad 0 \le x < \infty. \\ u(0,t) &= 0, \quad t > 0. \end{aligned} \tag{1.29}$$

This is a semi-infinite string with a fixed end. Initially it is plucked to produce a nonzero triangular profile for $0 \le x \le 2$. Since the initial velocity is zero, the general solution is found from d'Alembert's solution,

$$u(x,t) = \frac{1}{2}[f_o(x+ct) + f_o(x-ct)],$$

where $f_o(x)$ is the odd extension of $f(x) = u(x,0)$. In Figure 1.12 we show the initial condition and its odd extension. The odd extension is obtained through reflection of $f(x)$ about the origin.

The next step is to look at the horizontal shifts of $f_o(x)$. Several examples are shown in Figure 1.13.These show the left and right traveling waves.

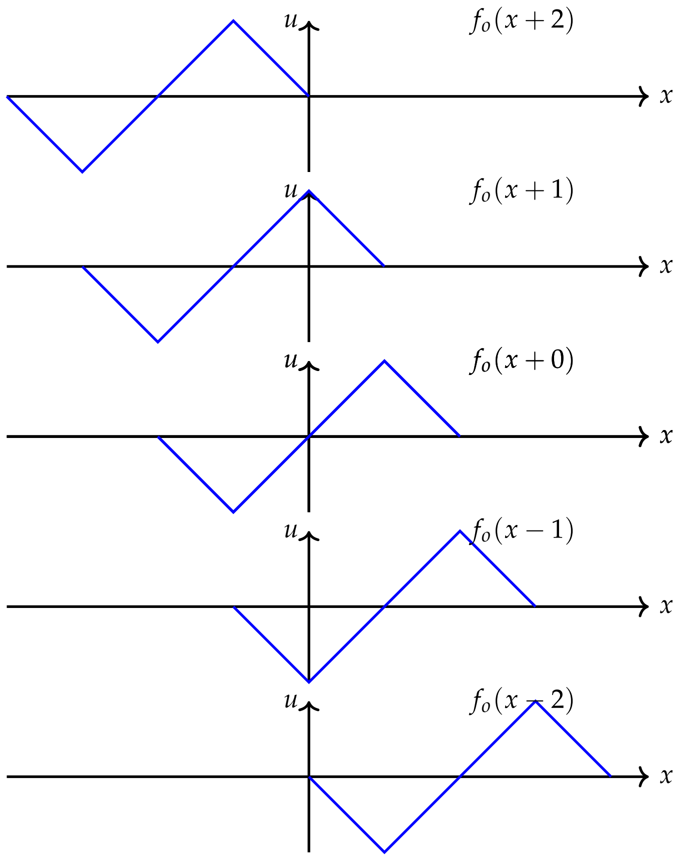
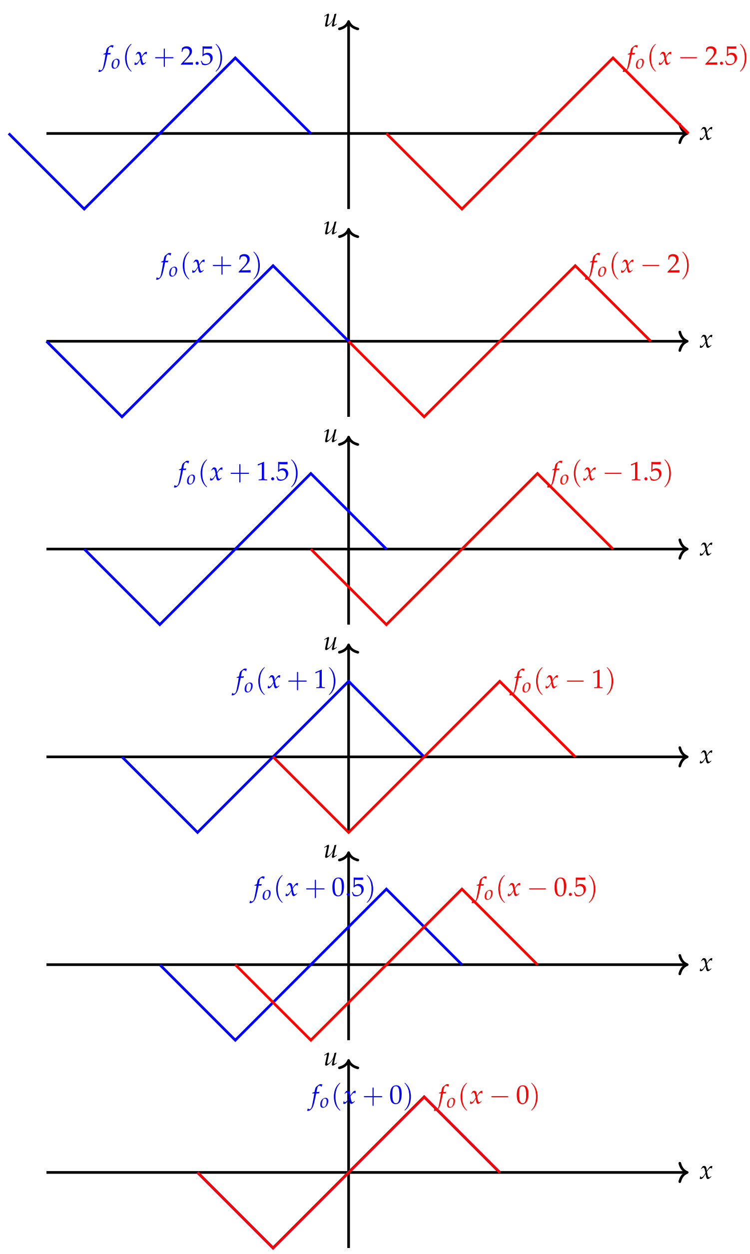


Figure 1.13: Examples of $f_o(x+ct)$ and $f_o(x-ct)$. We see that $f(x-1)$ and $f(x-2)$ have moved to the right, while $f(x+1)$ and $f(x+2)$ have moved to the left.

In Figure 1.14 we show superimposed plots of $f_o(x+ct)$ and $f_o(x-ct)$ for given times. The initial profile is at the bottom. By the time we have $ct = 2$, the full traveling wave has emerged. The solution to the problem emerges on the right side of Figure 1.23 by averaging each plot.

Figure 1.14: Superimposed plots of $f_o(x+ct)$ and $f_o(x-ct)$ for given times. The initial profile in at the bottom. By the time $ct = 2$ the full traveling wave has emerged.



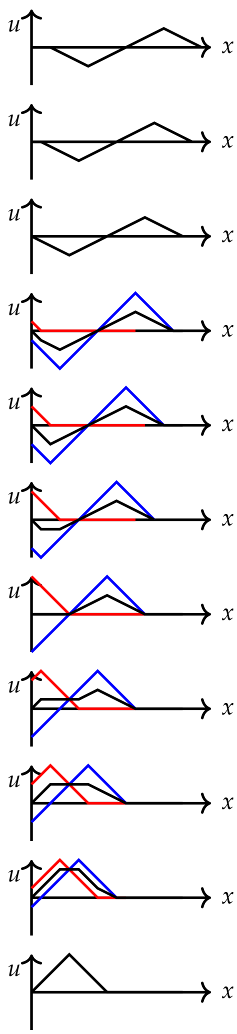


Figure 1.15: On the left is a plot of $f(x+ct)$, $f(x-ct)$ from Figure 1.14 and the average, $u(x,t)$. On the right the solution alone is shown for $ct = 0$ at bottom to $ct = 1$ at top for the semi-infinite string problem

Example 1.5. Use d'Alembert's solution to solve

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \leq x \leq \ell.$$

The general solution of the wave equation was found in the form

$$u(x,t) = F(x+ct) + G(x-ct).$$

However, for this problem we can only obtain information for values of x and t such that $0 \leq x+ct \leq \ell$ and $0 \leq x-ct \leq \ell$. In Figure 1.16 the characteristics $x = \xi + ct$ and $x = \eta - ct$ for $0 \leq \xi, \eta \leq \ell$ are shown. The main (gray) triangle, which is the domain of dependence of the point $(\ell, 2, \ell/2c)$, is the only region in which the solution can be found based solely on the initial conditions. As with the previous problem, boundary conditions will need to be given in order to extend the domain of the solution.

In the last example we saw that a fixed boundary at $x = 0$ could be satisfied when $f(x)$ and $g(x)$ are extended as odd functions. In Figure 1.17 we indicate how the characteristics are affected by drawing in the new ones as red dashed lines. This allows us to now construct solutions based on the initial conditions under the line $x = \ell - ct$ for $0 \leq x \leq \ell$. The new region for which we can construct solutions from the initial conditions is indicated in gray in Figure 1.17.

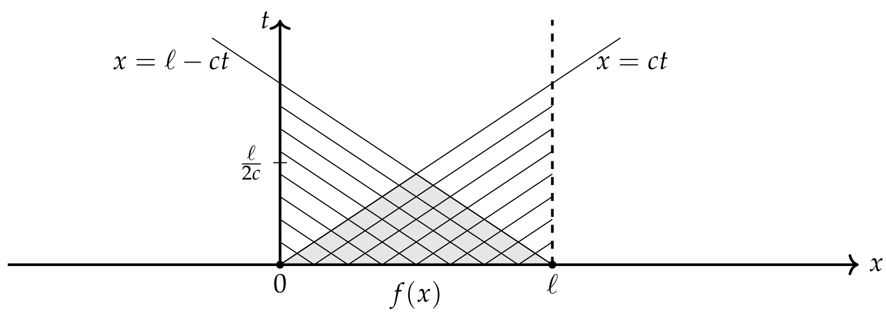


Figure 1.16: The characteristics emanating from the interval $0 \leq x \leq \ell$ for the finite string problem.

We can add characteristics on the right by adding a boundary condition at $x = \ell$. Again, we could use fixed $u(\ell, t) = 0$, or free, $u_x(\ell, t) = 0$, boundary conditions. This allows us to now construct solutions based on the initial conditions for $\ell \leq x \leq 2\ell$.

Let's consider a fixed boundary condition at $x = \ell$. Then, the solution must satisfy

$$u(\ell, t) = F(\ell + ct) + G(\ell - ct) = 0.$$

To see what this means, let $\zeta = \ell + ct$. Then, this condition becomes (since $ct = \zeta - \ell$)

$$F(\zeta) = -G(2\ell - \zeta), \quad \ell \leq \zeta \leq 2\ell.$$

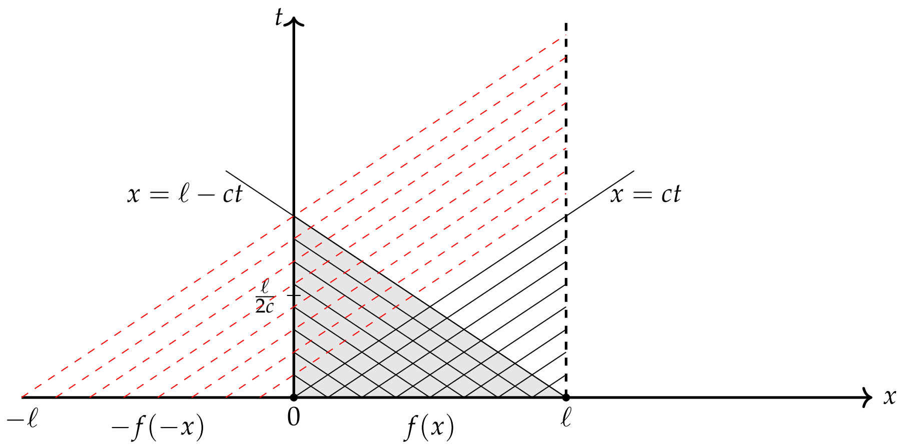


Figure 1.17: The red dashed lines are the characteristics from the interval $[-\ell, 0]$ from using the odd extension about $x = 0$.

Observe that $G(2\ell - \zeta)$ is defined for $0 \leq 2\ell - \zeta \leq \ell$. Therefore, this is a well-defined extension of the domain of $F(x)$.

Using the expressions for $F(x)$ and $G(x)$ in terms of $f(x)$ and $g(x)$, we have

$$\begin{aligned} F(\zeta) &= \frac{1}{2}f(\zeta) + \frac{1}{2c}\int_0^{\ell} g(s)\,ds. \\ -G(2\ell-\zeta) &= -\frac{1}{2}f(2\ell-\zeta) + \frac{1}{2c}\int_0^{2\ell-\zeta} g(s)\,ds \\ &= -\frac{1}{2}f(2\ell-\zeta) - \frac{1}{2c}\int_0^{\zeta} g(2\ell-\sigma)\,d\sigma \end{aligned} \tag{1.30}$$

Comparing the expressions for $G(\zeta)$ and $-G(2\ell - \zeta)$, we see that

$$f(\zeta) = -f(2\ell - \zeta), \quad g(\zeta) = -g(2\ell - \zeta).$$

These relations imply that we can extend the functions into the region $x > \ell$ if we consider an odd extension of $f(x)$ and $g(x)$ about $x = \ell$. This will give the blue dashed characteristics in Figure 1.18 and a larger gray region to construct the solution.

In these examples we are employing certain symmetries at one of the endpoints of the string. A function $f(x), x \in \mathbb{R}$, is symmetric about $x = x_0$ if and only if $f(x_0 - x) = f(x_0 + x)$ for all x in the domain of f. If we pick $x = y - x_0$, then $f(y) = f(2x_0 - y)$. $f(x)$ is antisymmetric about $x = x_0$ if and only if $f(x_0 - x) = -f(x_0 + x)$, or $f(y) = -f(2x_0 - y)$. We will say that the function is either even or odd about $x = x_0$.

So far we have extended $f(x)$ and $g(x)$ to the interval $-\ell \leq x \leq 2\ell$ in order to determine the solution over a larger xt-domain. For example, the function $f(x)$ has been extended to

$$f_{ext}(x) = \begin{cases} -f(-x), & -\ell < x < 0, \\ f(x), & 0 < x < \ell, \\ -f(2\ell - x), & \ell < x < 2\ell. \end{cases}$$

A similar extension is needed for $g(x)$. Inserting these extended functions into d'Alembert's solution, we can determine $u(x,t)$ in the region indicated in Figure 1.18.

Even though the original region has been expanded, we have not determined how to find the solution throughout the entire strip, $[0, \ell] \times [0, \infty)$.

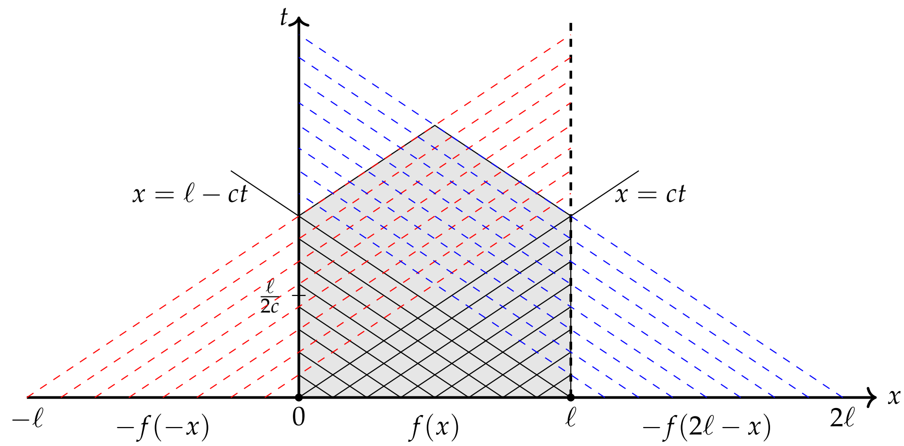


Figure 1.18: The red dashed lines are the characteristics from the interval $[-\ell, 0]$ from using the odd extension about $x = 0$ and the blue dashed lines are the characteristics from the interval $[\ell, 2\ell]$ from using the odd extension about $x = \ell$.

This is accomplished by periodically repeating these extended functions with period 2ℓ. This can be shown from the two conditions

$$\begin{aligned} f(x) &= -f(-x), \quad -\ell \le x \le 0, \\ f(x) &= -f(2\ell - x), \quad \ell \le x \le 2\ell. \end{aligned} \tag{1.31}$$

Now, consider

$$\begin{aligned} f(x+2\ell) &= -f(2\ell - (x - 2\ell)) \\ &= -f(-x) \\ &= f(x). \end{aligned} \tag{1.32}$$

This shows that $f(x)$ is periodic with period 2ℓ. Since $g(x)$ satisfies the same conditions, then it is as well.

In Figure 1.19 we show how the characteristics are extended throughout the domain strip using the periodicity of the extended initial conditions. The characteristics from the interval endpoints zig zag throughout the domain, filling it up. In the next example we show how to construct the odd periodic extension of a specific function.

Example 1.6. Construct the odd periodic extension of the plucked string initial profile given by

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{\ell}{2}, \\ \ell - x, & \frac{\ell}{2} \le x \le \ell, \end{cases}$$

satisfying fixed boundary conditions at $x = 0$ and $x = \ell$.

We first take the solution and add the odd extension about $x = 0$. Then we add an extension beyond $x = \ell$. This process is shown in Figure 1.20.

We can use the odd periodic function to construct solutions. In this case we use the result from the last example for obtaining the solution of the problem in which the initial velocity is zero, $u(x,t) = \frac{1}{2}[f(x+ct) + f(x -$

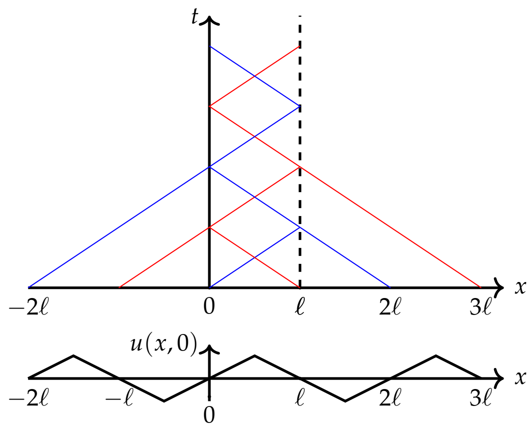


Figure 1.19: Extending the characteristics throughout the domain strip.

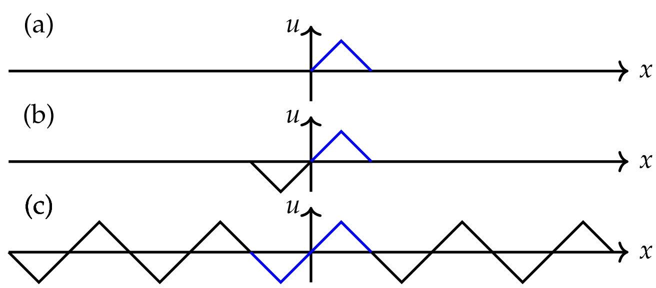


Figure 1.20: Construction of odd periodic extension for (a) The initial profile, $f(x)$. (b) Make $f(x)$ an odd function on $[-\ell, \ell]$. (c) Make the odd function periodic with period 2ℓ.

$ct)]$. Translations of the odd periodic extension are shown in Figure 1.21. In Figure 1.22 we show superimposed plots of $f(x+ct)$ and $f(x-ct)$ for different values of ct. A box is shown inside which the physical wave can be constructed. The solution is an average of these odd periodic extensions within this box. This is displayed in Figure 1.23.

1.6 *Classification of Second Order PDEs*

We have studied several examples of partial differential equations, the heat equation, the wave equation, and Laplace's equation. These equations are examples of parabolic, hyperbolic, and elliptic equations, respectively. Given a general second order linear partial differential equation, how can we tell what type it is? This is known as the classification of second order PDEs.

Let $u = u(x,y)$. Then, the general form of a linear second order partial differential equation is given by

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y). \tag{1.33}$$

In this section we will show that this equation can be transformed into one of three types of second order partial differential equations.

Let $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$ be an invertible transformation from

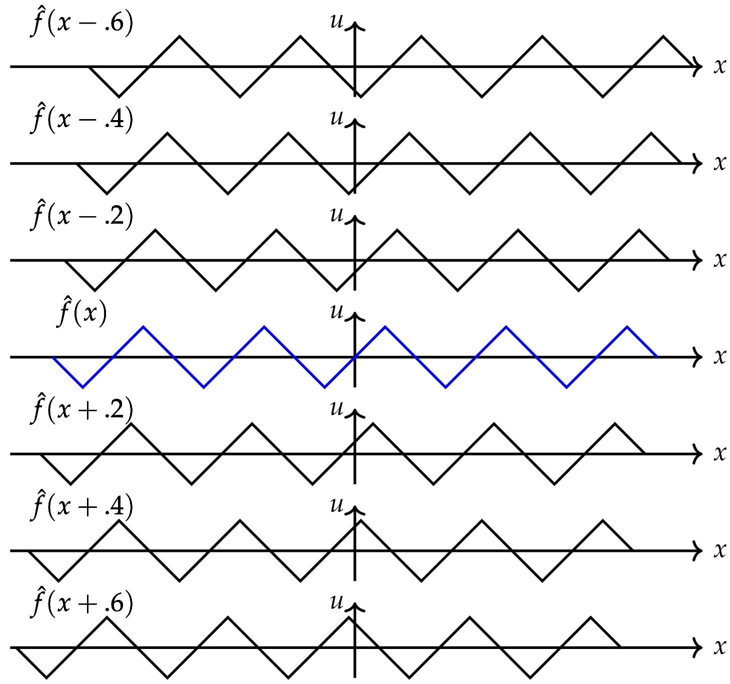


Figure 1.21: Translations of the odd periodic extension.

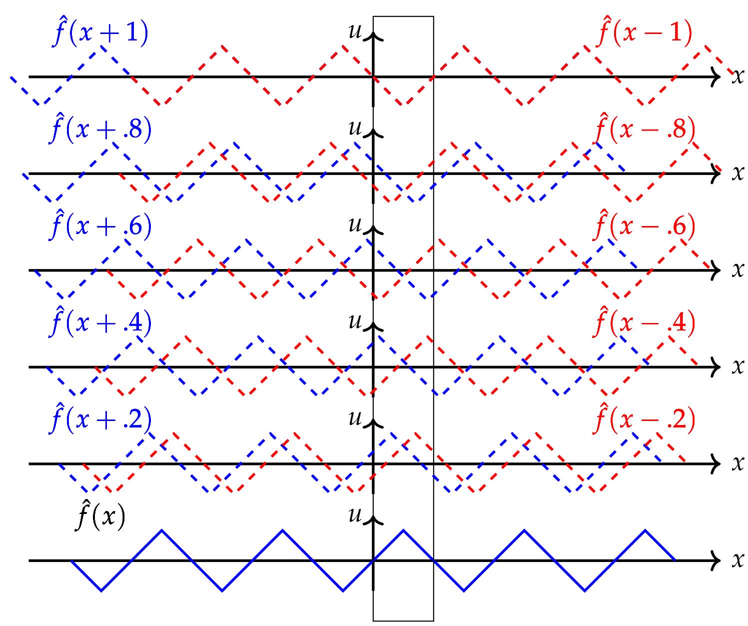


Figure 1.22: Superimposed translations of the odd periodic extension.

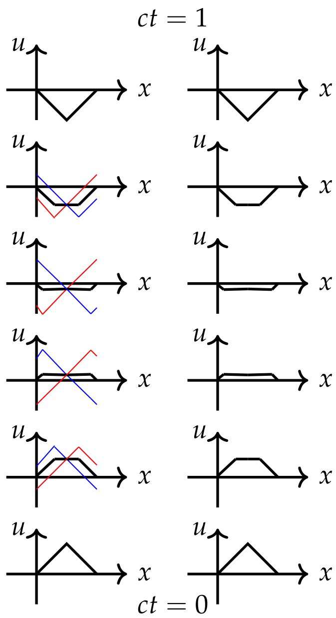


Figure 1.23: On the left is a plot of $f(x+ct)$, $f(x-ct)$ from Figure 1.22 and the average, $u(x,t)$. On the right the solution alone is shown for $ct = 0$ to $ct = 1$.

coordinates (ξ, η) to coordinates (x, y). Furthermore, let

$$u(x(\xi,\eta), y(\xi,\eta)) = U(\xi,\eta).$$

How does the partial differential equation (1.33) transform?

We first need to transform the derivatives of $u(x,t)$. We have

$$\begin{aligned}
u_x &= U_\xi \xi_x + U_\eta \eta_x, \\
u_y &= U_\xi \xi_y + U_\eta \eta_y, \\
u_{xx} &= \frac{\partial}{\partial x}(U_\xi \xi_x + U_\eta \eta_x), \\
&= U_{\xi\xi}\xi_x^2 + 2U_{\xi\eta}\xi_x\eta_x + U_{\eta\eta}\eta_x^2 + U_\xi \xi_{xx} + U_\eta \eta_{xx}, \\
u_{yy} &= \frac{\partial}{\partial y}(U_\xi \xi_y + U_\eta \eta_y), \\
&= U_{\xi\xi}\xi_y^2 + 2U_{\xi\eta}\xi_y\eta_y + U_{\eta\eta}\eta_y^2 + U_\xi \xi_{yy} + U_\eta \eta_{yy}, \\
u_{xy} &= \frac{\partial}{\partial y}(U_\xi \xi_x + U_\eta \eta_x), \\
&= U_{\xi\xi}\xi_x\xi_y + U_{\xi\eta}\xi_x\eta_y + U_{\xi\eta}\xi_y\eta_x + U_{\eta\eta}\eta_x\eta_y + U_\xi \xi_{xy} + U_\eta \eta_{xy}.
\end{aligned}$$

Inserting these derivatives into Equation (1.33), we have

$$\begin{aligned}
g - fU &= au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y \\
&= a\left(U_{\xi\xi}\xi_x^2 + 2U_{\xi\eta}\xi_x\eta_x + U_{\eta\eta}\eta_x^2 + U_\xi\xi_{xx} + U_\eta\eta_{xx}\right) \\
&\quad +2b\left(U_{\xi\xi}\xi_x\xi_y + U_{\xi\eta}\xi_x\eta_y + U_{\xi\eta}\xi_y\eta_x \right.\\
&\quad \left. + U_{\eta\eta}\eta_x\eta_y + U_\xi\xi_{xy} + U_\eta\eta_{xy}\right) \\
&\quad +c\left(U_{\xi\xi}\xi_y^2 + 2U_{\xi\eta}\xi_y\eta_y + U_{\eta\eta}\eta_y^2 + U_\xi\xi_{yy} + U_\eta\eta_{yy}\right) \\
&\quad +d\left(U_\xi\xi_x + U_\eta\eta_x\right)
\end{aligned}$$

$$
\begin{aligned}
& +e\left(U_\xi \xi_y + U_\eta \eta_y\right) \\
= \quad & (a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2)U_{\xi\xi} \\
& +(2a\xi_x\eta_x + 2b\xi_x\eta_y + 2b\xi_y\eta_x + 2c\xi_y\eta_y)U_{\xi\eta} \\
& +(a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2)U_{\eta\eta} \\
& +(a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y)U_\xi \\
& +(a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y)U_\eta .
\end{aligned}
$$

We see that this result takes the form of a linear second order differential equation for $U(\xi,\eta)$ with new coefficients,

$$
AU_{\xi\xi} + 2BU_{\xi\eta} + CU_{\eta\eta} + DU_\xi + EU_\eta + fU = g. \tag{1.34}
$$

Picking the right transformation, we can eliminate some of the second order derivative terms in Equation (1.34) depending on the type of differential equation we have. These choices can lead to one of three types of second order partial differential equations: elliptic, hyperbolic, or parabolic. We'll assume that at least one of the coefficients, A, B, C, in Equation (1.34) is not zero.

For example, if transformations can be found to make $A \equiv 0$ and $C \equiv 0$, then Equation (1.34) reduces to

$$
U_{\xi\eta} = \text{ lower order terms.}
$$

Hyperbolic case.

Such an equation is called hyperbolic. A generic example of a hyperbolic equation is the wave equation.

The conditions that $A \equiv 0$ and $C \equiv 0$ give the conditions

$$
\begin{aligned}
a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 &= 0. \\
a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 &= 0.
\end{aligned} \tag{1.35}
$$

We seek ξ and η satisfying these two equations, which are of the same form. Let's assume that $\xi = \xi(x,y)$ is a constant curve in the xy-plane. Furthermore, if this curve is the graph of a function, $y = y(x)$, then $\xi = \xi(x, y(x))$ and

$$
\frac{d\xi}{dx} = \xi_x + \frac{dy}{dx}\xi_y = 0.
$$

Then, we have

$$
\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}.
$$

Inserting this expression in the equation for $A = 0$, we find

$$
\begin{aligned}
A &= a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 \\
&= \xi_y^2\left(a\left(\frac{\xi_x}{\xi_y}\right)^2 + 2b\frac{\xi_x}{\xi_y} + c\right) \\
&= \xi_y^2\left(a\left(\frac{dy}{dx}\right)^2 - 2b\frac{dy}{dx} + c\right) = 0.
\end{aligned} \tag{1.36}
$$

Characteristic curves.

This equation is satisfied if $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}.$$

So, for $A = 0$, we choose ξ and η to be constant on these characteristic curves.

Example 1.7. Show that $u_{xx} - u_{yy} = 0$ is hyperbolic.

In this case we have $a = 1 = -c$ and $b = 0$. Then,

$$\frac{dy}{dx} = \pm 1.$$

This gives $y(x) = \pm x + c$. So, we choose ξ and η constant on these characteristic curves. Therefore, we let $\xi = x - y$, $\eta = x + y$.

Let's see if this transformation transforms the differential equation into a canonical form. Let $u(x, y) = U(\xi, \eta)$. Then, the needed derivatives become

$$\begin{aligned}
u_x &= U_\xi \xi_x + U_\eta \eta_x = U_\xi + U_\eta. \\
u_y &= U_\xi \xi_y + U_\eta \eta_y = -U_\xi + U_\eta. \\
u_{xx} &= \frac{\partial}{\partial x}(U_\xi + U_\eta) \\
&= U_{\xi\xi}\xi_x + U_{\xi\eta}\eta_x + U_{\eta\xi}\xi_x + U_{\eta\eta}\eta_x \\
&= U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}. \\
u_{yy} &= \frac{\partial}{\partial y}(-U_\xi + U_\eta) \\
&= -U_{\xi\xi}\xi_y - U_{\xi\eta}\eta_y + U_{\eta\xi}\xi_y + U_{\eta\eta}\eta_y \\
&= U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}. \qquad (1.37)
\end{aligned}$$

Inserting these derivatives into the differential equation, we have

$$0 = u_{xx} - u_{yy} = 4U_{\xi\eta}.$$

Thus, the transformed equation is $U_{\xi\eta} = 0$, showing it is a hyperbolic equation. In fact, we have already seen this form of equation when solving the wave equation in Section 1.5.

We have seen that A and C vanish for $\xi(x, y)$ and $\eta(x, y)$ constant along the characteristics

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}$$

for second order hyperbolic equations. This is possible when $b^2 - ac > 0$ since this leads to two characteristics.

In general, if we consider the second order operator

$$L[u] = a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy},$$

then this operator can be transformed to the new form

$$L'[U] = BU_{\xi\eta}$$

if $b^2 - ac > 0$. An example of a hyperbolic equation is the wave equation, $u_{tt} = u_{xx}$.

When $b^2 - ac = 0$, then there is only one characteristic solution, $\frac{dy}{dx} = \frac{b}{a}$. This is the parabolic case. But, $\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}$. So,

$$\frac{b}{a} = -\frac{\xi_x}{\xi_y},$$

which we can write as a first order partial differential equation for ξ,[4]

$$a\xi_x + b\xi_y = 0.$$

Also, $b^2 - ac = 0$ implies that $c = b^2/a$.

Inserting these expression into coefficient B, we have

$$\begin{aligned} B &= 2a\xi_x\eta_x + 2b\xi_x\eta_y + 2b\xi_y\eta_x + 2c\xi_y\eta_y \\ &= 2(a\xi_x + b\xi_y)\eta_x + 2(b\xi_x + c\xi_y)\eta_y \\ &= 2\frac{b}{a}(a\xi_x + b\xi_y)\eta_y = 0. \end{aligned} \tag{1.38}$$

Parabolic case.

Therefore, in the parabolic case, $A = 0$ and $B = 0$, and $L[u]$ transforms to

$$L'[U] = CU_{\eta\eta}$$

when $b^2 - ac = 0$. This is the canonical form for a parabolic operator. An example of a parabolic equation is the heat equation, $u_t = u_{xx}$.

Elliptic case.

Finally, when $b^2 - ac < 0$, we have the elliptic case. In this case we cannot force $A = 0$ or $C = 0$. However, we can force $B = 0$. As we just showed, we can write

$$B = 2(a\xi_x + b\xi_y)\eta_x + 2(b\xi_x + c\xi_y)\eta_y.$$

Letting $\eta_x = 0$, we can choose ξ to satisfy $b\xi_x + c\xi_y = 0$. This leads to

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = a\xi_x^2 - c\xi_y^2 = \frac{ac - b^2}{c}\xi_x^2$$

$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = c\eta_y^2$$

Furthermore, setting $\frac{ac - b^2}{c}\xi_x^2 = c\eta_y^2$, we can make $A = C$ and, finally, $L[u]$ transforms to

$$L'[U] = A[U_{\xi\xi} + U_{\eta\eta}]$$

when $b^2 - ac < 0$. This is the canonical form for an elliptic operator. An example of an elliptic equation is Laplace's equation, $u_{xx} + u_{yy} = 0$.

In summary, we have can classify second order partial differential equations by considering the sign of $b^2 - ac$. This is displayed in the box below.[5]

[4] We will see in Chapter 7 that this first order PDE can be solved using the method of characteristics. The result is that ξ is constant along the characteristics given by $\frac{dy}{dx} = \frac{b}{a}$.

[5] Be careful of the $2b$ in the form of the quadratic operator. This leads to a common error if you are not thinking.

Classification of Second Order PDEs
The second order differential operator

$$L[u] = a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy},$$

can be transformed to one of the following forms:

- $b^2 - ac > 0$. Hyperbolic: $L[u] = B(x,y)u_{xy}$
- $b^2 - ac = 0$. Parabolic: $L[u] = C(x,y)u_{yy}$
- $b^2 - ac < 0$. Elliptic: $L[u] = A(x,y)[u_{xx} + u_{yy}]$

As a final note,[6] the terminology used in this classification is borrowed from the general theory of quadratic equations which are the equations for translated and rotated conics. Recall from your calculus class that the general quadratic equation in two variable takes the form

[6] The rest of this section is about the classification of conics and is not essential to the study of PDEs.

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0. \tag{1.39}$$

One can complete the squares in x and y to obtain the new form

$$a(x-h)^2 + 2bxy + c(y-k)^2 + f' = 0.$$

So, translating points (x,y) using the transformations $x' = x - h$ and $y' = y - k$, we find the simpler form

$$ax^2 + 2bxy + cy^2 + f = 0.$$

Here we dropped all primes.

We can also introduce transformations to simplify the quadratic terms. Consider a rotation of the coordinate axes by θ, as shown in Figure 1.24,

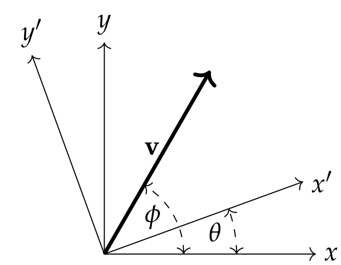


Figure 1.24: Comparison of the coordinate systems. Using the addition formula for trigonometric, functions we have $x' = r\cos\phi\cos\theta + v\sin\phi\sin\theta$ and $y' = r\sin\phi\cos\theta - v\cos\phi\sin\theta$. We can use the polar forms $x = r\cos\phi$ and $y = r\sin\phi$ to obtain system (1.40).

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta, \end{aligned} \tag{1.40}$$

or the inverse form,

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta. \end{aligned} \tag{1.41}$$

The resulting equation takes the form

$$Ax'^2 + 2Bx'y' + Cy'^2 + D = 0,$$

where

$$\begin{aligned} A &= a\cos^2\theta + 2b\sin\theta\cos\theta + c\sin^2\theta. \\ B &= (c-a)\sin\theta\cos\theta + b(\cos^2\theta - \sin^{\theta}). \\ C &= a\sin^2\theta - 2b\sin\theta\cos\theta + c\cos^2\theta. \end{aligned} \tag{1.42}$$

We can eliminate the $x'y'$ term by forcing $B = 0$. Since $\cos^2\theta - \sin^2\theta = \cos 2\theta$ and $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$, we have

$$B = \frac{(c-a)}{2}\sin 2\theta + b\cos 2\theta = 0.$$

Therefore, the condition for eliminating the $x'y'$ term is

$$\cot(2\theta) = \frac{a-c}{2b}.$$

Furthermore, one can show that the discriminant is an invariant, $b^2 - ac = B^2 - AC$. From the form $Ax'^2 + 2Bx'y' + Cy'^2 + D = 0$, the resulting quadratic equation takes one of the following forms:

- $b^2 - ac > 0$. Hyperbolic: $Ax^2 - Cy^2 + D = 0$.
- $b^2 - ac = 0$. Parabolic: $Ax^2 + By + D = 0$.
- $b^2 - ac < 0$. Elliptic: $Ax^2 + Cy^2 + D = 0$.

Thus, one can see the connection between the classification of quadratic equations and second order partial differential equations in two independent variables.

1.7 The Nonhomogeneous Heat Equation

IN THIS SECTION WE DISCUSS nonhomogeneous initial-boundary value problems in the form of the nonhomogeneous heat equation. Either the partial differential equation is nonhomogeneous, or the boundary conditions are nonhomogeneous. This will lead to the notion of a Green's function.[7] As with the earlier solutions of the heat and wave equations, we will come to a point where will will need to determine some Fourier coefficients, which we study in the next chapter. ,

[7] Green's functions were first introduced by the British mathematician George Green (1793-1841) in 1828 in a memoir which was hardly known. In 1845 a young William Thomson (1824-1907), later to be known as Lord Kelvin, saw a footnote in Robert Murphy's (1806-1843) 1832 paper, referring to Green's essay. When Thomson found the memoir he made Green known to the world.

1.7.1 Nonhomogeneous Time Independent Boundary Conditions

Consider the nonhomogeneous heat equation with nonhomogeneous boundary conditions:

$$\begin{aligned} u_t - ku_{xx} &= h(x), \qquad 0 \le x \le L, \quad t > 0, \\ u(0,t) &= a, \quad u(L,t) = b, \\ u(x,0) &= f(x). \end{aligned} \tag{1.43}$$

We are interested in finding a particular solution to this initial-boundary value problem. In fact, we can represent the solution to the general nonhomogeneous heat equation as the sum of two solutions that solve different problems.

First, we consider a function, $v(x,t)$, which satisfies the homogeneous problem,

$$\begin{aligned} v_t - kv_{xx} &= 0, \qquad 0 \leq x \leq L, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \\ v(x,0) &= g(x). \end{aligned} \tag{1.44}$$

Note that the differential equation as well as the boundary conditions are homogeneous. In this problem we will call $v(x,t)$ the transient solution.

We will also need a steady state solution to the original problem. A steady state solution is one that satisfies $u_t = 0$. Let $w(x)$ be the steady state solution. It satisfies the problem

The steady state solution, $w(t)$, satisfies a nonhomogeneous differential equation with nonhomogeneous boundary conditions. The transient solution, $v(t)$, satisfies the homogeneous heat equation with homogeneous boundary conditions and satisfies a modified initial condition.

$$\begin{aligned} -kw_{xx} &= h(x), \qquad 0 \leq x \leq L. \\ w(0) &= a, \quad w(L) = b. \end{aligned} \tag{1.45}$$

Now consider $u(x,t) = w(x) + v(x,t)$, the sum of the steady state solution, $w(x)$, and the transient solution, $v(x,t)$. We first note that $u(x,t)$ satisfies the nonhomogeneous heat equation,

The solution to the nonhomogeneous problem will be the sum of the steady state and transient solutions, $u(x,t) = w(x) + v(x,t)$.

$$\begin{aligned} u_t - ku_{xx} &= (w+v)_t - (w+v)_{xx} \\ &= v_t - kv_{xx} - kw_{xx} \equiv h(x). \end{aligned} \tag{1.46}$$

The boundary conditions are also satisfied by $u(x,t)$. Evaluating, $u(x,t)$ at $x = 0$ and $x = L$, we have

$$\begin{aligned} u(0,t) &= w(0) + v(0,t) = a, \\ u(L,t) &= w(L) + v(L,t) = b. \end{aligned} \tag{1.47}$$

Finally, the initial condition gives

$$u(x,0) = w(x) + v(x,0) = w(x) + g(x).$$

The transient solution satisfies

$$v(x,0) = f(x) - w(x).$$

Thus, if we set $g(x) = f(x) - w(x)$, then $u(x,t) = w(x) + v(x,t)$ will be the solution of the nonhomogeneous boundary value problem. We all ready know how to solve the homogeneous problem to obtain $v(x,t)$. So, we only need to find the steady state solution, $w(x)$.

There are several methods we could use to solve Equation (1.45) for the steady state solution. One is the Method of Variation of Parameters, which is closely related to the Green's function method for boundary value problems which we describe in the Appendix on ordinary differential equations. However, we will just integrate the differential equation for the steady state solution directly to find the solution. From this solution we will be able to read off the Green's function.

Integrating the steady state equation (1.45) once, yields

$$\frac{dw}{dx} = -\frac{1}{k}\int_0^x h(z)\,dz + A,$$

where we have been careful to include the integration constant, $A = w'(0)$. Integrating again, we obtain

$$w(x) = -\frac{1}{k}\int_0^x \left(\int_0^y h(z)\,dz\right)\,dy + Ax + B, \tag{1.48}$$

where a second integration constant has been introduced, $B = w(0)$. This gives the general solution for Equation (1.45).

The boundary conditions can now be used to determine the constants. It is clear that $B = a$ for the condition at $x = 0$ to be satisfied. The second condition gives

$$b = w(L) = -\frac{1}{k}\int_0^L \left(\int_0^y h(z)\,dz\right)\,dy + AL + a.$$

Solving for A, we have

$$A = \frac{1}{kL}\int_0^L \left(\int_0^y h(z)\,dz\right)\,dy + \frac{b-a}{L}.$$

The steady state solution.

Inserting the integration constants in Equation (1.48), the solution of the boundary value problem for the steady state solution is then

$$w(x) = -\frac{1}{k}\int_0^x \left(\int_0^y h(z)\,dz\right)\,dy + \frac{x}{kL}\int_0^L \left(\int_0^y h(z)\,dz\right)\,dy + \frac{b-a}{L}x + a.$$

This is sufficient for an answer, but it can be written in a more compact form. In fact, we will show that the solution can be written in a way that a Green's function can be identified.

First, we rewrite the double integrals as single integrals. We can do this using integration by parts. Consider integral in the first term of the solution,

$$I = \int_0^x \left(\int_0^y h(z)\,dz\right)\,dy.$$

Setting $u = \int_0^y h(z)\,dz$ and $dv = dy$ in the standard integration by parts formula, we obtain

$$\begin{aligned} I &= \int_0^x \left(\int_0^y h(z)\,dz\right)\,dy \\ &= y\int_0^y h(z)\,dz\Big|_0^x - \int_0^x yh(y)\,dy \\ &= \int_0^x (x-y)h(y)\,dy. \end{aligned} \tag{1.49}$$

Thus, the double integral has now collapsed to a single integral. Replacing the integral in the solution, the steady state solution becomes

$$w(x) = -\frac{1}{k}\int_0^x (x-y)h(y)\,dy + \frac{x}{kL}\int_0^L (L-y)h(y)\,dy + \frac{b-a}{L}x + a.$$

We can make a further simplification by combining these integrals. This can be done if the integration range, $[0, L]$, in the second integral is split into two pieces, $[0, x]$ and $[x, L]$. Writing the second integral as two integrals over these subintervals, we obtain

$$\begin{aligned} w(x) &= -\frac{1}{k}\int_0^x (x-y)h(y)\,dy + \frac{x}{kL}\int_0^x (L-y)h(y)\,dy \\ &\quad +\frac{x}{kL}\int_x^L (L-y)h(y)\,dy + \frac{b-a}{L}x + a. \end{aligned} \tag{1.50}$$

Next, we rewrite the integrands,

$$\begin{aligned} w(x) &= -\frac{1}{k}\int_0^x \frac{L(x-y)}{L}h(y)\,dy + \frac{1}{k}\int_0^x \frac{x(L-y)}{L}h(y)\,dy \\ &\quad +\frac{1}{k}\int_x^L \frac{x(L-y)}{L}h(y)\,dy + \frac{b-a}{L}x + a. \end{aligned} \tag{1.51}$$

It can now be seen how we can combine the first two integrals:

$$w(x) = -\frac{1}{k}\int_0^x \frac{y(L-x)}{L}h(y)\,dy + \frac{1}{k}\int_x^L \frac{x(L-y)}{L}h(y)\,dy + \frac{b-a}{L}x + a.$$

The resulting integrals now take on a similar form and this solution can be written compactly as

Writing the steady state solution in a compact form and introducing the Green's function.

$$w(x) = \int_0^L G(x,y)\left(-\frac{1}{k}h(y)\right)\,dy + \frac{b-a}{L}x + a, \tag{1.52}$$

where

$$G(x,y) = \begin{cases} -\dfrac{x(L-y)}{L}, & 0 \le x \le y, \\ -\dfrac{y(L-x)}{L}, & y \le x \le L, \end{cases} \tag{1.53}$$

is the Green's function for this problem.

The full solution to the original problem can be found by adding to this steady state solution a solution of the homogeneous problem,

$$\begin{aligned} u_t - ku_{xx} &= 0, \quad 0 \le x \le L, \quad t > 0, \\ u(0,t) &= 0, \quad u(L,t) = 0, \\ u(x,0) &= f(x) - w(x). \end{aligned} \tag{1.54}$$

Example 1.8. Solve the nonhomogeneous problem,

$$\begin{aligned} u_t - u_{xx} &= 10, \quad 0 \le x \le 1, \quad t > 0, \\ u(0,t) &= 20, \quad u(1,t) = 0, \\ u(x,0) &= 2x(1-x). \end{aligned} \tag{1.55}$$

In this problem we have a rod initially at a temperature of $u(x,0) = 2x(1-x)$. The ends of the rod are maintained at fixed temperatures and the bar is continually heated at a constant temperature, represented by the source term, 10.

First, we find the steady state temperature, $w(x)$, satisfying

$$\begin{aligned} -w_{xx} &= 10, \quad 0 \le x \le 1. \\ w(0,t) &= 20, \quad w(1,t) = 0. \end{aligned} \tag{1.56}$$

Using the general solution, we have

$$w(x) = -\int_0^1 10G(x,y)\,dy - 20x + 20,$$

where the Green's function in Equation (1.53) becomes

$$G(x,y) = \begin{cases} -x(1-y), & 0 \le x \le y, \\ -y(1-x), & y \le x \le 1, \end{cases}$$

we compute the solution

$$\begin{aligned} w(x) &= \int_0^x 10y(1-x)\,dy + \int_x^1 10x(1-y)\,dy - 20x + 20 \\ &= 5(x - x^2) - 20x + 20, \\ &= 20 - 15x - 5x^2. \end{aligned} \tag{1.57}$$

Checking this solution, it satisfies both the steady state equation and boundary conditions.

The transient solution satisfies

$$\begin{aligned} v_t - v_{xx} &= 0, \quad 0 \le x \le 1, \quad t > 0, \\ v(0,t) &= 0, \quad v(1,t) = 0, \\ v(x,0) &= x(1-x) - 10. \end{aligned} \tag{1.58}$$

Recall, that we have determined the solution of this problem as

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x,$$

where the Fourier coefficients, b_n, are given in terms of the initial temperature distribution. In the next chapter we will see that these are given by

$$b_n = 2\int_0^1 [x(1-x) - 10] \sin n\pi x\,dx, \quad n = 1, 2, \ldots.$$

Therefore, the full solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x + 20 - 15x - 5x^2.$$

Note that for large t, the transient solution tends to zero and we are left with the steady state solution as expected.

1.7.2 *The Green's Function*

While Green's functions are explored in the second semester of a PDE course or an ODE course, it is useful to point them out as we progress. A Green's function is the response to a point source. For

$$w_{xx} = f(x), \quad w(0) = a, \quad w(L) = b,$$

we seek a function $G(x,x')$ which satisfies

$$G_{xx}(x,x') = \delta(x-x'), \quad G(0,x') = 0, \quad G(L,x')) = 0,$$

where $\delta(x-x')$ is the Dirac delta function representing a unit impulse.

The Dirac delta function, which is described in more detail in the two semester text, satisfies two properties: $\delta(x) = 0$, $x \neq 0$, and

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

From this definition one can show that

$$\int_{-\infty}^{\infty} \delta(x-x') f(x)\, dx = f(x').$$

Using the differential equations satisfied by $w(x)$ and $G(x,x')$, we have

$$\begin{aligned}
\int_0^L \left(w''(x)G(x,x') - w(x)G_{xx}(x,x')\right)\, dx &= \int_0^L \frac{d}{dx}\left(w'(x)G(x,x') - w(x)G_x(x,x')\right)\, dx \\
\int_0^L \left(f(x)G(x,x') - w(x)\delta(x-x')\right)\, dx &= \left[w'(x)G(x,x') - w(x)G_x(x,x')\right]_0^L \\
\int_0^L f(x)G(x,x')\, dx - w(x') &= -w(L)G_x(L,x') + w(0)G_x(0,x') \\
w(x') &= \int_0^L f(x)G(x,x')\, dx + bG_x(L,x') - aG_x(0,x').
\end{aligned} \tag{1.59}$$

This tells us that if we know the Green's function, then we can write down the solution to the nonhomogeneous differential equation. But, first we need to interchange the independent variables.

Noting that when $f(x) = \delta(x-\xi)$, then $w(x) = G(x,\xi)$, and $a = b = 0$. So, for this special problem, the result gives

$$G(x',\xi) = \int_0^L \delta(x-\xi)G(x,x')\, dx = G(\xi,x').$$

This property is call reciprocality for the Green's function.

We can rewrite the solution for $w(x)$ by interchanging x and x' in the solution as

$$w(x) = \int_0^L f(x')G(x,x')\, dx' + bG_{x'}(x,L) - aG_{x'}(x,0).$$

If we use the Green's function in Equation (1.53) for the previous steady state problem, where $f(x) = -\frac{1}{k}h(x)$, then we have

$$w(x) = -\frac{1}{k}\int_0^L h(x')G(x,x')\, dx' + b\frac{x}{L} + a\frac{L-x}{L}).$$

since

$$G_y(x,y) = \begin{cases} \dfrac{x}{L}, & 0 \leq x \leq y, \\ -\dfrac{L-x}{L}, & y \leq x \leq L, \end{cases}$$

The solution is the same as Equation (1.52).

1.7.3 Time Dependent Boundary Conditions

In the last section we solved problems with time independent boundary conditions using equilibrium solutions satisfying the steady state heat equation and nonhomogeneous boundary conditions. When the boundary conditions are time dependent, we can also convert the problem to an auxiliary problem with homogeneous boundary conditions.

Consider the problem

$$\begin{aligned} u_t - ku_{xx} &= h(x), \quad 0 \le x \le L, \quad t > 0, \\ u(0,t) &= a(t), \quad u(L,t) = b(t), \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le L. \end{aligned} \tag{1.60}$$

We define $u(x,t) = v(x,t) + w(x,t)$, where $w(x,t)$ is a modified form of the steady state solution from the last section,

$$w(x,t) = a(t) + \frac{b(t) - a(t)}{L} x.$$

Noting that

$$\begin{aligned} u_t &= v_t + \dot{a} + \frac{\dot{b} - \dot{a}}{L} x, \\ u_{xx} &= v_{xx}, \end{aligned} \tag{1.61}$$

we find that $v(x,t)$ is a solution of the problem

$$\begin{aligned} v_t - kv_{xx} &= h(x) - \left[\dot{a}(t) + \frac{\dot{b}(t) - \dot{a}(t)}{L} x \right], \quad 0 \le x \le L, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \quad t > 0, \\ v(x,0) &= f(x) - \left[a(0) + \frac{b(0) - a(0)}{L} x \right], \quad 0 \le x \le L. \end{aligned} \tag{1.62}$$

Thus, we have converted the original problem into a nonhomogeneous heat equation with homogeneous boundary conditions and a new source term and new initial condition. [8]

[8] The bracketed expression look complicated, but we could write instead

$$\begin{aligned} v_t - kv_{xx} &= h(x) - w_t(x,t) \\ v(x,0) &= f(x) - w(x,0). \end{aligned}$$

Example 1.9. Solve the problem

$$\begin{aligned} u_t - u_{xx} &= x, \quad 0 \le x \le 1, \quad t > 0, \\ u(0,t) &= 2, \quad u(L,t) = t, \quad t > 0 \\ u(x,0) &= 3 \sin 2\pi x + 2(1 - x), \quad 0 \le x \le 1. \end{aligned} \tag{1.63}$$

We first define $w(x,t) = 2 + (t - 2)x$. Then, we have

$$u(x,t) = v(x,t) + 2 + (t - 2)x,$$

where $v(x,t)$ satisfies the problem

$$\begin{aligned} v_t - v_{xx} &= 0, \quad 0 \le x \le 1, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \quad t > 0, \\ v(x,0) &= 3 \sin 2\pi x, \quad 0 \le x \le 1. \end{aligned} \tag{1.64}$$

Note that this problem was rigged so that the initial value problem is easily solved. as we have seen earlier in the chapter, the general solution is given by

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x e^{-n^2\pi^2 t}.$$

Since $v(x,0) = 3\sin 2\pi x$, the b_n's all vanish except for $b_2 = 3$. This gives $v(x,t) = 3\sin 2\pi x e^{-4\pi^2 t}$. Therefore, we have found the solution

$$u(x,t) = 3\sin 2\pi x e^{-4\pi^2 t} + 2 + (t-2)x.$$

1.7.4 *Duhamel's Principle*

The idea that one can solve a nonhomogeneous partial differential equation by associating it with a related homogeneous problem with non-homogeneous initial conditions is know as Duhamel's Principle named after Jean-Marie Constant Duhamel (1797-1872). We apply this principle to the heat equation.

Recall the heated rod in Section 1.3.2 where we derived the heat equation without a heat source. When there is a heat source, the one-dimensional heat equation with homogeneous boundary conditions becomes

$$\begin{aligned} u_t - ku_{xx} &= Q(x,t), \qquad 0 \le x \le L, \quad t > 0, \\ u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le L. \end{aligned} \tag{1.65}$$

Let us consider a rod at initially zero temperature, $f(x) = 0$. We apply a source of heat energy $Q(x,0)$ to every point on the rod at $t = 0$. The temperature is then approximately $Q(x,0)\,\delta s$, assuming the time δs is too short for the energy to diffuse. At a later time, after diffusion and with no additional heat source, the temperature is $\tilde{v}(x,t)\,\delta s$, where $\tilde{v}(x,t)$ satisfies the (heat energy) diffusion equation, but with initial heat energy $Q(x,0)$,

$$\begin{aligned} \tilde{v}_t &= k\tilde{v}_{xx}, \quad 0 \le x \le L, \quad t > 0, \\ \tilde{v}(0,t) &= 0, \quad \tilde{v}(L,t) = 0, \quad t > 0, \\ \tilde{v}(x,0) &= Q(x,0), \quad 0 \le x \le L. \end{aligned} \tag{1.66}$$

Now, we consider what happens when the source is turned on at time $t = s - \delta s$ and turned off at $t = s$. This leads to a new solution, $\bar{v}(x,t;s)$, which satisfies

$$\begin{aligned} \bar{v}_t &= k\bar{v}_{xx}, \quad 0 \le x \le L, \quad t > 0, \\ \bar{v}(0,t;s) &= 0, \quad \bar{v}(L,t;s) = 0, \quad t > 0, \\ \bar{v}(x,s;s) &= Q(x,s), \quad 0 \le x \le L. \end{aligned} \tag{1.67}$$

Here we are given $Q(x,s)$ as the initial condition at $t = s$. Then, adding all of the sources before time t, we have the solution

$$u(x,t) = \int_0^t \bar{v}(x,t;s)\,ds.$$

The initial value for $\overline{v}(x,t;s)$ is given at $t = s$ while that for $\widetilde{v}(x,t)$ is at $t = 0$. So, we need to relate these functions. The relation between the solutions is found as

$$\overline{v}(x,t;s) \approx \widetilde{v}(x,t-s;s).$$

Therefore, $\widetilde{v}(x,0;s) = Q(x,s)$. Now, we can write the total solution over time as

$$u(x,t) = \int_0^t \widetilde{v}(x,t-s;s)\,ds,$$

where we have modified the problem solved by $\widetilde{v}(x,t;s)$ as

$$\begin{aligned} \widetilde{v}_t &= k\widetilde{v}_{xx}, \quad 0 \le x \le L, \quad t > 0, \\ \widetilde{v}(0,t;s) &= 0, \quad \widetilde{v}(L,t;s) = 0, \quad t > 0, \\ \widetilde{v}(x,0;s) &= Q(x,s), \quad 0 \le x \le L. \end{aligned} \tag{1.68}$$

Proof. Differentiating the solution for $u(x,t)$ with respect to t, we have

$$\begin{aligned} u_t(x,t) &= \widetilde{v}(x,0;t) + \int_0^t \widetilde{v}_t(x,t-s;s)\,ds \\ &= Q(x,t) + \int_0^t k\widetilde{v}_{xx}(x,t-s;s)\,ds \\ &= Q(x,t) + ku_{xx}(x,t). \end{aligned} \tag{1.69}$$

□

We see that the solution of a problem with a source can be converted to a related homogeneous differential equation with an initial condition. This is the essence of Duhamel's Principle. We demonstrate this in the following example.

Example 1.10. Use Duhamel's Principle to solve the initial-boundary value problem

$$\begin{aligned} u_t - ku_{xx} &= t \sin x, \qquad 0 \le x \le \pi, \quad t > 0, \\ u(0,t) &= 0, \quad u(\pi,t) = 0, \quad t > 0, \\ u(x,0) &= 0, \quad 0 \le x \le \pi. \end{aligned} \tag{1.70}$$

We first need to solve the related problem,

$$\begin{aligned} \widetilde{v}_t &= k\widetilde{v}_{xx}, \qquad 0 \le x \le \pi, \quad t > 0, \\ \widetilde{v}(0,t) &= 0, \quad \widetilde{v}(\pi,t) = 0, \quad t > 0, \\ \widetilde{v}(x,0) &= s \sin x, \quad 0 \le x \le \pi. \end{aligned} \tag{1.71}$$

One can show that the series solution is given by

$$\widetilde{v}(x,t;s) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx.$$

Setting $t = 0$,

$$\widetilde{v}(x,0;s) = \sum_{n=1}^{\infty} b_n \sin nx = s \sin x.$$

We can see that all of the b_n's vanish except that $n = 1$ case. Therefore, the solution is $\tilde{v}(x, 0) = se(-kt)\sin x$.

The final solution to the given problem is found through integration:

$$\begin{aligned} u(x,t) &= \int_0^t \tilde{v}(x, t-s; s)\, ds \\ &= \int_0^t se^{-k(t-s)} \sin x\, ds \\ &= \left(\frac{t}{k} - \frac{1-e^{-kt}}{k^2}\right) \sin x. \end{aligned} \tag{1.72}$$

1.8 Laplace's Equation in 2D

ANOTHER GENERIC PARTIAL DIFFERENTIAL EQUATION is Laplace's equation, $\nabla^2 u = 0$. Laplace's equation arises in many applications. As an example, consider a thin rectangular plate with boundaries set at fixed temperatures. Assume that any temperature changes of the plate are governed by the heat equation, $u_t = k\nabla^2 u$, subject to these boundary conditions. However, after a long period of time the plate may reach thermal equilibrium. If the boundary temperature is zero, then the plate temperature decays to zero across the plate. However, if the boundaries are maintained at a fixed nonzero temperature, which means energy is being put into the system to maintain the boundary conditions, the internal temperature may reach a nonzero equilibrium temperature. Reaching thermal equilibrium means that asymptotically in time the solution becomes time independent. Thus, the equilibrium state is a solution of the time independent heat equation, $\nabla^2 u = 0$.

Thermodynamic equilibrium, $\nabla^2 u = 0$.

A second example comes from electrostatics. Letting $\phi(\mathbf{r})$ be the electric potential, one has for a static charge distribution, $\rho(\mathbf{r})$, that the electric field, $\mathbf{E} = \nabla\phi$, satisfies one of Maxwell's equations, $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$. In regions devoid of charge, $\rho(\mathbf{r}) = 0$, the electric potential satisfies Laplace's equation, $\nabla^2\phi = 0$.

Incompressible, irrotational fluid flow, $\nabla^2\phi = 0$, for velocity $\mathbf{v} = \nabla\phi$. See more in Section 1.8.

As a final example, Laplace's equation appears in two-dimensional fluid flow described by the velocity field $\mathbf{v}(\mathbf{r})$. Assuming mass is conserved in the flow, then the fluid density, ρ, satisfies the continuity equation,

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0,$$

which is just another conservation law. If the density is constant, then the flow is called incompressible. For an incompressible flow, it follows that $\nabla \cdot \mathbf{v} = 0$.

An irrotational flow exists if you do not see any circulation in the fluid. If the flow is irrotational, then $\nabla \times \mathbf{v} = 0$. This means that the velocity field is a gradient field. So, we can introduce a velocity potential, $\mathbf{v} = \nabla\phi$. Thus, $\nabla \times \mathbf{v}$ vanishes by a vector identity and $\nabla \cdot \mathbf{v} = 0$ implies $\nabla^2\phi = 0$. So, once again we obtain Laplace's equation.

Solutions of Laplace's equation are called *harmonic functions*. These are often encountered in complex analysis where one applies complex variable techniques to solve the two-dimensional Laplace equation. In this section we will apply the Method of Separation of Variables to solve simple examples of Laplace's equation in two dimensions. Three-dimensional problems will be studied in Chapter 6. Parts of this discussion are repeated with more general boundary conditions in Section 6.3.

Example 1.11. Equilibrium Temperature Distribution for a Rectangular Plate

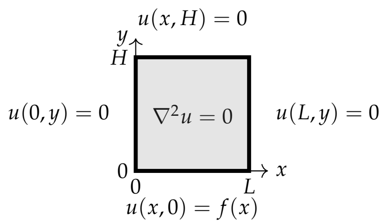


Figure 1.25: In this figure we show the domain and boundary conditions for the example of determining the equilibrium temperature distribution for a rectangular plate.

Let's consider Laplace's equation in Cartesian coordinates,

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, \quad 0 < y < H$$

with the boundary conditions

$$u(0,y) = 0, \quad u(L,y) = 0, \quad u(x,0) = f(x), \quad u(x,H) = 0.$$

The boundary conditions are shown in Figure 6.12. Note that there is no time-dependence and, therefore, no need for initial conditions.

As with the heat and wave equations, we can solve this problem using the method of separation of variables. Let $u(x,y) = X(x)Y(y)$. Then, Laplace's equation becomes

$$X''Y + XY'' = 0$$

and we can separate the x and y dependent functions and introduce a separation constant, λ,

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

Thus, we are led to two differential equations,

$$\begin{aligned} X'' + \lambda X &= 0, \\ Y'' - \lambda Y &= 0. \end{aligned} \tag{1.73}$$

From the pair of boundary conditions $u(0,y) = 0, u(L,y) = 0$, we have $X(0) = 0, X(L) = 0$. So, we have the usual eigenvalue problem for $X(x)$,

$$X'' + \lambda X = 0, \quad X(0) = 0, X(L) = 0.$$

The solutions to this problem are given by

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots.$$

The general solution of the equation for $Y(y)$ is given by

$$Y(y) = c_1 e^{\sqrt{\lambda} y} + c_2 e^{-\sqrt{\lambda} y}.$$

The boundary condition $u(x,H) = 0$ implies $Y(H) = 0$. So, we require that

$$Y(H) = c_1 e^{\sqrt{\lambda} H} + c_2 e^{-\sqrt{\lambda} H} = 0.$$

We solve this for c_2,

$$c_2 = -c_1 e^{2\sqrt{\lambda}H}.$$

Inserting this result into the expression for $Y(y)$, we have

$$\begin{aligned} Y(y) &= c_1 e^{\sqrt{\lambda}y} - c_1 e^{2\sqrt{\lambda}H} e^{-\sqrt{\lambda}y} \\ &= c_1 e^{\sqrt{\lambda}H}\left(e^{-\sqrt{\lambda}H} e^{\sqrt{\lambda}y} - e^{\sqrt{\lambda}H} e^{-\sqrt{\lambda}y}\right) \\ &= c_1 e^{\sqrt{\lambda}H}\left(e^{-\sqrt{\lambda}(H-y)} - e^{\sqrt{\lambda}(H-y)}\right) \\ &= -2c_1 e^{\sqrt{\lambda}H} \sinh \sqrt{\lambda}(H-y). \end{aligned} \tag{1.74}$$

Note: Having carried out this computation, we can now see that it would be better to guess this form in the future. So, for $Y(H) = 0$, one would guess a solution $Y(y) = \sinh \sqrt{\lambda}(H-y)$. For $Y(0) = 0$, one would guess a solution $Y(y) = \sinh \sqrt{\lambda}y$. Similarly, if $Y'(H) = 0$, one would guess a solution $Y(y) = \cosh \sqrt{\lambda}(H-y)$.

Since we already know the values of the eigenvalues λ_n from the eigenvalue problem for $X(x)$, we have that the y-dependence is given by

$$Y_n(y) = \sinh \frac{n\pi(H-y)}{L}.$$

So, the product solutions are given by

$$u_n(x,y) = \sin \frac{n\pi x}{L} \sinh \frac{n\pi(H-y)}{L}, \quad n = 1, 2, \ldots.$$

These solutions satisfy Laplace's equation and the three homogeneous boundary conditions and in the problem. We should note that the constant $-2c_1 e^{\sqrt{\lambda}H}$ can be absorbed into the final product solutions and, therefore, can be dropped. Another way to take care of this factor is to redefine $c_1 = -\frac{1}{2} e^{-\sqrt{\lambda}H}$.

The remaining boundary condition, $u(x,0) = f(x)$, still needs to be satisfied. Inserting $y = 0$ in the product solutions does not satisfy the boundary condition unless $f(x)$ is proportional to one of the eigenfunctions $X_n(x)$. So, we first write down the general solution as a linear combination of the product solutions,

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi(H-y)}{L}. \tag{1.75}$$

Now, we apply the boundary condition, $u(x,0) = f(x)$, to find that

$$f(x) = \sum_{n=1}^{\infty} a_n \sinh \frac{n\pi H}{L} \sin \frac{n\pi x}{L}. \tag{1.76}$$

Defining $b_n = a_n \sinh \frac{n\pi H}{L}$, this becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{1.77}$$

We see that the determination of the unknown coefficients, b_n, is simply done by recognizing that this is a Fourier sine series. We now move on to the study of Fourier series and provide more complete answers in Section 6.3.

Example 1.12. Consider the boundary value problem depicted in Figure 1.26

$$\begin{aligned} u_{xx}+u_{yy} &= 0, \quad 0\le x\le 1, \quad 0\le y\le 1, \\ u(0,y) &= 0, \quad u(1,y)=\sin 3\pi y, \quad 0\le y\le 1, \\ u(x,0) &= 2\sin \pi x, \quad u(x,1)=0, \quad 0\le x\le 1. \end{aligned} \tag{1.78}$$

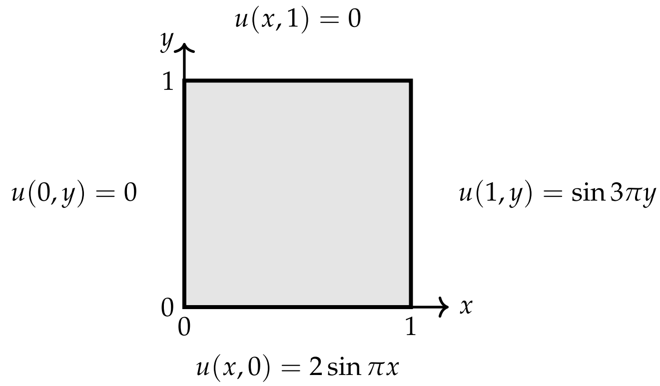


Figure 1.26: A boundary value problem for the 2D Laplace equation given by system (1.78).

In this problem there are two non-vanishing boundary conditions. But we can relate this problem to the problem with one non-vanishing boundary condition because we are dealing with a linear problem. In Figure 1.27 we depict how to split up a more general problem into two simpler problems. Thus, we have to solve Laplace's equation for solutions $u_1(x,y)$ and $u_2(x,y)$ which satisfy the boundary conditions

$$\begin{aligned} u_1(0,y) &= 0, \quad u_1(1,y)=0, \quad 0\le y\le 1, \\ u_1(x,0) &= f(x), \quad u_1(x,1)=0, \quad 0\le x\le 1, \\ u_2(0,y) &= 0, \quad u_2(1,y)=g(y), \quad 0\le y\le 1, \\ u_2(x,0) &= 0, \quad u_2(x,1)=0, \quad 0\le x\le 1. \end{aligned} \tag{1.79}$$

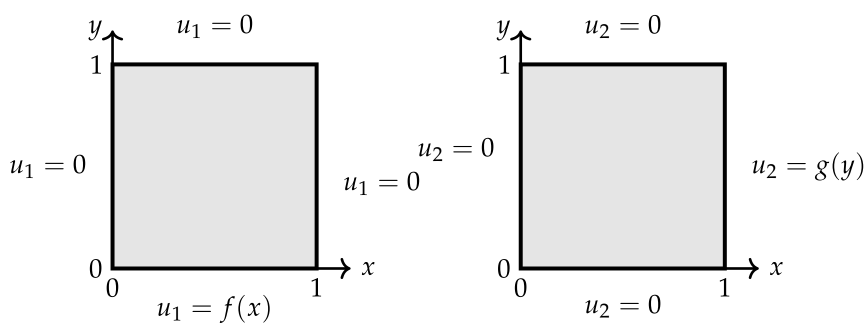


Figure 1.27: Creating two boundary value problems from that shown in Figure (1.26).

From the last example, we have

$$u_1(x,y)=\sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi(1-y).$$

Similarly, we can write the general solution for the second problem as

$$u_2(x,y) = \sum_{m=1}^{\infty} b_m \sin m\pi y \sinh m\pi x.$$

Then, the solution to the full problem is given by $u(x,y) = u_1(x,y) + u_2(x,y)$, or

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi(1-y) + \sum_{m=1}^{\infty} b_m \sin m\pi y \sinh m\pi x.$$

Applying this to the original problem, where $f(x) = 2\sin\pi x$ and $g(y) = \sin 3\pi y$, we see that the only nonzero coefficients are a_1 and b_3. In particular, we have

$$\begin{aligned} u(x,0) &= a_1 \sin\pi x \sinh\pi = 2\sin\pi x, \\ u(1,y) &= b_3 \sin 3\pi y \sinh 3\pi = \sin 3\pi y. \end{aligned} \tag{1.80}$$

Therefore, $a_1 = \dfrac{2}{\sinh\pi}$ and $b_3 = \dfrac{1}{\sinh 3\pi}$ and the solution is given as

$$u(x,y) = \frac{2}{\sinh\pi}\sin\pi x \sinh\pi(1-y) + \frac{1}{\sinh 3\pi}\sin 3\pi y \sinh 3\pi x. \tag{1.81}$$

A plot of this solution is shown in Figure 1.28.

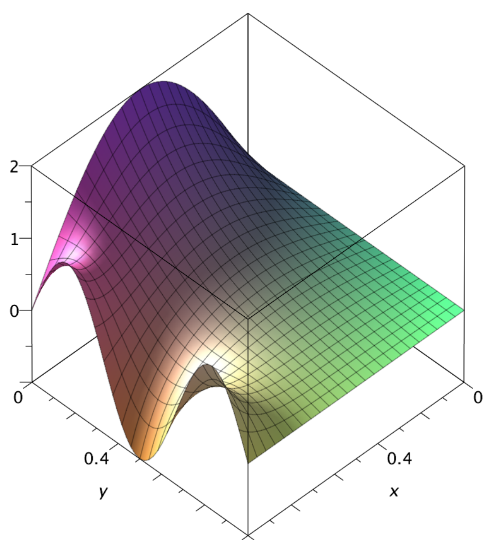


Figure 1.28: Plot of the solution in Equation (1.81) of the two-dimensional Laplace equation with boundary conditions $u(0,y) = 0$, $u(1,y) = \sin 3\pi y$, $u(x,0) = 2\sin\pi x$, $u(x,1) = 0$.

1.9 *Appendix: The Adomain Decomposition Method - Optional*

Throughout this course we are looking at classical methods for solving partial differential equations. The study of PDEs began in the 18th century with the work of Euler, d'Alembert, Lagrange and Laplace. Modeling the physics of continuous media, like vibrating strings, gravitation, electrostatics, fluid flows, heat conduction, electricity and magnetism lead to many linear and nonlinear partial differential equations. Linear PDEs were easiest to solve and during the 18th century the method of separation of variables was used starting with d'Alembert and Euler and the superpositions of solutions was reinforced through the work of Fourier, Dirichlet, and others. In the 20th century solution techniques for nonlinear PDEs evolved as well as the growth of numerical techniques due to the advent of computers. In this chapter we will introduce a relatively recent method which has appeared in many research papers in the last couple of decades, the Adomian Decomposition Method.

The Adomian Decomposition Method (ADM) is an analytical technique designed to solve a wide variety of linear and nonlinear differential equations, integral equations, and partial differential equations. Developed by George Adomian (1922-1996) in the 1980s, this method provides a systematic approach to decomposing complex equations into simpler subproblems, allowing for iterative refinement of the solution. ADM is particularly advantageous for tackling nonlinear problems directly, without the need for linearization or perturbative methods. In this section we will provide some examples of linear differential equations in order to demonstrate this method as another way to tackle some of the problems in this course.

The core idea of ADM involves expressing the solution of a given differential equation as an infinite series of unknown functions to be determined iteratively. If u denotes the solution, it is represented as:

$$u = \sum_{n=0}^{\infty} u_n, \tag{1.82}$$

where u_n are the components to be determined. For nonlinear PDEs, the method decomposes the nonlinear terms using Adomian polynomials. If $N(u)$ represents a nonlinear operator acting on u, it is expressed as

$$N(u) = \sum_{n=0}^{\infty} A_n, \tag{1.83}$$

where the A_n are the Adomian polynomials that depend on the components u_n.

The procedure begins by rewriting the differential equation in the form:

$$Lu + Ru + N(u) = g, \tag{1.84}$$

where L is a linear operator, R is a linear but potentially inhomogeneous operator, N is a nonlinear operator, and g is a source term.

Using the decomposed forms of u and $N(u)$, the equation becomes:

$$L\left(\sum_{n=0}^{\infty} u_n\right) + R\left(\sum_{n=0}^{\infty} u_n\right) + \sum_{n=0}^{\infty} A_n = g. \tag{1.85}$$

Equating terms of the same order, we derive a set of recursive relations for the components u_n. The zeroth component u_0 is obtained by solving:

$$u_0 = L^{-1}g + \text{initial/boundary values for} u, \tag{1.86}$$

and subsequent components are determined using:

$$u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n, \quad n \geq 0. \tag{1.87}$$

This is just a formal explanation of the process. We will give several examples implementing the process to see how it works in practice.

The Adomian Decomposition Method offers several advantages:

- **Direct Treatment of Nonlinearities:** ADM handles nonlinear terms directly without requiring linearization, preserving the inherent characteristics of the original problem.
- **Systematic Approach:** The method provides a clear, iterative process to approximate solutions, making it applicable to a broad range of problems.
- **Rapid Convergence:** For many well-behaved problems, the series solution converges quickly, offering accurate approximations with relatively few terms.

ADM has been successfully applied to various fields, including:

- **Engineering:** Problems involving heat transfer, fluid dynamics, and structural analysis.
- **Mathematical Physics:** Solutions to equations in quantum mechanics, general relativity, and field theory.
- **Biology and Medicine:** Models of population dynamics, disease spread, and other biological processes.

Example 1.13. Use the ADM to solve the first order (nonlinear) initial value problem

$$\frac{dy}{dt} = 1 + y^2, \quad y(0) = 0.$$

We identify the differential equation in operator form as

$$Ly + Ry = f(t), \tag{1.88}$$

where $L = \frac{d}{dt}$, $Ry = -y^2$, and $f(t) = 1$. Now, we prepare the problem for solution.

We write $Ly = 1 + y^2$. We want to solve for y. So, we apply L^{-1} to the equation,

$$L^{-1}Ly = L^{-1}(1) + L^{-1}(y^2).$$

But what is $L^{-1}Ly$? Since $L = \dfrac{d}{dt}$, then L^{-1} is just integration. We have

$$L^{-1}Ly = \int_0^t \frac{dy(\tau)}{d\tau}\, d\tau = y(t) - y(0). \tag{1.89}$$

[Note that we needed to distinguish between the integration variable τ and the independent variable t.] Also,

$$L^{-1}(1) = \int_0^t 1\, d\tau = t.$$

So, now the solution can be written

$$y(t) = y(0) + t + L^{-1}(y^2).$$

Next, we decompose the solution,

$$y(t) = \sum_{n=0}^{\infty} y_n(t) = y_0(t) + y_1(t) + y_2(t) + \dots. \tag{1.90}$$

Using the initial condition, $y(0) = 0$, we have

$$y_0(t) + y_1(t) + y_2(t) + \dots = t + L^{-1}[(y_0(t) + y_1(t) + \dots)^2].$$

Next, we formulate a recursive process to solve for the components, $y_n(t)$. First, we let $y_0(t) = t$. [If $y(0) \neq 0$, we would have included it as well.]

We then expand y^2 and group the terms so that

$$\begin{aligned} \sum_{n=0}^{\infty} A_n &= (y_0(t) + y_1(t) + y_2(t) + \dots)^2 \\ &= y_0^2 + 2y_0y_1 + y_1^2 + 2y_0y_2 + \dots. \end{aligned} \tag{1.91}$$

The A_n's are chosen so that they only contain y_k's with $k \leq n$. Then, we can solve for the components recursively as

$$y_{k+1}(t) = L^{-1}(A_k), \quad k \geq 0. \tag{1.92}$$

The A_k's are referred to as the Adomian polynomials. In this case we have

$$\begin{aligned} A_0 &= y_0^2 \\ A_1 &= 2y_0y_1 \\ A_2 &= y_1^2 + 2y_0y_2, \text{ etc.} \end{aligned} \tag{1.93}$$

There would be different polynomials for a nonlinear term of higher order.

Since $y_0(t) = t$, we can begin to find other components:

$$\begin{aligned} y_1(t) &= \int_0^t y_0^2\, d\tau = \frac{t^3}{3} \\ y_2(t) &= \int_0^t 2y_0y_1\, d\tau = \frac{2t^5}{15} \\ y_3(t) &= \int_0^t (y_1^2 + 2y_0y_2)\, d\tau = \frac{17t^7}{315} \end{aligned} \tag{1.94}$$

Summing the components, we have

$$y(t) = t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{17t^7}{315} + \dots. \tag{1.95}$$

This series can be truncated, giving an approximate solution, or it could be summed, if you are clever.

In this case, we could easily have integrated the separable equation.

$$t = \int_0^y \frac{dz}{1+z^2} = \tan^{-1} y,$$

or $y(t) = \tan t$. Looking up the series expansion for $\tan t$, we find agreement with the series found using the ADM.

Example 1.14. Forced Oscillator Problem

As the next example, we consider the second order differential equation for forced oscillations with no damping. It is given by

$$\frac{d^2u}{dt^2} + \omega_0^2 u = A\cos(\omega t), \tag{1.96}$$

where ω_0 is the natural frequency, and $f(t) = A\cos(\omega t)$ is the forcing function with amplitude A and forcing frequency ω.

We first rewrite the differential equation in operator form as

$$Lu + Ru = f(t), \tag{1.97}$$

where $L = \dfrac{d^2}{dt^2}$ and $Ru = \omega_0^2 u$.

Before proceeding, we need to be a little careful. We wish to solve for u. So, from $Lu = f(t) - Ru$, we obtain the solution by applying L^{-1} to both sides of the equation. This gives

$$L^{-1}Lu = L^{-1}(f(t) - Ru).$$

In the same way we inverted the first order derivative operator in the previous example, we need to be careful inverting the second derivative. We use τ and z for time integration variables to obtain

$$\begin{aligned} L^{-1}Lu &= \int_0^t \int_0^z \frac{d^2u(\tau)}{d\tau^2}\, d\tau dz \\ &= \int_0^t \frac{du(\tau)}{d\tau}\Big|_{\tau=0}^{z} dz \\ &= \int_0^t \left[\frac{du(z)}{dz} - u'(0)\right] dz \\ &= u(t) - u(0) - tu'(0). \end{aligned} \tag{1.98}$$

So, we now have the solution as

$$u(t) = u(0) + tu'(0) + L^{-1}(f(t) - Ru) \tag{1.99}$$

Then, we decompose the solution,

$$u(t) = \sum_{n=0}^{\infty} u_n(t). \tag{1.100}$$

In this case the forcing term does not need to be decomposed as it does not depend on u.

Finally, we finish setting up the problem by finding the needed recursive relations. The zeroth component is determined from using the source and initial values in Equation (A.34),

$$u_0 = u(0) + tu'(0) + L^{-1} f(t). \tag{1.101}$$

Then, subsequent components are found recursively using

$$u_{n+1} = -L^{-1} R u_n = -\omega_0^2 L^{-1} u_n. \tag{1.102}$$

The solution is found through an iterative procedure starting with the solution for u_0. For the forcing $f(t) = A\cos(\omega t)$, we have

$$u_0 = u(0) + tu'(0) + L^{-1}(A\cos(\omega t)).$$

Since, $L = \dfrac{d^2}{dt^2}$, we carefully integrate twice with respect to t as

$$L^{-1}(\cdot) = \int_0^t \int_0^z (\cdot)\, d\tau dz.$$

This gives

$$\begin{aligned} L^{-1}(A\cos(\omega t)) &= \int_0^t \int_0^z A\cos(\omega\tau)\, d\tau dz \\ &= \frac{A}{\omega}\int_0^t \sin(\omega z)\, dz \\ &= -\frac{A}{\omega^2}\cos(\omega u)\Big|_{z=0}^{t}. \\ &= -\frac{A}{\omega^2}(\cos(\omega t) - 1). \end{aligned} \tag{1.103}$$

Therefore,

$$u_0(t) = u(0) + tu'(0) - \frac{A}{\omega^2}(\cos(\omega t) - 1). \tag{1.104}$$

Next, we solve for the components $u_n(t)$, which satisfy the equations

$$\begin{aligned} u_1(t) &= -\omega_0^2 L^{-1} u_0, \\ u_2(t) &= -\omega_0^2 L^{-1} u_1, \\ u_3(t) &= -\omega_0^2 L^{-1} u_2, \text{ etc.} \end{aligned} \tag{1.105}$$

Carrying out the computations for these three terms, one finds

$$\begin{aligned} u_1(t) &= \frac{\omega_0^2\left(u'(0)\omega^4 t^3 + 3u(0)\omega^4 t^2 + 3A\omega^2 t^2 + 6A\cos(\omega t) - 6A\right)}{6\omega^4} \\ u_2(t) &= \frac{\omega_0^4\left(-u'(0)\omega^6 t^5 - 5\omega^6 u(0)\, t^4 - 5A\omega^4 t^4 + 60A\omega^2 t^2\right)}{120\omega^6} \\ &\quad + A\frac{\cos(\omega t) - 1}{\omega^6} \\ u_3(t) &= \frac{\omega_0^6\left(u'(0)\omega^8 t^7 + 7\omega^8 u(0)\, t^6 + 7A\omega^6 t^6 - 210A\omega^4 t^4 + 2520A\omega^2 t^2\right)}{5040\omega^8} \\ &\quad + A\frac{\cos(\omega t) - 1}{\omega^8}. \end{aligned} \tag{1.106}$$

Summing these, we get the approximate solution

$$\begin{aligned} u(t) &= \left(1-\frac{1}{2}\omega_0^2t^2+\frac{1}{24}\omega_0^4t^4-\frac{1}{720}\omega_0^6t^6\right)u(0) \\ &+\left(t-\frac{1}{6}\omega_0^2t^3+\frac{1}{120}\omega_0^4t^5-\frac{1}{5040}\omega_0^6t^7)\right)u'(0) \\ &-\frac{A\cos(\omega t)}{\omega^2}-\frac{\omega_0^2A\cos(\omega t)}{\omega^4}-\frac{\omega_0^4A\cos(\omega t)}{\omega^6}-\frac{\omega_0^6A\cos(\omega t)}{\omega^8} \\ &+\frac{\omega_0^4A\,t^4}{24\omega^2}-\frac{\omega_0^4A\,t^2}{2\omega^4}+\frac{\omega_0^6A}{\omega^8}+\frac{\omega_0^2A}{\omega^4}-\frac{\omega_0^2A\,t^2}{2\omega^2}-\frac{\omega_0^6A\,t^6}{720\omega^2} \\ &+\frac{\omega_0^6A\,t^4}{24\omega^4}-\frac{\omega_0^6A\,t^2}{2\omega^6}+\frac{\omega_0^4A}{\omega^6}+\frac{A}{\omega^2}. \end{aligned} \tag{1.107}$$

Of course, there is an easier way to solve this problem. The solution to the homogeneous problem is

$$u_h(t)=a\sin\omega_0t+b\cos\omega_0t.$$

The particular solution can be found using the Method of Undetermined Coefficients. We assume a form for the particular solution,

$$u_p(t)=B\cos\omega t.$$

[This is possible since there is no first order derivative term in the equation.] Inserting this guess, we find that $B=\dfrac{A}{\omega_0^2-\omega^2}$. So, the general solution is

$$u(t)=a\sin\omega_0t+b\cos\omega_0t+\frac{A}{\omega_0^2-\omega^2}\cos\omega t.$$

Finally, we need to impose the initial conditions.

$$u(0)=b+\frac{A}{\omega_0^2-\omega^2},\qquad u'(0)=a\omega_0.$$

Solving for the coefficients, gives

$$u(t)=\frac{u'(0)}{\omega_0}\sin\omega_0t+\left(u(0)-\frac{A}{\omega_0^2-\omega^2}\right)\cos\omega_0t+\frac{A}{\omega_0^2-\omega^2}\cos\omega t.$$

Do these solutions seem compatible? Is the ADM truncated series an approximation to the exact solution? We should recognize from the first two lines of Equation (1.107) as $u(0)\cos\omega_0t$ and $\dfrac{u'(0)}{\omega_0}\sin\omega_0t$, respectively. In order to obtain the third line, we consider an expansion for B. Namely,

$$\begin{aligned} \frac{A}{\omega_0^2-\omega^2} &= -\frac{A}{\omega^2}\frac{1}{1-\frac{\omega_0^2}{\omega^2}} \\ &= -\frac{A}{\omega^2}\left(1+\frac{\omega_0^2}{\omega^2}+\frac{\omega_0^4}{\omega^4}+\frac{\omega_0^6}{\omega^6}+\dots\right) \end{aligned} \tag{1.108}$$

This is what multiplies a $\cos \omega t$. All that is left to account for is the term

$$-\frac{A}{\omega_0^2 - \omega^2} \cos \omega_0 t.$$

However, a series expansion of this expression about $\omega_0 = 0$ captures the remaining terms.

Example 1.15. Obtain solutions to the Airy Equation,

$$u''(x) = xu(x), \quad u(0) = A, u'(0) = B. \tag{1.109}$$

This equation takes the form $Lu = xu$, where $L = \frac{d^2}{dx^2}$. Applying L^{-1} to the differential equation gives the solution

$$u(x) = L^{-1}(xu) + \text{terms involving initial conditions.}$$

As before, we find the terms involving initial conditions from

$$L^{-1}Lu = \int_0^x \int_0^z u''(\xi)\, d\xi dz = u(x) - u(0) - xu'(0).$$

Using the initial conditions, the solution takes the form

$$u(x) = \sum_{n=0}^{\infty} u_n(x) = A + Bx + L^{-1}\left(x \sum_{n=0}^{\infty} u_n(x)\right).$$

Then, the recursive scheme takes the form

$$\begin{aligned} u_0(x) &= A + Bx, \\ u_{k+1}(x) &= L^{-1}(xu_k(x)). \end{aligned} \tag{1.110}$$

We compute $u_1(x)$ as

$$\begin{aligned} u_1(x) &= \int_0^x \int_0^z \xi u_0(\xi)\, d\xi dz \\ &= \int_0^x \left(\frac{1}{3} B z^3 + \frac{1}{2} A z^2\right) dz \\ &= \frac{1}{12} B x^4 + \frac{1}{6} A x^3. \end{aligned} \tag{1.111}$$

Similarly, we find

$$\begin{aligned} u_2(x) &= \frac{1}{504} x^7 B + \frac{1}{180} x^6 A \\ u_3(x) &= \frac{1}{45360} x^{10} B + \frac{1}{12960} x^9 A. \end{aligned} \tag{1.112}$$

Adding the components, we obtain

$$\begin{aligned} u(x) &= A\left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \frac{x^9}{12960} + \ldots\right) \\ &\quad + B\left(x + \frac{x^4}{12} + \frac{x^7}{504} + \frac{x^{10}}{45360} + \ldots\right). \end{aligned} \tag{1.113}$$

Here we have the series solution that Airy and Stokes obtained using power series methods when trying to explain the superluminaries of a rainbow.

1.9.1 Heat Flow in a One-Dimensional Rod

Now it's time to use the ADM to solve a partial differential equation. We will show how useful it may be to solve an initial-boundary value problem for the heat equation. But first we will work out the notation needed for a PDE.

Consider the heat equation for a one-dimensional rod of length $L = 1$ with fixed zero temperature ends:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0, \tag{1.114}$$

with boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \tag{1.115}$$

and the initial condition

$$u(x,0) = f(x). \tag{1.116}$$

We have seen how to study such problems using the Method of Separation of Variables. We will solve this problem using the Adomian Decomposition Method. To apply this method to the heat equation, we first integrate the heat equation with respect to time:

$$u(x,t) - u(x,0) = k\int_0^t \frac{\partial^2 u(x,\tau)}{\partial x^2}\, d\tau.$$

Using the initial condition $u(x,0) = f(x)$, we get:

$$u(x,t) = f(x) + k\int_0^t \frac{\partial^2 u(x,\tau)}{\partial x^2}\, d\tau. \tag{1.117}$$

Next, we decompose the solution $u(x,t)$ into a series:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

We then apply the u-expansion, where the zeroth component u_0 satisfies the initial condition: $u_0(x,t) = f(x)$, and the subsequent components u_{n+1} are determined by:

$$u_{n+1}(x,t) = k\int_0^t \frac{\partial^2 u_n(x,\tau)}{\partial x^2}\, d\tau. \tag{1.118}$$

Example 1.16. Consider the initial condition $f(x) = \sin \pi x$.

We first set $u_0(x,t) = \sin \pi x$. Now, recursively find several of the component functions. First, we have

$$\begin{aligned} u_1(x,t) &= k\int_0^t \frac{\partial^2 u_0(x,\tau)}{\partial x^2}\, d\tau \\ &= k\int_0^t \frac{\partial^2 \sin \pi x}{\partial x^2}\, d\tau \\ &= -k\pi^2 t \sin \pi x. \end{aligned} \tag{1.119}$$

Similarly, we find

$$\begin{aligned} u_2(x,t) &= k^2\pi^4\frac{t^2}{2}\sin\pi x \\ u_3(x,t) &= -k^3\pi^6\frac{t^3}{6}\sin\pi x \\ u_4(x,t) &= k^4\pi^8\frac{t^4}{24}\sin\pi x. \end{aligned} \tag{1.120}$$

Then, the solution becomes

$$\begin{aligned} u(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+u_4(x,t)+\dots \\ &= \left(1-k\pi^2t+k^2\pi^4\frac{t^2}{2}-k^3\pi^6\frac{t^3}{6}+k^4\pi^8\frac{t^4}{24}-\dots\right)\sin\pi x \\ &= e^{-k\pi^2t}\sin\pi x. \end{aligned} \tag{1.121}$$

Based on what you know about solving the heat equation, does this solution look correct to you?

Example 1.17. $f(x) = x(1-x)$.

As we have seen in this chapter, we can find the general solution for this problem in terms of linear combinations of product solutions. However, we will need Fourier series from the next chapter to obtain a solution satisfying this initial condition. This can be done by the reader in Problem 16. Let's see how far we can proceed using the ADM.

We start with $u_0(x,t) = x(1-x)$. Now, we solve for u_1:

$$u_1(x,t) = k\int_0^t \frac{\partial^2 u_0(x,\tau)}{\partial x^2}\,d\tau.$$

Since $u_0(x,t) = x(1-x)$, we have

$$\frac{\partial^2 u_0}{\partial x^2} = \frac{\partial^2}{\partial x^2}(x(1-x)) = -2. \tag{1.122}$$

Therefore:

$$u_1(x,t) = k\int_0^t (-2)\,d\tau = -2\alpha t. \tag{1.123}$$

Next, we find u_2 as

$$u_2(x,t) = k\int_0^t \frac{\partial^2 u_1(x,\tau)}{\partial x^2}\,d\tau. = 0. \tag{1.124}$$

Therefore the series will truncate as all remaining components vanish. Combining the non-vanishing term, we find

$$u(x,t) \approx u_0(x,t)+u_1(x,t)+u_2(x,t) = x(1-x)-2\alpha t. \tag{1.125}$$

Does this look like it could be the correct solution? Adomian noted that in cases where one operator, $L_x = \frac{\partial^2}{\partial x^2}$, annilhilates the series in a finite number of terms, the ADM solution may not satisfy the given conditions. In 1992 Adomain and Rach noted that since the initial condition in the first example worked, then perhaps one needs to first expand $f(x) = x(1-x)$ in terms of sines and then apply this method.

