A

Ordinary Differential Equations Review

"The profound study of nature is the most fertile source of mathematical discoveries." - Joseph Fourier (1768-1830)

A.1 First Order Differential Equations

Before moving on, we first define an n-th order ordinary differential equation. It is an equation for an unknown function $y(x)$ that expresses a relationship between the unknown function and its first n derivatives. One could write this generally as

n-th order ordinary differential equation

$$F(y^{(n)}(x), y^{(n-1)}(x), \ldots, y'(x), y(x), x) = 0. \tag{A.1}$$

Here $y^{(n)}(x)$ represents the nth derivative of $y(x)$.

An initial value problem consists of the differential equation plus the values of the first $n-1$ derivatives at a particular value of the independent variable, say x_0:

Initial value problem.

$$y^{(n-1)}(x_0) = y_{n-1}, \quad y^{(n-2)}(x_0) = y_{n-2}, \quad \ldots, \quad y(x_0) = y_0. \tag{A.2}$$

A linear nth order differential equation takes the form

Linear nth order differential equation

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \ldots + a_1(x)y'(x) + a_0(x)y(x)) = f(x). \tag{A.3}$$

If $f(x) \equiv 0$, then the equation is said to be homogeneous, otherwise it is called nonhomogeneous.

Homogeneous and nonhomogeneous equations.

Typically, the first differential equations encountered are first order equations. A first order differential equation takes the form

First order differential equation

$$F(y', y, x) = 0. \tag{A.4}$$

There are two common first order differential equations for which one can formally obtain a solution. The first is the separable case and the second is a first order equation. We indicate that we can formally obtain solutions, as one can display the needed integration that leads to a solution. However, the resulting integrals are not always reducible to elementary functions nor does one obtain explicit solutions when the integrals are doable.

A.1.1 Separable Equations

A first order equation is separable if it can be written the form

$$\frac{dy}{dx} = f(x)g(y). \tag{A.5}$$

Special cases result when either $f(x) = 1$ or $g(y) = 1$. In the first case the equation is said to be autonomous.

Separable equations.

The general solution to equation (A.5) is obtained in terms of two integrals:

$$\boxed{\int \frac{dy}{g(y)} = \int f(x)\,dx + C,} \tag{A.6}$$

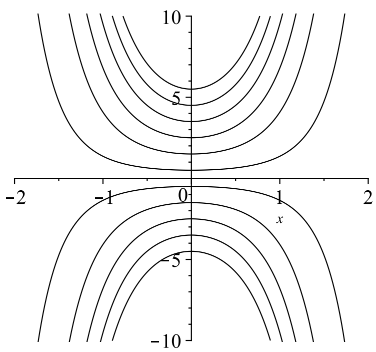

Figure A.1: Plots of solutions from the 1-parameter family of solutions of Example A.1 for several initial conditions.

where C is an integration constant. This yields a 1-parameter family of solutions to the differential equation corresponding to different values of C. If one can solve (A.6) for $y(x)$, then one obtains an explicit solution. Otherwise, one has a family of implicit solutions. If an initial condition is given as well, then one might be able to find a member of the family that satisfies this condition, which is often called a particular solution.

Example A.1. $y' = 2xy$, $y(0) = 2$.

Applying (A.6), one has

$$\int \frac{dy}{y} = \int 2x\,dx + C.$$

Integrating yields

$$\ln|y| = x^2 + C.$$

Exponentiating, one obtains the general solution,

$$y(x) = \pm e^{x^2+C} = Ae^{x^2}.$$

Here we have defined $A = \pm e^C$. Since C is an arbitrary constant, A is an arbitrary constant. Several solutions in this 1-parameter family are shown in Figure A.1.

Next, one seeks a particular solution satisfying the initial condition. For $y(0) = 2$, one finds that $A = 2$. So, the particular solution satisfying the initial condition is $y(x) = 2e^{x^2}$.