Luo et al. in 2006 suggested that if

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

for $x \in [0, L]$, the solution can be found without annihilation of the component series. If we define the operators $L_t = \dfrac{\partial}{\partial t}$ and $L_x = \dfrac{\partial^2}{\partial x^2}$, and consider the heat equation $u_t = u_{xx}$, then $L_t u = L_x u$ leads to the solution

$$u(x,t) = u(x,0) + L_t^{-1} L_x \sum_{n=0}^{\infty} u_n(x,t).$$

The recursive process with the sine series expansion of the initial condition yields

$$\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} \left[L_t^{-1} L_x\right]^k f(x) \\
&= \sum_{k=0}^{\infty} \left[L_t^{-1} L_x\right]^n \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\partial^{2k}}{\partial x^{2k}} \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^{2k} b_n \sin \frac{n\pi x}{L} \\
&= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \sum_{k=0}^{\infty} (-1)^k \left(\frac{n\pi}{L}\right)^{2k} \frac{t^k}{k!} \\
&= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} e^{-n^2\pi^2 t/L^2}. \qquad (1.126)
\end{aligned}$$

we have arrived at the solution of the one dimensional heat equation in Equation (1.12) obtained using the Method of Separation of Variables.

Before leaving this topic, we will consider a nonhomogenous heat equation example.

Example 1.18. Apply ADM to solve this nonhomogenous problem.

$$\begin{aligned}
u_t - k u_{xx} &= t \sin x, \qquad 0 \le x \le \pi, \quad t > 0, \\
u(0,t) &= 0, \quad u(\pi, t) = 0, \quad t > 0, \\
u(x,0) &= 0, \quad 0 \le x \le \pi. \qquad (1.127)
\end{aligned}$$

In Example 1.10 we used Duhamel's Principle to solve this initial-boundary value problem. Here we rewrite the PDE using the operators L_t and L_x from above:

$$L_t u = k L_x u + t \sin x.$$

Solving for $u(x,t)$,

$$u(x,t) = u(x,0) + L_t^{-1}(t \sin x) + k L_t^{-1} L_x u.$$

Introducing the decomposition

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

with

$$u_0(x,t) = u(x,0) + L_t^{-1}(t \sin x) = \frac{t^2}{2}\sin x,$$

we can proceed with the recursive evaluation of the other component functions. Namely,

$$\begin{aligned} u_n(x,t) &= kL_t^{-1}L_x u_{n-1}(x,t), \quad n \geq 1 \\ &= (kL_t^{-1}L_x)^2 u_{n-2}(x,t), \\ &= (kL_t^{-1}L_x)^p u_{n-p}(x,t), \\ &= (kL_t^{-1}L_x)^n u_0(x,t), \\ &= (kL_t^{-1}L_x)^n t \sin x \\ &= k^n L_t^{-n}(t) L_x^n(\sin x) \end{aligned} \tag{1.128}$$

.

Since $L_x(\sin x) = -\sin x$, $L_x^n(\sin x) = (-1)^n \sin x$. Also, we have

$$L_t^{-n}(t) = \frac{t^{n+2}}{(n+2)!}.$$

Then, the solution is

$$u(x,t) = \sum_{n=0}^{\infty} (-k)^n \frac{t^{n+2}}{(n+2)!} \sin x.$$

The coefficient sum looks similar to an exponential series, but it is slightly askew. If we re-indexed, using $j = n+2$, then

$$\begin{aligned} \sum_{n=0}^{\infty} (-k)^n \frac{t^{n+2}}{(n+2)!} &= \sum_{j=2}^{\infty} (-k)^{j-2} \frac{t^j}{j!} \\ &= \frac{1}{k^2} \sum_{j=2}^{\infty} \frac{(-kt)^j}{j!} \\ &= \frac{1}{k^2}\left(e^{-kt} - 1 + kt\right) \\ &= \frac{t}{k} - \frac{1-e^{-kt}}{k^2}. \end{aligned} \tag{1.129}$$

So, the final solution is

$$u(x,t) = \left(\frac{t}{k} - \frac{1-e^{-kt}}{k^2}\right)\sin x,$$

which is the same solution we found in Example 1.10.

In this section, we applied the Adomian Decomposition Method to solve the heat equation for a one-dimensional rod with fixed zero temperature ends and an initial temperature profile. The method involves integrating the heat equation with respect to time, applying the initial condition, and then using the u-expansion to iteratively determine the components of the solution. In some cases we need to be cautious as the component series may only have a finite number of terms, which then may need the introduction of a Fourier sine series.

Problems

1. Solve the following initial value problems.

a. $x'' + x = 0, \quad x(0) = 2, \quad x'(0) = 0.$

b. $y'' + 2y' - 8y = 0, \quad y(0) = 1, \quad y'(0) = 2.$

c. $x^2y'' - 2xy' - 4y = 0, \quad y(1) = 1, \quad y'(1) = 0.$

2. Solve the following boundary value problems directly, when possible.

a. $x'' + x = 2, \quad x(0) = 0, \quad x'(1) = 0.$

b. $y'' + 2y' - 8y = 0, \quad y(0) = 1, \quad y(1) = 0.$

c. $y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$

3. Consider the boundary value problem for the deflection of a horizontal beam fixed at one end,

$$\frac{d^4y}{dx^4} = C, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(L) = 0, \quad y'''(L) = 0.$$

Solve this problem assuming that C is a constant.

4. Consider the following boundary value problems. Determine the eigenvalues, λ, and eigenfunctions, $y(x)$ for each problem.[9]

a. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = 0.$

b. $y'' - \lambda y = 0, \quad y(-\pi) = 0, \quad y'(\pi) = 0.$

c. $x^2y'' + xy' + \lambda y = 0, \quad y(1) = 0, \quad y(2) = 0.$

d. $(x^2y')' + \lambda y = 0, \quad y(1) = 0, \quad y'(e) = 0.$

[9] In problem d you will not get exact eigenvalues. Show that you obtain a transcendental equation for the eigenvalues in the form $\tan z = 2z$. Find the first three eigenvalues numerically.

5. Find the product solutions, $u(x,t) = T(t)X(x)$, to the heat equation, $u_t - u_{xx} = 0$, on $[0, \pi]$ satisfying the boundary conditions $u_x(0,t) = 0$ and $u(\pi, t) = 0$.

6. Find the product solutions, $u(x,t) = T(t)X(x)$, to the wave equation $u_{tt} = 2u_{xx}$, on $[0, 2\pi]$ satisfying the boundary conditions $u(0,t) = 0$ and $u_x(2\pi, t) = 0$.

7. Find product solutions, $u(x,y) = X(x)Y(y)$, to Laplace's equation, $u_{xx} + u_{yy} = 0$, on the unit square satisfying the boundary conditions $u(0,y) = 0$, $u(1,y) = g(y)$, $u(x,0) = 0$, and $u(x,1) = 0$.

8. Classify the following equations as either hyperbolic, parabolic, or elliptic.

a. $u_{yy} + u_{xy} + u_{xx} = 0$.
b. $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 5$.
c. $3u_{xx} + 2u_{xy} + 5u_{yy} = 0$.
d. $u_{xx} - 3u_{xy} + u_{yy} = 4y$.
e. $u_{xx} + yu_{yy} = 0$.
f. $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$.
g. $y^2u_{xx} + 2xyu_{xy} + (x^2 + 4x^4)u_{yy} = 0$.

9. Use d'Alembert's solution to prove

$$f(-\zeta) = f(\zeta), \quad g(-\zeta) = g(\zeta)$$

for the semi-infinite string satisfying the free end condition $u_x(0,t) = 0$. This is similar to the derivation in Equation (1.28) for the case of a fixed boundary condition.

10. Derive a solution similar to d'Alembert's solution for the equation $u_{tt} + 2u_{xt} - 3u = 0$.

11. Consider the initial value problem for the wave equation:

PDE: $u_{tt} = u_{xx} = 0$, $-\infty < x < \infty$, $t > 0$,
IC: $u(x,0) = \sin 5x$, $u_t(x,0) = \frac{1}{5}\cos x$.

a. Use d'Alembert's formula to find the solution.
b. Show that the solution is of the form $u(x,t) = F(x+ct) + G(x-ct)$ by finding $F(\xi)$ and $G(\eta)$.

12. Solve the initial value problem for the wave equation using d'Alembert's solution.

PDE: $u_{tt} - 4u_{xx} = 0$, $-\infty < x < \infty$, $t > 0$,
IC: $u(x,0) = \frac{1}{1+x^2}$, $u_t(x,0) = \cos 2x$.

Plot the solution for $x \in [-50, 50]$ for several times and describe the behavior of the solution.

13. Sketch the appropriate periodic extension of the plucked string initial profile given by

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{\ell}{2}, \\ \ell - x, & \frac{\ell}{2} \le x \le \ell, \end{cases}$$

satisfying the boundary conditions at $u(0,t) = 0$ and $u_x(\ell,t) = 0$ for $t > 0$.

14. Find and sketch the solution of the problem

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 \le x \le 1, t > o \\ u(x,0) &= \begin{cases} 0, & 0 \le x < \frac{1}{4}, \\ 1, & \frac{1}{4} \le x \le \frac{3}{4}, \\ 0, & \frac{3}{4} < x \le 1, \end{cases} \\ u_t(x,0) &= 0, \\ u(0,t) &= 0, \quad t > 0, \\ u(1,t) &= 0, \quad t > 0, \end{aligned}$$

15. Find the solution to the heat equation:

PDE: $u_t = 5u_{xx}$, $0 \leq x \leq 1$, $t > 0$.

BC: $u(0,t) = 0$, $u(1,t) = 0$.

IC: $u(x,0) = \sin 7\pi x$.

16. Find the solution to the heat equation:

PDE: $u_t = 2u_{xx}$, $0 \leq x \leq 1$, $t > 0$.

BC: $u(0,t) = -1$, $u_x(1,t) = 1$.

IC: $u(x,0) = x + \sin \frac{3\pi x}{2} - 1$.

17. Find the solution to the heat equation:

PDE: $u_t = 5u_{xx}$, $0 \leq x \leq 10$, $t > 0$.

BC: $u_x(0,t) = 2$, $u_x(10,t) = 3$.

IC: $u(x,0) = \frac{x^2}{20} + 2x + \cos \pi x$.

18. Use Duhamel's Principle to find the solution to the nonhomogeneous heat equation:

PDE: $u_t - u_{xx} = t \sin x$, $0 \leq x \leq \pi$, $t > 0$.

BC: $u(0,t) = 0$, $u(\pi,t) = 0$.

IC: $u(x,0) = 0$.

19. Find the solution to the nonhomogeneous heat equation:

PDE: $u_t - u_{xx} = t(\sin 2\pi x + 2x)$, $0 \leq x \leq 1$, $t > 0$.

BC: $u(0,t) = 1$, $u(1,t) = t^2$.

IC: $u(x,0) = 1 + \sin 3\pi x - x$.

20. Solve the initial value problem for the wave equation,

PDE: $u_{tt} - 4u_{xx} = 0$, $0 < x < \pi$, $t > 0$,

BC: $u(0,t) = 0$, $u(\pi,t) = 0$.

IC: $u(x,0) = 5\sin 3x$, $u_t(x,0) = 0$.

21. Solve the initial value problem for the wave equation,

PDE: $u_{tt} - 9u_{xx} = 0$, $0 < x < 10$, $t > 0$,

BC: $u_x(0,t) = 0$, $u(10,t) = 0$.

IC: $u(x,0) = 5\cos \frac{\pi x}{4}$, $u_t(x,0) = 0$.

22. Solve the initial value problem for the wave equation,

PDE: $u_{tt} - u_{xx} = 0$, $0 < x < 1$, $t > 0$,

BC: $u(0,t) = 0$, $u(1,t) = 0$.

IC: $u(x,0) = \sin 4\pi x$, $u_t(x,0) = \sin 3\pi x$.

23. The nonhomogeneous problem for the wave equation,

PDE: $u_{tt} - c^2 u_{xx} = h(x,t)$, $-\infty < x < \infty$, $t > 0$,

IC: $u(x,0) = 0$, $u_t(x,0) = 0$,

can be solved using Duhamel's Principle for the wave equation. Namely, one solves the problem

PDE: $\tilde{v}_{tt} - c^2 \tilde{v}_{xx} = 0$, $-\infty < x < \infty$, $t > 0$,

IC: $\tilde{v}(x,0;s) = 0$, $\tilde{v}_t(x,0;s) = h(x,s)$,

and the solution to the nonhomogeneous equation is given by

$$u(x,t) = \int_0^t \tilde{v}(x,t-s;s)\,ds.$$

Verify that this is the solution.

24. Solve the initial value problem:

PDE: $u_{tt} - u_{xx} = x - t$, $-\infty < x < \infty$, $t > 0$,

IC: $u(x,0) = x^2$, $u_t(x,0) = \sin x$.

Hint: Split the problem into two terms, one for which you can use Duhamel's Principle from the previous problem and the other which can be solved using d'Alembert's solution.

25. Solve the initial-boundary value problem

$$\begin{aligned} u_t &= u_{xx} - u + 1, \qquad 0 \le x \le \pi, \quad t > 0, \\ u(0,t) &= 1, \quad u(\pi,t) = 1, \quad t > 0, \\ u(x,0) &= 1 + \sin x, \quad 0 \le x \le \pi. \end{aligned} \tag{1.130}$$

26. Solve the initial-boundary value problem

$$\begin{aligned} u_t - u_{xx} &= \cos x, \qquad 0 \le x \le \pi, \quad t > 0, \\ u(0,t) &= 1 - e^{-t}, \quad u(\pi,t) = e^{-t} - 1, \quad t > 0, \\ u(x,0) &= 0, \quad 0 \le x \le \pi. \end{aligned} \tag{1.131}$$

27. Use Duhamel's Principle to solve the initial-boundary value problem

$$\begin{aligned} u_t - ku_{xx} &= t \sin x, \qquad 0 \le x \le \pi, \quad t > 0, \\ u(0,t) &= 0, \quad u(\pi,t) = 0, \quad t > 0, \\ u(x,0) &= 0, \quad 0 \le x \le \pi. \end{aligned} \tag{1.132}$$

2

Trigonometric Fourier Series

"Fourier's theorem is not only one of the most beautiful results of modern analysis, but it may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics." William Thomson (Lord Kelvin, 1824-1907)

2.1 *Introduction to Fourier Series*

WE WILL NOW TURN TO THE STUDY of trigonometric series. You have seen that functions have series representations as expansions in powers of x, or $x - a$, in the form of Maclaurin and Taylor series. Recall that the Taylor series expansion is given by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

where the expansion coefficients are determined as

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

From the study of the heat equation and wave equation, we have found that there are infinite series expansions over other functions, such as sine functions. We now turn to such expansions and in the next chapter we will find out that expansions over special sets of functions are not uncommon in mathematics and physics. But, first we turn to Fourier trigonometric series.

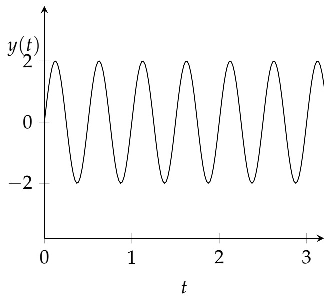
We will begin with the study of the Fourier trigonometric series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}.$$

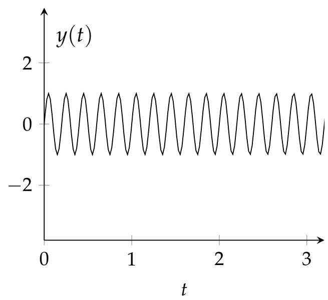
We will find expressions useful for determining the Fourier coefficients $\{a_n, b_n\}$ given a function $f(x)$ defined on $[-L, L]$. We will also see if the resulting infinite series reproduces $f(x)$. However, we first begin with some basic ideas involving simple sums of sinusoidal functions.

There is a natural appearance of such sums over sinusoidal functions in music. A pure note can be represented as

$$y(t) = A\sin(2\pi f t),$$



(a) $y(t) = 2\sin(4\pi f t)$



(b) $y(t) = \sin(10\pi f t)$

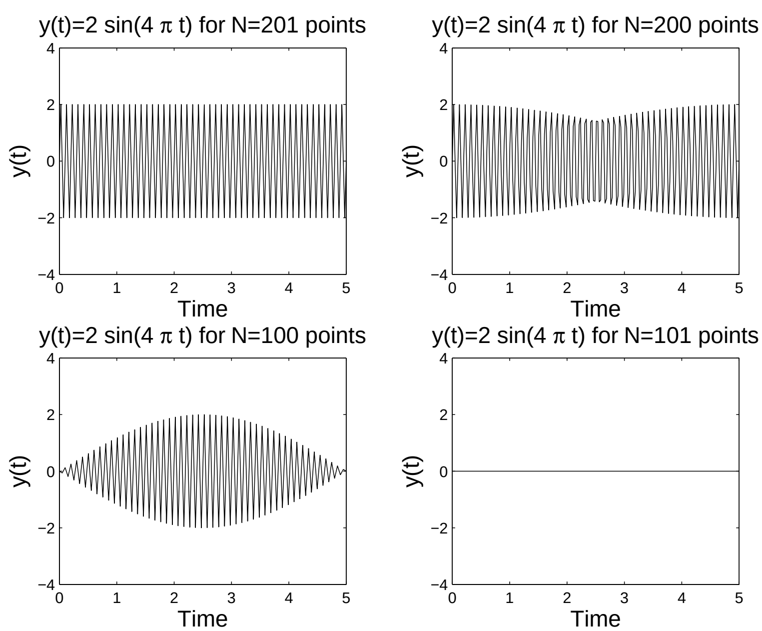
Figure 2.1: Plots of $y(t) = A\sin(2\pi f t)$ on $[0, 5]$ for $f = 2$ Hz and $f = 5$ Hz.

where A is the amplitude, f is the frequency in hertz (Hz), and t is time in seconds. The amplitude is related to the volume of the sound. The larger the amplitude, the louder the sound. In Figure 2.1 we show plots of two such tones with $f = 2$ Hz in the top plot and $f = 5$ Hz in the bottom one.

In these plots you should notice the difference due to the amplitudes and the frequencies. You can easily reproduce these plots and others in your favorite plotting utility.

As an aside, you should be cautious when plotting functions, or sampling data. The plots you get might not be what you expect, even for a simple sine function. In Figure 2.2 we show four plots of the function $y(t) = 2\sin(4\pi t)$. In the top left you see a proper rendering of this function. However, if you use a different number of points to plot this function, the results may be surprising. In this example we show what happens if you use $N = 200, 100, 101$ points instead of the 201 points used in the first plot. Such disparities are not only possible when plotting functions, but are also present when collecting data. Typically, when you sample a set of data, you only gather a finite amount of information at a fixed rate. This could happen when getting data on ocean wave heights, digitizing music and other audio to put on your computer, or any other process when you attempt to analyze a continuous signal.

Figure 2.2: Problems can occur while plotting. Here we plot the function $y(t) = 2\sin 4\pi t$ using $N = 201, 200, 100, 101$ points.



Next, we consider what happens when we add several pure tones. After all, most of the sounds that we hear are in fact a combination of pure tones with different amplitudes and frequencies. In Figure 2.3 we see what happens when we add several sinusoids. Note that as one adds more and more tones with different characteristics, the resulting signal gets more complicated. However, we still have a function of time. In this chapter we will ask, "Given a function $f(t)$, can we find a set of sinusoidal functions whose sum converges to $f(t)$?"

Looking at the superpositions in Figure 2.3, we see that the sums yield functions that appear to be periodic. This is not to be unexpected. We recall that a periodic function is one in which the function values repeat over the domain of the function. The length of the smallest part of the domain which repeats is called the period. We can define this more precisely: A function is said to be periodic with period T if $f(t+T) = f(t)$ for all t and the smallest such positive number T is called the period.

For example, we consider the functions used in Figure 2.3. We began with $y(t) = 2\sin(4\pi t)$. Recall from your first studies of trigonometric functions that one can determine the period by dividing the coefficient of t into 2π to get the period. In this case we have

$$T = \frac{2\pi}{4\pi} = \frac{1}{2}.$$

Looking at the top plot in Figure 2.1 we can verify this result. (You can count the full number of cycles in the graph and divide this into the total time to get a more accurate value of the period.)

In general, if $y(t) = A\sin(2\pi f t)$, the period is found as

$$T = \frac{2\pi}{2\pi f} = \frac{1}{f}.$$

Of course, this result makes sense, as the unit of frequency, the hertz, is also defined as s^{-1}, or cycles per second.

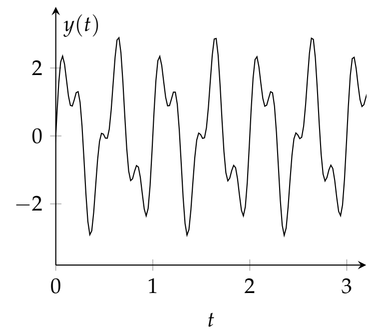
Returning to Figure 2.3, the functions $y(t) = 2\sin(4\pi t)$, $y(t) = \sin(10\pi t)$, and $y(t) = 0.5\sin(16\pi t)$ have periods of 0.5s, 0.2s, and 0.125s, respectively. Each superposition in Figure 2.3 retains a period that is the least common multiple of the periods of the signals added. For both plots, this is 1.0s $= 2(0.5)$s $= 5(.2)$s $= 8(.125)$s.

Our goal will be to start with a function and then determine the amplitudes of the simple sinusoids needed to sum to that function. We will see that this might involve an infinite number of such terms. Thus, we will be studying an infinite series of sinusoidal functions.

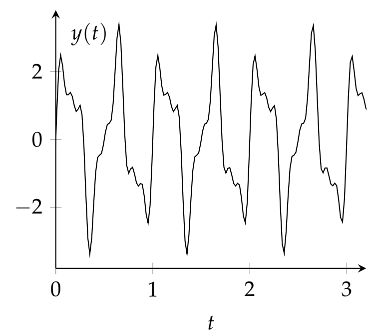
Secondly, we will find that using just sine functions will not be enough either. This is because we can add sinusoidal functions that do not necessarily peak at the same time. We will consider two signals that originate at different times. This is similar to when your music teacher would make sections of the class sing a song like "Row, Row, Row your Boat" starting at slightly different times.

We can easily add shifted sine functions. In Figure 2.4 we show the functions $y(t) = 2\sin(4\pi t)$ and $y(t) = 2\sin(4\pi t + 7\pi/8)$ and their sum. Note that this shifted sine function can be written as $y(t) = 2\sin(4\pi(t + 7/32))$. Thus, this corresponds to a time shift of $-7/32$.

So, we should account for shifted sine functions in the general sum. Of course, we would then need to determine the unknown time shift as well as the amplitudes of the sinusoidal functions that make up the signal, $f(t)$.

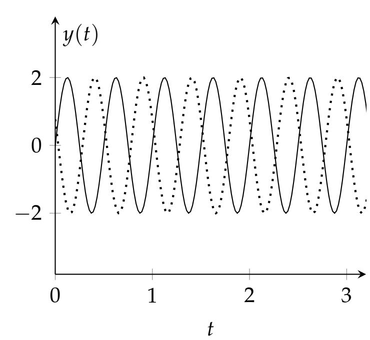


(a) Sum of signals with frequencies $f = 2$ Hz and $f = 5$ Hz.

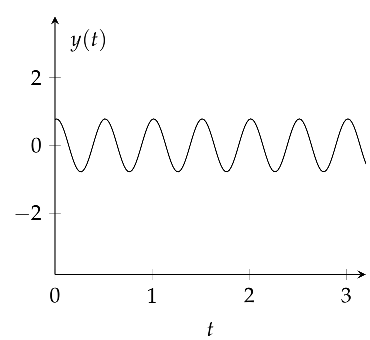


(b) Sum of signals with frequencies $f = 2$ Hz, $f = 5$ Hz, and $f = 8$ Hz.

Figure 2.3: Superposition of several sinusoids.



(a) Plot of each function.



(b) Plot of the sum of the functions.

Figure 2.4: Plot of the functions $y(t) = 2\sin(4\pi t)$ and $y(t) = 2\sin(4\pi t + 7\pi/8)$ and their sum.

We should note that the form in the lower plot of Figure 2.4 looks like a simple sinusoidal function for a reason. Let

$$y_1(t) = 2\sin(4\pi t),$$

$$y_2(t) = 2\sin(4\pi t + 7\pi/8).$$

Then,

$$\begin{aligned} y_1 + y_2 &= 2\sin(4\pi t + 7\pi/8) + 2\sin(4\pi t) \\ &= 2[\sin(4\pi t + 7\pi/8) + \sin(4\pi t)] \\ &= 4\cos\frac{7\pi}{16}\sin\left(4\pi t + \frac{7\pi}{16}\right). \end{aligned}$$

While this is one approach that some researchers use to analyze signals, there is a more common approach. This results from another reworking of the shifted function.

Consider the general shifted function

$$y(t) = A\sin(2\pi f t + \phi). \tag{2.1}$$

Note that $2\pi f t + \phi$ is called the phase of the sine function and ϕ is called the phase shift. We can use the trigonometric identity for the sine of the sum of two angles[1] to obtain

$$\begin{aligned} y(t) &= A\sin(2\pi f t + \phi) \\ &= A\sin(\phi)\cos(2\pi f t) + A\cos(\phi)\sin(2\pi f t). \end{aligned} \tag{2.2}$$

[1] Recall the identities

$$\begin{aligned} \sin(x+y) &= \sin x\cos y + \sin y\cos x, \\ \cos(x+y) &= \cos x\cos y - \sin x\sin y. \end{aligned}$$

Defining $a = A\sin(\phi)$ and $b = A\cos(\phi)$, we can rewrite this as

$$y(t) = a\cos(2\pi f t) + b\sin(2\pi f t).$$

Thus, we see that the signal in Equation (2.1) is a sum of sine and cosine functions with the same frequency and different amplitudes. If we can find a and b, then we can easily determine A and ϕ:

$$A = \sqrt{a^2 + b^2}, \quad \tan\phi = \frac{b}{a}.$$

We are now in a position to state our goal.

Goal - Fourier Analysis

Given a signal $f(t)$, we would like to determine its frequency content by finding out what combinations of sines and cosines of varying frequencies and amplitudes will sum to the given function. This is called Fourier Analysis.

2.2 *Fourier Trigonometric Series*

As we have seen in the last section, we are interested in finding representations of functions in terms of sines and cosines. Given a function $f(x)$ we seek a representation in the form

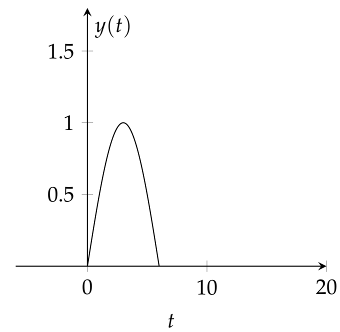
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n\cos nx + b_n\sin nx\right]. \tag{2.3}$$

Notice that we have opted to drop the references to the time-frequency form of the phase. This will lead to a simpler discussion for now and one can always make the transformation $nx = 2\pi f_n t$ when applying these ideas to applications.

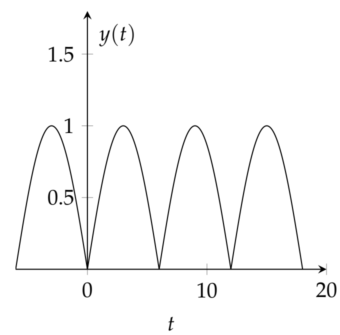
The series representation in Equation (2.3) is called a Fourier trigonometric series. We will simply refer to this as a Fourier series for now. The set of constants a_0, a_n, b_n, $n = 1, 2, \ldots$ are called the Fourier coefficients. The constant term is chosen in this form to make later computations simpler, though some other authors choose to write the constant term as a_0. Our goal is to find the Fourier series representation given $f(x)$. Having found the Fourier series representation, we will be interested in determining when the Fourier series converges and to what function it converges.

From our discussion in the last section, we see that The Fourier series is periodic. The periods of $\cos nx$ and $\sin nx$ are $\frac{2\pi}{n}$. Thus, the largest period, $T = 2\pi$, comes from the $n = 1$ terms and the Fourier series has period 2π. This means that the series should be able to represent functions that are periodic of period 2π.

While this appears restrictive, we could also consider functions that are defined over one period. In Figure 2.5 we show a function defined on $[0, 2\pi]$. In the same figure, we show its periodic extension. These are just copies of the original function shifted by the period and glued together. The extension can now be represented by a Fourier series and restricting the Fourier series to $[0, 2\pi]$ will give a representation of the original function. Therefore, we will first consider Fourier series representations of functions defined on this interval. Note that we could just as easily considered functions defined on $[-\pi, \pi]$ or any interval of length 2π. We will consider more general intervals later in the chapter.



(a) Plot of function $f(t)$.



(b) Periodic extension of $f(t)$.

Figure 2.5: Plot of the function $f(t)$ defined on $[0, 2\pi]$ and its periodic extension.

Fourier Coefficients

Theorem 2.1. *The Fourier series representation of $f(x)$ defined on $[0, 2\pi]$, when it exists, is given by (2.3) with Fourier coefficients*

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx, \quad n = 1, 2, \ldots. \end{aligned} \tag{2.4}$$

These expressions for the Fourier coefficients are obtained by considering special integrations of the Fourier series. We will now derive the a_n integrals in (2.4).

We begin with the computation of a_0. Integrating the Fourier series term by term in Equation (2.3), we have

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} \frac{a_0}{2}\,dx + \int_0^{2\pi} \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]\,dx. \tag{2.5}$$

We will assume that we can integrate the infinite sum term by term. Then we will need to compute

Evaluating the integral of an infinite series by integrating term by term depends on the convergence properties of the series.

$$\begin{aligned} \int_0^{2\pi} \frac{a_0}{2}\,dx &= \frac{a_0}{2}(2\pi) = \pi a_0, \\ \int_0^{2\pi} \cos nx\,dx &= \left[\frac{\sin nx}{n}\right]_0^{2\pi} = 0, \\ \int_0^{2\pi} \sin nx\,dx &= \left[\frac{-\cos nx}{n}\right]_0^{2\pi} = 0. \end{aligned} \tag{2.6}$$

From these results we see that only one term in the integrated sum does not vanish leaving

$$\int_0^{2\pi} f(x)\,dx = \pi a_0.$$

This confirms the value for a_0.[2]

[2] Note that $\frac{a_0}{2}$ is the average of $f(x)$ over the interval $[0, 2\pi]$. Recall from the first semester of calculus, that the average of a function defined on $[a, b]$ is given by

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

For $f(x)$ defined on $[0, 2\pi]$, we have

$$f_{\text{ave}} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{a_0}{2}.$$

Next, we will find the expression for a_n. We multiply the Fourier series (2.3) by $\cos mx$ for some positive integer m. This is like multiplying by $\cos 2x$, $\cos 5x$, etc. We are multiplying by all possible $\cos mx$ functions for different integers m all at the same time. We will see that this will allow us to solve for the a_n's.

We find the integrated sum of the series times $\cos mx$ is given by

$$\begin{aligned}\int_0^{2\pi} f(x)\cos mx\,dx &= \int_0^{2\pi} \frac{a_0}{2}\cos mx\,dx \\ &+ \int_0^{2\pi} \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]\cos mx\,dx.\end{aligned} \tag{2.7}$$

Integrating term by term, the right side becomes

$$\begin{aligned}\int_0^{2\pi} f(x)\cos mx\,dx = \frac{a_0}{2}\int_0^{2\pi}\cos mx\,dx \\ + \sum_{n=1}^{\infty}\left[a_n\int_0^{2\pi}\cos nx\cos mx\,dx + b_n\int_0^{2\pi}\sin nx\cos mx\,dx\right].\end{aligned} \tag{2.8}$$

We have already established that $\int_0^{2\pi}\cos mx\,dx = 0$, which implies that the first term vanishes.

Next we need to compute integrals of products of sines and cosines. This requires that we make use of some of the trigonometric identities listed in Chapter 1. For quick reference, we list these here.

Useful Trigonometric Identities

$$\sin(x \pm y) = \sin x\cos y \pm \sin y\cos x \tag{2.9}$$

$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y \tag{2.10}$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \tag{2.11}$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \tag{2.12}$$

$$\sin x\sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y)) \tag{2.13}$$

$$\cos x\cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y)) \tag{2.14}$$

$$\sin x\cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y)) \tag{2.15}$$

We first want to evaluate $\int_0^{2\pi}\cos nx\cos mx\,dx$. We do this by using the product identity (2.14). We have

$$\begin{aligned}\int_0^{2\pi} \cos nx \cos mx\, dx &= \frac{1}{2}\int_0^{2\pi} [\cos(m+n)x + \cos(m-n)x]\, dx \\ &= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_0^{2\pi} \\ &= 0. \end{aligned} \tag{2.16}$$

There is one caveat when doing such integrals. What if one of the denominators $m \pm n$ vanishes? For this problem $m+n \neq 0$, since both m and n are positive integers. However, it is possible for $m = n$. This means that the vanishing of the integral can only happen when $m \neq n$. So, what can we do about the $m = n$ case? One way is to start from scratch with our integration. (Another way is to compute the limit as n approaches m in our result and use L'Hopital's Rule. Try it!)

For $n = m$ we have to compute $\int_0^{2\pi} \cos^2 mx\, dx$. This can also be handled using a trigonometric identity. Using the half angle formula, (2.12), with $\theta = mx$, we find

$$\begin{aligned}\int_0^{2\pi} \cos^2 mx\, dx &= \frac{1}{2}\int_0^{2\pi} (1 + \cos 2mx)\, dx \\ &= \frac{1}{2}\left[x + \frac{1}{2m}\sin 2mx\right]_0^{2\pi} \\ &= \frac{1}{2}(2\pi) = \pi. \end{aligned} \tag{2.17}$$

To summarize, we have shown that

$$\int_0^{2\pi} \cos nx \cos mx\, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n. \end{cases} \tag{2.18}$$

This holds true for $m, n = 0, 1, \ldots$. [Why did we include $m, n = 0$?] When we have such a set of functions, they are said to be an orthogonal set over the integration interval. A set of (real) functions $\{\phi_n(x)\}$ is said to be orthogonal on $[a, b]$ if $\int_a^b \phi_n(x)\phi_m(x)\, dx = 0$ when $n \neq m$. Furthermore, if we also have that $\int_a^b \phi_n^2(x)\, dx = 1$, these functions are called orthonormal.

Definition of an orthogonal set of functions and orthonormal functions.

The set of functions $\{\cos nx\}_{n=0}^{\infty}$ are orthogonal on $[0, 2\pi]$. Actually, they are orthogonal on any interval of length 2π. We can make them orthonormal by dividing each function by $\sqrt{\pi}$ as indicated by Equation (2.17). This is sometimes referred to normalization of the set of functions.

The notion of orthogonality is actually a generalization of the orthogonality of vectors in finite dimensional vector spaces. The integral $\int_a^b f(x)f(x)\, dx$ is the generalization of the dot product, and is called the scalar product of $f(x)$ and $g(x)$, which are thought of as vectors in an infinite dimensional vector space spanned by a set of orthogonal functions. We will return to these ideas in the next chapter.

Returning to the integrals in equation (2.8), we still have to evaluate $\int_0^{2\pi} \sin nx \cos mx\, dx$. We can use the trigonometric identity involving products of sines and cosines, (2.15). Setting $A = nx$ and $B = mx$, we find that

$$\begin{aligned} \int_0^{2\pi} \sin nx \cos mx \, dx &= \frac{1}{2} \int_0^{2\pi} [\sin(n+m)x + \sin(n-m)x] \, dx \\ &= \frac{1}{2} \left[\frac{-\cos(n+m)x}{n+m} + \frac{-\cos(n-m)x}{n-m} \right]_0^{2\pi} \\ &= (-1+1) + (-1+1) = 0. \end{aligned} \tag{2.19}$$

So,

$$\int_0^{2\pi} \sin nx \cos mx \, dx = 0. \tag{2.20}$$

For these integrals we also should be careful about setting $n = m$. In this special case, we have the integrals

$$\int_0^{2\pi} \sin mx \cos mx \, dx = \frac{1}{2} \int_0^{2\pi} \sin 2mx \, dx = \frac{1}{2} \left[\frac{-\cos 2mx}{2m} \right]_0^{2\pi} = 0.$$

Finally, we can finish evaluating the expression in Equation (2.8). We have determined that all but one integral vanishes. In that case, $n = m$. This leaves us with

$$\int_0^{2\pi} f(x) \cos mx \, dx = a_m \pi.$$

Solving for a_m gives

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos mx \, dx.$$

Since this is true for all $m = 1, 2, \ldots$, we have proven this part of the theorem. The only part left is finding the b_n's This will be left as an exercise for the reader.

We now consider examples of finding Fourier coefficients for given functions. In all of these cases we define $f(x)$ on $[0, 2\pi]$.

Example 2.1. $f(x) = 3\cos 2x$, $x \in [0, 2\pi]$.

We first compute the integrals for the Fourier coefficients.

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} 3\cos 2x \, dx = 0. \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} 3\cos 2x \cos nx \, dx = 0, \quad n \neq 2. \\ a_2 &= \frac{1}{\pi} \int_0^{2\pi} 3\cos^2 2x \, dx = 3, \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} 3\cos 2x \sin nx \, dx = 0, \forall n. \end{aligned} \tag{2.21}$$

The integrals for a_0, $a_n, n \neq 2$, and b_n are the result of orthogonality. For a_2, the integral can be computed as follows:

$$a_2 = \frac{1}{\pi} \int_0^{2\pi} 3\cos^2 2x \, dx$$

$$= \frac{3}{2\pi}\int_0^{2\pi} [1+\cos 4x]\, dx$$

$$= \frac{3}{2\pi}\left[x + \underbrace{\frac{1}{4}\sin 4x}_{\text{This term vanishes!}}\right]_0^{2\pi} = 3. \tag{2.22}$$

Therefore, we have that the only nonvanishing coefficient is $a_2 = 3$. So there is one term and $f(x) = 3\cos 2x$.

Well, we should have known the answer to the last example before doing all of those integrals. If we have a function expressed simply in terms of sums of simple sines and cosines, then it should be easy to write down the Fourier coefficients without much work. This is seen by writing out the Fourier series,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]\,.$$

$$= \frac{a_0}{2} + a_1\cos x + b_1 \sin x + +a_2\cos 2x + b_2 \sin 2x + \ldots. \tag{2.23}$$

For the last problem, $f(x) = 3\cos 2x$. Comparing this to the expanded Fourier series, one can immediately read off the Fourier coefficients without doing any integration. In the next example we emphasize this point.

Example 2.2. $f(x) = \sin^2 x$, $x \in [0, 2\pi]$.

We could determine the Fourier coefficients by integrating as in the last example. However, it is easier to use trigonometric identities. We know that

$$\sin^2 x = \frac{1}{2}(1-\cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x.$$

There are no sine terms, so $b_n = 0$, $n = 1, 2, \ldots$. There is a constant term, implying $a_0/2 = 1/2$. So, $a_0 = 1$. There is a $\cos 2x$ term, corresponding to $n = 2$, so $a_2 = -\frac{1}{2}$. That leaves $a_n = 0$ for $n \neq 0, 2$. So, $a_0 = 1$, $a_2 = -\frac{1}{2}$, and all other Fourier coefficients vanish.

Example 2.3. $f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$.

This example will take a little more work. We cannot bypass evaluating any integrals this time. As seen in Figure 2.6, this function is discontinuous. So, we will break up any integration into two integrals, one over $[0, \pi]$ and the other over $[\pi, 2\pi]$.

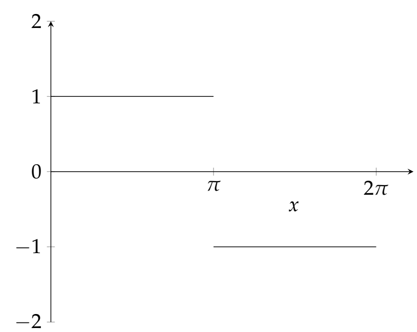


Figure 2.6: Plot of discontinuous function in Example 2.3.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} dx + \frac{1}{\pi}\int_{\pi}^{2\pi} (-1)\, dx$$

$$= \frac{1}{\pi}(\pi) + \frac{1}{\pi}(-2\pi + \pi) = 0. \tag{2.24}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$
\begin{aligned}
&= \frac{1}{\pi}\left[\int_0^{\pi} \cos nx\,dx - \int_{\pi}^{2\pi} \cos nx\,dx\right] \\
&= \frac{1}{\pi}\left[\left(\frac{1}{n}\sin nx\right)_0^{\pi} - \left(\frac{1}{n}\sin nx\right)_{\pi}^{2\pi}\right] \\
&= 0. \qquad (2.25)
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\left[\int_0^{\pi} \sin nx\,dx - \int_{\pi}^{2\pi} \sin nx\,dx\right] \\
&= \frac{1}{\pi}\left[\left(-\frac{1}{n}\cos nx\right)_0^{\pi} + \left(\frac{1}{n}\cos nx\right)_{\pi}^{2\pi}\right] \\
&= \frac{1}{\pi}\left[-\frac{1}{n}\cos n\pi + \frac{1}{n} + \frac{1}{n} - \frac{1}{n}\cos n\pi\right] \\
&= \frac{2}{n\pi}(1-\cos n\pi). \qquad (2.26)
\end{aligned}
$$

Often we see expressions involving $\cos n\pi = (-1)^n$ and $1 \pm \cos n\pi = 1 \pm (-1)^n$. This is an example showing how to re-index series containing $\cos n\pi$.

We have found the Fourier coefficients for this function. Before inserting them into the Fourier series (2.3), we note that $\cos n\pi = (-1)^n$. Therefore,

$$
1 - \cos n\pi = \begin{cases} 0, & n \text{ even} \\ 2, & n \text{ odd.} \end{cases} \qquad (2.27)
$$

So, half of the b_n's are zero. While we could write the Fourier series representation as

$$
f(x) \sim \frac{4}{\pi}\sum_{\substack{n=1\\ n\ odd}}^{\infty} \frac{1}{n}\sin nx,
$$

we could let $n = 2k-1$ in order to capture the odd numbers only. The answer can be written as

$$
f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1},
$$

Having determined the Fourier representation of a given function, we would like to know if the infinite series can be summed; i.e., does the series converge? Does it converge to $f(x)$? We will discuss this question later in the chapter after we generalize the Fourier series to intervals other than for $x \in [0, 2\pi]$.

2.3 *Fourier Series Over Other Intervals*

In many applications we are interested in determining Fourier series representations of functions defined on intervals other than $[0, 2\pi]$. In this section we will determine the form of the series expansion and the Fourier coefficients in these cases.

The most general type of interval is given as $[a, b]$. However, this often is too general. More common intervals are of the form $[-\pi, \pi]$, $[0, L]$, or $[-L/2, L/2]$. The simplest generalization is to the interval $[0, L]$. Such intervals arise often in applications. For example, for the problem of a one dimensional string of length L we set up the axes with the left end at $x = 0$ and the right end at $x = L$. Similarly for the temperature distribution along a one dimensional rod of length L we set the interval to $x \in [0, 2\pi]$. Such problems naturally lead to the study of Fourier series on intervals of length L. We will see later that symmetric intervals, $[-a, a]$, are also useful.

Given an interval $[0, L]$, we could apply a transformation to an interval of length 2π by simply rescaling the interval. Then, we could apply this transformation to the Fourier series representation to obtain an equivalent one useful for functions defined on $[0, L]$.

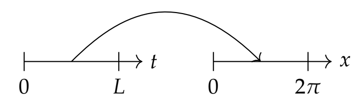


Figure 2.7: A sketch of the transformation between intervals $x \in [0, 2\pi]$ and $t \in [0, L]$.

We define $x \in [0, 2\pi]$ and $t \in [0, L]$. A linear transformation relating these intervals is simply $x = \dfrac{2\pi t}{L}$ as shown in Figure 2.7. So, $t = 0$ maps to $x = 0$ and $t = L$ maps to $x = 2\pi$. Furthermore, this transformation maps $f(x)$ to a new function $g(t) = f(x(t))$, which is defined on $[0, L]$. We will determine the Fourier series representation of this function using the representation for $f(x)$ from the last section.

Recall the form of the Fourier representation for $f(x)$ in Equation (2.3):

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]. \tag{2.28}$$

Inserting the transformation relating x and t, we have

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi t}{L} + b_n \sin \frac{2n\pi t}{L} \right]. \tag{2.29}$$

This gives the form of the series expansion for $g(t)$ with $t \in [0, L]$. But, we still need to rewrite the Fourier coefficients.

Recall, that

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx.$$

We need to make a substitution in the integral of $x = \dfrac{2\pi t}{L}$. We also will need to transform the differential, $dx = \dfrac{2\pi}{L} dt$. Thus, the resulting form for the Fourier coefficients is

$$a_n = \frac{2}{L} \int_0^L g(t) \cos \frac{2n\pi t}{L} \, dt. \tag{2.30}$$

Similarly, we find that

$$b_n = \frac{2}{L} \int_0^L g(t) \sin \frac{2n\pi t}{L} \, dt. \tag{2.31}$$

We note first that when $L = 2\pi$ we get back the series representation that we first studied. Also, the period of $\cos \dfrac{2n\pi t}{L}$ is $\dfrac{L}{n}$, which means that the representation for $g(t)$ has a period of L corresponding to $n = 1$.

At the end of this section we present the derivation of the Fourier series representation for a general interval for the interested reader. In Table 2.1 we summarize some commonly used Fourier series representations.

Integration of even and odd functions over symmetric intervals, $[-a, a]$.

At this point we need to remind the reader about the integration of even and odd functions on symmetric intervals.

Even Functions.

We first recall that $f(x)$ is an even function if $f(-x) = f(x)$ for all x. One can recognize even functions as they are symmetric with respect to the y-axis as shown in Figure 2.8.

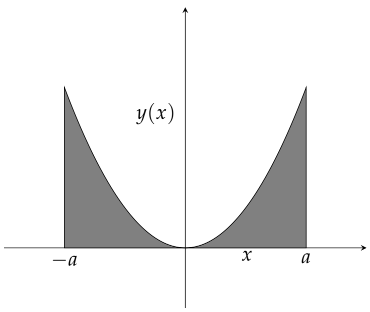


Figure 2.8: Area under an even function on a symmetric interval, $[-a, a]$.

If one integrates an even function over a symmetric interval, $[-a, a]$, then one has that

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx. \tag{2.32}$$

One can prove this by splitting off the integration over negative values of x, using the substitution $x = -y$, and employing the evenness of $f(x)$. Thus,

$$\begin{aligned} \int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= -\int_{a}^{0} f(-y)\,dy + \int_{0}^{a} f(x)\,dx \\ &= \int_{0}^{a} f(y)\,dy + \int_{0}^{a} f(x)\,dx \\ &= 2\int_{0}^{a} f(x)\,dx. \end{aligned} \tag{2.33}$$

Odd Functions.

This can be visually verified by looking at Figure 2.8.

A similar computation could be done for odd functions. $f(x)$ is an odd function if $f(-x) = -f(x)$ for all x. The graphs of such functions are symmetric with respect to the origin as shown in Figure 2.9. If one integrates an odd function over a symmetric interval, $[-a, a]$, then one has that

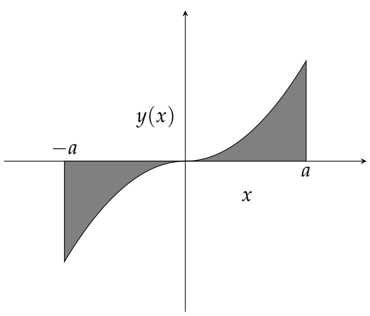


Figure 2.9: Area under an odd function on a symmetric interval, $[-a, a]$.

$$\int_{-a}^{a} f(x)\,dx = 0. \tag{2.34}$$

Example 2.4. Let $f(x) = |x|$ on $[-\pi, \pi]$ We compute the coefficients, beginning as usual with a_0. We have, using the fact that $|x|$ is an even function,

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x\,dx = \pi \end{aligned} \tag{2.35}$$

We continue with the computation of the general Fourier coefficients for $f(x) = |x|$ on $[-\pi, \pi]$. We have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x \cos nx\,dx. \tag{2.36}$$

Here we have made use of the fact that $|x| \cos nx$ is an even function.

In order to compute the resulting integral, we need to use integration by parts ,

$$\int_{a}^{b} u\,dv = uv\Big|_{a}^{b} - \int_{a}^{b} v\,du,$$

by letting $u = x$ and $dv = \cos nx\,dx$. Thus, $du = dx$ and $v = \int dv = \frac{1}{n}\sin nx$.

Fourier Series on $[0, L]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}\right]. \tag{2.37}$$

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}\,dx. \quad n = 0,1,2,\ldots, \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}\,dx. \quad n = 1,2,\ldots. \end{aligned} \tag{2.38}$$

Fourier Series on $[-\frac{L}{2}, \frac{L}{2}]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}\right]. \tag{2.39}$$

$$\begin{aligned} a_n &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\cos\frac{2n\pi x}{L}\,dx. \quad n = 0,1,2,\ldots, \\ b_n &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\sin\frac{2n\pi x}{L}\,dx. \quad n = 1,2,\ldots. \end{aligned} \tag{2.40}$$

Fourier Series on $[-\pi, \pi]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]. \tag{2.41}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx. \quad n = 0,1,2,\ldots, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx. \quad n = 1,2,\ldots. \end{aligned} \tag{2.42}$$

Table 2.1: Special Fourier Series Representations on Different Intervals

Continuing with the computation, we have

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx. \\ &= \frac{2}{\pi}\left[\frac{1}{n}x\sin nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right] \\ &= -\frac{2}{n\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} \\ &= -\frac{2}{\pi n^2}(1-(-1)^n). \end{aligned} \tag{2.43}$$

Here we have used the fact that $\cos n\pi = (-1)^n$ for any integer n.

This leads to a factor $(1-(-1)^n)$. This factor can be simplified as

$$1-(-1)^n = \begin{cases} 2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}. \tag{2.44}$$

So, $a_n = 0$ for n even and $a_n = -\frac{4}{\pi n^2}$ for n odd.

Computing the b_n's is simpler. We note that we have to integrate $|x| \sin nx$ from $x = -\pi$ to π. The integrand is an odd function and this is a symmetric interval. So, the result is that $b_n = 0$ for all n.

Putting this all together, the Fourier series representation of $f(x) = |x|$ on $[-\pi, \pi]$ is given as

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\cos nx}{n^2}. \tag{2.45}$$

While this is correct, we can rewrite the sum over only odd n by reindexing. We let $n = 2k-1$ for $k = 1, 2, 3, \ldots$. In this way, we only get the odd integers. The series can then be written as

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}. \tag{2.46}$$

Throughout our discussion we have referred to such results as Fourier representations. We have not looked at the convergence of these series. The previous series is example of an infinite series of functions. What does this series sum to? We show in Figure 2.10 the first few partial sums. They appear to be converging to $f(x) = |x|$ fairly quickly.

Even though $f(x)$ was defined on $[-\pi, \pi]$ we can still evaluate the Fourier series at values of x outside this interval. In Figure 2.11, we see that the representation agrees with $f(x)$ on the interval $[-\pi, \pi]$. Outside this interval we have a periodic extension of $f(x)$ with period 2π.

Another example is the Fourier series representation of $f(x) = x$ on $[-\pi, \pi]$ as left for Problem 7. This is determined to be

$$f(x) \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx. \tag{2.47}$$

As seen in Figure 2.12 we again obtain the periodic extension of the function. In this case we needed many more terms. Also, the vertical parts of the first plot are nonexistent. In the second plot we only plot the points and not the typical connected points that most software packages plot as the default style.

Example 2.5. It is interesting to note that one can use Fourier series to obtain sums of some infinite series on numbers. For example, in the previous example we found that

$$x \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

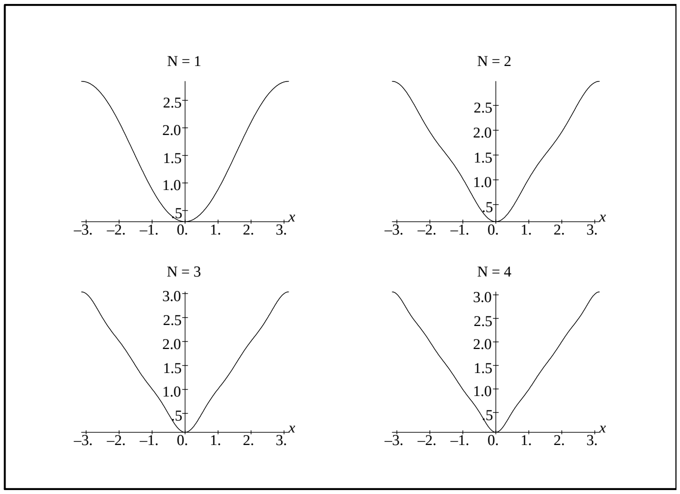


Figure 2.10: Plot of the first partial sums of the Fourier series representation for $f(x) = |x|$.

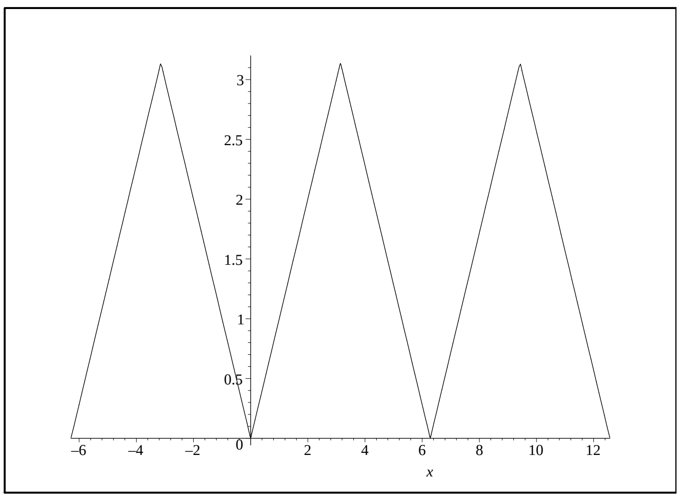


Figure 2.11: Plot of the first 10 terms of the Fourier series representation for $f(x) = |x|$ on the interval $[-2\pi, 4\pi]$.

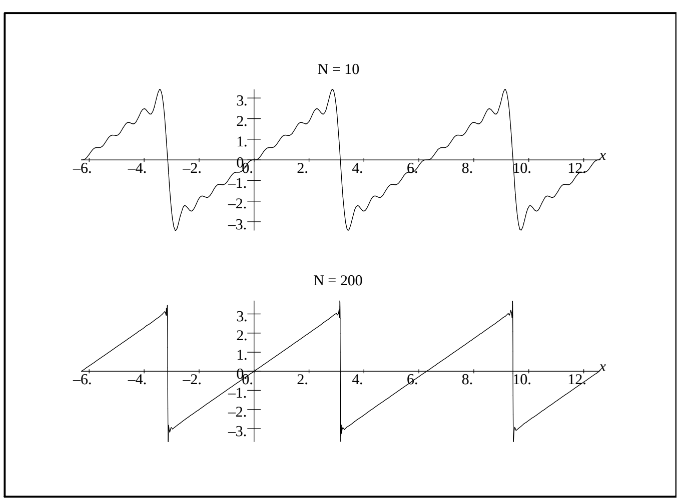


Figure 2.12: Plot of the first 10 terms and 200 terms of the Fourier series representation for $f(x) = x$ on the interval $[-2\pi, 4\pi]$.

Now, what if we chose $x = \frac{\pi}{2}$? Then, we have

$$\frac{\pi}{2} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi}{2} = 2\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right].$$

This gives a well known expression for π:

$$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right].$$

2.3.1 *Fourier Series on* $[a,b]$

This section can be skipped on first reading. It is here for completeness and the end result, Theorem 2.2 provides the result of the section.

A FOURIER SERIES REPRESENTATION is also possible for a general interval, $t \in [a,b]$. As before, we just need to transform this interval to $[0,2\pi]$. Let

$$x = 2\pi\frac{t-a}{b-a}.$$

Inserting this into the Fourier series (2.3) representation for $f(x)$ we obtain

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2n\pi(t-a)}{b-a} + b_n \sin\frac{2n\pi(t-a)}{b-a}\right]. \tag{2.48}$$

Well, this expansion is ugly. It is not like the last example, where the transformation was straightforward. If one were to apply the theory to applications, it might seem to make sense to just shift the data so that $a = 0$ and be done with any complicated expressions. However, some students enjoy the challenge of developing such generalized expressions. So, let's see what is involved.

First, we apply the addition identities for trigonometric functions and rearrange the terms.

$$\begin{aligned}
g(t) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2n\pi(t-a)}{b-a} + b_n \sin\frac{2n\pi(t-a)}{b-a}\right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\left(\cos\frac{2n\pi t}{b-a}\cos\frac{2n\pi a}{b-a} + \sin\frac{2n\pi t}{b-a}\sin\frac{2n\pi a}{b-a}\right)\right. \\
&\quad \left. + b_n\left(\sin\frac{2n\pi t}{b-a}\cos\frac{2n\pi a}{b-a} - \cos\frac{2n\pi t}{b-a}\sin\frac{2n\pi a}{b-a}\right)\right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\cos\frac{2n\pi t}{b-a}\left(a_n\cos\frac{2n\pi a}{b-a} - b_n \sin\frac{2n\pi a}{b-a}\right)\right. \\
&\quad \left. + \sin\frac{2n\pi t}{b-a}\left(a_n \sin\frac{2n\pi a}{b-a} + b_n\cos\frac{2n\pi a}{b-a}\right)\right].
\end{aligned} \tag{2.49}$$

Defining $A_0 = a_0$ and

$$\begin{aligned}
A_n &\equiv a_n\cos\frac{2n\pi a}{b-a} - b_n\sin\frac{2n\pi a}{b-a} \\
B_n &\equiv a_n\sin\frac{2n\pi a}{b-a} + b_n\cos\frac{2n\pi a}{b-a},
\end{aligned} \tag{2.50}$$

we arrive at the more desirable form for the Fourier series representation of a function defined on the interval $[a,b]$.

$$g(t) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos \frac{2n\pi t}{b-a} + B_n \sin \frac{2n\pi t}{b-a} \right]. \tag{2.51}$$

We next need to find expressions for the Fourier coefficients. We insert the known expressions for a_n and b_n and rearrange. First, we note that under the transformation $x = 2\pi \frac{t-a}{b-a}$ we have

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\ &= \frac{2}{b-a} \int_a^b g(t) \cos \frac{2n\pi(t-a)}{b-a} \, dt, \end{aligned} \tag{2.52}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\ &= \frac{2}{b-a} \int_a^b g(t) \sin \frac{2n\pi(t-a)}{b-a} \, dt. \end{aligned} \tag{2.53}$$

Then, inserting these integrals in A_n, combining integrals and making use of the addition formula for the cosine of the sum of two angles, we obtain

$$\begin{aligned} A_n &\equiv a_n \cos \frac{2n\pi a}{b-a} - b_n \sin \frac{2n\pi a}{b-a} \\ &= \frac{2}{b-a} \int_a^b g(t) \left[\cos \frac{2n\pi(t-a)}{b-a} \cos \frac{2n\pi a}{b-a} - \sin \frac{2n\pi(t-a)}{b-a} \sin \frac{2n\pi a}{b-a} \right] dt \\ &= \frac{2}{b-a} \int_a^b g(t) \cos \frac{2n\pi t}{b-a} \, dt. \end{aligned} \tag{2.54}$$

A similar computation gives

$$B_n = \frac{2}{b-a} \int_a^b g(t) \sin \frac{2n\pi t}{b-a} \, dt. \tag{2.55}$$

Summarizing, we have shown that:

Theorem 2.2. *The Fourier series representation of $f(x)$ defined on $[a,b]$, when it exists, is given by*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{b-a} + b_n \sin \frac{2n\pi x}{b-a} \right]. \tag{2.56}$$

with Fourier coefficients

$$\begin{aligned} a_n &= \frac{2}{b-a} \int_a^b f(x) \cos \frac{2n\pi x}{b-a} \, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{b-a} \int_a^b f(x) \sin \frac{2n\pi x}{b-a} \, dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{2.57}$$

2.4 Sine and Cosine Series

In the last two examples ($f(x) = |x|$ and $f(x) = x$ on $[-\pi, \pi]$) we have seen Fourier series representations that contain only sine or cosine terms. As we know, the sine functions are odd functions and thus sum to odd functions. Similarly, cosine functions sum to even functions. Such occurrences happen often in practice. Fourier representations involving just sines are called sine series and those involving just cosines (and the constant term) are called cosine series.

Another interesting result, based upon these examples, is that the original functions, $|x|$ and x agree on the interval $[0, \pi]$. Note from Figures 2.10-2.12 that their Fourier series representations do as well. Thus, more than one series can be used to represent functions defined on finite intervals. All they need to do is to agree with the function over that particular interval. Sometimes one of these series is more useful because it has additional properties needed in the given application.

We have made the following observations from the previous examples:

1. There are several trigonometric series representations for a function defined on a finite interval.

2. Odd functions on a symmetric interval are represented by sine series and even functions on a symmetric interval are represented by cosine series.

These two observations are related and are the subject of this section. We begin by defining a function $f(x)$ on interval $[0, L]$. We have seen that the Fourier series representation of this function appears to converge to a periodic extension of the function.

In Figure 2.13 we show a function defined on $[0, 1]$. To the right is its periodic extension to the whole real axis. This representation has a period of $L = 1$. The bottom left plot is obtained by first reflecting f about the y-axis to make it an even function and then graphing the periodic extension of this new function. Its period will be $2L = 2$. Finally, in the last plot we flip the function about each axis and graph the periodic extension of the new odd function. It will also have a period of $2L = 2$.

In general, we obtain three different periodic representations. In order to distinguish these we will refer to them simply as the periodic, even and odd extensions. Now, starting with $f(x)$ defined on $[0, L]$, we would like to determine the Fourier series representations leading to these extensions. [For easy reference, the results are summarized in Table 2.2]

We have already seen from Table 2.1 that the periodic extension of $f(x)$, defined on $[0, L]$, is obtained through the Fourier series representation

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right], \tag{2.58}$$

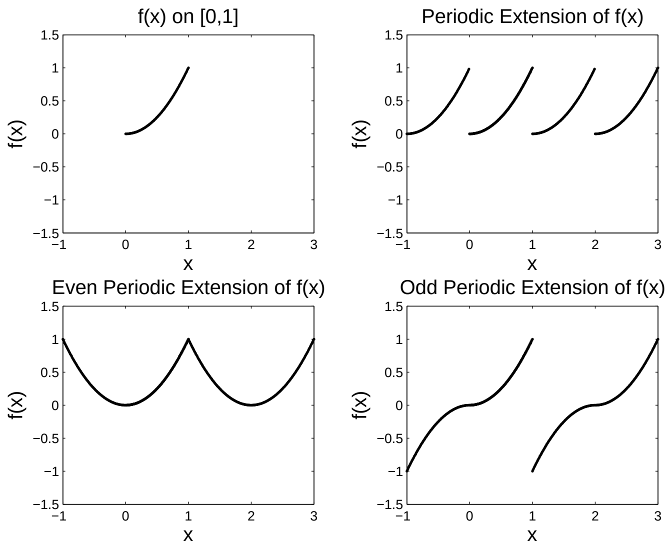


Figure 2.13: This is a sketch of a function and its various extensions. The original function $f(x)$ is defined on $[0,1]$ and graphed in the upper left corner. To its right is the periodic extension, obtained by adding replicas. The two lower plots are obtained by first making the original function even or odd and then creating the periodic extensions of the new function.

where

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}\,dx. \quad n=0,1,2,\ldots, \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}\,dx. \quad n=1,2,\ldots. \end{aligned} \tag{2.59}$$

Given $f(x)$ defined on $[0,L]$, the even periodic extension is obtained by simply computing the Fourier series representation for the even function

Even periodic extension.

$$f_e(x) \equiv \begin{cases} f(x), & 0<x<L, \\ f(-x) & -L<x<0. \end{cases} \tag{2.60}$$

Since $f_e(x)$ is an even function on a symmetric interval $[-L,L]$, we expect that the resulting Fourier series will not contain sine terms. Therefore, the series expansion will be given by [Use the general case in (2.56) with $a=-L$ and $b=L$.]:

$$f_e(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}. \tag{2.67}$$

with Fourier coefficients

$$a_n = \frac{1}{L}\int_{-L}^{L} f_e(x)\cos\frac{n\pi x}{L}\,dx. \quad n=0,1,2,\ldots. \tag{2.68}$$

However, we can simplify this by noting that the integrand is even and the interval of integration can be replaced by $[0,L]$. On this interval $f_e(x)=f(x)$. So, we have the Cosine Series Representation of $f(x)$ for $x\in[0,L]$ is given as

Fourier Cosine Series.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}. \tag{2.69}$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx. \quad n=0,1,2,\ldots. \tag{2.70}$$

Table 2.2: Fourier Cosine and Sine Series Representations on $[0, L]$

Fourier Series on $[0, L]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right]. \tag{2.61}$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{2.62}$$

Fourier Cosine Series on $[0, L]$

$$f(x) \sim a_0/2 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{2.63}$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots. \tag{2.64}$$

Fourier Sine Series on $[0, L]$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{2.65}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \tag{2.66}$$

Similarly, given $f(x)$ defined on $[0, L]$, the odd periodic extension is obtained by simply computing the Fourier series representation for the odd function

Odd periodic extension.

$$f_o(x) \equiv \begin{cases} f(x), & 0 < x < L, \\ -f(-x) & -L < x < 0. \end{cases} \tag{2.71}$$

The resulting series expansion leads to defining the Sine Series Representation of $f(x)$ for $x \in [0, L]$ as

Fourier Sine Series Representation.

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{2.72}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \tag{2.73}$$

Example 2.6. In Figure 2.13 we actually provided plots of the various extensions of the function $f(x) = x^2$ for $x \in [0, 1]$. Let's determine the representations of the periodic, even and odd extensions of this function.

For a change, we will use a CAS (Computer Algebra System) package to do the integrals. In this case we can use Maple. A general code for doing this for the periodic extension is shown in Table 2.3. See Section 2.8 for MATLAB and Python examples.

Example 2.7. Periodic Extension - Trigonometric Fourier Series Using the code in Table 2.3, we have that $a_0 = \frac{2}{3}$, $a_n = \frac{1}{n^2\pi^2}$, and $b_n = -\frac{1}{n\pi}$. Thus, the resulting series is given as

$$f(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty} \left[\frac{1}{n^2\pi^2} \cos 2n\pi x - \frac{1}{n\pi} \sin 2n\pi x \right].$$

In Figure 2.14 we see the sum of the first 50 terms of this series. Generally, we see that the series seems to be converging to the periodic extension of f. There appear to be some problems with the convergence around integer values of x. We will later see that this is because of the discontinuities in the periodic extension and the resulting overshoot is referred to as the Gibbs phenomenon which is discussed in the last section of this chapter.

Example 2.8. Even Periodic Extension - Cosine Series

In this case we compute $a_0 = \frac{2}{3}$ and $a_n = \frac{4(-1)^n}{n^2\pi^2}$. Therefore, we have

$$f(x) \sim \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x.$$

In Figure 2.15 we see the sum of the first 50 terms of this series. In this case the convergence seems to be much better than in the periodic extension case. We also see that it is converging to the even extension.

Figure 2.14: The periodic extension of $f(x) = x^2$ on $[0, 1]$.

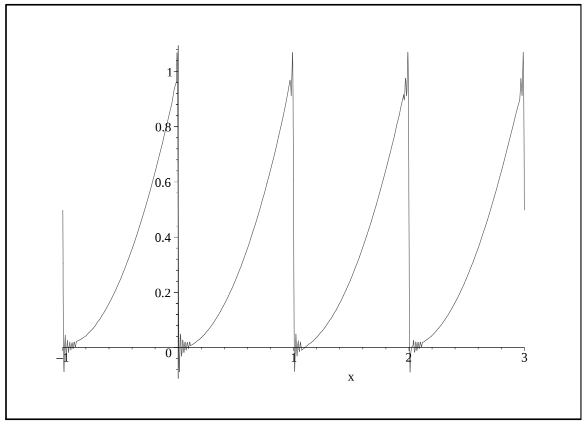
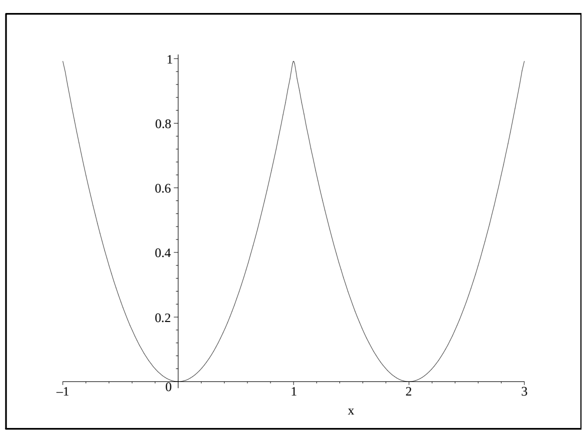


Table 2.3: Maple code for computing Fourier coefficients and plotting partial sums of the Fourier series.

```
> restart:
> L:=1:
> f:=x^2:
> assume(n,integer):
> a0:=2/L*int(f,x=0..L);
                            a0 := 2/3
> an:=2/L*int(f*cos(2*n*Pi*x/L),x=0..L);
                                   1
                          an := -------
                                  2   2
                                n~  Pi
> bn:=2/L*int(f*sin(2*n*Pi*x/L),x=0..L);
                                    1
                          bn := - -----
                                  n~ Pi
> F:=a0/2+sum((1/(k*Pi)^2)*cos(2*k*Pi*x/L)
     -1/(k*Pi)*sin(2*k*Pi*x/L),k=1..50):
> plot(F,x=-1..3,title=‘Periodic Extension‘,
      titlefont=[TIMES,ROMAN,14],font=[TIMES,ROMAN,14]);
```

Figure 2.15: The even periodic extension of $f(x) = x^2$ on $[0, 1]$.



Example 2.9. Odd Periodic Extension - Sine Series

Finally, we look at the sine series for this function. We find that

$$b_n = -\frac{2}{n^3\pi^3}(n^2\pi^2(-1)^n - 2(-1)^n + 2).$$

Therefore,

$$f(x) \sim -\frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}(n^2\pi^2(-1)^n - 2(-1)^n + 2)\sin n\pi x.$$

Once again we see discontinuities in the extension as seen in Figure 2.16. However, we have verified that our sine series appears to be converging to the odd extension as we first sketched in Figure 2.13.

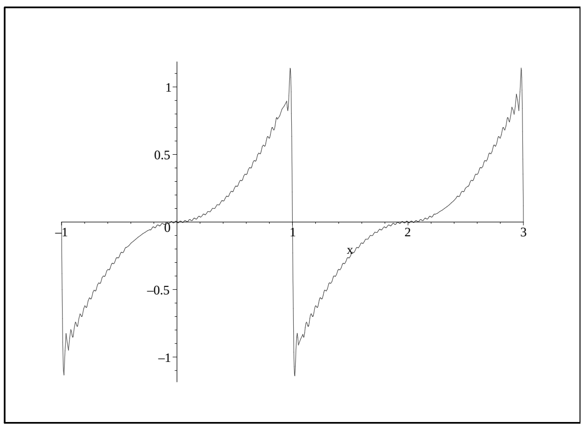


Figure 2.16: The odd periodic extension of $f(x) = x^2$ on $[0,1]$.

2.5 Solution of the Heat Equation

We started this chapter seeking solutions of initial-boundary value problems involving the heat equation and the wave equation. In particular, we found the general solution for the problem of heat flow in a one dimensional rod of length L with fixed zero temperature ends. The problem was given by

$$\begin{array}{llll} \text{PDE} & u_t = ku_{xx}, & 0 < t, \quad 0 \le x \le L, \\ \text{IC} & u(x,0) = f(x), & 0 < x < L, \\ \text{BC} & u(0,t) = 0, & t > 0, \\ & u(L,t) = 0, & t > 0. \end{array} \tag{2.74}$$

We found the solution using separation of variables. This resulted in a sum over various product solutions:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t}\sin\frac{n\pi x}{L},$$

where

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2.$$

This equation satisfies the boundary conditions. However, we had only gotten to state the initial condition using this solution. Namely,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

We were left with having to determine the constants b_n. Once we know them, we have the full solution to the original problem.

Now we can get the Fourier coefficients when we are given the initial condition, $f(x)$. They are given by

$$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \quad n = 1, 2, \ldots.$$

We will consider a couple of examples with different initial conditions.

Example 2.10. Find the solution of the heat equation with $f(x) = \sin x$ and $L = \pi$.

In this case the solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \sin nx.$$

However, the initial condition takes the form of the first term in the expansion; i.e., the $n = 1$ term. So, we need not carry out the integral because we can immediately write $b_1 = 1$ and $b_n = 0$, $n = 2, 3, \ldots$. Therefore, the solution consists of just one term,

$$u(x,t) = e^{-kt} \sin x.$$

In Figure 2.17 we see that how this solution behaves for $k = 1$ and $t \in [0, 1]$.

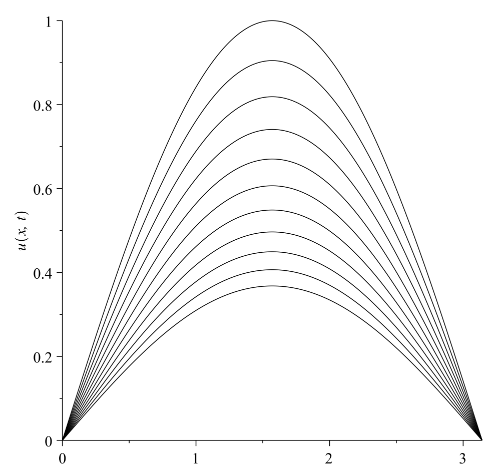


Figure 2.17: The evolution of the initial condition $f(x) = \sin x$ for $L = \pi$ and $k = 1$.

Example 2.11. Find solutions of the heat equation with $f(x) = x(1 - x)$ and $L = 1$.

This example requires a bit more work. The solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 kt} \sin n\pi x,$$

where

$$b_n = 2\int_0^1 f(x) \sin n\pi x\, dx.$$

This integral is easily computed using integration by parts

$$\begin{aligned}
b_n &= 2\int_0^1 x(1-x) \sin n\pi x\, dx \\
&= \left[2x(1-x)\left(-\frac{1}{n\pi}\cos n\pi x\right)\right]_0^1 + \frac{2}{n\pi}\int_0^1 (1-2x)\cos n\pi x\, dx \\
&= -\frac{2}{n^2\pi^2}\left\{[(1-2x)\sin n\pi x]_0^1 + 2\int_0^1 \sin n\pi x\, dx\right\} \\
&= \frac{4}{n^3\pi^3}\left[\cos n\pi x\right]_0^1 \\
&= \frac{4}{n^3\pi^3}\left(\cos n\pi - 1\right) \\
&= \begin{cases} 0, & n \text{ even} \\ -\dfrac{8}{n^3\pi^3}, & n \text{ odd} \end{cases}. \qquad (2.75)
\end{aligned}$$

So, the solution can be written as

$$u(x,t) = \frac{8}{\pi^3}\sum_{\ell=1}^{\infty} \frac{1}{(2\ell-1)^3} e^{-(2\ell-1)^2\pi^2 kt} \sin(2\ell-1)\pi x.$$

In Figure 2.18 we see that how this solution behaves for $k = 1$ and $t \in [0,1]$. Twenty terms were used. We see that this solution diffuses much faster than in the last example. Most of the terms damp out quickly as the solution asymptotically approaches the first term.

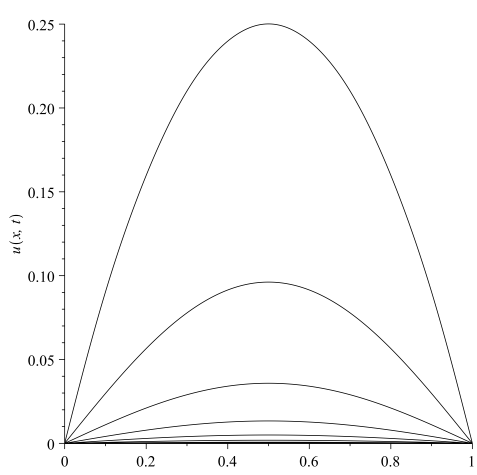


Figure 2.18: The evolution of the initial condition $f(x) = x(1-x)$ for $L = 1$ and $k = 1$. For large time $u(x,t) \sim \frac{8}{\pi^3}e^{-\pi^2 t}\sin \pi x$ since the other terms damp out quickly.

2.6 Finite Length Strings

We now return to the physical example of wave propagation in a string. We found that the general solution can be represented as a sum over product solutions. We will restrict our discussion to the special case that the initial velocity is zero, $g(x) = 0$, and the original profile is given by $u(x,0) = f(x)$. The solution is then

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L} \tag{2.76}$$

satisfying

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}. \tag{2.77}$$

We have seen that the Fourier sine series coefficients are given by

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \tag{2.78}$$

We can rewrite this solution in a more compact form. First, we define the wave numbers,

$$k_n = \frac{n\pi}{L}, \quad n = 1, 2, \ldots,$$

and the angular frequencies,

$$\omega_n = ck_n = \frac{n\pi c}{L}.$$

Then, the product solutions take the form

$$u_n(x,t) = \sin k_n x \cos \omega_n t.$$

Using trigonometric identities, these products can be written as

$$\sin k_n x \cos \omega_n t = \frac{1}{2}\left[\sin(k_n x + \omega_n t) + \sin(k_n x - \omega_n t)\right].$$

Inserting this expression into the solution, we have

$$u(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} A_n \left[\sin(k_n x + \omega_n t) + \sin(k_n x - \omega_n t)\right]. \tag{2.79}$$

Since $\omega_n = ck_n$, we can write this in the more suggestive form

$$u(x,t) = \frac{1}{2}\left[\sum_{n=1}^{\infty} A_n \sin k_n (x + ct) + \sum_{n=1}^{\infty} A_n \sin k_n (x - ct)\right]. \tag{2.80}$$

The solution of the wave equation can be written as the sum of right and left traveling waves.

We see that each sum is simply the sine series for $f(x)$ but evaluated at either $x + ct$ or $x - ct$. Thus, the solution takes the form

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]. \tag{2.81}$$

This should be reminiscent of d'Alembert's solution in the previous chapter. If $t = 0$, then we have $u(x,0) = \frac{1}{2}[f(x) + f(x)] = f(x)$. So, the solution satisfies the initial condition.

At $t = 1$, the sum has a term $f(x - c)$. Recall from your mathematics classes that this is simply a shifted version of $f(x)$. Namely, it is shifted to the right. For general times, the function is shifted by ct to the right. For larger values of t, this shift is further to the right. The function (wave) shifts to the right with velocity c. Similarly, $f(x + ct)$ is a wave traveling to the left with velocity $-c$.

Thus, the waves on the string consist of waves traveling to the right and to the left. However, the story does not stop here. We have a problem when needing to shift $f(x)$ across the boundaries. The original problem only defines $f(x)$ on $[0, L]$. If we are not careful, we would think that the function leaves the interval leaving nothing left inside. However, we have to recall that our sine series representation for $f(x)$ has a period of $2L$. So, before we apply this shifting, we need to account for its periodicity. In fact, being a sine series, we really have the odd periodic extension of $f(x)$ being shifted. The details of such analysis would take us too far from our current goal. However, we can illustrate this with a few figures.

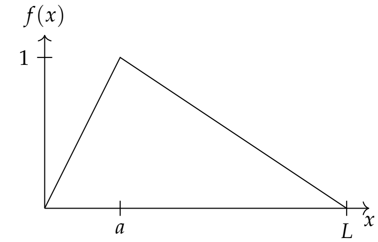


Figure 2.19: The initial profile for a string of length one plucked at $x = a$.

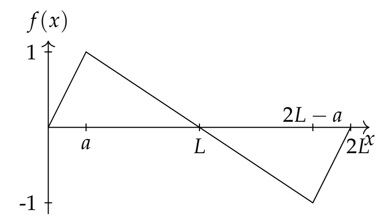


Figure 2.20: Odd extension about the right end of a plucked string.

We begin by plucking a string of length L. This can be represented by the function

$$f(x) = \begin{cases} \frac{x}{a} & 0 \le x \le a \\ \frac{L-x}{L-a} & a \le x \le L \end{cases} \tag{2.82}$$

where the string is pulled up one unit at $x = a$. This is shown in Figure 2.19.

Next, we create an odd function by extending the function to a period of $2L$. This is shown in Figure 2.20.

Finally, we construct the periodic extension of this to the entire line. In Figure 2.21 we show in the lower part of the figure copies of the periodic extension, one moving to the right and the other moving to the left. (Actually, the copies are $\dfrac{1}{2}f(x \pm ct)$.) The top plot is the sum of these solutions. The physical string lies in the interval [0,1]. Of course, this is better seen when the solution is animated.

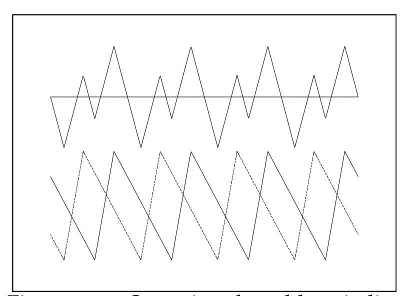
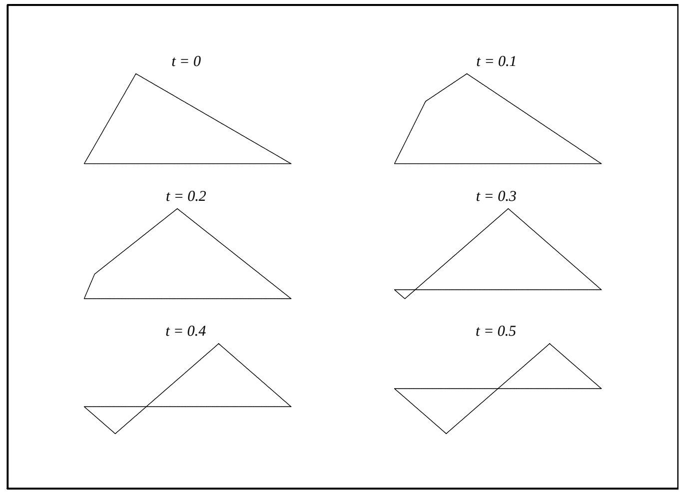


Figure 2.21: Summing the odd periodic extensions. The lower plot shows copies of the periodic extension, one moving to the right and the other moving to the left. The upper plot is the sum.

The time evolution for this plucked string is shown for several times in Figure 2.22. This results in a wave that appears to reflect from the ends as time increases.

The relation between the angular frequency and the wave number, $\omega = ck$, is called a dispersion relation. In this case ω depends on k linearly. If one knows the dispersion relation, then one can find the wave speed as $c = \dfrac{\omega}{k}$. In this case, all of the harmonics travel at the same speed. In cases where they do not, we have nonlinear dispersion, which we will discuss later.

Figure 2.22: This Figure shows the plucked string at six successive times.



2.7 *The Gibbs Phenomenon*

We have seen the Gibbs phenomenon when there is a jump discontinuity in the periodic extension of a function, whether the function originally had a discontinuity or developed one due to a mismatch in the values of the endpoints. This can be seen in Figures 2.12, 2.14 and 2.16. The Fourier series has a difficult time converging at the point of discontinuity and these graphs of the Fourier series show a distinct overshoot which does not go away. This is called the Gibbs phenomenon[3] and the amount of overshoot can be computed.

[3] The Gibbs phenomenon was named after Josiah Willard Gibbs (1839-1903) even though it was discovered earlier by the Englishman Henry Wilbraham (1825-1883). Wilbraham published a soon forgotten paper about the effect in 1848. In 1889 Albert Abraham Michelson (1852-1931), an American physicist, observed an overshoot in his mechanical graphing machine, the Harmonic Analyzer. Shortly afterwards J. Willard Gibbs published papers describing this phenomenon, which was later to be called the Gibbs phenomena. Gibbs was a mathematical physicist and chemist and is considered the father of physical chemistry.

In one of our first examples, Example 2.3, we found the Fourier series representation of the piecewise defined function

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$$

to be

$$f(x) \sim \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}.$$

In Figure 2.23 we display the sum of the first ten terms. Note the wiggles, overshoots and under shoots. These are seen more when we plot the representation for $x \in [-3\pi, 3\pi]$, as shown in Figure 2.24.

We note that the overshoots and undershoots occur at discontinuities in the periodic extension of $f(x)$. These occur whenever $f(x)$ has a discontinuity or if the values of $f(x)$ at the endpoints of the domain do not agree.

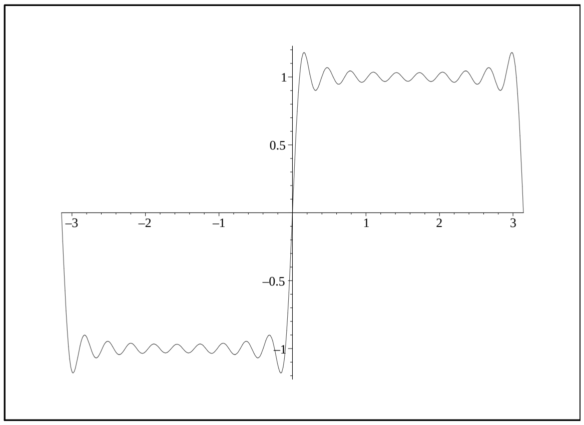


Figure 2.23: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 10$.

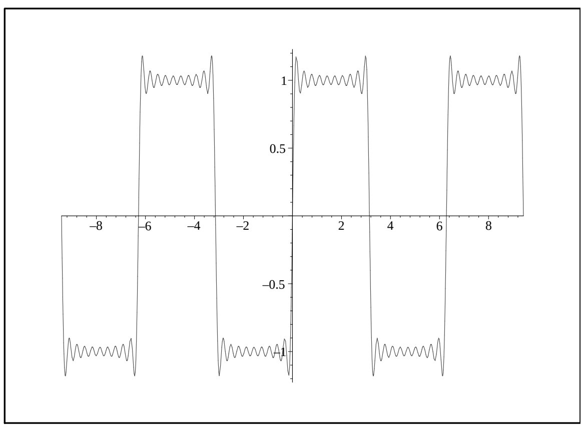


Figure 2.24: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 10$ plotted on $[-3\pi, 3\pi]$ displaying the periodicity.

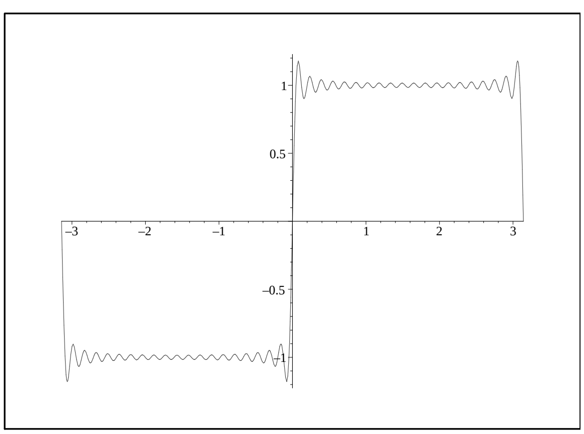


Figure 2.25: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 20$.

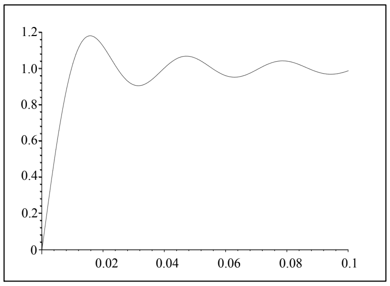


Figure 2.26: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 100$.

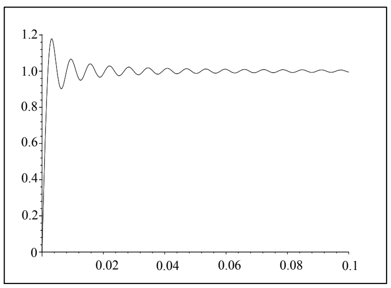


Figure 2.27: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 500$.

One might expect that we only need to add more terms. In Figure 2.25 we show the sum for twenty terms. Note the sum appears to converge better for points far from the discontinuities. But, the overshoots and undershoots are still present. In Figures 2.26 and 2.27 show magnified plots of the overshoot at $x = 0$ for $N = 100$ and $N = 500$, respectively. We see that the overshoot persists. The peak is at about the same height, but its location seems to be getting closer to the origin. We will show how one can estimate the size of the overshoot.

We can study the Gibbs phenomenon by looking at the partial sums of general Fourier trigonometric series for functions $f(x)$ defined on the interval $[-L, L]$. Writing out the partial sums, inserting the Fourier coefficients and rearranging, we have

$$\begin{aligned}
S_N(x) &= a_0 + \sum_{n=1}^{N}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right] \\
&= \frac{1}{2L}\int_{-L}^{L} f(y)\,dy + \sum_{n=1}^{N}\left[\left(\frac{1}{L}\int_{-L}^{L} f(y)\cos\frac{n\pi y}{L}\,dy\right)\cos\frac{n\pi x}{L}\right. \\
&\quad \left. + \left(\frac{1}{L}\int_{-L}^{L} f(y)\sin\frac{n\pi y}{L}\,dy.\right)\sin\frac{n\pi x}{L}\right] \\
&= \frac{1}{L}\int_{-L}^{L}\left\{\frac{1}{2}\right. \\
&\quad \left. + \sum_{n=1}^{N}\left(\cos\frac{n\pi y}{L}\cos\frac{n\pi x}{L} + \sin\frac{n\pi y}{L}\sin\frac{n\pi x}{L}\right)\right\} f(y)\,dy \\
&= \frac{1}{L}\int_{-L}^{L}\left\{\frac{1}{2} + \sum_{n=1}^{N}\cos\frac{n\pi(y-x)}{L}\right\} f(y)\,dy \\
&\equiv \frac{1}{L}\int_{-L}^{L} D_N(y-x) f(y)\,dy
\end{aligned}$$

We have defined

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N}\cos\frac{n\pi x}{L},$$

which is called the N-th Dirichlet kernel . We can sum this series as shown in the following Lemma.

Lemma 2.1. *The N-th Dirichlet kernel is given by*

$$D_N(x) = \begin{cases} \dfrac{\sin((N+\frac{1}{2})\frac{\pi x}{L})}{2\sin\frac{\pi x}{2L}}, & \sin\frac{\pi x}{2L} \neq 0, \\ N + \frac{1}{2}, & \sin\frac{\pi x}{2L} = 0. \end{cases}$$

Proof. Let $\theta = \dfrac{\pi x}{L}$ and multiply $D_N(x)$ by $2\sin\dfrac{\theta}{2}$ to obtain:

$$\begin{aligned}
2\sin\frac{\theta}{2}D_N(x) &= 2\sin\frac{\theta}{2}\left[\frac{1}{2}+\cos\theta+\cdots+\cos N\theta\right] \\
&= \sin\frac{\theta}{2}+2\cos\theta\sin\frac{\theta}{2}+2\cos2\theta\sin\frac{\theta}{2}+\cdots+2\cos N\theta\sin\frac{\theta}{2} \\
&= \sin\frac{\theta}{2}+\left(\sin\frac{3\theta}{2}-\sin\frac{\theta}{2}\right)+\left(\sin\frac{5\theta}{2}-\sin\frac{3\theta}{2}\right)+\cdots \\
&\quad +\left[\sin\left(N+\frac{1}{2}\right)\theta-\sin\left(N-\frac{1}{2}\right)\theta\right] \\
&= \sin\left(N+\frac{1}{2}\right)\theta. \qquad (2.83)
\end{aligned}$$

Thus,

$$2\sin\frac{\theta}{2}D_N(x)=\sin\left(N+\frac{1}{2}\right)\theta.$$

If $\sin\frac{\theta}{2}\neq 0$, then

$$D_N(x)=\frac{\sin\left(N+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}, \qquad \theta=\frac{\pi x}{L}.$$

If $\sin\frac{\theta}{2}=0$, then one needs to apply L'Hospital's Rule as $\theta\to 2m\pi$:

$$\begin{aligned}
\lim_{\theta\to2m\pi}\frac{\sin\left(N+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} &= \lim_{\theta\to2m\pi}\frac{(N+\frac{1}{2})\cos\left(N+\frac{1}{2}\right)\theta}{\cos\frac{\theta}{2}} \\
&= \frac{(N+\frac{1}{2})\cos\left(2m\pi N+m\pi\right)}{\cos m\pi} \\
&= \frac{(N+\frac{1}{2})(\cos2m\pi N\cos m\pi-\sin2m\pi N\sin m\pi)}{\cos m\pi} \\
&= N+\frac{1}{2}. \qquad (2.84)
\end{aligned}$$

□

We further note that $D_N(x)$ is periodic with period $2L$ and is an even function.

So far, we have found that the Nth partial sum is given by

$$S_N(x)=\frac{1}{L}\int_{-L}^{L}D_N(y-x)f(y)\,dy. \qquad (2.85)$$

Making the substitution $\xi=y-x$, we have

$$\begin{aligned}
S_N(x) &= \frac{1}{L}\int_{-L-x}^{L-x}D_N(\xi)f(\xi+x)\,d\xi \\
&= \frac{1}{L}\int_{-L}^{L}D_N(\xi)f(\xi+x)\,d\xi. \qquad (2.86)
\end{aligned}$$

In the second integral we have made use of the fact that $f(x)$ and $D_N(x)$ are periodic with period $2L$ and shifted the interval back to $[-L,L]$.

We now write the integral as the sum of two integrals over positive and negative values of ξ and use the fact that $D_N(x)$ is an even function. Then,

$$\begin{aligned} S_N(x) &= \frac{1}{L}\int_{-L}^{0} D_N(\xi) f(\xi+x)\,d\xi + \frac{1}{L}\int_{0}^{L} D_N(\xi) f(\xi+x)\,d\xi \\ &= \frac{1}{L}\int_{0}^{L} \left[f(x-\xi)+f(\xi+x)\right] D_N(\xi)\,d\xi. \end{aligned} \tag{2.87}$$

We can use this result to study the Gibbs phenomenon whenever it occurs. In particular, we will only concentrate on the earlier example. For this case, we have

$$S_N(x) = \frac{1}{\pi}\int_{0}^{\pi} \left[f(x-\xi)+f(\xi+x)\right] D_N(\xi)\,d\xi \tag{2.88}$$

for

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N} \cos nx.$$

Also, one can show that

$$f(x-\xi)+f(\xi+x) = \begin{cases} 2, & 0 \le \xi < x, \\ 0, & x \le \xi < \pi - x, \\ -2, & \pi - x \le \xi < \pi. \end{cases}$$

Thus, we have

$$\begin{aligned} S_N(x) &= \frac{2}{\pi}\int_{0}^{x} D_N(\xi)\,d\xi - \frac{2}{\pi}\int_{\pi-x}^{\pi} D_N(\xi)\,d\xi \\ &= \frac{2}{\pi}\int_{0}^{x} D_N(z)\,dz + \frac{2}{\pi}\int_{0}^{x} D_N(\pi - z)\,dz. \end{aligned} \tag{2.89}$$

Here we made the substitution $z = \pi - \xi$ in the second integral.

The Dirichlet kernel for $L = \pi$ is given by

$$D_N(x) = \frac{\sin(N+\frac{1}{2})x}{2\sin\frac{x}{2}}.$$

For N large, we have $N + \frac{1}{2} \approx N$, and for small x, we have $\sin\frac{x}{2} \approx \frac{x}{2}$. So, under these assumptions,

$$D_N(x) \approx \frac{\sin Nx}{x}.$$

Therefore,

$$S_N(x) \to \frac{2}{\pi}\int_{0}^{x} \frac{\sin N\xi}{\xi}\,d\xi \qquad \text{for large } N, \text{ and small } x.$$

If we want to determine the locations of the minima and maxima, where the undershoot and overshoot occur, then we apply the first derivative test for extrema to $S_N(x)$. Thus,

$$\frac{d}{dx}S_N(x) = \frac{2}{\pi}\frac{\sin Nx}{x} = 0.$$

The extrema occur for $Nx = m\pi$, $m = \pm 1, \pm 2, \ldots$. One can show that there is a maximum at $x = \dfrac{\pi}{N}$ and a minimum for $x = \dfrac{2\pi}{N}$. The value for the overshoot can be computed as

$$\begin{aligned} S_N\left(\frac{\pi}{N}\right) &= \frac{2}{\pi}\int_0^{\pi/N} \frac{\sin N\xi}{\xi}\, d\xi \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t}\, dt \\ &= \frac{2}{\pi}\mathrm{Si}(\pi) \\ &= 1.178979744\ldots. \end{aligned} \tag{2.90}$$

Note that this value is independent of N and is given in terms of the sine integral,

$$\mathrm{Si}(x) \equiv \int_0^x \frac{\sin t}{t}\, dt.$$

In Figure 2.28 is shown a plot of the sine integral.

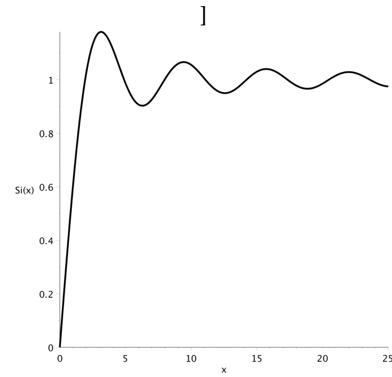


Figure 2.28: Plot of the sine integral, $\dfrac{2}{\pi}\mathrm{Si}(x)$.

2.8 *Plotting Fourier Series*

An example of Maple code was shown in Table 2.3 for finding Fourier coefficients and plotting Fourier series. If the Fourier coefficients are known, then the code is simpler to write:

```
plot((1/2)*Pi+2*(sum((cos(n*Pi)-1)*cos(n*x)/n^2, n = 1 .. 20))/Pi,
     x = -5*Pi .. 5*Pi)
```

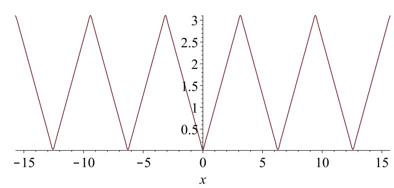


Figure 2.29: Nth Partial Sum in Maple.

However, if one does not have access to Maple, then one can use other software tools. For quick computations, one can be use online systems like WolframAlpha. For example, entering one of the following produces Fourier series plots.

```
plot sum_(k=1)^10(4/pi sin((2 k-1)*x)/(2 k-1)) x=0 to 2 pi
plot pi/2-sum_(k=1)^10(4/pi cos((2 k-1)*x)/(2 k-1)^2)
FourierSeries[abs(x), x, 10]
```

There are several other platforms like Sagemath with interactive Fourier Series `https://sage.brandoncurtis.com/fourier.html`, or at Desmos.com you can find several examples like `https://www.desmos.com/calculator/73hxmnkyvn` and `https://www.desmos.com/calculator/oowvp6lolf`. We now give some examples using MATLAB and Python.

2.8.1 *MATLAB and GNU Octave Files*

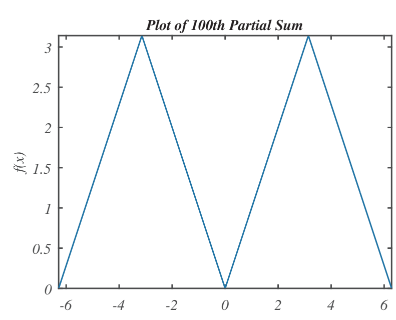


Figure 2.30: Nth Partial Sum done in MATLAB.

If one has access to MATLAB's symbolic toolset, then one way that the Nth partial sum can be plotted is by saving the m-file below as **plotfs.m** and running it. It produces the plot in Figure 2.30.

```
% Plot the Nth partial sum of a Fourier series
% of f on the interval [a,b].

clear
syms n x;
N=100;
a=-2*pi;
b=2*pi;
f = pi/2+symsum((2/(pi*n^2))*((-1)^n-1)*cos(n*x),n,1,N);
ezplot(f,[a,b])
xlabel('x')
ylabel('f(x)')
title(['Plot of ', num2str(N),'th Partial Sum'] )
axis([a,b,0,pi])
```

GNU Octave is mostly compatable with MATLAB and m-files can often be run in GNU Octave. However, to run these symbolic examples, one has to download and install the symbolic package first.

```
pkg install symbolic-3.2.1.tar.gz
```

Afterwards it needs to be loaded when it is needed using `pkg load symbolic`.

A version of MATLAB that does not need symbolic computation can be run with a slight modification. We can call this **plotfso.m**.

```
N=10;
a=-2*pi;
b=2*pi;

t=linspace(a,b,100);
fs=pi/2;
for k=1:N
    fs=fs+(2/(pi*k^2))*((-1)^k-1)*cos(k*t);
end

plot(t,fs)
xlabel('x')
ylabel('f(x)')
title(['Plot of ', num2str(N),'th Partial Sum'] )
axis([a,b,0,pi])
```

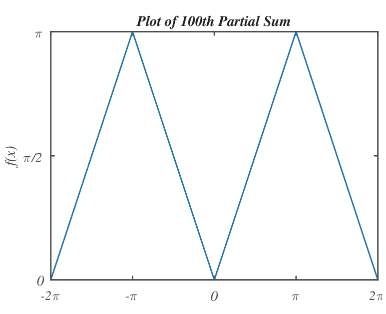


Figure 2.31: Nth Partial Sum showing labels in terms of π.

One can modify the plot labels to show increments of π or $\frac{\pi}{2}$ in MATLAB, as shown in Figure 2.31, by adding the following lines:

```
% Code to label axes with pi instead of the default
 xtick = [-2, -1, 0, 1, 2] * pi;
 xticklabel = { '-2 \pi', '-\pi', '0','\pi', '2\pi'};
 set (gca, 'xtick', xtick, 'xticklabel', xticklabel)
 ytick = [0 1/2 1]*pi;
```

```
yticklabel = {'0','\pi/2','\pi'};
set (gca, 'ytick', ytick, 'yticklabel', yticklabel)
```

The MATLAB code for **plotfso.m** will run in GNU Octave. A slight modification, as shown in the next section, will work in Python.

MATLAB now has a symbolic toolbox and one can do some computer algebra routines. The following code is used to find the Fourier coefficients for $f(x) = x^2$ on $x \in [0,1]$.

```
syms n x
n = sym('n','integer');
f = x^2;
L = 1;
a0 = simplify(2/L*int(f,0,L))
an = simplify(2/L*int(f*cos(2*n*pi*x/L),0,L))
bn = simplify(2/L*int(f*sin(2*n*pi*x/L),0,L))
```

Running this code produces

```
>> FScoeff
a0 = 2/3
an = 1/(pi^2*n^2)
bn = -1/(pi*n)
```

Adding the following lines,

```
N=100;
f = a0/2+symsum(an*cos(2*n*pi*x/L)+bn*sin(2*n*pi*x/L),n,1,N);
ezplot(f,[0,L])
xlabel('x')
ylabel('f(x)')
title(['Plot of ', num2str(N),'th Partial Sum'] )
axis([0,L,0,1])
```

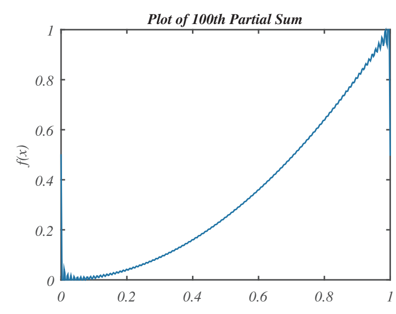


Figure 2.32: Nth Partial Sum for $f(x) = x^2$ on $x \in [0,1]$ as found in MATLAB using symbolic computation of the Fourier coefficients.

we can plot the partial sums for the corresponding Fourier series. We show a plot for $N = 100$ in Figure 2.32. A similar result can be obtained with Python as shown in the next section.

2.8.2 *Python Scripts*

Another environment for plotting Fourier series is in Python. There are several libraries which come with the installation of Python for doing scientific computation. The first program shows how one can use the Sympy library to compute the Fourier coefficients symbolically.

```
from sympy import *

n = symbols('n', integer=True, positive=True)
k = symbols('k', integer=True, positive=True)
x = Symbol('x')
```

```
a = 0
b = 1
L=b-a

f = x**2

a_0 = (2/L) * integrate(f, (x, a, b))
a_n = (2/L) * integrate(f*cos(2*n*pi*x/L), (x,a,b))
b_n = (2/L) * integrate(f*sin(2*n*pi*x/L), (x,a,b))

print "a_0 =", simplify(a_0)
print "a_n =", simplify(a_n)
print "b_n =", simplify(b_n)
```

Once one has the Fourier coefficients, then one would like to plot the series. There are a number of ways to do this, using the several different libraries for plots. An example for using the explicit series is

```
from pylab import *

N=10;
a=-2*pi;
b=2*pi;

t=linspace(a,b,100);
fs=pi/2;
for k in xrange(1,N):
    fs=fs+(2/(pi*k**2))*((-1)**k-1)*cos(k*t);

plot(t,fs)
xlabel('x')
ylabel('f(x)')
title('Plot of Partial Sum' )
axis([a,b,0,pi])
```

Using Python entirely to compute the coefficients and plot the series could be done by direct integration in the partial sum:

```
from sympy import *
import numpy as np
import matplotlib.pyplot as plt

n = symbols('n', integer=True, positive=True)
x = Symbol('x')
t = Symbol('t')

L=1
```

```
FS = (1/L*integrate(t**2,(t,0,L))
       +sum(2/L*integrate(t**2*cos(2*n*pi*t/L),(t,0,L))
       *cos(2*n*pi*x/L)+2/L*integrate(t**2*sin(2*n*pi*t/L),(t,0,L))
       *sin(2*n*pi*x/L) for n in range(1, 20, 1)) )

evFS = lambdify(x, FS, modules=['numpy'])
tt = np.linspace(0, L, 100)
plt.plot(tt, evFS(tt), 'b', label='Fourier Series')
plt.show()
```

Problems

1. Write $y(t) = 3\cos 2t - 4\sin 2t$ in the form $y(t) = A\cos(2\pi f t + \phi)$.

2. Derive the coefficients b_n in Equation (2.4).

3. Let $f(x)$ be defined for $x \in [-L, L]$. Parseval's identity is given by

$$\frac{1}{L}\int_{-L}^{L} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2.$$

Assuming the Fourier series of $f(x)$ converges uniformly in $(-L, L)$, prove Parseval's identity by multiplying the Fourier series representation by $f(x)$ and integrating from $x = -L$ to $x = L$. [Parseval's equality for Fourier transforms is a continuous version of this identity.]

4. Consider the square wave function

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi. \end{cases}$$

a. Find the Fourier series representation of this function and plot the first 50 terms.

b. Apply Parseval's identity in Problem 3 to the result in part a.

c. Use the result of part b to show $\dfrac{\pi^2}{8} = \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

5. For the following sets of functions: i) show that each is orthogonal on the given interval, and ii) determine the corresponding orthonormal set. [See page 69 for the definitions of orthogonal and orthonormal families of functions.]

a. $\{\sin 2nx\}, \quad n = 1, 2, 3, \ldots, \quad 0 \le x \le \pi.$

b. $\{\cos n\pi x\}, \quad n = 0, 1, 2, \ldots, \quad 0 \le x \le 2.$

c. $\{\sin \frac{n\pi x}{L}\}, \quad n = 1, 2, 3, \ldots, \quad x \in [-L, L].$

6. Consider $f(x) = 4\sin^3 2x$.

a. Derive the trigonometric identity giving $\sin^3\theta$ in terms of $\sin\theta$ and $\sin 3\theta$ using DeMoivre's Formula.

b. Find the Fourier series of $f(x) = 4\sin^3 2x$ on $[0, 2\pi]$ without computing any integrals.

7. Find the Fourier series representations of the following:

a. $f(x) = x$, $x \in [0, 2\pi]$.

b. $f(x) = \dfrac{x^2}{4}$, $|x| < \pi$.

c. $f(x) = \begin{cases} \dfrac{\pi}{2}, & 0 < x < \pi, \\ -\dfrac{\pi}{2}, & \pi < x < 2\pi. \end{cases}$

8. Find the Fourier Series of each function $f(x)$ of period 2π. For each series, plot the Nth partial sum,

$$S_N = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos nx + b_n \sin nx\right],$$

for $N = 5, 10, 50$ and describe the convergence (is it fast? what is it converging to, etc.) [Some sample codes for computing partial sums are shown in the Section 2.8.]

a. $f(x) = x$, $|x| < \pi$.

b. $f(x) = |x|$, $|x| < \pi$.

c. $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$

9. Find the Fourier series of $f(x) = x$ on the given interval. Plot the Nth partial sums and describe what you see.

a. $0 < x < 2$.

b. $-2 < x < 2$.

c. $1 < x < 2$.

10. The result in problem 7b above gives a Fourier series representation of $\dfrac{x^2}{4}$. By picking the right value for x and a little arrangement of the series, show that [See Example 2.5.]

a.

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots.$$

b.

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

Hint: Consider how the series in part a. can be used to do this.

11. Sketch (by hand) the graphs of each of the following functions over four periods. Then sketch the extensions each of the functions as both an even and odd periodic function. Determine the corresponding Fourier sine and cosine series and verify the convergence to the desired function using software such as Maple.

a. $f(x) = x^2, 0 < x < 1$.

b. $f(x) = x(2-x), 0 < x < 2$.

c. $f(x) = \begin{cases} 0, & 0 < x < 1, \\ 1, & 1 < x < 2. \end{cases}$

d. $f(x) = \begin{cases} \pi, & 0 < x < \pi, \\ 2\pi - x, & \pi < x < 2\pi. \end{cases}$

12. Consider the function $f(x) = x, -\pi < x < \pi$.

a. Show that $x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$.

b. Integrate the series in part a and show that

$$x^2 = \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos nx}{n^2}.$$

c. Find the Fourier cosine series of $f(x) = x^2$ on $[0,\pi]$ and compare it to the result in part b.

13. Consider the function $f(x) = x, 0 < x < 2$.

a. Find the Fourier sine series representation of this function and plot the first 50 terms.

b. Find the Fourier cosine series representation of this function and plot the first 50 terms.

c. Apply Parseval's identity in Problem 3 to the result in part b.

d. Use the result of part c to find the sum $\sum_{n=1}^{\infty}\frac{1}{n^4}$.

14. Differentiate the Fourier sine series term by term in Problem 13. Show that the result is not the derivative of $f(x) = x$.

15. Find the general solution to the heat equation, $u_t - u_{xx} = 0$, on $[0,\pi]$ satisfying the boundary conditions $u(0,t) = 0$ and $u(\pi,t) = 0$. Determine the solution satisfying the initial condition,

$$u(x,0) = \begin{cases} x, & 0 \leq x \leq \frac{\pi}{2}, \\ \pi - x, & \frac{\pi}{2} \leq x \leq \pi, \end{cases}$$

16. Find the general solution to the heat equation, $u_t = ku_{xx} = 0$, on $[0,1]$ satisfying the boundary conditions $u(0,t) = 0$ and $u(1,t) = 0$. Determine the solution satisfying the initial condition, $u(x,0) = x(1-x)$. Note: This problem is used as an example of the Adomian Decomposition Method in Section 1.9.1.

17. Find the general solution to the wave equation $u_{tt} = 2u_{xx}$, on $[0,2\pi]$ satisfying the boundary conditions $u(0,t) = 0$ and $u_x(2\pi,t) = 0$. Determine the solution satisfying the initial conditions, $u(x,0) = x(4\pi - x)$, and $u_t(x,0) = 0$.

18. Recall the plucked string initial profile example in the last chapter given by

$$f(x)=\begin{cases} x, & 0\le x\le \frac{\ell}{2}, \\ \ell-x, & \frac{\ell}{2}\le x\le \ell, \end{cases}$$

satisfying fixed boundary conditions at $x=0$ and $x=\ell$. Find and plot the solutions at $t=0,.2,...,1.0$, of $u_{tt}=u_{xx}$, for $u(x,0)=f(x)$, $u_t(x,0)=0$, with $x\in[0,1]$.

19. Find and plot the solutions at $t=0,.2,...,1.0$, of the problem

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0\le x\le 1, t>0 \\ u(x,0) &= \begin{cases} 0, & 0\le x<\frac{1}{4}, \\ 1, & \frac{1}{4}\le x\le \frac{3}{4}, \\ 0, & \frac{3}{4}<x\le 1, \end{cases} \\ u_t(x,0) &= 0, \\ u(0,t) &= 0, \quad t>0, \\ u(1,t) &= 0, \quad t>0. \end{aligned}$$

20. Find the solution to Laplace's equation, $u_{xx}+u_{yy}=0$, on the unit square, $[0,1]\times[0,1]$ satisfying the boundary conditions $u(0,y)=0$, $u(1,y)=y(1-y)$, $u(x,0)=0$, and $u(x,1)=0$.

21. Find the solution to Laplace's equation, $u_{xx}+u_{yy}=0$, inside the rectangle $0\le x\le 2$, $0\le y\le 1$, satisfying the boundary conditions $u_x(0,y)=0$, $u_x(2,y)=0$, $u(x,0)=0$, and $u(x,1)=2\cos 3\pi x$. Use Maple to verify your solution using the following commands:

```
restart;
pde := diff(f(x, y), x $ 2) + diff(f(x, y), y $ 2) = 0;
bc[1] := D[1](f)(0, y) = 0;
bc[2] := D[1](f)(2, y) = 0;
bc[3] := f(x, 0) = 0;
bc[4] := f(x, 1) = 2*cos(3*Pi*x);
sys := [pde, bc[1], bc[2], bc[3], bc[4]];
sol := pdsolve(sys, f(x, y));
```

3
Numerical Solutions of PDEs

There's no sense in being precise when you don't even know what you're talking about.- John von Neumann (1903-1957)

MUCH OF THE BOOK HAS DEALT WITH FINDING EXACT SOLUTIONS to some generic problems. However, many problems of interest cannot be solved exactly. The heat, wave, and Laplace equations are linear partial differential equations and can be solved using separation of variables in geometries in which the Laplacian is separable. However, once we introduce nonlinearities, or complicated non-constant coefficients into the equations, some of these methods do not work. Even when separation of variables or the method of eigenfunction expansions gave us exact results, the computation of the resulting series had to be done on a computer and inevitably one could only use a finite number of terms of the expansion. So, therefore, it is sometimes useful to be able to solve differential equations numerically. In this chapter we will introduce the idea of numerical solutions of partial differential equations. We will introduce the finite difference method and the idea of stability. Other common approaches may be added later.

3.1 *The Finite Difference Method*

THE HEAT EQUATION CAN BE SOLVED USING SEPARATION OF VARIABLES. However, many partial differential equations cannot be solved exactly and one needs to turn to numerical solutions. The heat equation is a simple test case for using numerical methods. Here we will use the simplest method, finite differences.

Let us consider the heat equation in one dimension,

$$u_t = ku_{xx}.$$

Boundary conditions and an initial condition will be applied later. The starting point is figuring out how to approximate the derivatives in this equation.

Recall that the partial derivative, u_t, is defined by

$$\frac{\partial u}{\partial t} = \lim_{\Delta t \to \infty} \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t}.$$

Therefore, we can use the approximation

$$\frac{\partial u}{\partial t} \approx \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t}. \tag{3.1}$$

Forward difference approximation.

This is called a forward difference approximation.

In order to find an approximation to the second derivative, u_{xx}, we start with the forward difference

$$\frac{\partial u}{\partial x} \approx \frac{u(x+\Delta x, t) - u(x,t)}{\Delta x}.$$

Then,

$$\frac{\partial u_x}{\partial x} \approx \frac{u_x(x+\Delta x, t) - u_x(x,t)}{\Delta x}.$$

We need to approximate the terms in the numerator. It is customary to use a backward difference approximation. This is given by letting $\Delta x \to -\Delta x$ in the forward difference form,

Backward difference approximation.

$$\frac{\partial u}{\partial x} \approx \frac{u(x,t) - u(x-\Delta x, t)}{\Delta x}. \tag{3.2}$$

Applying this to u_x evaluated at $x = x$ and $x = x + \Delta x$, we have

$$u_x(x,t) \approx \frac{u(x,t) - u(x-\Delta x, t)}{\Delta x},$$

and

$$u_x(x+\Delta x, t) \approx \frac{u(x+\Delta x, t) - u(x,t)}{\Delta x}.$$

Inserting these expressions into the approximation for u_{xx}, we have

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u_x}{\partial x} \\ &\approx \frac{u_x(x+\Delta x, t) - u_x(x,t)}{\Delta x} \\ &\approx \frac{\frac{u(x+\Delta x,t)-u(x,t)}{\Delta x}}{\Delta x} - \frac{\frac{u(x,t)-u(x-\Delta x,t)}{\Delta x}}{\Delta x} \\ &= \frac{u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t)}{(\Delta x)^2}. \end{aligned} \tag{3.3}$$

This approximation for u_{xx} is called the central difference approximation of u_{xx}.

Central difference approximation of u_{xx}.

Combining Equation (3.1) with (3.3) in the heat equation, we have

$$\frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} \approx k \frac{u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t)}{(\Delta x)^2}.$$

Solving for $u(x, t+\Delta t)$, we find

$$u(x, t+\Delta t) \approx u(x,t) + \alpha \left[u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t) \right], \tag{3.4}$$

where $\alpha = k\dfrac{\Delta t}{(\Delta x)^2}$.

In this equation we have a way to determine the solution at position x and time $t + \Delta t$ given that we know the solution at three positions, x, $x + \Delta x$, and $x + 2\Delta x$ at time t.

$$u(x, t + \Delta t) \approx u(x, t) + \alpha \left[u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t) \right]. \tag{3.5}$$

A shorthand notation is usually used to write out finite difference schemes. The domain of the solution is $x \in [a, b]$ and $t \geq 0$. We seek approximate values of $u(x, t)$ at specific positions and times. We first divide the interval $[a, b]$ into N subintervals of width $\Delta x = (b - a)/N$. Then, the endpoints of the subintervals are given by

$$x_i = a + i\Delta x, \quad i = 0, 1, \ldots, N.$$

Similarly, we take time steps of Δt, at times

$$t_j = j\Delta t, \quad j = 0, 1, 2, \ldots.$$

This gives a grid of points (x_i, t_j) in the domain.

At each grid point in the domain we seek an approximate solution to the heat equation, $u_{i,j} \approx u(x_i, t_j)$. Equation (3.5) becomes

$$u_{i,j+1} \approx u_{i,j} + \alpha \left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \right]. \tag{3.6}$$

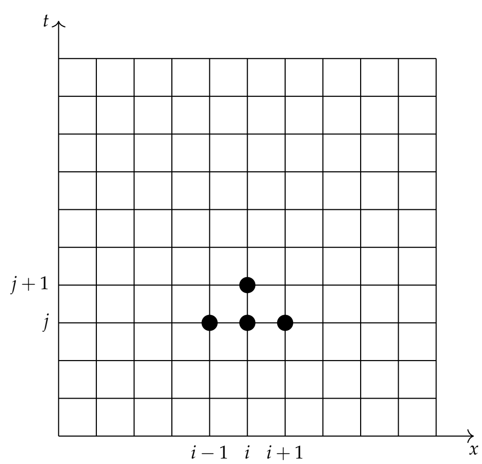


Figure 3.1: This *stencil* indicates the four types of terms in the finite difference scheme in Equation (3.6). The black circles represent the four terms in the equation, $u_{i,j}$ $u_{i-1,j}$ $u_{i+1,j}$ and $u_{i,j+1}$.

Equation (3.6) is the finite difference scheme for solving the heat equation. This equation is represented by the stencil shown in Figure 3.1. The black circles represent the four terms in the equation, $u_{i,j}$ $u_{i-1,j}$ $u_{i+1,j}$ and $u_{i,j+1}$.

Let's assume that the initial condition is given by

$$u(x, 0) = f(x).$$

Then, we have $u_{i,0} = f(x_i)$. Knowing these values, denoted by the open circles in Figure 3.2, we apply the stencil to generate the solution on the $j = 1$ row. This is shown in Figure 3.2.

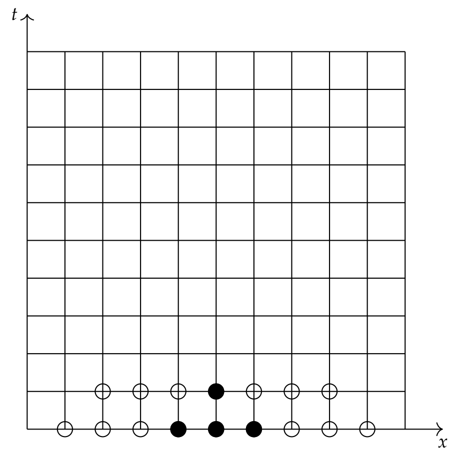


Figure 3.2: Applying the stencil to the row of initial values gives the solution at the next time step.

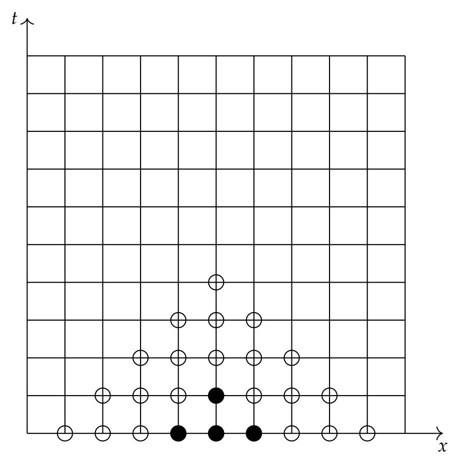


Figure 3.3: Continuation of the process provides solutions at the indicated points.

Further rows are generated by successively applying the stencil on each row, using the known approximations of $u_{i,j}$ at each level. This gives the values of the solution at the open circles shown in Figure 3.3. We notice that the solution can only be obtained at a finite number of points on the grid.

In order to obtain the missing values, we need to impose boundary conditions. For example, if we have Dirichlet conditions at $x = a$,

$$u(a,t) = 0,$$

or $u_{0,j} = 0$ for $j = 0, 1, \ldots$, then we can fill in some of the missing data points as seen in Figure 3.4.

The process continues until we again go as far as we can. This is shown in Figure 3.5.

We can fill in the rest of the grid using a boundary condition at $x = b$. For Dirichlet conditions at $x = b$,

$$u(b,t) = 0,$$

or $u_{N,j} = 0$ for $j = 0, 1, \ldots$, then we can fill in the rest of the missing data points as seen in Figure 3.6.

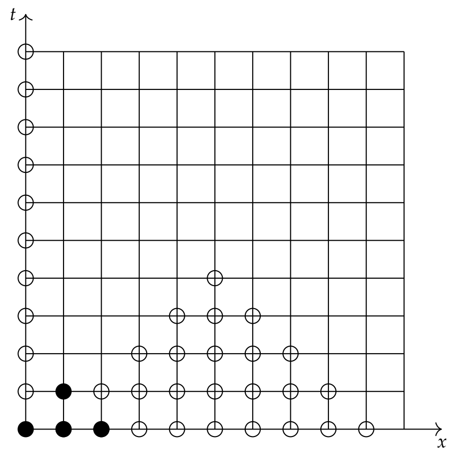


Figure 3.4: Knowing the values of the solution at $x = a$, we can fill in more of the grid.

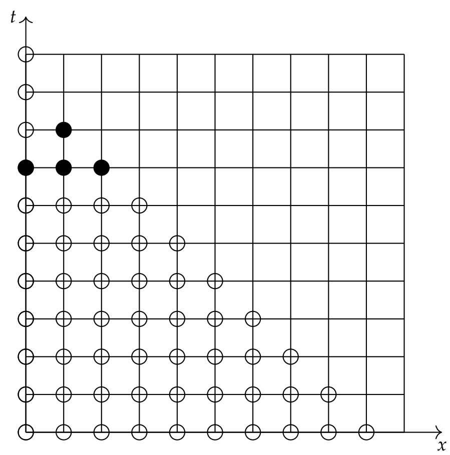


Figure 3.5: Knowing the values of the solution at other times, we continue to fill the grid as far as the stencil can go.

We could also use Neumann conditions. For example, let

$$u_x(a,t) = 0.$$

The approximation to the derivative gives

$$\left.\frac{\partial u}{\partial x}\right|_{x=a} \approx \frac{u(a+\Delta x, t) - u(a,t)}{\Delta x} = 0.$$

Then,

$$u(a+\Delta x, t) = u(a,t),$$

or $u_{0,j} = u_{1,j}$, for $j = 0, 1, \ldots$. Thus, we know the values at the boundary and can generate the solutions at the grid points as before.

We now have to code this using software. We can use MATLAB to do this. An example of the code is given below. In this example we specify the length of the rod, $L = 1$, and the heat constant, $k = 1$. The code is run for $t \in [0, 0.1]$.

The grid is created using $N = 10$ subintervals in space and $M = 50$ time steps. This gives dx = Δx and dt = Δt. Using these values, we find the numerical scheme constant $\alpha = k\Delta t/(\Delta x)^2$.

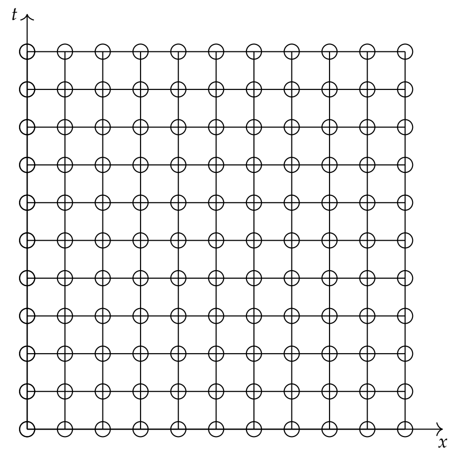


Figure 3.6: Using boundary conditions and the initial condition, the grid can be fill in through any time level.

Nest, we define $x_i = i * dx$, $i = 0, 1, \ldots, N$. However, in MATLAB, we cannot have an index of 0. We need to start with $i = 1$. Thus, $x_i = (i-1) * dx$, $i = 1, 2, \ldots, N+1$.

Next, we establish the initial condition. We take a simple condition of

$$u(x, 0) = \sin \pi x.$$

We have enough information to begin the numerical scheme as developed earlier. Namely, we cycle through the time steps using the scheme. There is one loop for each time step. We will generate the new time step from the last time step in the form

$$u_i^{new} = u_i^{old} + \alpha \left[u_{i+1}^{old} - 2u_i^{old} + u_{i-1}^{old} \right]. \tag{3.7}$$

This is done using $u0(i) = u_i^{new}$ and $u1(i) = u_i^{old}$.

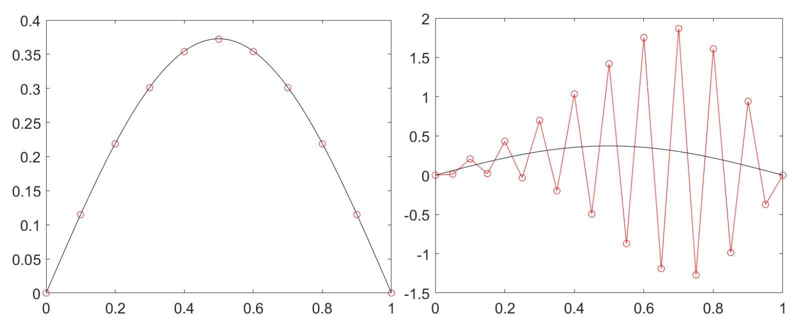


Figure 3.7: Plot of the exact solution and numerical solution of the heat equation for $\Delta t = 0.002$ for $\Delta x = 0.10$ (left) and $\Delta x = 0.20$ (right).

At the end of each time loop we update the boundary points so that the grid can be filled in as discussed. When done, we can plot the final solution. If we want to show solutions at intermediate steps, we can plot the solution earlier. In Figure 3.7 we plot the exact solution, $u(x,t) = e^{-\pi^2 t} \sin \pi x$, and numerical solution of the heat equation for $\Delta t = 0.002$ for $\Delta x = 0.10$ (left) and $\Delta x = 0.20$ (right). Even though the solution appears good for $\Delta x = 0.10$, doubling the number of points seems to have produced some strange result. This is due to the conditional stability of the scheme. In the next two sections we discuss the truncation error and stability of this scheme. We conclude with the MATLAB implementation.

```
% Solution of the Heat Equation Using a Forward Difference Scheme
% Initialize Data
%     Length of Rod, Time Interval
%     Number of Points in Space, Number of Time Steps
L=1;
T=0.1;
k=1;
N=10;
M=50;
dx=L/N;
dt=T/M;
alpha=k*dt/dx^2;

% Position
for i=1:N+1
    x(i)=(i-1)*dx;
end

% Initial Condition
for i=1:N+1
    u0(i)=sin(pi*x(i));
end

% Partial Difference Equation (Numerical Scheme)
for j=1:M
   for i=2:N
      u1(i)=u0(i)+alpha*(u0(i+1)-2*u0(i)+u0(i-1));
   end
   u1(1)=0;
   u1(N+1)=0;
   u0=u1;
end

% Plot solution
plot(x, u1);
```

3.2 *Truncation Error*

In the previous section we found a finite difference scheme for numerically solving the one dimensional heat equation. We have from Equations (3.5) and (3.6),

$$u(x,t+\Delta t) \approx u(x,t)+\alpha\left[u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)\right], \tag{3.8}$$

$$u_{i,j+1} \approx u_{i,j} + \alpha \left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \right], \tag{3.9}$$

where $\alpha = k\Delta t/(\Delta x)^2$. For points $x \in [a,b]$ and $t \geq 0$, we use the scheme to find approximate values of $u(x_i, t_j) = u_{i,j}$ at positions $x_i = a + i\Delta x, \quad i = 0, 1, \ldots, N$, and times $t_j = j\Delta t, \quad j = 0, 1, 2, \ldots$.

In implementing the scheme, we have found that there are errors introduced just like when using Euler's Method for ordinary differential equations. These truncations errors can be found by applying Taylor approximations just like we had for ordinary differential equations. In the schemes (3.8) and (3.9), we have not used equality. In order to replace the approximation by an equality, we need to estimate the order of the terms neglected in a Taylor series expansions of the time and space derivatives that we have approximated.

We begin with the time derivative approximation. We used the forward difference formula (3.1),

$$\frac{\partial u}{\partial t} \approx \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t}. \tag{3.10}$$

This formula can be derived from the Taylor series expansion of $u(x, t + \Delta t)$ about $\Delta t = 0$,

$$u(x, t+\Delta t) = u(x,t) + \frac{\partial u}{\partial t}(x,t)\Delta t + \frac{1}{2!}\frac{\partial^2 u}{\partial t^2}(x,t)(\Delta t)^2 + O((\Delta t)^3).$$

Here we use "big O" notation where $O((\Delta t)^3)$ indicates that the terms not listed are of the order $(\Delta t)^3$ or smaller.

Solving for $\frac{\partial u}{\partial t}(x,t)$, we obtain

$$\frac{\partial u}{\partial t}(x,t) = \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} - \frac{1}{2!}\frac{\partial^2 u}{\partial t^2}(x,t)\Delta t + O((\Delta t)^2).$$

We see that we have obtained the forward difference approximation (3.1) with the added benefit of knowing something about the error terms introduced in the approximation. Namely, when we approximate u_t with the forward difference approximation (3.1), we are making an error of

$$E(x,t,\Delta t) = -\frac{1}{2!}\frac{\partial^2 u}{\partial t^2}(x,t)\Delta t + O((\Delta t)^2).$$

We have truncated the Taylor series to obtain this approximation and we say that

$$\frac{\partial u}{\partial t} = \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} + O(\Delta t) \tag{3.11}$$

is a first order approximation in Δt.

In a similar manor, we can obtain the truncation error for the $u_{xx}-$ term. However, instead of starting with the approximation we used in Equation (3.3), we will derive a term using the Taylor series expansion of $u(x + \Delta x, t)$ about $\Delta x = 0$. Namely, we begin with the expansion

$$\begin{aligned} u(x+\Delta x, t) &= u(x,t) + u_x(x,t)\Delta x + \frac{1}{2!}u_{xx}(x,t)(\Delta x)^2 + \frac{1}{3!}u_{xxx}(x,t)(\Delta x)^3 \\ &\quad + \frac{1}{4!}u_{xxxx}(x,t)(\Delta x)^4 + \ldots. \end{aligned} \tag{3.12}$$

We want to solve this equation for u_{xx}. However, there are some obstructions, like needing to know the u_x term. So, we seek a way to eliminate lower order terms. One way is to note that replacing Δx by $-\Delta x$ gives

$$\begin{aligned} u(x-\Delta x,t) &= u(x,t) - u_x(x,t)\Delta x + \frac{1}{2!}u_{xx}(x,t)(\Delta x)^2 - \frac{1}{3!}u_{xxx}(x,t)(\Delta x)^3 \\ &\quad + \frac{1}{4!}u_{xxxx}(x,t)(\Delta x)^4 + \ldots . \end{aligned} \tag{3.13}$$

Adding these Taylor series, we have

$$\begin{aligned} u(x+\Delta x,t) + u(x-\Delta x,t) &= 2u(x,t) + u_{xx}(x,t)(\Delta x)^2 \\ &\quad + \frac{2}{4!}u_{xxxx}(x,t)(\Delta x)^4 + O((\Delta x)^6). \end{aligned} \tag{3.14}$$

We can now solve for u_{xx} to find

$$\begin{aligned} u_{xx}(x,t) &= \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{(\Delta x)^2} \\ &\quad + \frac{2}{4!}u_{xxxx}(x,t)(\Delta x)^2 + O((\Delta x)^4). \end{aligned} \tag{3.15}$$

Thus, we have that

$$u_{xx}(x,t) = \frac{u(x+\Delta x,t) - 2u(x,t) + u(x+\Delta x,t)}{(\Delta x)^2} + O((\Delta x)^2)$$

is the second order in Δx approximation of u_{xx}.

Combining these results, we find that the heat equation is approximated by

$$\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} = \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{(\Delta x)^2} + O\Big((\Delta x)^2, \Delta t\Big).$$

This has local truncation error that is first order in time and second order in space.

3.3 *Stability*

ANOTHER CONSIDERATION FOR NUMERICAL SCHEMES for the heat equation is the stability of the scheme. In implementing the finite difference scheme,

$$u_{m,j+1} = u_{m,j} + \alpha\left[u_{m+1,j} - 2u_{m,j} + u_{m-1,j}\right], \tag{3.16}$$

$\alpha = k\Delta t/(\Delta x)^2$, one finds that the solution goes crazy when α is too big. In other words, if you try to push the individual time steps too far into the future, then something goes haywire. We saw this in Figure 3.7. Even though Δx was halved, α went from 0.20 to 0.80. We can determine the onset of instability by looking at the solution of this equation for $u_{m,j}$. [Note: We changed index i to m to avoid confusion later in this section.] We will see that there needs to be a restriction placed on α.

The scheme is actually what is called a partial difference equation for $u_{m,j}$. We could write it in terms of differences, such as $u_{m+1,j} - u_{m,j}$ and $u_{m,j+1} - u_{m,j}$. The time steps are one unit and the spatial points are at most two units apart. We can see this in the stencils in Figure 3.1. So, this is a second order partial difference equation similar to the idea that the heat equation is a second order partial differential equation. The heat equation can be solved using the method of separation of variables. The difference scheme can also be solved in a similar fashion. We will show how this leads to product solutions.

We begin by assuming that $u_{mj} = X_m T_j$, a product of functions of the indices m and j. [Recall that sequences are functions whose domain consist of a subset of integers. For example, $X_m = X(m)$.] Inserting this guess into the finite difference equation, we have

$$\begin{aligned} u_{m,j+1} &= u_{m,j} + \alpha \left[u_{m+1,j} - 2u_{m,j} + u_{m-1,j} \right], \\ X_m T_{j+1} &= X_m T_j + \alpha \left[X_{m+1} - 2X_m + X_{m-1} \right] T_j, \\ \frac{T_{j+1}}{T_j} &= \frac{\alpha X_{m+1} + (1-2\alpha) X_m + \alpha X_{m-1}}{X_m}. \end{aligned} \tag{3.17}$$

Noting that we have a function of j equal to a function of m, then we can set each of these to a constant, λ. Then, we obtain two ordinary difference equations:

$$T_{j+1} = \lambda T_j, \tag{3.18}$$

$$\alpha X_{m+1} + (1-2\alpha) X_m + \alpha X_{m-1} = \lambda X_m. \tag{3.19}$$

The first equation is a simple first order difference equation and can be solved by iteration:

$$\begin{aligned} T_{j+1} &= \lambda T_j, \\ &= \lambda(\lambda T_{j-1}) = \lambda^2 T_{j-1}, \\ &= \lambda^3 T_{j-2}, \\ &= \lambda^{j+1} T_0, \end{aligned} \tag{3.20}$$

The second difference equation can be solved by making a guess in the same spirit as solving a second order constant coefficient differential equation. Namely, let $X_m = \xi^m$ for some number ξ. This gives

$$\begin{aligned} \alpha X_{m+1} + (1-2\alpha) X_m + \alpha X_{m-1} &= \lambda X_m, \\ \xi^{m-1} \left[\alpha \xi^2 + (1-2\alpha)\xi + \alpha \right] &= \lambda \xi^m \\ \alpha \xi^2 + (1-2\alpha-\lambda)\xi + \alpha &= 0. \end{aligned} \tag{3.21}$$

This is an equation for ξ in terms of α and λ. Due to the boundary conditions, we expect to have oscillatory solutions. So, we guess that $\xi = |\xi| e^{i\theta}$, where i is the imaginary unit.[1] We assume that $|\xi| = 1$, and thus $\xi = e^{i\theta}$ and $X_m = \xi^m = e^{im\theta}$. Since $x_m = m\Delta x$, we have $X_m = e^{ix_m \theta/\Delta x}$. We define $\beta = \theta/\Delta x$, to give $X_m = e^{i\beta x_m}$ and $\xi = e^{i\beta \Delta x}$.

[1] Recall Euler's Formula,

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

So, for real θ, $e^{i\theta}$ represents oscillations. Also, note that $|e^{i\theta}| = 1$.

Inserting this value for ξ into the quadratic equation for ξ, we have

$$\begin{aligned} 0 &= \alpha\xi^2 + (1-2\alpha-\lambda)\xi + \alpha \\ &= \alpha e^{2i\beta\Delta x} + (1-2\alpha-\lambda)e^{i\beta\Delta x} + \alpha \\ &= e^{i\beta\Delta x}\left[\alpha(e^{i\beta\Delta x} + e^{-i\beta\Delta x}) + (1-2\alpha-\lambda)\right] \\ &= e^{i\beta\Delta x}\left[2\alpha\cos(\beta\Delta x) + (1-2\alpha-\lambda)\right]. \end{aligned}$$

Solving for λ, we have

$$\lambda = 2\alpha\cos(\beta\Delta x) + 1 - 2\alpha. \tag{3.22}$$

So, we have found that

$$u_{mj} = X_m T_j = \lambda^m(a\cos\alpha x_m + b\sin\alpha x_m), \quad a^2 + b^2 = h_0^2,$$

with λ given by Equation (3.22). For the solution to remain bounded, or stable, we need $|\lambda| \le 1$.

Therefore, we have the inequality

$$-1 \le 2\alpha\cos(\beta\Delta x) + 1 - 2\alpha \le 1.$$

Since $\cos(\beta\Delta x) \le 1$, the upper bound is obviously satisfied. Since $-1 \le \cos(\beta\Delta x)$, the lower bound is satisfied for $-1 \le -2\alpha + 1 - 2\alpha$, or $\alpha \le \frac{1}{2}$. Therefore, the stability criterion is satisfied when

Stability criterion.

$$\alpha = k\frac{\Delta t}{\Delta x^2} \le \frac{1}{2}. \tag{3.23}$$

3.4 *Discretization of Laplace Equation*

Let's consider Laplace's equation in Cartesian coordinates,

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, \quad 0 < y < H$$

with the boundary conditions

$$u(0,y) = g_1(y), \quad u(L,y) = g_2(y), \quad u(x,0) = f_1(x), \quad u(x,H) = f_2(x).$$

The boundary conditions are shown in Figure 6.12. In Chapter 1 we learned how to seek a solution of this problem using the Method of Separation of Variables. This generally leads to needing to compute a Fourier series and represent the solution in an infinite series whose coefficients we hopefully can compute. However, it might not always be possible. So, seeking a numerical solution may be an option. In this section we explore one method for numerically solving Laplace's equation on a rectangular domain.

We use finite difference approximations to approximate the second order derivatives at grid points. For a function $f(x)$, we can use the central difference approximation,

$$f''(x) = \frac{y(x+\Delta x) - 2y(x) + y(x-\Delta x)}{\Delta x^2} + O(\Delta x^2).$$

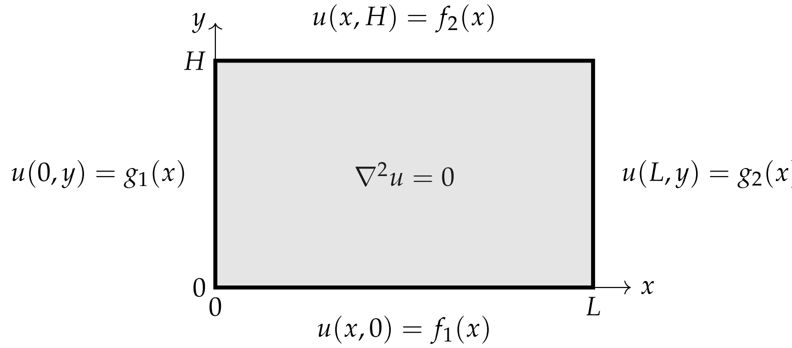


Figure 3.8: In this figure we show the domain and boundary conditions for the example of determining the solution of Laplace's equation in a rectangular region.

The domain of the solution is $x \in [0, L]$ and $y \in [0, H]$. We seek approximate values of $u(x, y)$ at specific positions in the domain. We first divide the intervals into N_x and N_y subintervals of width $\Delta x = L/N_x$. and $\Delta y = H/N_y$, respectively. This gives a grid of points (x_i, y_j) in the domain, where

$$x_i = i\Delta x, \quad i = 0, 1, \ldots, N_x, \quad y_j = j\Delta y, \quad i = 0, 1, \ldots, N_y.$$

At each grid point in the domain we seek an approximate solution to the heat equation, $u_{i,j} \approx u(x_i, y_j)$. Then, the finite difference form of Laplace's equation for the interior points ($i = 1, 2, \ldots, N_x - 1$, $j = 1, 2, \ldots, N_y - 1$) is given by

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = 0.$$

Rearranging,

$$\begin{aligned} 0 &= \frac{\Delta y^2(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + \Delta x^2(u_{i,j+1} - 2u_{i,j} + u_{i,j-1})}{\Delta x^2 \Delta y^2} \\ &= \Delta y^2(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + \Delta x^2(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) \\ &= \Delta y^2(u_{i+1,j} + u_{i-1,j}) + \Delta x^2(u_{i,j+1} + u_{i,j-1}) - 2(\Delta x^2 + \Delta y^2)u_{i,j} \end{aligned} \tag{3.24}$$

In Figure 3.9 we show the relationships between the terms of this discrete version of Laplace's equation.

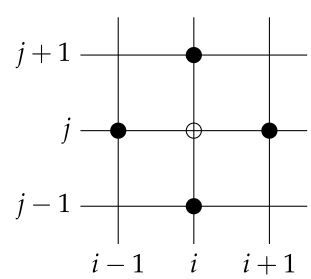


Figure 3.9: This stencil indicates the five types of terms in the finite difference scheme in Equation (3.24). The black circles represent the four terms in the equation, $u_{i-1,j}$ $u_{i+1,j}$ $u_{i,j-1}$ and $u_{i,j+1}$ and the empty circle represents $u_{i,j}$.

The method of solution differs from that of the heat equation. The only information we have is on the boundary of the domain. In Figure 3.10 we

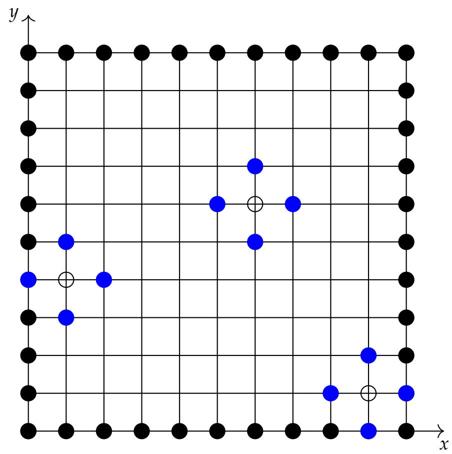


Figure 3.10: In this figure we indicate with black circles that the boundary values are known. Applying the stencil at various locations in the grid, we see that we cannot use the boundary values to approximate a value at other points.

see that we cannot use the boundary values to approximate a value at other points.

However, we can rearrange Equation (3.24) to get a more suggestive method. Namely, if we guess the solution across the grid, we can hopefully get better approximations to the solution. We assume we know approximate values at the blue circles in Figure 3.10 to compute a new values at the open circle in the center of the stencil. Thus, defining $n_x = N_x - 1$ and $n_y = N_y - 1$, we have for the interior points

$$u_{i,j}^{new} = \frac{\Delta y^2(u_{i+1,j} + u_{i-1,j}) + \Delta x^2(u_{i,j+1} + u_{i,j-1})}{2(\Delta x^2 + \Delta y^2)}, \quad i = 1, \ldots, n_x, j = 1, \ldots, n_y. \tag{3.25}$$

For now, we can simplify things a lot by letting $\Delta x = \Delta y$. Then, Equation (3.24) becomes

$$-4u_{i,j} + u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}, \quad i = 1, \ldots, n_x, j = 1, \ldots, n_y. \tag{3.26}$$

Let's consider what this means for $n_x = n_y = n$ for different n and develop a matrix form of the scheme.

3.5 *Matrix Formulation of the Scheme*

For $n = 2$, we see from Figure 3.11 that there are four interior points. Making these points the first term in the scheme (3.26), we arrive at four equations for the four unknowns, $u_{1,1}$, $u_{1,2}$, $u_{2,1}$, and $u_{2,2}$,

$$\begin{aligned}
0 &= -4u_{1,1} + u_{i2,1} + u_{0,1} + u_{1,2} + u_{1,0} \\
0 &= -4u_{1,2} + u_{i2,2} + u_{0,2} + u_{1,3} + u_{1,1} \\
0 &= -4u_{2,1} + u_{3,1} + u_{1,1} + u_{2,2} + u_{2,0} \\
0 &= -4u_{2,2} + u_{3,2} + u_{1,2} + u_{2,3} + u_{2,1}.
\end{aligned} \tag{3.27}$$

The highlighted u's lie on the boundary and their values are known.

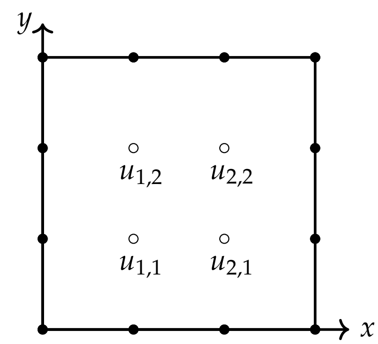


Figure 3.11: Example of a 4×4 grid with a 2×2 inner region. The goal is that given boundary values on can find approximate solutions to Laplace's Equation using Equation (3.26).

We can rearrange these linear equations into a matrix equation, $S\Phi = b$

$$\left[\begin{array}{cc|cc} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ \hline 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{array}\right] \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{2,1} \\ u_{2,2} \end{bmatrix} = - \begin{bmatrix} u_{0,1} \\ u_{0,2} \\ u_{3,1} \\ u_{3,2} \end{bmatrix} - \begin{bmatrix} u_{1,0} \\ u_{2,0} \\ u_{1,3} \\ u_{2,3} \end{bmatrix}$$

We have moved the boundary values to the right hand side and arranged them into two vectors. The first vector contains boundary conditions on the sides of the region and the second vector contains values of the solution along the bottom and top of the region.

The 4×4 matrix shows some symmetries. We can write the S matrix using a tensor product, also known as a direct produce or Kronecker product. The tensor product between an $m \times n$ matrix A and a $p \times q$ matrix B gives a $np \times mq$ matrix $C = A \otimes B$ with elements

$$c_{rs} = a_{ij} b_{k\ell}, \quad r = p(i-1) + k, \quad s = q(j-1) + \ell. \tag{3.28}$$

For example, if A is a 2×2 matrix, then

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}$$

In particular, consider the matrices

$$M = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We can compute two tensor products,

$$\begin{aligned} I \otimes M &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \\ M \otimes I &= \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}. \tag{3.29}$$

Adding these, we have the S matrix for the $n = 2$ case:

$$\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} + \begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix}.$$

More compactly, we have shown that

$$S = M \otimes I + I \otimes M.$$

Similarly, for $n = 3$ we obtain

$$\left[\begin{array}{ccc|ccc|ccc} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -4 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{array}\right] \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{1,3} \\ u_{2,1} \\ u_{2,2} \\ u_{2,3} \\ u_{3,1} \\ u_{3,2} \\ u_{3,3} \end{bmatrix} = - \begin{bmatrix} u_{0,1} \\ u_{0,2} \\ u_{0,3} \\ 0 \\ 0 \\ 0 \\ u_{4,1} \\ u_{4,2} \\ u_{4,3} \end{bmatrix} - \begin{bmatrix} u_{1,0} \\ 0 \\ u_{1,3} \\ u_{2,0} \\ 0 \\ u_{2,3} \\ u_{3,0} \\ 0 \\ u_{3,3} \end{bmatrix} \tag{3.30}$$

We see that these matrix equations take the form

$$S\Phi = b, \tag{3.31}$$

where S is a block tridiagonal matrix and b involves the values of the solution on the boundary. Again we have separated out the vertical and horizontal values as two column vectors for clarity. We have also defined the vector of unknowns

$$\Phi = \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix}.$$

It is a simple matter to generalize the process when $\Delta x = \Delta y$ and $n_x = n_y = n$. In the case for $n = 4$, we have the 4×4 blocks

$$M = \begin{bmatrix} -4 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & -4 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

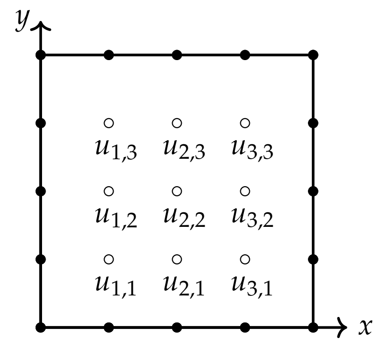


Figure 3.12: Example of a 5×5 grid with a 3×3 inner region. The numerical scheme for solving Laplace's equation is in Equation (3.30)

The S matrix is then,

$$S = \begin{bmatrix} M & I & 0 & 0 \\ I & M & I & 0 \\ 0 & I & M & I \\ 0 & 0 & I & M \end{bmatrix},$$

or written out A is given as

$$\left[\begin{array}{cccc|cccc|cccc|cccc}
-4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & & & & & & & & \\
1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & & & & & & & & \\
0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & & & & & & & & \\
0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 & & & & & & & & \\
\hline
1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & & & & \\
0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & & & & \\
0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & & & & \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 & & & & \\
\hline
 & & & & 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & & & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 \\
 & & & & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 \\
\hline
 & & & & & & & & 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 \\
 & & & & & & & & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 \\
 & & & & & & & & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 \\
 & & & & & & & & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4
\end{array}\right]$$

Again, we can use the tensor product of the 4×4 identity matrix with

$$M = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

to find that $S = M \otimes I + I \otimes M$.

3.6 *Numerical Solution of the 2D Laplace Equation*

The goal is to solve Equation (3.31). Since it is a linear system, there are several ways to approach the solution. We can solve it directly,

$$\Phi = S^{-1} b,$$

using Gaussian elimination. We could also use iterative methods as we alluded to earlier before Equation (3.25). It is best to do a simple example.

Example:

We consider the solution of Laplace's equation on the unit square,

$$u_{xx} + u_{yy} = 0, \quad 0 < x < 1, \quad 0 < y < 1,$$

with the boundary conditions

$$u(0, y) = 0, \quad u(1, y) = 1, \quad u(x, 0) = \sin \pi x, \quad u(x, 1) = 0.$$

We learned how to solve Laplace's equation in the first chapter using separation of variables. In this problem we have to solve two separate problems to accommodate the boundary conditions.

$$\begin{aligned} w_{xx} + w_{yy} &= 0, \quad w(0, y) = w(1, y) = w(x, 1) = 0, w(x, 0) = \sin \pi x, \\ v_{xx} + v_{yy} &= 0, \quad v(0, y) = v(1, y) = v(x, 1) = 0, v(1, y) = 1. \end{aligned} \tag{3.32}$$

Then, $u(x, y) = w(x, y) + v(x, y)$ will solve the original problem.

The product solutions for these two problems are

$$\begin{aligned} w_n(x, y) &= \sin n\pi x \sinh n\pi(1 - y), \\ v_n(x, y) &= \sin n\pi y \sinh n\pi x. \end{aligned} \tag{3.33}$$

This gives the general solution to the full problem as[2]

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi(1 - y) + \sum_{n=1}^{\infty} b_n \sin n\pi y \sinh n\pi x.$$

You can verify that $u(0, y) = 0$ and $u(x, 1) = 0$. We still need to satisfy $u(1, y) = 1$ and $u(x, 0) = \sin \pi x$. For the latter condition, we see that $a_n = 0$ for $n > 1$. This leaves for $n = 1$

$$u(x, 0) = a_1 \sin \pi x \sinh \pi = \sin \pi x,$$

or $a_1 = (\sinh \pi)^{-1}$.

For the other condition, we have

$$u(1, y) = \sum_{n=1}^{\infty} (b_n \sinh n\pi) \sin n\pi y = 1.$$

This is a Fourier sine series for $y \in [0, 1]$. The expansion coefficient for this sine series is $b_n \sinh n\pi$. We learned from the previous chapter that the coefficients are found by integration giving

$$\begin{aligned} b_n \sinh n\pi &= 2 \int_0^1 \sin n\pi y \, dy \\ &= \frac{2}{n\pi}(1 - \cos n\pi). \end{aligned} \tag{3.34}$$

So, we have the exact solution

$$u(x, y) = \sin \pi x \frac{\sinh \pi(1 - y)}{\sinh \pi} + \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - \cos n\pi) \sin n\pi y \frac{\sinh n\pi x}{\sinh n\pi}. \tag{3.35}$$

[2] This problem could have been applied more generally to a rectangle $[0, L] \times [0, H]$ and not a unit square. In that case, the solution and boundary conditions would need to be modified to give a rescaled model. The solution subject to the boundary conditions Here $u(x, 0) = \sin \frac{\pi x}{L}$ and $u(L, y) = 1$, takes the form

$$u(x, y) = \sin \frac{\pi x}{L} \frac{\sinh \frac{\pi}{L}(H - y)}{\sinh \frac{\pi H}{L}} + \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - \cos n\pi) \sin \frac{n\pi y}{H} \frac{\sinh \frac{n\pi x}{H}}{\sinh \frac{n\pi L}{H}}.$$

We can now numerically integrate this problem. We do this in MATLAB. We begin by setting up the problem as seen in the code below. Should be able to provide Python code.

```
clear
global L H dx dy Nx Ny  x y

% Inside dimensions - so far nx=ny, dx=dy
nx=3;
ny=3;
% Outside dimensions
Nx=nx+1;
Ny=ny+1;
% Domain size
L=1;
H=0.75;
% Increments
dx=L/Nx;
dy=H/Ny;
% Independent variables
x=0:dx:L;
y=0:dy:H;
% Initialize u(x,y) plus Boundary Conditions
U=zeros(Nx+1,Ny+1);
U(1,:)=0;                 % left
U(Nx+1,:)=1;              % right
U(:,1)=sin(pi*x/L);       % bottom
U(:,Ny+1)=0;              % top
\end{verbatim}
%\end{quote}

Next, we obtain matrices $A$ and $b.$

%\begin{quote}
\begin{verbatim}
% A matrix
k=nx;                % Need to adjust for nonsquare grid
A=getA(k);

% Inside boundary values
bvert=zeros(1,k^2);
bhor=zeros(1,k^2);

for i=1:k
    bvert(i) = U(1,i+1);
    bvert(k^2-i+1) = U(k+2, k-i+2);
    bhor((i-1)*k+1) = U(i+1, 1);
```

```
    bhor((i-1)*k+k) = U(i+1, k+2);
end

b=-bhor-bvert;
```

Here we used a function to generate matrix A, called `getA`. It is given by

```
function A=getA(k)
e = ones(k,1);
a = spdiags([e -2*e e] , [-1 0 1], k, k);
I = speye (k, k);
A = kron(a,I) + kron(I,a);
end
```

This code relies on the Kronecker product (tensor product) in Equation (3.28). In MATLAB is is implemented using the `kron` function.

Now, we are in a position to solve Equation (3.31). One could compute A^{-1}, if it exists. But, in MATLAB one can use the backslash to solve a linear system. This is equivalent to using Gaussian elimination when A is a square matrix. Also, since b in the above code is a row vector, we first turn it into a column vector, `b'`. Then, `Phi=A'` produces a column vector. In order to related the solution to the inner grid points, we reshape `Phi` into a $k \times k$ matrix. %beginquote

```
Phi=A\b';
Phi=reshape(Phi,k,k);
U(2:k+1,2:k+1)=Phi';
Ugaussian=U';
myplot(Ugaussian,'Gaussian Solution')
\end\begin{lstlisting}
%\end{quote}

Finally, we plot the solution using {\tt myplot} which provides a
     plot of the solution normalized so that the maximum value is
     set to one. The resulting plot is in Figure \ref{fig:
    gaussian}.
%\begin{quote}

 \begin{lstlisting}[style=Matlab-editor]
function myplot(U,mytitle)
global L H dx dy Nx Ny x y
maxU=max(U,[],'all');
nU=U/maxU;
contourf(x,y,nU,200,'linecolor','none')
colormap(jet)
colorbar
caxis([-1,1])
xlabel('$x$','Interpreter','latex','Fontsize',14)
ylabel('$y$','Interpreter','latex','Fontsize',14)
```

```
title(mytitle,'Interpreter','latex','Fontsize',14)
axis equal
end
```

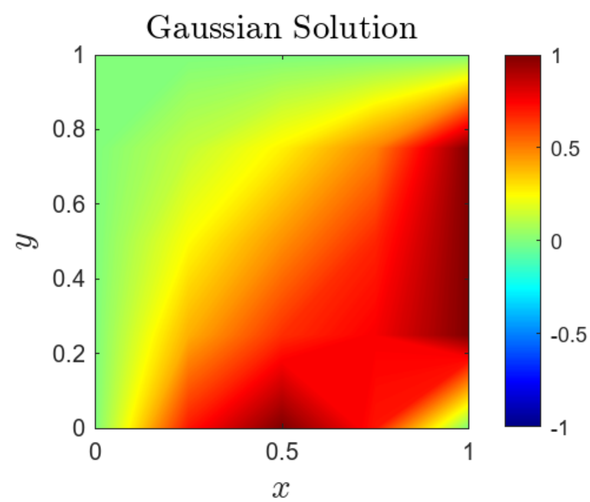


Figure 3.13: Solution of the Laplace equation example for $n = 3$ using Gaussian elimination through MATLAB's backslash operation.

We can compare the solution in Figure 3.13 with the exact solution in Equation (3.34). That solution involves an infinite sum. So, we plot a partial sum keeping enough terms to indicate an approximation to the full sum. One such function to produce the exact solution is below.

```
function v=exact(N)
global L H dx dy Nx Ny x y
Mx=length(x);
My=length(y);
v=zeros(Mx,My);
for i=1:Mx
    for j=1:My
        v(i,j)=sin(pi*x(i)/L).*sinh(pi*(H-y(j))/L)/sinh(pi*H/L);
        for n=1:N
            v(i,j)=v(i,j)...
            +2*(1-cos(n*pi))/(n*pi)*sin(n*pi*y(j)/H)...
            *sinh(n*pi*x(i)/H)/sinh(n*pi*L/H);
        end
    end
end
end
```

In Figure 3.14 we see the exact solution using $N = 100$ terms of the Fourier sine series. We see similarities with the plot in Figure 3.13. In order to compare solutions, we should compute some type of quantitative difference between the two results.

Another method for solving the finite difference scheme (3.26) is to use an iterative method. We rewrite the system as[3]

[3] From Equation (3.24), we can also write

$$u_{i,j} = \frac{\Delta y^2(u_{i+1,j} + u_{i-1,j}) + \Delta x^2(u_{i,j+1} + u_{i,j-1})}{2(\Delta x^2 + \Delta y^2)}$$

This would work with a more general grid.

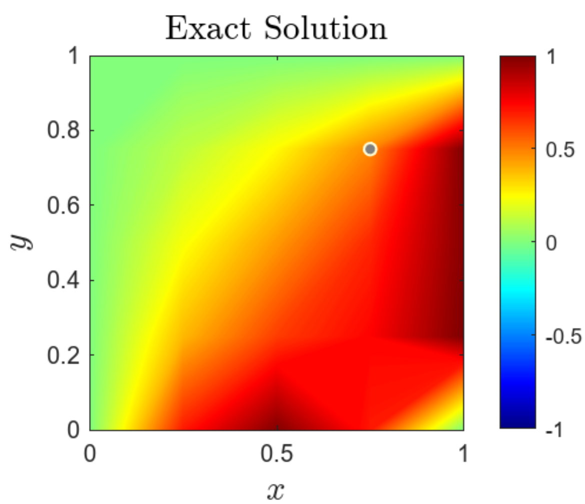


Figure 3.14: Solution of the Laplace equation example for $n = 3$ using the exact solution in Equation (3.34).

$$u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right), \quad i = 1,\dots,n_x, j = 1,\dots,n_y. \tag{3.36}$$

The idea is to guess $u_{i,j}$, insert the guess on the right hand side of the equation and output a better guess. Then, take the new approximation and keep going until the procedure seems to converge to a solution. If we let the kth iteration produce $u_{i,j}^{(k)}$, then the scheme could be written as

$$u_{i,j}^{(k+1)} = \frac{1}{4}\left(u_{i+1,j}^{(k)} + u_{i-1,j}^{(k)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k)}\right), \quad i = 1,\dots,n_x, j = 1,\dots,n_y, \tag{3.37}$$

where we begin with a guess $u_{i,j}^{(0)}$. The only thing we currently know are the boundary values. So, we can use those to generate a better approximation. For the inside points we can set the values to some approximation based on the characteristics of the problem or simply set all values to zero.

We will call this process a Jacobi iterative method. The MATLAB code that captures this with an input of the boundary values and number of iterations is shown in the function `Jacobi` and implemented by calling

```
Ujacobi=Jacobi(U(1,:), U(Nx+1,:),U(:,1),U(:,Ny+1),10);
```

```
function W=Jacobi(bcl, bcr,bcb,bct,N)
global L H dx dy Nx Ny x y

W=zeros(Nx+1,Ny+1);
den=2*(dx^2+dy^2);

W(1,:)=bcl;          % left
W(Ny+1,:)=bcr;       % right
W(:,1)=bcb;          % bottom
W(:,Nx+1)=bct;       % top

for k=1:N;
for i=2:Nx
```

```
        for j=2:Ny
            W(i,j)=(dy^2*(W(i+1,j)+W(i-1,j)) ...
                +dx^2*(W(i,j+1)+W(i,j-1)))/den;
        end
    end
end
end
```

For the $n = 3$ grid and boundary conditions, we iterate the scheme ten times. In Figure 3.15 we show the results. Again, they look similar to those in Figures 3.14 and 3.13.

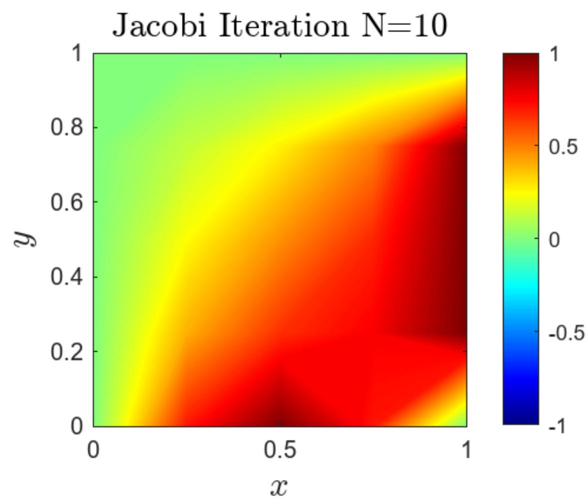


Figure 3.15: Solution of the Laplace equation example for $n = 3$ using the Jacobi iteration scheme for $N = 10$ iterations.

If we wish to provide more accurate solutions, then we need a finer grid. So, we let $n = 50$ and do not change anything else. We show in Figure 3.16 these using the exact solution in Equation (3.34), the Gaussian solution, and the Jacobi iteration for $N = 10, 50$. The Gaussian solution has the same features as the exact solution. However, we see that the Jacobi scheme has not yet converged.

So, we continue iterating the Jacobi scheme. In Figure 3.17 several plots as the iteration proceeds. We see that for $N = 1000$ iterations, the solution is looking more like the exact solution.

3.7 *Jacobi, Gauss-Seidel, and SOR Iterative Schemes*

The Jacobi iterative scheme is just one possible iterative scheme. It seems to converge slowly. Will it always converge? Can we speed up the convergence? Are there better schemes? These are just some of the questions a numerical analyst might ask. But first, we need to recast the Jacobi scheme in matrix form.

We begin with the numerical scheme in the form $A\Phi = b$, where b contains the boundary values and A is the full $n_x \times n_y$ matrix. The equivalent of solving the system for u_{ij} would be to move the diagonal elements to

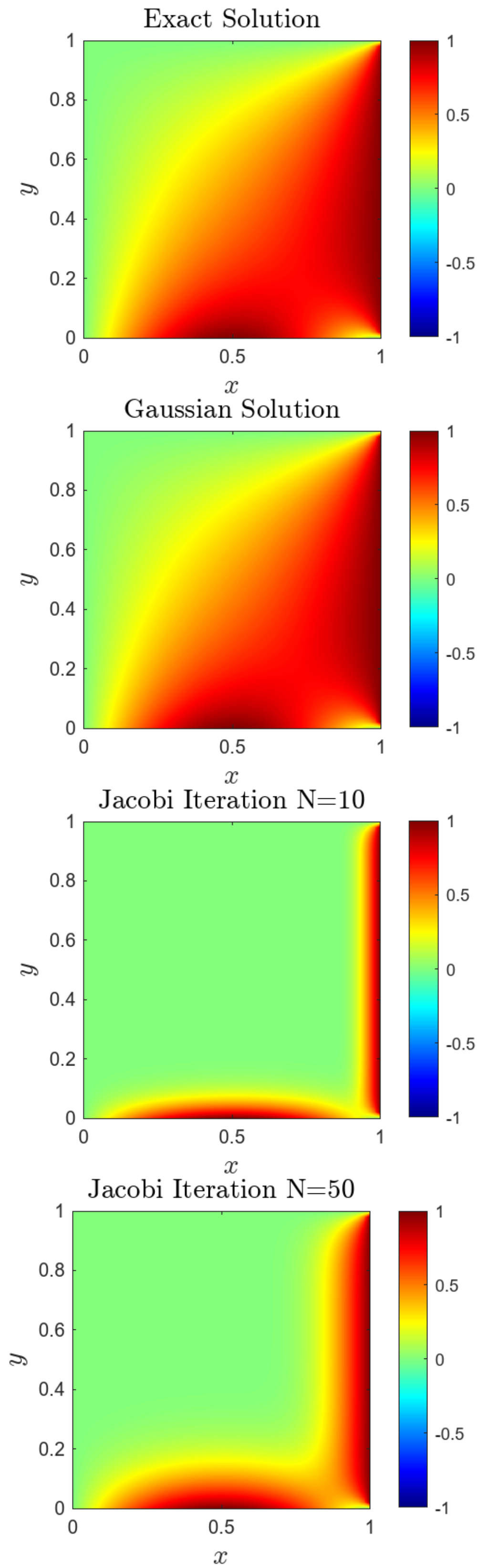


Figure 3.16: Solution of the Laplace equation example for $n = 50$ using the exact solution in Equation (3.34), the Gaussian solution, and the Jacobi iteration for $N = 10, 50$. The Gaussian solution has the same features as the exact siolution. However, we see that 10 or 50 iterations of the Jacobi scheme do not appear correct.

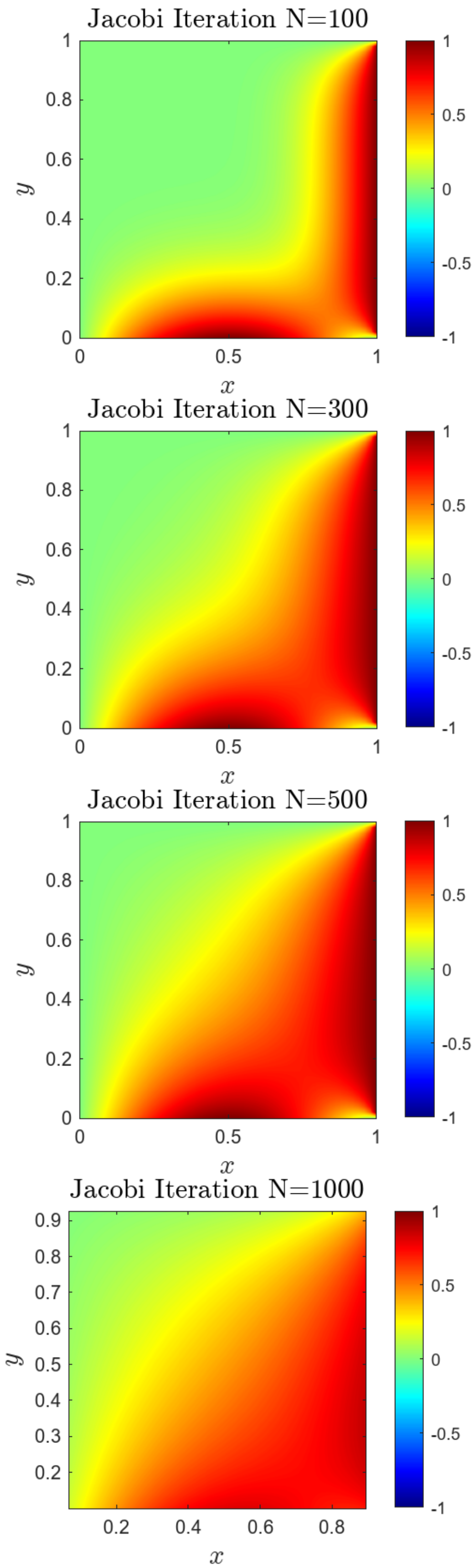


Figure 3.17: Several iterations of the Jacobi scheme for the solution of the 2D Laplace equation.

diagonal matrix D. Removing the diagonal will split the elements of A into those above the diagonal, part of an upper triangular matrix U, and those below the diagonal, part of an upper triangular matrix L. Thus, we can write $A = L + D + U$. Now, we can rewrite $A\Phi = b$ as

$$\begin{aligned} (L+D+U)\Phi &= b \\ D\Phi &= b-(L+U)\Phi \\ \Phi &= D^{-1}\left[b-(L+U)\Phi\right]. \end{aligned} \tag{3.38}$$

The iterative process then follows as

$$\Phi^{(k+1)} = D^{-1}\left[b-(L+U)\Phi^{(k)}\right], \quad k = 0,1,2,\ldots.$$

This is the Jacobi matrix iteration scheme. Convergence is based on fixed point theorems and we will not go into this here.

To implement this in MATLAB we first obtain the decomposition of matrix A. This is done with the following code:

```
% Obtain upper/lower/diagonal parts of A
[c,r] = meshgrid(1:size(A,1),1:size(A,2));
diagA =A;
diagA(c ~= r) = 0;
trilA = A-diagA;
trilA(c>r) = 0;
triuA = A-diagA;
triuA(c<r) = 0;
triuA;
```

Now, we can set up the scheme assuming we have already defined the boundary values as seen previously.

```
W=zeros(Nx+1,Ny+1);
W(1,:)=U(1,:);          % left
W(Ny+1,:)=U(Ny+1,:);    % right
W(:,1)=U(:,1);          % bottom
W(:,Nx+1)=U(:,Nx+1);    % top

% Jacobi
T = -diagA^(-1)*(trilA+triuA);
P=zeros(k*k,1);
M=1000;
for i=1:M
    P=T*P+diagA^(-1)*b';
end
W(2:k+1,2:k+1)=reshape(P,k,k)';
myplot(W',['Jacobi Matrix Iteration  N=',num2str(M)])
```

There are other ways to rearrange the matrices before performing the iteration. Consider the following:

$$(L+D+U)\Phi = b$$

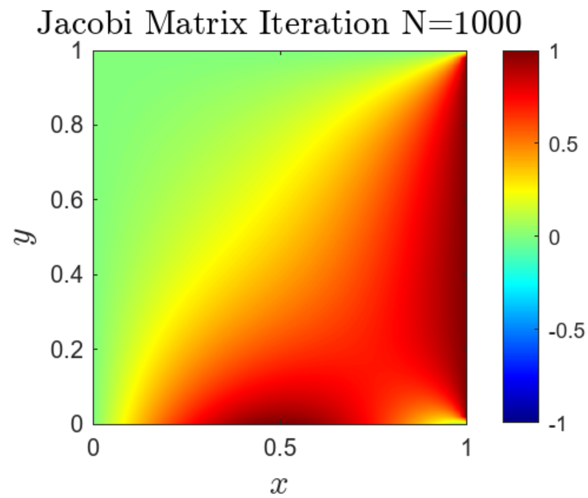


Figure 3.18: Solution of the Laplace equation example for $n = 50$ using the Jacobi Matrix iteration scheme for $N = 1000$ iterations.

$$\begin{aligned} (L+D)\Phi &= b - U\Phi \\ \Phi &= (L+D)^{-1}(b-U\Phi). \end{aligned} \tag{3.39}$$

The iterative process that results is called the Gauss-Seidel iterative scheme and is given by

$$\Phi^{(k+1)} = (L+D)^{-1}(b - U\Phi^{(k)}), \quad k = 0, 1, 2, \ldots.$$

We implement this in MATLAB:

```
% Gauss-Seidel
P=zeros(k*k,1);
M=500;
for i=1:M
    P=(trilA+diagA)^(-1)*(b'-triuA*P);
end
W(2:k+1,2:k+1)=reshape(P,k,k)';
myplot(W',['Gauss-Seidel Iteration  N=',num2str(M)])
```

We get the plot in Figure 3.19.

Finally, we move around the Gauss-Seidel terms.

$$\begin{aligned} \Phi^{(k+1)} &= (L+D)^{-1}(b - U\Phi^{(k)}) \\ (L+D)\Phi^{(k+1)} &= b - U\Phi^{(k)} \\ D\Phi^{(k+1)} &= -L\Phi^{(k+1)} + b - U\Phi^{(k)} \\ \Phi^{(k+1)} &= D^{-1}\left[b - L\Phi^{(k+1)} + b - U\Phi^{(k)}\right]. \end{aligned} \tag{3.40}$$

Now, subtract $\Phi^{(k)}$ to obtain the change in the approximation at each step of the iteration process.

$$\begin{aligned} \Phi^{(k+1)} - \Phi^{(k)} &= D^{-1}\left[b - L\Phi^{(k+1)} + b - U\Phi^{(k)}\right] - \Phi^{(k)} \\ \Phi^{(k+1)} - \Phi^{(k)} &= D^{-1}\left[b - L\Phi^{(k+1)} - U\Phi^{(k)} - D\Phi^{(k)}\right]. \end{aligned} \tag{3.41}$$

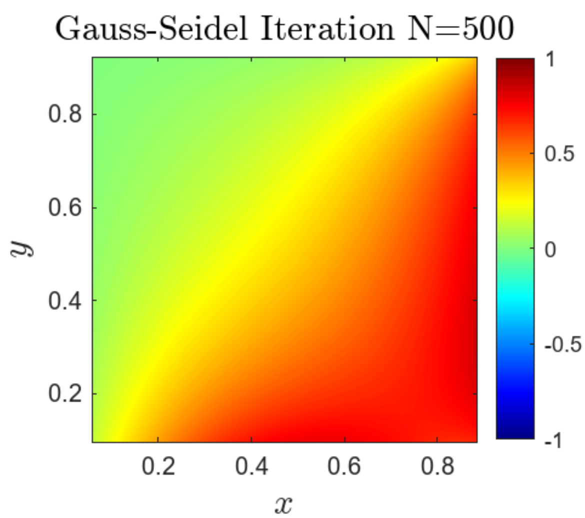


Figure 3.19: Solution of the Laplace equation example for $n = 50$ using the Gauss-Seidel iteration scheme for $N = 10$ iterations.

Instead of adding this correction to the previous step to get $\Phi^{(k+1)}$,

$$\Phi^{(k+1)} = \Phi^{(k)} + \left(\Phi^{(k+1)} - \Phi^{(k)}\right),$$

we can add a multiple of it,

$$\Phi^{(k+1)} = \Phi^{(k)} + \omega\left(\Phi^{(k+1)} - \Phi^{(k)}\right),$$

where if $0 < \omega < 1$ it is called under-relaxation and if $1 < \omega < 2$, it is called over-relaxation. In the later case, we develop a scheme called successive over-relaxation, or SOR.It is given in terms of matrices as

$$\Phi^{(k+1)} = \Phi^{(k)} + \omega D^{-1}\left[b - L\Phi^{(k+1)} - U\Phi^{(k)} - D\Phi^{(k)}\right]. \tag{3.42}$$

An equivalent approach is through a manipulation of the matrix equations. From $A\Phi = b$, we have

$$D\Phi = b - (L+U)\Phi$$

and for some constant ω we have $\omega A\Phi = \omega b$. Then, we write

$$\omega L\Phi = \omega b - \omega(D+U)\Phi.$$

Combining these two expressions, we obtain

$$\begin{aligned}(D+\omega L)\Phi &= \omega b + D\Phi - \omega(D+U)\Phi \\ &= \omega b - [\omega U + (\omega-1)D]\,\Phi\end{aligned}$$

$$\Phi = (D+\omega L)^{-1}\omega b - (D+\omega L)^{-1}[\omega U + (\omega-1)D]\,\Phi. \tag{3.43}$$

Writing $T = -(D+\omega L)^{-1}[\omega U + (\omega-1)D]$ and $c = (D+\omega L)^{-1}\omega b$, we can write the scheme as

$$\Phi^{(k+1)} = T\Phi^{(k)} + c.$$

The implementation is shown in the code and the solution plot is seen in Figure 3.20.

```
% SOR
omega=1.25;
c = (diagA+omega*trilA)^(-1)*b';
T = -(diagA+omega*trilA)^(-1)*(omega*triuA+(omega-1)*diagA);
P=zeros(k*k,1);
M=1000;
for i=1:M
    P=T*P+c;
end
W(2:k+1,2:k+1)=reshape(P,k,k)';
myplot(W',['SOR  ','$\omega =$',num2str(omega),' N=',num2str(M)])
```

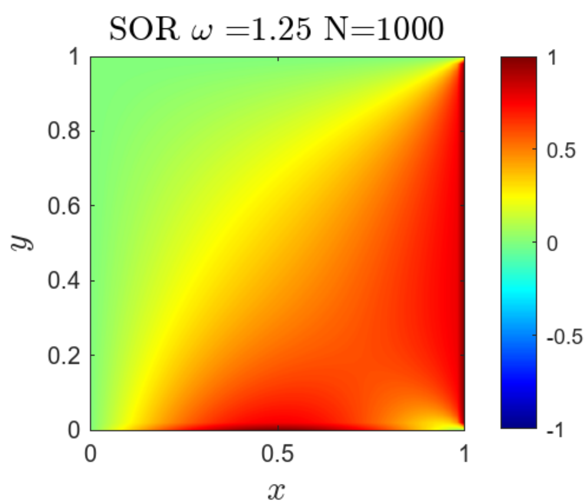


Figure 3.20: Solution of the Laplace equation example for $n = 3$ using the SOR iteration scheme for $N = 10$ iterations.

3.8 *Heat Equation Project*

Each group will be assigned a specific set of initial and boundary conditions and solve the two-dimensional heat equation both analytically and numerically. Let $u = u(x, y, t)$ satisfy the following:

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad 0 < x < L, \quad 0 < y < W, t > 0 \tag{3.44}$$

$$u(x, y, 0) = f(x, y) \qquad 0 < x < L, \quad 0 < y < W. \tag{3.45}$$

3.8.1 *1D Heat Equation*

For the first part of the project you will numerically solve the one-dimensional heat equation. Below is a copy of the MATLAB code you will be given to carry out this part of the project.

```
% Solution of the Heat Equation Using a Forward Difference Scheme

% Initialize Data
```

Group	L	W	k	$x=0$	$x=L$	$y=0$	$y=W$	$f(x,y)$
A	1	1	$\frac{1}{10}$	$u=0$	$u=0$	$u=0$	$u=0$	$xy(1-x)(1-y)$
B	2	1	$\frac{1}{5}$	$u=0$	$u=0$	$u=0$	$u_y=0$	$xy(2-x)\left(1-\frac{y}{2}\right)$
C	1	2	$\frac{1}{10}$	$u_x=0$	$u=0$	$u=0$	$u=0$	$y(1-x^2)(2-y)$
D	1	2	$\frac{1}{5}$	$u_x=0$	$u_x=0$	$u=0$	$u=0$	$x^2y^2\left(1-\frac{2}{3}x\right)(2-y)$
E	1	1	$\frac{1}{10}$	$u_x=0$	$u=0$	$u_y=0$	$u=0$	$(1-x^2)(1-y^2)$
F	1	1	$\frac{1}{5}$	$u_x=0$	$u=0$	$u=0$	$u_y=0$	$y(1-x^2)\left(1-\frac{y}{2}\right)$
G	1	1	$\frac{1}{5}$	$u_x=0$	$u_x=0$	$u=0$	$u_y=0$	$x^2y\left(1-\frac{2}{3}x\right)\left(1-\frac{y}{2}\right)$
H	2	2	$\frac{1}{15}$	$u=0$	$u_x=0$	$u=0$	$u=0$	$xy(4-x)(2-y)$
I	2	1	$\frac{1}{15}$	$u=0$	$u=0$	$u_y=0$	$u=0$	$x(2-x)(1-y^2)$

Table 3.1: These are the parameters and conditions needed for your assigned group: length, L, width, W, heat constant, k, boundary conditions, and initial condition.

```
%     Length of Rod, Time Interval
%     Number of Points in Space, Number of Time Steps
clear
L=1;
T=0.1;
k=1;
N=50;
M=500;
dx=L/N;
dt=T/M;
alpha=k*dt/dx^2;

t0 = cputime;  % Combine with t1 to time the routine

% Position
for i=1:N+1
    x(i)=(i-1)*dx;
end

% Initial Condition
for i=1:N+1
    u0(i)=x(i)*(1-x(i));
end

% Partial Difference Equation (Numerical Scheme)
for j=1:M
   for i=2:N
      u1(i)=u0(i)+alpha*(u0(i+1)-2*u0(i)+u0(i-1));
   end
```

```
   u1(1)=0;
   u1(N+1)=0;
   u0=u1;
   % Plot solution
   hold on
   if mod(j,10)==0
       plot(x, u1);
   end
   hold off
end

t1=cputime;
telapsed = t1-t0
```

3.8.2 2D Heat Equation

For the second part of the project you will solve the two-dimensional heat equation by constructing the solution to the initial-boundary value problem you are assigned from Table 1. You will find the product solutions $\phi_{n,m}(x,y,t)$ and the Fourier coefficients, $c_{n,m}$. Be careful as problems D and G also have coefficients $c_{0,m}$.

Below is a copy of MATLAB code for plotting the exact solution. Besides generating a 3D plot of the solution evolving in time, frames are captured and placed in a movie file. The movie can be played using one of the commands at the end of the file.

```
% Solution of the 2D Heat Equation Using the series solution.
% u_t = k (u_xx + u_yy)

% Initialize Data
%      L, W   = Length and Width of Playe,
%      T      = for Time Interval [0, T]
%      Nx, Ny = Number of Points in Space Grid,
%      M      = Number of Time Steps
%      dx, dy = Delta x and Delta y.
% Set your own values of L, W, T, k
clear
L=1;
W=2;
T=1;
k=1/10;
Nx=20;
Ny=20;
dx=L/Nx;
dy=W/Ny;
M=20;
dt = T/M;
```

```
% Spatial grid
[x,y]=meshgrid(0:dx:L,0:dy:W);

% Initialize u % Change to your initial condition
u=zeros(Nx+1,Ny+1);
for n=1:10
    for m=1:10
        u=u+sin(n*pi*x/L).*sin(m*pi*y/W)/n^2/m^2;
    end
end
H=max(max(u));

% Plot initial condition
surf(x,y,u,'FaceColor','red','EdgeColor','none')
camlight left;
lighting phong
xlabel('x')
ylabel('y')
title('Solution at t = 0')
axis([0,L,0,W,0,H])
frame=1;
Mov(frame)=getframe(gcf);
pause(0.5)

% Time evolution
for j=1:M
    u=zeros(Nx+1,Ny+1);
    t=j*dt;
    for n=1:10
        for m=1:10
            lambda=(n*pi/L)^2+(m*pi/W)^2;   % Change lambda
            u=u+sin(n*pi*x/L).*sin(m*pi*y/W)/n^2/m^2*exp(-k*
                lambda*t);
        end
    end

   % Plot 3D solution every 10 time steps
   %if mod(m,10)==0
       frame=frame+1;
       surf(x,y,u,'FaceColor','red','EdgeColor','none')
       camlight left;
       lighting phong
       xlabel('x')
       ylabel('y')
       title(['Solution at t = ' num2str(t)])
```

```
       axis([0,L,0,W,0,H])
       Mov(frame)=getframe(gcf);
       pause (0.5)
   %end
end

% Extra - show or create a movie
%    movie(gcf, Mov)         % plays movie
%    movie(gcf, Mov,1,2)     % plays at 2 fps
%    movie(gcf, Mov,10,5)    % repeats 10 times at 5 fps
% Create movie
%    movie2avi(Mov, 'heat2d.avi', 'compression', 'None');
```

One can also make use of the Symbolic Toolbox in MATLAB. The following defines the product solutions and computes the Fourier coefficients.

```
clear
% Problem Test
syms n x y
n = sym('n','integer');
m = sym('m','integer');

L = 1;
W = 1;
Kx = n*pi/L;
Ky = m*pi/W;

f = sin(pi*x/L)*sin(3*pi*y/W);
phi = sin(Kx*x)*sin(Ky*y);

% Fourier Coefficients
c=simplify(4/L/W*int(int(f*phi, x, [0 L]), y, [0 W]) );
\end{minted}

Now one can plot the initial condition
\begin{minted}{matlab}
% Initial Condition
N=5;
M=5;
H=.07;
ff=symsum(symsum(c*phi,n,1,N),m,1,M);
h=fsurf(ff,[0,L,0,W],'FaceColor','red','EdgeColor','none');
    camlight left;
    lighting phong
    xlabel('x')
    ylabel('y')
    title(['Solution at t = ' num2str(0)])
```

```
    axis([0,L,0,W,-1,1])
%frame=1;
%Mov(frame)=getframe(gcf);
\end{minted}

The solution can then be evolved in time.
\begin{minted}{matlab}
k=1/200;    % Heat constant

% Time Evolution
T =   1;   % Final time
Nt = 20;   % Number of steps

dt=T/Nt;
lambda = Kx^2+Ky^2;

for j=1:20
    t=j*dt;
%   frame=frame+1;
    ff=symsum(symsum(c*phi*exp(-k*lambda*t),n,1,N),m,1,M);
    %fsurf(ff,[0,L,0,W],'FaceColor','red','EdgeColor','none')
    h.Function=ff; % Alternative to using fsurf by updating the
        function plotted

    title(['Solution at t = ' num2str(t)])

%    Mov(frame)=getframe(gcf);
%    pause(.05)
end
```

Problems

1. Use the forward and backward difference formulae to find approximate values of $f'(x)$ given the following data.

a.

x	$f(x)$
1.1	0.4990
1.2	0.4348
1.3	0.3477
1.4	0.2380
1.5	0.1061

b.

x	$f(x)$
2.1	-1.7098
2.2	-1.3738
2.3	-1.1192
2.4	-0.9160
2.5	-0.747

c. The functions used to generate the tables we $f(x) = x \cos x$ and $\tan x$, respectively. What were the errors made in parts a and b?

2. Use the central difference approximation to find $f''(x)$ given the data in Problem 1. What error is made using this approximation?

3. Use $f(x_0)$, $f(x_0 \pm h)$, and $f(x_0 \pm 2h)$, for $h = \Delta x$, to find the most accurate approximation for $f'(x_0)$. What is the truncation error?

4. What does the finite difference expression $\dfrac{2u_{n,j} - 5u_{n-1,j} + 4u_{n-2,j} - u_{n-3,j}}{h^2}$ approximate?

5. Consider using a finite difference approximation for the wave equation, $u_{tt} = c^2 u_{xx}$ on $x \in [0, L]$.

a. Use centered differences in space and time to derive a finite difference model of the wave equation.

b. Draw the appropriate stencil for this difference equation and describe what conditions would be needed to proceed with numerically solving the wave equation.

c. Use the initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$ and boundary conditions $u(0, t) = u(L, t) = 0$ to fill in the missing details of the solution in part b.

d. What is the order of the truncation error for this scheme?

4
Sturm-Liouville Boundary Value Problems

"Ordinary language is totally unsuited for expressing what physics really asserts, since the words of everyday life are not sufficiently abstract. Only mathematics and mathematical logic can say as little as the physicist means to say." Bertrand Russell (1872-1970)

We have seen that trigonometric functions and special functions are the solutions of differential equations. These solutions give orthogonal sets of functions which can be used to represent functions in generalized Fourier series expansions. At the same time we would like to generalize the techniques we had first used to solve the heat equation in order to solve more general initial-boundary value problems. Namely, we use separation of variables to separate the given partial differential equation into a set of ordinary differential equations. A subset of those equations provide us with a set of boundary value problems whose eigenfunctions are useful in representing solutions of the partial differential equation. Hopefully, those solutions will form a useful basis in some function space.

A class of problems to which our previous examples belong are the Sturm-Liouville eigenvalue problems. These problems involve self-adjoint (differential) operators which play an important role in the spectral theory of linear operators and the existence of the eigenfunctions needed to solve the interesting physics problems described by the above initial-boundary value problems. In this section we will introduce the Sturm-Liouville eigenvalue problem as a general class of boundary value problems containing the Legendre and Bessel equations and supplying the theory needed to solve a variety of problems.

4.1 *Sturm-Liouville Operators*

In physics many problems arise in the form of boundary value problems involving second order ordinary differential equations. For example, in Chapter 6 we will explore the wave equation and the heat equation in three dimensions. Separating out the time dependence leads to a three dimensional boundary value problem in both cases. Further separation of variables leads to a set of boundary value problems involving second order ordinary differential equations.

In general, we might obtain equations of the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \tag{4.1}$$

for $y = y(x)$ subject to boundary conditions. We can write such an equation in operator form by defining the differential operator

$$L = a_2(x)D^2 + a_1(x)D + a_0(x),$$

where $D = d/dx$. Then, Equation (4.1) takes the form

$$Ly = f.$$

Recall that we had solved such nonhomogeneous differential equations in Chapter 1 and a review is in Section A.3. In this section we will show that these equations can be solved using eigenfunction expansions. Namely, we seek solutions to the eigenvalue problem

$$L\phi = \lambda\phi$$

with homogeneous boundary conditions on ϕ and then seek a solution of the nonhomogeneous problem, $Ly = f$, as an expansion over these eigenfunctions. Formally, we let

$$y(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

However, we are not guaranteed a nice set of eigenfunctions. We need an appropriate set to form a basis in the function space. Also, it would be nice to have orthogonality so that we can easily solve for the expansion coefficients.

It turns out that any linear second order differential operator can be turned into an operator that possesses just the right properties (self-adjointedness) to carry out this procedure. The resulting operator is referred to as a Sturm-Liouville operator. We will highlight some of the properties of these operators and see how they are used in applications.

The Sturm-Liouville operator.

We define the Sturm-Liouville operator as

$$\mathcal{L} = \frac{d}{dx} p(x) \frac{d}{dx} + q(x). \tag{4.2}$$

The Sturm-Liouville eigenvalue problem.

The Sturm-Liouville eigenvalue problem is given by the differential equation

$$\mathcal{L}y = -\lambda\sigma(x)y,$$

or

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y + \lambda\sigma(x)y = 0, \tag{4.3}$$

for $x \in (a,b)$, $y = y(x)$, plus boundary conditions. The functions $p(x)$, $p'(x)$, $q(x)$ and $\sigma(x)$ are assumed to be continuous on (a,b) and $p(x) > 0$, $\sigma(x) > 0$ on $[a,b]$. If the interval is finite and these assumptions on the coefficients are true on $[a,b]$, then the problem is said to be a regular Sturm-Liouville problem. Otherwise, it is called a singular Sturm-Liouville problem.

We also need to impose the set of homogeneous boundary conditions

$$\begin{aligned} \alpha_1 y(a) + \beta_1 y'(a) = 0, \\ \alpha_2 y(b) + \beta_2 y'(b) = 0. \end{aligned} \tag{4.4}$$

The α's and β's are constants. For different values, one has special types of boundary conditions. For $\beta_i = 0$, we have what are called Dirichlet boundary conditions. Namely, $y(a) = 0$ and $y(b) = 0$. For $\alpha_i = 0$, we have Neumann boundary conditions. In this case, $y'(a) = 0$ and $y'(b) = 0$. In terms of the heat equation example, Dirichlet conditions correspond to maintaining a fixed temperature at the ends of the rod. The Neumann boundary conditions would correspond to no heat flow across the ends, or insulating conditions, as there would be no temperature gradient at those points. The more general boundary conditions allow for partially insulated boundaries.

Types of boundary conditions:
Dirichlet boundary conditions - the solution takes fixed values on the boundary. These are named after Gustav Lejeune Dirichlet (1805-1859).

Neumann boundary conditions - the derivative of the solution takes fixed values on the boundary. These are named after Carl Neumann (1832-1925).

Robin boundary conditions, $ay(0) + by'(0) = 0$, named after Victor Gustave Robin (1855–1897).

Another type of boundary condition that is often encountered is the periodic boundary condition. Consider the heated rod that has been bent to form a circle. Then the two end points are physically the same. So, we would expect that the temperature and the temperature gradient should agree at those points. For this case we write $y(a) = y(b)$ and $y'(a) = y'(b)$. Boundary value problems using these conditions have to be handled differently than the above homogeneous conditions. These conditions leads to different types of eigenfunctions and eigenvalues.

Differential equations of Sturm-Liouville form.

As previously mentioned, equations of the form (4.1) occur often. We now show that any second order linear operator can be put into the form of the Sturm-Liouville operator. In particular, equation (4.1) can be put into the form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = F(x). \tag{4.5}$$

The proof of this is straight forward as we soon show. Let's first consider the equation (4.1) for the case that $a_1(x) = a_2'(x)$. Then, we can write the equation in a form in which the first two terms combine,

$$\begin{aligned} f(x) &= a_2(x)y'' + a_1(x)y' + a_0(x)y \\ &= (a_2(x)y')' + a_0(x)y. \end{aligned} \tag{4.6}$$

The resulting equation is now in Sturm-Liouville form. We just identify $p(x) = a_2(x)$ and $q(x) = a_0(x)$.

Not all second order differential equations are as simple to convert. Consider the differential equation

$$x^2y'' + xy' + 2y = 0.$$

In this case $a_2(x) = x^2$ and $a_2'(x) = 2x \neq a_1(x)$. So, this does not fall into this case. However, we can change the operator in this equation, $x^2D + xD$, to a Sturm-Liouville operator, $Dp(x)D$ for a $p(x)$ that depends on the coefficients x^2 and x.

In the Sturm Liouville operator the derivative terms are gathered together into one exact derivative, $Dp(x)D$. This is similar to what we saw for ODE [see Section A.1.2] when we solved linear first order equations. In that case we sought an integrating factor. We can do the same thing here. We seek a multiplicative function $\mu(x)$ that we can multiply through (4.1) so that it can be written in Sturm-Liouville form.

We first divide out the $a_2(x)$, giving

$$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = \frac{f(x)}{a_2(x)}.$$

Next, we multiply this differential equation by and integrating factor, μ,

$$\mu(x)y'' + \mu(x)\frac{a_1(x)}{a_2(x)}y' + \mu(x)\frac{a_0(x)}{a_2(x)}y = \mu(x)\frac{f(x)}{a_2(x)}.$$

The first two terms can now be combined into an exact derivative $(\mu y')'$ if the second coefficient is $\mu'(x)$. Therefore, $\mu(x)$ satisfies a first order, separable differential equation:

$$\frac{d\mu}{dx} = \mu(x)\frac{a_1(x)}{a_2(x)}.$$

This is formally solved to give the sought integrating factor[1]

$$\mu(x) = e^{\int \frac{a_1(x)}{a_2(x)}\,dx}.$$

Thus, the original equation can be multiplied by factor

$$\frac{\mu(x)}{a_2(x)} = \frac{1}{a_2(x)}e^{\int \frac{a_1(x)}{a_2(x)}\,dx}$$

to turn it into Sturm-Liouville form.

In summary,

Equation (4.1),

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x), \tag{4.7}$$

can be put into the Sturm-Liouville form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = F(x), \tag{4.8}$$

where

$$\begin{aligned} p(x) &= e^{\int \frac{a_1(x)}{a_2(x)}\,dx}, \\ q(x) &= p(x)\frac{a_0(x)}{a_2(x)}, \\ F(x) &= p(x)\frac{f(x)}{a_2(x)}. \end{aligned} \tag{4.9}$$

[1] Actually, we should write this as

$$\mu(x) = e^{\int^x \frac{a_1(\xi)}{a_2(\xi)}\,d\xi}.$$

This distinguishes the variable of integration ξ from the evaluation at x.

Example 4.1. Convert $x^2y'' + xy' + 2y = 0$ into Sturm-Liouville form.

We multiply this equation by

$$\frac{\mu(x)}{a_2(x)} = \frac{1}{x^2}e^{\int \frac{dx}{x}} = \frac{1}{x},$$

to put the equation in Sturm-Liouville form:

$$\begin{aligned} 0 &= xy'' + y' + \frac{2}{x}y \\ &= (xy')' + \frac{2}{x}y. \end{aligned} \quad (4.10)$$

Conversion of a linear second order differential equation to Sturm Liouville form.

We could also proceed as follows. Divide by x^2, giving

$$y'' + \frac{1}{x}y' + \frac{2}{x}y = 0.$$

Now find

$$\mu(x) = \int^x \frac{dx}{x} = \frac{1}{x}.$$

Multiply the original equation by $\mu(x)$ and rewrite the equation.

4.2 *Properties of Sturm-Liouville Eigenvalue Problems*

There are several properties that can be proven for the (regular) Sturm-Liouville eigenvalue problem in (4.3). However, we will not prove them all here. We will merely list some of the important facts and focus on a few of the properties.

Real, countable eigenvalues.

1. The eigenvalues are real, countable, ordered and there is a smallest eigenvalue. Thus, we can write them as $\lambda_1 < \lambda_2 < \ldots$. However, there is no largest eigenvalue and $n \to \infty$, $\lambda_n \to \infty$.

Oscillatory eigenfunctions.

2. For each eigenvalue λ_n there exists an eigenfunction ϕ_n with $n-1$ zeros on (a, b).
3. Eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the weight function, $\sigma(x)$. Defining the inner product of $f(x)$ and $g(x)$ as

$$\langle f, g \rangle = \int_a^b f(x)g(x)\sigma(x)\,dx, \quad (4.11)$$

   then the orthogonality of the eigenfunctions can be written in the form

$$\langle \phi_n, \phi_m \rangle = \langle \phi_n, \phi_n \rangle \delta_{nm}, \quad n, m = 1, 2, \ldots. \quad (4.12)$$

Orthogonality of eigenfunctions.

4. The set of eigenfunctions is complete; i.e., any piecewise smooth function can be represented by a generalized Fourier series expansion of the eigenfunctions,

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x),$$

   where

$$c_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

   Actually, one needs $f(x) \in L^2_\sigma(a, b)$, the set of square integrable functions over $[a, b]$ with weight function $\sigma(x)$. By square integrable, we mean that $\langle f, f \rangle < \infty$. One can show that such a space is isomorphic to a Hilbert space, a complete inner product space. Hilbert spaces play a special role in quantum mechanics.

Complete basis of eigenfunctions.

5. The eigenvalues satisfy the Rayleigh quotient

$$\lambda_n = \frac{-p\phi_n \frac{d\phi_n}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{d\phi_n}{dx}\right)^2 - q\phi_n^2 \right] dx}{\langle \phi_n, \phi_n \rangle}.$$

The Rayleigh quotient is named after Lord Rayleigh, John William Strutt, 3rd Baron Raleigh (1842-1919).

This is verified by multiplying the eigenvalue problem

$$\mathcal{L}\phi_n = -\lambda_n \sigma(x) \phi_n$$

by ϕ_n and integrating. Solving this result for λ_n, we obtain the Rayleigh quotient. The Rayleigh quotient is useful for getting estimates of eigenvalues and proving some of the other properties.

Example 4.2. Verify some of these properties for the eigenvalue problem

$$y'' = -\lambda y, \quad y(0) = y(\pi) = 0.$$

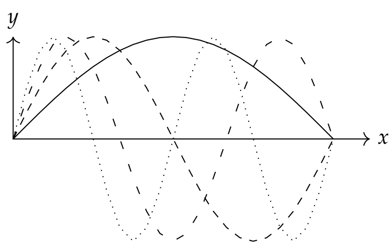


Figure 4.1: Plot of the eigenfunctions $\phi_n(x) = \sin nx$ for $n = 1, 2, 3, 4$.

This is a problem we had seen many times. The eigenfunctions for this eigenvalue problem are $\phi_n(x) = \sin nx$, with eigenvalues $\lambda_n = n^2$ for $n = 1, 2, \ldots$. These satisfy the properties listed above.

First of all, the eigenvalues are real, countable and ordered, $1 < 4 < 9 < \ldots$. There is no largest eigenvalue and there is a first one.

The eigenfunctions corresponding to each eigenvalue have $n - 1$ zeros on $(0, \pi)$. This is demonstrated for several eigenfunctions in Figure 4.1.

We also know from Chapter 2 that the set $\{\sin nx\}_{n=1}^{\infty}$ is an orthogonal set of basis functions of length[2]

$$\|\phi_n\| = \sqrt{\frac{\pi}{2}}.$$

[2] Here the length, or norm, of ϕ is defined by

$$\|\phi\|^2 = \langle \phi, \phi \rangle.$$

Thus, the Rayleigh quotient can be computed using $p(x) = 1$, $q(x) = 0$, and the eigenfunctions. It is given by

$$\begin{aligned} R &= \frac{-\phi_n \phi_n' \Big|_0^\pi + \int_0^\pi (\phi_n')^2 \, dx}{\frac{\pi}{2}} \\ &= \frac{2}{\pi} \int_0^\pi \left(-n^2 \cos nx\right)^2 dx = n^2. \end{aligned} \tag{4.13}$$

Therefore, knowing the eigenfunction, the Rayleigh quotient returns the eigenvalues as expected.

Example 4.3. 4.9 We seek the eigenfunctions of the operator found in Example 4.1. Namely, we want to solve the eigenvalue problem

$$\mathcal{L}y = (xy')' + \frac{2}{x} y = -\lambda \sigma y \tag{4.14}$$

subject to a set of homogeneous boundary conditions. Let's use the boundary conditions

$$y'(1) = 0, \quad y'(2) = 0.$$

[Note that we do not know $\sigma(x)$ yet, but will choose an appropriate function to obtain solutions. Why didn't we give a condition at $x = 0$?]

Expanding the derivative, we have

$$xy'' + y' + \frac{2}{x}y = -\lambda\sigma y.$$

Multiply through by x to obtain

$$x^2y'' + xy' + (2 + \lambda x\sigma)\,y = 0.$$

Notice that if we choose $\sigma(x) = x^{-1}$, then this equation can be made a Cauchy-Euler type equation. [see Section A.4.] Thus, we have

$$x^2y'' + xy' + (\lambda + 2)\,y = 0.$$

The characteristic equation, found assuming solutions of the form $y(x) = x^r$, is

$$r^2 + \lambda + 2 = 0.$$

For oscillatory solutions, we need $\lambda + 2 > 0$. Thus, the general solution is

$$y(x) = c_1\cos(\sqrt{\lambda+2}\ln|x|) + c_2\sin(\sqrt{\lambda+2}\ln|x|). \qquad (4.15)$$

See the Appendix for a review of Cauchy-Euler equations.

Next we apply the boundary conditions. $y'(1) = 0$ forces $c_2 = 0$. This leaves

$$y(x) = c_1\cos(\sqrt{\lambda+2}\ln x).$$

The second condition, $y'(2) = 0$, yields

$$\sin(\sqrt{\lambda+2}\ln 2) = 0.$$

This will give nontrivial solutions when

$$\sqrt{\lambda+2}\ln 2 = n\pi, \quad n = 0, 1, 2, 3 \ldots.$$

In summary, the eigenfunctions for this eigenvalue problem are

$$y_n(x) = \cos\left(\frac{n\pi}{\ln 2}\ln x\right), \quad 1 \le x \le 2$$

and the eigenvalues are $\lambda_n = \left(\frac{n\pi}{\ln 2}\right)^2 - 2$ for $n = 0, 1, 2, \ldots$.

Note: We include the $n = 0$ case because $y(x) =$ constant is a solution of the $\lambda = -2$ case. More specifically, in this case the characteristic equation reduces to $r^2 = 0$. Thus, the general solution of this Cauchy-Euler equation is

$$y(x) = c_1 + c_2\ln|x|.$$

Setting $y'(1) = 0$, forces $c_2 = 0$. $y'(2)$ automatically vanishes, leaving the solution in this case as $y(x) = c_1$.

We note that some of the properties listed in the beginning of the section hold for this example. The eigenvalues are seen to be real, countable and ordered. There is a least one, $\lambda_0 = -2$. Next, one can find the zeros of each eigenfunction on [1,2]. Then the argument of the cosine, $\frac{n\pi}{\ln 2}\ln x$, takes values 0 to $n\pi$ for $x \in [1,2]$. The cosine function has $n-1$ roots on this interval.

Orthogonality can be checked as well. We set up the integral and use the substitution $y = \pi \ln x / \ln 2$. This gives the inner product 4.11 as

$$\begin{aligned}\langle y_n, y_m \rangle &= \int_1^2 \cos\left(\frac{n\pi}{\ln 2}\ln x\right)\cos\left(\frac{m\pi}{\ln 2}\ln x\right)\frac{dx}{x} \\ &= \frac{\ln 2}{\pi}\int_0^\pi \cos ny \cos my\, dy \\ &= \frac{\ln 2}{2}\delta_{n,m}. \end{aligned} \tag{4.16}$$

4.2.1 *Adjoint Operators*

In the study of the spectral theory of matrices, one learns about the adjoint of the matrix, $A^\dagger$, and the role that self-adjoint, or Hermitian, matrices play in diagonalization. Also, one needs the concept of adjoint to discuss the existence of solutions to the matrix problem $\mathbf{y} = A\mathbf{x}$. In the same spirit, one is interested in the existence of solutions of the operator equation $Lu = f$ and solutions of the corresponding eigenvalue problem. The study of linear operators on a Hilbert space is a generalization of what the reader had seen in a linear algebra course.

Just as one can find a basis of eigenvectors and diagonalize Hermitian, or self-adjoint, matrices (or, real symmetric in the case of real matrices), we will see that the Sturm-Liouville operator is self-adjoint. In this section we will define the domain of an operator and introduce the notion of adjoint operators. In the last section we discuss the role the adjoint plays in the existence of solutions to the operator equation $Lu = f$.

Adjoint of an operator.

We begin by defining the adjoint of an operator. The adjoint, $L^\dagger$, of operator L satisfies

$$\langle u, Lv \rangle = \langle L^\dagger u, v \rangle$$

for all v in the domain of L and u in the domain of $L^\dagger$. Here the domain of a differential operator L is the set of all $u \in L^2_\sigma(a,b)$ satisfying a given set of homogeneous boundary conditions. This is best understood through example.

Example 4.4. Find the adjoint of $L = a_2(x)D^2 + a_1(x)D + a_0(x)$ for $D = d/dx$.

In order to find the adjoint, we place the operator inside an integral. Consider the inner product

$$\langle u, Lv \rangle = \int_a^b u(a_2 v'' + a_1 v' + a_0 v)\, dx.$$

We have to move the operator L from v and determine what operator is acting on u in order to formally preserve the inner product. For a simple operator like $L = \frac{d}{dx}$, this is easily done using integration by parts. For the given operator, we will need to apply several integrations by parts to the individual terms. We consider each derivative term in the integrand separately.

For the $a_1 v'$ term, we integrate by parts to find

$$\int_a^b u(x)a_1(x)v'(x)\,dx = a_1(x)u(x)v(x)\Big|_a^b - \int_a^b (u(x)a_1(x))'v(x)\,dx. \quad (4.17)$$

Now, we consider the $a_2 v''$ term. In this case it will take two integrations by parts:

$$\begin{aligned} \int_a^b u(x)a_2(x)v''(x)\,dx &= a_2(x)u(x)v'(x)\Big|_a^b - \int_a^b (u(x)a_2(x))'v(x)'\,dx \\ &= \left[a_2(x)u(x)v'(x) - (a_2(x)u(x))'v(x)\right]\Big|_a^b \\ &\quad + \int_a^b (u(x)a_2(x))''v(x)\,dx. \end{aligned} \quad (4.18)$$

Combining these results, we obtain

$$\begin{aligned} \langle u, Lv\rangle &= \int_a^b u(a_2 v'' + a_1 v' + a_0 v)\,dx \\ &= \left[a_1(x)u(x)v(x) + a_2(x)u(x)v'(x) - (a_2(x)u(x))'v(x)\right]\Big|_a^b \\ &\quad + \int_a^b \left[(a_2 u)'' - (a_1 u)' + a_0 u\right] v\,dx. \end{aligned} \quad (4.19)$$

Before inserting the boundary conditions for v, one has to determine boundary conditions for u such that

$$\left[a_1(x)u(x)v(x) + a_2(x)u(x)v'(x) - (a_2(x)u(x))'v(x)\right]\Big|_a^b = 0.$$

This leaves

$$\langle u, Lv\rangle = \int_a^b \left[(a_2 u)'' - (a_1 u)' + a_0 u\right] v\,dx \equiv \langle L^\dagger u, v\rangle.$$

Therefore,

$$L^\dagger = \frac{d^2}{dx^2}a_2(x) - \frac{d}{dx}a_1(x) + a_0(x). \quad (4.20)$$

Self-adjoint operators.

When $L^\dagger = L$, the operator is called formally self-adjoint, or Hermitian. When the domain of L is the same as the domain of $L^\dagger$, the term self-adjoint is used. As the domain is important in establishing self-adjointness, we need to do a complete example in which the domain of the adjoint is found.

Example 4.5. Determine $L^\dagger$ and its domain for operator $Lu = \frac{du}{dx}$ where u satisfies the boundary conditions $u(0) = 2u(1)$ on $[0, 1]$.

We need to find the adjoint operator satisfying $\langle v, Lu\rangle = \langle L^\dagger v, u\rangle$. Therefore, we rewrite the integral

$$\langle v, Lu\rangle = \int_0^1 v\frac{du}{dx}\,dx = uv\big|_0^1 - \int_0^1 u\frac{dv}{dx}\,dx = \langle L^\dagger v, u\rangle.$$

From this we have the adjoint problem consisting of an adjoint operator and the associated boundary condition (or, domain of $L^\dagger$.):

1. $L^\dagger = -\frac{d}{dx}$.
2. $uv\Big|_0^1 = 0 \Rightarrow 0 = u(1)[v(1) - 2v(0)] \Rightarrow v(1) = 2v(0)$.

4.2.2 *Lagrange's and Green's Identities*

Before turning to the proofs that the eigenvalues of a Sturm-Liouville problem are real and the associated eigenfunctions orthogonal, we will first need to introduce two important identities. For the Sturm-Liouville operator,

$$\mathcal{L} = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q,$$

we have the two identities:

Lagrange's Identity:	$u\mathcal{L}v - v\mathcal{L}u$	$= [p(uv' - vu')]'$.
Green's Identity:	$\int_a^b (u\mathcal{L}v - v\mathcal{L}u)\,dx$	$= [p(uv' - vu')]\vert_a^b$.

The proof of Lagrange's identity follows by a simple manipulations of the operator:

$$\begin{aligned} u\mathcal{L}v - v\mathcal{L}u &= u\left[\frac{d}{dx}\left(p\frac{dv}{dx}\right) + qv\right] - v\left[\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu\right] \\ &= u\frac{d}{dx}\left(p\frac{dv}{dx}\right) - v\frac{d}{dx}\left(p\frac{du}{dx}\right) \\ &= u\frac{d}{dx}\left(p\frac{dv}{dx}\right) + p\frac{du}{dx}\frac{dv}{dx} - v\frac{d}{dx}\left(p\frac{du}{dx}\right) - p\frac{du}{dx}\frac{dv}{dx} \\ &= \frac{d}{dx}\left[pu\frac{dv}{dx} - pv\frac{du}{dx}\right]. \end{aligned} \tag{4.21}$$

Green's identity is simply proven by integrating Lagrange's identity.

4.2.3 *Orthogonality and Reality*

We are now ready to prove that the eigenvalues of a Sturm-Liouville problem are real and the corresponding eigenfunctions are orthogonal. These are easily established using Green's identity, which in turn is a statement about the Sturm-Liouville operator being self-adjoint.

Example 4.6. The eigenvalues of the Sturm-Liouville problem (4.3) are real.

Let $\phi_n(x)$ be a solution of the eigenvalue problem associated with λ_n:

$$\mathcal{L}\phi_n = -\lambda_n\sigma\phi_n.$$

We want to show that the eigenvalues are real. Namely, we show that $\overline{\lambda}_n = \lambda_n$, where the bar means complex conjugate. So, we also consider the complex conjugate of this eigenvalue problem,

$$\mathcal{L}\overline{\phi}_n = -\overline{\lambda}_n \sigma \overline{\phi}_n.$$

Now, multiply the first equation by $\overline{\phi}_n$, the second equation by ϕ_n, and then subtract the results. We obtain

$$\overline{\phi}_n \mathcal{L}\phi_n - \phi_n \mathcal{L}\overline{\phi}_n = (\overline{\lambda}_n - \lambda_n)\sigma\phi_n\overline{\phi}_n.$$

Integrating both sides of this equation, we have

$$\int_a^b \left(\overline{\phi}_n \mathcal{L}\phi_n - \phi_n \mathcal{L}\overline{\phi}_n\right) dx = (\overline{\lambda}_n - \lambda_n) \int_a^b \sigma\phi_n\overline{\phi}_n \, dx.$$

We apply Green's identity to the left hand side to find

$$[p(\overline{\phi}_n\phi_n' - \phi_n\overline{\phi}_n')]|_a^b = (\overline{\lambda}_n - \lambda_n) \int_a^b \sigma\phi_n\overline{\phi}_n \, dx.$$

Using the homogeneous boundary conditions (4.4) for a self-adjoint operator, the left side vanishes. This leaves

$$0 = (\overline{\lambda}_n - \lambda_n) \int_a^b \sigma \left|\phi_n\right|^2 \, dx.$$

The integral is positive, so we must have $\overline{\lambda}_n = \lambda_n$. Therefore, the eigenvalues are real.

Example 4.7. The eigenfunctions corresponding to different eigenvalues of the Sturm-Liouville problem (4.3) are orthogonal.

This is proven similar to the last example. Let $\phi_n(x)$ be a solution of the eigenvalue problem associated with λ_n,

$$\mathcal{L}\phi_n = -\lambda_n\sigma\phi_n,$$

and let $\phi_m(x)$ be a solution of the eigenvalue problem associated with $\lambda_m \neq \lambda_n$,

$$\mathcal{L}\phi_m = -\lambda_m\sigma\phi_m,$$

Now, multiply the first equation by ϕ_m and the second equation by ϕ_n. Subtracting these results, we obtain

$$\phi_m\mathcal{L}\phi_n - \phi_n\mathcal{L}\phi_m = (\lambda_m - \lambda_n)\sigma\phi_n\phi_m$$

Integrating both sides of the equation, using Green's identity with homogeneous boundary conditions, we obtain

$$0 = (\lambda_m - \lambda_n) \int_a^b \sigma\phi_n\phi_m \, dx.$$

Since the eigenvalues are distinct, we can divide by $\lambda_m - \lambda_n$, leaving the desired result,

$$\int_a^b \sigma\phi_n\phi_m \, dx = 0.$$

Therefore, the eigenfunctions are orthogonal with respect to the weight function $\sigma(x)$.

4.2.4 *The Rayleigh Quotient*

The Rayleigh quotient is useful for getting estimates of eigenvalues and proving some of the other properties associated with Sturm-Liouville eigenvalue problems. The Rayleigh quotient is general and finds applications for both matrix eigenvalue problems as well as self-adjoint operators. For a Hermitian matrix M the Rayleigh quotient is given by

$$R(\mathbf{v}) = \frac{\langle \mathbf{v}, M\mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

One can show that the critical values of the Rayleigh quotient, as a function of $\mathbf{v}$, are the eigenvectors of M and the values of R at these critical values are the corresponding eigenvectors. In particular, minimizing $R(\mathbf{v})$ over the vector space will give the lowest eigenvalue. This leads to the Rayleigh-Ritz method for computing the lowest eigenvalues when the eigenvectors are not known.

This definition can easily be extended to Sturm-Liouville operators,

$$R(\phi_n) = \frac{\langle \phi_n \mathcal{L} \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

We begin by multiplying the eigenvalue problem,

$$\mathcal{L}\phi_n = -\lambda_n \sigma(x) \phi_n,$$

by ϕ_n and integrating. This gives

$$\int_a^b \left[\phi_n \frac{d}{dx} \left(p \frac{d\phi_n}{dx} \right) + q\phi_n^2 \right] dx = -\lambda_n \int_a^b \phi_n^2 \sigma \, dx.$$

One can solve the last equation for λ to find

$$\lambda_n = \frac{-\int_a^b \left[\phi_n \frac{d}{dx} \left(p \frac{d\phi_n}{dx} \right) + q\phi_n^2 \right] dx}{\int_a^b \phi_n^2 \sigma \, dx} = R(\phi_n).$$

It appears that we have solved for the eigenvalues and have not needed the machinery used in studying boundary value problems. However, we really cannot evaluate this expression when we do not know the eigenfunctions, $\phi_n(x)$. Nevertheless, we will see what we can determine from the Rayleigh quotient.

One can rewrite this result by performing an integration by parts on the first term in the numerator. Namely, pick $u = \phi_n$ and $dv = \frac{d}{dx}\left(p \frac{d\phi_n}{dx} \right) dx$ for the standard integration by parts formula. Then, we have

$$\int_a^b \phi_n \frac{d}{dx} \left(p \frac{d\phi_n}{dx} \right) dx = p\phi_n \frac{d\phi_n}{dx} \Big|_a^b - \int_a^b \left[p \left(\frac{d\phi_n}{dx} \right)^2 - q\phi_n^2 \right] dx.$$

Inserting the new formula into the expression for λ, leads to the Rayleigh Quotient

$$\lambda_n = \frac{-p\phi_n \frac{d\phi_n}{dx} \Big|_a^b + \int_a^b \left[p \left(\frac{d\phi_n}{dx} \right)^2 - q\phi_n^2 \right] dx}{\int_a^b \phi_n^2 \sigma \, dx}. \tag{4.22}$$

In many applications the sign of the eigenvalue is important. As we had seen in the solution of the heat equation, $T' + k\lambda T = 0$. Since we expect the heat energy to diffuse, the solutions should decay in time. Thus, we would expect $\lambda > 0$. In studying the wave equation, one expects vibrations and these are only possible with the correct sign of the eigenvalue (positive again). Thus, in order to have nonnegative eigenvalues, we see from (4.22) that

a. $q(x) \leq 0$, and

b. $-p\phi_n \frac{d\phi_n}{dx}\Big|_a^b \geq 0$.

Furthermore, if λ is a zero eigenvalue, then $q(x) \equiv 0$ and $\alpha_1 = \alpha_2 = 0$ in the homogeneous boundary conditions. This can be seen by setting the numerator equal to zero. Then, $q(x) = 0$ and $\phi_n'(x) = 0$. The second of these conditions inserted into the boundary conditions forces the restriction on the type of boundary conditions.

One of the properties of Sturm-Liouville eigenvalue problems with homogeneous boundary conditions is that the eigenvalues are ordered, $\lambda_1 < \lambda_2 < \ldots$. Thus, there is a smallest eigenvalue. It turns out that for any continuous function, $y(x)$,

$$\lambda_1 = \min_{y(x)} \frac{-py\frac{dy}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{dy}{dx}\right)^2 - qy^2\right] dx}{\int_a^b y^2\sigma\, dx} \tag{4.23}$$

and this minimum is obtained when $y(x) = \phi_1(x)$. This result can be used to get estimates of the minimum eigenvalue by using trial functions which are continuous and satisfy the boundary conditions, but do not necessarily satisfy the differential equation.

Example 4.8. We have already solved the eigenvalue problem $\phi'' + \lambda\phi = 0$, $\phi(0) = 0$, $\phi(1) = 0$. In this case, the lowest eigenvalue is $\lambda_1 = \pi^2$. We can pick a nice function satisfying the boundary conditions, say $y(x) = x - x^2$. Inserting this into Equation (4.23), we find

$$\lambda_1 \leq \frac{\int_0^1 (1-2x)^2\, dx}{\int_0^1 (x-x^2)^2\, dx} = 10.$$

Indeed, $10 \geq \pi^2$.

The only condition we placed on our trial function was that it satisfied the boundary conditions. Thus, this procedure would work with other functions obtaining bounds on the eigenvalue. With a good choice, one might get a better estimate of the true value.

4.3 The Eigenfunction Expansion Method

In this section we solve the nonhomogeneous problem $\mathcal{L}y = f$ using expansions over the basis of Sturm-Liouville eigenfunctions. We have seen that Sturm-Liouville eigenvalue problems have the requisite set of orthogonal eigenfunctions. In this section we will apply the eigenfunction expansion method to solve a particular nonhomogeneous boundary value problem.

Recall that one starts with a nonhomogeneous differential equation

$$\mathcal{L}y = f,$$

where $y(x)$ is to satisfy given homogeneous boundary conditions. The method makes use of the eigenfunctions satisfying the eigenvalue problem

$$\mathcal{L}\phi_n = -\lambda_n \sigma \phi_n$$

subject to the given boundary conditions. Then, one assumes that $y(x)$ can be written as an expansion in the eigenfunctions,

$$y(x) = \sum_{n=1}^{\infty} c_n \phi_n(x),$$

and inserts the expansion into the nonhomogeneous equation. This gives

$$f(x) = \mathcal{L}\left(\sum_{n=1}^{\infty} c_n \phi_n(x)\right) = -\sum_{n=1}^{\infty} c_n \lambda_n \sigma(x) \phi_n(x).$$

The expansion coefficients are then found by making use of the orthogonality of the eigenfunctions. Namely, we multiply the last equation by $\phi_m(x)$ and integrate. We obtain

$$\int_a^b f(x)\phi_m(x)\,dx = -\sum_{n=1}^{\infty} c_n \lambda_n \int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx.$$

Orthogonality yields

$$\int_a^b f(x)\phi_m(x)\,dx = -c_m \lambda_m \int_a^b \phi_m^2(x)\sigma(x)\,dx.$$

Solving for c_m, we have

$$c_m = -\frac{\int_a^b f(x)\phi_m(x)\,dx}{\lambda_m \int_a^b \phi_m^2(x)\sigma(x)\,dx}.$$

Example 4.9. As an example, we consider the solution of the boundary value problem

$$(xy')' + \frac{y}{x} = \frac{1}{x}, \quad x \in [1, e], \tag{4.24}$$

$$y(1) = 0 = y(e). \tag{4.25}$$

This equation is already in self-adjoint form. So, we know that the associated Sturm-Liouville eigenvalue problem has an orthogonal set of eigenfunctions. We first determine this set. Namely, we need to solve the eigenvalue problem

$$(x\phi')' + \frac{\phi}{x} = -\lambda\sigma\phi, \quad \phi(1) = 0 = \phi(e). \tag{4.26}$$

Rearranging the terms and multiplying by x, we have that

$$x^2\phi'' + x\phi' + (1 + \lambda\sigma x)\phi = 0.$$

This is almost an equation of Cauchy-Euler type similar to Example . Picking the weight function $\sigma(x) = \frac{1}{x}$, we have

$$x^2\phi'' + x\phi' + (1 + \lambda)\phi = 0.$$

This is easily solved. From Section A.4, the characteristic equation is

$$r^2 + (1 + \lambda) = 0.$$

One obtains nontrivial solutions [3] of the eigenvalue problem satisfying the boundary conditions when $\lambda > -1$. The solutions are

$$\phi_n(x) = A\sin(n\pi\ln x), \quad n = 1, 2, \ldots.$$

where $\lambda_n = n^2\pi^2 - 1$.

[3] The general solution is

$$\phi(x) = A\sin(\beta\ln x) + A\cos(\beta\ln x),$$

where $\beta = \sqrt{(1+\lambda)}$. The boundary conditions given that $B = 0$ and $\sin\beta = 0$. So, $\beta = n\pi$, $n = 1, 2, \ldots$.

It is often useful to normalize the eigenfunctions. This means that one chooses A so that the norm of each eigenfunction is one. Thus, we have

$$\begin{aligned} 1 &= \int_1^e \phi_n(x)^2\sigma(x)\,dx \\ &= A^2\int_1^e \sin(n\pi\ln x)\frac{1}{x}\,dx \\ &= A^2\int_0^1 \sin(n\pi y)\,dy = \frac{1}{2}A^2. \end{aligned} \tag{4.27}$$

Thus, $A = \sqrt{2}$. Several of these eigenfunctions are show in Figure 4.2.

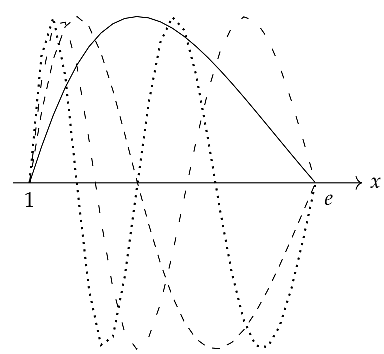


Figure 4.2: Plots of the first five eigenfunctions, $y(x) = \sqrt{2}\sin(n\pi\ln x)$.

We now turn towards solving the nonhomogeneous problem, $\mathcal{L}y = \frac{1}{x}$. We first expand the unknown solution in terms of the normalized eigenfunctions,

$$y(x) = \sum_{n=1}^{\infty} c_n\sqrt{2}\sin(n\pi\ln x).$$

Inserting this solution into the differential equation, we have

$$\frac{1}{x} = \mathcal{L}y = -\sum_{n=1}^{\infty} c_n\lambda_n\sqrt{2}\sin(n\pi\ln x)\frac{1}{x}.$$

Next, we make use of orthogonality. Multiplying both sides by the eigenfunction $\phi_m(x) = \sqrt{2}\sin(m\pi\ln x)$ and integrating, gives

$$\lambda_m c_m = \int_1^e \sqrt{2}\sin(m\pi\ln x)\frac{1}{x}\,dx = \frac{\sqrt{2}}{m\pi}[(-1)^m - 1].$$

Solving for c_m, we have

$$c_m = \frac{\sqrt{2}}{m\pi}\frac{[(-1)^m - 1]}{m^2\pi^2 - 1}.$$

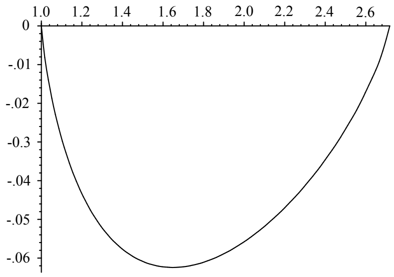


Figure 4.3: Plot of the solution in Example 4.9.

Finally, we insert these coefficients into the expansion for $y(x)$. The solution is then

$$y(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \frac{[(-1)^n - 1]}{n^2\pi^2 - 1} \sin(n\pi \ln(x)).$$

We plot this solution in Figure 4.3. In Problem 11 you can directly solve this problem and compare your solution with the solution in Figure 4.3.

4.4 *Appendix: The Fredholm Alternative Theorem - Optional*

Given that $Ly = f$, when can one expect to find a solution? Is it unique? These questions are answered by the Fredholm Alternative Theorem. This theorem occurs in many forms from a statement about solutions to systems of algebraic equations to solutions of boundary value problems and integral equations. The theorem comes in two parts, thus the term "alternative". Either the equation has exactly one solution for all f, or the equation has many solutions for some f's and none for the rest.

The reader is familiar with the statements of the Fredholm Alternative for the solution of systems of algebraic equations. One seeks solutions of the system $Ax = b$ for A an $n \times m$ matrix. Defining the matrix adjoint, A^* through $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x, y, \in \mathcal{C}^n$, then either

Theorem 4.1. First Alternative

The equation $Ax = b$ has a solution if and only if $\langle b, v\rangle = 0$ for all v satisfying $A^*v = 0$.

or

Theorem 4.2. Second Alternative

A solution of $Ax = b$, if it exists, is unique if and only if $x = 0$ is the only solution of $Ax = 0$.

The second alternative is more familiar when given in the form: The solution of a nonhomogeneous system of n equations and n unknowns is unique if the only solution to the homogeneous problem is the zero solution. Or, equivalently, A is invertible, or has nonzero determinant.

Proof. We prove the second theorem first. Assume that $Ax = 0$ for $x \neq 0$ and $Ax_0 = b$. Then $A(x_0 + \alpha x) = b$ for all α. Therefore, the solution is not unique. Conversely, if there are two different solutions, x_1 and x_2, satisfying $Ax_1 = b$ and $Ax_2 = b$, then one has a nonzero solution $x = x_1 - x_2$ such that $Ax = A(x_1 - x_2) = 0$.

The proof of the first part of the first theorem is simple. Let $A^*v = 0$ and $Ax_0 = b$. Then we have

$$\langle b, v\rangle = \langle Ax_0, v\rangle = \langle x_0, A^*v\rangle = 0.$$

For the second part we assume that $\langle b, v\rangle = 0$ for all v such that $A^*v = 0$. Write b as the sum of a part that is in the range of A and a part that in the space orthogonal to the range of A, $b = b_R + b_O$. Then, $0 = \langle b_O, Ax\rangle = \langle A^*b, x\rangle$ for all x. Thus, $A^*b_O = 0$. Since $\langle b, v\rangle = 0$ for all v in the nullspace of A^*, then $\langle b, b_O\rangle = 0$.

Therefore, $\langle b, v\rangle = 0$ implies that

$$0 = \langle b, b_O\rangle = \langle b_R + b_O, b_O\rangle = \langle b_O, b_O\rangle.$$

This means that $b_O = 0$, giving $b = b_R$ is in the range of A. So, $Ax = b$ has a solution. □

Example 4.10. Determine the allowed forms of $\mathbf{b}$ for a solution of $A\mathbf{x} = \mathbf{b}$ to exist, where

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}.$$

First note that $A^* = \overline{A}^T$. This is seen by looking at

$$\begin{aligned} \langle A\mathbf{x}, \mathbf{y}\rangle &= \langle \mathbf{x}, A^*\mathbf{y}\rangle \\ \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_j \bar{y}_i &= \sum_{j=1}^n x_j \sum_{j=1}^n a_{ij} \bar{y}_i \\ &= \sum_{j=1}^n x_j \overline{\sum_{j=1}^n (\bar{a}^T)_{ji}\, y_i}. \end{aligned} \tag{4.28}$$

For this example,

$$A^* = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}.$$

We next solve $A^*\mathbf{v} = 0$. This means, $v_1 + 3v_2 = 0$. So, the nullspace of A^* is spanned by $\mathbf{v} = (3, -1)^T$. For a solution of $A\mathbf{x} = \mathbf{b}$ to exist, $\mathbf{b}$ would have to be orthogonal to $\mathbf{v}$. Therefore, a solution exists when

$$\mathbf{b} = \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

So, what does the Fredholm Alternative say about solutions of boundary value problems? We extend the Fredholm Alternative for linear operators. A more general statement would be

Theorem 4.3. *If L is a bounded linear operator on a Hilbert space, then $Ly = f$ has a solution if and only if $\langle f, v\rangle = 0$ for every v such that $L^\dagger v = 0$.*

The statement for boundary value problems is similar. However, we need to be careful to treat the boundary conditions in our statement. As we have seen, after several integrations by parts we have that

$$\langle \mathcal{L}u, v\rangle = S(u, v) + \langle u, \mathcal{L}^\dagger v\rangle,$$

where $S(u, v)$ involves the boundary conditions on u and v. Note that for nonhomogeneous boundary conditions, this term may no longer vanish.

Theorem 4.4. *The solution of the boundary value problem $\mathcal{L}u = f$ with boundary conditions $Bu = g$ exists if and only if*

$$\langle f, v\rangle - S(u, v) = 0$$

for all v satisfying $\mathcal{L}^\dagger v = 0$ and $B^\dagger v = 0$.

Example 4.11. Consider the problem

$$u'' + u = f(x), \quad u(0) - u(2\pi) = \alpha, u'(0) - u'(2\pi) = \beta.$$

Only certain values of α and β will lead to solutions. We first note that

$$L = L^\dagger = \frac{d^2}{dx^2} + 1.$$

Solutions of

$$L^\dagger v = 0, \quad v(0) - v(2\pi) = 0, v'(0) - v'(2\pi) = 0$$

are easily found to be linear combinations of $v = \sin x$ and $v = \cos x$.

Next, one computes

$$\begin{aligned} S(u, v) &= \left[u'v - uv'\right]_0^{2\pi} \\ &= u'(2\pi)v(2\pi) - u(2\pi)v'(2\pi) - u'(0)v(0) + u(0)v'(0). \end{aligned} \tag{4.29}$$

For $v(x) = \sin x$, this yields

$$S(u, \sin x) = -u(2\pi) + u(0) = \alpha.$$

Similarly,

$$S(u, \cos x) = \beta.$$

Using $\langle f, v\rangle - S(u, v) = 0$, this leads to the conditions that we were seeking,

$$\int_0^{2\pi} f(x) \sin x \, dx = \alpha,$$

$$\int_0^{2\pi} f(x) \cos x \, dx = \beta.$$

Problems

1. Prove the if $u(x)$ and $v(x)$ satisfy the general homogeneous boundary conditions

$$\begin{aligned} \alpha_1 u(a) + \beta_1 u'(a) &= 0, \\ \alpha_2 u(b) + \beta_2 u'(b) &= 0 \end{aligned} \tag{4.30}$$

at $x = a$ and $x = b$, then

$$p(x)[u(x)v'(x) - v(x)u'(x)]_{x=a}^{x=b} = 0.$$

2. Prove Green's Identity $\int_a^b (u\mathcal{L}v - v\mathcal{L}u)\,dx = [p(uv' - vu')]\Big|_a^b$ for the general Sturm-Liouville operator $\mathcal{L}$.

3. Find the adjoint operator and its domain for $Lu = u'' + 4u' - 3u$, $u'(0) + 4u(0) = 0$, $u'(1) + 4u(1) = 0$.

4. Show that a Sturm-Liouville operator with periodic boundary conditions on $[a, b]$ is self-adjoint if and only if $p(a) = p(b)$. [Recall, periodic boundary conditions are given as $u(a) = u(b)$ and $u'(a) = u'(b)$.]

5. The Hermite differential equation is given by $y'' - 2xy' + \lambda y = 0$. Rewrite this equation in self-adjoint form. From the Sturm-Liouville form obtained, verify that the differential operator is self adjoint on $(-\infty, \infty)$. Give the integral form for the orthogonality of the eigenfunctions.

6. Find the eigenvalues and eigenfunctions of the given Sturm-Liouville problems.

a. $y'' + \lambda y = 0$, $y'(0) = 0 = y'(\pi)$.

b. $y'' + 2y' + y + \lambda y = 0$, $y(0) = 0 = y(1)$.

c. $(xy')' + \frac{\lambda}{x} y = 0$, $y(1) = y(e^2) = 0$.

d. $y'' + \lambda y = 0$, $2y(0) - y'(0) = 0$, $y'(1) = 0$. In this problem you will need a computer to find the first two eigenvalues.

7. Consider the Sturm-Liouville problem $y'' + \lambda y = 0$, $y(0) + y'(0) = 0$, $y(1) + 2y'(1) = 0$.

a. Can $\lambda = 0$?

b. Show that for $\lambda < 0$ there is only one solution. Determine this solution numerically to three decimal places.

c. Show that for $\lambda > 0$ there are an infinite number of eigenvalues. Numerically find the first three to three decimal places.

8. The eigenvalue problem $x^2 y'' - \lambda x y' + \lambda y = 0$ with $y(1) = y(2) = 0$ is not a Sturm-Liouville eigenvalue problem. Show that none of the eigenvalues are real by solving this eigenvalue problem.

9. Consider the eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = y'(0), \quad y(1) = \alpha y'(1).$$

For what value of α is $\lambda = 0$ an eigenvalue? What is the corresponding eigenfunction?

10. In Example 4.8 we found a bound on the lowest eigenvalue for the given eigenvalue problem.

a. Verify the computation in the example.

b. Apply the method using

$$y(x) = \begin{cases} x, & 0 < x < \frac{1}{2} \\ 1 - x, & \frac{1}{2} < x < 1. \end{cases}$$

Is this an upper bound on λ_1

c. Use the Rayleigh quotient to obtain a good upper bound for the lowest eigenvalue of the eigenvalue problem: $\phi'' + (\lambda - x^2)\phi = 0$, $\phi(0) = 0$, $\phi'(1) = 0$.

11. One can rewrite the problem in Example 4.9 as

$$x^2 y''(x) + xy'(x) + y(x) = 1, \quad y(1) = y(e) = 0.$$

Find the closed form solution and plot the solutions with the eigenfunction expansion in Example 4.9 to verify that the solutions are the same.

12. Use the method of eigenfunction expansions to solve the problems:

a. $y'' = x^2, \quad y(0) = y(1) = 0.$

b. $y'' + 4y = x^2, \quad y'(0) = y'(1) = 0.$

Note that these problems can also be solved directly. Find the solutions and plot these solutions with the eigenfunction expansions to verify that the solutions are the same.

13. Determine the solvability conditions for the nonhomogeneous boundary value problem: $u'' + 4u = f(x)$, $u(0) = \alpha$, $u'(\pi/4) = \beta$.

5
Non-sinusoidal Harmonics and Special Functions

"To the pure geometer the radius of curvature is an incidental characteristic - like the grin of the Cheshire cat. To the physicist it is an indispensable characteristic. It would be going too far to say that to the physicist the cat is merely incidental to the grin. Physics is concerned with interrelatedness such as the interrelatedness of cats and grins. In this case the "cat without a grin" and the "grin without a cat" are equally set aside as purely mathematical phantasies." Sir Arthur Stanley Eddington (1882-1944)

In this chapter we provide a glimpse into generalized Fourier series in which the normal modes of oscillation are not sinusoidal. For vibrating strings, we saw that the harmonics were sinusoidal basis functions for a large, infinite dimensional, function space. Now, we will extend these ideas to non-sinusoidal harmonics and explore the underlying structure behind these ideas. In particular, we will explore Legendre polynomials and Bessel functions which will later arise in problems having cylindrical or spherical symmetry.

The background for the study of generalized Fourier series is that of function spaces. We begin by exploring the general context in which one finds oneself when discussing Fourier series and (later) Fourier transforms. We can view the sine and cosine functions in the Fourier trigonometric series representations as basis vectors in an infinite dimensional function space. A given function in that space may then be represented as a linear combination over this infinite basis. With this in mind, we might wonder

- Do we have enough basis vectors for the function space?
- Are the infinite series expansions convergent?
- What functions can be represented by such expansions?

In the context of the boundary value problems which typically appear in mathematics and physics, one is led to the study of boundary value problems in the form of Sturm-Liouville eigenvalue problems. These lead to an appropriate set of basis vectors for the function space under consideration. We will touch a little on these ideas, leaving some of the deeper results for more advanced courses in mathematics. For now, we will turn to function spaces and explore some typical basis functions, many which originated from the study of physical problems. The common basis functions are often referred to as special functions in physics. Examples are the classical orthogonal polynomials (Legendre, Hermite, Laguerre, Tchebychef) and Bessel functions. But first we will introduce function spaces.

5.1 Function Spaces

In a course on linear algebra one studies finite dimensional vector spaces. Given a set of basis vectors, $\{\mathbf{a}_k\}_{k=1}^n$, in vector space V, we can expand any vector $\mathbf{v} \in V$ in terms of this basis, $\mathbf{v} = \sum_{k=1}^n v_k \mathbf{a}_k$. We can then extract the components v_k of the vector. The keys to doing this simply are to have a scalar product and an orthogonal basis set. We define the scalar product between two vectors $\mathbf{u}$ and $\mathbf{v}$ in V as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{k=1}^{n} u_k v_k.$$

We assume the basis is an orthogonal basis,

$$\langle \mathbf{a}_i, \mathbf{a}_j \rangle = N_i \delta_{ij},$$

where the Kronecker delta is defined as

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases} \tag{5.1}$$

We can extract the components v_k of the vector $\mathbf{v}$ by taking scalar product with the basis vectors. Thus, for $j = 1, 2, \ldots, n$,

$$\begin{aligned} \langle \mathbf{a}_j, \mathbf{v} \rangle &= \langle \mathbf{a}_j, \sum_{k=1}^{n} v_k \mathbf{a}_k. \rangle \\ &= \sum_{k=1}^{n} v_k \langle \mathbf{a_j}, \mathbf{a_k} \rangle \\ &= \sum_{k=1}^{n} v_k N_k \delta_{jk} \\ &= N_j v_j. \end{aligned} \tag{5.2}$$

Therefore, $v_j = N_j^{-1} \langle \mathbf{a}_j, \mathbf{v} \rangle$, for $j = 1, 2, \ldots, n$. These are the key ingredients that we will need in the infinite dimensional case. In fact, we had already done this when we studied Fourier series as we will show.

We note that the above determination of vector components for finite dimensional spaces is precisely what we had done to compute the Fourier coefficients using trigonometric bases. Reading further, you will see how this works.

Recall when we found Fourier trigonometric series representations of functions, we started with a function (vector) that we wanted to expand in a set of trigonometric functions (basis) and we sought the Fourier coefficients (components). In this section we will extend our notions from finite dimensional spaces to infinite dimensional spaces and we will develop the needed background in which to think about more general Fourier series expansions. This conceptual framework is very important in other areas in mathematics (such as ordinary and partial differential equations) and physics (such as quantum mechanics and electrodynamics).

We will consider various infinite dimensional function spaces. Functions in these spaces would differ by their properties. For example, we could consider the space of continuous functions on [0,1], the space of differentiably continuous functions, or the set of functions integrable from a to b. As you will see, there are many types of function spaces . In order to view these spaces as vector spaces, we will need to be able to add functions and multiply them by scalars in such as way that they satisfy the definition of a vector space.[1]

[1] A vector space V over a field F is a set that is closed under addition and scalar multiplication and satisfies the following conditions:

For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $a, b \in F$

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
3. There exists a $\mathbf{0}$ such that $\mathbf{0} + \mathbf{v} = \mathbf{v}$.
4. There exists an additive inverse, $-\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

   There are also several distributive properties:
5. $a(b\mathbf{v}) = (ab)\mathbf{v}$.
6. $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.
7. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
8. There exists a multiplicative identity, 1, such that $1(\mathbf{v}) = \mathbf{v}$.

We will also need a scalar product defined on this space of functions. Recall that a scalar product of two vectors is the generalization of the dot product and is given by

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{k=1}^{n} u_k v_k. \tag{5.3}$$

Since $\langle \mathbf{u}, \mathbf{u} \rangle$ gives the square of the length of $\mathbf{u}$, we define the length, or Euclidean norm, of $\mathbf{u}$ as $\|\mathbf{u}\|^2 = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$. We will develop analogous definitions for what are termed inner product spaces.

There are several types of scalar products, or inner products, that we can define. An inner product $\langle , \rangle$ on a real vector space V is a mapping from $V \times V$ into R such that for $u, v, w \in V$ and $\alpha \in R$ one has

1. $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$.
2. $\langle v, w \rangle = \langle w, v \rangle$.
3. $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$.
4. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$.

A real vector space equipped with the above inner product leads to what is called a real inner product space. For complex inner product spaces the above properties hold with the third property replaced with $\langle v, w \rangle = \overline{\langle w, v \rangle}$.

For the time being, we will only deal with real valued functions and, thus, we will need an inner product appropriate for such spaces. One such definition is the following. Let $f(x)$ and $g(x)$ be functions defined on $[a, b]$ and introduce the weight function $\sigma(x) > 0$. Then, we define the inner product, if the integral exists, as

$$\langle f, g \rangle = \int_a^b f(x) g(x) \sigma(x)\, dx. \tag{5.4}$$

The space of square integrable functions.

Spaces in which $\langle f, f \rangle < \infty$ under this inner product are called the space of square integrable functions on (a, b) under weight σ and denoted as $L^2_\sigma(a, b)$. In what follows, we will assume for simplicity that $\sigma(x) = 1$. This is possible to do by using a change of variables.

Now that we have function spaces equipped with an inner product, we seek a basis for the space. For an n-dimensional space we need n basis vectors. For an infinite dimensional space, how many will we need? How do we know when we have enough? We will provide some answers to these questions later.

Let's assume that we have a basis of functions $\{\phi_n(x)\}_{n=1}^{\infty}$. Given a function $f(x)$, how can we go about finding the components of f in this basis? In other words, let

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

How do we find the c_n's? Does it remind you of the problem we had earlier for finite dimensional spaces? Does this remind you of Fourier series expansions in Chapter 2?

Formally, we take the inner product of f with each ϕ_j and use the properties of the inner product to find

$$\begin{aligned} \langle \phi_j, f \rangle &= \langle \phi_j, \sum_{n=1}^{\infty} c_n \phi_n \rangle \\ &= \sum_{n=1}^{\infty} c_n \langle \phi_j, \phi_n \rangle. \end{aligned} \tag{5.5}$$

If the basis is an orthogonal basis, then we can write

$$\langle \phi_j, \phi_n \rangle = N_j \delta_{jn}, \tag{5.6}$$

where δ_{jn} is the Kronecker delta.

For the generalized Fourier series expansion $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$, we have determined the generalized Fourier coefficients to be $c_j = \langle \phi_j, f \rangle / \langle \phi_j, \phi_j \rangle$.

Continuing with the derivation, we have

$$\begin{aligned} \langle \phi_j, f \rangle &= \sum_{n=1}^{\infty} c_n \langle \phi_j, \phi_n \rangle \\ &= \sum_{n=1}^{\infty} c_n N_j \delta_{jn} \end{aligned} \tag{5.7}$$

Expanding the sum, we see that the Kronecker delta picks out one nonzero term:

$$\begin{aligned} \langle \phi_j, f \rangle &= c_1 N_j \delta_{j1} + c_2 N_j \delta_{j2} + \ldots + c_j N_j \delta_{jj} + \ldots \\ &= c_j N_j. \end{aligned} \tag{5.8}$$

So, the expansion coefficients are

$$c_j = \frac{\langle \phi_j, f \rangle}{N_j} = \frac{\langle \phi_j, f \rangle}{\langle \phi_j, \phi_j \rangle} \quad j = 1, 2, \ldots.$$

We summarize this important result:

Generalized Basis Expansion

Let $f(x)$ be represented by an expansion over a basis of orthogonal functions, $\{\phi_n(x)\}_{n=1}^{\infty}$,

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

Then, the expansion coefficients are formally determined as

$$c_n = \frac{\langle \phi_n, f \rangle}{\langle \phi_n, \phi_n \rangle}.$$

This will be referred to as the general Fourier series expansion and the c_j's are called the (generalized) Fourier coefficients. Technically, equality only holds when the infinite series converges to the given function on the interval of interest.

Example 5.1. Find the coefficients of the Fourier sine series expansion of $f(x)$, given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad x \in [-\pi, \pi].$$

In a previous chapter we established that the set of functions $\phi_n(x) = \sin nx$ for $n = 1, 2, \ldots$ is orthogonal on the interval $[-\pi, \pi]$. Recall that using trigonometric identities, we have for $n \neq m$

$$\langle \phi_n, \phi_m \rangle = \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \pi \delta_{nm}. \tag{5.9}$$

Therefore, the set $\phi_n(x) = \sin nx$ for $n = 1, 2, \ldots$ is an orthogonal set of functions on the interval $[-\pi, \pi]$.

We determine the expansion coefficients using

$$b_n = \frac{\langle \phi_n, f \rangle}{N_n} = \frac{\langle \phi_n, f \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

Does this result look familiar?

Just as with vectors in three dimensions, we can normalize these basis functions to arrive at an orthonormal basis. This is simply done by dividing by the length of the vector. Recall from earlier that the length of a vector is obtained as $v = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. In the same way, we define the norm of a function by

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

Note, there are many types of norms, but this induced norm will be sufficient.[2]

For this example, the norms of the basis functions in this example are $\|\phi_n\| = \sqrt{\pi}$. Defining $\psi_n(x) = \frac{1}{\sqrt{\pi}} \phi_n(x)$, we can normalize the ϕ_n's and obtain an orthonormal basis of functions $\psi_n(x)$ on $[-\pi, \pi]$.

[2] The norm defined here is the natural, or induced, norm on the inner product space. Norms are a generalization of the concept of lengths of vectors. Denoting $\|\mathbf{v}\|$ the norm of $\mathbf{v}$, it needs to satisfy the properties

1. $\|\mathbf{v}\| \geq 0$. $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
2. $\|\alpha \mathbf{v}\| = |\alpha| \|\mathbf{v}\|$.
3. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

Examples of common norms are

1. Euclidean norm:
$$\|\mathbf{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}.$$
2. Taxicab norm:
$$\|\mathbf{v}\| = |v_1| + \cdots + |v_n|.$$
3. L^p norm:
$$\|f\| = \left(\int [f(x)]^p \, dx \right)^{\frac{1}{p}}.$$

We can also use the normalized basis to determine the expansion coefficients. In this case, we begin with

$$f(x) = \sum_{n=1}^{\infty} c_n \psi_n(x) = \sum_{n=1}^{\infty} \frac{c_n}{\sqrt{\pi}} \sin nx, \quad x \in [-\pi, \pi].$$

Since $N_n = 1$, we have

$$c_n = \langle \psi_n, f \rangle = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

In both cases we have found that $f(x)$ can be written as

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(\xi) \sin n\xi \, d\xi \right) \sin nx.$$

5.2 *Classical Orthogonal Polynomials*

There are other basis functions that can be used to develop series representations of functions. In this section we introduce the classical orthogonal polynomials. We begin by noting that the sequence of functions $\{1, x, x^2, \ldots\}$ is a basis of linearly independent functions. In fact, by the Stone-Weierstraß Approximation Theorem[3] this set is a basis of $L^2_\sigma(a, b)$, the space of square integrable functions over the interval $[a, b]$ relative to weight $\sigma(x)$. However, we will show that the sequence of functions $\{1, x, x^2, \ldots\}$ does not provide an orthogonal basis for these spaces. We will then proceed to find an appropriate orthogonal basis of functions.

[3] **Stone-Weierstraß Approximation Theorem** Suppose f is a continuous function defined on the interval $[a, b]$. For every $\epsilon > 0$, there exists a polynomial function $P(x)$ such that for all $x \in [a, b]$, we have $|f(x) - P(x)| < \epsilon$. Therefore, every continuous function defined on $[a, b]$ can be uniformly approximated as closely as we wish by a polynomial function.

We are familiar with being able to expand functions over a basis such as $\{1, x, x^2, \ldots\}$, since these expansions are just Maclaurin series representations of the functions about $x = 0$,

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^n.$$

However, this basis is not an orthogonal basis of functions. One can easily see this by integrating the product of two even, or two odd, basis functions with $\sigma(x) = 1$ and (a, b)=$(-1, 1)$. For example,

$$\int_{-1}^{1} x^0 x^2 \, dx = \frac{2}{3}.$$

The Gram-Schmidt Orthogonalization Process.

Since we have found that orthogonal bases have been useful in determining the coefficients for expansions of given functions, we might ask, "Given a set of linearly independent basis vectors, can one find an orthogonal basis of the given space?" The answer is yes. We recall from introductory linear algebra, which mostly covers finite dimensional vector spaces, that there is a method for carrying this out called the Gram-Schmidt Orthogonalization Process. We will review this process for finite dimensional vectors and then generalize to function spaces.

Let's assume that we have three vectors that span the usual three dimensional space, $\mathbf{R}^3$, given by $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ and shown in Figure 5.1. We seek an orthogonal basis $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, beginning one vector at a time.

First we take one of the original basis vectors, say $\mathbf{a}_1$, and define

$$\mathbf{e}_1 = \mathbf{a}_1.$$

It is sometimes useful to normalize these basis vectors, denoting such a normalized vector with a "hat":

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_1}{e_1},$$

where $e_1 = \sqrt{\mathbf{e}_1 \cdot \mathbf{e}_1}$.

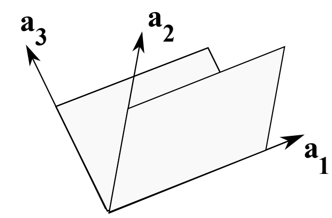


Figure 5.1: The basis $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, of $\mathbf{R}^3$.

Next, we want to determine an $\mathbf{e}_2$ that is orthogonal to $\mathbf{e}_1$. We take another element of the original basis, $\mathbf{a}_2$. In Figure 5.2 we show the orientation of the vectors. Note that the desired orthogonal vector is $\mathbf{e}_2$. We can now write $\mathbf{a}_2$ as the sum of $\mathbf{e}_2$ and the projection of $\mathbf{a}_2$ on $\mathbf{e}_1$. Denoting this projection by $\mathbf{pr}_1\mathbf{a}_2$, we then have

$$\mathbf{e}_2 = \mathbf{a}_2 - \mathbf{pr}_1\mathbf{a}_2. \tag{5.10}$$

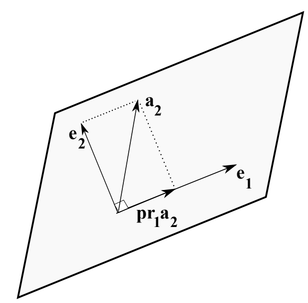


Figure 5.2: A plot of the vectors $\mathbf{e}_1$, $\mathbf{a}_2$, and $\mathbf{e}_2$ needed to find the projection of $\mathbf{a}_2$, on $\mathbf{e}_1$.

Recall the projection of one vector onto another from your vector calculus class.

$$\mathbf{pr}_1\mathbf{a}_2 = \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1. \tag{5.11}$$

This is easily proven by writing the projection as a vector of length $a_2 \cos\theta$ in direction $\hat{\mathbf{e}}_1$, where θ is the angle between $\mathbf{e}_1$ and $\mathbf{a}_2$. Using the definition of the dot product, $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$, the projection formula follows.

Combining Equations (5.10)-(5.11), we find that

$$\mathbf{e}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1. \tag{5.12}$$

It is a simple matter to verify that $\mathbf{e}_2$ is orthogonal to $\mathbf{e}_1$:

$$\begin{aligned} \mathbf{e}_2 \cdot \mathbf{e}_1 &= \mathbf{a}_2 \cdot \mathbf{e}_1 - \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1 \cdot \mathbf{e}_1 \\ &= \mathbf{a}_2 \cdot \mathbf{e}_1 - \mathbf{a}_2 \cdot \mathbf{e}_1 = 0. \end{aligned} \tag{5.13}$$

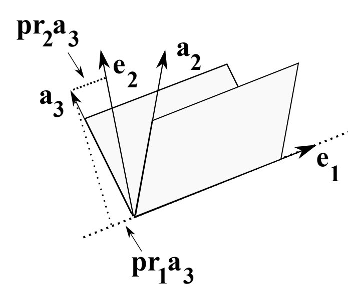


Figure 5.3: A plot of vectors for determining $\mathbf{e}_3$.

Next, we seek a third vector $\mathbf{e}_3$ that is orthogonal to both $\mathbf{e}_1$ and $\mathbf{e}_2$. Pictorially, we can write the given vector $\mathbf{a}_3$ as a combination of vector projections along $\mathbf{e}_1$ and $\mathbf{e}_2$ with the new vector. This is shown in Figure 5.3. Thus, we can see that

$$\mathbf{e}_3 = \mathbf{a}_3 - \frac{\mathbf{a}_3 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1 - \frac{\mathbf{a}_3 \cdot \mathbf{e}_2}{e_2^2}\mathbf{e}_2. \tag{5.14}$$

Again, it is a simple matter to compute the scalar products with $\mathbf{e}_1$ and $\mathbf{e}_2$ to verify orthogonality.

The Gram-Schmidt Orthogonalization for a vector basis.

We can easily generalize this procedure to the N-dimensional case. Let $\mathbf{a}_n$, $n = 1, ..., N$ be a set of linearly independent vectors in $\mathbf{R}^N$. Then, an orthogonal basis can be found by setting $\mathbf{e}_1 = \mathbf{a}_1$ and defining

$$\mathbf{e}_n = \mathbf{a}_n - \sum_{j=1}^{n-1} \frac{\mathbf{a}_n \cdot \mathbf{e}_j}{e_j^2} \mathbf{e}_j, \quad n = 2, 3, \ldots, N. \tag{5.15}$$

Now, we can generalize this idea to (real) function spaces. Let $f_n(x)$, $n \in N_0 = \{0, 1, 2, \ldots\}$, be a linearly independent sequence of continuous functions defined for $x \in [a, b]$. Then, an orthogonal basis of functions, $\phi_n(x)$, $n \in N_0$ can be found and is given by

$$\phi_0(x) = f_0(x)$$

The Gram-Schmidt Orthogonalization for a basis of functions

and

$$\phi_n(x) = f_n(x) - \sum_{j=0}^{n-1} \frac{\langle f_n, \phi_j \rangle}{\|\phi_j\|^2} \phi_j(x), \quad n = 1, 2, \ldots. \tag{5.16}$$

Here we are using inner products relative to weight $\sigma(x)$,

$$\langle f, g \rangle = \int_a^b f(x) g(x) \sigma(x)\, dx. \tag{5.17}$$

Note the similarity between the orthogonal basis in (5.16) and the expression for the finite dimensional case in Equation (5.15).

Example 5.2. Apply the Gram-Schmidt Orthogonalization process to the set $f_n(x) = x^n$, $n \in N_0$, when $x \in (-1, 1)$ and $\sigma(x) = 1$.

First, we have $\phi_0(x) = f_0(x) = 1$. Note that

$$\int_{-1}^{1} \phi_0^2(x)\, dx = 2.$$

We could use this result to fix the normalization of the new basis, but we will hold off doing that for now.

Now, we compute the second basis element:

$$\begin{aligned} \phi_1(x) &= f_1(x) - \frac{\langle f_1, \phi_0 \rangle}{\|\phi_0\|^2} \phi_0(x) \\ &= x - \frac{\langle x, 1 \rangle}{\|1\|^2} 1 = x, \end{aligned} \tag{5.18}$$

since $\langle x, 1 \rangle$ is the integral of an odd function over a symmetric interval.

For $\phi_2(x)$, we have

$$\begin{aligned} \phi_2(x) &= f_2(x) - \frac{\langle f_2, \phi_0 \rangle}{\|\phi_0\|^2} \phi_0(x) - \frac{\langle f_2, \phi_1 \rangle}{\|\phi_1\|^2} \phi_1(x) \\ &= x^2 - \frac{\langle x^2, 1 \rangle}{\|1\|^2} 1 - \frac{\langle x^2, x \rangle}{\|x\|^2} x \\ &= x^2 - \frac{\int_{-1}^{1} x^2\, dx}{\int_{-1}^{1} dx} \\ &= x^2 - \frac{1}{3}. \end{aligned} \tag{5.19}$$

So far, we have the orthogonal set $\{1, x, x^2 - \frac{1}{3}\}$. If one chooses to normalize these by forcing $\phi_n(1) = 1$, then one obtains the classical Legendre polynomials, $P_n(x)$. Thus,

$$P_2(x) = \frac{1}{2}(3x^2 - 1).$$

Note that this normalization is different than the usual one. In fact, we see the $P_2(x)$ does not have a unit norm,

$$\|P_2\|^2 = \int_{-1}^{1} P_2^2(x)\,dx = \frac{2}{5}.$$

The set of Legendre[4] polynomials is just one set of classical orthogonal polynomials that can be obtained in this way. Many of these special functions had originally appeared as solutions of important boundary value problems in physics. They all have similar properties and we will just elaborate some of these for the Legendre functions in the next section. Others in this group are shown in Table 5.1.

[4] Adrien-Marie Legendre (1752-1833) was a French mathematician who made many contributions to analysis and algebra.

Polynomial	Symbol	Interval	$\sigma(x)$
Hermite	$H_n(x)$	$(-\infty, \infty)$	e^{-x^2}
Laguerre	$L_n^\alpha(x)$	$[0, \infty)$	e^{-x}
Legendre	$P_n(x)$	(-1,1)	1
Gegenbauer	$C_n^\lambda(x)$	(-1,1)	$(1-x^2)^{\lambda-1/2}$
Tchebychef of the 1st kind	$T_n(x)$	(-1,1)	$(1-x^2)^{-1/2}$
Tchebychef of the 2nd kind	$U_n(x)$	(-1,1)	$(1-x^2)^{-1/2}$
Jacobi	$P_n^{(\nu,\mu)}(x)$	(-1,1)	$(1-x)^\nu(1-x)^\mu$

Table 5.1: Common classical orthogonal polynomials with the interval and weight function used to define them.

5.3 *Fourier-Legendre Series*

In the last chapter we saw how useful Fourier series expansions were for solving the heat and wave equations. In Chapter 6 we will investigate partial differential equations in higher dimensions and find that problems with spherical symmetry may lead to the series representations in terms of a basis of Legendre polynomials. For example, we could consider the steady state temperature distribution inside a hemispherical igloo, which takes the form

$$\phi(r, \theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

in spherical coordinates. Evaluating this function at the surface $r = a$ as $\phi(a, \theta) = f(\theta)$, leads to a Fourier-Legendre series expansion of function f :

$$f(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta),$$

where $c_n = A_n a^n$

In this section we would like to explore Fourier-Legendre series expansions of functions $f(x)$ defined on $(-1,1)$:

$$f(x) \sim \sum_{n=0}^{\infty} c_n P_n(x). \tag{5.20}$$

As with Fourier trigonometric series, we can determine the expansion coefficients by multiplying both sides of Equation (5.20) by $P_m(x)$ and integrating for $x \in [-1,1]$. Orthogonality gives the usual form for the generalized Fourier coefficients,

$$c_n = \frac{\langle f, P_n \rangle}{\|P_n\|^2}, n = 0, 1, \ldots.$$

We will later show that

$$\|P_n\|^2 = \frac{2}{2n+1}.$$

Therefore, the Fourier-Legendre coefficients are

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx. \tag{5.21}$$

5.3.1 *Properties of Legendre Polynomials*

We can do examples of Fourier-Legendre expansions given just a few facts about Legendre polynomials. The first property that the Legendre polynomials have is the Rodrigues formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n, \quad n \in N_0. \tag{5.22}$$

The Rodrigues Formula is credited to Benjamin Olinde Rodrigues (1795-1851) who discovered it in 1816 according to Hermite in 1865 and named by Heinrich Heine in 1878. The formula was also independently discovered by Sir James Ivory (1824) and Carl Gustav Jacobi (1827). Similar formulae give the other classical orthogonal polynomials.

From the Rodrigues formula, one can show that $P_n(x)$ is an nth degree polynomial. Also, for n odd, the polynomial is an odd function and for n even, the polynomial is an even function.

Example 5.3. Determine $P_2(x)$ from Rodrigues formula:

$$\begin{aligned}
P_2(x) &= \frac{1}{2^2 2!} \frac{d^2}{dx^2} (x^2-1)^2 \\
&= \frac{1}{8} \frac{d^2}{dx^2} (x^4 - 2x^2 + 1) \\
&= \frac{1}{8} \frac{d}{dx} (4x^3 - 4x) \\
&= \frac{1}{8} (12x^2 - 4) \\
&= \frac{1}{2} (3x^2 - 1).
\end{aligned} \tag{5.23}$$

Note that we get the same result as we found in the last section using orthogonalization.

n	$(x^2-1)^n$	$\frac{d^n}{dx^n}(x^2-1)^n$	$\frac{1}{2^n n!}$	$P_n(x)$
0	1	1	1	1
1	x^2-1	$2x$	$\frac{1}{2}$	x
2	x^4-2x^2+1	$12x^2-4$	$\frac{1}{8}$	$\frac{1}{2}(3x^2-1)$
3	$x^6-3x^4+3x^2-1$	$120x^3-72x$	$\frac{1}{48}$	$\frac{1}{2}(5x^3-3x)$

Table 5.2: Tabular computation of the Legendre polynomials using the Rodrigues formula.

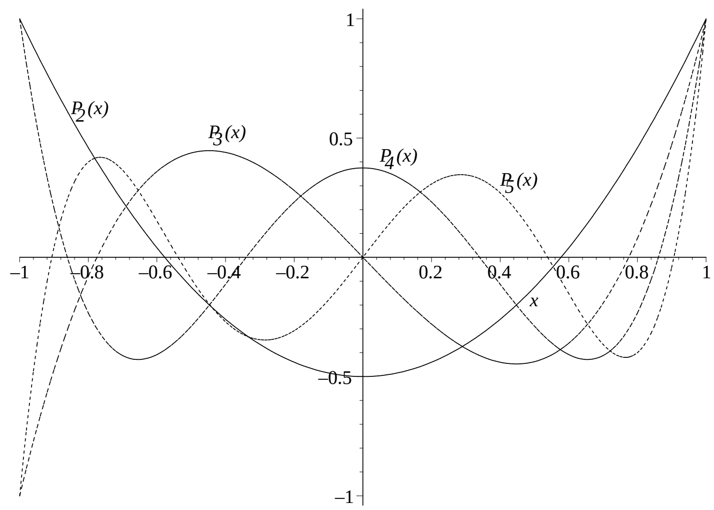


Figure 5.4: Plots of the Legendre polynomials $P_2(x)$, $P_3(x)$, $P_4(x)$, and $P_5(x)$.

The first several Legendre polynomials are computed using the Rodrigues formula in Table 5.2. In Figure 5.4 we show plots of these Legendre polynomials.

All of the classical orthogonal polynomials satisfy a three term recursion formula (or, recurrence relation or formula). In the case of the Legendre polynomials, we have

The Three Term Recursion Formula.

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad n = 1, 2, \ldots. \tag{5.24}$$

This can also be rewritten by replacing n with $n-1$ as

$$(2n-1)xP_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x), \quad n = 1, 2, \ldots. \tag{5.25}$$

Example 5.4. Use the recursion formula to find $P_2(x)$ and $P_3(x)$, given that $P_0(x) = 1$ and $P_1(x) = x$.

We first begin by inserting $n = 1$ into Equation (5.24):

$$2P_2(x) = 3xP_1(x) - P_0(x) = 3x^2 - 1.$$

So, $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

For $n = 2$, we have

$$\begin{aligned} 3P_3(x) &= 5xP_2(x) - 2P_1(x) \\ &= \frac{5}{2}x(3x^2-1) - 2x \end{aligned}$$

$$= \frac{1}{2}(15x^3 - 9x). \qquad (5.26)$$

This gives $P_3(x) = \frac{1}{2}(5x^3 - 3x)$. These expressions agree with the earlier results.

We will prove the three term recursion relation in two ways. The first proof we mention is using the orthogonality of the Legendre polynomials and is provided in Appendix 5.6. A more algebraic proof relies on the generating function for Legendre polynomials.

5.3.2 *The Generating Function for Legendre Polynomials*

A second proof of the three term recursion formula can be obtained from the generating function of the Legendre polynomials. Many special functions have such generating functions. In this case it is given by

Legendre said that Pierre-Simon Laplace (1749-1827) introduced the potential function and Legendre provided the expansion. The polynomials were named in 1875 by Todhunter as Legendre coefficients. Laplace and Legendre had written memoirs which came out in 1783 and 1785, respectively. But Legendre's work was published several years after it was written.

$$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \quad |x| \le 1, |t| < 1. \qquad (5.27)$$

This generating function occurs often in applications. In particular, it arises in potential theory, such as electromagnetic or gravitational potentials. These potential functions are $\frac{1}{r}$ type functions.

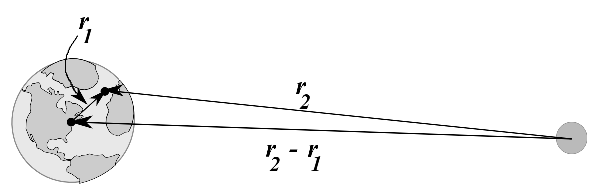


Figure 5.5: The position vectors used to describe the tidal force on the Earth due to the moon.

For example, the gravitational potential between the Earth and the moon is proportional to the reciprocal of the magnitude of the difference between their positions relative to some coordinate system. An even better example, would be to place the origin at the center of the Earth and consider the forces on the non-pointlike Earth due to the moon. Consider a piece of the Earth at position $\mathbf{r}_1$ and the moon at position $\mathbf{r}_2$ as shown in Figure 5.5. The tidal potential Φ is proportional to

$$\Phi \propto \frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|} = \frac{1}{\sqrt{(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{r}_2 - \mathbf{r}_1)}} = \frac{1}{\sqrt{r_1^2 - 2r_1r_2\cos\theta + r_2^2}},$$

where θ is the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$.

Typically, one of the position vectors is much larger than the other. Let's assume that $r_1 \ll r_2$. Then, one can write

$$\Phi \propto \frac{1}{\sqrt{r_1^2 - 2r_1r_2\cos\theta + r_2^2}} = \frac{1}{r_2}\frac{1}{\sqrt{1 - 2\frac{r_1}{r_2}\cos\theta + \left(\frac{r_1}{r_2}\right)^2}}.$$

Now, define $x = \cos\theta$ and $t = \frac{r_1}{r_2}$. We then have that the tidal potential is proportional to the generating function for the Legendre polynomials! So, we can write the tidal potential as

$$\Phi \propto \frac{1}{r_2} \sum_{n=0}^{\infty} P_n(\cos\theta) \left(\frac{r_1}{r_2}\right)^n .$$

The first term in the expansion, $\frac{1}{r_2}$, is the gravitational potential that gives the usual force between the Earth and the moon. [Recall that the gravitational potential for mass m at distance r from M is given by $\Phi = -\frac{GMm}{r}$ and that the force is the gradient of the potential, $\mathbf{F} = -\nabla\Phi \propto \nabla\left(\frac{1}{r}\right)$.] The next terms will give expressions for the tidal effects.

Now that we have some idea as to where this generating function might have originated, we can proceed to use it. First of all, the generating function can be used to obtain special values of the Legendre polynomials.

Example 5.5. Evaluate $P_n(0)$ using the generating function.

$P_n(0)$ is found by considering $g(0,t)$. Setting $x = 0$ in Equation (5.27), we have

$$\begin{aligned} g(0,t) &= \frac{1}{\sqrt{1+t^2}} \\ &= \sum_{n=0}^{\infty} P_n(0) t^n \\ &= P_0(0) + P_1(0)t + P_2(0)t^2 + P_3(0)t^3 + \dots . \end{aligned} \tag{5.28}$$

We can use the binomial expansion to find the final answer. Namely, we have

$$\frac{1}{\sqrt{1+t^2}} = 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 + \dots .$$

Comparing these expansions, we have the $P_n(0) = 0$ for n odd and for even integers one can show (see Problem 15) that[5]

$$P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}, \tag{5.29}$$

where $n!!$ is the *double factorial*,

$$n!! = \begin{cases} n(n-2)\dots(3)1, & n > 0, \text{odd}, \\ n(n-2)\dots(4)2, & n > 0, \text{even}, \\ 1 & n = 0, -1 \end{cases} .$$

[5] This example can be finished by first proving that

$$(2n)!! = 2^n n!$$

and

$$(2n-1)!! = \frac{(2n)!}{(2n)!!} = \frac{(2n)!}{2^n n!}.$$

Example 5.6. Evaluate $P_n(-1)$.

This is a simpler problem. In this case we have

$$g(-1,t) = \frac{1}{\sqrt{1+2t+t^2}} = \frac{1}{1+t} = 1 - t + t^2 - t^3 + \dots .$$

Therefore, $P_n(-1) = (-1)^n$.

Example 5.7. Prove the three term recursion formula,

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0, \quad k = 1, 2, \ldots,$$

using the generating function.

Proof of the three term recursion formula using the generating function.

We can also use the generating function to find recurrence relations. To prove the three term recursion (5.24) that we introduced above, then we need only differentiate the generating function with respect to t in Equation (5.27) and rearrange the result. First note that

$$\frac{\partial g}{\partial t} = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \frac{x-t}{1-2xt+t^2} g(x,t).$$

Combining this with

$$\frac{\partial g}{\partial t} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1},$$

we have

$$(x-t)g(x,t) = (1-2xt+t^2)\sum_{n=0}^{\infty} nP_n(x)t^{n-1}.$$

Inserting the series expression for $g(x,t)$ and distributing the sum on the right side, we obtain

$$(x-t)\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} nP_n(x)t^{n-1} - \sum_{n=0}^{\infty} 2nxP_n(x)t^n + \sum_{n=0}^{\infty} nP_n(x)t^{n+1}.$$

Multiplying out the $x-t$ factor and rearranging, leads to three separate sums:

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} - \sum_{n=0}^{\infty}(2n+1)xP_n(x)t^n + \sum_{n=0}^{\infty}(n+1)P_n(x)t^{n+1} = 0. \quad (5.30)$$

Each term contains powers of t that we would like to combine into a single sum. This is done by reindexing. For the first sum, we could use the new index $k = n-1$. Then, the first sum can be written

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} = \sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k.$$

Using different indices is just another way of writing out the terms. Note that

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} = 0 + P_1(x) + 2P_2(x)t + 3P_3(x)t^2 + \ldots$$

and

$$\sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k = 0 + P_1(x) + 2P_2(x)t + 3P_3(x)t^2 + \ldots$$

actually give the same sum. The indices are sometimes referred to as *dummy indices* because they do not show up in the expanded expression and can be replaced with another letter.

If we want to do so, we could now replace all of the k's with n's. However, we will leave the k's in the first term and now reindex the next sums in Equation (5.30). The second sum just needs the replacement $n = k$ and the last sum we reindex using $k = n + 1$. Therefore, Equation (5.30) becomes

$$\sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k - \sum_{k=0}^{\infty}(2k+1)xP_k(x)t^k + \sum_{k=1}^{\infty}kP_{k-1}(x)t^k = 0. \quad (5.31)$$

We can now combine all of the terms, noting the $k = -1$ term is automatically zero and the $k = 0$ terms give

$$P_1(x) - xP_0(x) = 0. \quad (5.32)$$

Of course, we know this already. So, that leaves the $k > 0$ terms:

$$\sum_{k=1}^{\infty}\left[(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x)\right]t^k = 0. \quad (5.33)$$

Since this is true for all t, the coefficients of the t^k's are zero, or

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0, \quad k = 1, 2, \ldots.$$

While this is the standard form for the three term recurrence relation, the earlier form is obtained by setting $k = n - 1$.

There are other recursion relations which we list in the box below. Equation (5.34) was derived using the generating function. Differentiating it with respect to x, we find Equation (5.35). Equation (5.36) can be proven using the generating function by differentiating $g(x,t)$ with respect to x and rearranging the resulting infinite series just as in this last manipulation. This will be left as Problem 4. Combining this result with Equation (5.34), we can derive Equations (5.37)-(5.38). Adding and subtracting these equations yields Equations (5.39)-(5.40).

Recursion Formulae for Legendre Polynomials for $n = 1, 2, \ldots$.

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \quad (5.34)$$

$$(n+1)P'_{n+1}(x) = (2n+1)[P_n(x) + xP'_n(x)] - nP'_{n-1}(x) \quad (5.35)$$

$$P_n(x) = P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x) \quad (5.36)$$

$$P'_{n-1}(x) = xP'_n(x) - nP_n(x) \quad (5.37)$$

$$P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x) \quad (5.38)$$

$$P'_{n+1}(x) + P'_{n-1}(x) = 2xP'_n(x) + P_n(x). \quad (5.39)$$

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x). \quad (5.40)$$

$$(x^2-1)P'_n(x) = nxP_n(x) - nP_{n-1}(x) \quad (5.41)$$

Finally, Equation (5.41) can be obtained using Equations (5.37) and (5.38). Just multiply Equation (5.37) by x,

$$x^2P'_n(x) - nxP_n(x) = xP'_{n-1}(x).$$

Now use Equation (5.38), but first replace n with $n-1$ to eliminate the $xP'_{n-1}(x)$ term:

$$x^2P'_n(x) - nxP_n(x) = P'_n(x) - nP_{n-1}(x).$$

Rearranging gives the Equation (5.41).

Example 5.8. Use the generating function to prove

$$\|P_n\|^2 = \int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}.$$

Another use of the generating function is to obtain the normalization constant. This can be done by first squaring the generating function in order to get the products $P_n(x)P_m(x)$, and then integrating over x.

The normalization constant.

Squaring the generating function has to be done with care, as we need to make proper use of the dummy summation index. So, we first write

$$\begin{aligned}\frac{1}{1-2xt+t^2} &= \left[\sum_{n=0}^{\infty} P_n(x)t^n\right]^2 \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(x)P_m(x)t^{n+m}. \qquad (5.42)\end{aligned}$$

Integrating from $x=-1$ to $x=1$ and using the orthogonality of the Legendre polynomials, we have

$$\begin{aligned}\int_{-1}^{1}\frac{dx}{1-2xt+t^2} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t^{n+m}\int_{-1}^{1} P_n(x)P_m(x)\,dx \\ &= \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1} P_n^2(x)\,dx. \qquad (5.43)\end{aligned}$$

However, one can show that[6]

$$\int_{-1}^{1}\frac{dx}{1-2xt+t^2} = \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right).$$

Expanding this expression about $t=0$, we obtain[7]

$$\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right) = \sum_{n=0}^{\infty}\frac{2}{2n+1}t^{2n}.$$

Comparing this result with Equation (5.43), we find that

$$\|P_n\|^2 = \int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}. \qquad (5.44)$$

[6] You will need the integral

$$\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx) + C.$$

[7] You will need the series expansion

$$\begin{aligned}\ln(1+x) &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots.\end{aligned}$$

5.3.3 *The Differential Equation for Legendre Polynomials*

The Legendre polynomials satisfy a second order linear differential equation. This differential equation occurs naturally in the solution of initial-boundary value problems in three dimensions which possess some spherical symmetry. We will see this in the next chapter. There are two approaches we could take in showing that the Legendre polynomials satisfy a particular differential equation. Either we can write down the equations and attempt to solve it,[8] or we could use the above properties to obtain the equation. For now, we will seek the differential equation satisfied by $P_n(x)$ using the above recursion relations.

[8] The standard approach to solving Legendre's Equation (5.46) is to use power series. This is typically seen in an introductory differential equations course. We insert

$$y(x) = \sum_{k=0}^{\infty} c_k x^k, \quad x \in (-1,1)$$

into Equation (5.46) to find that the coefficient satisfy a recurrence relation,

$$c_{k+1} = \frac{(n-k)(n+k+1)}{(k+1)(k+2)} c_k, \quad k \geq 0.$$

When $k = n$, the series truncates to a polynomial.

We begin by differentiating Equation (5.41) and using Equation (5.37) to simplify:

$$\begin{aligned} \frac{d}{dx}\left((x^2-1)P_n'(x)\right) &= nP_n(x) + nxP_n'(x) - nP_{n-1}'(x) \\ &= nP_n(x) + n^2P_n(x) \\ &= n(n+1)P_n(x). \end{aligned} \tag{5.45}$$

Therefore, Legendre polynomials, or Legendre functions of the first kind, are solutions of the differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0. \tag{5.46}$$

As this is a linear second order differential equation, we expect two linearly independent solutions. The second solution, called the Legendre function of the second kind, is given by $Q_n(x)$ and is not well behaved at $x = \pm 1$. For example,

$$Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x}.$$

We will not need these for physically interesting examples in this book.

A generalization of the Legendre equation is given by $(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0$. Solutions to this equation, $P_n^m(x)$ and $Q_n^m(x)$, are called the associated Legendre functions of the first and second kind.

5.3.4 *Fourier-Legendre Series*

WITH THESE PROPERTIES OF LEGENDRE FUNCTIONS we are now prepared to compute the expansion coefficients for the Fourier-Legendre series representation of a given function.

Example 5.9. Expand $f(x) = x^3$ in a Fourier-Legendre series.

We simply need to compute

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} x^3 P_n(x)\, dx. \tag{5.47}$$

We first note that

$$\int_{-1}^{1} x^m P_n(x)\, dx = 0 \quad \text{for } m > n.$$

As a result, we have that $c_n = 0$ for $n > 3$. We could just compute $\int_{-1}^{1} x^3 P_m(x)\, dx$ for $m = 0, 1, 2, \ldots$ outright by looking up Legendre polynomials. But, note that x^3 is an odd function. So, $c_0 = 0$ and $c_2 = 0$.

This leaves us with only two coefficients to compute. We refer to Table 5.2 and find that

$$c_1 = \frac{3}{2}\int_{-1}^{1} x^4\, dx = \frac{3}{5}$$

$$c_3 = \frac{7}{2}\int_{-1}^{1} x^3 \left[\frac{1}{2}(5x^3-3x)\right]\, dx = \frac{2}{5}.$$

Thus,

$$x^3 = \frac{3}{5}P_1(x) + \frac{2}{5}P_3(x).$$

Of course, this is simple to check using Table 5.2:

$$\frac{3}{5}P_1(x) + \frac{2}{5}P_3(x) = \frac{3}{5}x + \frac{2}{5}\left[\frac{1}{2}(5x^3-3x)\right] = x^3.$$

We could have obtained this result without doing any integration. Write x^3 as a linear combination of $P_1(x)$ and $P_3(x)$:

$$\begin{aligned} x^3 &= c_1 x + \frac{1}{2}c_2(5x^3-3x) \\ &= (c_1 - \frac{3}{2}c_2)x + \frac{5}{2}c_2 x^3. \end{aligned} \tag{5.48}$$

Equating coefficients of like terms, we have that $c_2 = \frac{2}{5}$ and $c_1 = \frac{3}{2}c_2 = \frac{3}{5}$.

Example 5.10. Expand the Heaviside[9] function in a Fourier-Legendre series.

The Heaviside function is defined as

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \tag{5.49}$$

In this case, we cannot find the expansion coefficients without some integration. We have to compute

$$\begin{aligned} c_n &= \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\, dx \\ &= \frac{2n+1}{2}\int_{0}^{1} P_n(x)\, dx. \end{aligned} \tag{5.50}$$

We can make use of identity (5.40),

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x), \quad n > 0. \tag{5.51}$$

We have for $n > 0$

$$c_n = \frac{1}{2}\int_0^1 [P'_{n+1}(x) - P'_{n-1}(x)]\, dx = \frac{1}{2}[P_{n-1}(0) - P_{n+1}(0)].$$

For $n = 0$, we have

$$c_0 = \frac{1}{2}\int_0^1 dx = \frac{1}{2}.$$

This leads to the expansion

$$f(x) \sim \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}[P_{n-1}(0) - P_{n+1}(0)]P_n(x).$$

[9] Oliver Heaviside (1850-1925) was an English mathematician, physicist and engineer who used complex analysis to study circuits and was a co-founder of vector analysis. The Heaviside function is also called the step function. The modern idea of the step function is credited to Bernhard Riemann (1826-1866) and was used by Fourier and Augustin-Louis Cauchy (1789-1857). However, Heaviside's usage in his operational calculus carried over to electrical engineering and Laplace transforms.

We still need to evaluate the Fourier-Legendre coefficients

$$c_n = \frac{1}{2}[P_{n-1}(0) - P_{n+1}(0)].$$

Since $P_n(0) = 0$ for n odd, the c_n's vanish for n even. Letting $n = 2k-1$, we re-index the sum, obtaining

$$f(x) \sim \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{\infty}[P_{2k-2}(0) - P_{2k}(0)]P_{2k-1}(x).$$

We can compute the nonzero Fourier coefficients, $c_{2k-1} = \frac{1}{2}[P_{2k-2}(0) - P_{2k}(0)]$, using a result from Problem 15:

$$P_{2k}(0) = (-1)^k \frac{(2k-1)!!}{(2k)!!}. \tag{5.52}$$

Namely, we have

$$\begin{aligned} c_{2k-1} &= \frac{1}{2}[P_{2k-2}(0) - P_{2k}(0)] \\ &= \frac{1}{2}\left[(-1)^{k-1}\frac{(2k-3)!!}{(2k-2)!!} - (-1)^k \frac{(2k-1)!!}{(2k)!!}\right] \\ &= -\frac{1}{2}(-1)^k \frac{(2k-3)!!}{(2k-2)!!}\left[1 + \frac{2k-1}{2k}\right] \\ &= -\frac{1}{2}(-1)^k \frac{(2k-3)!!}{(2k-2)!!}\frac{4k-1}{2k}. \end{aligned} \tag{5.53}$$

Thus, the Fourier-Legendre series expansion for the Heaviside function is given by

$$f(x) \sim \frac{1}{2} - \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n \frac{(2n-3)!!}{(2n-2)!!}\frac{4n-1}{2n}P_{2n-1}(x). \tag{5.54}$$

The sum of the first 21 terms of this series are shown in Figure 5.6. We note the slow convergence to the Heaviside function. Also, we see that the Gibbs phenomenon is present due to the jump discontinuity at $x = 0$. [See Section 2.7.]

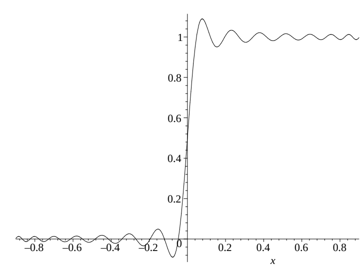


Figure 5.6: Sum of first 21 terms for Fourier-Legendre series expansion of Heaviside function.

5.4 *Gamma Function*

The name and symbol for the Gamma function were first given by Legendre in 1811. However, the search for a generalization of the factorial extends back to the 1720's when Euler provided the first representation of the factorial as an infinite product, later to be modified by others like Gauss, Weierstrass, and Legendre.

A FUNCTION THAT OFTEN OCCURS IN THE STUDY OF SPECIAL FUNCTIONS is the Gamma function. We will need the Gamma function in the next section on Fourier-Bessel series.

For $x > 0$, we define the Gamma function as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \quad x > 0. \tag{5.55}$$

The Gamma function is a generalization of the factorial function and a plot is shown in Figure 5.7. In fact, we have

$$\Gamma(1) = 1$$

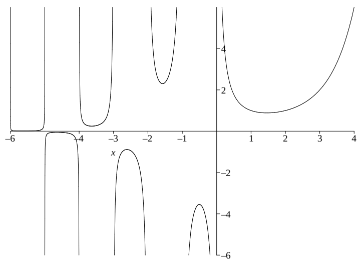


Figure 5.7: Plot of the Gamma function.

and

$$\Gamma(x+1) = x\Gamma(x).$$

The reader can prove this identity by simply performing an integration by parts. (See Problem 7.) In particular, for integers $n \in Z^+$, we then have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-2) = n(n-1)\cdots 2\Gamma(1) = n!.$$

We can also define the Gamma function for negative, non-integer values of x. We first note that by iteration on $n \in Z^+$, we have

$$\Gamma(x+n) = (x+n-1)\cdots(x+1)x\Gamma(x), \quad x+n > 0.$$

Solving for $\Gamma(x)$, we then find

$$\Gamma(x) = \frac{\Gamma(x+n)}{(x+n-1)\cdots(x+1)x}, \quad -n < x < 0$$

Note that the Gamma function is undefined at zero and the negative integers.

Example 5.11. We now prove that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

This is done by direct computation of the integral:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt.$$

Letting $t = z^2$, we have

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-z^2}\, dz.$$

Due to the symmetry of the integrand, we obtain the classic integral

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^\infty e^{-z^2}\, dz,$$

which can be performed using a standard trick. Consider the integral

$$I = \int_{-\infty}^\infty e^{-x^2}\, dx.$$

Then,

$$I^2 = \int_{-\infty}^\infty e^{-x^2}\, dx \int_{-\infty}^\infty e^{-y^2}\, dy.$$

Note that we changed the integration variable. This will allow us to write this product of integrals as a double integral:

$$I^2 = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)}\, dxdy.$$

This is an integral over the entire xy-plane. We can transform this Cartesian integration to an integration over polar coordinates. The integral becomes

$$I^2 = \int_0^{2\pi} \int_0^\infty e^{-r^2}\, rdrd\theta.$$

This is simple to integrate and we have $I^2 = \pi$. So, the final result is found by taking the square root of both sides:

$$\Gamma\left(\frac{1}{2}\right) = I = \sqrt{\pi}.$$

In Problem 15 the reader will prove the identity

$$\Gamma(n+\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}.$$

Another useful relation, which we only state, is

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

There are many other important relations, including infinite products, which we will not need at this point. The reader is encouraged to read about these elsewhere. In the meantime, we move on to the discussion of another important special function in physics and mathematics.

More generally, we have

$$\int_{-\infty}^{\infty} e^{-\beta y^2}\, dy = \sqrt{\frac{\pi}{\beta}}.$$

For $\beta = 1$ this is referred to as the Gauss error integral, or Gauss-Poisson integral. Gauss published this result in 1809, but it was Abraham de Moivre (1667-1754) who originally discovered these types of integrals in 1733. More general Gaussian integrals are reserved for the homework such as in Problems 2 and 11.

5.5 *Fourier-Bessel Series*

Bessel functions arise in many problems in physics possessing cylindrical symmetry such as the vibrations of circular drumheads and the radial modes in optical fibers. They also provide us with another orthogonal set of basis functions.

The first occurrence of Bessel functions (zeroth order) was in the work of Daniel Bernoulli on heavy chains (1738). More general Bessel functions were studied by Leonhard Euler in 1781 and in his study of the vibrating membrane in 1764. Joseph Fourier found them in the study of heat conduction in solid cylinders and Siméon Poisson (1781-1840) in heat conduction of spheres (1823).

Bessel functions have a long history and were named after Friedrich Wilhelm Bessel (1784-1846).

The history of Bessel functions does not just originate in the study of the wave and heat equations. These solutions originally came up in the study of the Kepler problem, describing planetary motion. According to G. N. Watson in his *Treatise on Bessel Functions*, the formulation and solution of Kepler's Problem was discovered by Joseph-Louis Lagrange (1736-1813), in 1770. Namely, the problem was to express the radial coordinate and what is called the eccentric anomaly, E, as functions of time. Lagrange found expressions for the coefficients in the expansions of r and E in trigonometric functions of time. However, he only computed the first few coefficients. In 1816 Friedrich Wilhelm Bessel (1784-1846) had shown that the coefficients in the expansion for r could be given an integral representation. In 1824 he presented a thorough study of these functions, which are now called Bessel functions.

You might have seen Bessel functions in a course on differential equations as solutions of the differential equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0. \tag{5.56}$$

Solutions to this equation are obtained in the form of series expansions.[10] Namely, one seeks solutions of the form

$$y(x) = \sum_{j=0}^{\infty} a_j x^{j+n}$$

by determining the form the coefficients must take. We will leave this for a homework exercise and simply report the results.

[10] Since $x = 0$ is a regular singular point, we solve Bessel's equation using the Method of Frobenius. This differs from the series solution of Legendre's differential equation.

One solution of the differential equation is the *Bessel function of the first kind of order p*, given as

$$y(x) = J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}. \qquad (5.57)$$

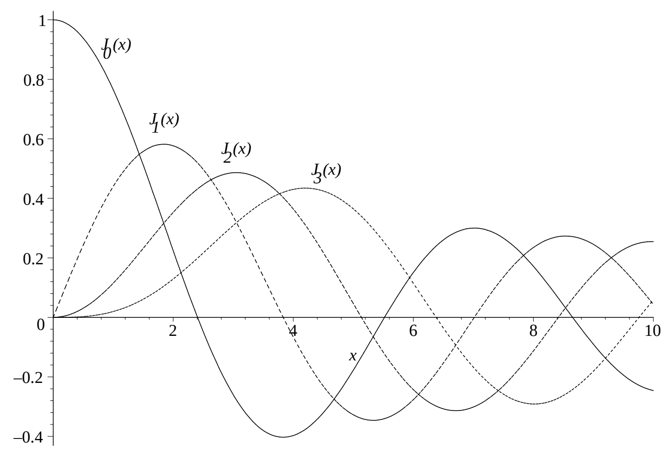


Figure 5.8: Plots of the Bessel functions $J_0(x)$, $J_1(x)$, $J_2(x)$, and $J_3(x)$.

In Figure 5.8 we display the first few Bessel functions of the first kind of integer order. Note that these functions can be described as decaying oscillatory functions.

A second linearly independent solution is obtained for p not an integer as $J_{-p}(x)$. However, for p an integer, the $\Gamma(n+p+1)$ factor leads to evaluations of the Gamma function at zero, or negative integers, when p is negative. Thus, the above series is not defined in these cases.

Another method for obtaining a second linearly independent solution is through a linear combination of $J_p(x)$ and $J_{-p}(x)$ as

$$N_p(x) = Y_p(x) = \frac{\cos \pi p J_p(x) - J_{-p}(x)}{\sin \pi p}. \qquad (5.58)$$

These functions are called the Neumann functions, or Bessel functions of the second kind of order p.

In Figure 5.9 we display the first few Bessel functions of the second kind of integer order. Note that these functions are also decaying oscillatory functions. However, they are singular at $x = 0$.

In many applications one desires bounded solutions at $x = 0$. These functions do not satisfy this boundary condition. For example, we will

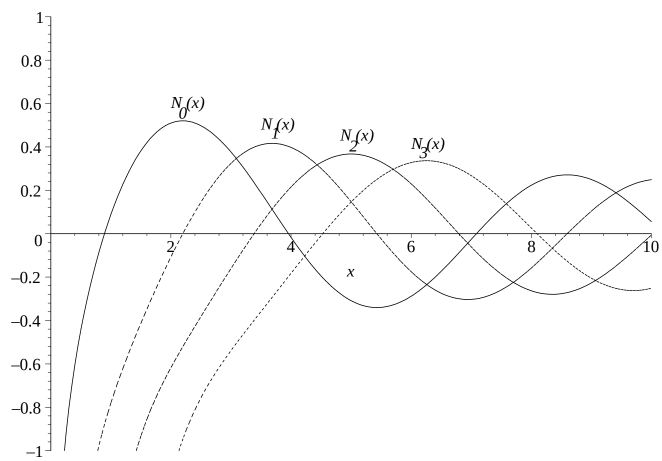


Figure 5.9: Plots of the Neumann functions $N_0(x)$, $N_1(x)$, $N_2(x)$, and $N_3(x)$.

later study one standard problem is to describe the oscillations of a circular drumhead. For this problem one solves the two dimensional wave equation using separation of variables in cylindrical coordinates. The radial equation leads to a Bessel equation. The Bessel function solutions describe the radial part of the solution and one does not expect a singular solution at the center of the drum. The amplitude of the oscillation must remain finite. Thus, only Bessel functions of the first kind can be used.

Bessel functions satisfy a variety of properties, which we will only list at this time for Bessel functions of the first kind. The reader will have the opportunity to prove these for homework.

Derivative Identities These identities follow directly from the manipulation of the series solution.

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x). \tag{5.59}$$

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x). \tag{5.60}$$

Recursion Formulae The next identities follow from adding, or subtracting, the derivative identities.

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x). \tag{5.61}$$

$$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x). \tag{5.62}$$

Orthogonality As we will see in the next chapter, one can recast the Bessel equation into an eigenvalue problem whose solutions form an orthogonal basis of functions on $L_x^2(0,a)$. Using Sturm-Liouville theory, one can show that

$$\int_0^a x J_p(j_{pn}\frac{x}{a}) J_p(j_{pm}\frac{x}{a})\, dx = \frac{a^2}{2}\left[J_{p+1}(j_{pn})\right]^2 \delta_{n,m}, \tag{5.63}$$

where j_{pn} is the nth root of $J_p(x)$, $J_p(j_{pn}) = 0$, $n = 1, 2, \ldots$. A list of some of these roots are provided in Table 5.3.

Table 5.3: The zeros of Bessel Functions, $J_m(j_{mn}) = 0$.

n	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
1	2.405	3.832	5.136	6.380	7.588	8.771
2	5.520	7.016	8.417	9.761	11.065	12.339
3	8.654	10.173	11.620	13.015	14.373	15.700
4	11.792	13.324	14.796	16.223	17.616	18.980
5	14.931	16.471	17.960	19.409	20.827	22.218
6	18.071	19.616	21.117	22.583	24.019	25.430
7	21.212	22.760	24.270	25.748	27.199	28.627
8	24.352	25.904	27.421	28.908	30.371	31.812
9	27.493	29.047	30.569	32.065	33.537	34.989

Generating Function

$$e^{x(t-\frac{1}{t})/2} = \sum_{n=-\infty}^{\infty} J_n(x)t^n, \quad x > 0, t \neq 0. \tag{5.64}$$

Integral Representation

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - n\theta)\, d\theta, \quad x > 0, n \in \mathrm{Z}. \tag{5.65}$$

Fourier-Bessel Series

Since the Bessel functions are an orthogonal set of functions of a Sturm-Liouville problem, we can expand square integrable functions in this basis. In fact, the Sturm-Liouville problem is given in the form

$$x^2y'' + xy' + (\lambda x^2 - p^2)y = 0, \quad x \in [0, a], \tag{5.66}$$

satisfying the boundary conditions: $y(x)$ is bounded at $x = 0$ and $y(a) = 0$. The solutions are then of the form $J_p(\sqrt{\lambda}x)$, as can be shown by making the substitution $t = \sqrt{\lambda}x$ in the differential equation. Namely, we let $y(x) = u(t)$ and note that

$$\frac{dy}{dx} = \frac{dt}{dx}\frac{du}{dt} = \sqrt{\lambda}\frac{du}{dt}.$$

Then,

$$t^2u'' + tu' + (t^2 - p^2)u = 0,$$

which has a solution $u(t) = J_p(t)$.

Using Sturm-Liouville theory, one can show that $J_p(j_{pn}\frac{x}{a})$ is a basis of eigenfunctions and the resulting *Fourier-Bessel series expansion* of $f(x)$ defined on $x \in [0,a]$ is

$$f(x) = \sum_{n=1}^{\infty} c_n J_p(j_{pn}\frac{x}{a}), \tag{5.67}$$

where the Fourier-Bessel coefficients are found using the orthogonality relation as

$$c_n = \frac{2}{a^2\left[J_{p+1}(j_{pn})\right]^2}\int_0^a xf(x)J_p(j_{pn}\frac{x}{a})\,dx. \tag{5.68}$$

In the study of boundary value problems in differential equations, Sturm-Liouville problems are a bountiful source of basis functions for the space of square integrable functions.

Example 5.12. Expand $f(x) = 1$ for $0 < x < 1$ in a Fourier-Bessel series of the form

$$f(x) = \sum_{n=1}^{\infty} c_n J_0(j_{0n}x)$$

.

We need only compute the Fourier-Bessel coefficients in Equation (5.68):

$$c_n = \frac{2}{[J_1(j_{0n})]^2}\int_0^1 xJ_0(j_{0n}x)\,dx. \tag{5.69}$$

From the identity

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x). \tag{5.70}$$

we have

$$\begin{aligned}\int_0^1 xJ_0(j_{0n}x)\,dx &= \frac{1}{j_{0n}^2}\int_0^{j_{0n}} yJ_0(y)\,dy \\ &= \frac{1}{j_{0n}^2}\int_0^{j_{0n}} \frac{d}{dy}\left[yJ_1(y)\right]\,dy \\ &= \frac{1}{j_{0n}^2}\left[yJ_1(y)\right]_0^{j_{0n}} \\ &= \frac{1}{j_{0n}}J_1(j_{0n}). \end{aligned}\tag{5.71}$$

As a result, the desired Fourier-Bessel expansion is given as

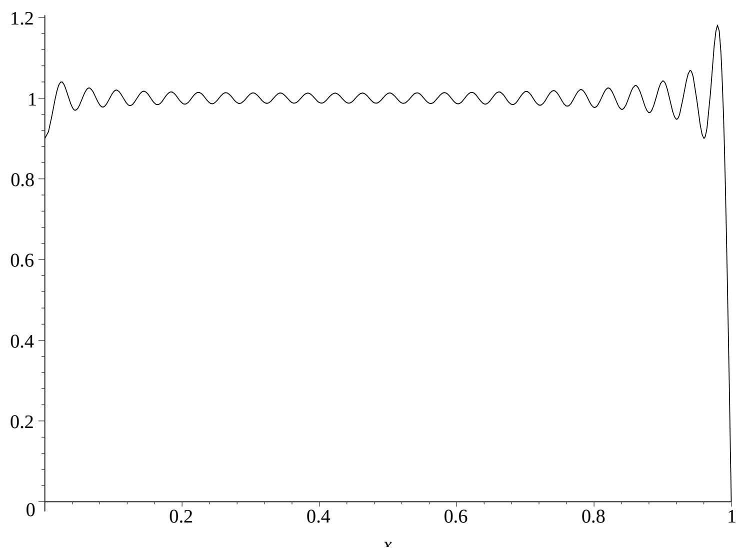
$$1 = 2\sum_{n=1}^{\infty}\frac{J_0(j_{0n}x)}{j_{0n}J_1(j_{0n})}, \quad 0 < x < 1. \tag{5.72}$$

In Figure 5.10 we show the partial sum for the first fifty terms of this series. Note once again the slow convergence due to the Gibbs phenomenon.

5.6 Appendix: Orthogonality Proof of Three Term Recursion Formula - Optional

WE WILL PROVE THE THREE TERM RECURSION FORMULA using the orthogonality properties of Legendre polynomials and the following lemma.

Figure 5.10: Plot of the first 50 terms of the Fourier-Bessel series in Equation (5.72) for $f(x) = 1$ on $0 < x < 1$.



Lemma 5.1. *The leading coefficient of x^n in $P_n(x)$ is* $\frac{1}{2^n n!}\frac{(2n)!}{n!}$.

The first proof of the three term recursion formula is based upon the nature of the Legendre polynomials as an orthogonal basis, while the second proof is derived using generating functions.

Proof. We can prove this using the Rodrigues formula. First, we focus on the leading coefficient of $(x^2-1)^n$, which is x^{2n}. The first derivative of x^{2n} is $2nx^{2n-1}$. The second derivative is $2n(2n-1)x^{2n-2}$. The jth derivative is

$$\frac{d^j x^{2n}}{dx^j} = [2n(2n-1)\dots(2n-j+1)]x^{2n-j}.$$

Thus, the nth derivative is given by

$$\frac{d^n x^{2n}}{dx^n} = [2n(2n-1)\dots(n+1)]x^n.$$

This proves that $P_n(x)$ has degree n. The leading coefficient of $P_n(x)$ can now be written as

$$\begin{aligned} \frac{[2n(2n-1)\dots(n+1)]}{2^n n!} &= \frac{[2n(2n-1)\dots(n+1)]}{2^n n!}\frac{n(n-1)\dots 1}{n(n-1)\dots 1} \\ &= \frac{1}{2^n n!}\frac{(2n)!}{n!}. \end{aligned} \tag{5.73}$$

□

Theorem 5.1. *Legendre polynomials satisfy the three term recursion formula*

$$(2n-1)xP_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x), \quad n = 1, 2, \dots. \tag{5.74}$$

Proof. In order to prove the three term recursion formula we consider the expression $(2n-1)xP_{n-1}(x) - nP_n(x)$. While each term is a polynomial of degree n, the leading order terms cancel. We need only look at the coefficient of the leading order term first expression. It is

$$\frac{2n-1}{2^{n-1}(n-1)!}\frac{(2n-2)!}{(n-1)!} = \frac{1}{2^{n-1}(n-1)!}\frac{(2n-1)!}{(n-1)!} = \frac{(2n-1)!}{2^{n-1}\left[(n-1)!\right]^2}.$$

The coefficient of the leading term for $nP_n(x)$ can be written as

$$n\frac{1}{2^n n!}\frac{(2n)!}{n!} = n\left(\frac{2n}{2n^2}\right)\left(\frac{1}{2^{n-1}(n-1)!}\right)\frac{(2n-1)!}{(n-1)!}\frac{(2n-1)!}{2^{n-1}\left[(n-1)!\right]^2}.$$

It is easy to see that the leading order terms in the expression $(2n-1)xP_{n-1}(x) - nP_n(x)$ cancel.

The next terms will be of degree $n-2$. This is because the P_n's are either even or odd functions, thus only containing even, or odd, powers of x. We conclude that

$$(2n-1)xP_{n-1}(x) - nP_n(x) = \text{ polynomial of degree } n-2.$$

Therefore, since the Legendre polynomials form a basis, we can write this polynomial as a linear combination of Legendre polynomials:

$$(2n-1)xP_{n-1}(x) - nP_n(x) = c_0P_0(x) + c_1P_1(x) + \ldots + c_{n-2}P_{n-2}(x). \quad (5.75)$$

Multiplying Equation (5.75) by $P_m(x)$ for $m = 0, 1, \ldots, n-3$, integrating from -1 to 1, and using orthogonality, we obtain

$$0 = c_m\|P_m\|^2, \quad m = 0, 1, \ldots, n-3.$$

[Note: $\int_{-1}^{1} x^k P_n(x)\,dx = 0$ for $k \leq n-1$. Thus, $\int_{-1}^{1} xP_{n-1}(x)P_m(x)\,dx = 0$ for $m \leq n-3$.]

Thus, all of these c_m's are zero, leaving Equation (5.75) as

$$(2n-1)xP_{n-1}(x) - nP_n(x) = c_{n-2}P_{n-2}(x).$$

The final coefficient can be found by using the normalization condition, $P_n(1) = 1$. Thus, $c_{n-2} = (2n-1) - n = n-1$. □

5.7 *Appendix: The Least Squares Approximation - Optional*

In the first section of this chapter we showed that we can expand functions over an infinite set of basis functions as

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$$

and that the generalized Fourier coefficients are given by

$$c_n = \frac{\langle\phi_n, f\rangle}{\langle\phi_n, \phi_n\rangle}.$$

In this section we turn to a discussion of approximating $f(x)$ by the partial sums $\sum_{n=1}^{N} c_n\phi_n(x)$ and showing that the Fourier coefficients are the best coefficients minimizing the deviation of the partial sum from $f(x)$. This will lead us to a discussion of the convergence of Fourier series.

More specifically, we set the following goal:

The Method of Least Squares is often attributed to Gauss. The history is discussed by O. B. Sheynin's *C. F. Gauss and the Theory of Errors*. According to Lagrange, Gauss used the method [starting in about 1795] but Legendre was the first to publish it in 1805. This led to a few interesting exchanges.

It is further interesting that Sommerfeld, in his book on *Partial Differential Equations*, used the method of least squares to derive the Legendre polynomials and to obtain the Fourier coefficients. One thinks of approximating $f(x)$ with the partial sums

$$S_N(x) = \sum_{n=N}^{\infty} c_n\phi_n(x)$$

making an error $\varepsilon_N(x)$. Then, as Sommerfeld says, "Following Gauss we consider the mean square error,"

$$M = \frac{1}{b-a}\int_a^b \varepsilon_N^2(x)\,dx,$$

"and reduce M to a minimum ...."

Goal

To find the best approximation of $f(x)$ on $[a,b]$ by $S_N(x) = \sum_{n=1}^{N} c_n\phi_n(x)$ for a set of fixed functions $\phi_n(x)$; i.e., to find the expansion coefficients, c_n, such that $S_N(x)$ approximates $f(x)$ in the least squares sense.

We want to measure the deviation of the finite sum from the given function. Essentially, we want to look at the error made in the approximation. This is done by introducing the mean square deviation:

$$E_N = \int_a^b [f(x) - S_N(x)]^2 \rho(x)\,dx,$$

where we have introduced the weight function $\rho(x) > 0$. It gives us a sense as to how close the Nth partial sum is to $f(x)$.

The mean square deviation.

We want to minimize this deviation by choosing the right c_n's. We begin by inserting the partial sums and expand the square in the integrand:

$$\begin{aligned} E_N &= \int_a^b [f(x) - S_N(x)]^2 \rho(x)\,dx \\ &= \int_a^b \left[f(x) - \sum_{n=1}^{N} c_n\phi_n(x) \right]^2 \rho(x)\,dx \\ &= \int_a^b f^2(x)\rho(x)\,dx - 2\int_a^b f(x) \sum_{n=1}^{N} c_n\phi_n(x)\rho(x)\,dx \\ &\quad + \int_a^b \sum_{n=1}^{N} c_n\phi_n(x) \sum_{m=1}^{N} c_m\phi_m(x)\rho(x)\,dx \end{aligned} \tag{5.76}$$

Looking at the three resulting integrals, we see that the first term is just the inner product of f with itself. The other integrations can be rewritten after interchanging the order of integration and summation. The double sum can be reduced to a single sum using the orthogonality of the ϕ_n's. Thus, we have

$$\begin{aligned} E_N &= \langle f, f\rangle - 2\sum_{n=1}^{N} c_n\langle f, \phi_n\rangle + \sum_{n=1}^{N}\sum_{m=1}^{N} c_n c_m \langle \phi_n, \phi_m\rangle \\ &= \langle f, f\rangle - 2\sum_{n=1}^{N} c_n\langle f, \phi_n\rangle + \sum_{n=1}^{N} c_n^2 \langle \phi_n, \phi_n\rangle. \end{aligned} \tag{5.77}$$

We are interested in finding the coefficients, so we will complete the square in c_n. Focusing on the last two terms, we have

$$\begin{aligned} & -2\sum_{n=1}^{N} c_n\langle f, \phi_n\rangle + \sum_{n=1}^{N} c_n^2 \langle \phi_n, \phi_n\rangle \\ =\ & \sum_{n=1}^{N} \langle \phi_n, \phi_n\rangle c_n^2 - 2\langle f, \phi_n\rangle c_n \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{N}\langle\phi_n,\phi_n\rangle\left[c_n^2 - \frac{2\langle f,\phi_n\rangle}{\langle\phi_n,\phi_n\rangle}c_n\right] \\
&= \sum_{n=1}^{N}\langle\phi_n,\phi_n\rangle\left[\left(c_n - \frac{\langle f,\phi_n\rangle}{\langle\phi_n,\phi_n\rangle}\right)^2 - \left(\frac{\langle f,\phi_n\rangle}{\langle\phi_n,\phi_n\rangle}\right)^2\right].
\end{aligned} \tag{5.78}
$$

To this point we have shown that the mean square deviation is given as

$$E_N = \langle f,f\rangle + \sum_{n=1}^{N}\langle\phi_n,\phi_n\rangle\left[\left(c_n - \frac{\langle f,\phi_n\rangle}{\langle\phi_n,\phi_n\rangle}\right)^2 - \left(\frac{\langle f,\phi_n\rangle}{\langle\phi_n,\phi_n\rangle}\right)^2\right].$$

So, E_N is minimized by choosing

$$c_n = \frac{\langle f,\phi_n\rangle}{\langle\phi_n,\phi_n\rangle}.$$

However, these are the Fourier Coefficients. This minimization is often referred to as Minimization in Least Squares Sense.

Minimization in Least Squares Sense

Inserting the Fourier coefficients into the mean square deviation yields

Bessel's Inequality.

$$0 \le E_N = \langle f,f\rangle - \sum_{n=1}^{N} c_n^2\langle\phi_n,\phi_n\rangle.$$

Thus, we obtain Bessel's Inequality:

$$\langle f,f\rangle \ge \sum_{n=1}^{N} c_n^2\langle\phi_n,\phi_n\rangle.$$

Convergence in the mean.

For convergence, we next let N get large and see if the partial sums converge to the function. In particular, we say that the infinite series converges in the mean if

$$\int_a^b [f(x) - S_N(x)]^2\rho(x)\,dx \;\to 0 \text{ as } N\to\infty.$$

Letting N get large in Bessel's inequality shows that the sum $\sum_{n=1}^N c_n^2\langle\phi_n,\phi_n\rangle$ converges if

$$\langle f,f\rangle = \int_a^b f^2(x)\rho(x)\,dx < \infty.$$

The space of all such f is denoted $L^2_\rho(a,b)$, the space of square integrable functions on (a,b) with weight $\rho(x)$.

From the nth term divergence test from calculus we know that $\sum a_n$ converges implies that $a_n \to 0$ as $n\to\infty$. Therefore, in this problem the terms $c_n^2\langle\phi_n,\phi_n\rangle$ approach zero as n gets large. This is only possible if the c_n's go to zero as n gets large. Thus, if $\sum_{n=1}^N c_n\phi_n$ converges in the mean to f, then $\int_a^b[f(x)-\sum_{n=1}^N c_n\phi_n]^2\rho(x)\,dx$ approaches zero as $N\to\infty$. This implies from the above derivation of Bessel's inequality that

$$\langle f,f\rangle - \sum_{n=1}^{N} c_n^2(\phi_n,\phi_n) \to 0.$$

This leads to Parseval's equality:

$$\langle f, f \rangle = \sum_{n=1}^{\infty} c_n^2 \langle \phi_n, \phi_n \rangle.$$

Parseval's equality holds if and only if

Parseval's equality.

$$\lim_{N\to\infty} \int_a^b \left(f(x) - \sum_{n=1}^{N} c_n \phi_n(x) \right)^2 \rho(x)\, dx = 0.$$

If this is true for every square integrable function in $L_\rho^2(a,b)$, then the set of functions $\{\phi_n(x)\}_{n=1}^{\infty}$ is said to be complete. One can view these functions as an infinite dimensional basis for the space of square integrable functions on (a,b) with weight $\rho(x) > 0$.

One can extend the above limit $c_n \to 0$ as $n \to \infty$, by assuming that $\frac{\phi_n(x)}{\|\phi_n\|}$ is uniformly bounded and that $\int_a^b |f(x)|\rho(x)\, dx < \infty$. This is the Riemann-Lebesgue Lemma, but will not be proven here.

Riemann-Lebesgue Lemma.

Problems

1. Consider the set of vectors $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$.

a. Use the Gram-Schmidt process to find an orthonormal basis for R^3 using this set in the given order.

b. What do you get if you do reverse the order of these vectors?

2. Use the Gram-Schmidt process to find the first four orthogonal polynomials satisfying the following:

a. Interval: $(-\infty, \infty)$ Weight Function: e^{-x^2}.

b. Interval: $(0, \infty)$ Weight Function: e^{-x}.

3. Find $P_4(x)$ using

a. The Rodrigues' Formula in Equation (5.22).

b. The three term recursion formula in Equation (5.24).

4. In Equations (5.34)-(5.41) we provide several identities for Legendre polynomials. Derive the results in Equations (5.35)-(5.41) as described in the text. Namely,

a. Differentiating Equation (5.34) with respect to x, derive Equation (5.35).

b. Derive Equation (5.36) by differentiating $g(x,t)$ with respect to x and rearranging the resulting infinite series.

c. Combining the last result with Equation (5.34), derive Equations (5.37)-(5.38).

d. Adding and subtracting Equations (5.37)-(5.38), obtain Equations (5.39)-(5.40).

e. Derive Equation (5.41) using some of the other identities.

5. Use the recursion relation (5.24) to evaluate $\int_{-1}^{1} xP_n(x)P_m(x)\,dx$, $n \leq m$. Namely, insert $xP_n(x) = \frac{n+1}{(2n+1)}P_{n+1}(x) + \frac{n}{(2n+1)}P_{n-1}(x)$ in the integral and use the orthogonality of the Legendre polynomials to evaluate the integral.

6. Expand the following in a Fourier-Legendre series for $x \in (-1, 1)$.

a. $f(x) = x^2$.

b. $f(x) = 5x^4 + 2x^3 - x + 3$.

c. $f(x) = \begin{cases} -1, & -1 < x < 0, \\ 1, & 0 < x < 1. \end{cases}$

d. $f(x) = \begin{cases} x, & -1 < x < 0, \\ 0, & 0 < x < 1. \end{cases}$

7. Use integration by parts to show $\Gamma(x+1) = x\Gamma(x)$.

8. Prove the double factorial identities:

$$(2n)!! = 2^n n!$$

and

$$(2n-1)!! = \frac{(2n)!}{2^n n!}.$$

9. Express the following as Gamma functions. Namely, noting the form $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt$ and using an appropriate substitution, each expression can be written in terms of a Gamma function.

a. $\int_0^\infty x^{2/3} e^{-x}\,dx$.

b. $\int_0^\infty x^5 e^{-x^2}\,dx$

c. $\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^n\,dx$

10. Show that

$$-\int_0^1 x^k \ln x\,dx = \frac{1}{(k+1)^2}, \quad k > -1$$

by first transforming the integral into a Gamma function.

11. Gaussian integrals are important in statistical mechanics and quantum mechanics. Show that

a. $\int_0^\infty x^{2n+1} e^{-ax^2}\,dx = \frac{n!}{2a^{n+1}}$.

b. $\int_0^\infty x^{2n} e^{-ax^2}\,dx = \frac{(2n-1)!!}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$.

12. Prove the double factorial identities:

$$\int_0^\infty e^{-x^4}\,dx = \Gamma\left(\frac{5}{4}\right) = \left(\frac{1}{4}\right)!.$$

13. The coefficients C_k^p in the binomial expansion for $(1+x)^p$ are given by

$$C_k^p = \frac{p(p-1)\cdots(p-k+1)}{k!}.$$

a. Write C_k^p in terms of Gamma functions.

b. For $p = 1/2$ use the properties of Gamma functions to write $C_k^{1/2}$ in terms of factorials.

c. Confirm you answer in part b by deriving the Maclaurin series expansion of $(1+x)^{1/2}$.

14. The Hermite polynomials, $H_n(x)$, satisfy the following:

i. $< H_n, H_m >= \int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x)\,dx = \sqrt{\pi}2^n n!\delta_{n,m}$.

ii. $H_n'(x) = 2nH_{n-1}(x)$.

iii. $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$.

iv. $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$.

Using these, show that

a. $H_n'' - 2xH_n' + 2nH_n = 0$. [Use properties ii. and iii.]

b. $\int_{-\infty}^{\infty} xe^{-x^2} H_n(x)H_m(x)\,dx = \sqrt{\pi}2^{n-1}n!\left[\delta_{m,n-1} + 2(n+1)\delta_{m,n+1}\right]$. [Use properties i. and iii.]

c. $H_n(0) = \begin{cases} 0, & n \text{ odd}, \\ (-1)^m \frac{(2m)!}{m!}, & n = 2m. \end{cases}$ [Let $x = 0$ in iii. and iterate. Note from iv. that $H_0(x) = 1$ and $H_1(x) = 2x$.]

15. In Maple one can type **simplify(LegendreP(2*n-2,0)-LegendreP(2*n,0));** to find a value for $P_{2n-2}(0) - P_{2n}(0)$. It gives the result in terms of Gamma functions. However, in Example 5.10 for Fourier-Legendre series, the value is given in terms of double factorials! So, we have

$$P_{2n-2}(0) - P_{2n}(0) = \frac{\sqrt{\pi}(4n-1)}{2\Gamma(n+1)\Gamma\left(\frac{3}{2}-n\right)} = (-1)^n \frac{(2n-3)!!}{(2n-2)!!}\frac{4n-1}{2n}.$$

You will verify that both results are the same by doing the following:

a. Prove that $P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}$ using the generating function and a binomial expansion.

b. Prove that $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$ using $\Gamma(x) = (x-1)\Gamma(x-1)$ and iteration.

c. Verify the result from Maple that $P_{2n-2}(0) - P_{2n}(0) = \frac{\sqrt{\pi}(4n-1)}{2\Gamma(n+1)\Gamma\left(\frac{3}{2}-n\right)}$.

d. Can either expression for $P_{2n-2}(0) - P_{2n}(0)$ be simplified further?

16. A solution Bessel's equation, $x^2y'' + xy' + (x^2 - n^2)y = 0,$, can be found using the guess $y(x) = \sum_{j=0}^{\infty} a_j x^{j+n}$. One obtains the recurrence relation $a_j = \frac{-1}{j(2n+j)} a_{j-2}$. Show that for $a_0 = (n!2^n)^{-1}$ we get the Bessel function of the first kind of order n from the even values $j = 2k$:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k}.$$

17. Use the infinite series in the last problem to derive the derivative identities (5.70) and (5.60):

a. $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$.

b. $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$.

18. Prove the following identities based on those in the last problem.

a. $J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$.

b. $J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x)$.

19. Use the derivative identities of Bessel functions,(5.70)-(5.60), and integration by parts to show that

$$\int x^3 J_0(x)\, dx = x^3 J_1(x) - 2x^2 J_2(x) + C.$$

20. Use the generating function to find $J_n(0)$ and $J_n'(0)$.

21. Bessel functions $J_p(\lambda x)$ are solutions of $x^2y'' + xy' + (\lambda^2 x^2 - p^2)y = 0$. Assume that $x \in (0,1)$ and that $J_p(\lambda) = 0$ and $J_p(0)$ is finite.

a. Show that this equation can be written in the form

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (\lambda^2 x - \frac{p^2}{x})y = 0.$$

This is the standard Sturm-Liouville form for Bessel's equation.

b. Prove that

$$\int_0^1 x J_p(\lambda x) J_p(\mu x)\, dx = 0, \quad \lambda \neq \mu$$

by considering

$$\int_0^1 \left[J_p(\mu x)\frac{d}{dx}\left(x\frac{d}{dx}J_p(\lambda x)\right) - J_p(\lambda x)\frac{d}{dx}\left(x\frac{d}{dx}J_p(\mu x)\right)\right] dx.$$

Thus, the solutions corresponding to different eigenvalues (λ, μ) are orthogonal.

c. Prove that

$$\int_0^1 x\left[J_p(\lambda x)\right]^2 dx = \frac{1}{2}J_{p+1}^2(\lambda) = \frac{1}{2}J_p'^2(\lambda).$$

22. We can rewrite Bessel functions, $J_\nu(x)$, in a form which will allow the order to be non-integer by using the gamma function. You will need the results from Problem 15b for $\Gamma\left(k + \frac{1}{2}\right)$.

a. Extend the series definition of the Bessel function of the first kind of order ν, $J_\nu(x)$, for $\nu \geq 0$ by writing the series solution for $y(x)$ in Problem 16 using the gamma function.

b. Extend the series to $J_{-\nu}(x)$, for $\nu \geq 0$. Discuss the resulting series and what happens when ν is a positive integer.

c. Use these results to obtain the closed form expressions

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x,$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

d. Use the results in part c with the recursion formula for Bessel functions to obtain a closed form for $J_{3/2}(x)$.

23. In this problem you will derive the expansion

$$x^2 = \frac{c^2}{2} + 4 \sum_{j=2}^{\infty} \frac{J_0(\alpha_j x)}{\alpha_j^2 J_0(\alpha_j c)}, \quad 0 < x < c,$$

where the $\alpha_j' s$ are the positive roots of $J_1(\alpha c) = 0$, by following the below steps.

a. List the first five values of α for $J_1(\alpha c) = 0$ using the Table 5.3 and Figure 5.8. [Note: Be careful determining α_1.]

b. Show that $\|J_0(\alpha_1 x)\|^2 = \frac{c^2}{2}$. Recall,

$$\|J_0(\alpha_j x)\|^2 = \int_0^c x J_0^2(\alpha_j x)\, dx.$$

c. Show that $\|J_0(\alpha_j x)\|^2 = \frac{c^2}{2} \left[J_0(\alpha_j c)\right]^2$, $j = 2, 3, \ldots$. (This is the most involved step.) First note from Problem 21 that $y(x) = J_0(\alpha_j x)$ is a solution of

$$x^2 y'' + x y' + \alpha_j^2 x^2 y = 0.$$

i. Verify the Sturm-Liouville form of this differential equation: $(xy')' = -\alpha_j^2 x y$.

ii. Multiply the equation in part i. by $y(x)$ and integrate from $x = 0$ to $x = c$ to obtain

$$\begin{aligned} \int_0^c (xy')' y\, dx &= -\alpha_j^2 \int_0^c x y^2\, dx \\ &= -\alpha_j^2 \int_0^c x J_0^2(\alpha_j x)\, dx. \end{aligned} \tag{5.79}$$

iii. Noting that $y(x) = J_0(\alpha_j x)$, integrate the left hand side by parts and use the following to simplify the resulting equation.

1. $J_0'(x) = -J_1(x)$ from Equation (5.60).
2. Equation (5.63).
3. $J_2(\alpha_j c) + J_0(\alpha_j c) = 0$ from Equation (5.61).

iv. Now you should have enough information to complete this part.

d. Use the results from parts b and c and Problem 16 to derive the expansion coefficients for

$$x^2 = \sum_{j=1}^{\infty} c_j J_0(\alpha_j x)$$

in order to obtain the desired expansion.

6
Problems in Higher Dimensions

"Equations of such complexity as are the equations of the gravitational field can be found only through the discovery of a logically simple mathematical condition that determines the equations completely or at least almost completely."

"What I have to say about this book can be found inside this book." Albert Einstein (1879-1955)

In this chapter we will explore several examples of the solution of initial-boundary value problems involving higher spatial dimensions. These are described by higher dimensional partial differential equations, such as the ones presented in Table 1.1 in Chapter 1. The spatial domains of the problems span many different geometries, which will necessitate the use of rectangular, polar, cylindrical, or spherical coordinates.

We will solve many of these problems using the method of separation of variables, which we first saw in Chapter 1. Using separation of variables will result in a system of ordinary differential equations for each problem. Adding the boundary conditions, we will need to solve a variety of eigenvalue problems. The product solutions that result will involve trigonometric or some of the special functions that we had encountered in Chapter 5. These methods are used in solving vibrations of membranes in different geometries, cake baking, the hydrogen atom in quantum mechanics, and electrostatic problems in electrodynamics. We will bring to this discussion many of the tools from earlier in this book showing how much of what we have seen can be used to solve some generic partial differential equations which describe oscillation and diffusion type problems.

As we proceed through the examples in this chapter, we will see some common features. For example, the two key equations that we have studied are the heat equation and the wave equation. For higher dimensional problems these take the form

$$u_t = k\nabla^2 u, \tag{6.1}$$

$$u_{tt} = c^2\nabla^2 u. \tag{6.2}$$

We can separate out the time dependence in each equation. Inserting a guess of $u(\mathbf{r},t) = \phi(\mathbf{r})T(t)$ into the heat and wave equations, we obtain

$$T'\phi = kT\nabla^2\phi, \tag{6.3}$$

$$T''\phi = c^2 T\nabla^2\phi. \tag{6.4}$$

Dividing each equation by $\phi(\mathbf{r})T(t)$, we can separate the time and space dependence just as we had in Chapter 2. In each case we find that a function of time equals a function of the spatial variables. Thus, these functions must be constant functions. We set these equal to the constant $-\lambda$ and find the respective equations

$$\frac{1}{k}\frac{T'}{T} = \frac{\nabla^2\phi}{\phi} = -\lambda, \tag{6.5}$$

$$\frac{1}{c^2}\frac{T''}{T} = \frac{\nabla^2\phi}{\phi} = -\lambda. \tag{6.6}$$

The sign of λ will be taken to be positive ($\lambda > 0$) since we expect decaying solutions in time for the heat equation and oscillations in time for the wave equation.

The respective equations for the temporal functions $T(t)$ are given by

$$T' = -\lambda k T, \tag{6.7}$$

$$T'' + c^2\lambda T = 0. \tag{6.8}$$

These are easily solved as we had seen in Chapter 2. We have

$$T(t) = T(0)e^{-\lambda k t}, \tag{6.9}$$

$$T(t) = a\cos\omega t + b\sin\omega t, \quad \omega = c\sqrt{\lambda}, \tag{6.10}$$

respectively, ,where $T(0)$, a, and b are integration constants and ω is the angular frequency of vibration.

The Helmholtz equation.

In both cases the spatial equation is of the same form,

$$\nabla^2\phi + \lambda\phi = 0. \tag{6.11}$$

The Helmholtz equation is named after Hermann Ludwig Ferdinand von Helmholtz (1821-1894). He was both a physician and a physicist and made significant contributions in physiology, optics, acoustics, and electromagnetism.

This equation is called the Helmholtz equation. For one dimensional problems, which we have already solved, the Helmholtz equation takes the form $\phi'' + \lambda\phi = 0$. We had to impose the boundary conditions and found that there were a discrete set of eigenvalues, λ_n, and associated eigenfunctions, ϕ_n.

In higher dimensional problems we need to further separate out the spatial dependence. We will again use the boundary conditions to find the eigenvalues, λ, and eigenfunctions, $\phi(\mathbf{r})$, for the Helmholtz equation, though the eigenfunctions will be labeled with more than one index. The resulting boundary value problems are often second order ordinary differential equations, which can be set up as Sturm-Liouville problems. We know from Chapter 5 that such problems possess an orthogonal set of eigenfunctions. These can then be used to construct a general solution from the product solutions which may involve elementary, or special, functions, such as Legendre polynomials and Bessel functions.

We will begin our study of higher dimensional problems by considering the vibrations of two dimensional membranes. First we will solve the problem of a vibrating rectangular membrane and then we will turn our attention to a vibrating circular membrane. The rest of the chapter will be devoted to the study of other two and three dimensional problems possessing cylindrical or spherical symmetry.

6.1 Vibrations of Rectangular Membranes

OUR FIRST EXAMPLE WILL BE THE STUDY of the vibrations of a rectangular membrane. You can think of this as a drumhead with a rectangular cross section as shown in Figure 6.1. We stretch the membrane over the drumhead and fasten the material to the boundary of the rectangle. The height of the vibrating membrane is described by its height from equilibrium, $u(x,y,t)$.

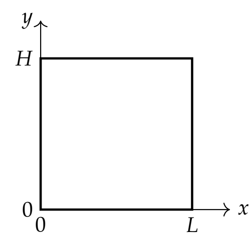


Figure 6.1: The rectangular membrane of length L and width H. There are fixed boundary conditions along the edges.

Example 6.1. The vibrating rectangular membrane.

The problem is given by the two dimensional wave equation in Cartesian coordinates,

$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad t > 0, 0 < x < L, 0 < y < H, \tag{6.12}$$

a set of boundary conditions,

$$\begin{aligned} u(0,y,t) = 0, \quad u(L,y,t) = 0, \quad t > 0, \quad 0 < y < H, \\ u(x,0,t) = 0, \quad u(x,H,t) = 0, \quad t > 0, \quad 0 < x < L, \end{aligned} \tag{6.13}$$

and a pair of initial conditions (since the equation is second order in time),

$$u(x,y,0) = f(x,y), \quad u_t(x,y,0) = g(x,y). \tag{6.14}$$

The first step is to separate the variables: $u(x,y,t) = X(x)Y(y)T(t)$. Inserting the guess, $u(x,y,t)$ into the wave equation, we have

$$X(x)Y(y)T''(t) = c^2 \left(X''(x)Y(y)T(t) + X(x)Y''(y)T(t) \right).$$

Dividing by both $u(x,y,t)$ and c^2, we obtain

$$\underbrace{\frac{1}{c^2}\frac{T''}{T}}_{\text{Function of } t} = \underbrace{\frac{X''}{X} + \frac{Y''}{Y}}_{\text{Function of } x \text{ and } y} = -\lambda. \tag{6.15}$$

We see that we have a function of t equals a function of x and y. Thus, both expressions are constant. We expect oscillations in time, so we choose the constant λ to be positive, $\lambda > 0$. (Note: As usual, the primes mean differentiation with respect to the specific dependent variable. So, there should be no ambiguity.)

These lead to two equations:

$$T'' + c^2\lambda T = 0, \tag{6.16}$$

and

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda. \tag{6.17}$$

We note that the spatial equation is just the separated form of Helmholtz's equation with $\phi(x,y) = X(x)Y(y)$.

The first equation is easily solved. We have

$$T(t) = a\cos\omega t + b\sin\omega t, \tag{6.18}$$

where

$$\omega = c\sqrt{\lambda}. \tag{6.19}$$

This is the angular frequency in terms of the separation constant, or eigenvalue. It leads to the frequency of oscillations for the various harmonics of the vibrating membrane as

$$\nu = \frac{\omega}{2\pi} = \frac{c}{2\pi}\sqrt{\lambda}. \tag{6.20}$$

Once we know λ, we can compute these frequencies.

Next we solve the spatial equation. We need carry out another separation of variables. Rearranging the spatial equation, we have

$$\underbrace{\frac{X''}{X}}_{\text{Function of } x} = \underbrace{-\frac{Y''}{Y} - \lambda}_{\text{Function of } y} = -\mu. \tag{6.21}$$

Here we have a function of x equal to a function of y. So, the two expressions are constant, which we indicate with a second separation constant, $-\mu < 0$. We pick the sign in this way because we expect oscillatory solutions for $X(x)$. This leads to two equations:

$$\begin{aligned} X'' + \mu X &= 0, \\ Y'' + (\lambda - \mu)Y &= 0. \end{aligned} \tag{6.22}$$

We now impose the boundary conditions. We have $u(0,y,t) = 0$ for all $t > 0$ and $0 < y < H$. This implies that $X(0)Y(y)T(t) = 0$ for all t and y in the domain. This is only true if $X(0) = 0$. Similarly, from the other boundary conditions we find that $X(L) = 0$, $Y(0) = 0$, and $Y(H) = 0$. We note that homogeneous boundary conditions are important in carrying out this process. Nonhomogeneous boundary conditions could be imposed just like we had in Section 1.7, but we still need the solutions for homogeneous boundary conditions before tackling the more general problems.

In summary, the boundary value problems we need to solve are:

$$\begin{aligned} X'' + \mu X = 0, &\quad X(0) = 0, X(L) = 0. \\ Y'' + (\lambda - \mu)Y = 0, &\quad Y(0) = 0, Y(H) = 0. \end{aligned} \tag{6.23}$$

We have seen boundary value problems of these forms in Chapter 2. The solutions of the first eigenvalue problem are

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad \mu_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$

The second eigenvalue problem is solved in the same manner. The differences from the first problem are that the "eigenvalue" is $\lambda - \mu$, the independent variable is y, and the interval is $[0, H]$. Thus, we can quickly write down the solutions as

$$Y_m(y) = \sin\frac{m\pi x}{H}, \quad \lambda - \mu_m = \left(\frac{m\pi}{H}\right)^2, \quad m = 1, 2, 3, \ldots.$$

At this point we need to be careful about the indexing of the separation constants. So far, we have seen that μ depends on n and that the quantity $\kappa = \lambda - \mu$ depends on m. Solving for λ, we should write $\lambda_{nm} = \mu_n + \kappa_m$, or

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2, \quad n, m = 1, 2, \ldots. \tag{6.24}$$

The harmonics for the vibrating rectangular membrane are given by the frequencies

$$\nu_{nm} = \frac{c}{2}\sqrt{\left(\frac{n}{L}\right)^2 + \left(\frac{m}{H}\right)^2},$$

for $n, m = 1, 2, \ldots$.

Since $\omega = c\sqrt{\lambda}$, we have that the discrete frequencies of the harmonics are given by

$$\omega_{nm} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}, \quad n, m = 1, 2, \ldots. \tag{6.25}$$

We have successfully carried out the separation of variables for the wave equation for the vibrating rectangular membrane. The product solutions can be written as

$$u_{nm} = (a\cos\omega_{nm}t + b\sin\omega_{nm}t)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H} \tag{6.26}$$

and the most general solution is written as a linear combination of the product solutions,

$$u(x, y, t) = \sum_{n,m=1}^{\infty}(a_{nm}\cos\omega_{nm}t + b_{nm}\sin\omega_{nm}t)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}.$$

Here we used $\sum_{n,m}$ to indicate a double sum.

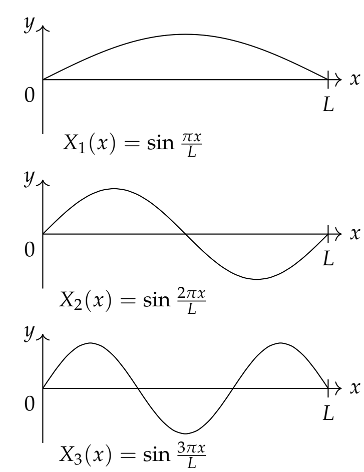


Figure 6.2: The first harmonics of the vibrating string

Before we carry the general solution any further, we will first concentrate on the two dimensional harmonics of this membrane. For the vibrating string the nth harmonic corresponds to the function $\sin\frac{n\pi x}{L}$ and several are shown in Figure 6.2. The various harmonics correspond to the pure tones supported by the string. These then lead to the corresponding frequencies that one would hear. The actual shapes of the harmonics are sketched by locating the nodes, or places on the string that do not move.

In the same way, we can explore the shapes of the harmonics of the vibrating membrane. These are given by the spatial functions

$$\phi_{nm}(x, y) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}. \tag{6.27}$$

A discussion of the nodal lines.

Instead of nodes, we will look for the nodal curves, or nodal lines. These are the points (x, y) at which $\phi_{nm}(x, y) = 0$. Of course, these depend on the indices, n and m.

For example, when $n = 1$ and $m = 1$, we have

$$\sin\frac{\pi x}{L}\sin\frac{\pi y}{H} = 0.$$

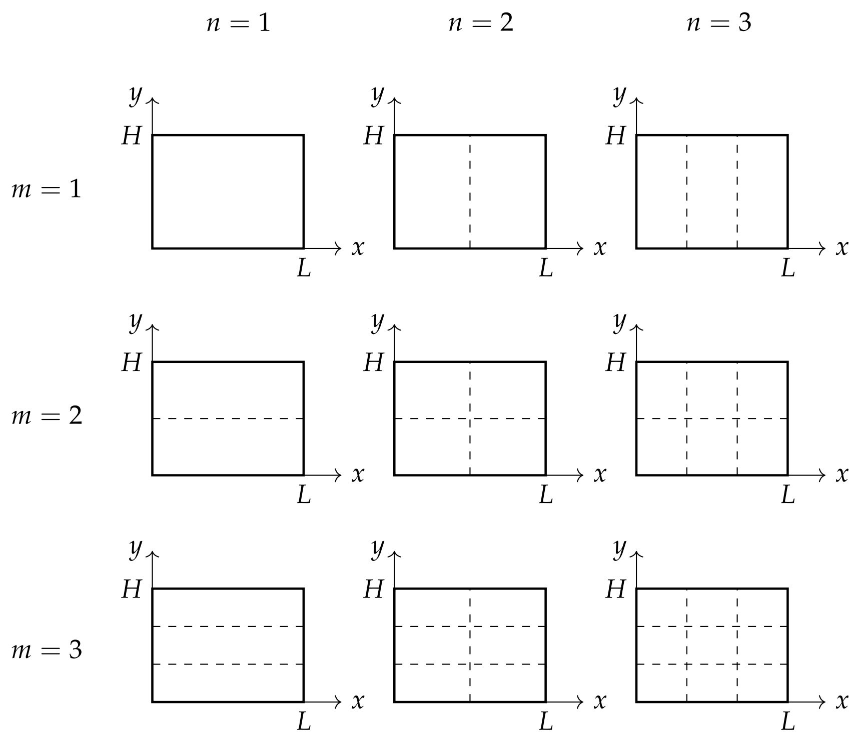


Figure 6.3: The first few modes of the vibrating rectangular membrane. The dashed lines show the nodal lines indicating the points that do not move for the particular mode. Compare these the nodal lines to the 3D view in Figure 6.1

These are zero when either

$$\sin \frac{\pi x}{L} = 0, \quad \text{or} \quad \sin \frac{\pi y}{H} = 0.$$

Of course, this can only happen for $x = 0, L$ and $y = 0, H$. Thus, there are no interior nodal lines as seen in the first membrane in Figure 6.3.

When $n = 2$ and $m = 1$, we have $y = 0, H$ and

$$\sin \frac{2\pi x}{L} = 0, \quad \text{or} \quad \sin \frac{\pi y}{H} = 0.$$

There are no horizontal interior nodal lines. When $x = 0, \frac{L}{2}, L$, $\sin \dfrac{2\pi x}{L} = 0$. Therefore, there is one interior nodal line at $x = \frac{L}{2}$. These points stay fixed during the oscillation and all other points oscillate on either side of this line. A similar solution shape results for the (1,2)-mode; i.e., $n = 1$ and $m = 2$. These can be seen in Figure 6.3.

In Figure 6.3 we show the nodal lines for several modes for $n, m = 1, 2, 3$ with different columns corresponding to different n-values while the rows are labeled with different m-values. The blocked regions appear to vibrate independently. A better view is the three dimensional view depicted in Figure 6.1 . The frequencies of vibration are easily computed using the formula for ω_{nm}.

The vibrations of the rectangular membrane differ from other examples, such as the vibrating string or circular membrane in the next section, in that it is possible for two different mode shapes to have exactly the same frequency. We see that in Equation (6.25) two frequencies would be the

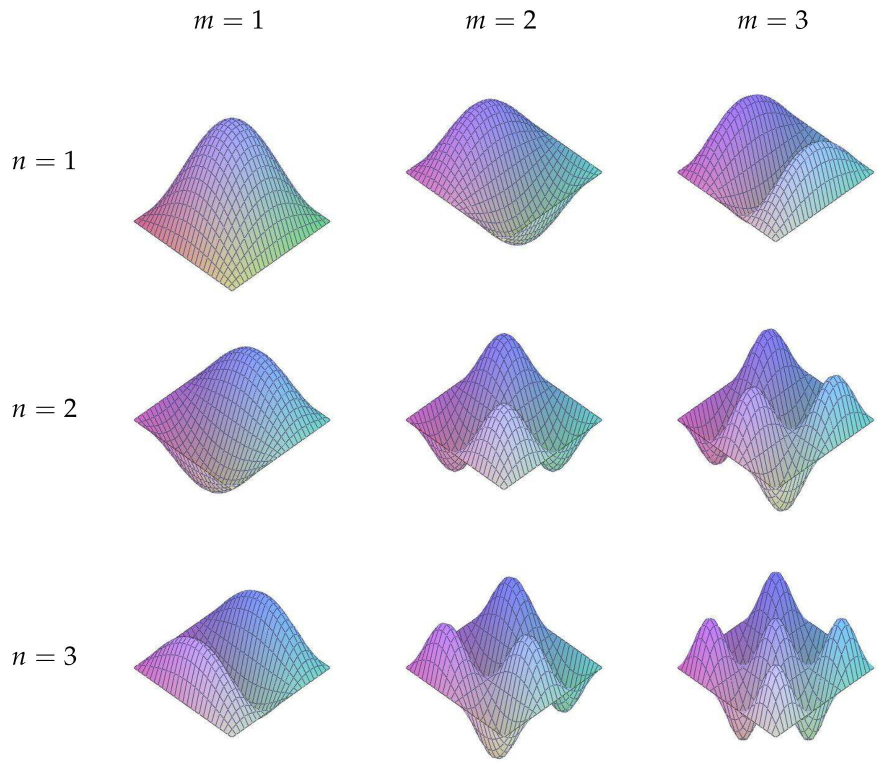


Table 6.1: A three dimensional view of the vibrating rectangular membrane for the lowest modes. Compare these images with the nodal lines in Figure 6.3

same if

$$\frac{n}{L} = \frac{m}{H}.$$

In the case of a square membrane,$H = L$, we could have $m = n$. In such a case the product solutions $phi_{nm}(x, y)$ and $phi_{mn}(x, y)$ would oscillate at the same frequency. Then, the product of spatial solutions would take the form

$$\Phi_{nm}(x, y) = \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{L} + \sin \frac{m\pi x}{L} \sin \frac{n\pi y}{L}.$$

In this case, the parts of the membrane for which $\Phi_{nm} = 0$, do not move for any time. These nodal curves give rise to what are called degenerate modes. In Figure 6.4 we show examples of degenerate modes for a square rectangular membrane.

Notice the different features as compared to those in Figure 6.3. There is also some symmetry in the grid of modes, as may be expected. In particular the 3-1 and 1-3 modes appear to be almost circular looking in appearance. In Figure 6.5 we see two snapshots of the vibrating membrane for this degenerate mode which are separated in time by half a period. Compare these with the nodal curve in Figure 6.4. Can one make out the curve in these figures?

As an aside, you might ask if this curve can be described by an equation. We simplify the problem by taking $L = H = 1$. Then, the nodal curves for the 3-1 and 1-3 modes is given by

$$\Phi_{31} = \sin 3\pi x \sin \pi y + \sin \pi x \sin 3\pi y = 0.$$

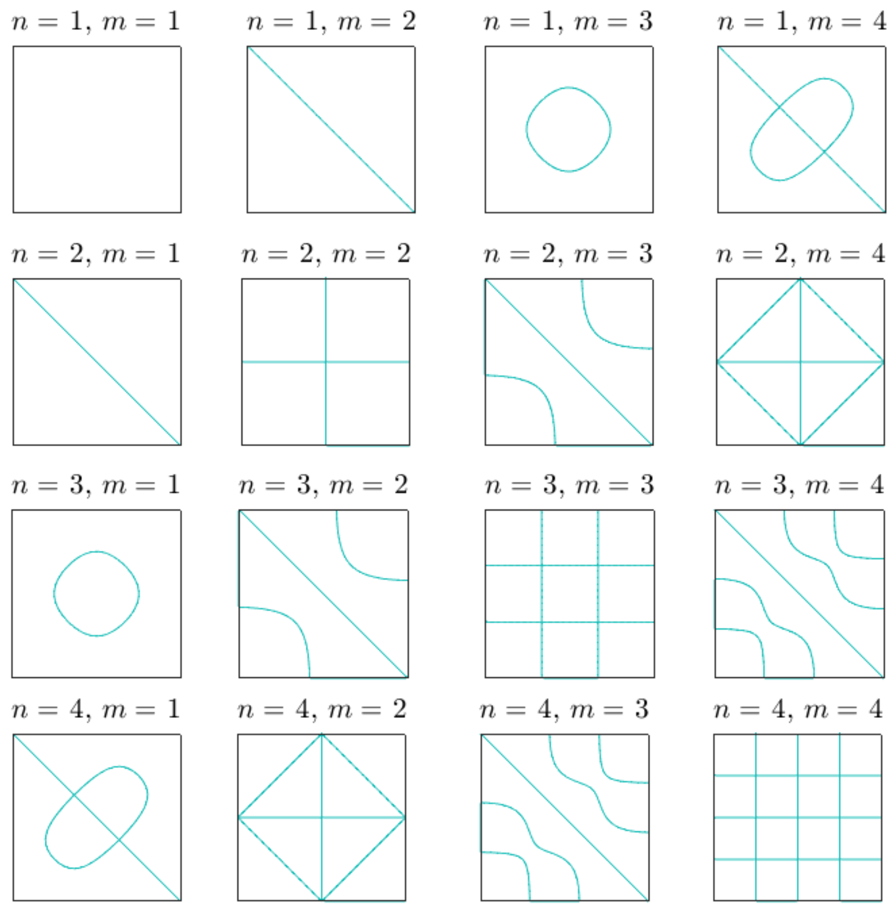


Figure 6.4: Examples of nodal curves for degenerate modes for a square rectangular membrane.

Can we solve this equation? We can make an attempt by recalling the trigonometric identity

$$\sin 3x = 3\sin x - 4\sin^3 x.$$

Then, after substitution, we have

$$\begin{aligned}
\Phi_{31}(x,y) &= \sin 3\pi x \sin \pi y + \sin \pi x \sin 3\pi y \\
&= \left(3\sin \pi x - 4\sin^3 \pi x\right) \sin \pi y + \sin \pi x \left(3 \sin \pi y - 4\sin^3 \pi y\right) \\
&= -2\sin(\pi y)\sin(\pi x)\left(2\sin(\pi x)^2 + 2\sin(\pi y)^2 - 3\right). \qquad (6.28)
\end{aligned}$$

We see that $\Phi_{31}(x,y) = 0$ if x is an integer, y is an integer (corresponding to the boundary of the membrane), or

$$2\sin(\pi x)^2 + 2\sin(\pi y)^2 = 3. \qquad (6.29)$$

It is this curve which is seen in Figure 6.4. This curve is graphed in Figure 6.6.

For completeness, we now return to the general solution and apply the initial conditions, $u(x,y,0) = f(x,y)$, and $u_t(x,y,0) = g(x,y)$. The general

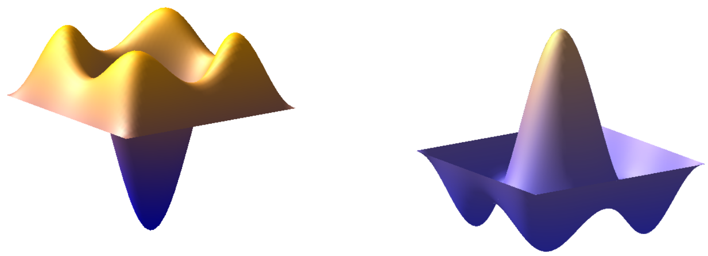


Figure 6.5: Plot of the degenerate mode for a square rectangular membrane with $n = 1$ and $m = 3$ separated a half period apart in its oscillation.

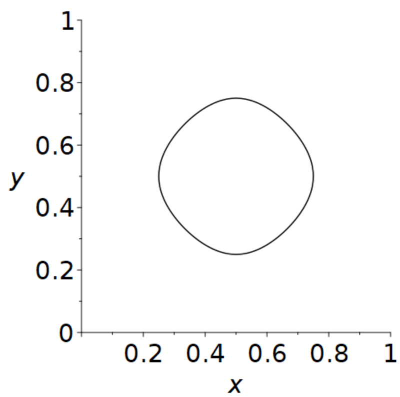


Figure 6.6: Plot of the degenerate mode nodal curve $n = 1$ and $m = 3$ using algebraic form for the curve in Equation (6.29).

solution is given by a linear superposition of the product solutions. There are two indices to sum over. Thus, the general solution is

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(a_{nm}\cos\omega_{nm}t + b_{nm}\sin\omega_{nm}t)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}, \quad (6.30)$$

The general solution for the vibrating rectangular membrane.

where

$$\omega_{nm} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}. \quad (6.31)$$

The first initial condition is $u(x,y,0) = f(x,y)$. Setting $t = 0$ in the general solution, we obtain

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}. \quad (6.32)$$

This is a double Fourier sine series. The goal is to find the unknown coefficients a_{nm}.

The coefficients a_{nm} can be found knowing what we already know about Fourier sine series. We can write the initial condition in Equation (6.32) as the single sum

$$f(x,y) = \sum_{n=1}^{\infty} A_n(y)\sin\frac{n\pi x}{L}, \quad (6.33)$$

where

$$A_n(y) = \sum_{m=1}^{\infty} a_{nm} \sin \frac{m\pi y}{H}. \tag{6.34}$$

We now have two Fourier sine series. Recall from Chapter 2 in Equation (2.73), the coefficients of Fourier sine series give us

$$\begin{aligned} A_n(y) &= \frac{2}{L}\int_0^L f(x,y) \sin \frac{n\pi x}{L}\, dx, \\ a_{nm} &= \frac{2}{H}\int_0^H A_n(y) \sin \frac{m\pi y}{H}\, dy. \end{aligned} \tag{6.35}$$

The Fourier coefficients for the double Fourier sine series.

Inserting the integral for $A_n(y)$ into that for a_{nm}, we have an integral representation for the Fourier coefficients in the double Fourier sine series,

$$a_{nm} = \frac{4}{LH}\int_0^H \int_0^L f(x,y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}\, dxdy. \tag{6.36}$$

We can carry out the same process for satisfying the second initial condition, $u_t(x,y,0) = g(x,y)$ for the initial velocity of each point. Inserting the general solution into this initial condition, we obtain

$$g(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\omega_{nm} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}. \tag{6.37}$$

Again, we have a double Fourier sine series. But, now we can quickly determine the Fourier coefficients using the expression (6.36) for a_{nm} to find that

$$b_{nm} = \frac{4}{\omega_{nm} LH}\int_0^H \int_0^L g(x,y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}\, dxdy. \tag{6.38}$$

The full solution of the vibrating rectangular membrane.

This completes the full solution of the vibrating rectangular membrane problem. Namely, we have obtained the solution

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (a_{nm} \cos \omega_{nm} t + b_{nm} \sin \omega_{nm} t) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}, \tag{6.39}$$

where

$$a_{nm} = \frac{4}{LH}\int_0^H \int_0^L f(x,y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}\, dxdy, \tag{6.40}$$

$$b_{nm} = \frac{4}{\omega_{nm} LH}\int_0^H \int_0^L g(x,y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}\, dxdy, \tag{6.41}$$

and the angular frequencies are given by

$$\omega_{nm} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}. \tag{6.42}$$

6.2 *Vibrations of a Kettle Drum*

In this section we consider the vibrations of a circular membrane of radius a as shown in Figure 6.7. Again we are looking for the harmonics of the vibrating membrane, but with the membrane fixed around the circular boundary given by $x^2 + y^2 = a^2$. However, expressing the boundary condition in Cartesian coordinates is awkward. Namely, we can only write $u(x, y, t) = 0$ for $x^2 + y^2 = a^2$. It is more natural to use polar coordinates as indicated in Figure 6.7. Let the height of the membrane be given by $u = u(r, \theta, t)$ at time t and position (r, θ). Now the boundary condition is given as $u(a, \theta, t) = 0$ for all $t > 0$ and $\theta \in [0, 2\pi]$.

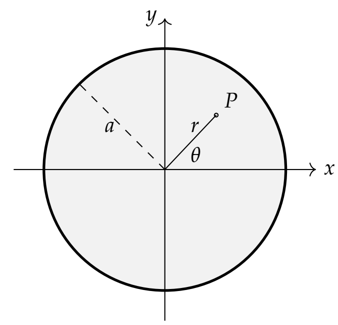


Figure 6.7: The circular membrane of radius a. A general point P on the membrane is given by the distance from the center, r, and the angle, θ. There are fixed boundary conditions along the edge at $r = a$.

Before solving the initial-boundary value problem, we have to cast the full problem in polar coordinates. This means that we need to rewrite the Laplacian in r and θ. To do so would require that we know how to transform derivatives in x and y into derivatives with respect to r and θ. There are general results using curvilinear coordinates for writing the Laplacian in polar coordinates which can be found in standard texts in mathematical method in physics such as seen in Section 6.9. We will use direct methods in cylindrical coordinates for functions, $f = f(r, \theta)$, which are z-independent to show that the Laplacian is given by

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}. \tag{6.43}$$

Derivation of Laplacian in polar coordinates.

We will obtain this result by applying the Chain Rule in higher dimensions. First recall the transformations between polar and Cartesian coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta$$

and

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$

Now, consider a function $f = f(x(r, \theta), y(r, \theta)) = g(r, \theta)$. (Technically, once we transform a given function of Cartesian coordinates we obtain a new function g of the polar coordinates. Many texts do not rigorously distinguish between the two functions.) Thinking of $x = x(r, \theta)$ and $y = y(r, \theta)$, we have from the chain rule for functions of two variables:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial g}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial x} \\ &= \frac{\partial g}{\partial r}\frac{x}{r} - \frac{\partial g}{\partial \theta}\frac{y}{r^2} \\ &= \cos\theta\frac{\partial g}{\partial r} - \frac{\sin\theta}{r}\frac{\partial g}{\partial \theta}. \end{aligned} \tag{6.44}$$

Here we have used

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r};$$

and

$$\frac{\partial \theta}{\partial x} = \frac{d}{dx}\left(\tan^{-1}\frac{y}{x}\right) = \frac{-y/x^2}{1 + \left(\frac{y}{x}\right)^2} = -\frac{y}{r^2}.$$

Similarly,

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\begin{aligned} &= \frac{\partial g}{\partial r}\frac{y}{r} + \frac{\partial g}{\partial \theta}\frac{x}{r^2} \\ &= \sin\theta\frac{\partial g}{\partial r} + \frac{\cos\theta}{r}\frac{\partial g}{\partial \theta}. \end{aligned} \tag{6.45}$$

The 2D Laplacian can now be computed as

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right) \\ &\quad + \sin\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right) \\ &= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial g}{\partial r} - \frac{\sin\theta}{r}\frac{\partial g}{\partial \theta}\right) \\ &\quad - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial g}{\partial r} - \frac{\sin\theta}{r}\frac{\partial g}{\partial \theta}\right) \\ &\quad + \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial g}{\partial r} + \frac{\cos\theta}{r}\frac{\partial g}{\partial \theta}\right) \\ &\quad + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial g}{\partial r} + \frac{\cos\theta}{r}\frac{\partial g}{\partial \theta}\right) \\ &= \cos\theta\left(\cos\theta\frac{\partial^2 g}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial g}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 g}{\partial r\partial \theta}\right) \\ &\quad - \frac{\sin\theta}{r}\left(\cos\theta\frac{\partial^2 g}{\partial \theta\partial r} - \frac{\sin\theta}{r}\frac{\partial^2 g}{\partial \theta^2} - \sin\theta\frac{\partial g}{\partial r} - \frac{\cos\theta}{r}\frac{\partial g}{\partial \theta}\right) \\ &\quad + \sin\theta\left(\sin\theta\frac{\partial^2 g}{\partial r^2} + \frac{\cos\theta}{r}\frac{\partial^2 g}{\partial r\partial \theta} - \frac{\cos\theta}{r^2}\frac{\partial g}{\partial \theta}\right) \\ &\quad + \frac{\cos\theta}{r}\left(\sin\theta\frac{\partial^2 g}{\partial \theta\partial r} + \frac{\cos\theta}{r}\frac{\partial^2 g}{\partial \theta^2} + \cos\theta\frac{\partial g}{\partial r} - \frac{\sin\theta}{r}\frac{\partial g}{\partial \theta}\right) \\ &= \frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}. \end{aligned} \tag{6.46}$$

We have left the result in the form of a Sturm-Liouville operator. Now that we have written the Laplacian in polar coordinates, we can pose the problem of a vibrating circular membrane.

Example 6.2. The vibrating circular membrane.

This problem is given by a partial differential equation,[1]

$$u_{tt} = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right], \tag{6.47}$$

$$t > 0, \quad 0 < r < a, \quad -\pi < \theta < \pi,$$

the boundary condition,

$$u(a,\theta,t) = 0, \quad t > 0, \quad -\pi < \theta < \pi, \tag{6.48}$$

and the initial conditions,

$$\begin{aligned} u(r,\theta,0) &= f(r,\theta), \quad 0 < r < a, -\pi < \theta < \pi, \\ u_t(r,\theta,0) &= g(r,\theta),, \quad 0 < r < a, -\pi < \theta < \pi. \end{aligned} \tag{6.49}$$

[1] Here we state the problem of a vibrating circular membrane. We have chosen $-\pi < \theta < \pi$, but could have just as easily used $0 < \theta < 2\pi$. The symmetric interval about $\theta = 0$ will make the use of boundary conditions simpler.

Now, we are ready to solve this problem using separation of variables. As before, we can separate out the time dependence. Let $u(r,\theta,t) = T(t)\phi(r,\theta)$. As usual, $T(t)$ can be written in terms of sines and cosines. This leads to the Helmholtz equation for $\phi = \phi(r,\theta)$,

$$\nabla^2\phi + \lambda\phi = 0.$$

We now separate the Helmholtz equation by letting $\phi(r,\theta) = R(r)\Theta(\theta)$. This gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R\Theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 R\Theta}{\partial\theta^2} + \lambda R\Theta = 0. \tag{6.50}$$

Dividing by $u = R\Theta$, as usual, leads to

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} + \lambda = 0. \tag{6.51}$$

The last term is a constant. The first term is a function of r. However, the middle term involves both r and θ. This can be remedied by multiplying the equation by r^2. Rearranging the resulting equation, we can separate out the θ-dependence from the radial dependence. Letting μ be another separation constant, we have

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda r^2 = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \mu. \tag{6.52}$$

This gives us two ordinary differential equations:

$$\begin{aligned} \frac{d^2\Theta}{d\theta^2} + \mu\Theta &= 0, \\ r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (\lambda r^2 - \mu)R &= 0. \end{aligned} \tag{6.53}$$

Let's consider the first of these equations. It should look familiar by now. For $\mu > 0$, the general solution is

$$\Theta(\theta) = a\cos\sqrt{\mu}\theta + b\sin\sqrt{\mu}\theta.$$

The next step typically is to apply the boundary conditions in θ. However, when we look at the given boundary conditions in the problem, we do not see anything involving θ. This is a case for which the boundary conditions that are needed are implied and not stated outright.

We can determine the hidden boundary conditions by making some observations. Let's consider the solution corresponding to the endpoints $\theta = \pm\pi$. We note that at these θ-values we are at the same physical point for any $r < a$. So, we would expect the solution to have the same value at $\theta = -\pi$ as it has at $\theta = \pi$. Namely, the solution is continuous at these physical points. Similarly, we expect the slope of the solution to be the same at these points. This can be summarized using the boundary conditions

The boundary conditions in θ are periodic boundary conditions.

$$\Theta(\pi) = \Theta(-\pi), \quad \Theta'(\pi) = \Theta'(-\pi).$$

Such boundary conditions are called periodic boundary conditions.

Let's apply these conditions to the general solution for $\Theta(\theta)$. First, we set $\Theta(\pi) = \Theta(-\pi)$ and use the symmetries of the sine and cosine functions to obtain

$$a\cos\sqrt{\mu}\pi + b\sin\sqrt{\mu}\pi = a\cos\sqrt{\mu}\pi - b\sin\sqrt{\mu}\pi.$$

This implies that

$$\sin\sqrt{\mu}\pi = 0.$$

This can only be true for $\sqrt{\mu} = m$, for $m = 0, 1, 2, 3, \ldots$. Therefore, the eigenfunctions are given by

$$\Theta_m(\theta) = a\cos m\theta + b\sin m\theta, \quad m = 0, 1, 2, 3, \ldots.$$

For the other half of the periodic boundary conditions, $\Theta'(\pi) = \Theta'(-\pi)$, we have that

$$-am\sin m\pi + bm\cos m\pi = am\sin m\pi + bm\cos m\pi.$$

But, this gives no new information since this equation boils down to $bm = bm$..

To summarize what we know at this point, we have found the general solutions to the temporal and angular equations. The product solutions will have various products of $\{\cos\omega t, \sin\omega t\}$ and $\{\cos m\theta, \sin m\theta\}_{m=0}^{\infty}$. We also know that $\mu = m^2$ and $\omega = c\sqrt{\lambda}$.

We still need to solve the radial equation. Inserting $\mu = m^2$, the radial equation has the form

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (\lambda r^2 - m^2)R = 0. \tag{6.54}$$

Expanding the derivative term, we have

$$r^2R''(r) + rR'(r) + (\lambda r^2 - m^2)R(r) = 0. \tag{6.55}$$

The reader should recognize this differential equation from Equation (5.66).[2] It is a Bessel equation with bounded solutions $R(r) = J_m(\sqrt{\lambda}r)$.

[2] You might want to do a change of variables to verify this. Let $x = \sqrt{\lambda}r$ and $y(x) = R(r)$. Then,

$$\frac{d}{dr} = \frac{dx}{dr}\frac{d}{dx} = \sqrt{\lambda}\frac{d}{dx}.$$

Inserting this into the differential equation, we find

$$x^2y'' + xy' + (x^2 - m^2)y = 0.$$

Recall there are two linearly independent solutions of this second order equation: $J_m(\sqrt{\lambda}r)$, the Bessel function of the first kind of order m, and $N_m(\sqrt{\lambda}r)$, the Bessel function of the second kind of order m, or Neumann functions. Plots of these functions are shown in Figures 5.8 and 5.9. So, we have the general solution of the radial equation is

$$R(r) = c_1J_m(\sqrt{\lambda}r) + c_2N_m(\sqrt{\lambda}r).$$

Now we are ready to apply the boundary conditions to the radial factor in the product solutions. Looking at the original problem we find only one condition: $u(a, \theta, t) = 0$ for $t > 0$ and $-\pi < \theta < \pi$. This implies that $R(a) = 0$. But where is the second condition?

This is another unstated boundary condition. Look again at the plots of the Bessel functions. Notice that the Neumann functions are not well behaved at the origin. Do you expect that the solution will become infinite at the center of the drum? No, the solutions should be finite at the center. So, this observation leads to the second boundary condition. Namely, $|R(0)| < \infty$. This implies that $c_2 = 0$.

Now we are left with

$$R(r) = J_m(\sqrt{\lambda} r).$$

We have set $c_1 = 1$ for simplicity. We can apply the vanishing condition at $r = a$. This gives

$$J_m(\sqrt{\lambda} a) = 0.$$

Looking again at the plots of $J_m(x)$, we see that there are an infinite number of zeros, but they are not as easy as π! In Table 6.2 we list the nth zeros of J_m, which were first seen in Table 5.3.

n	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
1	2.405	3.832	5.136	6.380	7.588	8.771
2	5.520	7.016	8.417	9.761	11.065	12.339
3	8.654	10.173	11.620	13.015	14.373	15.700
4	11.792	13.324	14.796	16.223	17.616	18.980
5	14.931	16.471	17.960	19.409	20.827	22.218
6	18.071	19.616	21.117	22.583	24.019	25.430
7	21.212	22.760	24.270	25.748	27.199	28.627
8	24.352	25.904	27.421	28.908	30.371	31.812
9	27.493	29.047	30.569	32.065	33.537	34.989

Table 6.2: The zeros of Bessel Functions, $J_m(j_{mn}) = 0$.

Let's denote the nth zero of $J_m(x)$ by j_{mn}. Then, the boundary condition tells us that

$$\sqrt{\lambda} a = j_{mn}, \quad m = 0, 1, \ldots, \quad n = 1, 2, \ldots.$$

This gives the eigenvalues as

$$\lambda_{mn} = \left(\frac{j_{mn}}{a}\right)^2, \quad m = 0, 1, \ldots, \quad n = 1, 2, \ldots.$$

Thus, the radial function satisfying the boundary conditions is

$$R_{mn}(r) = J_m\left(\frac{j_{mn}}{a} r\right).$$

We are finally ready to write out the product solutions for the vibrating circular membrane. They are given by

Product solutions for the vibrating circular membrane.

$$u(r, \theta, t) = \left\{ \begin{array}{c} \cos \omega_{mn} t \\ \sin \omega_{mn} t \end{array} \right\} \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} J_m\left(\frac{j_{mn}}{a} r\right). \qquad (6.56)$$

Here we have indicated choices with the braces, leading to four different types of product solutions. Also, the angular frequency depends on the zeros of the Bessel functions,

$$\omega_{mn} = \frac{j_{mn}}{a}c, \quad m = 0, 1, \ldots, \quad n = 1, 2, \ldots.$$

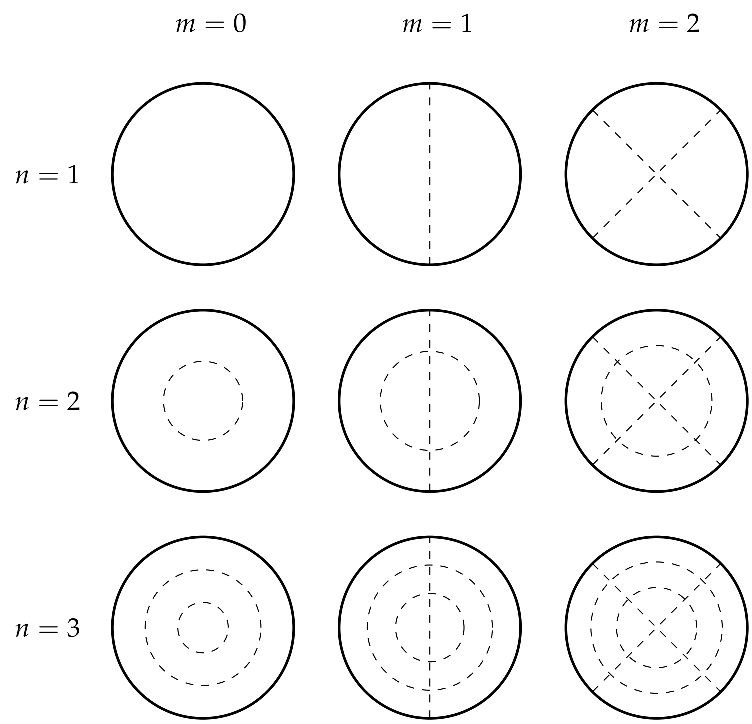


Figure 6.8: The first few modes of the vibrating circular membrane. The dashed lines show the nodal lines indicating the points that do not move for the particular mode. Compare these nodal lines with the three dimensional images in Figure 6.3.

As with the rectangular membrane, we are interested in the shapes of the harmonics. So, we consider the spatial solution ($t = 0$)

$$\phi(r, \theta) = (\cos m\theta) J_m\left(\frac{j_{mn}}{a}r\right).$$

Including the solutions involving $\sin m\theta$ will only rotate these modes. The nodal curves are given by $\phi(r, \theta) = 0$. This can be satisfied if $\cos m\theta = 0$, or $J_m\left(\frac{j_{mn}}{a}r\right) = 0$. The various nodal curves which result are shown in Figure 6.8.

For the angular part, we easily see that the nodal curves are radial lines, $\theta =$const. For $m = 0$, there are no solutions, since $\cos m\theta = 1$ for $m = 0$. in Figure 6.8 this is seen by the absence of radial lines in the first column.

For $m = 1$, we have $\cos\theta = 0$. This implies that $\theta = \pm\frac{\pi}{2}$. These values give the vertical line as shown in the second column in Figure 6.8. For $m = 2$, $\cos 2\theta = 0$ implies that $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$. This results in the two lines shown in the last column of Figure 6.8.

We can also consider the nodal curves defined by the Bessel functions. We seek values of r for which $\frac{j_{mn}}{a}r$ is a zero of the Bessel function and lies in the interval $[0, a]$. Thus, we have

$$\frac{j_{mn}}{a} r = j_{mj}, \quad 1 \le j \le n,$$

or

$$r = \frac{j_{mj}}{j_{mn}} a, \quad 1 \le j \le n.$$

These will give circles of these radii with $j_{mj} \le j_{mn}$, or $j \le n$. For $m = 0$ and $n = 1$, there is only one zero and $r = a$. In fact, for all $n = 1$ modes, there is only one zero giving $r = a$. Thus, the first row in Figure 6.8 shows no interior nodal circles.

For a three dimensional view, one can look at Figure 6.3. Imagine that the various regions are oscillating independently and that the points on the nodal curves are not moving.

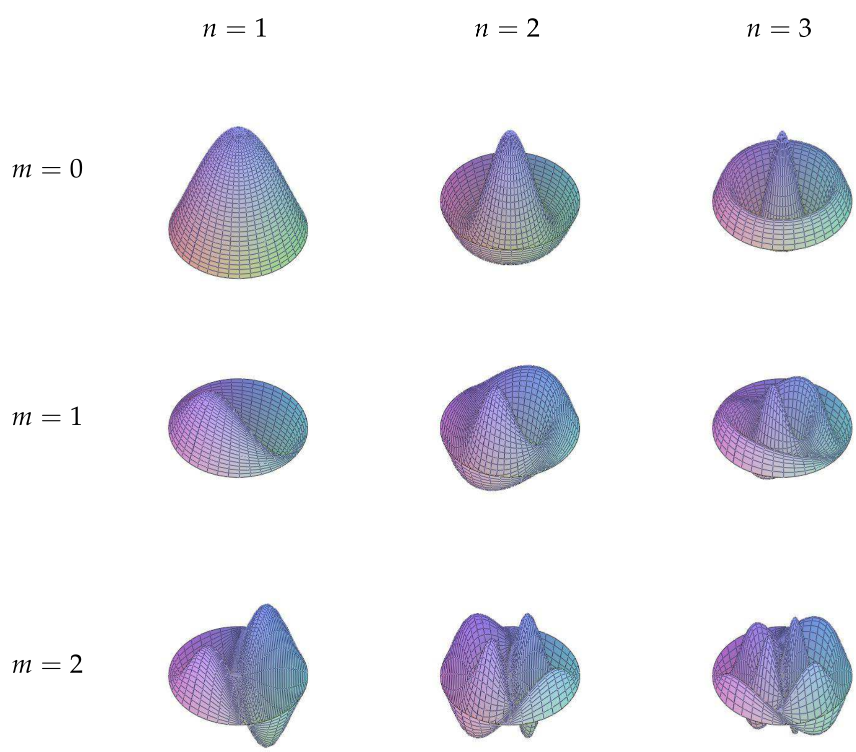


Table 6.3: A three dimensional view of the vibrating circular membrane for the lowest modes. Compare these images with the nodal line plots in Figure 6.8.

We should note that the nodal circles are not evenly spaced and that the radii can be computed relatively easily. For the $n = 2$ modes, we have two circles, $r = a$ and $r = \frac{j_{m1}}{j_{m2}} a$ as shown in the second row of Figure 6.8. For $m = 0$,

$$r = \frac{2.405}{5.520} a \approx 0.4357a$$

for the inner circle. For $m = 1$,

$$r = \frac{3.832}{7.016} a \approx 0.5462a,$$

and for $m = 2$,

$$r = \frac{5.136}{8.417}a \approx 0.6102a.$$

For $n = 3$ we obtain circles of radii

$$r = a, \quad r = \frac{j_{m1}}{j_{m3}}a, \text{ and } r = \frac{j_{m2}}{j_{m3}}a.$$

For $m = 0$,

$$r = a, \quad \frac{5.520}{8.654}a \approx 0.6379a, \quad \frac{2.405}{8.654}a \approx 0.2779a.$$

Similarly, for $m = 1$,

$$r = a, \quad \frac{3.832}{10.173}a \approx 0.3767a, \quad \frac{7.016}{10.173}a \approx 0.6897a,$$

and for $m = 2$,

$$r = a, \quad \frac{5.136}{11.620}a \approx 0.4420a, \quad \frac{8.417}{11.620}a \approx 0.7224a.$$

Example 6.3. Vibrating Annulus

More complicated vibrations can be dreamt up for this geometry. Consider an annulus in which the drum is formed from two concentric circular cylinders and the membrane is stretch between the two with an annular cross section as shown in Figure 6.9. The separation would follow as before except now the boundary conditions are that the membrane is fixed around the two circular boundaries. In this case we cannot toss out the Neumann functions because the origin is not part of the drum head.

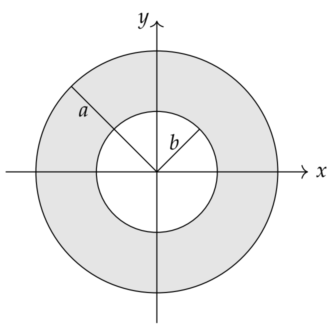


Figure 6.9: An annular membrane with radii a and $b > a$. There are fixed boundary conditions along the edges at $r = a$ and $r = b$.

The domain for this problem is shown in Figure 6.9 and the problem is given by the partial differential equation

$$u_{tt} = c^2 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right], \tag{6.57}$$

$$t > 0, \quad b < r < a, \quad -\pi < \theta < \pi,$$

the boundary conditions,

$$u(b, \theta, t) = 0, \quad u(a, \theta, t) = 0, \quad t > 0, \quad -\pi < \theta < \pi, \tag{6.58}$$

and the initial conditions,

$$\begin{aligned} u(r, \theta, 0) &= f(r, \theta), \quad b < r < a, -\pi < \theta < \pi, \\ u_t(r, \theta, 0) &= g(r, \theta),, \quad b < r < a, -\pi < \theta < \pi. \end{aligned} \tag{6.59}$$

Since we cannot dispose of the Neumann functions, the product solutions take the form

$$u(r, \theta, t) = \left\{ \begin{array}{c} \cos \omega t \\ \sin \omega t \end{array} \right\} \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} R_m(r), \tag{6.60}$$

where

$$R_m(r) = c_1 J_m(\sqrt{\lambda} r) + c_2 N_m(\sqrt{\lambda} r)$$

and $\omega = c\sqrt{\lambda}, \quad m = 0, 1, \ldots.$

For this problem the radial boundary conditions are that the membrane is fixed at $r = a$ and $r = b$. Taking $b < a$, we then have to satisfy the conditions

$$\begin{aligned} R(a) &= c_1 J_m(\sqrt{\lambda} a) + c_2 N_m(\sqrt{\lambda} a) = 0, \\ R(b) &= c_1 J_m(\sqrt{\lambda} b) + c_2 N_m(\sqrt{\lambda} b) = 0. \end{aligned} \tag{6.61}$$

This leads to two homogeneous equations for c_1 and c_2. The coefficient determinant of this system has to vanish if there are to be nontrivial solutions. This gives the eigenvalue equation for λ :

$$J_m(\sqrt{\lambda} a) N_m(\sqrt{\lambda} b) - J_m(\sqrt{\lambda} b) N_m(\sqrt{\lambda} a) = 0.$$

There are an infinite number of zeros of the function

$$F(\lambda) = J_m(\sqrt{\lambda} a) N_m(\sqrt{\lambda} b) - J_m(\sqrt{\lambda} b) N_m(\sqrt{\lambda} a).$$

In Figure 6.10 we show a plot of $F(\lambda)$ for $a = 4$, $b = 2$ and $m = 0, 1, 2, 3$.

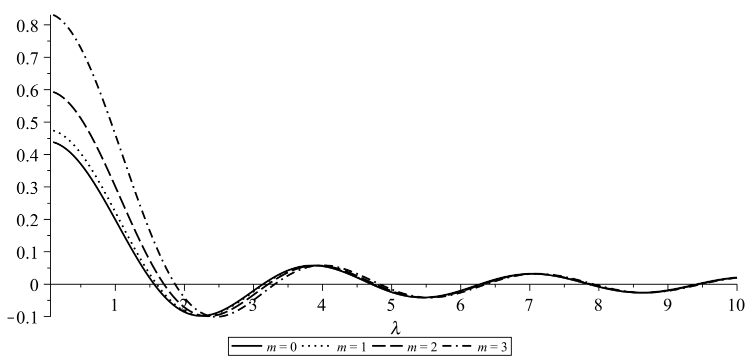


Figure 6.10: Plot of the function $F(\lambda) = J_m(\sqrt{\lambda} a) N_m(\sqrt{\lambda} b) - J_m(\sqrt{\lambda} b) N_m(\sqrt{\lambda} a)$ for $a = 4$ and $b = 2$ and $m = 0, 1, 2, 3$.

This eigenvalue equation needs to be solved numerically. Choosing $a = 2$ and $b = 4$, we have for the first few modes

$$\begin{aligned} \sqrt{\lambda_{mn}} &\approx 1.562, \quad 3.137, \quad 4.709, \quad m = 0 \\ &\approx 1.598, \quad 3.156, \quad 4.722, \quad m = 1 \\ &\approx 1.703, \quad 3.214, \quad 4.761, \quad m = 2. \end{aligned} \tag{6.62}$$

Note, since $\omega_{mn} = c\sqrt{\lambda_{mn}}$, these numbers essentially give us the frequencies of oscillation.

For these particular roots, we can solve for c_1 and c_2 up to a multiplicative constant. A simple solution is to set

$$c_1 = N_m(\sqrt{\lambda_{mn}} b), \quad c_2 = J_m(\sqrt{\lambda_{mn}} b).$$

This leads to the basic modes of vibration,

$$R_{mn}(r)\Theta_m(\theta) = \cos m\theta \left(N_m(\sqrt{\lambda_{mn}} b) J_m(\sqrt{\lambda_{mn}} r) - J_m(\sqrt{\lambda_{mn}} b) N_m(\sqrt{\lambda_{mn}} r) \right),$$

for $m = 0, 1, \ldots,$ and $n = 1, 2, \ldots$. In Figure 6.4 we show various modes for the particular choice of annular membrane dimensions, $a = 2$ and $b = 4$.

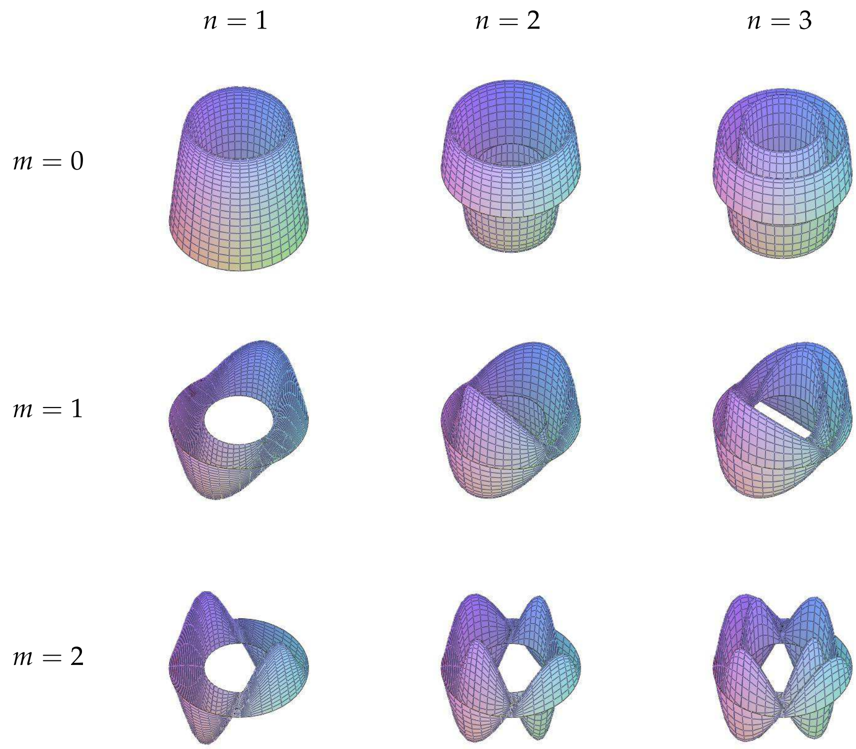


Table 6.4: A three dimensional view of the vibrating annular membrane for the lowest modes.

Example 6.4. Vibrating Elliptical Membrane

Another variation on the circular membrane is the elliptical membrane. Instead of polar coordinates, one needs an elliptic coordinate system. If the boundary is described by the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

with $a > b$, then the foci are located at $(\pm c, 0)$. where $c^2 = a^2 - b^2$. The elliptic coordinates are defined by

$$\begin{aligned} x &= c \cosh \xi \cos \eta \\ y &= c \sinh \xi \sin \eta \end{aligned} \qquad (6.63)$$

where $0 < \xi < \infty$, $0 \leq \eta\, 2\pi$. The boundary of the membrane is defined by $\xi_0 = \sinh^{-1}(b/c)$. Thus, we have

$$a = c \cosh \xi_0, \quad b = c \sinh \xi_0.$$

In Figure 6.11 an elliptical membrane with $a = 2$ and $b = 1$ is covered with an elliptical coordinate system consisting of curves of constant ξ and η similar to how the Cartsian coordinate grid can be drawn with lines of constant x and y. Using the identities

$$\cosh^2\xi - \sinh^2\xi = 1 \quad (6.64)$$
$$\cos^2\eta + \sin^2\eta = 1, \quad (6.65)$$

one can show that curves of constant ξ are (confocal) ellipses and curves of constant η are hyperbolae in an elliptical coordinate system.

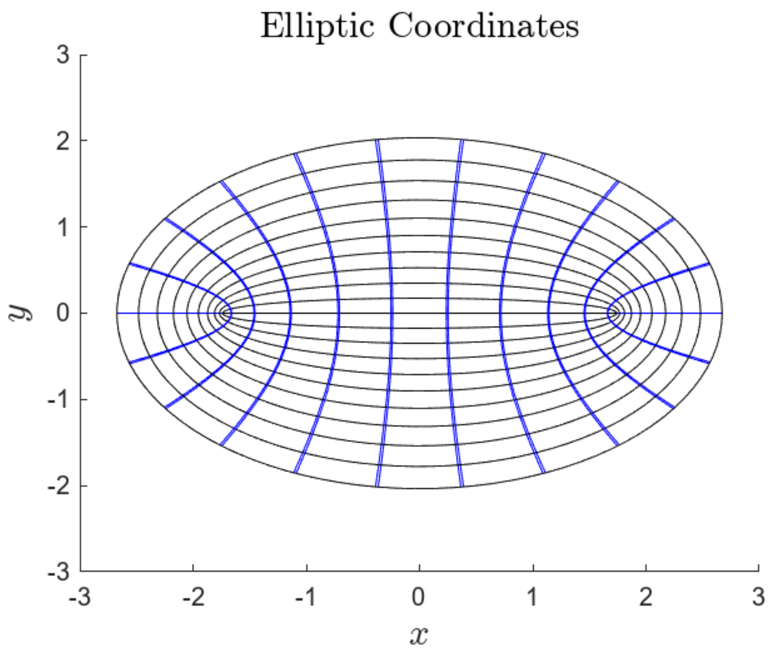


Figure 6.11: Curves of constant *xi* (ellipses) and η (hyperbolae) in an elliptical coordinate system constrained to an elliptical membrane with $a = 2$ and $b = 1$.

In these coordinates, the Helmholtz equation becomes

$$\frac{\partial^2\phi(\xi,\eta)}{\partial\xi^2} + \frac{\partial^2\phi(\xi,\eta)}{\partial\eta^2} + (kc)^2(\cosh^2\xi - \cos^2\eta)\phi(\xi,\eta) = 0. \quad (6.66)$$

Separation of variables, $\phi(\xi,\eta) = u(\xi)v(\eta)$ leads to two ordinary differential equations,

$$v''(\eta) + (\alpha - 2q\cos 2\eta)v(\eta) = 0,$$
$$u''(\xi) - (\alpha - 2q\cosh 2\xi)u(\xi) = 0, \quad (6.67)$$

where $q = \frac{1}{4}k^2c^2$. The first of these equations is called the Mathieu equation, named after Émile Léonard Mathieu (1835-1890), who originally studies vibrating elliptical membranes. The solutions are known as Mathieu functions. The second equation is the modified Mathieu equation. The next step would be to determine the associated eigenvalues so that we know what the modes of vibrations are and the frequencies of oscillation. These are too complicated for our class to put into MATLAB, so we will leave this topic to the interested reader to explore.

The Mathieu functions satisfy the Mathieu equation,

$$y'' + (a - 2qcos(2x))y = 0.)$$

6.3 Laplace's Equation in 2D

Another of the generic partial differential equations is Laplace's equation,[3] $\nabla^2 u = 0$. When studying functions of a complex variable, one might learn that functions which satisfy Laplace's equation are called harmonic functions. Another example is the electric potential for electrostatics. For static electromagnetic fields, the divergence of the electric filed vanishes,

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0.$$

However, the electric field is a gradient field. So, we can write

$$\mathbf{E} = \nabla\phi,$$

where ϕ is the electric potential. Combining these equations, we obtain the Laplace equation, $\nabla^2\phi = 0$.

[3] The first part of this section is the same as in Section 1.8.

Another example comes from studying temperature distributions. Consider a thin rectangular plate with the boundaries set at fixed temperatures. Temperature changes of the plate are governed by the heat equation. The solution of the heat equation subject to these boundary conditions is time dependent. In fact, after a long period of time the plate will reach thermal equilibrium. If the boundary temperature is zero, then the plate temperature decays to zero across the plate. However, if the boundaries are maintained at a fixed nonzero temperature, which means energy is being put into the system to maintain the boundary conditions, the internal temperature may reach a nonzero equilibrium temperature. Reaching thermal equilibrium means that asymptotically in time the solution becomes time independent. Thus, the equilibrium state is a steady state solution of the heat equation. So, it satisfies a time-independent heat equation, which is just Laplace's equation, $\nabla^2 u = 0$.

Thermodynamic equilibrium, $\nabla^2 u = 0$.

Finally, we consider fluid flow. For an incompressible flow, $\nabla \cdot \mathbf{v} = 0$. If fluid particles do not rotate about a centre of mass ,the flow is irrotational and $\nabla \times \mathbf{v} = 0$. Since the curl of a gradient is zero, we can introduce a velocity potential, ϕ, such that $\mathbf{v} = \nabla\phi$. Thus, $\nabla \times \mathbf{v}$ vanishes by a vector identity and $\nabla \cdot \mathbf{v} = 0$ implies $\nabla^2\phi = 0$. So, once again we obtain Laplace's equation.

Incompressible, irrotational fluid flow, $\nabla^2\phi = 0$, for velocity $\mathbf{v} = \nabla\phi$. This comes from the assumption that mass is conserved in fluid flow and satisfies the continuity equation,

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0.$$

Here ρ is the density, $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ is the velocity, and $\rho\mathbf{v}$ is the flux (mass per area per time). For incompressible flow, such as water waves, ρ is constant giving

$$\nabla \cdot \mathbf{v} = 0.$$

In this section we will look at examples of Laplace's equation in two dimensions. The solutions in these examples could be examples from any of the application in the above physical situations and the solutions can be applied appropriately.

Example 6.5. Equilibrium Temperature Distribution for a Rectangular Plate.

Let's consider Laplace's equation in Cartesian coordinates,

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, \quad 0 < y < H$$

with the boundary conditions

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) = f(x), \quad u(x, H) = 0.$$

The boundary conditions are shown in Figure 6.12

As with the heat and wave equations, we can solve this problem using the method of separation of variables. Let $u(x,y) = X(x)Y(y)$. Then, Laplace's equation becomes

$$X''Y + XY'' = 0$$

and we can separate the x and y dependent functions and introduce a separation constant, λ,

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

Thus, we are led to two differential equations,

$$\begin{aligned} X'' + \lambda X &= 0, \\ Y'' - \lambda Y &= 0. \end{aligned} \quad (6.68)$$

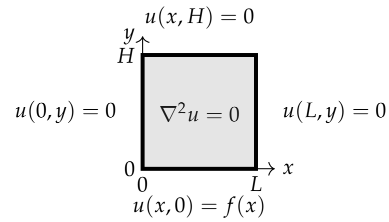


Figure 6.12: In this figure we show the domain and boundary conditions for the example of determining the equilibrium temperature distribution for a rectangular plate.

From the boundary condition $u(0,y) = 0, u(L,y) = 0$, we have $X(0) = 0, X(L) = 0$. So, we have the usual eigenvalue problem for $X(x)$,

$$X'' + \lambda X = 0, \quad X(0) = 0, X(L) = 0.$$

The solutions to this problem are given by

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots.$$

The general solution of the equation for $Y(y)$ is given by

$$Y(y) = c_1 e^{\sqrt{\lambda}y} + c_2 e^{-\sqrt{\lambda}y}.$$

The boundary condition $u(x,H) = 0$ implies $Y(H) = 0$. So, we have

$$c_1 e^{\sqrt{\lambda}H} + c_2 e^{-\sqrt{\lambda}H} = 0.$$

Thus,

$$c_2 = -c_1 e^{2\sqrt{\lambda}H}.$$

Inserting this result into the expression for $Y(y)$, we have

$$\begin{aligned} Y(y) &= c_1 e^{\sqrt{\lambda}y} - c_1 e^{2\sqrt{\lambda}H} e^{-\sqrt{\lambda}y} \\ &= c_1 e^{\sqrt{\lambda}H}\left(e^{-\sqrt{\lambda}H} e^{\sqrt{\lambda}y} - e^{\sqrt{\lambda}H} e^{-\sqrt{\lambda}y}\right) \\ &= c_1 e^{\sqrt{\lambda}H}\left(e^{-\sqrt{\lambda}(H-y)} - e^{\sqrt{\lambda}(H-y)}\right) \\ &= -2c_1 e^{\sqrt{\lambda}H} \sinh\sqrt{\lambda}(H-y). \end{aligned} \quad (6.69)$$

Note: Having carried out this computation, we can now see that it would be better to guess this form in the future. So, for $Y(H) = 0$, one would guess a solution $Y(y) = \sinh\sqrt{\lambda}(H-y)$. For $Y(0) = 0$, one would guess a solution $Y(y) = \sinh\sqrt{\lambda}y$. Similarly, if $Y'(H) = 0$, one would guess a solution $Y(y) = \cosh\sqrt{\lambda}(H-y)$.

Since we already know the values of the eigenvalues λ_n from the eigenvalue problem for $X(x)$, we have that the y-dependence is given by

$$Y_n(y) = \sinh\frac{n\pi(H-y)}{L}.$$

So, the product solutions are given by

$$u_n(x,y) = \sin\frac{n\pi x}{L}\sinh\frac{n\pi(H-y)}{L}, \quad n = 1,2,\ldots.$$

These solutions satisfy Laplace's equation and the three homogeneous boundary conditions and in the problem.

The remaining boundary condition, $u(x,0) = f(x)$, still needs to be satisfied. Inserting $y = 0$ in the product solutions does not satisfy the boundary condition unless $f(x)$ is proportional to one of the eigenfunctions $X_n(x)$. So, we first write down the general solution as a linear combination of the product solutions,

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi(H-y)}{L}. \tag{6.70}$$

Now we apply the boundary condition, $u(x,0) = f(x)$, to find that

$$f(x) = \sum_{n=1}^{\infty} a_n \sinh\frac{n\pi H}{L}\sin\frac{n\pi x}{L}. \tag{6.71}$$

Defining $b_n = a_n \sinh\frac{n\pi H}{L}$, this becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}. \tag{6.72}$$

We see that the determination of the unknown coefficients, b_n, is simply done by recognizing that this is a Fourier sine series. The Fourier coefficients are easily found as

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \tag{6.73}$$

Since $a_n = \dfrac{b_n}{\sinh\frac{n\pi H}{L}}$, we can finish solving the problem. The solution is

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi(H-y)}{L}, \tag{6.74}$$

where

$$a_n = \frac{2}{L\sinh\frac{n\pi H}{L}}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \tag{6.75}$$

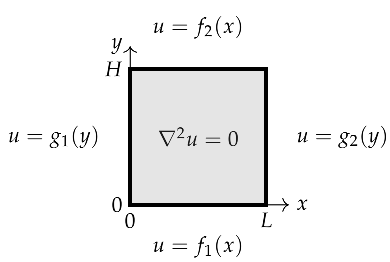


Figure 6.13: In this figure we show the domain and general boundary conditions for the example of determining the equilibrium temperature distribution for a rectangular plate.

Example 6.6. Equilibrium Temperature Distribution for a Rectangular Plate for General Boundary Conditions

A more general problem is to seek solutions to Laplace's equation in Cartesian coordinates,

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, 0 < y < H$$

with non-zero boundary conditions on more than one side of the domain,

$$u(0,y) = g_1(y), \quad u(L,y) = g_2(y), \quad 0 < y < H,$$

$$u(x,0) = f_1(x), \quad u(x,H) = f_2(x), \quad 0 < x < L.$$

These boundary conditions are shown in Figure 6.13

The problem with this example is that none of the boundary conditions are homogeneous. This means that the corresponding eigenvalue problems will not have the homogeneous boundary conditions which Sturm-Liouville theory in Section 4 needs. However, we can express this problem in terms of four different problems with nonhomogeneous boundary conditions on only one side of the rectangle.

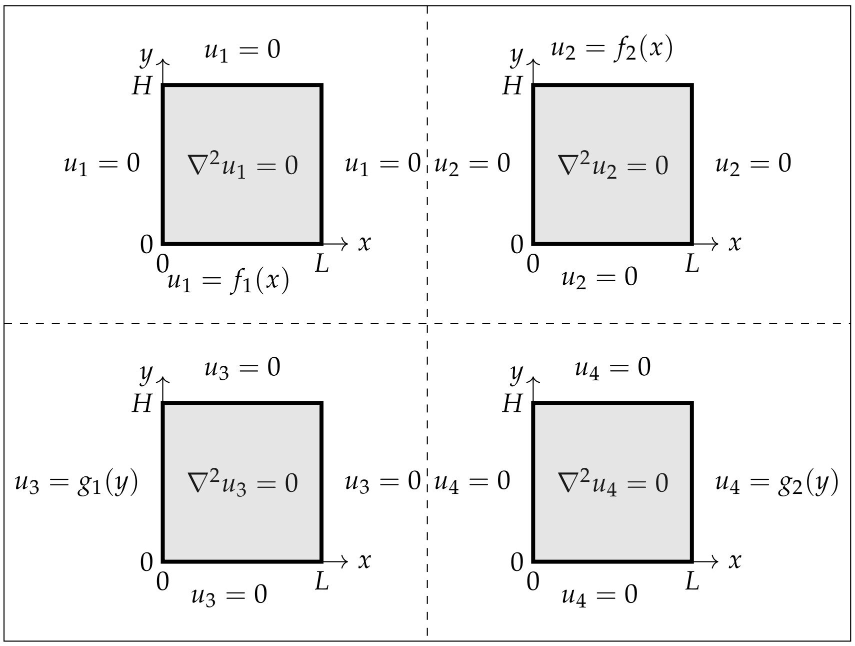


Figure 6.14: The general boundary value problem for a rectangular plate can be written as the sum of these four separate problems.

In Figure 6.14 we show how the problem can be broken up into four separate problems for functions $u_i(x,y)$, $i = 1,\ldots,4$. Since the boundary conditions and Laplace's equation are linear, the solution to the general problem is simply the sum of the solutions to these four problems,

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y).$$

Then, this solution satisfies Laplace's equation,

$$\nabla^2 u(x,y) = \nabla^2 u_1(x,y) + \nabla^2 u_2(x,y) + \nabla^2 u_3(x,y) + \nabla^2 u_4(x,y) = 0,$$

and the boundary conditions. For example, using the boundary conditions defined in Figure 6.14, we have for $y = 0$,

$$u(x,0) = u_1(x,0) + u_2(x,0) + u_3(x,0) + u_4(x,0) = f_1(x).$$

The other boundary conditions can also be shown to hold.

We can solve each of the problems in Figure 6.14 quickly based on the solution we obtained in the last example. The solution for $u_1(x,y)$, which satisfies the boundary conditions

$$u_1(0,y) = 0, \quad u_1(L,y) = 0, \quad 0 < y < H,$$

$$u_1(x,0) = f_1(x), \quad u_1(x,H) = 0, \quad 0 < x < L,$$

is the easiest to write down. It is given by

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi(H-y)}{L}. \tag{6.76}$$

where

$$a_n = \frac{2}{L \sinh \frac{n\pi H}{L}} \int_0^L f_1(x) \sin \frac{n\pi x}{L}\, dx. \tag{6.77}$$

For the boundary conditions

$$u_2(0,y) = 0, \quad u_2(L,y) = 0, \quad 0 < y < H,$$

$$u_2(x,0) = 0, \quad u_2(x,H) = f_2(x), \quad 0 < x < L.$$

the boundary conditions for $X(x)$ are $X(0) = 0$ and $X(L) = 0$. So, we get the same form for the eigenvalues and eigenfunctions as before:

$$X_n(x) = \sin \frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, \ldots.$$

The remaining homogeneous boundary condition is now $Y(0) = 0$. Recalling that the equation satisfied by $Y(y)$ is

$$Y'' - \lambda Y = 0,$$

we can write the general solution as

$$Y(y) = c_1 \cosh \sqrt{\lambda} y + c_2 \sinh \sqrt{\lambda} y.$$

Requiring $Y(0) = 0$, we have $c_1 = 0$, or

$$Y(y) = c_2 \sinh \sqrt{\lambda} y.$$

Then, the general solution is

$$u_2(x,y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}. \tag{6.78}$$

We now force the nonhomogeneous boundary condition, $u_2(x,H) = f_2(x)$,

$$f_2(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi H}{L}. \tag{6.79}$$

Once again we have a Fourier sine series. The Fourier coefficients are given by

$$b_n = \frac{2}{L \sinh \frac{n\pi H}{L}} \int_0^L f_2(x) \sin \frac{n\pi x}{L}\, dx. \tag{6.80}$$

Next we turn to the problem with the boundary conditions

$$u_3(0,y) = g_1(y), \quad u_3(L,y) = 0, \quad 0 < y < H,$$
$$u_3(x,0) = 0, \quad u_3(x,H) = 0, \quad 0 < x < L.$$

In this case the pair of homogeneous boundary conditions $u_3(x,0) = 0, \quad u_3(x,H) = 0$ lead to solutions

$$Y_n(y) = \sin\frac{n\pi y}{H}, \quad \lambda_n = -\left(\frac{n\pi}{H}\right)^2, \quad n = 1,2\ldots.$$

The condition $u_3(L,0) = 0$ gives $X(x) = \sinh\frac{n\pi(L-x)}{H}$.

The general solution satisfying the homogeneous conditions is

$$u_3(x,y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi y}{H} \sinh\frac{n\pi(L-x)}{H}. \tag{6.81}$$

Applying the nonhomogeneous boundary condition, $u_3(0,y) = g_1(y)$, we obtain the Fourier sine series

$$g_1(y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi y}{H} \sinh\frac{n\pi L}{H}. \tag{6.82}$$

The Fourier coefficients are found as

$$c_n = \frac{2}{H\sinh\frac{n\pi L}{H}} \int_0^H g_1(y) \sin\frac{n\pi y}{H}\, dy. \tag{6.83}$$

Finally, we can find the solution

$$u_4(0,y) = 0, \quad u_4(L,y) = g_2(y), \quad 0 < y < H,$$
$$u_4(x,0) = 0, \quad u_4(x,H) = 0, \quad 0 < x < L.$$

Following the above analysis, we find the general solution

$$u_4(x,y) = \sum_{n=1}^{\infty} d_n \sin\frac{n\pi y}{H} \sinh\frac{n\pi x}{H}. \tag{6.84}$$

The nonhomogeneous boundary condition, $u(L,y) = g_2(y)$, is satisfied if

$$g_2(y) = \sum_{n=1}^{\infty} d_n \sin\frac{n\pi y}{H} \sinh\frac{n\pi L}{H}. \tag{6.85}$$

The Fourier coefficients, d_n, are given by

$$d_n = \frac{2}{H\sinh\frac{n\pi L}{H}} \int_0^H g_1(y) \sin\frac{n\pi y}{H}\, dy. \tag{6.86}$$

The solution to the general problem is given by the sum of these four solutions.

$$\begin{aligned} u(x,y) \quad = \quad & \sum_{n=1}^{\infty} \left[\left(a_n \sinh\frac{n\pi(H-y)}{L} + b_n \sinh\frac{n\pi y}{L} \right) \sin\frac{n\pi x}{L} \right. \\ & \left. + \left(c_n \sinh\frac{n\pi(L-x)}{H} + d_n \sinh\frac{n\pi x}{H} \right) \sin\frac{n\pi y}{H} \right], \end{aligned} \tag{6.87}$$

where the coefficients are given by the above Fourier integrals.

Example 6.7. Laplace's Equation on a Disk

We now turn to solving Laplace's equation on a disk of radius a as shown in Figure 6.15. Laplace's equation in polar coordinates is given in Equation (6.43) by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0, \quad 0<r<a, \quad -\pi<\theta<\pi. \tag{6.88}$$

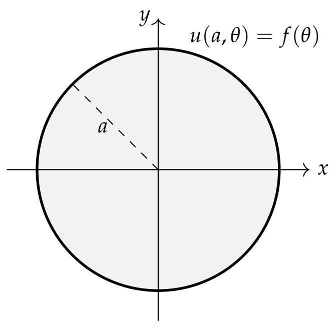


Figure 6.15: The disk of radius a with boundary condition along the edge at $r=a$.

The boundary conditions are given as

$$u(a,\theta)=f(\theta), \quad -\pi<\theta<\pi, \tag{6.89}$$

plus periodic boundary conditions in θ.

Separation of variable proceeds as usual. Let $u(r,\theta)=R(r)\Theta(\theta)$. Then

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial (R\Theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 (R\Theta)}{\partial \theta^2}=0, \tag{6.90}$$

or

$$\Theta\frac{1}{r}(rR')'+\frac{1}{r^2}R\Theta''=0. \tag{6.91}$$

Dividing by $u(r,\theta)=R(r)\Theta(\theta)$, multiplying by r^2, and rearranging, we have

$$\frac{r}{R}(rR')'=-\frac{\Theta''}{\Theta}=\lambda. \tag{6.92}$$

Since this equation gives a function of r equal to a function of θ, we set the equation equal to a constant. Thus, we have obtained two differential equations, which can be written as

$$r(rR')'-\lambda R = 0, \tag{6.93}$$
$$\Theta''+\lambda\Theta = 0. \tag{6.94}$$

We can solve the second equation subject to the periodic boundary conditions in the θ variable. The reader should be able to confirm that

$$\Theta(\theta)=a_n\cos n\theta+b_n\sin n\theta, \quad \lambda=n^2, \quad n=0,1,2,\ldots$$

is the solution. Note that the $n=0$ case just leads to a constant solution.

Inserting $\lambda=n^2$ into the radial equation, we find

$$r^2R''+rR'-n^2R=0.$$

This is a Cauchy-Euler type of ordinary differential equation. Recall that we solve such equations by guessing a solution of the form $R(r)=r^m$. This leads to the characteristic equation $m^2-n^2=0$. Therefore, $m=\pm n$. So,

$$R(r)=c_1r^n+c_2r^{-n}.$$

Since we expect finite solutions at the origin, $r=0$, we can set $c_2=0$.

Thus, the general solution is

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)\, r^n. \tag{6.95}$$

Note that we have taken the constant term out of the sum and put it into a familiar form.

Now we can impose the remaining boundary condition, $u(a,\theta) = f(\theta)$, or

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)\, a^n. \tag{6.96}$$

This is a Fourier trigonometric series. The Fourier coefficients can be determined using the results from Chapter 4:

$$\begin{aligned} a_n &= \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta, \quad n = 0, 1, \ldots, & (6.97) \\ b_n &= \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta \quad n = 1, 2 \ldots. & (6.98) \end{aligned}$$

6.3.1 *Poisson Integral Formula*

We can put the solution from the last example in a more compact form by inserting the Fourier coefficients into the general solution. Doing this, we have

$$\begin{aligned} u(r,\theta) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)\, r^n \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi)\, d\phi \\ &\quad + \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} [\cos n\phi \cos n\theta + \sin n\phi \sin n\theta] \left(\frac{r}{a}\right)^n f(\phi)\, d\phi \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \cos n(\theta - \phi) \left(\frac{r}{a}\right)^n \right] f(\phi)\, d\phi. \end{aligned} \tag{6.99}$$

The term in the brackets can be summed. We note that[4]

$$\begin{aligned} \cos n(\theta - \phi) \left(\frac{r}{a}\right)^n &= \operatorname{Re}\left(e^{in(\theta-\phi)} \left(\frac{r}{a}\right)^n \right) \\ &= \operatorname{Re}\left(\frac{r}{a} e^{i(\theta-\phi)} \right)^n. \end{aligned} \tag{6.100}$$

[4] Here we are using Euler's formula, $e^{i\theta} = \cos\theta + i \sin\theta$. So, the real part of this is just the cosine function.

Therefore,

$$\sum_{n=1}^{\infty} \cos n(\theta - \phi) \left(\frac{r}{a}\right)^n = \operatorname{Re}\left(\sum_{n=1}^{\infty} \left(\frac{r}{a} e^{i(\theta-\phi)} \right)^n \right).$$

The right hand side of this equation is a geometric series with common ratio of $\frac{r}{a} e^{i(\theta-\phi)}$, which is also the first term of the series. Recall that

$$\alpha + \alpha z + \alpha z^2 + \ldots = \frac{\alpha}{1-z}, \quad |z| < 1.$$

Since $\left|\frac{r}{a}e^{i(\theta-\phi)}\right| = \frac{r}{a} < 1$, the series converges. Summing the series, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{r}{a}e^{i(\theta-\phi)}\right)^n &= \frac{\frac{r}{a}e^{i(\theta-\phi)}}{1-\frac{r}{a}e^{i(\theta-\phi)}} \\ &= \frac{re^{i(\theta-\phi)}}{a-re^{i(\theta-\phi)}} \end{aligned} \tag{6.101}$$

We need to rewrite this result so that we can easily take the real part. Thus, we multiply and divide by the complex conjugate of the denominator to obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{r}{a}e^{i(\theta-\phi)}\right)^n &= \frac{re^{i(\theta-\phi)}}{a-re^{i(\theta-\phi)}}\frac{a-re^{-i(\theta-\phi)}}{a-re^{-i(\theta-\phi)}} \\ &= \frac{are^{-i(\theta-\phi)}-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}. \end{aligned} \tag{6.102}$$

The real part of the sum is given as

$$\operatorname{Re}\left(\sum_{n=1}^{\infty}\left(\frac{r}{a}e^{i(\theta-\phi)}\right)^n\right) = \frac{ar\cos(\theta-\phi)-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}.$$

Therefore, the factor in the brackets under the integral in Equation (6.99) is

$$\begin{aligned}\frac{1}{2}+\sum_{n=1}^{\infty}\cos n(\theta-\phi)\left(\frac{r}{a}\right)^n &= \frac{1}{2}+\frac{ar\cos(\theta-\phi)-r^2}{a^2+r^2-2ar\cos(\theta-\phi)} \\ &= \frac{a^2-r^2}{2(a^2+r^2-2ar\cos(\theta-\phi))}. \end{aligned} \tag{6.103}$$

Thus, we have shown that the solution of Laplace's equation on a disk of radius a with boundary condition $u(a,\theta) = f(\theta)$ can be written in the closed form

Poisson Integral Formula

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{a^2-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}f(\phi)\,d\phi. \tag{6.104}$$

This result is called the Poisson Integral Formula and

$$K(\theta,\phi) = \frac{a^2-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}$$

is called the Poisson kernel.

Example 6.8. Evaluate the solution (6.104) at the center of the disk.

We insert $r = 0$ into the solution (6.104) to obtain

$$u(0,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,d\phi.$$

Recalling that the average of a function $g(x)$ on $[a,b]$ is given by

$$g_{ave} = \frac{1}{b-a}\int_a^b g(x)\,dx,$$

we see that the value of the solution u at the center of the disk is the average of the boundary values. This is sometimes referred to as the mean value theorem.

Example 6.9. Consider a disk of radius a which is allowed to cool off while the temperature of the boundary is maintained at $u(a, \theta, t) = T$, a constant. Find the steady state temperature.

The steady state temperature occurs when $u(r, \theta, t)$ no longer depends on time. Thus, $u = u(r, \theta)$ satisfies Laplace's equation with $f(\theta) = T$ on the boundary. We can find the solution using the Poisson Integral Formula in Equation (6.104). We have seen that it is useful for proving that the value at the center of the disc is the average of $f(\theta)$ over the boundary. But, how useful is it at providing the solution given $f(\theta)$?

If $f(\theta)$ can be written as a function of $\sin\theta$ and/or $\cos\theta$, then one may be able to evaluate this integral. We can use the tangent half angle substitution, $\tan\frac{\theta}{2} = t$. This is derivable from a rational parametrizaton of the circle,

There is an ambiguous history to this substitution. It is sometime referred to as the Weierstrass substitution. However, according to Wikipedia, Euler used it in his calculus textbook in 1768 and Legendre Legendre, Adrien-Marie in 1817. The author recalls it in Thomas' calculus text in the 1960s and Michael Spivak is quoted as referring to it as, the "world's sneakiest substitution."

$$(x, y) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) = (\cos\theta, \sin\theta).$$

For $t \in (-\infty, \infty)$, $\theta \in [-\pi, \pi]$. From the half angle identity for the tangent, we have

$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta} = t.$$

Geometric relations between t and θ for the Weierstrass substitution are shown in Figure 6.16.

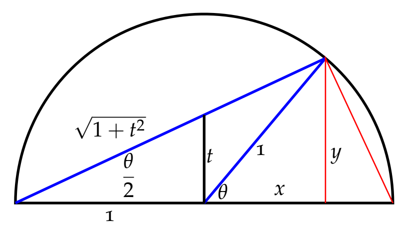


Figure 6.16: Geometric relations between t and θ for the Weierstrass substitution.

Returning to the Poisson's integral formula, we let

$$t = \tan\frac{\phi}{2}, \quad d\phi = \frac{2\,dt}{1+t^2}.$$

Then, $u(r, \theta)$

$$\begin{aligned}
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{a^2-r^2}{a^2+r^2-2ar\cos(\theta-\phi)} f(\phi)\,d\phi \\
&= \frac{a^2-r^2}{2\pi}\int_{-\pi}^{\pi} \frac{f(\phi)}{a^2+r^2-2ar[\cos\theta\cos\phi+\sin\theta\sin\phi)]}\,d\phi \\
&= \frac{a^2-r^2}{\pi}\int_{-\infty}^{\infty} \frac{g(t)}{(a^2+r^2)(1+t^2)-2ar[(1-t^2)\cos\theta+2t\sin\theta)]}\,dt.
\end{aligned}$$

Here we defined $g(t) = f(\phi)$.

The denominator in the integrand is quadratic in t. So, we write

$$(a^2+r^2)(1+t^2) - 2ar[(1-t^2)\cos\theta + 2t\sin\theta)] = \alpha t^2 + \beta t + \gamma,$$

where

$$\begin{aligned}
\alpha &= a^2+r^2+2ar\cos\theta \\
\beta &= -4ar\sin\theta \\
\gamma &= a^2+r^2-2ar\cos\theta.
\end{aligned}$$

In this example $g(t) = T$, a constant. So, we need to evaluate

$$u(r,\theta) = \frac{T}{\pi}(a^2 - r^2)\int_{-\infty}^{\infty} \frac{dt}{\alpha t^2 + \beta t + \gamma}$$

One can show that the roots of $\alpha t^2 + \beta t + \gamma = 0$ are complex, so we need to complete the square. This is given by

$$\begin{aligned} \alpha t^2 + \beta t + \gamma &= \alpha\left[t^2 + \frac{\beta}{\alpha}t + \frac{\gamma}{\alpha}\right] \\ &= \alpha\left[\left(t + \frac{\beta}{2\alpha}\right)^2 + \frac{\gamma}{\alpha} - \left(\frac{\beta}{2\alpha}\right)^2\right] \\ &= \alpha\left[\left(t + \frac{\beta}{2\alpha}\right)^2 + \frac{4\alpha\gamma - \beta^2}{4\alpha^2}\right] \\ &= \alpha\left[\left(t + \frac{\beta}{2\alpha}\right)^2 + \frac{(a^2 - r^2)^2}{\alpha^2}\right] \end{aligned} \tag{6.105}$$

Now, we can carry out the integration, leading to the steady state solution

$$\begin{aligned} u(r,\theta) &= \frac{T}{\pi}(a^2 - r^2)\int_{-\infty}^{\infty} \frac{dt}{\alpha t^2 + \beta t + \gamma} \\ &= \frac{T}{\pi}(a^2 - r^2)\int_{-\infty}^{\infty} \frac{dt}{\alpha\left[\left(t + \frac{\beta}{2\alpha}\right)^2 + \frac{(a^2-r^2)^2}{\alpha^2}\right]} \\ &= \frac{T}{\pi\alpha}(a^2 - r^2)\int_{-\infty}^{\infty} \frac{dy}{y^2 + \frac{(a^2-r^2)^2}{\alpha^2}} \\ &= \frac{T}{\pi\alpha}(a^2 - r^2)\left[\frac{|\alpha|}{a^2 - r^2}\tan^{-1}\left(\frac{|\alpha|}{a^2 - r^2}y\right)\right]_{-\infty}^{\infty} \\ &= T. \end{aligned} \tag{6.106}$$

6.4 Three Dimensional Cake Baking

In the rest of the chapter we will extend our studies to three dimensional problems. In this section we will solve the heat equation as we look at examples of baking cakes.

We consider cake batter, which is at room temperature of $T_i = 80^\circ$F. It is placed into an oven, also at a fixed temperature, $T_b = 350^\circ$F. For simplicity, we will assume that the thermal conductivity and cake density are constant. Of course, this is not quite true. However, it is an approximation which simplifies the model. We will consider two cases, one in which the cake is a rectangular solid, such as baking it in a $13'' \times 9'' \times 2''$ baking pan. The other case will lead to a cylindrical cake, such as you would obtain from a round cake pan.

This discussion of cake baking is adapted from R. Wilkinson's 2007 thesis. That in turn was inspired by work done by Dr. Olszewski, (2006), From baking a cake to solving the diffusion equation. *American Journal of Physics* 74(6).

Assuming that the heat constant k is indeed constant and the temperature is given by $T(\mathbf{r}, t)$, we begin with the heat equation in three dimensions,

$$\frac{\partial T}{\partial t} = k\nabla^2 T. \tag{6.107}$$

We will need to specify initial and boundary conditions. Let T_i be the initial batter temperature, $T(x,y,z,0) = T_i$.

We choose the boundary conditions to be fixed at the oven temperature T_b. However, these boundary conditions are not homogeneous and would lead to problems when carrying out separation of variables. This is easily remedied by subtracting the oven temperature from all temperatures involved and defining $u(\mathbf{r},t) = T(\mathbf{r},t) - T_b$. The heat equation then becomes

$$\frac{\partial u}{\partial t} = k\nabla^2 u \tag{6.108}$$

with initial condition

$$u(\mathbf{r},0) = T_i - T_b.$$

The boundary conditions are now homogeneous. We cannot be any more specific than this until we specify the geometry.

Example 6.10. Temperature of a Rectangular Cake

We will consider a rectangular cake with dimensions $0 \le x \le W$, $0 \le y \le L$, and $0 \le z \le H$ as show in Figure 6.17. For this problem, we seek solutions of the heat equation plus the conditions

$$\begin{aligned} u(x,y,z,0) &= T_i - T_b, \\ u(0,y,z,t) = u(W,y,z,t) &= 0, \\ u(x,0,z,t) = u(x,L,z,t) &= 0, \\ u(x,y,0,t) = u(x,y,H,t) &= 0. \end{aligned}$$

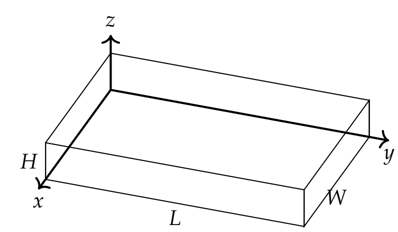


Figure 6.17: The dimensions of a rectangular cake.

Using the method of separation of variables, we seek solutions of the form

$$u(x,y,z,t) = X(x)Y(y)Z(z)G(t). \tag{6.109}$$

Substituting this form into the heat equation, we get

$$\frac{1}{k}\frac{G'}{G} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}. \tag{6.110}$$

Setting these expressions equal to $-\lambda$, we get

$$\frac{1}{k}\frac{G'}{G} = -\lambda \quad \text{and} \quad \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -\lambda. \tag{6.111}$$

Therefore, the equation for $G(t)$ is given by

$$G' + k\lambda G = 0.$$

We further have to separate out the functions of x, y, and z. We anticipate that the homogeneous boundary conditions will lead to oscillatory solutions in these variables. Therefore, we expect separation of variables will lead to the eigenvalue problems

$$\begin{aligned} X'' + \mu^2 X &= 0, \quad X(0) = X(W) = 0, \\ Y'' + \nu^2 Y &= 0, \quad Y(0) = Y(L) = 0, \\ Z'' + \kappa^2 Z &= 0, \quad Z(0) = Z(H) = 0. \end{aligned} \tag{6.112}$$

Noting that

$$\frac{X''}{X} = -\mu^2, \quad \frac{Y''}{Y} = -\nu^2, \quad \frac{Z''}{Z} = -\kappa^2,$$

we find from the heat equation that the separation constants are related,

$$\lambda^2 = \mu^2 + \nu^2 + \kappa^2.$$

We could have gotten to this point quicker by writing the first separated equation labeled with the separation constants as

$$\underbrace{\frac{1}{k}\frac{G'}{G}}_{-\lambda} = \underbrace{\frac{X''}{X}}_{-\mu} + \underbrace{\frac{Y''}{Y}}_{-\nu} + \underbrace{\frac{Z''}{Z}}_{-\kappa}.$$

Then, we can read off the eigenvalues problems and determine that $\lambda^2 = \mu^2 + \nu^2 + \kappa^2$.

From the boundary conditions, we get product solutions for $u(x, y, z, t)$ in the form

$$u_{mn\ell}(x, y, z, t) = \sin \mu_m x \sin \nu_n y \sin \kappa_\ell z \, e^{-\lambda_{mn\ell} kt},$$

for

$$\lambda_{mnl} = \mu_m^2 + \nu_n^2 + \kappa_\ell^2 = \left(\frac{m\pi}{W}\right)^2 + \left(\frac{n\pi}{L}\right)^2 + \left(\frac{\ell\pi}{H}\right)^2,$$

where $m, n, \ell = 1, 2, \ldots$.

The general solution is a linear combination of all of the product solutions, summed over three different indices,

$$u(x, y, z, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} A_{mnl} \sin \mu_m x \sin \nu_n y \sin \kappa_\ell z \, e^{-\lambda_{mn\ell} kt}, \tag{6.113}$$

where the $A_{mn\ell}$'s are arbitrary constants.

We can use the initial condition $u(x, y, z, 0) = T_i - T_b$ to determine the $A_{mn\ell}$'s. We find

$$T_i - T_b = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} A_{mnl} \sin \mu_m x \sin \nu_n y \sin \kappa_\ell z. \tag{6.114}$$

Triple Fourier sine series.

This is a triple Fourier sine series.

We can determine these coefficients in a manner similar to how we handled double Fourier sine series earlier in the chapter. [See Equation (6.36).] Defining

$$b_m(y, z) = \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} A_{mnl} \sin \nu_n y \sin \kappa_\ell z,$$

we obtain a simple Fourier sine series:

$$T_i - T_b = \sum_{m=1}^{\infty} b_m(y, z) \sin \mu_m x. \tag{6.115}$$

The Fourier coefficients can then be found as

$$b_m(y,z) = \frac{2}{W}\int_0^W (T_i - T_b)\sin\mu_m x\,dx.$$

Using the same technique for the remaining sine series and noting that $T_i - T_b$ is constant, we can determine the general coefficients A_{mnl} by carrying out the needed integrations:

$$\begin{aligned}
A_{mnl} &= \frac{8}{WLH}\int_0^H\int_0^L\int_0^W (T_i - T_b)\sin\mu_m x\sin\nu_n y\sin\kappa_\ell z\,dxdydz \\
&= (T_i - T_b)\frac{8}{\pi^3}\left[\frac{\cos\left(\frac{m\pi x}{W}\right)}{m}\right]_0^W\left[\frac{\cos\left(\frac{n\pi y}{L}\right)}{n}\right]_0^L\left[\frac{\cos\left(\frac{\ell\pi z}{H}\right)}{\ell}\right]_0^H \\
&= (T_i - T_b)\frac{8}{\pi^3}\left[\frac{\cos m\pi - 1}{m}\right]\left[\frac{\cos n\pi - 1}{n}\right]\left[\frac{\cos\ell\pi - 1}{\ell}\right] \\
&= (T_i - T_b)\frac{8}{\pi^3}\begin{cases} 0, & \text{for at least one } m,n,\ell \text{ even,} \\ \left[\frac{-2}{m}\right]\left[\frac{-2}{n}\right]\left[\frac{-2}{\ell}\right], & \text{for } m,n,\ell \text{ all odd.}\end{cases}
\end{aligned}$$

Since only the odd multiples yield non-zero $A_{mn\ell}$ we let $m = 2m' - 1$, $n = 2n' - 1$, and $\ell = 2\ell' - 1$ for $m', n', \ell' = 1, 2, \ldots$. The expansion coefficients can now be written in the simpler form

$$A_{mnl} = \frac{64(T_b - T_i)}{(2m'-1)\,(2n'-1)\,(2\ell'-1)\,\pi^3}.$$

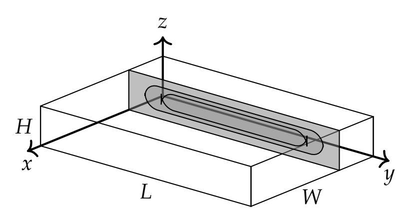


Figure 6.18: Rectangular cake showing a vertical slice.

Substituting this result into general solution and dropping the primes, we find

$$u(x,y,z,t) = \frac{64(T_b - T_i)}{\pi^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{\sin\mu_m x\sin\nu_n y\sin\kappa_\ell z\,e^{-\lambda_{mn\ell}kt}}{(2m-1)(2n-1)(2\ell-1)},$$

where

$$\lambda_{mn\ell} = \left(\frac{(2m-1)\pi}{W}\right)^2 + \left(\frac{(2n-1)\pi}{L}\right)^2 + \left(\frac{(2\ell-1)\pi}{H}\right)^2$$

for $m, n, \ell = 1, 2, \ldots$.

Recalling that the solution to the physical problem is

$$T(x,y,z,t) = u(x,y,z,t) + T_b,$$

we have the final solution is given by

$$T(x,y,z,t) = T_b + \frac{64(T_b - T_i)}{\pi^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{\sin\hat{\mu}_m x\sin\hat{\nu}_n y\sin\hat{\kappa}_\ell z\,e^{-\hat{\lambda}_{mn\ell}kt}}{(2m-1)(2n-1)(2\ell-1)}.$$

We show some temperature distributions in Figure 6.19. Since we cannot capture the entire cake, we show vertical slices such as depicted in Figure 6.18. Vertical slices are taken at the positions and times indicated for a $13'' \times 9'' \times 2''$ cake. Obviously, this is not accurate because the cake consistency is changing and this will affect the parameter k. A more realistic model would be to allow $k = k(T(x,y,z,t))$. However, such problems are beyond the simple methods described in this book.

Example 6.11. Circular Cakes

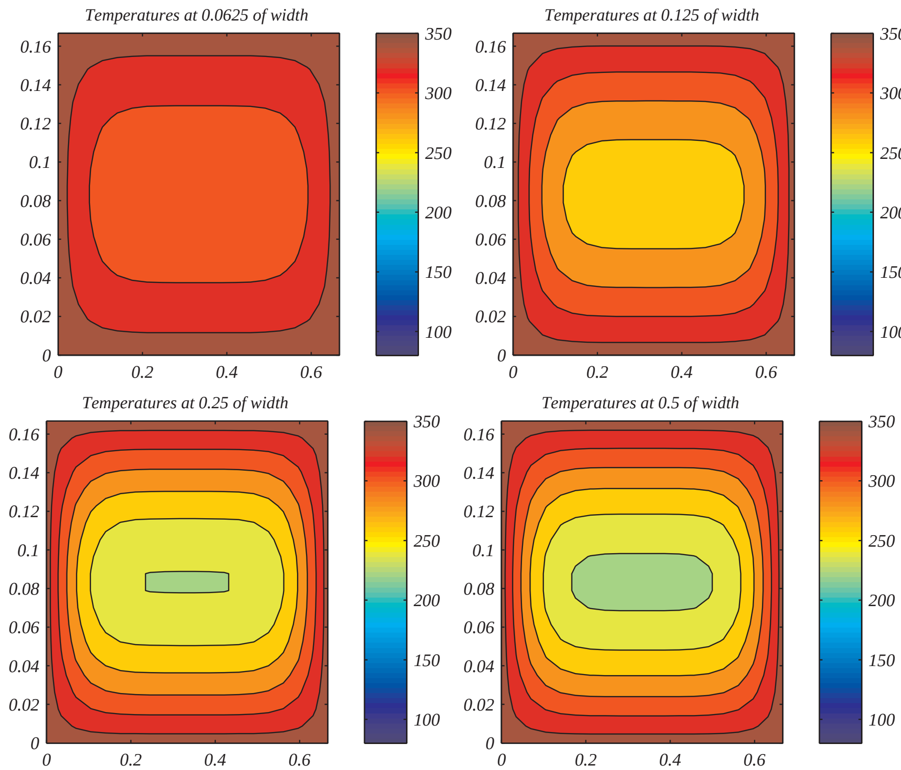


Figure 6.19: Temperature evolution for a $13'' \times 9'' \times 2''$ cake shown as vertical slices at the indicated length in feet.

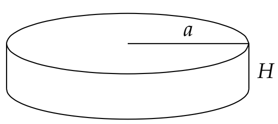


Figure 6.20: Geometry for a cylindrical cake.

In this case the geometry of the cake is cylindrical as show in Figure 6.20. Therefore, we need to express the boundary conditions and heat equation in cylindrical coordinates. Also, we will assume that the solution, $u(r,z,t) = T(r,z,t) - T_b$, is independent of θ due to axial symmetry. This gives the heat equation in θ-independent cylindrical coordinates as

$$\frac{\partial u}{\partial t} = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}\right), \tag{6.116}$$

where $0 \le r \le a$ and $0 \le z \le Z$. The initial condition is

$$u(r,z,0) = T_i - T_b,$$

and the homogeneous boundary conditions on the side, top, and bottom of the cake are

$$u(a,z,t) = 0,$$
$$u(r,0,t) = u(r,Z,t) = 0.$$

Again, we seek solutions of the form $u(r,z,t) = R(r)H(z)G(t)$. Separation of variables leads to

$$\underbrace{\frac{1}{k}\frac{G'}{G}}_{-\lambda} = \underbrace{\frac{1}{rR}\frac{d}{dr}\left(rR'\right)}_{-\mu^2} + \underbrace{\frac{H''}{H}}_{-\nu^2}. \tag{6.117}$$

Here we have indicated the separation constants, which lead to three ordinary differential equations. These equations and the boundary conditions are

$$\begin{aligned} G' + k\lambda G &= 0, \\ \frac{d}{dr}\left(rR'\right) + \mu^2 rR &= 0, \quad R(a) = 0, \quad R(0) \text{ is finite}, \\ H'' + \nu^2 H &= 0, \quad H(0) = H(Z) = 0. \end{aligned} \tag{6.118}$$

We further note that the separation constants are related by the expression $\lambda = \mu^2 + \nu^2$.

We can easily write down the solutions for $G(t)$ and $H(z)$,

$$G(t) = Ae^{-\lambda kt}$$

and

$$H_n(z) = \sin\frac{n\pi z}{Z}, \quad n = 1, 2, 3, \ldots,$$

where $\nu = \frac{n\pi}{Z}$. Recalling from the rectangular case that only odd terms arise in the Fourier sine series coefficients for the constant initial condition, we proceed by rewriting $H(z)$ as

$$H_n(z) = \sin\frac{(2n-1)\,\pi z}{Z}, \quad n = 1, 2, 3, \ldots \tag{6.119}$$

with $\nu = \frac{(2n-1)\,\pi}{Z}$.

The radial equation can be written in the form

$$r^2 R'' + rR' + \mu^2 r^2 R = 0.$$

This is a Bessel equation of the first kind of order zero which we had seen in Section 5.5. Therefore, the general solution is a linear combination of Bessel functions of the first and second kind,

$$R(r) = c_1 J_0(\mu r) + c_2 N_0(\mu r). \tag{6.120}$$

Since $R(r)$ is bounded at $r = 0$ and $N_0(\mu r)$ is not well behaved at $r = 0$, we set $c_2 = 0$. Up to a constant factor, the solution becomes

$$R(r) = J_0(\mu r). \tag{6.121}$$

The boundary condition $R(a) = 0$ gives the eigenvalues as

$$\mu_m = \frac{j_{0m}}{a}, \quad m = 1, 2, 3, \ldots,$$

where j_{0m} are the m^{th} roots of the zero-order Bessel function, $J_0(j_{0m}) = 0$.

Therefore, we have found the product solutions

$$H_n(z)R_m(r)G(t) = \sin\frac{(2n-1)\,\pi z}{Z} J_0\left(\frac{r}{a}j_{0m}\right) e^{-\lambda_{nm}kt}, \tag{6.122}$$

where $m = 1, 2, 3, \ldots, n = 1, 2, \ldots$. Combining the product solutions, the general solution is found as

$$u(r, z, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin \frac{(2n-1)\pi z}{Z} J_0\left(\frac{r}{a} j_{0m}\right) e^{-\lambda_{nm} k t} \tag{6.123}$$

with

$$\lambda_{nm} = \left(\frac{(2n-1)\pi}{Z}\right)^2 + \left(\frac{j_{0m}}{a}\right)^2,$$

for $n, m = 1, 2, 3, \ldots$.

Inserting the solution into the constant initial condition, we have

$$T_i - T_b = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin \frac{(2n-1)\pi z}{Z} J_0\left(\frac{r}{a} j_{0m}\right).$$

This is a double Fourier series but it involves a Fourier-Bessel expansion. Writing

$$b_n(r) = \sum_{m=1}^{\infty} A_{nm} J_0\left(\frac{r}{a} j_{0m}\right),$$

the condition becomes

$$T_i - T_b = \sum_{n=1}^{\infty} b_n(r) \sin \frac{(2n-1)\pi z}{Z}.$$

As seen previously, this is a Fourier sine series and the Fourier coefficients are given by

$$\begin{aligned} b_n(r) &= \frac{2}{Z} \int_0^Z (T_i - T_b) \sin \frac{(2n-1)\pi z}{Z} dz \\ &= \frac{2(T_i - T_b)}{Z} \left[-\frac{Z}{(2n-1)\pi} \cos \frac{(2n-1)\pi z}{Z} \right]_0^Z \\ &= \frac{4(T_i - T_b)}{(2n-1)\pi}. \end{aligned}$$

We insert this result into the Fourier-Bessel series,

$$\frac{4(T_i - T_b)}{(2n-1)\pi} = \sum_{m=1}^{\infty} A_{nm} J_0\left(\frac{r}{a} j_{0m}\right),$$

and recall from Section 5.5 that we can determine the Fourier coefficients A_{nm} using the Fourier-Bessel series,

$$f(x) = \sum_{n=1}^{\infty} c_n J_p(j_{pn} \frac{x}{a}), \tag{6.124}$$

where the Fourier-Bessel coefficients are found as

$$c_n = \frac{2}{a^2 \left[J_{p+1}(j_{pn})\right]^2} \int_0^a x f(x) J_p(j_{pn} \frac{x}{a})\, dx. \tag{6.125}$$

Comparing these series expansions, we have

$$A_{nm} = \frac{2}{a^2 J_1^2(j_{0m})} \frac{4(T_i - T_b)}{(2n-1)\pi} \int_0^a J_0(\mu_m r) r\, dr. \tag{6.126}$$

In order to evaluate $\int_0^a J_0(\mu_m r) r\,dr$, we let $y = \mu_m r$ and get

$$\begin{aligned}
\int_0^a J_0(\mu_m r) r dr &= \int_0^{\mu_m a} J_0(y) \frac{y}{\mu_m} \frac{dy}{\mu_m} \\
&= \frac{1}{\mu_m^2} \int_0^{\mu_m a} J_0(y) y\, dy \\
&= \frac{1}{\mu_m^2} \int_0^{\mu_m a} \frac{d}{dy} \left(y J_1(y) \right)\, dy \\
&= \frac{1}{\mu_m^2} (\mu_m a) J_1(\mu_m a) = \frac{a^2}{j_{0m}} J_1(j_{0m}). \qquad (6.127)
\end{aligned}$$

Here we have made use of the identity $\frac{d}{dx}\left(x J_1(x) \right) = J_0(x)$ from Section 5.5.

Substituting the result of this integral computation into the expression for A_{nm}, we find

$$A_{nm} = \frac{8(T_i - T_b)}{(2n-1)\pi} \frac{1}{j_{0m} J_1(j_{0m})}.$$

Substituting this result into the original expression for $u(r,z,t)$, gives

$$u(r,z,t) = \frac{8(T_i - T_b)}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin \frac{(2n-1)\pi z}{Z}}{(2n-1)} \frac{J_0(\frac{r}{a} j_{0m}) e^{-\lambda_{nm} kt}}{j_{0m} J_1(j_{0m})}.$$

Therefore, $T(r,z,t)$ is found as

$$T(r,z,t) = T_b + \frac{8(T_i - T_b)}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin \frac{(2n-1)\pi z}{Z}}{(2n-1)} \frac{J_0(\frac{r}{a} j_{0m}) e^{-\lambda_{nm} kt}}{j_{0m} J_1(j_{0m})},$$

where

$$\lambda_{nm} = \left(\frac{(2n-1)\pi}{Z} \right)^2 + \left(\frac{j_{0m}}{a} \right)^2, \quad n, m = 1, 2, 3, \ldots.$$

We have therefore found the general solution for the three-dimensional heat equation in cylindrical coordinates with constant diffusivity. Similar to the solutions shown in Figure 6.19 of the previous section, we show in Figure 6.22 the temperature evolution throughout a standard 9″ round cake pan. These are vertical slices similar to what is depicted in Figure 6.21.

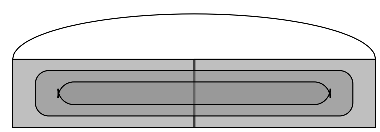


Figure 6.21: Depiction of a sideview of a vertical slice of a circular cake.

Again, one could generalize this example to considerations of other types of cakes with cylindrical symmetry. For example, there are muffins, Boston steamed bread which is steamed in tall cylindrical cans. One could also consider an annular pan, such as a bundt cake pan. In fact, such problems extend beyond baking cakes to possible heating molds in manufacturing.

6.5 Laplace's Equation and Spherical Symmetry

WE HAVE SEEN THAT LAPLACE'S EQUATION, $\nabla^2 u = 0$, arises in electrostatics as an equation for electric potential outside a charge distribution

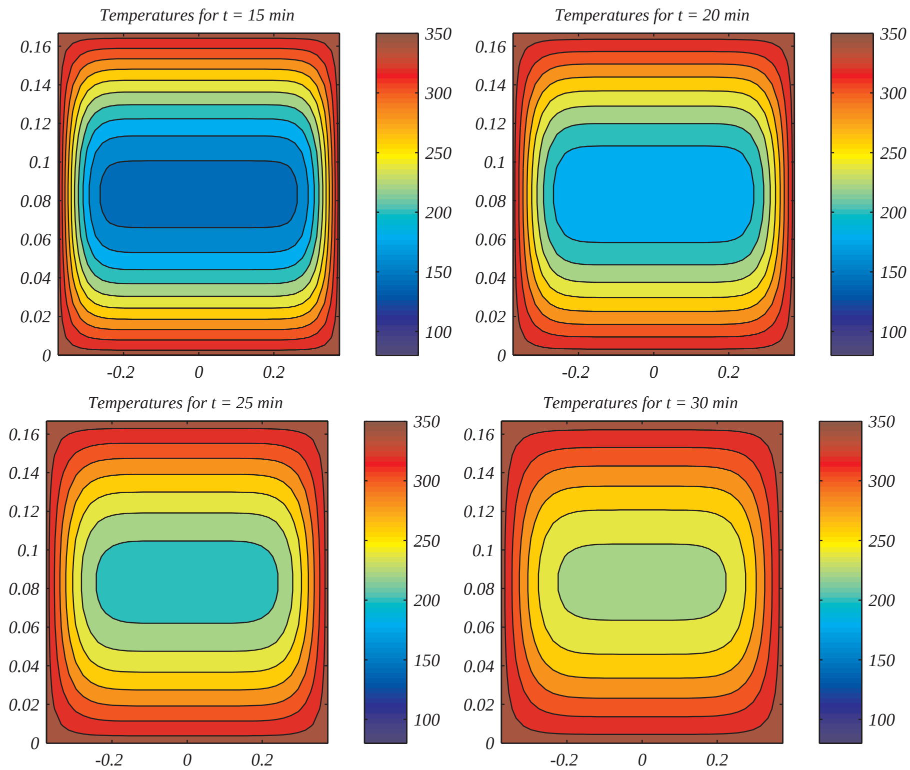


Figure 6.22: Temperature evolution for a standard 9″ cake shown as vertical slices through the center.

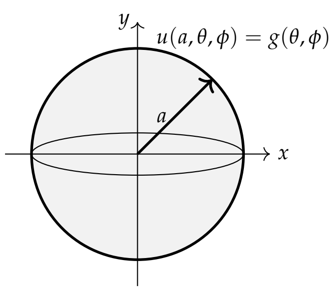


Figure 6.23: A sphere of radius r with the boundary condition $u(r,\theta,\phi) = g(\theta,\phi)$.

and it occurs as the equation governing equilibrium temperature distributions. Laplace's equation originally occurred in the study of potential theory, which also includes the study of gravitational and fluid potentials. The equation is named after Pierre-Simon Laplace (1749-1827) who had studied the properties of this equation.

Example 6.12. Solve Laplace's equation in spherical coordinates.

We seek solutions of this equation inside a sphere of radius a subject to the boundary condition as shown in Figure 6.23. The problem is given by Laplace's equation Laplace's equation in spherical coordinates.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0, \qquad (6.128)$$

where $u = u(r,\theta,\phi)$.

The boundary conditions are given by

$$u(a,\theta,\phi) = g(\theta,\phi), \quad 0 < \phi < 2\pi, \quad 0 < \theta < \pi,$$

and the periodic boundary conditions

$$u(r,\theta,0) = u(r,\theta,2\pi), \quad u_\phi(r,\theta,0) = u_\phi(r,\theta,2\pi),$$

where $0 < r < \infty$, and $\quad 0 < \theta < \pi$.

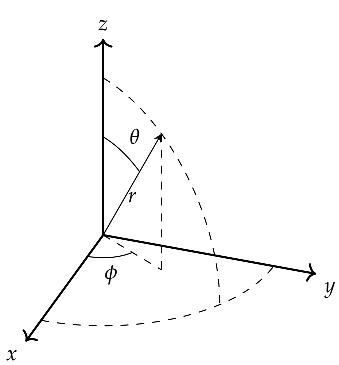


Figure 6.24: Definition of spherical coordinates (r,θ,ϕ). Note that there are different conventions for labeling spherical coordinates. This labeling is used often in physics.

As before, we perform a separation of variables by seeking product solutions of the form $u(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$. Inserting this form into the Laplace equation, we obtain

$$\frac{\Theta\Phi}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{R\Phi}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{R\Theta}{r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2}=0. \qquad (6.129)$$

Multiplying this equation by r^2 and dividing by $R\Theta\Phi$, yields

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2}=0. \qquad (6.130)$$

Note that the first term is the only term depending upon r. Thus, we can separate out the radial part. However, there is still more work to do on the other two terms, which give the angular dependence. Thus, we have

$$-\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2}=-\lambda, \qquad (6.131)$$

where we have introduced the first separation constant. This leads to two equations:

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\lambda R=0 \qquad (6.132)$$

and

$$\frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2}=-\lambda. \qquad (6.133)$$

The final separation can be performed by multiplying the last equation by $\sin^2\theta$, rearranging the terms, and introducing a second separation constant:

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\lambda\sin^2\theta=-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=\mu. \qquad (6.134)$$

Equation (6.133) is a key equation which occurs when studying problems possessing spherical symmetry. It is an eigenvalue problem for $Y(\theta,\phi)=\Theta(\theta)\Phi(\phi)$, $LY=-\lambda Y$, where

$$L=\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

The eigenfunctions of this operator are referred to as spherical harmonics.

From this expression we can determine the differential equations satisfied by $\Theta(\theta)$ and $\Phi(\phi)$:

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+(\lambda\sin^2\theta-\mu)\Theta=0, \qquad (6.135)$$

and

$$\frac{d^2\Phi}{d\phi^2}+\mu\Phi=0. \qquad (6.136)$$

We now have three ordinary differential equations to solve. These are the radial equation (6.132) and the two angular equations (6.135)-(6.136). We note that all three are in Sturm-Liouville form. We will solve each eigenvalue problem subject to appropriate boundary conditions.

The simplest of these differential equations is Equation (6.136) for $\Phi(\phi)$. We have seen equations of this form many times and the general solution is a linear combination of sines and cosines. Furthermore, in this problem $u(r,\theta,\phi)$ is periodic in ϕ,

$$u(r,\theta,0)=u(r,\theta,2\pi),\quad u_\phi(r,\theta,0)=u_\phi(r,\theta,2\pi).$$

Since these conditions hold for all r and θ, we must require that $\Phi(\phi)$ satisfy the periodic boundary conditions

$$\Phi(0)=\Phi(2\pi),\quad \Phi'(0)=\Phi'(2\pi).$$

The eigenfunctions and eigenvalues for Equation (6.136) are then found as

$$\Phi(\phi) = \{\cos m\phi, \sin m\phi\}, \quad \mu = m^2, \quad m = 0, 1, \ldots. \tag{6.137}$$

Next we turn to solving equation, (6.135). We first transform this equation in order to identify the solutions. Let $x = \cos\theta$. Then the derivatives with respect to θ transform as

$$\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = -\sin\theta \frac{d}{dx}.$$

Letting $y(x) = \Theta(\theta)$ and noting that $\sin^2\theta = 1 - x^2$, Equation (6.135) becomes

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \left(\lambda - \frac{m^2}{1-x^2}\right) y = 0. \tag{6.138}$$

We further note that $x \in [-1, 1]$, as can be easily confirmed by the reader.

This is a Sturm-Liouville eigenvalue problem. The solutions consist of a set of orthogonal eigenfunctions. For the special case that $m = 0$ Equation (6.138) becomes

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \lambda y = 0. \tag{6.139}$$

In a course in differential equations one learns to seek solutions of this equation in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

This leads to the recursion relation

$$a_{n+2} = \frac{n(n+1) - \lambda}{(n+2)(n+1)} a_n.$$

Setting $n = 0$ and seeking a series solution, one finds that the resulting series does not converge for $x = \pm 1$. This is remedied by choosing $\lambda = \ell(\ell+1)$ for $\ell = 0, 1, \ldots$, leading to the differential equation

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \ell(\ell+1)y = 0. \tag{6.140}$$

We saw this equation in Chapter 5 in the form

$$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0.$$

The solutions of this differential equation are Legendre polynomials, denoted by $P_\ell(x)$.

Associated Legendre functions.

For the more general case, $m \neq 0$, the differential equation (6.138) with $\lambda = \ell(\ell+1)$ becomes

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right) y = 0. \tag{6.141}$$

The solutions of this equation are called the associated Legendre functions. The two linearly independent solutions are denoted by $P_\ell^m(x)$ and $Q_\ell^m(x)$. The latter functions are not well behaved at $x = \pm 1$, corresponding to the north and south poles of the original problem. So, we can throw out these solutions in many physical cases, leaving

$$\Theta(\theta) = P_\ell^m(\cos\theta)$$

as the needed solutions. In Table 6.5 we list a few of these.

Table 6.5: Associated Legendre Functions, $P_n^m(x)$.

	$P_n^m(x)$	$P_n^m(\cos\theta)$
$P_0^0(x)$	1	1
$P_1^0(x)$	x	$\cos\theta$
$P_1^1(x)$	$-(1-x^2)^{\frac{1}{2}}$	$-\sin\theta$
$P_2^0(x)$	$\frac{1}{2}(3x^2-1)$	$\frac{1}{2}(3\cos^2\theta-1)$
$P_2^1(x)$	$-3x(1-x^2)^{\frac{1}{2}}$	$-3\cos\theta\sin\theta$
$P_2^2(x)$	$3(1-x^2)$	$3\sin^2\theta$
$P_3^0(x)$	$\frac{1}{2}(5x^3-3x)$	$\frac{1}{2}(5\cos^3\theta-3\cos\theta)$
$P_3^1(x)$	$-\frac{3}{2}(5x^2-1)(1-x^2)^{\frac{1}{2}}$	$-\frac{3}{2}(5\cos^2\theta-1)\sin\theta$
$P_3^2(x)$	$15x(1-x^2)$	$15\cos\theta\sin^2\theta$
$P_3^3(x)$	$-15(1-x^2)^{\frac{3}{2}}$	$-15\sin^3\theta$

The associated Legendre functions are related to the Legendre polynomials by[5]

$$P_\ell^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x), \tag{6.142}$$

for $\ell = 0, 1, 2, , \ldots$ and $m = 0, 1, \ldots, \ell$. We further note that $P_\ell^0(x) = P_\ell(x)$, as one can see in the table. Since $P_\ell(x)$ is a polynomial of degree ℓ, then for $m > \ell$, $\frac{d^m}{dx^m}P_\ell(x) = 0$ and $P_\ell^m(x) = 0$.

[5] The factor of $(-1)^m$ is known as the Condon-Shortley phase and is useful in quantum mechanics in the treatment of agular momentum. It is sometimes omitted by some

Furthermore, since the differential equation only depends on m^2, $P_\ell^{-m}(x)$ is proportional to $P_\ell^m(x)$. One normalization is given by

$$P_\ell^{-m}(x) = (-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(x).$$

The associated Legendre functions also satisfy the orthogonality condition

Orthogonality relation.

$$\int_{-1}^{1} P_\ell^m(x)P_{\ell'}^m(x)\,dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell\ell'}. \tag{6.143}$$

The last differential equation we need to solve is the radial equation. With $\lambda = \ell(\ell+1)$, $\ell = 0, 1, 2, \ldots$, the radial equation (6.132) can be written as

$$r^2R'' + 2\,rR' - \ell(\ell+1)R = 0. \tag{6.144}$$

The radial equation is a Cauchy-Euler type of equation. So, we can guess the form of the solution to be $R(r) = r^s$, where s is a yet to be determined constant. Inserting this guess into the radial equation, we obtain the characteristic equation

$$s(s+1) = \ell(\ell+1).$$

Solving for s, we have

$$s = \ell, -(\ell+1).$$

Thus, the general solution of the radial equation is

$$R(r) = \alpha r^{\ell} + \beta r^{-(\ell+1)}. \tag{6.145}$$

When seeking solutions outside the sphere, one considers the boundary condition $R(r) \to 0$ as $r \to \infty$. In this case, $R(r) = r^{-(\ell+1)}$.

We would normally apply boundary conditions at this point. The boundary condition $u(a,\theta,\phi) = g(\theta,\phi)$ is not a homogeneous boundary condition, so we will need to hold off using it until we have the general solution to the three dimensional problem. However, we do have a hidden condition. Since we are interested in solutions inside the sphere, we need to consider what happens at $r = 0$. Note that $r^{-(\ell+1)}$ is not defined at the origin. Since the solution is expected to be bounded at the origin, we can set $\beta = 0$. So, in the current problem we have established that

$$R(r) = \alpha r^{\ell}.$$

We have carried out the full separation of Laplace's equation in spherical coordinates. The product solutions consist of the forms

$$u^{(c)}_{\ell m}(r,\theta,\phi) = r^{\ell} P^m_{\ell}(\cos\theta)\cos m\phi$$

and

$$u^{(s)}_{\ell m}(r,\theta,\phi) = r^{\ell} P^m_{\ell}(\cos\theta)\sin m\phi$$

for $\ell = 0, 1, 2, \ldots$ and $m = 0, \pm 1, , \ldots, \pm\ell$. These solutions can be combined to give a complex representation of the product solutions as

$$u_{\ell m}(r,\theta,\phi) = r^{\ell} P^m_{\ell}(\cos\theta) e^{im\phi}.$$

The general solution is then given as a linear combination of these product solutions. As there are two indices, we have a double sum:[6]

$$u(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell m}\, r^{\ell} P^m_{\ell}(\cos\theta) e^{im\phi}. \tag{6.146}$$

[6] While this appears to be a complex-valued solution, it can be rewritten as a sum over real functions. The inner sum contains terms for both $m = k$ and $m = -k$. Adding these contributions, we have that

$$c_{\ell k}\, r^{\ell} P^k_{\ell}(\cos\theta) e^{ik\phi} + c_{\ell(-k)}\, r^{\ell} P^{-k}_{\ell}(\cos\theta) e^{-ik\phi}$$

can be rewritten as

$$(A_{\ell k}\cos k\phi + B_{\ell k}\sin k\phi)\, r^{\ell} P^k_{\ell}(\cos\theta).$$

Example 6.13. Laplace's Equation with Azimuthal Symmetry

As a simple example we consider the solution of Laplace's equation in which there is azimuthal symmetry. Let

$$u(a,\theta,\phi) = g(\theta) = 1 - \cos 2\theta.$$

This function is zero at the poles and has a maximum at the equator. So, this could be a crude model of the temperature distribution of the Earth with zero temperature at the poles and a maximum near the equator.

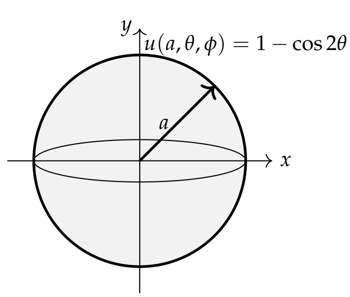


Figure 6.25: A sphere of radius a with the boundary condition

$$u(a,\theta,\phi) = 1 - \cos 2\theta.$$

In problems in which there is no ϕ-dependence, only the $m = 0$ terms of the general solution survives. Thus, we have that

$$u(r,\theta,\phi) = \sum_{\ell=0}^{\infty} c_\ell \, r^\ell P_\ell(\cos\theta). \tag{6.147}$$

Here we have used the fact that $P_\ell^0(x) = P_\ell(x)$. We just need to determine the unknown expansion coefficients, c_ℓ. Imposing the boundary condition at $r = a$, we are lead to

$$g(\theta) = \sum_{\ell=0}^{\infty} c_\ell a^\ell P_\ell(\cos\theta). \tag{6.148}$$

This is a Fourier-Legendre series representation of $g(\theta)$. Since the Legendre polynomials are an orthogonal set of eigenfunctions, we can extract the coefficients.

In Chapter 5 we had proven that

$$\int_0^\pi P_n(\cos\theta)P_m(\cos\theta)\sin\theta\,d\theta = \int_{-1}^1 P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{nm}.$$

So, multiplying the expression for $g(\theta)$ by $P_m(\cos\theta)\sin\theta$ and integrating, we obtain the expansion coefficients:

$$c_\ell = \frac{2\ell+1}{2a^\ell}\int_0^\pi g(\theta)P_\ell(\cos\theta)\sin\theta\,d\theta. \tag{6.149}$$

Sometimes it is easier to rewrite $g(\theta)$ as a polynomial in $\cos\theta$ and avoid the integration. For this example we see that

$$\begin{aligned} g(\theta) &= 1-\cos 2\theta \\ &= 2\sin^2\theta \\ &= 2-2\cos^2\theta. \end{aligned} \tag{6.150}$$

Thus, setting $x = \cos\theta$ and $G(x) = g(\theta(x))$, we have $G(x) = 2 - 2x^2$.

We seek the form

$$G(x) = d_0P_0(x) + d_1P_1(x) + d_2P_2(x),$$

where $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = \frac{1}{2}(3x^2-1)$. Since $G(x) = 2-2x^2$ does not have any x terms, we know that $d_1 = 0$. So,

$$2-2x^2 = d_0(1) + d_2\frac{1}{2}(3x^2-1) = d_0 - \frac{1}{2}d_2 + \frac{3}{2}d_2x^2.$$

By observation we have $d_2 = -\frac{4}{3}$ and thus, $d_0 = 2 + \frac{1}{2}d_2 = \frac{4}{3}$. Therefore, $G(x) = \frac{4}{3}P_0(x) - \frac{4}{3}P_2(x)$.

We have found the expansion of $g(\theta)$ in terms of Legendre polynomials,

$$g(\theta) = \frac{4}{3}P_0(\cos\theta) - \frac{4}{3}P_2(\cos\theta). \tag{6.151}$$

Therefore, the nonzero coefficients in the general solution become

$$c_0 = \frac{4}{3}, \quad c_2 = \frac{4}{3}\frac{1}{a^2},$$

and the rest of the coefficients are zero. Inserting these into the general solution, we have the final solution

$$\begin{aligned} u(r,\theta,\phi) &= \frac{4}{3}P_0(\cos\theta) - \frac{4}{3}\left(\frac{r}{a}\right)^2 P_2(\cos\theta) \\ &= \frac{4}{3} - \frac{2}{3}\left(\frac{r}{a}\right)^2 (3\cos^2\theta - 1). \end{aligned} \tag{6.152}$$

6.5.1 *Spherical Harmonics*

$Y_{\ell m}(\theta,\phi)$, are the spherical harmonics. Spherical harmonics are important in applications from atomic electron configurations to gravitational fields, planetary magnetic fields, and the cosmic microwave background radiation.

The solutions of the angular parts of the problem are often combined into one function of two variables, as problems with spherical symmetry arise often, leaving the main differences between such problems confined to the radial equation. These functions are referred to as spherical harmonics, $Y_{\ell m}(\theta,\phi)$, which are defined with a special normalization as

$$Y_{\ell m}(\theta,\phi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta) e^{im\phi}. \tag{6.153}$$

These satisfy the simple orthogonality relation

$$\int_0^\pi \int_0^{2\pi} Y_{\ell m}(\theta,\phi) Y^*_{\ell' m'}(\theta,\phi) \sin\theta \, d\phi \, d\theta = \delta_{\ell\ell'}\delta_{mm'}.$$

As seen earlier in the chapter, the spherical harmonics are eigenfunctions of the eigenvalue problem $LY = -\lambda Y$, where

$$L = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

This operator appears in many problems in which there is spherical symmetry, such as obtaining the solution of Schrödinger's equation for the hydrogen atom as we will see later. Therefore, it is customary to plot spherical harmonics. Because the $Y_{\ell m}$'s are complex functions, one typically plots either the real part or the modulus squared. One rendition of $|Y_{\ell m}(\theta,\phi)|^2$ is shown in Figure 6.6 for $\ell, m = 0, 1, 2, 3$.

We could also look for the nodal curves of the spherical harmonics like we had for vibrating membranes. Such surface plots on a sphere are shown in Figure 6.7. The colors provide for the amplitude of the $|Y_{\ell m}(\theta,\phi)|^2$. We can match these with the shapes in Figure 6.6 by coloring the plots with some of the same colors as shown in Figure 6.7. However, by plotting just the sign of the spherical harmonics, as in Figure 6.8, we can pick out the nodal curves much easier.

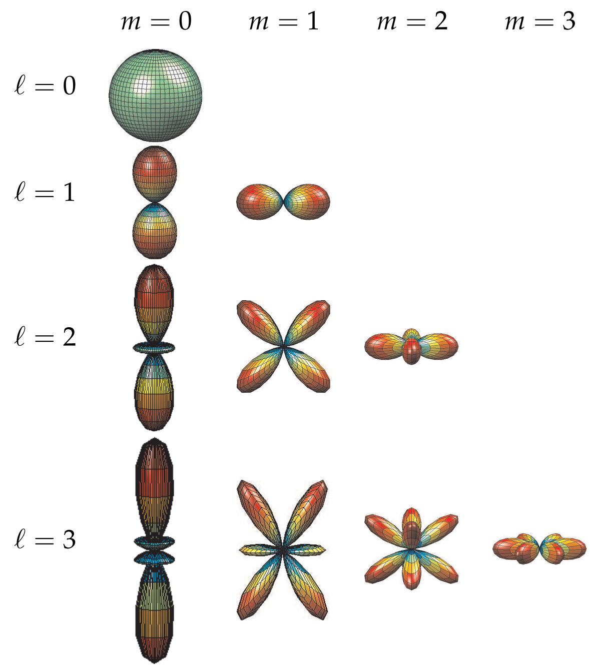


Table 6.6: The first few spherical harmonics, $|Y_{\ell m}(\theta, \phi)|^2$

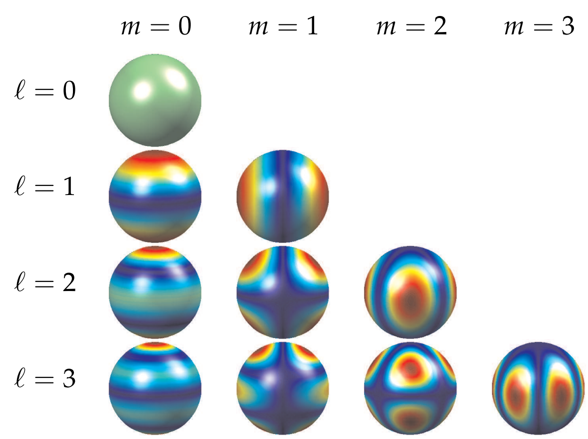


Table 6.7: Spherical harmonic contours for $|Y_{\ell m}(\theta, \phi)|^2$.

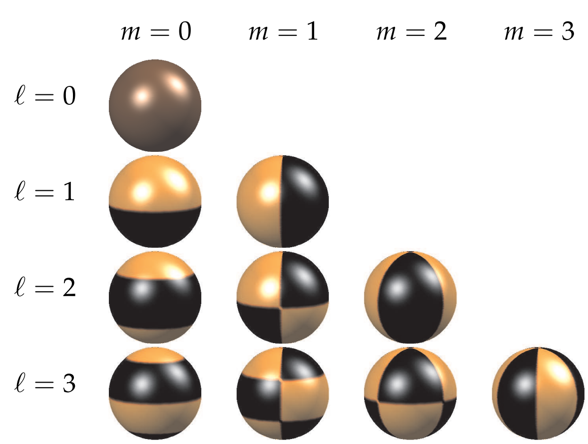


Table 6.8: In these figures we show the nodal curves of $|Y_{\ell m}(\theta, \phi)|^2$ Along the first column ($m = 0$) are the zonal harmonics seen as ℓ horizontal circles. Along the top diagonal ($m = \ell$) are the sectional harmonics. These look like orange sections formed from m vertical circles. The remaining harmonics are tesseral harmonics. They look like a checkerboard pattern formed from intersections of $\ell - m$ horizontal circles and m vertical circles.

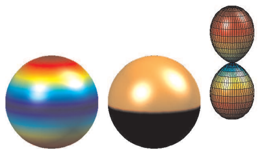


Figure 6.26: Zonal harmonics, $\ell = 1$, $m = 0$.

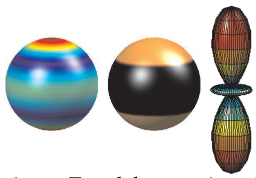


Figure 6.27: Zonal harmonics, $\ell = 2$, $m = 0$.

Figure 6.28: Sectoral harmonics, $\ell = 2$, $m = 2$.

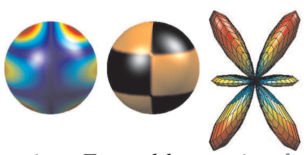


Figure 6.29: Tesseral harmonics, $\ell = 3$, $m = 1$.

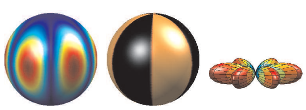


Figure 6.30: Sectoral harmonics, $\ell = 3$, $m = 3$.

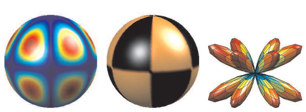


Figure 6.31: Tesseral harmonics, $\ell = 4$, $m = 3$.

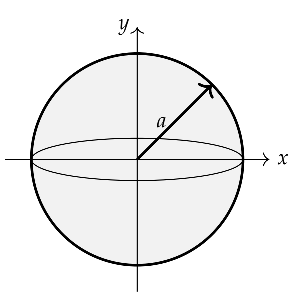


Figure 6.32: A vibrating sphere of radius a with the initial conditions

$$u(\theta, \phi, 0) = f(\theta, \phi),$$

$$u_t(\theta, \phi, 0) = g(\theta, \phi).$$

Spherical, or surface, harmonics can be further grouped into zonal, sectoral, and tesseral harmonics. Zonal harmonics correspond to the $m = 0$ modes. In this case, one seeks nodal curves for which $P_\ell(\cos\theta) = 0$. Solutions of this equation lead to constant θ values such that $\cos\theta$ is a zero of the Legendre polynomial, $P_\ell(x)$. The zonal harmonics correspond to the first column in Figure 6.8. Since $P_\ell(x)$ is a polynomial of degree ℓ, the zonal harmonics consist of ℓ latitudinal circles.

Sectoral, or meridional, harmonics result for the case that $m = \pm\ell$. For this case, we note that $P_\ell^{\pm\ell}(x) \propto (1 - x^2)^{m/2}$. This function vanishes for $x = \pm 1$, or $\theta = 0, \pi$. Therefore, the spherical harmonics can only produce nodal curves for $e^{im\phi} = 0$. Thus, one obtains the meridians satisfying the condition $A\cos m\phi + B\sin m\phi = 0$. Solutions of this equation are of the form $\phi =$ constant. These modes can be seen in Figure 6.8 in the top diagonal and can be described as m circles passing through the poles, or longitudinal circles.

Tesseral harmonics consist of the rest of the modes, which typically look like a checker board glued to the surface of a sphere. Examples can be seen in the pictures of nodal curves, such as Figure 6.8. Looking in Figure 6.8 along the diagonals going downward from left to right, one can see the same number of latitudinal circles. In fact, there are $\ell - m$ latitudinal nodal curves in these figures

In summary, the spherical harmonics have several representations, as show in Figures 6.7-6.8. Note that there are ℓ nodal lines, m meridional curves, and $\ell - m$ horizontal curves in these figures. The plots in Figure 6.6 are the typical plots shown in physics for discussion of the wavefunctions of the hydrogen atom. Those in 6.7 are useful for describing gravitational or electric potential functions, temperature distributions, or wave modes on a spherical surface. The relationships between these pictures and the nodal curves can be better understood by comparing respective plots. Several modes were separated out in Figures 6.26-6.31 to make this comparison easier.

6.6 *Spherically Symmetric Vibrations*

Another application of spherical harmonics is a vibrating spherical membrane, such as a balloon. Just as for the two-dimensional membranes encountered earlier, we let $u(\theta, \phi, t)$ represent the vibrations of the surface about a fixed radius obeying the wave equation, $u_{tt} = c^2\nabla^2 u$, and satisfying the initial conditions

$$u(\theta, \phi, 0) = f(\theta, \phi), \quad u_t(\theta, \phi, 0) = g(\theta, \phi).$$

In spherical coordinates, we have (for $r = a =$ constant.)

$$u_{tt} = \frac{c^2}{a^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right), \tag{6.154}$$

where $u = u(\theta, \phi, t)$.

The boundary conditions are given by the periodic boundary conditions

$$u(\theta, 0, t) = u(\theta, 2\pi, t), \quad u_\phi(\theta, 0, t) = u_\phi(\theta, 2\pi, t),$$

where $0 < t$, and $0 < \theta < \pi$, and that $u = u(\theta, \phi, t)$ should remain bounded.

Noting that the wave equation takes the form

$$u_{tt} = \frac{c^2}{a^2} Lu, \quad \text{where} \quad LY_{\ell m} = -\ell(\ell+1)Y_{\ell m}$$

for the spherical harmonics $Y_{\ell m}(\theta, \phi) = P_\ell^m(\cos\theta)e^{im\phi}$, we can seek product solutions of the form

$$u_{\ell m}(\theta, \phi, t) = T(t)Y_{\ell m}(\theta, \phi).$$

Inserting this form into the wave equation in spherical coordinates, we find

$$T''Y_{\ell m} = -\frac{c^2}{a^2} T(t)\ell(\ell+1)Y_{\ell m},$$

or

$$T'' + \ell(\ell+1)\frac{c^2}{a^2} T(t) = 0.$$

The solutions of this equation are easily found as

$$T(t) = A\cos\omega_\ell t + B\sin\omega_\ell t, \quad \omega_\ell = \sqrt{\ell(\ell+1)}\frac{c}{a}.$$

Therefore, the product solutions are given by

$$u_{\ell m}(\theta, \phi, t) = [A\cos\omega_\ell t + B\sin\omega_\ell t]\, Y_{\ell m}(\theta, \phi)$$

for $\ell = 0, 1, \ldots$, $m = -\ell, -\ell+1, \ldots, \ell$.

In Figure 6.33 we show several solutions for $a = c = 1$ at $t = 10$.

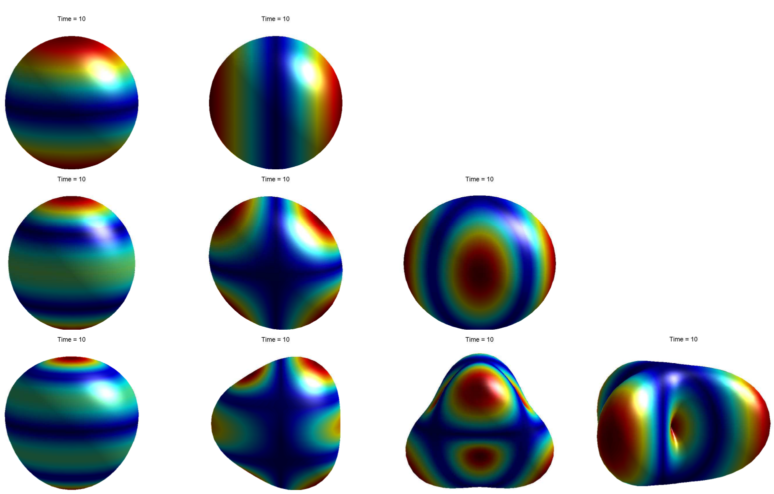


Figure 6.33: Modes for a vibrating spherical membrane:
Row 1: $(1,0), (1,1)$;
Row 2: $(2,0), (2,1), (2,2)$;
Row 3 $(3,0), (3,1), (3,2), (3,3)$.

The general solution is found as

$$u(\theta, \phi, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} [A_{\ell m}\cos\omega_\ell t + B_{\ell m}\sin\omega_\ell t]\, Y_{\ell m}(\theta, \phi).$$

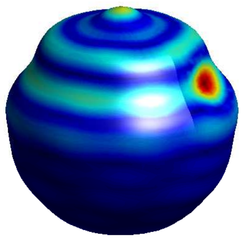


Figure 6.34: A moment captured from a simulation of a spherical membrane after hit with a velocity impulse.

An interesting problem is to consider hitting the balloon with a velocity impulse while at rest. An example of such a solution is shown in Figure 6.34. In this images several modes are excited after the impulse.

6.7 *Baking a Spherical Turkey*

During one year as this course was being taught, an instructor returned from the American holiday of Thanksgiving, where it is customary to cook a turkey. Such a turkey is shown in Figure 6.35. This reminded the instructor of a typical problem, such as in Weinberger, (1995, p. 92.), where one is given a roast of a certain volume and one is asked to find the time it takes to cook one double the size. In this section, we explore a similar problem for cooking a turkey.

Figure 6.35: A 12-lb turkey leaving the oven.

Often during this time of the year, November, articles appear with some scientific evidence as to how to gauge how long it takes to cook a turkey of a given weight. Inevitably it refers to the story, as told in `http://today.slac.stanford.edu/a/2008/11-26.htm`http://today.slac.stanford.edu/a/2008/11-26.htm that Pief Panofsky, a former SLAC Director, was determined to find a nonlinear equation for determining cooking times instead of using the rule of thumb of 30 minutes per pound of turkey. He had arrived at the form,

$$t = \frac{W^{2/3}}{1.5},$$

where t is the cooking time and W is the weight of the turkey in pounds. Nowadays, one can go to Wolframalpha.com and enter the question "how long should you cook a turkey" and get results based on a similar formula.

Before turning to the solution of the heat equation for a turkey, let's consider a simpler problem.

Example 6.14. If it takes 4 hours to cook a 10 pound turkey in a 350^0 F oven, then how long would it take to cook a 20 pound turkey at the same conditions?

In all of our analysis, we will consider a spherical turkey. While the turkey in Figure 6.35 is not quite spherical, we are free to approximate the turkey as such. If you prefer, we could imagine a spherical turkey like the one shown in Figure 6.36.

This problem is one of scaling. Thinking of the turkey as being spherically symmetric, then the baking follows the heat equation in the form

$$u_t = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right).$$

We can rescale the variables from coordinates (r,t) to (r,τ) as $r = \beta r$, and $t = \alpha\tau$. Then the derivatives transform as

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial r}{\partial r}\frac{\partial}{\partial r} = \frac{1}{\beta}\frac{\partial}{\partial r}, \\ \frac{\partial}{\partial t} &= \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = \frac{1}{\alpha}\frac{\partial}{\partial \tau}. \end{aligned} \tag{6.155}$$

Figure 6.36: The depiction of a spherical turkey.

Inserting these transformations into the heat equation, we have

$$u_\tau = \frac{\alpha}{\beta^2}\frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right).$$

To keep conditions the same, then we need $\alpha = \beta^2$. So, the transformation that keeps the form of the heat equation the same, or makes it invariant, is $r = \beta\, r$, and $t = \beta^2\tau$. This is also known as a self-similarity transformation.

So, if the radius increases by a factor of β, then the time to cook the turkey (reaching a given temperature, u), would increase by β^2. Returning to the problem, if the weight of the doubles, then the volume doubles, assuming that the density is held constant. However, the volume is proportional to r^3. So, r increases by a factor of $2^{1/3}$. Therefore, the time increases by a factor of $2^{2/3} \approx 1.587$. This give the time for cooking a 20 lb turkey as $t = 4(2^{2/3}) = 2^{8/3} \approx 6.35$ hours.

The previous example shows the power of using similarity transformations to get general information about solutions of differential equations. However, we have focussed on using the method of separation of variables for most of the book so far. We should be able to find a solution to the spherical turkey model using these methods as well. This will be shown in the next example.

Example 6.15. Find the temperature, $T(r,t)$ inside a spherical turkey, initially at 40°, which is F placed in a 350° F. Assume that the turkey is of constant density and that the surface of the turkey is maintained at the oven temperature. [We will also neglect convection and radiation processes inside the oven.]

The problem can be formulated as a heat equation problem for $T(r,t)$:

$$\begin{aligned} T_t &= \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right), \quad 0< r<a, t>0, \\ T(a,t) &= 350, \quad T(r,t) \quad \text{finite at } r=0, \quad t>0, \\ T(r,0) &= 40. \end{aligned} \tag{6.156}$$

We note that the boundary condition is not homogeneous. However, we can fix that by introducing the auxiliary function (the difference between the turkey and oven temperatures) $u(r,t) = T(r,t) - T_a$, where $T_a = 350$. Then, the problem to be solved becomes

$$\begin{aligned} u_t &= \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right), \quad 0< r<a, t>0, \\ u(a,t) &= 0, \quad u(\,r,t) \quad \text{finite at } r=0, \quad t>0, \\ u(r,0) &= T_i - T_a = -310, \end{aligned} \tag{6.157}$$

where $T_i = 40$.

We can now employ the method of separation of variables. Let $u(r,t) = R(r)G(t)$. Inserting into the heat equation for u, we have

$$\frac{1}{k}\frac{G'}{G} = \frac{1}{R}\left(R'' + \frac{2}{r}R'\right) = -\lambda.$$

This give the two ordinary differential equations, the temporal equation,

$$G' = -k\lambda G, \tag{6.158}$$

and the radial equation,

$$rR'' + 2R' + \lambda rR = 0. \tag{6.159}$$

The temporal equation is easy to solve,

$$G(t) = G_0 e^{-\lambda kt}.$$

However, the radial equation is slightly more difficult. But, making the substitution $R(r) = y(r)/r$, it is readily transformed into a simpler form:[7]

$$y'' + \lambda y = 0.$$

The boundary conditions on $u(r,t) = R(r)G(t)$ transfer to $R(a) = 0$ and $R(r)$ finite at the origin. In turn, this means that $y(a) = 0$ and $y(r)$ has to vanish near the origin. If $y(\,r)$ does not vanish near the origin, then $R(\,r)$ is not finite as $r \to 0$.

[7] The radial equation almost looks familiar when it is multiplied by r :

$$r^2R'' + 2rR' + \lambda r^2 R = 0.$$

If it were not for the '2', it would be the zeroth order Bessel equation. This is actually the zeroth order spherical Bessel equation. In general, the spherical Bessel functions, $j_n(x)$ and $y_n(x)$, satisfy

$$x^2y'' + 2xy' + [x^2 - n(n+1)]y = 0.$$

So, the radial solution of the turkey problem is

$$R(r) = j_n(z) = (-z)^n\left(\frac{1}{z}\frac{d}{dz}\right)^n \frac{\sin z}{z}$$

for $z = \sqrt{\lambda}r$ and

$$j_0(\sqrt{\lambda}r) = \frac{\sin\sqrt{\lambda}r}{\sqrt{\lambda}r}.$$

We further note that

$$j_n(x) = \sqrt{\frac{\pi}{2x}}J_{n+\frac{1}{2}}(x)$$

So, we need to solve the boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(a) = 0.$$

This gives the well-known set of eigenfunctions

$$y(r) = \sin \frac{n\pi r}{a}, \quad \lambda_n = \left(\frac{n\pi}{a}\right)^2, \quad n = 1, 2, 3, \ldots.$$

Therefore, we have found

$$R(r) = \frac{\sin \frac{n\pi r}{a}}{r}, \quad \lambda_n = \left(\frac{n\pi}{a}\right)^2, \quad n = 1, 2, 3, \ldots.$$

The general solution to the auxiliary problem is

$$u(r,t) = \sum_{n=1}^{\infty} A_n \frac{\sin \frac{n\pi r}{a}}{r} e^{-(n\pi/a)^2 kt}.$$

This gives the general solution for the temperature as

$$T(r,t) = T_a + \sum_{n=1}^{\infty} A_n \frac{\sin \frac{n\pi r}{a}}{r} e^{-(n\pi/a)^2 kt}.$$

All that remains is to find the solution satisfying the initial condition, $T(r,0) = 40$. Inserting $t = 0$, we have

$$T_i - T_a = \sum_{n=1}^{\infty} A_n \frac{\sin \frac{n\pi r}{a}}{r}.$$

This is almost a Fourier sine series. Multiplying by r, we have

$$(T_i - T_a)r = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi r}{a}.$$

Now, we can solve for the coefficients,

$$\begin{aligned} A_n &= \frac{2}{a} \int_0^a (T_i - T_a) r \sin \frac{n\pi r}{a}\, d\, r \\ &= \frac{2a}{n\pi} (T_i - T_a)(-1)^{n+1}. \end{aligned} \tag{6.160}$$

This gives the final solution,

$$T(r,t) = T_a + \frac{2a(T_i - T_a)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{\sin \frac{n\pi r}{a}}{r} e^{-(n\pi/a)^2 kt}.$$

For generality, the ambient and initial temperature were left in terms of T_a and T_i, respectively.

It is interesting to use the above solution to compare roasting different turkeys. We take the same conditions as above. Let the radius of the spherical turkey be six inches. We will assume that such a turkey takes four hours to cook, i.e., reach a temperature of 180° F. Plotting the solution with 400 terms, one finds that $k \approx 0.000089$. This gives a "baking time" of $t_1 = 239.63$.

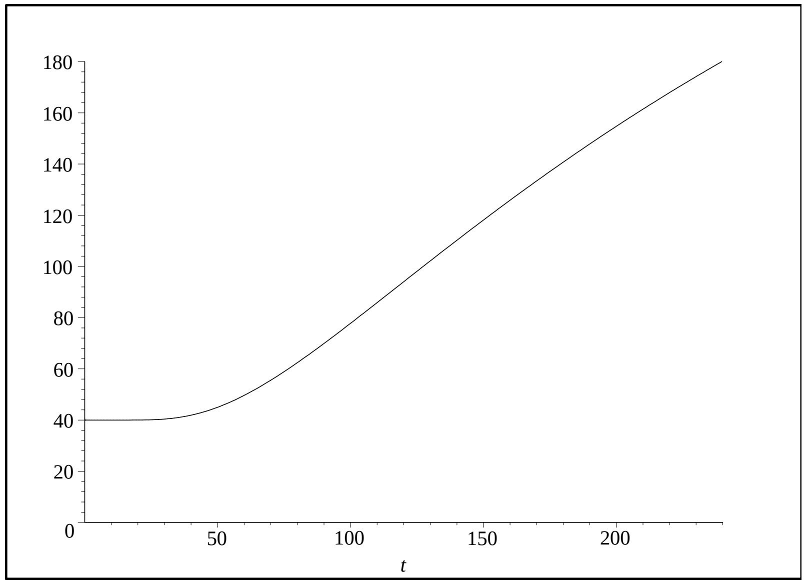


Figure 6.37: The temperature at the center of a turkey with radius $a = 0.5$ ft and $k \approx 0.000089$.

A plot of the temperature at the center point ($r = a/2$) of the bird is in Figure 6.37.

Using the same constants, but increasing the radius of a turkey to $a = 0.5(2^{1/3})$ ft, we obtain the temperature plot in Figure 6.38. This radius corresponds to doubling the volume of the turkey. Solving for the time at which the center temperature (at $r = a/2$) reaches 180° F, we obtained $t_2 = 380.38$. Comparing the two temperatures, we find the ratio (using the full computation of the solution in Maple)

$$\frac{t_2}{t_1} = \frac{380.3813709}{239.6252478} \approx 1.587401054.$$

The compares well to

$$2^{2/3} \approx 1.587401052.$$

Of course, the temperature is not quite the center of the spherical turkey. The reader can work out the details for other locations. Perhaps other interesting models would be a spherical shell of turkey with bread stuffing. Or, one might consider an ellipsoidal geometry.

6.8 Schrödinger Equation in Spherical Coordinates - Optional

Another important eigenvalue problem in physics is the Schrödinger equation. The time-dependent Schrödinger equation is given by

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi. \qquad (6.161)$$

Here $\Psi(\mathbf{r}, t)$ is the wave function, which determines the quantum state of a particle of mass m subject to a (time independent) potential, $V(\mathbf{r})$. From

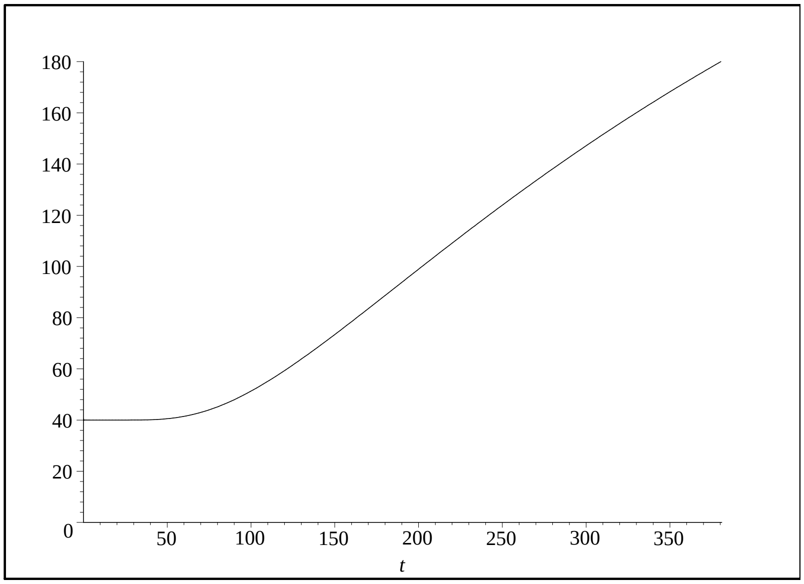


Figure 6.38: The temperature at the center of a turkey with radius $a = 0.5(2^{1/3})$ ft and $k \approx 0.000089$.

Planck's constant, h, one defines $\hbar = \frac{h}{2\pi}$. The probability of finding the particle in an infinitesimal volume, dV, is given by $|\Psi(\mathbf{r},t)|^2\,dV$, assuming the wave function is normalized,

$$\int_{\text{all space}} |\Psi(\mathbf{r},t)|^2\,dV = 1.$$

One can separate out the time dependence by assuming a special form, $\Psi(\mathbf{r},t) = \psi(\mathbf{r})e^{-iEt/\hbar}$, where E is the energy of the particular stationary state solution, or product solution. Inserting this form into the time-dependent equation, one finds that $\psi(\mathbf{r})$ satisfies the time-independent Schrödinger equation,

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi. \tag{6.162}$$

Assuming that the potential depends only on the distance from the origin, $V = V(\,r)$, we can further separate out the radial part of this solution using spherical coordinates. Recall that the Laplacian in spherical coordinates is given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{6.163}$$

Then, the time-independent Schrödinger equation can be written as

$$\begin{aligned} &- \frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] \\ &= [E - V(r)]\psi. \end{aligned} \tag{6.164}$$

Let's continue with the separation of variables. Assuming that the wave function takes the form $\psi(r,\theta,\phi) = R(r)Y(\theta,\phi)$, we obtain

$$\begin{aligned} &- \frac{\hbar^2}{2m}\left[\frac{Y}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{R}{r^2\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] \\ &= RY[E - V(r)]\psi. \end{aligned} \tag{6.165}$$

Dividing by $\psi = RY$, multiplying by $-\frac{2mr^2}{\hbar^2}$, and rearranging, we have

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right] = -\frac{1}{Y}LY,$$

where

$$L = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

We have a function of r equal to a function of the angular variables. So, we set each side equal to a constant. We will judiciously write the separation constant as $\ell(\ell+1)$. The resulting equations are then

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]R = \ell(\ell+1)R, \qquad (6.166)$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -\ell(\ell+1)Y. \qquad (6.167)$$

The second of these equations should look familiar from the last section. This is the equation for spherical harmonics,

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{2}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\phi}. \qquad (6.168)$$

So, any further analysis of the problem depends upon the choice of potential, $V(r)$, and the solution of the radial equation. For this, we turn to the determination of the wave function for an electron in orbit about a proton.

Example 6.16. The Hydrogen Atom - $\ell = 0$ States

Historically, the first test of the Schrödinger equation was the determination of the energy levels in a hydrogen atom. This is modeled by an electron orbiting a proton. The potential energy is provided by the Coulomb potential,

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r}.$$

Solution of the hydrogen problem.

Thus, the radial equation becomes

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2mr^2}{\hbar^2}\left[\frac{e^2}{4\pi\epsilon_0 r} + E\right]R = \ell(\ell+1)R. \qquad (6.169)$$

Before looking for solutions, we need to simplify the equation by absorbing some of the constants. One way to do this is to make an appropriate change of variables. Let $r = a\rho$. Then, by the Chain Rule we have

$$\frac{d}{dr} = \frac{d\rho}{dr}\frac{d}{d\rho} = \frac{1}{a}\frac{d}{d\rho}.$$

Under this transformation, the radial equation becomes

$$\frac{d}{d\rho}\left(\rho^2\frac{du}{d\rho}\right) + \frac{2ma^2\rho^2}{\hbar^2}\left[\frac{e^2}{4\pi\epsilon_0 a\rho} + E\right]u = \ell(\ell+1)u, \qquad (6.170)$$

where $u(\rho) = R(r)$. Expanding the second term,

$$\frac{2ma^2\rho^2}{\hbar^2}\left[\frac{e^2}{4\pi\epsilon_0 a\rho} + E\right] u = \left[\frac{mae^2}{2\pi\epsilon_0\hbar^2}\rho + \frac{2mEa^2}{\hbar^2}\rho^2\right] u,$$

we see that we can define

$$a = \frac{2\pi\epsilon_0\hbar^2}{me^2}, \tag{6.171}$$

$$\begin{aligned} \epsilon &= -\frac{2mEa^2}{\hbar^2} \\ &= -\frac{2(2\pi\epsilon_0)^2\hbar^2}{me^4}E. \end{aligned} \tag{6.172}$$

Using these constants, the radial equation becomes

$$\frac{d}{d\rho}\left(\rho^2\frac{du}{d\rho}\right) + \rho u - \ell(\ell+1)u = \epsilon\rho^2 u. \tag{6.173}$$

Expanding the derivative and dividing by ρ^2,

$$u'' + \frac{2}{\rho}u' + \frac{1}{\rho}u - \frac{\ell(\ell+1)}{\rho^2}u = \epsilon u. \tag{6.174}$$

The first two terms in this differential equation came from the Laplacian. The third term came from the Coulomb potential. The fourth term can be thought to contribute to the potential and is attributed to angular momentum. Thus, ℓ is called the angular momentum quantum number. This is an eigenvalue problem for the radial eigenfunctions $u(\rho)$ and energy eigenvalues ϵ.

The solutions of this equation are determined in a quantum mechanics course. In order to get a feeling for the solutions, we will consider the zero angular momentum case, $\ell = 0$:

$$u'' + \frac{2}{\rho}u' + \frac{1}{\rho}u = \epsilon u. \tag{6.175}$$

Even this equation is one we have not encountered in this book. Let's see if we can find some of the solutions.

First, we consider the behavior of the solutions for large ρ. For large ρ the second and third terms on the left hand side of the equation are negligible. So, we have the approximate equation

$$u'' - \epsilon u = 0. \tag{6.176}$$

Therefore, the solutions behave like $u(\rho) = e^{\pm\sqrt{\epsilon}\rho}$ for large ρ. For bounded solutions, we choose the decaying solution.

This suggests that solutions take the form $u(\rho) = v(\rho)e^{-\sqrt{\epsilon}\rho}$ for some unknown function, $v(\rho)$. Inserting this guess into Equation (6.175), gives an equation for $v(\rho)$:

$$\rho v'' + 2\left(1 - \sqrt{\epsilon}\rho\right) v' + (1 - 2\sqrt{\epsilon})v = 0. \tag{6.177}$$

Next we seek a series solution to this equation. Let

$$v(\rho) = \sum_{k=0}^{\infty} c_k \rho^k.$$

Inserting this series into Equation (6.177), we have

$$\sum_{k=1}^{\infty} [k(k-1) + 2k] c_k \rho^{k-1} + \sum_{k=1}^{\infty} [1 - 2\sqrt{\epsilon}(k+1)] c_k \rho^k = 0.$$

We can re-index the dummy variable in each sum. Let $k = m$ in the first sum and $k = m - 1$ in the second sum. We then find that

$$\sum_{k=1}^{\infty} \left[m(m+1) c_m + (1 - 2m\sqrt{\epsilon}) c_{m-1} \right] \rho^{m-1} = 0.$$

Since this has to hold for all $m \geq 1$,

$$c_m = \frac{2m\sqrt{\epsilon} - 1}{m(m+1)} c_{m-1}.$$

Further analysis indicates that the resulting series leads to unbounded solutions unless the series terminates. This is only possible if the numerator, $2m\sqrt{\epsilon} - 1$, vanishes for $m = n$, $n = 1, 2 \ldots$. Thus,

$$\epsilon = \frac{1}{4n^2}.$$

Since ϵ is related to the energy eigenvalue, E, we have

$$E_n = -\frac{me^4}{2(4\pi\epsilon_0)^2 \hbar^2 n^2}.$$

Inserting the values for the constants, this gives

$$E_n = -\frac{13.6 \text{ eV}}{n^2}.$$

Energy levels for the hydrogen atom.

This is the well known set of energy levels for the hydrogen atom.

The corresponding eigenfunctions are polynomials, since the infinite series was forced to terminate. We could obtain these polynomials by iterating the recursion equation for the c_m's. However, we will instead rewrite the radial equation (6.177).

Let $x = 2\sqrt{\epsilon}\rho$ and define $y(x) = v(\rho)$. Then

$$\frac{d}{d\rho} = 2\sqrt{\epsilon} \frac{d}{dx}.$$

This gives

$$2\sqrt{\epsilon} x y'' + (2 - x) 2\sqrt{\epsilon} y' + (1 - 2\sqrt{\epsilon}) y = 0.$$

Rearranging, we have

$$xy'' + (2 - x) y' + \frac{1}{2\sqrt{\epsilon}} (1 - 2\sqrt{\epsilon}) y = 0.$$

Noting that $2\sqrt{\epsilon} = \frac{1}{n}$, this equation becomes

$$xy'' + (2 - x)y' + (n - 1)y = 0. \tag{6.178}$$

The resulting equation is well known. It takes the form

$$xy'' + (\alpha + 1 - x)y' + ny = 0. \tag{6.179}$$

Solutions of this equation are the associated Laguerre polynomials. The solutions are denoted by $L_n^\alpha(x)$. They can be defined in terms of the Laguerre polynomials,

$$L_n(x) = e^x \left(\frac{d}{dx}\right)^n (e^{-x}x^n).$$

The associated Laguerre polynomials are defined as

$$L_{n-m}^m(x) = (-1)^m \left(\frac{d}{dx}\right)^m L_n(x).$$

Note: The Laguerre polynomials were first encountered in Problem 2 in Chapter 5 as an example of a classical orthogonal polynomial defined on $[0, \infty)$ with weight $w(x) = e^{-x}$. Some of these polynomials are listed in Table 6.9 and several Laguerre polynomials are shown in Figure 6.39.

The associated Laguerre polynomials are named after the French mathematician Edmond Laguerre (1834-1886).

Comparing Equation (6.178) with Equation (6.179), we find that $y(x) = L_{n-1}^1(x)$.

Table 6.9: Associated Laguerre Functions, $L_n^m(x)$. Note that $L_n^0(x) = L_n(x)$.

	$L_n^m(x)$
$L_0^0(x)$	1
$L_1^0(x)$	$1 - x$
$L_2^0(x)$	$\frac{1}{2}(x^2 - 4x + 2)$
$L_3^0(x)$	$\frac{1}{6}(-x^3 + 9x^2 - 18x + 6)$
$L_0^1(x)$	1
$L_1^1(x)$	$2 - x$
$L_2^1(x)$	$\frac{1}{2}(x^2 - 6x + 6)$
$L_3^1(x)$	$\frac{1}{6}(-x^3 + 3x^2 - 36x + 24)$
$L_0^2(x)$	1
$L_1^2(x)$	$3 - x$
$L_2^2(x)$	$\frac{1}{2}(x^2 - 8x + 12)$
$L_3^2(x)$	$\frac{1}{12}(-2x^3 + 30x^2 - 120x + 120)$

In summary, we have made the following transformations:

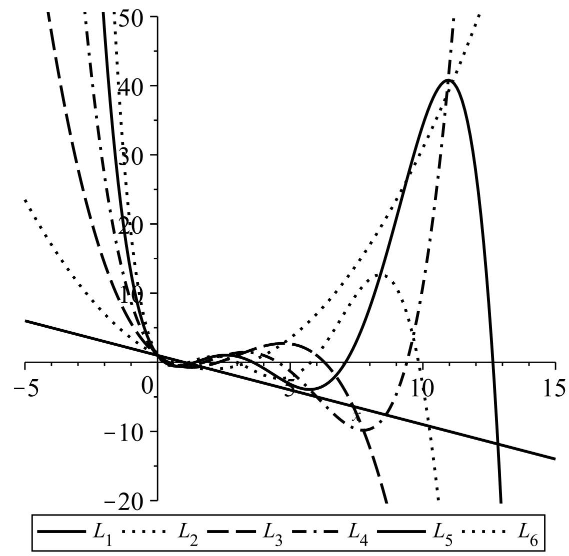


Figure 6.39: Plots of the first few Laguerre polynomials.

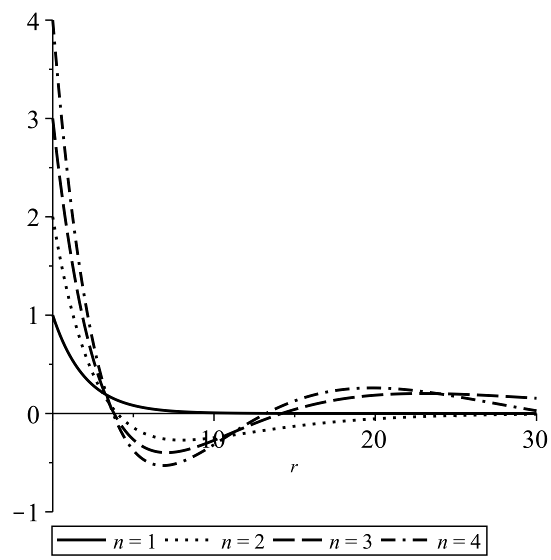


Figure 6.40: Plots of $R(r)$ for $a = 1$ and $n = 1, 2, 3, 4$ for the $\ell = 0$ states.

1. $R(r) = u(\rho)$, $r = a\rho$.
2. $u(\rho) = v(\rho)e^{-\sqrt{\epsilon}\rho}$.
3. $v(\rho) = y(x) = L^1_{n-1}(x)$, $x = 2\sqrt{\epsilon}\rho$.

Therefore,

$$R(r) = e^{-\sqrt{\epsilon}r/a} L^1_{n-1}(2\sqrt{\epsilon}r/a).$$

In quantum mechanics $a = \frac{a_0}{2}$. where $a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}$ is the Bohr radius and $a_0 = 5.2917 \times 10^{-11}$m.

However, we also found that $2\sqrt{\epsilon} = 1/n$. So,

$$R(r) = e^{-r/2na} L^1_{n-1}(r/na).$$

In Figure 6.40 we show a few of these solutions.

Example 6.17. Find the $\ell \geq 0$ solutions of the radial equation.

For the general case, for all $\ell \geq 0$, we need to solve the differential equation

$$u'' + \frac{2}{\rho}u' + \frac{1}{\rho}u - \frac{\ell(\ell+1)}{\rho^2}u = \epsilon u. \tag{6.180}$$

Instead of letting $u(\rho) = v(\rho)e^{-\sqrt{\epsilon}\rho}$, we let

$$u(\rho) = v(\rho)\rho^\ell e^{-\sqrt{\epsilon}\rho}.$$

This led to the differential equation

$$\rho v'' + 2(\ell + 1 - \sqrt{\epsilon}\rho)v' + (1 - 2(\ell+1)\sqrt{\epsilon})v = 0. \tag{6.181}$$

as before, we let $x = 2\sqrt{\epsilon}\rho$ to obtain

$$xy'' + 2\left[\ell + 1 - \frac{x}{2}\right]v' + \left[\frac{1}{2\sqrt{\epsilon}} - \ell(\ell+1)\right]v = 0.$$

Noting that $2\sqrt{\epsilon} = 1/n$, we have

$$xy'' + 2\left[2(\ell+1) - x\right]v' + (n - \ell(\ell+1))v = 0.$$

We see that this is once again in the form of the associate Laguerre equation and the solutions are

$$y(x) = L^{2\ell+1}_{n-\ell-1}(x).$$

So, the solution to the radial equation for the hydrogen atom is given by

$$\begin{aligned} R(r) &= \rho^\ell e^{-\sqrt{\epsilon}\rho} L^{2\ell+1}_{n-\ell-1}(2\sqrt{\epsilon}\rho) \\ &= \left(\frac{r}{2na}\right)^\ell e^{-r/2na} L^{2\ell+1}_{n-\ell-1}\left(\frac{r}{na}\right). \end{aligned} \tag{6.182}$$

Interpretations of these solutions will be left for your quantum mechanics course.

6.9 Appendix: Curvilinear Coordinates

In order to study solutions of the wave equation, the heat equation, or even Schrödinger's equation in different geometries, we need to see how differential operators, such as the Laplacian, appear in these geometries. The most common coordinate systems arising in physics are polar coordinates, cylindrical coordinates, and spherical coordinates. These reflect the common geometrical symmetries often encountered in physics.

In such systems it is easier to describe boundary conditions and to make use of these symmetries. For example, specifying that the electric potential is 10.0 V on a spherical surface of radius one, we would say $\phi(x,y,z) = 10$ for $x^2 + y^2 + z^2 = 1$. However, if we use spherical coordinates, (r, θ, ϕ), then we would say $\phi(r, \theta, \phi) = 10$ for $r = 1$, or $\phi(1, \theta, \phi) = 10$. This is a much simpler representation of the boundary condition.

However, this simplicity in boundary conditions leads to a more complicated looking partial differential equation in spherical coordinates. In this section we will consider general coordinate systems and how the differential operators are written in the new coordinate systems. This is a more general approach than that taken earlier in the chapter. For a more modern and elegant approach, one can use differential forms.

We begin by introducing the general coordinate transformations between Cartesian coordinates and the more general curvilinear coordinates. Let the Cartesian coordinates be designated by (x_1, x_2, x_3) and the new coordinates by (u_1, u_2, u_3). We will assume that these are related through the transformations

$$\begin{aligned} x_1 &= x_1(u_1, u_2, u_3), \\ x_2 &= x_2(u_1, u_2, u_3), \\ x_3 &= x_3(u_1, u_2, u_3). \end{aligned} \tag{6.183}$$

Thus, given the curvilinear coordinates (u_1, u_2, u_3) for a specific point in space, we can determine the Cartesian coordinates, (x_1, x_2, x_3), of that point. We will assume that we can invert this transformation: Given the Cartesian coordinates, one can determine the corresponding curvilinear coordinates.

In the Cartesian system we can assign an orthogonal basis, $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. As a particle traces out a path in space, one locates its position by the coordinates (x_1, x_2, x_3). Picking x_2 and x_3 constant, the particle lies on the curve $x_1 =$ value of the x_1 coordinate. This line lies in the direction of the basis vector $\mathbf{i}$. We can do the same with the other coordinates and essentially map out a grid in three dimensional space as sown in Figure 6.41. All of the x_i-curves intersect at each point orthogonally and the basis vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ lie along the grid lines and are mutually orthogonal. We would like to mimic this construction for general curvilinear coordinates. Requiring the orthogonality of the resulting basis vectors leads to orthogonal curvilinear coordinates.

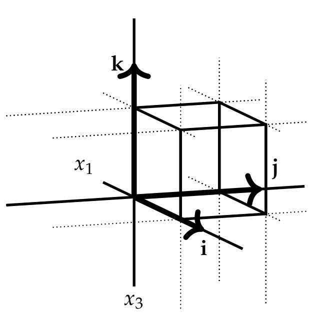


Figure 6.41: Plots of x_i-curves forming an orthogonal Cartesian grid.

As for the Cartesian case, we consider u_2 and u_3 constant. This leads to a curve parametrized by $u_1 : \mathbf{r} = x_1(u_1)\mathbf{i} + x_2(u_1)\mathbf{j} + x_3(u_1)\mathbf{k}$. We call this the u_1-curve. Similarly, when u_1 and u_3 are constant we obtain a u_2-curve and for u_1 and u_2 constant we obtain a u_3-curve. We will assume that these curves intersect such that each pair of curves intersect orthogonally as seen in Figure 6.42. Furthermore, we will assume that the unit tangent vectors to these curves form a right handed system similar to the $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ systems for Cartesian coordinates. We will denote these as $\{\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3\}$.

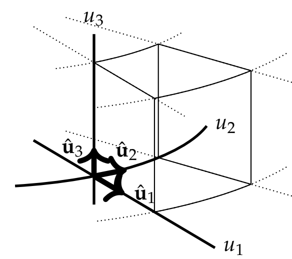


Figure 6.42: Plots of general u_i-curves forming an orthogonal grid.

We can determine these tangent vectors from the coordinate transformations. Consider the position vector as a function of the new coordinates,

$$\mathbf{r}(u_1, u_2, u_3) = x_1(u_1, u_2, u_3)\mathbf{i} + x_2(u_1, u_2, u_3)\mathbf{j} + x_3(u_1, u_2, u_3)\mathbf{k}.$$

Then, the infinitesimal change in position is given by

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3 = \sum_{i=1}^{3} \frac{\partial \mathbf{r}}{\partial u_i} du_i.$$

We note that the vectors $\frac{\partial \mathbf{r}}{\partial u_i}$ are tangent to the u_i-curves. Thus, we define the unit tangent vectors

$$\hat{\mathbf{u}}_i = \frac{\frac{\partial \mathbf{r}}{\partial u_i}}{\left|\frac{\partial \mathbf{r}}{\partial u_i}\right|}.$$

Solving for the original tangent vector, we have

$$\frac{\partial \mathbf{r}}{\partial u_i} = h_i \hat{\mathbf{u}}_i,$$

where

$$h_i \equiv \left|\frac{\partial \mathbf{r}}{\partial u_i}\right|.$$

The scale factors, $h_i \equiv \left|\frac{\partial \mathbf{r}}{\partial u_i}\right|$.

The h_i's are called the scale factors for the transformation. The infinitesimal change in position in the new basis is then given by

$$d\mathbf{r} = \sum_{i=1}^{3} h_i u_i \hat{\mathbf{u}}_i.$$

Example 6.18. Determine the scale factors for the polar coordinate transformation.

The transformation for polar coordinates is

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Here we note that $x_1 = x$, $x_2 = y$, $u_1 = r$, and $u_2 = \theta$. The u_1-curves are curves with $\theta =$ const. Thus, these curves are radial lines. Similarly, the u_2-curves have $r =$ const. These curves are concentric circles about the origin as shown in Figure 6.43.

The unit vectors are easily found. We will denote them by $\hat{\mathbf{u}}_r$ and $\hat{\mathbf{u}}_\theta$. We can determine these unit vectors by first computing $\frac{\partial \mathbf{r}}{\partial u_i}$. Let

$$\mathbf{r} = x(r, \theta)\mathbf{i} + y(r, \theta)\mathbf{j} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}.$$

Then,

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial r} &= \cos\theta \mathbf{i} + \sin\theta \mathbf{j} \\ \frac{\partial \mathbf{r}}{\partial \theta} &= -r\sin\theta \mathbf{i} + r\cos\theta \mathbf{j}. \end{aligned} \tag{6.184}$$

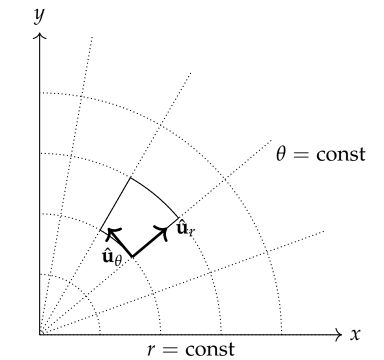


Figure 6.43: Plots an orthogonal polar grid.

The first vector already is a unit vector. So,

$$\hat{\mathbf{u}}_r = \cos\theta \mathbf{i} + \sin\theta \mathbf{j}.$$

The second vector has length r since $|-r\sin\theta \mathbf{i} + r\cos\theta \mathbf{j}| = r$. Dividing $\frac{\partial \mathbf{r}}{\partial \theta}$ by r, we have

$$\hat{\mathbf{u}}_\theta = -\sin\theta \mathbf{i} + \cos\theta \mathbf{j}.$$

We can see these vectors are orthogonal ($\hat{\mathbf{u}}_r \cdot \hat{\mathbf{u}}_\theta = 0$) and form a right hand system. That they form a right hand system can be seen by either drawing the vectors, or computing the cross product,

$$\begin{aligned} (\cos\theta \mathbf{i} + \sin\theta \mathbf{j}) \times (-\sin\theta \mathbf{i} + \cos\theta \mathbf{j}) &= \cos^2\theta \mathbf{i} \times \mathbf{j} - \sin^2\theta \mathbf{j} \times \mathbf{i} \\ &= \mathbf{k}. \end{aligned} \tag{6.185}$$

Since

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial r} &= \hat{\mathbf{u}}_r, \\ \frac{\partial \mathbf{r}}{\partial \theta} &= r\hat{\mathbf{u}}_\theta, \end{aligned}$$

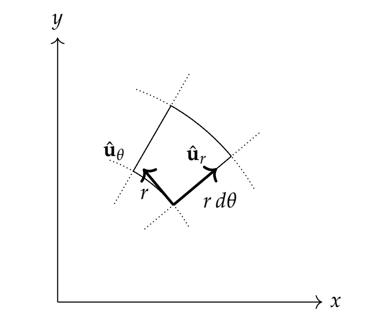


Figure 6.44: Infinitesimal area in polar coordinates.

The scale factors are $h_r = 1$ and $h_\theta = r$.

Once we know the scale factors, we have that

$$d\mathbf{r} = \sum_{i=1}^{3} h_i du_i \hat{\mathbf{u}}_i.$$

The infinitesimal arclength is then given by the Euclidean line element

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = \sum_{i=1}^{3} h_i^2 du_i^2$$

when the system is orthogonal. The h_i^2 are referred to as the metric coefficients.

Example 6.19. Verify that $d\mathbf{r} = dr\hat{\mathbf{u}}_r + r\,d\theta\hat{\mathbf{u}}_\theta$ directly from $\mathbf{r} = r\cos\theta \mathbf{i} + r\sin\theta \mathbf{j}$ and obtain the Euclidean line element for polar coordinates.

We begin by computing

$$\begin{aligned} d\mathbf{r} &= d(r\cos\theta \mathbf{i} + r\sin\theta \mathbf{j}) \\ &= (\cos\theta \mathbf{i} + \sin\theta \mathbf{j})\,dr + r(-\sin\theta \mathbf{i} + \cos\theta \mathbf{j})\,d\theta \\ &= dr\hat{\mathbf{u}}_r + r\,d\theta\hat{\mathbf{u}}_\theta. \end{aligned} \tag{6.186}$$

This agrees with the form $d\mathbf{r} = \sum_{i=1}^{3} h_i du_i \hat{\mathbf{u}}_i$ when the scale factors for polar coordinates are inserted.

The line element is found as

$$\begin{aligned} ds^2 &= d\mathbf{r}\cdot d\mathbf{r} \\ &= (dr\hat{\mathbf{u}}_r + r\,d\theta\hat{\mathbf{u}}_\theta)\cdot(dr\hat{\mathbf{u}}_r + r\,d\theta\hat{\mathbf{u}}_\theta) \\ &= dr^2 + r^2\,d\theta^2. \end{aligned} \tag{6.187}$$

This is the Euclidean line element in polar coordinates.

Also, along the u_i-curves,

$$d\mathbf{r} = h_i du_i \hat{\mathbf{u}}_i, \quad \text{(no summation)}.$$

This can be seen in Figure 6.45 by focusing on the u_1 curve. Along this curve, u_2 and u_3 are constant. So, $du_2 = 0$ and $du_3 = 0$. This leaves $d\mathbf{r} = h_1 du_1 \hat{\mathbf{u}}_1$ along the u_1-curve. Similar expressions hold along the other two curves.

We can use this result to investigate infinitesimal volume elements for general coordinate systems as shown in Figure 6.45. At a given point (u_1, u_2, u_3) we can construct an infinitesimal parallelepiped of sides $h_i du_i$, $i = 1, 2, 3$. This infinitesimal parallelepiped has a volume of size

$$dV = \left|\frac{\partial \mathbf{r}}{\partial u_1}\cdot\frac{\partial \mathbf{r}}{\partial u_2}\times\frac{\partial \mathbf{r}}{\partial u_3}\right| du_1 du_2 du_3.$$

The triple scalar product can be computed using determinants and the resulting determinant is call the Jacobian, and is given by

$$\begin{aligned} J &= \left|\frac{\partial(x_1,x_2,x_3)}{\partial(u_1,u_2,u_3)}\right| \\ &= \left|\frac{\partial \mathbf{r}}{\partial u_1}\cdot\frac{\partial \mathbf{r}}{\partial u_2}\times\frac{\partial \mathbf{r}}{\partial u_3}\right| \\ &= \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \frac{\partial x_3}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_3}{\partial u_2} \\ \frac{\partial x_1}{\partial u_3} & \frac{\partial x_2}{\partial u_3} & \frac{\partial x_3}{\partial u_3} \end{vmatrix}. \end{aligned} \tag{6.188}$$

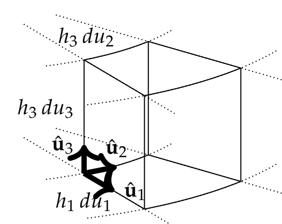


Figure 6.45: Infinitesimal volume element with sides of length $h_i\,du_i$.

Therefore, the volume element can be written as

$$dV = J\,du_1 du_2 du_3 = \left|\frac{\partial(x_1,x_2,x_3)}{\partial(u_1,u_2,u_3)}\right| du_1 du_2 du_3.$$

Example 6.20. Determine the volume element for cylindrical coordinates (r, θ, z), given by

$$x = r\cos\theta, \tag{6.189}$$
$$y = r\sin\theta, \tag{6.190}$$
$$z = z. \tag{6.191}$$

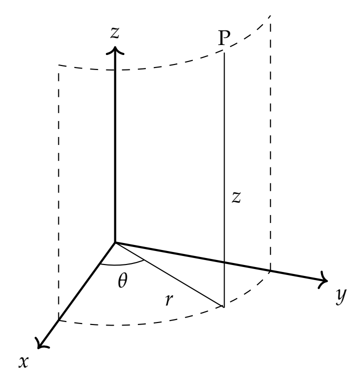


Figure 6.46: Cylindrical coordinate system.

Here, we have $(u_1, u_2, u_3) = (r, \theta, z)$ as displayed in Figure 6.46. Then, the Jacobian is given by

$$J = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right|$$

$$
\begin{aligned}
&= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} \\
&= \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
&= r \qquad (6.192)
\end{aligned}
$$

Thus, the volume element is given as

$$dV = rdrd\theta dz.$$

This result should be familiar from multivariate calculus.

Another approach is to consider the geometry of the infinitesimal volume element. The directed edge lengths are given by $d\mathbf{s}_i = h_i du_i \hat{\mathbf{u}}_i$ as seen in Figure 6.42. The infinitesimal area element of for the face in direction $\hat{\mathbf{u}}_k$ is found from a simple cross product,

$$d\mathbf{A}_k = d\mathbf{s}_i \times d\mathbf{s}_j = h_i h_j du_i du_j \hat{\mathbf{u}}_i \times \hat{\mathbf{u}}_j.$$

Since these are unit vectors, the areas of the faces of the infinitesimal volumes are $dA_k = h_i h_j du_i du_j$.

The infinitesimal volume is then obtained as

$$dV = |d\mathbf{s}_k \cdot d\mathbf{A}_k| = h_i h_j h_k du_i du_j du_k |\hat{\mathbf{u}}_i \cdot (\hat{\mathbf{u}}_k \times \hat{\mathbf{u}}_j)|.$$

Thus, $dV = h_1 h_2 h_3 du_1 du_1 du_3$. Of course, this should not be a surprise since

$$J = \left| \frac{\partial \mathbf{r}}{\partial u_1} \cdot \frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} \right| = |h_1 \hat{\mathbf{u}}_1 \cdot h_2 \hat{\mathbf{u}}_2 \times h_3 \hat{\mathbf{u}}_3| = h_1 h_2 h_3.$$

Example 6.21. For polar coordinates, determine the infinitesimal area element.

In an earlier example, we found the scale factors for polar coordinates as $h_r = 1$ and $h_\theta = r$. Thus, $dA = h_r h_\theta \, drd\theta = r \, drd\theta$. Also, the last example for cylindrical coordinates will yield similar results if we already know the scales factors without having to compute the Jacobian directly. Furthermore, the area element perpendicular to the z-coordinate gives the polar coordinate system result.

Next we will derive the forms of the gradient, divergence, and curl in curvilinear coordinates using several vector identities. The results are given here for quick reference.

$$\nabla\phi = \sum_{i=1}^{3} \frac{\hat{\mathbf{u}}_i}{h_i}\frac{\partial\phi}{\partial u_i} = \frac{\hat{\mathbf{u}}_1}{h_1}\frac{\partial\phi}{\partial u_1} + \frac{\hat{\mathbf{u}}_2}{h_2}\frac{\partial\phi}{\partial u_2} + \frac{\hat{\mathbf{u}}_3}{h_3}\frac{\partial\phi}{\partial u_3}. \tag{6.193}$$

$$\nabla\cdot\mathbf{F} = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u_1}(h_2h_3F_1) + \frac{\partial}{\partial u_2}(h_1h_3F_2) + \frac{\partial}{\partial u_3}(h_1h_2F_3)\right). \tag{6.194}$$

$$\nabla\times\mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\mathbf{u}}_1 & h_2\hat{\mathbf{u}}_2 & h_3\hat{\mathbf{u}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ F_1h_1 & F_2h_2 & F_3h_3 \end{vmatrix}. \tag{6.195}$$

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right)\right) \tag{6.196}$$

Gradient, divergence and curl in orthogonal curvilinear coordinates.

Derivation of the gradient form.

We begin the derivations of these formulae by looking at the gradient, $\nabla\phi$, of the scalar function $\phi(u_1, u_2, u_3)$. We recall that the gradient operator appears in the differential change of a scalar function,

$$d\phi = \nabla\phi\cdot d\mathbf{r} = \sum_{i=1}^{3}\frac{\partial\phi}{\partial u_i}du_i.$$

Since

$$d\mathbf{r} = \sum_{i=1}^{3} h_i du_i \hat{\mathbf{u}}_i, \tag{6.197}$$

we also have that

$$d\phi = \nabla\phi\cdot d\mathbf{r} = \sum_{i=1}^{3}(\nabla\phi)_i h_i du_i.$$

Comparing these two expressions for $d\phi$, we determine that the components of the del operator can be written as

$$(\nabla\phi)_i = \frac{1}{h_i}\frac{\partial\phi}{\partial u_i}$$

and thus the gradient is given by

$$\nabla\phi = \frac{\hat{\mathbf{u}}_1}{h_1}\frac{\partial\phi}{\partial u_1} + \frac{\hat{\mathbf{u}}_2}{h_2}\frac{\partial\phi}{\partial u_2} + \frac{\hat{\mathbf{u}}_3}{h_3}\frac{\partial\phi}{\partial u_3}. \tag{6.198}$$

Derivation of the divergence form.

Next we compute the divergence,

$$\nabla\cdot\mathbf{F} = \sum_{i=1}^{3}\nabla\cdot(F_i\hat{\mathbf{u}}_i).$$

We can do this by computing the individual terms in the sum. We will compute $\nabla\cdot(F_1\hat{\mathbf{u}}_1)$.

Using Equation (6.198), we have that

$$\nabla u_i = \frac{\hat{\mathbf{u}}_i}{h_i}.$$

Then

$$\nabla u_2 \times \nabla u_3 = \frac{\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3}{h_2 h_3} = \frac{\hat{\mathbf{u}}_1}{h_2 h_3}.$$

Solving for $\hat{\mathbf{u}}_1$, gives

$$\hat{\mathbf{u}}_1 = h_2 h_3 \nabla u_2 \times \nabla u_3.$$

Inserting this result into $\nabla \cdot (F_1 \hat{\mathbf{u}}_1)$ and using the vector identity,

$$\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f,$$

we have

$$\begin{aligned} \nabla \cdot (F_1 \hat{\mathbf{u}}_1) &= \nabla \cdot (F_1 h_2 h_3 \nabla u_2 \times \nabla u_3) \\ &= \nabla (F_1 h_2 h_3) \cdot \nabla u_2 \times \nabla u_3 + F_1 h_2 h_2 \nabla \cdot (\nabla u_2 \times \nabla u_3). \end{aligned} \tag{6.199}$$

The second term of this result vanishes by the vector identity

$$\nabla \cdot (\nabla f \times \nabla g) = 0.$$

Since $\nabla u_2 \times \nabla u_3 = \dfrac{\hat{\mathbf{u}}_1}{h_2 h_3}$, the first term can be evaluated as

$$\nabla \cdot (F_1 \hat{\mathbf{u}}_1) = \nabla (F_1 h_2 h_3) \cdot \frac{\hat{\mathbf{u}}_1}{h_2 h_3} = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} (F_1 h_2 h_3).$$

Similar computations can be carried out for the remaining components, leading to the sought expression for the divergence in curvilinear coordinates:

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} (h_2 h_3 F_1) + \frac{\partial}{\partial u_2} (h_1 h_3 F_2) + \frac{\partial}{\partial u_3} (h_1 h_2 F_3) \right). \tag{6.200}$$

Example 6.22. Write the divergence operator in cylindrical coordinates.

In this case we have

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \frac{1}{h_r h_\theta h_z} \left(\frac{\partial}{\partial r} (h_\theta h_z F_r) + \frac{\partial}{\partial \theta} (h_r h_z F_\theta) + \frac{\partial}{\partial \theta} (h_r h_\theta F_z) \right) \\ &= \frac{1}{r} \left(\frac{\partial}{\partial r} (r F_r) + \frac{\partial}{\partial \theta} (F_\theta) + \frac{\partial}{\partial \theta} (r F_z) \right) \\ &= \frac{1}{r} \frac{\partial}{\partial r} (r F_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (F_\theta) + \frac{\partial}{\partial \theta} (F_z). \end{aligned} \tag{6.201}$$

Derivation of the curl form.

We now turn to the curl operator. In this case, we need to evaluate

$$\nabla \times \mathbf{F} = \sum_{i=1}^{3} \nabla \times (F_i \hat{\mathbf{u}}_i).$$

Again we focus on one term, $\nabla \times (F_1 \hat{\mathbf{u}}_1)$. Using the vector identity

$$\nabla \times (f\mathbf{A}) = f\nabla \times \mathbf{A} - \mathbf{A} \times \nabla f,$$

we have

$$\begin{aligned} \nabla \times (F_1 \hat{\mathbf{u}}_1) &= \nabla \times (F_1 h_1 \nabla u_1) \\ &= F_1 h_1 \nabla \times \nabla u_1 - \nabla (F_1 h_1) \times \nabla u_1. \end{aligned} \tag{6.202}$$

The curl of the gradient vanishes, leaving

$$\nabla \times (F_1 \hat{\mathbf{u}}_1) = \nabla (F_1 h_1) \times \nabla u_1.$$

Since $\nabla u_1 = \dfrac{\hat{\mathbf{u}}_1}{h_1}$, we have

$$\begin{aligned} \nabla \times (F_1 \hat{\mathbf{u}}_1) &= \nabla (F_1 h_1) \times \frac{\hat{\mathbf{u}}_1}{h_1} \\ &= \left(\sum_{i=1}^{3} \frac{\hat{\mathbf{u}}_i}{h_i} \frac{\partial (F_1 h_1)}{\partial u_i} \right) \times \frac{\hat{\mathbf{u}}_1}{h_1} \\ &= \frac{\hat{\mathbf{u}}_2}{h_3 h_1} \frac{\partial (F_1 h_1)}{\partial u_3} - \frac{\hat{\mathbf{u}}_3}{h_1 h_2} \frac{\partial (F_1 h_1)}{\partial u_2}. \end{aligned} \tag{6.203}$$

The other terms can be handled in a similar manner. The overall result is that

$$\begin{aligned} \nabla \times \mathbf{F} &= \frac{\hat{\mathbf{u}}_1}{h_2 h_3} \left(\frac{\partial (h_3 F_3)}{\partial u_2} - \frac{\partial (h_2 F_2)}{\partial u_3} \right) + \frac{\hat{\mathbf{u}}_2}{h_1 h_3} \left(\frac{\partial (h_1 F_1)}{\partial u_3} - \frac{\partial (h_3 F_3)}{\partial u_1} \right) \\ &\quad + \frac{\hat{\mathbf{u}}_3}{h_1 h_2} \left(\frac{\partial (h_2 F_2)}{\partial u_1} - \frac{\partial (h_1 F_1)}{\partial u_2} \right) \end{aligned} \tag{6.204}$$

This can be written more compactly as

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{u}}_1 & h_2 \hat{\mathbf{u}}_2 & h_3 \hat{\mathbf{u}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ F_1 h_1 & F_2 h_2 & F_3 h_3 \end{vmatrix} \tag{6.205}$$

Example 6.23. Write the curl operator in cylindrical coordinates.

$$\begin{aligned} \nabla \times \mathbf{F} &= \frac{1}{r} \begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix} \\ &= \left(\frac{1}{r} \frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \right) \hat{\mathbf{e}}_r + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \right) \hat{\mathbf{e}}_\theta \\ &\quad + \frac{1}{r} \left(\frac{\partial (rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} \right) \hat{\mathbf{e}}_z. \end{aligned} \tag{6.206}$$

Finally, we turn to the Laplacian. In this chapter we have solved higher dimensional problems in various geometric settings such as the wave equation, the heat equation, and Laplace's equation. These all involved knowing how to write the Laplacian in different coordinate systems. Since $\nabla^2\phi = \nabla \cdot \nabla\phi$, we need only combine the results from Equations (6.198) and (6.200) for the gradient and the divergence in curvilinear coordinates. This is straight forward and gives

$$\begin{aligned}\nabla^2\phi &= \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial u_2}\right)\right.\\ &\left.+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right)\right). \end{aligned} \tag{6.207}$$

The Laplacians in cylindrical and spherical coordinates are shown below.

Cylindrical Coordinates:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}. \tag{6.208}$$

Spherical Coordinates:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial\, r}\left(r^2\frac{\partial f}{\partial\, r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}. \tag{6.209}$$

Problems

1. A rectangular plate $0 \le x \le L$ $0 \le y \le H$ with heat diffusivity constant k is insulated on the edges $y = 0, H$ and is kept at constant zero temperature on the other two edges. Assuming an initial temperature of $u(x, y, 0) = f(x, y)$, use separation of variables t find the general solution.

2. Solve the following problem.

$$\begin{gathered} u_{xx} + u_{yy} + u_{zz} = 0, \quad 0 < x < 2\pi, \quad 0 < y < \pi, \quad 0 < z < 1, \\ u(x, y, 0) = \sin x \sin y, \quad u(x, y, z) = 0 \text{ on other faces.} \end{gathered}$$

3. Consider Laplace's equation on the unit square, $u_{xx} + u_{yy} = 0, 0 \le x, y \le 1$. Let $u(0, y) = 0, u(1, y) = 0$ for $0 < y < 1$ and $u_y(x, 0) = 0$ for $0 < y < 1$. Carry out the needed separation of variables and write down the product solutions satisfying these boundary conditions.

4. Consider a cylinder of height H and radius a.

a. Write down Laplace's Equation for this cylinder in cylindrical coordinates.

b. Carry out the separation of variables and obtain the three ordinary differential equations that result from this problem.

c. What kind of boundary conditions could be satisfied in this problem in the independent variables?

5. Consider a square drum of side s and a circular drum of radius a.

a. Rank the modes corresponding to the first 6 frequencies for each.

b. Write each frequency (in Hz) in terms of the fundamental (i.e., the lowest frequency.)

c. What would the lengths of the sides of the square drum have to be to have the same fundamental frequency? (Assume that $c = 1.0$ for each one.)

6. We presented the full solution of the vibrating rectangular membrane in Equation 6.39. Finish the solution to the vibrating circular membrane by writing out a similar full solution.

7. A copper cube 10.0 cm on a side is heated to 100° C. The block is placed on a surface that is kept at 0° C. The sides of the block are insulated, so the normal derivatives on the sides are zero. Heat flows from the top of the block to the air governed by the gradient $u_z = -10^\circ\text{C/m}$. Determine the temperature of the block at its center after 1.0 minutes. Consider the following hints:

a. This is a heat conduction problem with nonhomogeneous boundary conditions. Assume $u(x,y,z,t) = v(x,y,z,t) + f(z)$, where $v(x,y,z,t)$ satisfies homogeneous boundary conditions. Find $v(x,y,z,t)$ and $f(z)$.

b. In order to get a numerical value for the temperature, you will need the thermal diffusivity, which is given by $k = \frac{K}{\rho c_p}$, where K is the thermal conductivity, ρ is the density, and c_p is the specific heat capacity. Look up any needed properties of copper.

8. Consider a spherical balloon of radius a. Small deformations on the surface can produce waves on the balloon's surface.

a. Write the wave equation in spherical polar coordinates. (Note: r is constant!)

b. Carry out a separation of variables and find the product solutions for this problem.

c. Describe the nodal curves for the first six modes.

d. For each mode determine the frequency of oscillation in Hz assuming $c = 1.0$ m/s.

9. Consider a circular cylinder of radius $R = 4.00$ cm and height $H = 20.0$ cm which obeys the steady state heat equation

$$u_{rr} + \frac{1}{r}u_r + u_{zz}.$$

Find the temperature distribution, $u(r,z)$, given that $u(r,0) = 0^\circ\text{C}$, $u(r,20) = 20^\circ\text{C}$, and heat is lost through the sides due to Newton's Law of Cooling

$$[u_r + hu]_{r=4} = 0,$$

for $h = 1.0\ \text{cm}^{-1}$.

a. Show that the product solutions are $\sinh \lambda_n H) J_0(\lambda_n r)$, where $\lambda J_0'(\lambda R) + J_0(\lambda R) = 0$.

b. For mixed boundary conditions, if

$$f(r) = \sum_{n=1}^{\infty} a_n J_0(\lambda_n r) r\, dr$$

where $j_n J_0'(j_n) + J_0(j_n) = 0$ and $\lambda_n = j_n/R$, then the Fourier-Bessel coefficients are given by

$$a_n = \frac{2}{R^2 J_0^2(j_n)} \frac{j_n^2}{j_n^2 + R} \int_0^R f(r) J_0(\frac{j_n}{R} r) r\, dr.$$

c. The eigenvalues, λ_n, are found by solving a transcendental equation. Plotting the function

$$f(x) = x J_0'(x) + J_0(x) = -x J_1(x) + J_0(x)$$

will aid in finding the first several roots, j_n, satisfying $-j_n J_0'(j_n) + J_0(j_n) = 0$. The eigenvalues are given by $\lambda_n = j_n/R$. Numerically find several of these and obtain an approximate solution to the problem.

10. The spherical surface of a homogeneous ball of radius one in maintained at zero temperature. It has an initial temperature distribution $u(r, 0) = 100^o$ C. Assuming a heat diffusivity constant k, find the temperature throughout the sphere, $u(r, \theta, \phi, t)$.

11. Determine the steady state temperature of a spherical ball maintained at the temperature

$$u(x, y, z) = x^2 + 2y^2 + 3z^2, \quad r = 1.$$

[Hint - Rewrite the problem in spherical coordinates, $u(r, \theta, \phi)$, compare it to Equation (6.146) with $r = 1$, and use the properties of spherical harmonics. Table 6.5 may be useful.]

12. A hot dog initially at temperature 50°C is put into boiling water at 100°C. Assume the hot dog is 12.0 cm long, has a radius of 2.00 cm, and the heat constant is 2.0×10^{-5} cm^2/s.

a. Find the general solution for the temperature. [Hint: Solve the heat equation for $u(r, z, t) = T(r, z, t) - 100$, where $T(r, z, t)$ is the temperature of the hot dog.]

b. Indicate how one might proceed with the remaining information in order to determine when the hot dog is cooked; i.e., when the center temperature is 80°C.

7
First Order Partial Differential Equations

"The profound study of nature is the most fertile source of mathematical discoveries." - Joseph Fourier (1768-1830)

7.1 Introduction

We begin our study of partial differential equations with *first order partial differential equations.* Before doing so, we need to define a few terms.

Recall (see the appendix on differential equations) that an n-th order ordinary differential equation is an equation for an unknown function $y(x)$ that expresses a relationship between the unknown function and its first n derivatives. One could write this generally as

n-th order ordinary differential equation

$$F(y^{(n)}(x), y^{(n-1)}(x), \ldots, y'(x), y(x), x) = 0. \tag{7.1}$$

Here $y^{(n)}(x)$ represents the nth derivative of $y(x)$. Furthermore, and initial value problem consists of the differential equation plus the values of the first $n-1$ derivatives at a particular value of the independent variable, say x_0:

Initial value problem.

$$y^{(n-1)}(x_0) = y_{n-1}, \quad y^{(n-2)}(x_0) = y_{n-2}, \quad \ldots, \quad y(x_0) = y_0. \tag{7.2}$$

If conditions are instead provided at more than one value of the independent variable, then we have a boundary value problem. .

If the unknown function is a function of several variables, then the derivatives are partial derivatives and the resulting equation is a partial differential equation. Thus, if $u = u(x, y, \ldots)$, a general partial differential equation might take the form

$$F\left(x, y, \ldots, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \ldots, \frac{\partial^2 u}{\partial x^2}, \ldots\right) = 0. \tag{7.3}$$

Since the notation can get cumbersome, there are different ways to write the partial derivatives. First order derivatives could be written as

$$\frac{\partial u}{\partial x}, u_x, \partial_x u, D_x u.$$

Second order partial derivatives could be written in the forms

$$\frac{\partial^2 u}{\partial x^2}, u_{xx}, \partial_{xx} u, D_x^2 u.$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}, u_{xy}, \partial_{xy} u, D_y D_x u.$$

Note, we are assuming that $u(x, y, \ldots)$ has continuous partial derivatives. Then, according to Clairaut's Theorem (Alexis Claude Clairaut, 1713-1765) , mixed partial derivatives are the same.

Examples of some of the partial differential equation treated in this book are shown in Table 1.1. However, being that the highest order derivatives in these equation are of second order, these are second order partial differential equations. In this chapter we will focus on first order partial differential equations. Examples are given by

$$\begin{aligned} u_t + u_x &= 0. \\ u_t + u u_x &= 0. \\ u_t + u u_x &= u. \\ 3u_x - 2u_y + u &= x. \end{aligned}$$

For function of two variables, which the above are examples, a general first order partial differential equation for $u = u(x, y)$ is given as

$$F(x, y, u, u_x, u_y) = 0, \quad (x, y) \in D \subset R^2. \tag{7.4}$$

This equation is too general. So, restrictions can be placed on the form, leading to a classification of first order equations. A linear first order partial differential equation is of the form

Linear first order partial differential equation.

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y). \tag{7.5}$$

Note that all of the coefficients are independent of u and its derivatives and each term in linear in u, u_x, or u_y.

We can relax the conditions on the coefficients a bit. Namely, we could assume that the equation is linear only in u_x and u_y. This gives the quasilinear first order partial differential equation in the form

Quasilinear first order partial differential equation.

$$a(x, y, u)u_x + b(x, y, u)u_y = f(x, y, u). \tag{7.6}$$

Note that the u-term was absorbed by $f(x, y, u)$.

In between these two forms we have the semilinear first order partial differential equation in the form

Semilinear first order partial differential equation.

$$a(x, y)u_x + b(x, y)u_y = f(x, y, u). \tag{7.7}$$

Here the left side of the equation is linear in u, u_x and u_y. However, the right hand side can be nonlinear in u.

For the most part, we will introduce the Method of Characteristics for solving quasilinear equations. But, let us first consider the simpler case of linear first order constant coefficient partial differential equations.

7.2 Linear Constant Coefficient Equations

Let's consider the linear first order constant coefficient partial differential equation

$$au_x + bu_y + cu = f(x,y), \tag{7.8}$$

for a, b, and c constants with $a^2 + b^2 > 0$. We will consider how such equations might be solved. We do this by considering two cases, $b = 0$ and $b \neq 0$.

For the first case, $b = 0$, we have the equation

$$au_x + cu = f.$$

We can view this as a first order linear (ordinary) differential equation with y a parameter. Recall that the solution of such equations can be obtained using an integrating factor. [See the discussion after Equation (A.7).] First rewrite the equation as

$$u_x + \frac{c}{a}u = \frac{f}{a}.$$

Introducing the integrating factor

$$\mu(x) = \exp\left(\int^x \frac{c}{a}\,d\xi\right) = e^{\frac{c}{a}x},$$

the differential equation can be written as

$$(\mu u)_x = \frac{f}{a}\mu.$$

Integrating this equation and solving for $u(x,y)$, we have

$$\begin{aligned}
\mu(x)u(x,y) &= \frac{1}{a}\int f(\xi,y)\mu(\xi)\,d\xi + g(y) \\
e^{\frac{c}{a}x}u(x,y) &= \frac{1}{a}\int f(\xi,y)e^{\frac{c}{a}\xi}\,d\xi + g(y) \\
u(x,y) &= \frac{1}{a}\int f(\xi,y)e^{\frac{c}{a}(\xi - x)}\,d\xi + g(y)e^{-\frac{c}{a}x}. \qquad (7.9)
\end{aligned}$$

Here $g(y)$ is an arbitrary function of y.

For the second case, $b \neq 0$, we have to solve the equation

$$au_x + bu_y + cu = f.$$

It would help if we could find a transformation which would eliminate one of the derivative terms reducing this problem to the previous case. That is what we will do.

We first note that

$$\begin{aligned}
au_x + bu_y &= (a\mathbf{i} + b\mathbf{j}) \cdot (u_x\mathbf{i} + u_y\mathbf{j}) \\
&= (a\mathbf{i} + b\mathbf{j}) \cdot \nabla u. \qquad (7.10)
\end{aligned}$$

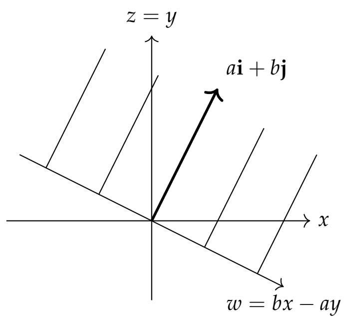


Figure 7.1: Coordinate systems for transforming $au_x + bu_y + cu = f$ into $bv_z + cv = f$ using the transformation $w = bx - ay$ and $z = y$.

Recall from multivariable calculus that the last term is nothing but a directional derivative of $u(x, y)$ in the direction $a\mathbf{i} + b\mathbf{j}$. [Actually, it is proportional to the directional derivative if $a\mathbf{i} + b\mathbf{j}$ is not a unit vector.]

Therefore, we seek to write the partial differential equation as involving a derivative in the direction $a\mathbf{i} + b\mathbf{j}$ but not in a directional orthogonal to this. In Figure 7.1 we depict a new set of coordinates in which the w direction is orthogonal to $a\mathbf{i} + b\mathbf{j}$.

We consider the transformation

$$\begin{aligned} w &= bx - ay, \\ z &= y. \end{aligned} \tag{7.11}$$

We first note that this transformation is invertible,

$$\begin{aligned} x &= \frac{1}{b}(w + az), \\ y &= z. \end{aligned} \tag{7.12}$$

Next we consider how the derivative terms transform. Let $u(x, y) = v(w, z)$. Then, we have

$$\begin{aligned} au_x + bu_y &= a\frac{\partial}{\partial x}v(w, z) + b\frac{\partial}{\partial y}v(w, z), \\ &= a\left[\frac{\partial v}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right] \\ &\quad +b\left[\frac{\partial v}{\partial w}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right] \\ &= a[bv_w + 0 \cdot v_z] + b[-av_w + v_z] \\ &= bv_z. \end{aligned} \tag{7.13}$$

Therefore, the partial differential equation becomes

$$bv_z + cv = f\left(\frac{1}{b}(w + az), z\right).$$

This is now in the same form as in the first case and can be solved using an integrating factor.

Example 7.1. Find the general solution of the equation $3u_x - 2u_y + u = x$.

First, we transform the equation into new coordinates.

$$w = bx - ay = -2x - 3y,$$

and $z = y$. The,

$$\begin{aligned} u_x - 2u_y &= 3[-2v_w + 0 \cdot v_z] - 2[-3v_w + v_z] \\ &= -2v_z. \end{aligned} \tag{7.14}$$

The new partial differential equation for $v(w, z)$ is

$$-2\frac{\partial v}{\partial z} + v = x = -\frac{1}{2}(w + 3z).$$

Rewriting this equation,

$$\frac{\partial v}{\partial z} - \frac{1}{2}v = \frac{1}{4}(w+3z),$$

we identify the integrating factor

$$\mu(z) = \exp\left[-\int^z \frac{1}{2}\,d\zeta\right] = e^{-z/2}.$$

Using this integrating factor, we can solve the differential equation for $v(w,z)$.

$$\begin{aligned}
\frac{\partial}{\partial z}\left(e^{-z/2}v\right) &= \frac{1}{4}(w+3z)e^{-z/2}, \\
e^{-z/2}v(w,z) &= \frac{1}{4}\int^z (w+3\zeta)e^{-\zeta/2}\,d\zeta \\
&= -\frac{1}{2}(w+6+3z)e^{-z/2} + c(w) \\
v(w,z) &= -\frac{1}{2}(w+6+3z) + c(w)e^{z/2} \\
u(x,y) &= x-3+c(-2x-3y)e^{y/2}.
\end{aligned} \tag{7.15}$$

7.3 Quasilinear Equations: The Method of Characteristics

7.3.1 Geometric Interpretation

WE CONSIDER THE QUASILINEAR PARTIAL DIFFERENTIAL EQUATION in two independent variables,

$$a(x,y,u)u_x + b(x,y,u)u_y - c(x,y,u) = 0. \tag{7.16}$$

Let $u = u(x,y)$ be a solution of this equation. Then,

$$f(x,y,u) = u(x,y) - u = 0$$

describes the solution surface, or integral surface,

Integral surface.

We recall from multivariable, or vector, calculus that the normal to the integral surface is given by the gradient function,

$$\nabla f = (u_x, u_y, -1).$$

Now consider the vector of coefficients, $\mathbf{v} = (a,b,c)$ and the dot product with the gradient above:

$$\mathbf{v}\cdot\nabla f = au_x + bu_y - c.$$

This is the left hand side of the partial differential equation. Therefore, for the solution surface we have

$$\mathbf{v}\cdot\nabla f = 0,$$

or $\mathbf{v}$ is perpendicular to ∇f. Since ∇f is normal to the surface, $\mathbf{v} = (a,b,c)$ is tangent to the surface. Geometrically, $\mathbf{v}$ defines a direction field, called the characteristic field. These are shown in Figure 7.2.

The characteristic field.

7.3.2 Characteristics

We seek the forms of the characteristic curves such as the one shown in Figure 7.2. Recall that one can parametrize space curves,

$$\mathbf{c}(t) = (x(t), y(t), u(t)), \quad t \in [t_1, t_2].$$

The tangent to the curve is then

$$\mathbf{v}(t) = \frac{d\mathbf{c}(t)}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{du}{dt}\right).$$

However, in the last section we saw that $\mathbf{v}(t) = (a, b, c)$ for the partial differential equation $a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) = 0$. This gives the parametric form of the characteristic curves as

$$\frac{dx}{dt} = a, \frac{dy}{dt} = b, \frac{du}{dt} = c. \tag{7.17}$$

Another form of these equations is found by relating the differentials, dx, dy, du, to the coefficients in the differential equation. Since $x = x(t)$ and $y = y(t)$, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b}{a}.$$

Similarly, we can show that

$$\frac{du}{dx} = \frac{c}{a}, \quad \frac{du}{dy} = \frac{c}{b}.$$

All of these relations can be summarized in the form

$$dt = \frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}. \tag{7.18}$$

How do we use these characteristics to solve quasilinear partial differential equations? Consider the next example.

Example 7.2. Find the general solution: $u_x + u_y - u = 0$.

We first identify $a = 1$, $b = 1$, and $c = u$. The relations between the differentials is

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{u}.$$

We can pair the differentials in three ways:

$$\frac{dy}{dx} = 1, \quad \frac{du}{dx} = u, \quad \frac{du}{dy} = u.$$

Only two of these relations are independent. We focus on the first pair.

The first equation gives the characteristic curves in the xy-plane. This equation is easily solved to give

$$y = x + c_1.$$

The second equation can be solved to give $u = c_2 e^x$.

The goal is to find the general solution to the differential equation. Since $u = u(x,y)$, the integration "constant" is not really a constant, but is constant with respect to x. It is in fact an arbitrary constant function. In fact, we could view it as a function of c_1, the constant of integration in the first equation. Thus, we let $c_2 = G(c_1)$ for G and arbitrary function. Since $c_1 = y - x$, we can write the general solution of the differential equation as

$$u(x,y) = G(y-x)e^x.$$

Example 7.3. Solve the advection equation, $u_t + cu_x = 0$, for c a constant, and $u = u(x,t)$, $|x| < \infty$, $t > 0$.

The characteristic equations are

$$d\tau = \frac{dt}{1} = \frac{dx}{c} = \frac{du}{0} \tag{7.19}$$

and the parametric equations are given by

$$\frac{dx}{d\tau} = c, \quad \frac{du}{d\tau} = 0. \tag{7.20}$$

These equations imply that

- $u =$ const. $= c_1$.
- $x = ct +$ const. $= ct + c_2$.

Traveling waves.

As before, we can write c_1 as an arbitrary function of c_2. However, before doing so, let's replace c_1 with the variable ξ and then we have that

$$\xi = x - ct, \quad u(x,t) = f(\xi) = f(x-ct)$$

where f is an arbitrary function. Furthermore, we see that $u(x,t) = f(x-ct)$ indicates that the solution is a wave moving in one direction in the shape of the initial function, $f(x)$. This is known as a traveling wave. A typical traveling wave is shown in Figure 7.3.

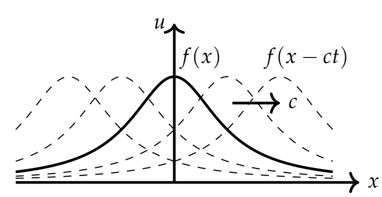


Figure 7.3: Depiction of a traveling wave. $u(x,t) = f(x)$ at $t = 0$ travels without changing shape.

Note that since $u = u(x,t)$, we have

$$\begin{aligned} 0 &= u_t + cu_x \\ &= \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} \\ &= \frac{du(x(t),t}{dt}. \end{aligned} \tag{7.21}$$

This implies that $u(x,t) =$ constant along the characteristics, $\frac{dx}{dt} = c$.

As with ordinary differential equations, the general solution provides an infinite number of solutions of the differential equation. If we want to pick out a particular solution, we need to specify some side conditions. We investigate this by way of examples.

Side conditions.

Example 7.4. Find solutions of $u_x + u_y - u = 0$ subject to $u(x,0) = 1$.

We found the general solution to the partial differential equation as $u(x,y) = G(y-x)e^x$. The side condition tells us that $u = 1$ along $y = 0$. This requires

$$1 = u(x,0) = G(-x)e^x.$$

Thus, $G(-x) = e^{-x}$. Replacing x with $-z$, we find

$$G(z) = e^z.$$

Thus, the side condition has allowed for the determination of the arbitrary function $G(y-x)$. Inserting this function, we have

$$u(x,y) = G(y-x)e^x = e^{y-x}e^x = e^y.$$

Side conditions could be placed on other curves. For the general line, $y = mx + d$, we have $u(x, mx+d) = g(x)$ and for $x = d$, $u(d,y) = g(y)$. As we will see, it is possible that a given side condition may not yield a solution. We will see that conditions have to be given on non-characteristic curves in order to be useful.

Example 7.5. Find solutions of $3u_x - 2u_y + u = x$ for a) $u(x,x) = x$ and b) $u(x,y) = 0$ on $3y + 2x = 1$.

Before applying the side condition, we find the general solution of the partial differential equation. Rewriting the differential equation in standard form, we have

$$3u_x - 2u_y = x = u.$$

The characteristic equations are

$$\frac{dx}{3} = \frac{dy}{-2} = \frac{du}{x-u}. \tag{7.22}$$

These equations imply that

- $-2dx = 3dy$

  This implies that the characteristic curves (lines) are $2x + 3y = c_1$.

- $\frac{du}{dx} = \frac{1}{3}(x-u)$.

  This is a linear first order differential equation, $\frac{du}{dx} + \frac{1}{3}u = \frac{1}{3}x$. It can be solved using the integrating factor,

$$\mu(x) = \exp\left(\frac{1}{3}\int^x d\xi\right) = e^{x/3}.$$

$$\begin{aligned}
\frac{d}{dx}\left(ue^{x/3}\right) &= \frac{1}{3}xe^{x/3} \\
ue^{x/3} &= \frac{1}{3}\int^x \xi e^{\xi/3}\,d\xi + c_2 \\
&= (x-3)e^{x/3} + c_2 \\
u(x,y) &= x - 3 + c_2 e^{-x/3}.
\end{aligned} \tag{7.23}$$

As before, we write c_2 as an arbitrary function of $c_1 = 2x + 3y$. This gives the general solution

$$u(x, y) = x - 3 + G(2x + 3y)e^{-x/3}.$$

Note that this is the same answer that we had found in Example 7.1

Now we can look at any side conditions and use them to determine particular solutions by picking out specific G's.

a $u(x, x) = x$

This states that $u = x$ along the line $y = x$. Inserting this condition into the general solution, we have

$$x = x - 3 + G(5x)e^{-x/3},$$

or

$$G(5x) = 3e^{x/3}.$$

Letting $z = 5x$,

$$G(z) = 3e^{z/15}.$$

The particular solution satisfying this side condition is

$$\begin{aligned} u(x, y) &= x - 3 + G(2x + 3y)e^{-x/3} \\ &= x - 3 + 3e^{(2x+3y)/15}e^{-x/3} \\ &= x - 3 + 3e^{(y-x)/5}. \end{aligned} \tag{7.24}$$

This surface is shown in Figure 7.5.

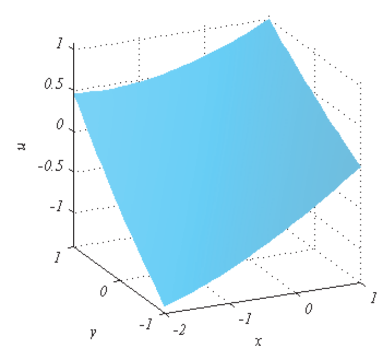


Figure 7.4: Integral surface found in Example 7.5.

In Figure 7.5 we superimpose the values of $u(x, y)$ along the characteristic curves. The characteristic curves are the red lines and the images of these curves are the black lines. The side condition is indicated with the blue curve drawn along the surface.

The values of $u(x, y)$ are found from the side condition as follows. For $x = \xi$ on the blue curve, we know that $y = \xi$ and $u(\xi, \xi) = \xi$. Now, the characteristic lines are given by $2x + 3y = c_1$. The constant c_1 is found on the blue curve from the point of intersection with one of the black characteristic lines. For $x = y = \xi$, we have $c_1 = 5\xi$. Then, the equation of the characteristic line, which is red in Figure 7.5, is given by $y = \frac{1}{3}(5\xi - 2x)$.

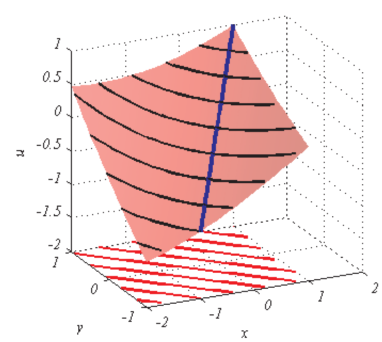


Figure 7.5: Integral surface with side condition and characteristics for Example 7.5.

Along these lines we need to find $u(x, y) = x - 3 + c_2e^{-x/3}$. First we have to find c_2. We have on the blue curve, that

$$\begin{aligned} \xi &= u(\xi, \xi) \\ &= \xi - 3 + c_2e^{-\xi/3}. \end{aligned} \tag{7.25}$$

Therefore, $c_2 = 3e^{\xi/3}$. Inserting this result into the expression for the solution, we have

$$u(x, y) = x - 3 + e^{(\xi - x)/3}.$$

So, for each ξ, one can draw a family of spacecurves

$$\left(x, \frac{1}{3}(5\xi - 2x), x - 3 + e^{(\xi - x)/3}\right)$$

yielding the integral surface.

b $u(x, y) = 0$ on $3y + 2x = 1$.
For this condition, we have

$$0 = x - 3 + G(1)e^{-x/3}.$$

We note that G is not a function in this expression. We only have one value for G. So, we cannot solve for $G(x)$. Geometrically, this side condition corresponds to one of the black curves in Figure 7.5.

7.4 *Applications*

7.4.1 *Conservation Laws*

There are many applications of quasilinear equations, especially in fluid dynamics. The advection equation is one such example and generalizations of this example to nonlinear equations leads to some interesting problems. These equations fall into a category of equations called conservation laws. We will first discuss one-dimensional (in space) conservations laws and then look at simple examples of nonlinear conservation laws.

Conservation laws are useful in modeling several systems. They can be boiled down to determining the rate of change of some stuff, $Q(t)$, in a region, $a \le x \le b$, as depicted in Figure 7.6. The simples model is to think of fluid flowing in one dimension, such as water flowing in a stream. Or, it could be the transport of mass, such as a pollutant. One could think of traffic flow down a straight road. We have seen such ideas encapsulated in the continuity equation.

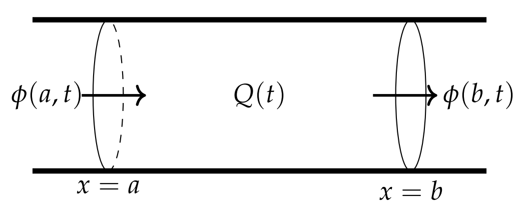


Figure 7.6: The rate of change of Q between $x = a$ and $x = b$ depends on the rates of flow through each end.

This is an example of a typical mixing problem. The rate of change of $Q(t)$ is given as

$$\text{the rate of change of } Q = \text{Rate in} - \text{Rate Out} + \text{ source term.}$$

Here the "Rate in" is how much is flowing into the region in Figure 7.6 from the $x = a$ boundary. Similarly, the "Rate out" is how much is flowing into the region from the $x = b$ boundary. [Of course, this could be the other way, but we can imagine for now that q is flowing from left to right.] We can describe this flow in terms of the flux, $\phi(x,t)$ over the ends of the region. On the left side we have a gain of $\phi(a,t)$ and on the right side of the region there is a loss of $\phi(b,t)$.

The source term would be some other means of adding or removing Q from the region. In terms of fluid flow, there could be a source of fluid inside the region such as a faucet adding more water. Or, there could be a drain letting water escape. We can denote this by the total source over the interval, $\int_a^b f(x,t)\,dx$. Here $f(x,t)$ is the source density.

In summary, the rate of change of $Q(x,t)$ can be written as

$$\frac{dQ}{dt} = \phi(a,t) - \phi(b,t) + \int_a^b f(x,y)\,dx.$$

We can write this in a slightly different form by noting that $\phi(a,t) - \phi(b,t)$ can be viewed as the evaluation of antiderivatives in the Fundamental Theorem of Calculus. Namely, we can recall that

$$\int_a^b \frac{\partial \phi(x,t)}{\partial x}\,dx = \phi(b,t) - \phi(a,t).$$

The difference is not exactly in the order that we desire, but it is easy to see that

Integral form of conservation law.

$$\frac{dQ}{dt} = -\int_a^b \frac{\partial \phi(x,t)}{\partial x}\,dx + \int_a^b f(x,t)\,dx. \tag{7.26}$$

This is the integral form of the conservation law.

We can rewrite the conservation law in differential form. First, we introduce the density function, $u(x,t)$, so that the total amount of stuff at a given time is

$$Q(t) = \int_a^b u(x,t)\,dx.$$

Introducing this form into the integral conservation law, we have

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = -\int_a^b \frac{\partial \phi}{\partial x}\,dx + \int_a^b f(x,t)\,dx. \tag{7.27}$$

Assuming that a and b are fixed in time and that the integrand is continuous, we can bring the time derivative inside the integrand and collect the three terms into one to find

$$\int_a^b \left(u_t(x,t) + \phi_x(x,t) - f(x,t)\right)\,dx = 0, \quad \forall x \in [a,b].$$

We cannot simply set the integrant to zero just because the integral vanishes. However, if this result holds for every region $[a,b]$, then we can conclude the integrand vanishes. So, under that assumption, we have the local conservation law,

Differential form of conservation law.

$$u_t(x,t) + \phi_x(x,t) = f(x,t). \tag{7.28}$$

This partial differential equation is actually an equation in terms of two unknown functions, assuming we know something about the source function. We would like to have a single unknown function. So, we need some additional information. This added information comes from the constitutive relation, a function relating the flux to the density function. Namely, we will assume that we can find the relationship $\phi = \phi(u)$. If so, then we can write

$$\frac{\partial \phi}{\partial x} = \frac{d\phi}{du}\frac{\partial u}{\partial x},$$

or $\phi_x = \phi'(u)u_x$.

Example 7.6. Inviscid Burgers' Equation Find the equation satisfied by $u(x,t)$ for $\phi(u) = \frac{1}{2}u^2$ and $f(x,t) \equiv 0$.

For this flux function we have $\phi_x = \phi'(u)u_x = uu_x$. The resulting equation is then $u_t + uu_x = 0$. This is the inviscid Burgers' equation. We will later discuss Burgers' equation.

Example 7.7. Traffic Flow

This is a simple model of one-dimensional traffic flow. Let $u(x,t)$ be the density of cars. Assume that there is no source term. For example, there is no way for a car to disappear from the flow by turning off the road or falling into a sinkhole. Also, there is no source of additional cars.

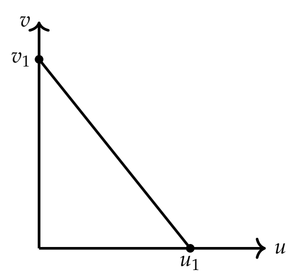


Figure 7.7: Car velocity as a function of car density.

Let $\phi(x,t)$ denote the number of cars per hour passing position x at time t. Note that the units are given by cars/mi times mi/hr. Thus, we can write the flux as $\phi = uv$, where v is the velocity of the carts at position x and time t.

In order to continue we need to assume a relationship between the car velocity and the car density. Let's assume the simplest form, a linear relationship. The more dense the traffic, we expect the speeds to slow down. So, a function similar to that in Figure 7.7 is in order. This is a straight line between the two intercepts $(0, v_1)$ and $(u_1, 0)$. It is easy to determine the equation of this line. Namely the relationship is given as

$$v = v_1 - \frac{v_1}{u_1}u.$$

This gives the flux as

$$\phi = uv = v_1\left(u - \frac{u^2}{u_1}\right).$$

We can now write the equation for the car density,

$$\begin{aligned} 0 &= u_t + \phi' u_x \\ &= u_t + v_1\left(1 - \frac{2u}{u_1}\right)u_x. \end{aligned} \tag{7.29}$$

7.4.2 *Nonlinear Advection Equations*

In this section we consider equations of the form $u_t + c(u)u_x = 0$. When $c(u)$ is a constant function, we have the advection equation. In the last two examples we have seen cases in which $c(u)$ is not a constant function.

We will apply the method of characteristics to these equations. First, we will recall how the method works for the advection equation.

The advection equation is given by $u_t + cu_x = 0$. The characteristic equations are given by

$$\frac{dx}{dt} = c, \quad \frac{du}{dt} = 0.$$

These are easily solved to give the result that

$$u(x,t) = \text{constant along the lines } x = ct + x_0,$$

where x_0 is an arbitrary constant.

The characteristic lines are shown in Figure 7.8. We note that $u(x,t) = u(x_0, 0) = f(x_0)$. So, if we know u initially, we can determine what u is at a later time.

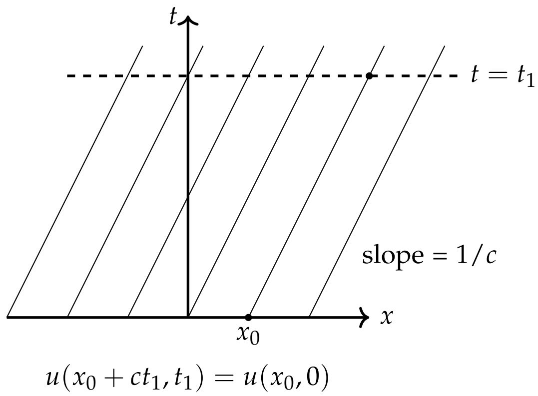


Figure 7.8: The characteristics lines the xt-plane.

In Figure 7.8 we see that the value of $u(x_0,)$ at $t = 0$ and $x = x_0$ propagates along the characteristic to a point at time $t = t_1$. From $x - ct = x_0$, we can solve for x in terms of t_1 and find that $u(x_0 + ct_1, t_1) = u(x_0, 0)$.

Plots of solutions $u(x,t)$ versus x for specific times give traveling waves as shown in Figure 7.3. In Figure 7.9 we show how each wave profile for different times are constructed for a given initial condition.

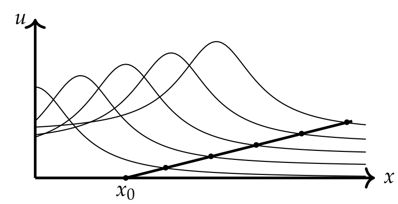


Figure 7.9: For each $x = x_0$ at $t = 0$, $u(x_0 + ct, t) = u(x_0, 0)$.

The nonlinear advection equation is given by $u_t + c(u)u_x = 0$, $|x| < \infty$. Let $u(x,0) = u_0(x)$ be the initial profile. The characteristic equations are given by

$$\frac{dx}{dt} = c(u), \quad \frac{du}{dt} = 0.$$

These are solved to give the result that

$$u(x,t) = \text{constant},$$

along the characteristic curves $x'(t) = c(u)$. The lines passing though $u(x_0,) = u_0(x_0)$ have slope $1/c(u_0(x_0))$.

Example 7.8. Solve $u_t + uu_x = 0$, $u(x,0) = e^{-x^2}$.

For this problem $u = $ constant along

$$\frac{dx}{dt} = u.$$

Since u is constant, this equation can be integrated to yield $x = u(x_0, 0)t + x_0$. Inserting the initial condition, $x = e^{-x_0^2}t + x_0$. Therefore, the solution is

$$u(x,t) = e^{-x_0^2} \text{ along } x = e^{-x_0^2}t + x_0.$$

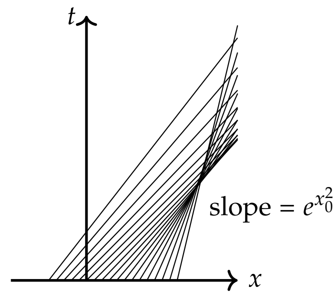


Figure 7.10: The characteristics lines the xt-plane for the nonlinear advection equation.

In Figure 7.10 the characteristics a shown. In this case we see that the characteristics intersect. In Figure charlines3 we look more specifically at the intersection of the characteristic lines for $x_0 = 0$ and $x_0 = 1$. These are approximately the first lines to intersect; i.e., there are (almost) no intersections at earlier times. At the intersection point the function $u(x,t)$ appears to take on more than one value. For the case shown, the solution wants to take the values $u = 0$ and $u = 1$.

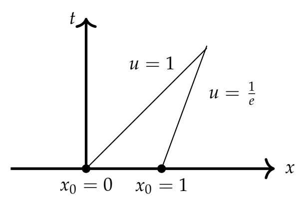


Figure 7.11: The characteristics lines for $x_0 = 0, 1$ in the xt-plane for the nonlinear advection equation.

In Figure 7.12 we see the development of the solution. This is found using a parametric plot of the points $(x_0 + te^{-x_0^2}, e^{-x_0^2})$ for different times. The initial profile propagates to the right with the higher points traveling faster than the lower points since $x'(t) = u > 0$. Around $t = 1.0$ the wave breaks and becomes multivalued. The time at which the function becomes multivalued is called the breaking time.

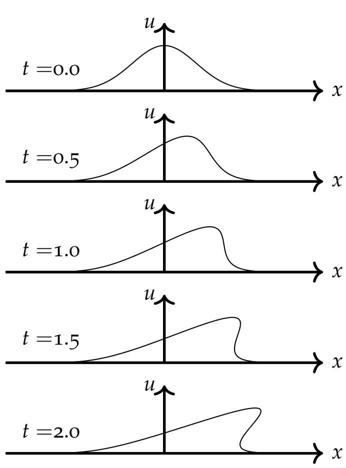


Figure 7.12: The development of a gradient catastrophe in Example 7.8 leading to a multivalued function.

7.4.3 *The Breaking Time*

In the last example we saw that for nonlinear wave speeds a gradient catastrophe might occur. The first time at which a catastrophe occurs is called the breaking time. We will determine the breaking time for the nonlinear advection equation, $u_t + c(u)u_x = 0$. For the characteristic corresponding to $x_0 = \xi$, the wavespeed is given by

$$F(\xi) = c(u_0(\xi))$$

and the characteristic line is given by

$$x = \xi + tF(\xi).$$

The value of the wave function along this characteristic is

$u_0(\xi) = u(\xi, 0)$.

$$\begin{aligned} u(x,t) &= u(\xi + tF(\xi), t) \\ &= . \end{aligned} \quad (7.30)$$

Therefore, the solution is

$$u(x,t) = u_0(\xi) \text{ along } x = \xi + tF(\xi).$$

This means that

$$u_x = u_0'(\xi)\xi_x \quad \textit{and} \quad u_t = u_0'(\xi)\xi_t.$$

We can determine ξ_x and ξ_t using the characteristic line

$$\xi = x - tF(\xi).$$

Then, we have

$$\begin{aligned} \xi_x &= 1 - tF'(\xi)\xi_x \\ &= \frac{1}{1 + tF'(\xi)}. \\ \xi_t &= \frac{\partial}{\partial t}(x - tF(\xi)) \\ &= -F(\xi) - tF'(\xi)\xi_t \\ &= \frac{-F(\xi)}{1 + tF'(\xi)}. \end{aligned} \quad (7.31)$$

Note that ξ_x and ξ_t are undefined if the denominator in both expressions vanishes, $1 + tF'(\xi) = 0$, or at time

$$t = -\frac{1}{F'(\xi)}.$$

The minimum time for this to happen in the breaking time,

The breaking time.

$$t_b = \min\left\{-\frac{1}{F'(\xi)}\right\}. \quad (7.32)$$

Example 7.9. Find the breaking time for $u_t + uu_x = 0$, $u(x,0) = e^{-x^2}$.

Since $c(u) = u$, we have

$$F(\xi) = c(u_0(\xi)) = e^{-\xi^2}$$

and

$$F'(\xi) = -2\xi e^{-\xi^2}.$$

This gives

$$t = \frac{1}{2\xi e^{-\xi^2}}.$$

We need to find the minimum time. Thus, we set the derivative equal to zero and solve for ξ.

$$\begin{aligned} 0 &= \frac{d}{d\xi}\left(\frac{e^{\xi^2}}{2\xi}\right) \\ &= \left(2 - \frac{1}{\xi^2}\right)\frac{e^{\xi^2}}{2}. \end{aligned} \tag{7.33}$$

Thus, the minimum occurs for $2 - \frac{1}{\xi^2} = 0$, or $\xi = 1/\sqrt{2}$. This gives

$$t_b = t\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\frac{2}{\sqrt{2}e^{-1/2}}} = \sqrt{\frac{e}{2}} \approx 1.16. \tag{7.34}$$

7.4.4 *Shock Waves*

Solutions of nonlinear advection equations can become multivalued due to a gradient catastrophe. Namely, the derivatives u_t and u_x become undefined. We would like to extend solutions past the catastrophe. However, this leads to the possibility of discontinuous solutions. Such solutions which may not be differentiable or continuous in the domain are known as weak solutions. In particular, consider the initial value problem

Weak solutions.

$$u_t + \phi_x = 0, \quad x \in R, \quad t > 0, \quad u(x,0) = u_0(x).$$

Then, $u(x,t)$ is a weak solution of this problem if

$$\int_0^\infty \int_{-\infty}^\infty [uv_t + \phi v_x]\, dxdt + \int_{-\infty}^\infty u_0(x)v(x,0)\, dx = 0$$

for all smooth functions $v \in C^\infty(R \times [0,\infty))$ with compact support, i.e., $v \equiv= 0$ outside some compact subset of the domain.

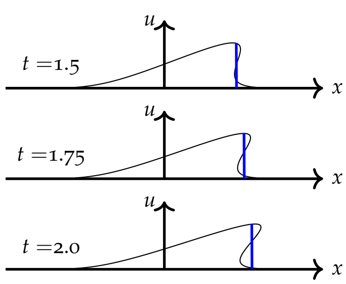


Figure 7.13: The shock solution after the breaking time.

Effectively, the weak solution that evolves will be a piecewise smooth function with a discontinuity, the shock wave, that propagates with shock speed. It can be shown that the form of the shock will be the discontinuity shown in Figure 7.13 such that the areas cut from the solutions will cancel leaving the total area under the solution constant. [See G. B. Whitham's *Linear and Nonlinear Waves*, 1973.] We will consider the discontinuity as shown in Figure 7.14.

We can find the equation for the shock path by using the integral form of the conservation law,

$$\frac{d}{dt}\int_a^b u(x,t)\, dx = \phi(a,t) - \phi(b,t).$$

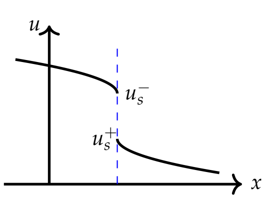


Figure 7.14: Depiction of the jump discontinuity at the shock position.

Recall that one can differentiate under the integral if $u(x,t)$ and $u_t(x,t)$ are continuous in x and t in an appropriate subset of the domain. In particular, we will integrate over the interval $[a,b]$ as shown in Figure 7.15. The domains on either side of shock path are denoted as R^+ and R^- and the limits of $x(t)$ and $u(x,t)$ as one approaches from the left of the shock are denoted by $x_s^-(t)$ and $u^- = u(x_s^-, t)$. Similarly, the limits of $x(t)$ and $u(x,t)$ as one approaches from the right of the shock are denoted by $x_s^+(t)$ and $u^+ = u(x_s^+, t)$.

We need to be careful in differentiating under the integral,

$$\begin{aligned}\frac{d}{dt}\int_a^b u(x,t)\,dx &= \frac{d}{dt}\left[\int_a^{x_s^-(t)} u(x,t)\,dx + \int_{x_s^+(t)}^b u(x,t)\,dx\right] \\ &= \int_a^{x_s^-(t)} u_t(x,t)\,dx + \int_{x_s^+(t)}^b u_t(x,t)\,dx \\ &\quad + u(x_s^-,t)\frac{dx_s^-}{dt} - u(x_s^+,t)\frac{dx_s^+}{dt} \\ &= \phi(a,t) - \phi(b,t). \end{aligned} \tag{7.35}$$

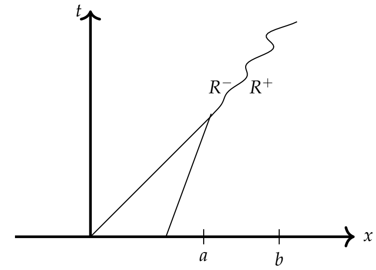


Figure 7.15: Domains on either side of shock path are denoted as R^+ and R^-.

Taking the limits $a \to x_s^-$ and $b \to x_s^+$, we have that

$$\left(u(x_s^-,t) - u(x_s^+,t)\right)\frac{dx_s}{dt} = \phi(x_s^-,t) - \phi(x_s^+,t).$$

Adopting the notation

$$[f] = f(x_s^+) - f(x_s^-),$$

we arrive at the Rankine-Hugoniot jump condition

The Rankine-Hugoniot jump condition.

$$\frac{dx_s}{dt} = \frac{[\phi]}{[u]}. \tag{7.36}$$

This gives the equation for the shock path as will be shown in the next example.

Example 7.10. Consider the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 1, & x \le 0, \\ 0, & x > 0. \end{cases}$$

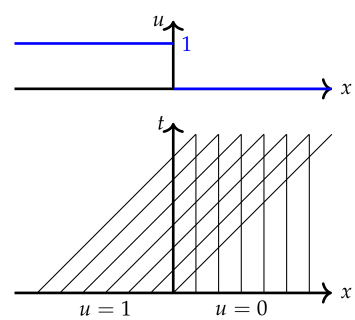


Figure 7.16: Initial condition and characteristics for Example 7.10.

The characteristics for this partial differential equation are familiar by now. The initial condition and characteristics are shown in Figure 7.16. From $x'(t) = u$, there are two possibilities. If $u = 0$, then we have a constant. If $u = 1$ along the characteristics, the we have straight lines of slope one. Therefore, the characteristics are given by

$$x(t) = \begin{cases} x_0, & x > 0, \\ t + x_0, & x < 0. \end{cases}$$

As seen in Figure 7.16 the characteristics intersect immediately at $t = 0$. The shock path is found from the Rankine-Hugoniot jump condition. We first note that $\phi(u) = \frac{1}{2}u^2$, since $\phi_x = uu_x$. Then, we have

$$\frac{dx_s}{dt} = \frac{[\phi]}{[u]}$$

$$
\begin{aligned}
&= \frac{\frac{1}{2}u^{+2} - \frac{1}{2}u^{-2}}{u^+ - u^-} \\
&= \frac{1}{2}\frac{(u^+ + u^-)(u^+ - u^-)}{u^+ - u^-} \\
&= \frac{1}{2}(u^+ + u^-) \\
&= \frac{1}{2}(0+1) = \frac{1}{2}. \qquad (7.37)
\end{aligned}
$$

Now we need only solve the ordinary differential equation $x_s'(t) = \frac{1}{2}$ with initial condition $x_s(0) = 0$. This gives $x_s(t) = \frac{t}{2}$. This line separates the characteristics on the left and right side of the shock solution. The solution is given by

$$
u(x,t) = \begin{cases} 1, & x \le t/2, \\ 0, & x > t/2. \end{cases}
$$

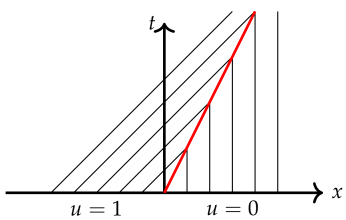


Figure 7.17: The characteristic lines end at the shock path (in red). On the left $u = 1$ and on the right $u = 0$.

In Figure 7.17 we show the characteristic lines ending at the shock path (in red) with $u = 0$ and on the right and $u = 1$ on the left of the shock path. This is consistent with the solution. One just sees the initial step function moving to the right with speed $1/2$ without changing shape.

7.4.5 *Rarefaction Waves*

Shocks are not the only type of solutions encountered when the velocity is a function of u. There may sometimes be regions where the characteristic lines do not appear. A simple example is the following.

Example 7.11. Draw the characteristics for the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$
u(x,0) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0. \end{cases}
$$

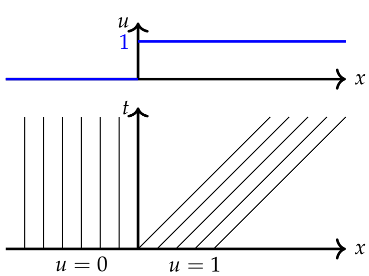


Figure 7.18: Initial condition and characteristics for Example 7.14.

In this case the solution is zero for negative values of x and positive for positive values of x as shown in Figure 7.18. Since the wavespeed is given by u, the $u = 1$ initial values have the waves on the right moving to the right and the values on the left stay fixed. This leads to the characteristics in Figure 7.18 showing a region in the xt-plane that has no characteristics. In this section we will discover how to fill in the missing characteristics and, thus, the details about the solution between the $u = 0$ and $u = 1$ values.

As motivation, we consider a smoothed out version of this problem.

Example 7.12. Draw the characteristics for the initial condition

$$u(x,0) = \begin{cases} 0, & x \le -\epsilon, \\ \frac{x+\epsilon}{2\epsilon}, & |x| \le \epsilon, \\ 1, & x > \epsilon. \end{cases}$$

The function is shown in the top graph in Figure 7.19. The leftmost and rightmost characteristics are the same as the previous example. The only new part is determining the equations of the characteristics for $|x| \le \epsilon$. These are found using the method of characteristics as

$$x = \xi + u_0(\xi)t, \quad u_0(\xi) = \frac{\xi+\epsilon}{2\epsilon}t.$$

These characteristics are drawn in Figure 7.19 in red. Note that these lines take on slopes varying from infinite slope to slope one, corresponding to speeds going from zero to one.

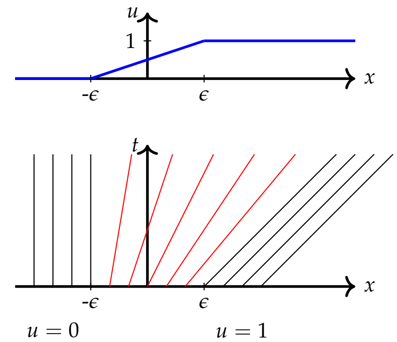


Figure 7.19: The function and characteristics for the smoothed step function.

Characteristics for rarefaction, or expansion, waves are fan-like characteristics.

Comparing the last two examples, we see that as ϵ approaches zero, the last example converges to the previous example. The characteristics in the region where there were none become a "fan". We can see this as follows. Since $|\xi| < \epsilon$ for the fan region, as ϵ gets small, so does this interval. Let's scale ξ as $\xi = \sigma\epsilon$, $\sigma \in [-1,1]$. Then,

$$x = \sigma\epsilon + u_0(\sigma\epsilon)t, \quad u_0(\sigma\epsilon) = \frac{\sigma\epsilon+\epsilon}{2\epsilon}t = \frac{1}{2}(\sigma+1)t.$$

For each $\sigma \in [-1,1]$ there is a characteristic. Letting $\epsilon \to 0$, we have

$$x = ct, \quad c = \frac{1}{2}(\sigma+1)t.$$

Thus, we have a family of straight characteristic lines in the xt-plane passing through $(0,0)$ of the form $x = ct$ for c varying from $c = 0$ to $c = 1$. These are shown as the red lines in Figure 7.20.

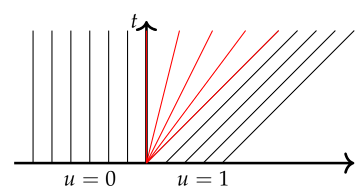


Figure 7.20: The characteristics for Example 7.14 showing the "fan" characteristics.

The fan characteristics can be written as $x/t = \text{constant}$. So, we can seek to determine these characteristics analytically and in a straight forward manner by seeking solutions of the form $u(x,t) = g(\frac{x}{t})$.

Example 7.13. Determine solutions of the form $u(x,t) = g(\frac{x}{t})$ to $u_t + uu_x = 0$.

Inserting this guess into the differential equation, we have

$$\begin{aligned} 0 &= u_t + uu_x \\ &= \frac{1}{t}g'\left(g - \frac{x}{t}\right). \end{aligned} \tag{7.38}$$

Seek rarefaction fan waves using $u(x,t) = g(\frac{x}{t})$.

Thus, either $g' = 0$ or $g = \frac{x}{t}$. The first case will not work since this gives constant solutions. The second solution is exactly what we had obtained before. Recall that solutions along characteristics give $u(x,t) = \frac{x}{t} = \text{constant}$. The characteristics and solutions for $t = 0,1,2$ are shown in Figure rarefactionfig4. At a specific time one can draw a line (dashed lines in figure) and follow the characteristics back to the $t = 0$ values, $u(\xi,0)$ in order to construct $u(x,t)$.

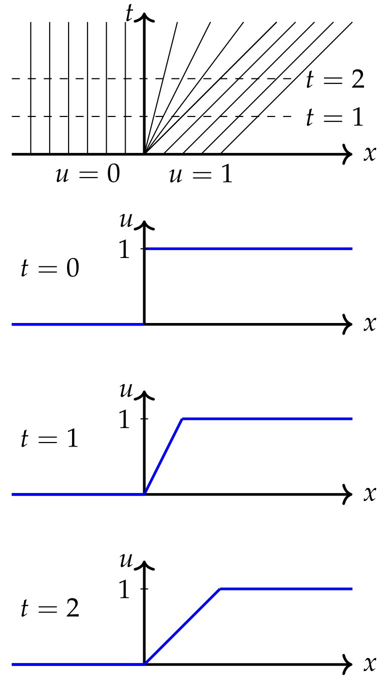


Figure 7.21: The characteristics and solutions for $t = 0, 1, 2$ for Example 7.14

As a last example, let's investigate a nonlinear model which possesses both shock and rarefaction waves.

Example 7.14. Solve the initial value problem $u_t + u^2 u_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 0, & x \leq 0, \\ 1, & 0 < x < 2, \\ 0, & x \geq 2. \end{cases}$$

The method of characteristics gives

$$\frac{dx}{dt} = u^2, \quad \frac{du}{dt} = 0.$$

Therefore,

$$u(x,t) = u_0(\xi) = \text{ const. along the lines } x(t) = u_0^2(\xi)t + \xi.$$

There are three values of $u_0(\xi)$,

$$u_0(\xi) = \begin{cases} 0, & \xi \leq 0, \\ 1, & 0 < \xi < 2, \\ 0, & \xi \geq 2. \end{cases}$$

In Figure 7.22 we see that there is a rarefaction and a gradient catastrophe.

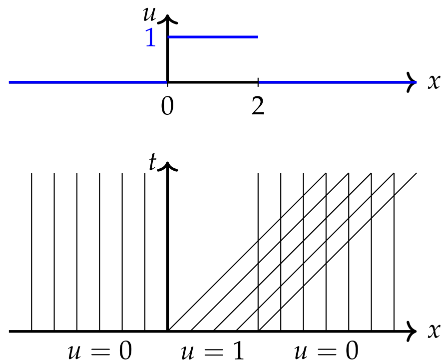


Figure 7.22: In this example there occurs a rarefaction and a gradient catastrophe.

In order to fill in the fan characteristics, we need to find solutions $u(x,t) = g(x/t)$. Inserting this guess into the differential equation, we have

$$\begin{aligned} 0 &= u_t + u^2 u_x \\ &= \frac{1}{t} g' \left(g^2 - \frac{x}{t} \right). \end{aligned} \tag{7.39}$$

Thus, either $g' = 0$ or $g^2 = \frac{x}{t}$. The first case will not work since this gives constant solutions. The second solution gives

$$g\left(\frac{x}{t}\right) = \sqrt{\frac{x}{t}}.$$

. Therefore, along the fan characteristics the solutions are $u(x,t) = \sqrt{\frac{x}{t}} =$ constant. These fan characteristics are added in Figure 7.23.

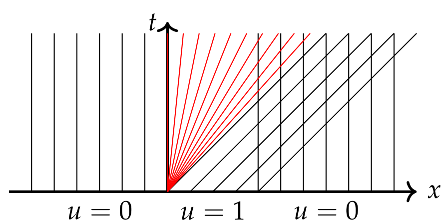


Figure 7.23: The fan characteristics are added to the other characteristic lines.

Next, we turn to the shock path. We see that the first intersection occurs at the point $(x,t) = (2,0)$. The Rankine-Hugonoit condition gives

$$\begin{aligned} \frac{dx_s}{dt} &= \frac{[\phi]}{[u]} \\ &= \frac{\frac{1}{3}u^{+3} - \frac{1}{3}u^{-3}}{u^+ - u^-} \\ &= \frac{1}{3}\frac{(u^+ - u^-)(u^{+2} + u^+u^- + u^{-2})}{u^+ - u^-} \\ &= \frac{1}{3}(u^{+2} + u^+u^- + u^{-2}) \\ &= \frac{1}{3}(0 + 0 + 1) = \frac{1}{3}. \end{aligned} \tag{7.40}$$

Thus, the shock path is given by $x_s'(t) = \frac{1}{3}$ with initial condition $x_s(0) = 2$. This gives $x_s(t) = \frac{t}{3} + 2$. In Figure 7.24 the shock path is shown in red with the fan characteristics and vertical lines meeting the path. Note that the fan lines and vertical lines cross the shock path. This leads to a change in the shock path.

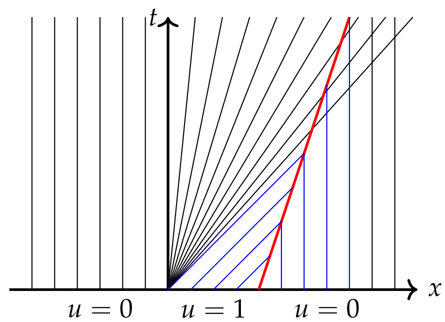


Figure 7.24: The shock path is shown in red with the fan characteristics and vertical lines meeting the path.

The new path is found using the Rankine-Hugonoit condition with $u^+ = 0$ and $u^- = \sqrt{\frac{x}{t}}$. Thus,

$$\begin{aligned} \frac{dx_s}{dt} &= \frac{[\phi]}{[u]} \\ &= \frac{\frac{1}{3}u^{+3} - \frac{1}{3}u^{-3}}{u^+ - u^-} \\ &= \frac{1}{3}\frac{(u^+ - u^-)(u^{+2} + u^+u^- + u^{-2})}{u^+ - u^-} \\ &= \frac{1}{3}(u^{+2} + u^+u^- + u^{-2}) \\ &= \frac{1}{3}(0 + 0 + \sqrt{\frac{x_s}{t}}) = \frac{1}{3}\frac{x_s}{t}. \end{aligned} \tag{7.41}$$

We need to solve the initial value problem

$$\frac{dx_s}{dt} = \frac{1}{3}\frac{x_s}{t}, \quad x_s(3) = 3.$$

This can be done using separation of variables. Namely,

$$\int \frac{dx_s}{x_s} = \frac{1}{3}\int \frac{dt}{t}.$$

This gives the solution

$$\ln x_s = \frac{1}{3}\ln t + c \quad \Rightarrow \quad x_s = At^{1/3}.$$

Since the second shock solution starts at the point $(3,3)$, we can determine $A = 3^{2/3}$. This gives the shock path as

$$x_s(t) = 3^{2/3}t^{1/3}.$$

In Figure 7.25 we show this shock path and the other characteristics ending on the path.

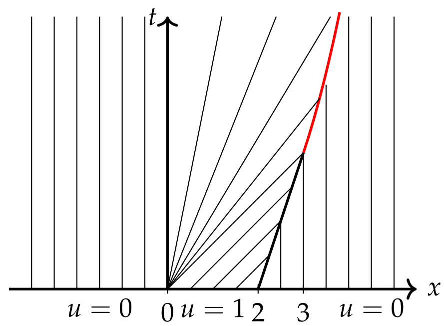


Figure 7.25: The second shock path is shown in red with the characteristics shown in all regions.

It is interesting to construct the solution at different times based on the characteristics. For a given time, t, one draws a horizontal line in the xt-plane and reads off the values of $u(x,t)$ using the values at $t=0$ and the rarefaction solutions. This is shown in Figure 7.26. The right discontinuity in the initial profile continues as a shock front until $t=3$. At that time the back rarefaction wave has caught up to the shock. After $t=3$, the shock propagates forward slightly slower and the height of the shock begins to decrease. Due to the fact that the partial differential equation is a conservation law, the area under the shock remains constant as it stretches and decays in amplitude.

7.4.6 *Traffic Flow*

An interesting application is that of traffic flow. We had already derived the flux function. Let's investigate examples with varying initial conditions that lead to shock or rarefaction waves. As we had seen earlier in modeling traffic flow, we can consider the flux function

$$\phi = uv = v_1\left(u - \frac{u^2}{u_1}\right),$$

which leads to the conservation law

$$u_t + v_1(1 - \frac{2u}{u_1})u_x = 0.$$

Here $u(x,t)$ represents the density of the traffic and u_1 is the maximum density and v_1 is the initial velocity.

First, consider the flow of traffic vas it approaches a red light as shown in Figure 7.27. The traffic that is stopped has reached the maximum density u_1. The incoming traffic has a lower density, u_0. For this red light problem, we consider the initial condition

$$u(x,0) = \begin{cases} u_0, & x < 0, \\ u_1, & x \ge 0. \end{cases}$$

Figure 7.26: Solutions for the shock-rarefaction example.

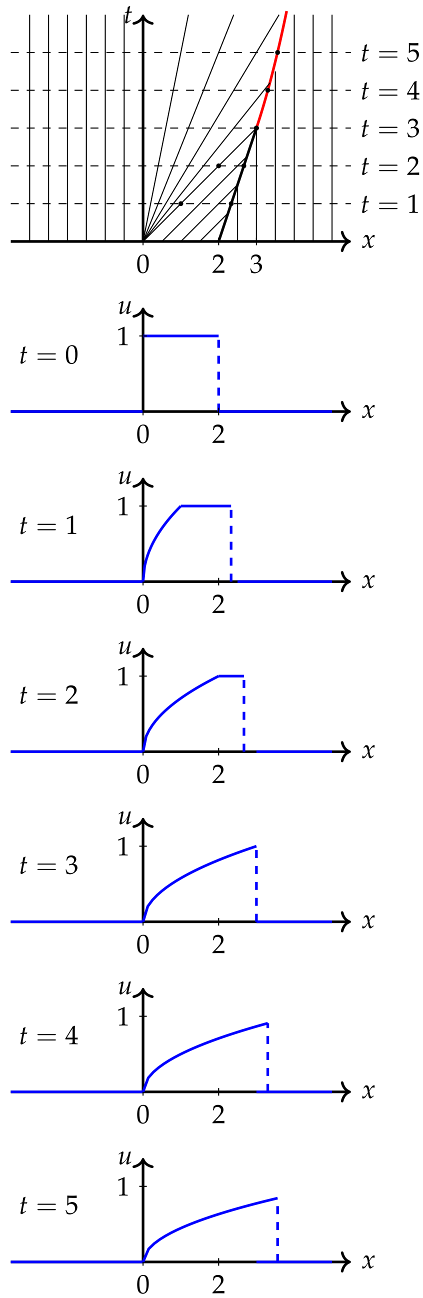
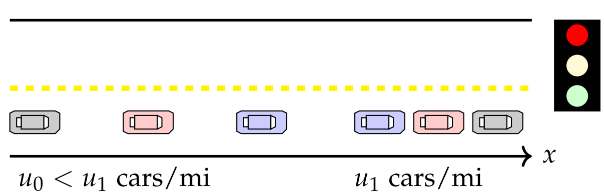


Figure 7.27: Cars approaching a red light.



The characteristics for this problem are given by

$$x = c(u(x_0, t))t + x_0,$$

where

$$c(u(x_0, t)) = v_1 \left(1 - \frac{2u(x_0, 0)}{u_1}\right).$$

Since the initial condition is a piecewise-defined function, we need to consider two cases.

First, for $x \geq 0$, we have

$$c(u(x_0, t)) = c(u_1) = v_1 \left(1 - \frac{2u_1}{u_1}\right) = -v_1.$$

Therefore, the slopes of the characteristics, $x = -v_1 t + x_0$ are $-1/v_1$.

For $x_0 < 0$, we have

$$c(u(x_0, t)) = c(u_0) = v_1 \left(1 - \frac{2u_0}{u_1}\right).$$

So, the characteristics are $x = -v_1 \left(1 - \frac{2u_0}{u_1}\right) t + x_0$.

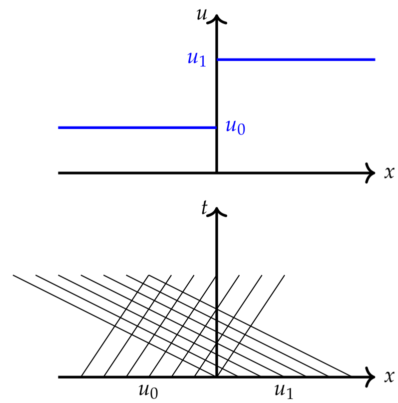


Figure 7.28: Initial condition and characteristics for the red light problem.

In Figure 7.28 we plot the initial condition and the characteristics for $x < 0$ and $x > 0$. We see that there are crossing characteristics and the begin crossing at $t = 0$. Therefore, the breaking time is $t_b = 0$. We need to find the shock path satisfying $x_s(0) = 0$. The Rankine-Hugonoit conditions give

$$\begin{aligned} \frac{dx_s}{dt} &= \frac{[\phi]}{[u]} \\ &= \frac{\frac{1}{2}u^{+2} - \frac{1}{2}u^{-2}}{u^+ - u^-} \\ &= \frac{1}{2}\frac{0 - v_1 \frac{u_0^2}{u_1}}{u_1 - u_0} \\ &= -v_1 \frac{u_0}{u_1}. \end{aligned} \tag{7.42}$$

Thus, the shock path is found as $x_s(t) = -v_1 \frac{u_0}{u_1} t$.

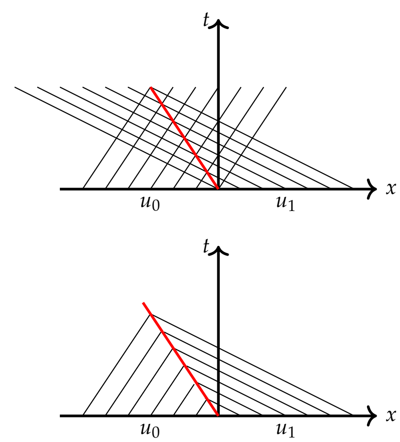


Figure 7.29: The addition of the shock path for the red light problem.

In Figure 7.29 we show the shock path. In the top figure the red line shows the path. In the lower figure the characteristics are stopped on the shock path to give the complete picture of the characteristics. The picture was drawn with $v_1 = 2$ and $u_0/u_1 = 1/3$.

The next problem to consider is stopped traffic as the light turns green. The cars in Figure 7.30 begin to fan out when the traffic light turns green. In this model the initial condition is given by

$$u(x, 0) = \begin{cases} u_1, & x \leq 0, \\ 0, & x > 0. \end{cases}$$

Again,

$$c(u(x_0, t)) = v_1 \left(1 - \frac{2u(x_0, 0)}{u_1}\right).$$

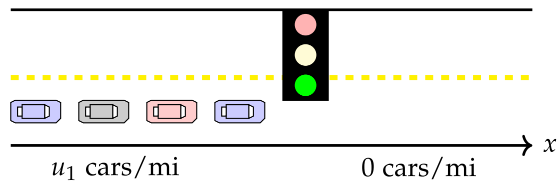


Figure 7.30: Cars begin to fan out when the traffic light turns green.

Inserting the initial values of u into this expression, we obtain constant speeds, $\pm v_1$. The resulting characteristics are given by

$$x(t) = \begin{cases} -v_1 t + x_0, & x \leq 0, \\ v_1 t + x_0, & x > 0. \end{cases}$$

This leads to a rarefaction wave with the solution in the rarefaction region given by

$$u(x,t) = g(x/t) = \frac{1}{2} u_1 \left(1 - \frac{1}{v_1} \frac{x}{t} \right).$$

The characteristics are shown in Figure 7.30. The full solution is then

$$u(x,t) = \begin{cases} u_1, & x \leq -v_1 t, \\ g(x/t), & |x| < v_1 t, \\ 0, & x > v_1 t. \end{cases}$$

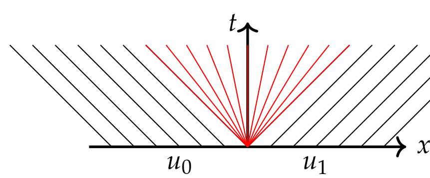


Figure 7.31: The characteristics for the green light problem.

7.5 General First Order PDEs

We have spent time solving quasilinear first order partial differential equations. We now turn to nonlinear first order equations of the form

$$F(x, y, u, u_x, u_y) = 0,$$

for $u = u(x, y)$.

If we introduce new variables, $p = u_x$ and $q = u_y$, then the differential equation takes the form

$$F(x, y, u, p, q) = 0.$$

Note that for $u(x,t)$ a function with continuous derivatives, we have

$$p_y = u_{xy} = u_{yx} = q_x.$$

We can view $F = 0$ as a surface in a five dimensional space. Since the arguments are functions of x and y, we have from the multivariable Chain Rule that

$$\begin{aligned} \frac{dF}{dx} &= F_x + F_u \frac{\partial u}{\partial x} + F_p \frac{\partial p}{\partial x} + F_q \frac{\partial q}{\partial x} \\ 0 &= F_x + pF_u + p_x F_p + p_y F_q. \end{aligned} \tag{7.43}$$

This can be rewritten as a quasilinear equation for $p(x, y)$:

$$F_p p_x + F_q p_x = -F_x - pF_u.$$

The characteristic equations are

$$\frac{dx}{F_p} = \frac{dy}{F_q} = -\frac{dp}{F_x + pF_u}.$$

Similarly, from $\frac{dF}{dy} = 0$ we have that

$$\frac{dx}{F_p} = \frac{dy}{F_q} = -\frac{dq}{F_y + qF_u}.$$

Furthermore, since $u = u(x, y)$,

$$\begin{aligned} du &= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \\ &= pdx + qdy \\ &= pdx + q\frac{F_q}{F_p} dx \\ &= \left(p + q\frac{F_q}{F_p} \right). \end{aligned} \tag{7.44}$$

Therefore,

$$\frac{dx}{F_p} = \frac{du}{pF_p + qF_q}.$$

Combining these results we have the Charpit Equations

$$\boxed{\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q} = -\frac{dp}{F_x + pF_u} = -\frac{dq}{F_y + qF_u}.} \tag{7.45}$$

The Charpit equations. These were named after the French mathematician Paul Charpit Villecourt, who was probably the first to present the method in his thesis the year of his death, 1784. His work was further extended in 1797 by Lagrange and given a geometric explanation by Gaspard Monge (1746-1818) in 1808. This method is often called the Lagrange-Charpit method.

These equations can be used to find solutions of nonlinear first order partial differential equations as seen in the following examples.

Example 7.15. Find the general solutio of $u_x^2 + yu_y - u = 0$.

First, we introduce $u_x = p$ and $u_y = q$. Then,

$$F(x, y, u, p, q) = p^2 + qy - u = 0.$$

Next we identify

$$F_p = 2p, \quad F_q = y, \quad F_u = -1, \quad F_x = 0, \quad , F_y = q.$$

Then,

$$\begin{aligned} pF_p + qF_q &= 2p^2 + qy, \\ F_x + pF_u &= -p, \\ F_y + qF_u &= q - q = 0. \end{aligned}$$

The Charpit equations are then

$$\frac{dx}{2p} = \frac{dy}{y} = \frac{du}{2p^2 + qy} = \frac{dp}{p} = \frac{dq}{0}.$$

The first conclusion is that $q = c_1 =$ constant. So, from the partial differential equation we have $u = p^2 + c_1 y$.

Since $du = pdx + qdy = pdx + c_1 dy$, then

$$du - cdy = \sqrt{u - c_1 y}\, dx.$$

Therefore,

$$\begin{aligned} \int \frac{d(u - c_1 y)}{\sqrt{u - c_1 y}} &= \int dx \\ \int \frac{z}{\sqrt{z}} &= x + c_2 \\ 2\sqrt{u - c_1 y} &= x + c_2. \end{aligned} \tag{7.46}$$

Solving for u, we have

$$u(x, y) = \frac{1}{4}(x + c_2)^2 + c_1 y.$$

This example required a few tricks to implement the solution. Sometimes one needs to find parametric solutions. Also, if an initial condition is given, one needs to find the particular solution. In the next example we show how parametric solutions are found to the initial value problem.

Example 7.16. Solve the initial value problem $u_x^2 + u_y + u = 0$, $u(x, 0) = x$.

We consider the parametric form of the Charpit equations,

$$dt = \frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q} = -\frac{dp}{F_x + pF_u} = -\frac{dq}{F_y + qF_u}. \tag{7.47}$$

This leads to the system of equations

$$\begin{aligned} \frac{dx}{dt} &= F_p = 2p. \\ \frac{dy}{dt} &= F_q = 1. \\ \frac{du}{dt} &= pF_p + qF_q = 2p^2 + q. \\ \frac{dp}{dt} &= -(F_x + pF_u) = -p. \\ \frac{dq}{dt} &= -(F_y + qF_u) = -q. \end{aligned}$$

The second, fourth, and fifth equations can be solved to obtain

$$y = t + c_1.$$
$$p = c_2 e^{-t}.$$
$$q = c_3 e^{-t}.$$

Inserting these results into the remaining equations, we have

$$\begin{aligned} \frac{dx}{dt} &= 2c_2 e^{-t}. \\ \frac{du}{dt} &= 2c_2^2 e^{-2t} + c_3 e^{-t}. \end{aligned}$$

These equations can be integrated to find Inserting these results into the remaining equations, we have

$$\begin{aligned} x &= -2c_2 e^{-t} + c_4. \\ u &= -c_2^2 e^{-2t} - c_3 e^{-t} + c_5. \end{aligned}$$

This is a parametric set of equations for $u(x,t)$. Since

$$e^{-t} = \frac{x - c_4}{-2c_2},$$

we have

$$\begin{aligned} u(x,y) &= -c_2^2 e^{-2t} - c_3 e^{-t} + c_5. \\ &= -c_2^2 \left(\frac{x - c_4}{-2c_2} \right)^2 - c_3 \left(\frac{x - c_4}{-2c_2} \right) + c_5 \\ &= \frac{1}{4}(x - c_4)^2 + \frac{c_3}{2c_2}(x - c_4). \end{aligned} \tag{7.48}$$

We can use the initial conditions by first parametrizing the conditions. Let $x(s,0) = s$ and $y(s,0) = 0$, Then, $u(s,0) = s$. Since $u(x,0) = x$, $u_x(x,0) = 1$, or $p(s,0) = 1$.

From the partial differential equation, we have $p^2 + q + u = 0$. Therefore,

$$q(s,0) = -p^2(s,0) - u(s,0) = -(1+s).$$

These relations imply that

$$y(s,t)|_{t-0} = 0 \Rightarrow c_1 = 0.$$
$$p(s,t)|_{t-0} = 1 \Rightarrow c_2 = 1.$$
$$q(s,t)|_{t-0} = -(1+s) = c_3.$$

So,

$$y(s,t) = t.$$
$$p(s,t) = e^{-t}.$$
$$q(s,t) = -(1+s)e^{-t}.$$

The conditions on x and u give

$$x(s,t) = (s+2) - 2e^{-t},$$
$$u(s,t) = (s+1)e^{-t} - e^{-2t}.$$

7.6 Modern Nonlinear PDEs

The study of nonlinear partial differential equations is a hot research topic. We will (eventually) describe some examples of important evolution equations and discuss their solutions in the last chapter.

Problems

1. Write the following equations in conservation law form, $u_t + \phi_x = 0$ by finding the flux function $\phi(u)$.

   a. $u_t + cu_x = 0$.

   b. $u_t + uu_x - \mu u_{xx} = 0$.

   c. $u_t + 6uu_x + u_{xxx} = 0$.

   d. $u_t + u^2 u_x + u_{xxx} = 0$.

2. Consider the Klein-Gordon equation, $u_{tt} - au_{xx} = bu$ for a and b constants. Find traveling wave solutions $u(x,t) = f(x-ct)$.

3. Find the general solution $u(x,y)$ to the following problems.

   a. $u_x = 0$.

   b. $yu_x - xu_y = 0$.

   c. $2u_x + 3u_y = 1$.

   d. $u_x + u_y = u$.

4. Solve the following problems.

   a. $u_x + 2u_y = 0$, $u(x,0) = \sin x$.

   b. $u_t + 4u_x = 0$, $u(x,0) = \frac{1}{1+x^2}$.

   c. $yu_x - xu_y = 0$, $u(x,0) = x$.

   d. $u_t + xtu_x = 0$, $u(x,0) = \sin x$.

   e. $yu_x + xu_y = 0$, $u(0,y) = e^{-y^2}$.

   f. $xu_t - 2xtu_x = 2tu$, $u(x,0) = x^2$.

   g. $(y-u)u_x + (u-x)u_y = x - y$, $u = 0$ on $xy = 1$.

   h. $yu_x + xu_y = xy$, $x, y > 0$, for $u(x,0) = e^{-x^2}, x > 0$ and $u(0,y) = e^{-y^2}, y > 0$.

5. Consider the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition $u(x,0) = \frac{1}{1+x^2}$.

   a. Find and plot the characteristics.

   b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.

   c. Analytically determine the breaking time.

d. Plot solutions $u(x,t)$ at times before and after the breaking time.

6. Consider the problem $u_t + u^2 u_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition $u(x,0) = \frac{1}{1+x^2}$.

a. Find and plot the characteristics.

b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.

c. Analytically determine the breaking time.

d. Plot solutions $u(x,t)$ at times before and after the breaking time.

7. Consider the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 2, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

a. Find and plot the characteristics.

b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.

c. Analytically determine the breaking time.

d. Find the shock wave solution.

8. Consider the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 1, & x \leq 0, \\ 2, & x > 0. \end{cases}$$

a. Find and plot the characteristics.

b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.

c. Analytically determine the breaking time.

d. Find the shock wave solution.

9. Consider the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 0, & x \leq -1, \\ 2, & |x| < 1, \\ 1, & x > 1. \end{cases}$$

a. Find and plot the characteristics.

b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.

c. Analytically determine the breaking time.

d. Find the shock wave solution.

10. Solve the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 1, & x \le 0, \\ 1 - \frac{x}{a}, & 0 < x < a, \\ 0, & x \ge a. \end{cases}$$

11. Solve the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 0, & x \le 0, \\ \frac{x}{a}, & 0 < x < a, \\ 1, & x \ge a. \end{cases}$$

12. Consider the problem $u_t + u^2 u_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 2, & x \le 0, \\ 1, & x > 0. \end{cases}$$

a. Find and plot the characteristics.

b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.

c. Analytically determine the breaking time.

d. Find the shock wave solution.

13. Consider the problem $u_t + u^2 u_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 1, & x \le 0, \\ 2, & x > 0. \end{cases}$$

a. Find and plot the characteristics.

b. Find and plot the fan characteristics.

c. Write out the rarefaction wave solution for all regions of the xt-plane.

14. Solve the initial-value problem $u_t + uu_x = 0$ $|x| < \infty$, $t > 0$ satisfying

$$u(x,0) = \begin{cases} 1, & x \le 0, \\ 1 - x, & 0 \le x \le 1, \\ 0, & x \ge 1. \end{cases}$$

15. Consider the stopped traffic problem in a situation where the maximum car density is 200 cars per mile and the maximum speed is 50 miles per hour. Assume that the cars are arriving at 30 miles per hour. Find the solution of this problem and determine the rate at which the traffic is backing up. How does the answer change if the cars were arriving at 15 miles per hour.

16. Solve the following nonlinear equations where $p = u_x$ and $q = u_y$.

a. $p^2 + q^2 = 1$, $u(x, x) = x$.

b. $pq = u$, $u(0, y) = y^2$.

c. $p + q = pq$, $u(x, 0) = x$.

d. $pq = u^2$

e. $p^2 + qy = u$.

17. Find the solution of $xp + qy - p^2q - u = 0$ in parametric form for the initial conditions at $t = 0$:

$$x(t, s) = s, \quad y(t, s) = 2, \quad u(t, s) = s + 1$$

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A
Ordinary Differential Equations Review

"The profound study of nature is the most fertile source of mathematical discoveries." - Joseph Fourier (1768-1830)

A.1 First Order Differential Equations

BEFORE MOVING ON, WE FIRST DEFINE an n-th order ordinary differential equation. It is an equation for an unknown function $y(x)$ that expresses a relationship between the unknown function and its first n derivatives. One could write this generally as

n-th order ordinary differential equation

$$F(y^{(n)}(x), y^{(n-1)}(x), \ldots, y'(x), y(x), x) = 0. \tag{A.1}$$

Here $y^{(n)}(x)$ represents the nth derivative of $y(x)$.

An initial value problem consists of the differential equation plus the values of the first $n-1$ derivatives at a particular value of the independent variable, say x_0:

Initial value problem.

$$y^{(n-1)}(x_0) = y_{n-1}, \quad y^{(n-2)}(x_0) = y_{n-2}, \quad \ldots, \quad y(x_0) = y_0. \tag{A.2}$$

A linear nth order differential equation takes the form

Linear nth order differential equation

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \ldots + a_1(x)y'(x) + a_0(x)y(x)) = f(x). \tag{A.3}$$

If $f(x) \equiv 0$, then the equation is said to be homogeneous, otherwise it is called nonhomogeneous.

Homogeneous and nonhomogeneous equations.

Typically, the first differential equations encountered are first order equations. A first order differential equation takes the form

First order differential equation

$$F(y', y, x) = 0. \tag{A.4}$$

There are two common first order differential equations for which one can formally obtain a solution. The first is the separable case and the second is a first order equation. We indicate that we can formally obtain solutions, as one can display the needed integration that leads to a solution. However, the resulting integrals are not always reducible to elementary functions nor does one obtain explicit solutions when the integrals are doable.

A.1.1 Separable Equations

A first order equation is separable if it can be written the form

$$\frac{dy}{dx} = f(x)g(y). \tag{A.5}$$

Special cases result when either $f(x) = 1$ or $g(y) = 1$. In the first case the equation is said to be autonomous.

Separable equations.

The general solution to equation (A.5) is obtained in terms of two integrals:

$$\int \frac{dy}{g(y)} = \int f(x)\, dx + C, \tag{A.6}$$

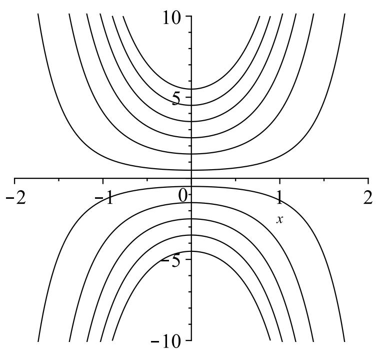

Figure A.1: Plots of solutions from the 1-parameter family of solutions of Example A.1 for several initial conditions.

where C is an integration constant. This yields a 1-parameter family of solutions to the differential equation corresponding to different values of C. If one can solve (A.6) for $y(x)$, then one obtains an explicit solution. Otherwise, one has a family of implicit solutions. If an initial condition is given as well, then one might be able to find a member of the family that satisfies this condition, which is often called a particular solution.

Example A.1. $y' = 2xy$, $y(0) = 2$.

Applying (A.6), one has

$$\int \frac{dy}{y} = \int 2x\, dx + C.$$

Integrating yields

$$\ln |y| = x^2 + C.$$

Exponentiating, one obtains the general solution,

$$y(x) = \pm e^{x^2+C} = Ae^{x^2}.$$

Here we have defined $A = \pm e^C$. Since C is an arbitrary constant, A is an arbitrary constant. Several solutions in this 1-parameter family are shown in Figure A.1.

Next, one seeks a particular solution satisfying the initial condition. For $y(0) = 2$, one finds that $A = 2$. So, the particular solution satisfying the initial condition is $y(x) = 2e^{x^2}$.

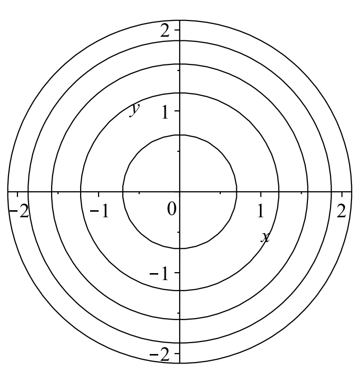

Figure A.2: Plots of solutions of Example A.2 for several initial conditions.

Example A.2. $yy' = -x$. Following the same procedure as in the last example, one obtains:

$$\int y\, dy = -\int x\, dx + C \Rightarrow y^2 = -x^2 + A, \quad \text{where} \quad A = 2C.$$

Thus, we obtain an implicit solution. Writing the solution as $x^2 + y^2 = A$, we see that this is a family of circles for $A > 0$ and the origin for $A = 0$. Plots of some solutions in this family are shown in Figure A.2.

A.1.2 Linear First Order Equations

THE SECOND TYPE OF FIRST ORDER EQUATION encountered is the linear first order differential equation in the standard form

$$y'(x) + p(x)y(x) = q(x). \tag{A.7}$$

In this case one seeks an integrating factor, $\mu(x)$, which is a function that one can multiply through the equation making the left side a perfect derivative. Thus, obtaining,

$$\frac{d}{dx}[\mu(x)y(x)] = \mu(x)q(x). \tag{A.8}$$

The integrating factor that works is $\mu(x) = \exp(\int^x p(\xi)\,d\xi)$. One can derive $\mu(x)$ by expanding the derivative in Equation (A.8),

$$\mu(x)y'(x) + \mu'(x)y(x) = \mu(x)q(x), \tag{A.9}$$

and comparing this equation to the one obtained from multiplying (A.7) by $\mu(x)$:

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \mu(x)q(x). \tag{A.10}$$

Note that these last two equations would be the same if the second terms were the same. Thus, we will require that

$$\frac{d\mu(x)}{dx} = \mu(x)p(x).$$

This is a separable first order equation for $\mu(x)$ whose solution is the integrating factor:

Integrating factor.

$$\mu(x) = \exp\left(\int^x p(\xi)\,d\xi\right). \tag{A.11}$$

Equation (A.8) is now easily integrated to obtain the general solution to the linear first order differential equation:

$$y(x) = \frac{1}{\mu(x)}\left[\int^x \mu(\xi)q(\xi)\,d\xi + C\right]. \tag{A.12}$$

Example A.3. $xy' + y = x, \quad x > 0,\ y(1) = 0.$

One first notes that this is a linear first order differential equation. Solving for y', one can see that the equation is not separable. Furthermore, it is not in the standard form (A.7). So, we first rewrite the equation as

$$\frac{dy}{dx} + \frac{1}{x}y = 1. \tag{A.13}$$

Noting that $p(x) = \frac{1}{x}$, we determine the integrating factor

$$\mu(x) = \exp\left[\int^x \frac{d\xi}{\xi}\right] = e^{\ln x} = x.$$

Multiplying equation (A.13) by $\mu(x) = x$, we actually get back the original equation! In this case we have found that $xy' + y$ must have been the derivative of something to start. In fact, $(xy)' = xy' + x$. Therefore, the differential equation becomes

$$(xy)' = x.$$

Integrating, one obtains

$$xy = \frac{1}{2}x^2 + C,$$

or

$$y(x) = \frac{1}{2}x + \frac{C}{x}.$$

Inserting the initial condition into this solution, we have $0 = \frac{1}{2} + C$. Therefore, $C = -\frac{1}{2}$. Thus, the solution of the initial value problem is

$$y(x) = \frac{1}{2}(x - \frac{1}{x}).$$

We can verify that this is the solution. Since $y' = \frac{1}{2} + \frac{1}{2x^2}$, we have

$$xy' + y = \frac{1}{2}x + \frac{1}{2x} + \frac{1}{2}\left(x - \frac{1}{x}\right) = x.$$

Also, $y(1) = \frac{1}{2}(1-1) = 0$.

Example A.4. $(\sin x)y' + (\cos x)y = x^2$.

Actually, this problem is easy if you realize that the left hand side is a perfect derivative. Namely,

$$\frac{d}{dx}((\sin x)y) = (\sin x)y' + (\cos x)y.$$

But, we will go through the process of finding the integrating factor for practice.

First, we rewrite the original differential equation in standard form. We divide the equation by $\sin x$ to obtain

$$y' + (\cot x)y = x^2 \csc x.$$

Then, we compute the integrating factor as

$$\mu(x) = \exp\left(\int^x \cot \xi \, d\xi\right) = e^{\ln(\sin x)} = \sin x.$$

Using the integrating factor, the standard form equation becomes

$$\frac{d}{dx}\left((\sin x)y\right) = x^2.$$

Integrating, we have

$$y \sin x = \frac{1}{3}x^3 + C.$$

So, the solution is

$$y(x) = \left(\frac{1}{3}x^3 + C\right) \csc x.$$

A.2 Second Order Linear Differential Equations

SECOND ORDER DIFFERENTIAL EQUATIONS ARE TYPICALLY HARDER than first order. In most cases students are only exposed to second order linear differential equations. A general form for a *second order linear differential equation* is given by

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{A.14}$$

One can rewrite this equation using operator terminology. Namely, one first defines the differential operator $L = a(x)D^2 + b(x)D + c(x)$, where $D = \frac{d}{dx}$. Then equation (A.14) becomes

$$Ly = f. \tag{A.15}$$

The solutions of linear differential equations are found by making use of the linearity of L. Namely, we consider the *vector space*[1] consisting of real-valued functions over some domain. Let f and g be vectors in this function space. L is a *linear operator* if for two vectors f and g and scalar a, we have that

a. $L(f + g) = Lf + Lg$

b. $L(af) = aLf$.

[1] We assume that the reader has been introduced to concepts in linear algebra. Later in the text we will recall the definition of a vector space and see that linear algebra is in the background of the study of many concepts in the solution of differential equations.

One typically solves (A.14) by finding the general solution of the homogeneous problem,

$$Ly_h = 0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p = f.$$

Then, the general solution of (A.14) is simply given as $y = y_h + y_p$. This is true because of the linearity of L. Namely,

$$\begin{aligned} Ly &= L(y_h + y_p) \\ &= Ly_h + Ly_p \\ &= 0 + f = f. \end{aligned} \tag{A.16}$$

There are methods for finding a particular solution of a nonhomogeneous differential equation. These methods range from pure guessing, the Method of Undetermined Coefficients, the Method of Variation of Parameters, or Green's functions. We will review these methods later in the chapter.

Determining solutions to the homogeneous problem, $Ly_h = 0$, is not always easy. However, many now famous mathematicians and physicists have studied a variety of second order linear equations and they have saved us the trouble of finding solutions to the differential equations that often appear in applications. We will encounter many of these in the following chapters. We will first begin with some simple homogeneous linear differential equations.

Linearity is also useful in producing the general solution of a homogeneous linear differential equation. If y_1 and y_2 are solutions of the homogeneous equation, then the *linear combination* $y = c_1y_1 + c_2y_2$ is also a solution of the homogeneous equation. In fact, if y_1 and y_2 are *linearly independent*,[2] then $y = c_1y_1 + c_2y_2$ is the general solution of the homogeneous problem.

[2] A set of functions $\{y_i(x)\}_{i=1}^n$ is a linearly independent set if and only if

$$c_1y_1(x) + \ldots + c_ny_n(x) = 0$$

implies $c_i = 0$, for $i = 1, \ldots, n$.

For $n = 2$, $c_1y_1(x) + c_2y_2(x) = 0$. If y_1 and y_2 are linearly dependent, then the coefficients are not zero and $y_2(x) = -\frac{c_1}{c_2}y_1(x)$ and is a multiple of $y_1(x)$.

Linear independence can also be established by looking at the Wronskian of the solutions. For a second order differential equation the Wronskian is defined as

$$W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x). \tag{A.17}$$

The solutions are linearly independent if the Wronskian is not zero.

A.2.1 Constant Coefficient Equations

The simplest second order differential equations are those with constant coefficients. The general form for a homogeneous constant coefficient second order linear differential equation is given as

$$ay''(x) + by'(x) + cy(x) = 0, \tag{A.18}$$

where a, b, and c are constants.

Solutions to (A.18) are obtained by making a guess of $y(x) = e^{rx}$. Inserting this guess into (A.18) leads to the characteristic equation

$$ar^2 + br + c = 0. \tag{A.19}$$

The characteristic equation for $ay'' + by' + cy = 0$ is $ar^2 + br + c = 0$. Solutions of this quadratic equation lead to solutions of the differential equation.

Namely, we compute the derivatives of $y(x) = e^{rx}$, to get $y(x) = re^{rx}$, and $y(x) = r^2e^{rx}$. Inserting into (A.18), we have

$$0 = ay''(x) + by'(x) + cy(x) = (ar^2 + br + c)e^{rx}.$$

Since the exponential is never zero, we find that $ar^2 + br + c = 0$.

Two real, distinct roots, r_1 and r_2, give solutions of the form

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x}.$$

The roots of this equation, r_1, r_2, in turn lead to three types of solutions depending upon the nature of the roots. In general, we have two linearly independent solutions, $y_1(x) = e^{r_1x}$ and $y_2(x) = e^{r_2x}$, and the general solution is given by a linear combination of these solutions,

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x}.$$

For two real distinct roots, we are done. However, when the roots are real, but equal, or complex conjugate roots, we need to do a little more work to obtain usable solutions.

Example A.5. $y'' - y' - 6y = 0$ $y(0) = 2, y'(0) = 0$.

The characteristic equation for this problem is $r^2 - r - 6 = 0$. The roots of this equation are found as $r = -2, 3$. Therefore, the general solution can be quickly written down:

$$y(x) = c_1e^{-2x} + c_2e^{3x}.$$

Note that there are two arbitrary constants in the general solution. Therefore, one needs two pieces of information to find a particular solution. Of course, we have the needed information in the form of the initial conditions.

One also needs to evaluate the first derivative

$$y'(x) = -2c_1e^{-2x} + 3c_2e^{3x}$$

in order to attempt to satisfy the initial conditions. Evaluating y and y' at $x = 0$ yields

$$\begin{aligned} 2 &= c_1 + c_2 \\ 0 &= -2c_1 + 3c_2 \end{aligned} \tag{A.20}$$

These two equations in two unknowns can readily be solved to give $c_1 = 6/5$ and $c_2 = 4/5$. Therefore, the solution of the initial value problem is obtained as $y(x) = \frac{6}{5}e^{-2x} + \frac{4}{5}e^{3x}$.

Repeated roots, $r_1 = r_2 = r$, give solutions of the form

$$y(x) = (c_1 + c_2x)e^{rx}.$$

In the case when there is a repeated real root, one has only one solution, $y_1(x) = e^{rx}$. The question is how does one obtain the second linearly independent solution? Since the solutions should be independent, we must have that the ratio $y_2(x)/y_1(x)$ is not a constant. So, we guess the form $y_2(x) = v(x)y_1(x) = v(x)e^{rx}$. (This process is called the Method of Reduction of Order.)

For constant coefficient second order equations, we can write the equation as

$$(D-r)^2y = 0,$$

where $D = \frac{d}{dx}$. We now insert $y_2(x) = v(x)e^{rx}$ into this equation. First we compute

$$(D-r)ve^{rx} = v'e^{rx}.$$

Then,

$$0 = (D-r)^2ve^{rx} = (D-r)v'e^{rx} = v''e^{rx}.$$

So, if $y_2(x)$ is to be a solution to the differential equation, then $v''(x)e^{rx} = 0$ for all x. So, $v''(x) = 0$, which implies that

$$v(x) = ax + b.$$

So,

$$y_2(x) = (ax + b)e^{rx}.$$

Without loss of generality, we can take $b = 0$ and $a = 1$ to obtain the second linearly independent solution, $y_2(x) = xe^{rx}$. The general solution is then

$$y(x) = c_1e^{rx} + c_2xe^{rx}.$$

Example A.6. $y'' + 6y' + 9y = 0$.

In this example we have $r^2 + 6r + 9 = 0$. There is only one root, $r = -3$. From the above discussion, we easily find the solution $y(x) = (c_1 + c_2x)e^{-3x}$.

When one has complex roots in the solution of constant coefficient equations, one needs to look at the solutions

$$y_{1,2}(x) = e^{(\alpha \pm i\beta)x}.$$

We make use of Euler's formula (See Chapter 6 for more on complex variables)

$$e^{i\beta x} = \cos\beta x + i\sin\beta x. \tag{A.21}$$

Then, the linear combination of $y_1(x)$ and $y_2(x)$ becomes

$$\begin{aligned} Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} &= e^{\alpha x}\left[Ae^{i\beta x} + Be^{-i\beta x}\right] \\ &= e^{\alpha x}\left[(A+B)\cos\beta x + i(A-B)\sin\beta x\right] \\ &\equiv e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x). \end{aligned} \tag{A.22}$$

Thus, we see that we have a linear combination of two real, linearly independent solutions, $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$.

Complex roots, $r = \alpha \pm i\beta$, give solutions of the form

$$y(x) = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x).$$

Example A.7. $y'' + 4y = 0$.

The characteristic equation in this case is $r^2 + 4 = 0$. The roots are pure imaginary roots, $r = \pm 2i$, and the general solution consists purely of sinusoidal functions, $y(x) = c_1\cos(2x) + c_2\sin(2x)$, since $\alpha = 0$ and $\beta = 2$.

Example A.8. $y'' + 2y' + 4y = 0$.

The characteristic equation in this case is $r^2 + 2r + 4 = 0$. The roots are complex, $r = -1 \pm \sqrt{3}i$ and the general solution can be written as

$$y(x) = \left[c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x)\right]e^{-x}.$$

Example A.9. $y'' + 4y = \sin x$.

This is an example of a nonhomogeneous problem. The homogeneous problem was actually solved in Example A.7. According to the theory, we need only seek a particular solution to the nonhomogeneous problem and add it to the solution of the last example to get the general solution.

The particular solution can be obtained by purely guessing, making an educated guess, or using the Method of Variation of Parameters. We will not review all of these techniques at this time. Due to the simple form of the driving term, we will make an intelligent guess of $y_p(x) = A\sin x$ and determine what A needs to be. Inserting this guess into the differential equation gives $(-A + 4A)\sin x = \sin x$. So, we see that $A = 1/3$ works. The general solution of the nonhomogeneous problem is therefore $y(x) = c_1\cos(2x) + c_2\sin(2x) + \frac{1}{3}\sin x$.

The three cases for constant coefficient linear second order differential equations are summarized below.

Classification of Roots of the Characteristic Equation for Second Order Constant Coefficient ODEs

1. **Real, distinct roots** r_1, r_2. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.

2. **Real, equal roots** $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the *Method of Reduction of Order*. This gives the second solution as xe^{rx}. Therefore, the general solution is found as $y(x) = (c_1 + c_2 x)e^{rx}$.

3. **Complex conjugate roots** $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. Making use of Euler's identity, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, these complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$.

A.3 Forced Systems

Many problems can be modeled by nonhomogeneous second order equations. Thus, we want to find solutions of equations of the form

$$Ly(x) = a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{A.23}$$

As noted in Section A.2, one solves this equation by finding the general solution of the homogeneous problem,

$$Ly_h = 0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p = f.$$

Then, the general solution of (A.14) is simply given as $y = y_h + y_p$.

So far, we only know how to solve constant coefficient, homogeneous equations. So, by adding a nonhomogeneous term to such equations we will need to find the particular solution to the nonhomogeneous equation.

We could guess a solution, but that is not usually possible without a little bit of experience. So, we need some other methods. There are two main methods. In the first case, the Method of Undetermined Coefficients, one makes an intelligent guess based on the form of $f(x)$. In the second method, one can systematically developed the particular solution. We will come back to the Method of Variation of Parameters and we will also introduce the powerful machinery of Green's functions later in this section.

A.3.1 Method of Undetermined Coefficients

Let's solve a simple differential equation highlighting how we can handle nonhomogeneous equations.

Example A.10. Consider the equation

$$y'' + 2y' - 3y = 4. \tag{A.24}$$

The first step is to determine the solution of the homogeneous equation. Thus, we solve

$$y_h'' + 2y_h' - 3y_h = 0. \tag{A.25}$$

The characteristic equation is $r^2 + 2r - 3 = 0$. The roots are $r = 1, -3$. So, we can immediately write the solution

$$y_h(x) = c_1 e^x + c_2 e^{-3x}.$$

The second step is to find a particular solution of (A.24). What possible function can we insert into this equation such that only a 4 remains? If we try something proportional to x, then we are left with a linear function after inserting x and its derivatives. Perhaps a constant function you might think. $y = 4$ does not work. But, we could try an arbitrary constant, $y = A$.

Let's see. Inserting $y = A$ into (A.24), we obtain

$$-3A = 4.$$

Ah ha! We see that we can choose $A = -\frac{4}{3}$ and this works. So, we have a particular solution, $y_p(x) = -\frac{4}{3}$. This step is done.

Combining the two solutions, we have the general solution to the original nonhomogeneous equation (A.24). Namely,

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}.$$

Insert this solution into the equation and verify that it is indeed a solution. If we had been given initial conditions, we could now use them to determine the arbitrary constants.

Example A.11. What if we had a different source term? Consider the equation

$$y'' + 2y' - 3y = 4x. \tag{A.26}$$

The only thing that would change is the particular solution. So, we need a guess.

We know a constant function does not work by the last example. So, let's try $y_p = Ax$. Inserting this function into Equation (A.26), we obtain

$$2A - 3Ax = 4x.$$

Picking $A = -4/3$ would get rid of the x terms, but will not cancel everything. We still have a constant left. So, we need something more general.

Let's try a linear function, $y_p(x) = Ax + B$. Then we get after substitution into (A.26)

$$2A - 3(Ax + B) = 4x.$$

Equating the coefficients of the different powers of x on both sides, we find a system of equations for the undetermined coefficients:

$$\begin{aligned} 2A - 3B &= 0 \\ -3A &= 4. \end{aligned} \tag{A.27}$$

These are easily solved to obtain

$$\begin{aligned} A &= -\frac{4}{3} \\ B &= \frac{2}{3}A = -\frac{8}{9}. \end{aligned} \tag{A.28}$$

So, the particular solution is

$$y_p(x) = -\frac{4}{3}x - \frac{8}{9}.$$

This gives the general solution to the nonhomogeneous problem as

$$y(x) = y_h(x) + y_p(x) = c_1e^x + c_2e^{-3x} - \frac{4}{3}x - \frac{8}{9}.$$

There are general forms that you can guess based upon the form of the driving term, $f(x)$. Some examples are given in Table A.1. More general applications are covered in a standard text on differential equations. However, the procedure is simple. Given $f(x)$ in a particular form, you make an appropriate guess up to some unknown parameters, or coefficients. Inserting the guess leads to a system of equations for the unknown coefficients. Solve the system and you have the solution. This solution is then added to the general solution of the homogeneous differential equation.

$f(x)$	Guess
$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$	$A_nx^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0$
ae^{bx}	Ae^{bx}
$a\cos\omega x + b\sin\omega x$	$A\cos\omega x + B\sin\omega x$

Table A.1: Forms used in the Method of Undetermined Coefficients.

Example A.12. Solve

$$y'' + 2y' - 3y = 2e^{-3x}. \tag{A.29}$$

According to the above, we would guess a solution of the form $y_p = Ae^{-3x}$. Inserting our guess, we find

$$0 = 2e^{-3x}.$$

Oops! The coefficient, A, disappeared! We cannot solve for it. What went wrong?

The answer lies in the general solution of the homogeneous problem. Note that e^x and e^{-3x} are solutions to the homogeneous problem. So, a multiple of e^{-3x} will not get us anywhere. It turns out that there is one further modification of the method. If the driving term contains terms that are solutions of the homogeneous problem, then we need to make a guess consisting of the smallest possible power of x times the function which is no longer a solution of the homogeneous problem. Namely, we guess $y_p(x) = Axe^{-3x}$ and differentiate this guess to obtain the derivatives $y_p' = A(1-3x)e^{-3x}$ and $y_p'' = A(9x-6)e^{-3x}$.

Inserting these derivatives into the differential equation, we obtain

$$[(9x-6) + 2(1-3x) - 3x]Ae^{-3x} = 2e^{-3x}.$$

Comparing coefficients, we have

$$-4A = 2.$$

So, $A = -1/2$ and $y_p(x) = -\frac{1}{2}xe^{-3x}$. Thus, the solution to the problem is

$$y(x) = \left(2 - \frac{1}{2}x\right)e^{-3x}.$$

Modified Method of Undetermined Coefficients

In general, if any term in the guess $y_p(x)$ is a solution of the homogeneous equation, then multiply the guess by x^k, where k is the smallest positive integer such that no term in $x^k y_p(x)$ is a solution of the homogeneous problem.

A.3.2 Periodically Forced Oscillations

A SPECIAL TYPE OF FORCING is periodic forcing. Realistic oscillations will dampen and eventually stop if left unattended. For example, mechanical clocks are driven by compound or torsional pendula and electric oscillators are often designed with the need to continue for long periods of time. However, they are not perpetual motion machines and will need a periodic injection of energy. This can be done systematically by adding periodic forcing. Another simple example is the motion of a child on a swing in the park. This simple damped pendulum system will naturally slow down to equilibrium (stopped) if left alone. However, if the child pumps energy into the swing at the right time, or if an adult pushes the child at the right time, then the amplitude of the swing can be increased.

There are other systems, such as airplane wings and long bridge spans, in which external driving forces might cause damage to the system. A well know example is the wind induced collapse of the Tacoma Narrows Bridge due to strong winds. Of course, if one is not careful, the child in the

The Tacoma Narrows Bridge opened in Washington State (U.S.) in mid 1940. However, in November of the same year the winds excited a transverse mode of vibration, which eventually (in a few hours) lead to large amplitude oscillations and then collapse.

last example might get too much energy pumped into the system causing a similar failure of the desired motion.

While there are many types of forced systems, and some fairly complicated, we can easily get to the basic characteristics of forced oscillations by modifying the mass-spring system by adding an external, time-dependent, driving force. Such as system satisfies the equation

$$m\ddot{x} + b\dot{(x)} + kx = F(t), \quad \text{(A.30)}$$

where m is the mass, b is the damping constant, k is the spring constant, and $F(t)$ is the driving force. If $F(t)$ is of simple form, then we can employ the Method of Undetermined Coefficients. Since the systems we have considered so far are similar, one could easily apply the following to pendula or circuits.

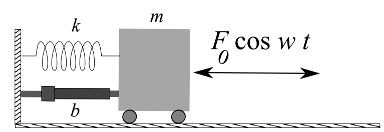

Figure A.3: An external driving force is added to the spring-mass-damper system.

As the damping term only complicates the solution, we will consider the simpler case of undamped motion and assume that $b = 0$. Furthermore, we will introduce a sinusoidal driving force, $F(t) = F_0 \cos \omega t$ in order to study periodic forcing. This leads to the simple periodically driven mass on a spring system

$$m\ddot{x} + kx = F_0 \cos \omega t. \quad \text{(A.31)}$$

In order to find the general solution, we first obtain the solution to the homogeneous problem,

$$x_h = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t,$$

where $\omega_0 = \sqrt{\frac{k}{m}}$. Next, we seek a particular solution to the nonhomogeneous problem. We will apply the Method of Undetermined Coefficients.

A natural guess for the particular solution would be to use $x_p = A \cos \omega t + B\ sin \omega t$. However, recall that the guess should not be a solution of the homogeneous problem. Comparing x_p with x_h, this would hold if $\omega \neq \omega_0$. Otherwise, one would need to use the Modified Method of Undetermined Coefficients as described in the last section. So, we have two cases to consider.

Dividing through by the mass, we solve the simple driven system,

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t.$$

Example A.13. Solve $\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$, for $\omega \neq \omega_0$.

In this case we continue with the guess $x_p = A \cos \omega t + B\ sin \omega t$. Since there is no damping term, one quickly finds that $B = 0$. Inserting $x_p = A \cos \omega t$ into the differential equation, we find that

$$\left(-\omega^2 + \omega_0^2\right) A \cos \omega t = \frac{F_0}{m} \cos \omega t.$$

Solving for A, we obtain

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)}.$$

The general solution for this case is thus,

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t. \quad \text{(A.32)}$$

Example A.14. Solve $\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega_0 t$.

In this case, we need to employ the Modified Method of Undetermined Coefficients. So, we make the guess $x_p = t\,(A \cos \omega_0 t + B\ sin \omega_0 t)$. Since there is no damping term, one finds that $A = 0$. Inserting the guess in to the differential equation, we find that

$$B = \frac{F_0}{2m\omega_0},$$

or the general solution is

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega} t \sin \omega t. \qquad \text{(A.33)}$$

The general solution to the problem is thus

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \begin{cases} \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t, & \omega \neq \omega_0, \\ \frac{F_0}{2m\omega_0} t \sin \omega_0 t, & \omega = \omega_0. \end{cases} \qquad \text{(A.34)}$$

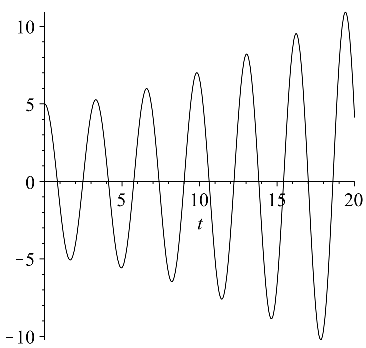

Figure A.4: Plot of

$$x(t) = 5 \cos 2t + \frac{1}{2} t \sin 2t,$$

Special cases of these solutions provide interesting physics, which can be explored by the reader in the homework. In the case that $\omega = \omega_0$, we see that the solution tends to grow as t gets large. This is what is called a resonance. Essentially, one is driving the system at its natural frequency. As the system is moving to the left, one pushes it to the left. If it is moving to the right, one is adding energy in that direction. This forces the amplitude of oscillation to continue to grow until the system breaks. An example of such an oscillation is shown in Figure A.4.

In the case that $\omega \neq \omega_0$, one can rewrite the solution in a simple form. Let's choose the initial conditions that $c_1 = -F_0/(m(\omega_0^2 - \omega^2))$, $c_2 = 0$. Then one has

$$x(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2}. \qquad \text{(A.35)}$$

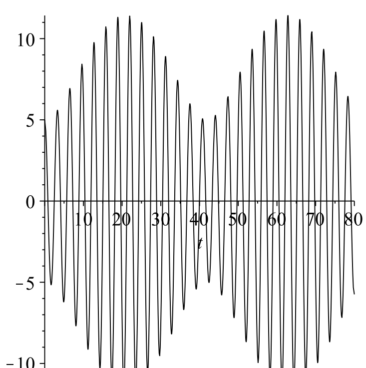

Figure A.5: Plot of

$$x(t) = \frac{1}{249}\left(2045 \cos 2t - 800 \cos \frac{43}{20} t\right),$$

a solution of $\ddot{x} + 4x = 2 \cos 2.15t$.

For values of ω near ω_0, one finds the solution consists of a rapid oscillation, due to the $\sin \frac{(\omega_0+\omega)t}{2}$ factor, with a slowly varying amplitude, $\frac{2F_0}{m(\omega_0^2-\omega^2)} \sin \frac{(\omega_0-\omega)t}{2}$. The reader can investigate this solution.

This slow variation is called a beat and the beat frequency is given by $f = \frac{|\omega_0 - \omega|}{4\pi}$. In Figure A.5 we see the high frequency oscillations are contained by the lower beat frequency, $f = \frac{0.15}{4\pi}$s. This corresponds to a period of $T = 1/f \approx 83.7$ Hz, which looks about right from the figure.

Example A.15. Solve $\ddot{x} + x = 2 \cos \omega t$, $x(0) = 0$, $\dot{x}(0) = 0$, for $\omega = 1, 1.15$. For each case, we need the solution of the homogeneous problem,

$$x_h(t) = c_1 \cos t + c_2 \sin t.$$

The particular solution depends on the value of ω.

For $\omega = 1$, the driving term, $2 \cos \omega t$, is a solution of the homogeneous problem. Thus, we assume

$$x_p(t) = At \cos t + Bt \sin t.$$

Inserting this into the differential equation, we find $A = 0$ and $B = 1$. So, the general solution is

$$x(t) = c_1 \cos t + c_2 \sin t + t \sin t.$$

Imposing the initial conditions, we find

$$x(t) = t \sin t.$$

This solution is shown in Figure A.6.

For $\omega = 1.15$, the driving term, $2\cos\omega 1.15t$, is not a solution of the homogeneous problem. Thus, we assume

$$x_p(t) = A\cos 1.15t + B\sin 1.15t.$$

Inserting this into the differential equation, we find $A = -\frac{800}{129}$ and $B = 0$. So, the general solution is

$$x(t) = c_1 \cos t + c_2 \sin t - \frac{800}{129}\cos t.$$

Imposing the initial conditions, we find

$$x(t) = \frac{800}{129}\left(\cos t - \cos 1.15t\right).$$

This solution is shown in Figure A.7. The beat frequency in this case is the same as with Figure A.5.

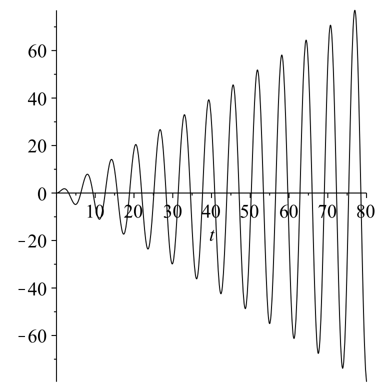

Figure A.6: Plot of

$$x(t) = t \sin 2t,$$

a solution of $\ddot{x} + x = 2\cos t$.

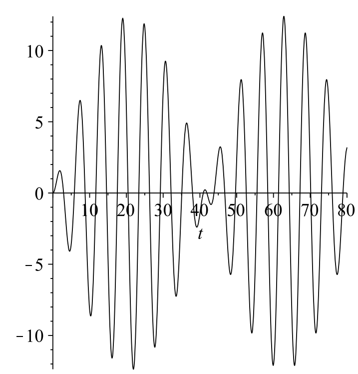

Figure A.7: Plot of

$$x(t) = \frac{800}{129}\left(\cos t - \cos\frac{23}{20}t\right),$$

a solution of $\ddot{x} + x = 2\cos 1.15t$.

A.3.3 Method of Variation of Parameters

A MORE SYSTEMATIC WAY to find particular solutions is through the use of the Method of Variation of Parameters. The derivation is a little detailed and the solution is sometimes messy, but the application of the method is straight forward if you can do the required integrals. We will first derive the needed equations and then do some examples.

We begin with the nonhomogeneous equation. Let's assume it is of the standard form

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{A.36}$$

We know that the solution of the homogeneous equation can be written in terms of two linearly independent solutions, which we will call $y_1(x)$ and $y_2(x)$:

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x).$$

Replacing the constants with functions, then we no longer have a solution to the homogeneous equation. Is it possible that we could stumble across the right functions with which to replace the constants and somehow end up with $f(x)$ when inserted into the left side of the differential equation? It turns out that we can.

So, let's assume that the constants are replaced with two unknown functions, which we will call $c_1(x)$ and $c_2(x)$. This change of the parameters is where the name of the method derives. Thus, we are assuming that a particular solution takes the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x). \tag{A.37}$$

If this is to be a solution, then insertion into the differential equation should make the equation hold. To do this we will first need to compute some derivatives.

We assume the nonhomogeneous equation has a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

The first derivative is given by

$$y_p'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x) + c_1'(x)y_1(x) + c_2'(x)y_2(x). \tag{A.38}$$

Next we will need the second derivative. But, this will yield eight terms. So, we will first make a simplifying assumption. Let's assume that the last two terms add to zero:

$$c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0. \tag{A.39}$$

It turns out that we will get the same results in the end if we did not assume this. The important thing is that it works!

Under the assumption the first derivative simplifies to

$$y_p'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x). \tag{A.40}$$

The second derivative now only has four terms:

$$y_p'(x) = c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x). \tag{A.41}$$

Now that we have the derivatives, we can insert the guess into the differential equation. Thus, we have

$$\begin{aligned} f(x) \quad = \quad & a(x)\left[c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right] \\ & + b(x)\left[c_1(x)y_1'(x) + c_2(x)y_2'(x)\right] \\ & + c(x)\left[c_1(x)y_1(x) + c_2(x)y_2(x)\right]. \end{aligned} \tag{A.42}$$

Regrouping the terms, we obtain

$$\begin{aligned} f(x) \quad = \quad & c_1(x)\left[a(x)y_1''(x) + b(x)y_1'(x) + c(x)y_1(x)\right] \\ & + c_2(x)\left[a(x)y_2''(x) + b(x)y_2'(x) + c(x)y_2(x)\right] \\ & + a(x)\left[c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right]. \end{aligned} \tag{A.43}$$

Note that the first two rows vanish since y_1 and y_2 are solutions of the homogeneous problem. This leaves the equation

$$f(x) = a(x)\left[c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right],$$

which can be rearranged as

$$c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = \frac{f(x)}{a(x)}. \tag{A.44}$$

In summary, we have assumed a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

This is only possible if the unknown functions $c_1(x)$ and $c_2(x)$ satisfy the system of equations

$$\begin{aligned} c_1'(x)y_1(x) + c_2'(x)y_2(x) &= 0 \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) &= \frac{f(x)}{a(x)}. \end{aligned} \qquad (A.45)$$

In order to solve the differential equation $Ly = f$, we assume

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x),$$

for $Ly_{1,2} = 0$. Then, one need only solve a simple system of equations (A.45). System (A.45) can be solved as

$$c_1'(x) = -\frac{fy_2}{aW(y_1, y_2)},$$

$$c_1'(x) = \frac{fy_1}{aW(y_1, y_2)},$$

where $W(y_1, y_2) = y_1y_2' - y_1'y_2$ is the Wronskian. We use this solution in the next section.

It is standard to solve this system for the derivatives of the unknown functions and then present the integrated forms. However, one could just as easily start from this system and solve the system for each problem encountered.

Example A.16. Find the general solution of the nonhomogeneous problem: $y'' - y = e^{2x}$.

The general solution to the homogeneous problem $y_h'' - y_h = 0$ is

$$y_h(x) = c_1e^x + c_2e^{-x}.$$

In order to use the Method of Variation of Parameters, we seek a solution of the form

$$y_p(x) = c_1(x)e^x + c_2(x)e^{-x}.$$

We find the unknown functions by solving the system in (A.45), which in this case becomes

$$\begin{aligned} c_1'(x)e^x + c_2'(x)e^{-x} &= 0 \\ c_1'(x)e^x - c_2'(x)e^{-x} &= e^{2x}. \end{aligned} \qquad (A.46)$$

Adding these equations we find that

$$2c_1'e^x = e^{2x} \to c_1' = \frac{1}{2}e^x.$$

Solving for $c_1(x)$ we find

$$c_1(x) = \frac{1}{2}\int e^x\, dx = \frac{1}{2}e^x.$$

Subtracting the equations in the system yields

$$2c_2'e^{-x} = -e^{2x} \to c_2' = -\frac{1}{2}e^{3x}.$$

Thus,

$$c_2(x) = -\frac{1}{2}\int e^{3x}\, dx = -\frac{1}{6}e^{3x}.$$

The particular solution is found by inserting these results into y_p:

$$\begin{aligned} y_p(x) &= c_1(x)y_1(x) + c_2(x)y_2(x) \\ &= (\frac{1}{2}e^x)e^x + (-\frac{1}{6}e^{3x})e^{-x} \\ &= \frac{1}{3}e^{2x}. \end{aligned} \qquad (A.47)$$

Thus, we have the general solution of the nonhomogeneous problem as

$$y(x) = c_1 e^x + c_2 e^{-x} + \frac{1}{3} e^{2x}.$$

Example A.17. Now consider the problem: $y'' + 4y = \sin x$.

The solution to the homogeneous problem is

$$y_h(x) = c_1 \cos 2x + c_2 \sin 2x. \tag{A.48}$$

We now seek a particular solution of the form

$$y_h(x) = c_1(x) \cos 2x + c_2(x) \sin 2x.$$

We let $y_1(x) = \cos 2x$ and $y_2(x) = \sin 2x$, $a(x) = 1$, $f(x) = \sin x$ in system (A.45):

$$\begin{aligned} c_1'(x) \cos 2x + c_2'(x) \sin 2x &= 0 \\ -2c_1'(x) \sin 2x + 2c_2'(x) \cos 2x &= \sin x. \end{aligned} \tag{A.49}$$

Now, use your favorite method for solving a system of two equations and two unknowns. In this case, we can multiply the first equation by $2 \sin 2x$ and the second equation by $\cos 2x$. Adding the resulting equations will eliminate the c_1' terms. Thus, we have

$$c_2'(x) = \frac{1}{2} \sin x \cos 2x = \frac{1}{2}(2\cos^2 x - 1) \sin x.$$

Inserting this into the first equation of the system, we have

$$c_1'(x) = -c_2'(x) \frac{\sin 2x}{\cos 2x} = -\frac{1}{2} \sin x \sin 2x = -\sin^2 x \cos x.$$

These can easily be solved:

$$c_2(x) = \frac{1}{2} \int (2\cos^2 x - 1) \sin x \, dx = \frac{1}{2}(\cos x - \frac{2}{3} \cos^3 x).$$

$$c_1(x) = -\int \sin^x \cos x \, dx = -\frac{1}{3} \sin^3 x.$$

The final step in getting the particular solution is to insert these functions into $y_p(x)$. This gives

$$\begin{aligned} y_p(x) &= c_1(x) y_1(x) + c_2(x) y_2(x) \\ &= (-\frac{1}{3} \sin^3 x) \cos 2x + (\frac{1}{2} \cos x - \frac{1}{3} \cos^3 x) \sin x \\ &= \frac{1}{3} \sin x. \end{aligned} \tag{A.50}$$

So, the general solution is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3} \sin x. \tag{A.51}$$

A.4 Cauchy-Euler Equations

Another class of solvable linear differential equations that is of interest are the Cauchy-Euler type of equations, also referred to in some books as Euler's equation. These are given by

$$ax^2y''(x) + bxy'(x) + cy(x) = 0. \tag{A.52}$$

Note that in such equations the power of x in each of the coefficients matches the order of the derivative in that term. These equations are solved in a manner similar to the constant coefficient equations.

One begins by making the guess $y(x) = x^r$. Inserting this function and its derivatives,

$$y'(x) = rx^{r-1}, \qquad y''(x) = r(r-1)x^{r-2},$$

into Equation (A.52), we have

$$[ar(r-1) + br + c]\, x^r = 0.$$

Since this has to be true for all x in the problem domain, we obtain the characteristic equation

$$ar(r-1) + br + c = 0. \tag{A.53}$$

The solutions of Cauchy-Euler equations can be found using the characteristic equation $ar(r-1) + br + c = 0$.

Just like the constant coefficient differential equation, we have a quadratic equation and the nature of the roots again leads to three classes of solutions. If there are two real, distinct roots, then the general solution takes the form $y(x) = c_1x^{r_1} + c_2x^{r_2}$.

For two real, distinct roots, the general solution takes the form

$$y(x) = c_1x^{r_1} + c_2x^{r_2}.$$

Example A.18. Find the general solution: $x^2y'' + 5xy' + 12y = 0$.

As with the constant coefficient equations, we begin by writing down the characteristic equation. Doing a simple computation,

$$\begin{aligned} 0 &= r(r-1) + 5r + 12 \\ &= r^2 + 4r + 12 \\ &= (r+2)^2 + 8, \\ -8 &= (r+2)^2, \end{aligned} \tag{A.54}$$

one determines the roots are $r = -2 \pm 2\sqrt{2}i$. Therefore, the general solution is $y(x) = \left[c_1 \cos(2\sqrt{2}\ln|x|) + c_2 \sin(2\sqrt{2}\ln|x|)\right] x^{-2}$

Deriving the solution for Case 2 for the Cauchy-Euler equations works in the same way as the second for constant coefficient equations, but it is a bit messier. First note that for the real root, $r = r_1$, the characteristic equation has to factor as $(r - r_1)^2 = 0$. Expanding, we have

$$r^2 - 2r_1r + r_1^2 = 0.$$

The general characteristic equation is

$$ar(r-1) + br + c = 0.$$

Dividing this equation by a and rewriting, we have

$$r^2 + (\frac{b}{a} - 1)r + \frac{c}{a} = 0.$$

Comparing equations, we find

$$\frac{b}{a} = 1 - 2r_1, \qquad \frac{c}{a} = r_1^2.$$

So, the Cauchy-Euler equation for this case can be written in the form

$$x^2 y'' + (1 - 2r_1)xy' + r_1^2 y = 0.$$

Now we seek the second linearly independent solution in the form $y_2(x) = v(x)x^{r_1}$. We first list this function and its derivatives,

$$\begin{aligned} y_2(x) &= vx^{r_1}, \\ y_2'(x) &= (xv' + r_1 v)x^{r_1 - 1}, \\ y_2''(x) &= (x^2 v'' + 2r_1 xv' + r_1(r_1 - 1)v)x^{r_1 - 2}. \end{aligned} \tag{A.55}$$

Inserting these forms into the differential equation, we have

$$\begin{aligned} 0 &= x^2 y'' + (1 - 2r_1)xy' + r_1^2 y \\ &= (xv'' + v')x^{r_1 + 1}. \end{aligned} \tag{A.56}$$

Thus, we need to solve the equation

$$xv'' + v' = 0,$$

or

$$\frac{v''}{v'} = -\frac{1}{x}.$$

Integrating, we have

$$\ln |v'| = -\ln |x| + C,$$

where $A = \pm e^C$ absorbs C and the signs from the absolute values. Exponentiating, we obtain one last differential equation to solve,

$$v' = \frac{A}{x}.$$

Thus,

$$v(x) = A \ln |x| + k.$$

For one root, $r_1 = r_2 = r$, the general solution is of the form

$$y(x) = (c_1 + c_2 \ln |x|)x^r.$$

So, we have found that the second linearly independent equation can be written as

$$y_2(x) = x^{r_1} \ln |x|.$$

Therefore, the general solution is found as $y(x) = (c_1 + c_2 \ln |x|)x^r$.

Example A.19. Solve the initial value problem: $t^2 y'' + 3ty' + y = 0$, with the initial conditions $y(1) = 0$, $y'(1) = 1$.

For this example the characteristic equation takes the form

$$r(r - 1) + 3r + 1 = 0,$$

or

$$r^2 + 2r + 1 = 0.$$

There is only one real root, $r = -1$. Therefore, the general solution is

$$y(t) = (c_1 + c_2 \ln |t|)t^{-1}.$$

However, this problem is an initial value problem. At $t = 1$ we know the values of y and y'. Using the general solution, we first have that

$$0 = y(1) = c_1.$$

Thus, we have so far that $y(t) = c_2 \ln |t| t^{-1}$. Now, using the second condition and

$$y'(t) = c_2(1 - \ln |t|)t^{-2},$$

we have

$$1 = y(1) = c_2.$$

Therefore, the solution of the initial value problem is $y(t) = \ln |t| t^{-1}$.

For complex conjugate roots, $r = \alpha \pm i\beta$, the general solution takes the form

$$y(x) = x^\alpha (c_1 \cos(\beta \ln |x|) + c_2 \sin(\beta \ln |x|)).$$

We now turn to the case of complex conjugate roots, $r = \alpha \pm i\beta$. When dealing with the Cauchy-Euler equations, we have solutions of the form $y(x) = x^{\alpha + i\beta}$. The key to obtaining real solutions is to first rewrite x^y :

$$x^y = e^{\ln x^y} = e^{y \ln x}.$$

Thus, a power can be written as an exponential and the solution can be written as

$$y(x) = x^{\alpha + i\beta} = x^\alpha e^{i\beta \ln x}, \quad x > 0.$$

Recalling that

$$e^{i\beta \ln x} = \cos(\beta \ln |x|) + i \sin(\beta \ln |x|),$$

we can now find two real, linearly independent solutions, $x^\alpha \cos(\beta \ln |x|)$ and $x^\alpha \sin(\beta \ln |x|)$ following the same steps as earlier for the constant coefficient case. This gives the general solution as

$$y(x) = x^\alpha (c_1 \cos(\beta \ln |x|) + c_2 \sin(\beta \ln |x|)).$$

Example A.20. Solve: $x^2 y'' - xy' + 5y = 0$.

The characteristic equation takes the form

$$r(r-1) - r + 5 = 0,$$

or

$$r^2 - 2r + 5 = 0.$$

The roots of this equation are complex, $r_{1,2} = 1 \pm 2i$. Therefore, the general solution is $y(x) = x(c_1 \cos(2 \ln |x|) + c_2 \sin(2 \ln |x|))$.

The three cases are summarized in the table below.

Classification of Roots of the Characteristic Equation for Cauchy-Euler Differential Equations

1. Real, distinct roots r_1, r_2. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1 x^{r_1} + c_2 x^{r_2}$.

2. Real, equal roots $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the Method of Reduction of Order. This gives the second solution as $x^r \ln|x|$. Therefore, the general solution is found as $y(x) = (c_1 + c_2 \ln|x|)x^r$.

3. Complex conjugate roots $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. These complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $x^\alpha \cos(\beta \ln|x|)$ and $x^\alpha \sin(\beta \ln|x|)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = x^\alpha(c_1 \cos(\beta \ln|x|) + c_2 \sin(\beta \ln|x|))$.

Nonhomogeneous Cauchy-Euler Equations

We can also solve some nonhomogeneous Cauchy-Euler equations using the Method of Undetermined Coefficients or the Method of Variation of Parameters. We will demonstrate this with a couple of examples.

Example A.21. Find the solution of $x^2y'' - xy' - 3y = 2x^2$.

First we find the solution of the homogeneous equation. The characteristic equation is $r^2 - 2r - 3 = 0$. So, the roots are $r = -1, 3$ and the solution is $y_h(x) = c_1 x^{-1} + c_2 x^3$.

We next need a particular solution. Let's guess $y_p(x) = Ax^2$. Inserting the guess into the nonhomogeneous differential equation, we have

$$\begin{aligned} 2x^2 &= x^2y'' - xy' - 3y = 2x^2 \\ &= 2Ax^2 - 2Ax^2 - 3Ax^2 \\ &= -3Ax^2. \end{aligned} \tag{A.57}$$

So, $A = -2/3$. Therefore, the general solution of the problem is

$$y(x) = c_1 x^{-1} + c_2 x^3 - \frac{2}{3}x^2.$$

Example A.22. Find the solution of $x^2y'' - xy' - 3y = 2x^3$.

In this case the nonhomogeneous term is a solution of the homogeneous problem, which we solved in the last example. So, we will need a modification of the method. We have a problem of the form

$$ax^2y'' + bxy' + cy = dx^r,$$

where r is a solution of $ar(r-1)+br+c=0$. Let's guess a solution of the form $y = Ax^r \ln x$. Then one finds that the differential equation reduces to $Ax^r(2ar - a + b) = dx^r$. [You should verify this for yourself.]

With this in mind, we can now solve the problem at hand. Let $y_p = Ax^3 \ln x$. Inserting into the equation, we obtain $4Ax^3 = 2x^3$, or $A = 1/2$. The general solution of the problem can now be written as

$$y(x) = c_1 x^{-1} + c_2 x^3 + \frac{1}{2}x^3 \ln x.$$

Example A.23. Find the solution of $x^2y'' - xy' - 3y = 2x^3$ using Variation of Parameters.

As noted in the previous examples, the solution of the homogeneous problem has two linearly independent solutions, $y_1(x) = x^{-1}$ and $y_2(x) = x^3$. Assuming a particular solution of the form $y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$, we need to solve the system (A.45):

$$\begin{aligned} c_1'(x)x^{-1} + c_2'(x)x^3 &= 0 \\ -c_1'(x)x^{-2} + 3c_2'(x)x^2 &= \frac{2x^3}{x^2} = 2x. \end{aligned} \tag{A.58}$$

From the first equation of the system we have $c_1'(x) = -x^4 c_2'(x)$. Substituting this into the second equation gives $c_2'(x) = \frac{1}{2x}$. So, $c_2(x) = \frac{1}{2}\ln|x|$ and, therefore, $c_1(x) = \frac{1}{8}x^4$. The particular solution is

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x) = \frac{1}{8}x^3 + \frac{1}{2}x^3 \ln|x|.$$

Adding this to the homogeneous solution, we obtain the same solution as in the last example using the Method of Undetermined Coefficients. However, since $\frac{1}{8}x^3$ is a solution of the homogeneous problem, it can be absorbed into the first terms, leaving

$$y(x) = c_1 x^{-1} + c_2 x^3 + \frac{1}{2}x^3 \ln x.$$

Problems

1. Find all of the solutions of the first order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $\dfrac{dy}{dx} = \dfrac{e^x}{2y}$.

b. $\dfrac{dy}{dt} = y^2(1+t^2)$, $y(0) = 1$.

c. $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{x}$.

d. $xy' = y(1-2y)$, $\quad y(1) = 2$.

e. $y' - (\sin x)y = \sin x$.

f. $xy' - 2y = x^2, y(1) = 1$.

g. $\dfrac{ds}{dt} + 2s = st^2, \quad , s(0) = 1.$

h. $x' - 2x = te^{2t}.$

i. $\dfrac{dy}{dx} + y = \sin x,\ y(0) = 0.$

j. $\dfrac{dy}{dx} - \dfrac{3}{x}y = x^3,\ y(1) = 4.$

2. Consider the differential equation

$$\frac{dy}{dx} = \frac{x}{y} - \frac{x}{1+y}.$$

a. Find the 1-parameter family of solutions (general solution) of this equation.

b. Find the solution of this equation satisfying the initial condition $y(0) = 1$. Is this a member of the 1-parameter family?

3. Identify the type of differential equation. Find the general solution and plot several particular solutions. Also, find the singular solution if one exists.

a. $y = xy' + \frac{1}{y'}.$

b. $y = 2xy' + \ln y'.$

c. $y' + 2xy = 2xy^2.$

d. $y' + 2xy = y^2e^{x^2}.$

4. Find all of the solutions of the second order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $y'' - 9y' + 20y = 0.$

b. $y'' - 3y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1.$

c. $8y'' + 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0.$

d. $x'' - x' - 6x = 0$ for $x = x(t)$.

5. Verify that the given function is a solution and use Reduction of Order to find a second linearly independent solution.

a. $x^2y'' - 2xy' - 4y = 0, \quad y_1(x) = x^4.$

b. $xy'' - y' + 4x^3y = 0, \quad y_1(x) = \sin(x^2).$

6. Prove that $y_1(x) = \sinh x$ and $y_2(x) = 3\sinh x - 2\cosh x$ are linearly independent solutions of $y'' - y = 0$. Write $y_3(x) = \cosh x$ as a linear combination of y_1 and y_2.

7. Consider the nonhomogeneous differential equation $x'' - 3x' + 2x = 6e^{3t}$.

a. Find the general solution of the homogenous equation.

b. Find a particular solution using the Method of Undetermined Coefficients by guessing $x_p(t) = Ae^{3t}$.

c. Use your answers in the previous parts to write down the general solution for this problem.

8. Find the general solution of the given equation by the method given.

a. $y'' - 3y' + 2y = 10$. Method of Undetermined Coefficients.
b. $y'' + y' = 3x^2$. Variation of Parameters.

9. Use the Method of Variation of Parameters to determine the general solution for the following problems.

a. $y'' + y = \tan x$.
b. $y'' - 4y' + 4y = 6xe^{2x}$.

10. Instead of assuming that $c_1'y_1 + c_2'y_2 = 0$ in the derivation of the solution using Variation of Parameters, assume that $c_1'y_1 + c_2'y_2 = h(x)$ for an arbitrary function $h(x)$ and show that one gets the same particular solution.

11. Find all of the solutions of the second order differential equations for $x > 0.$. When an initial condition is given, find the particular solution satisfying that condition.

a. $x^2y'' + 3xy' + 2y = 0$.
b. $x^2y'' - 3xy' + 3y = 0$.
c. $x^2y'' + 5xy' + 4y = 0$.
d. $x^2y'' - 2xy' + 3y = 0$.
e. $x^2y'' + 3xy' - 3y = x^2$.

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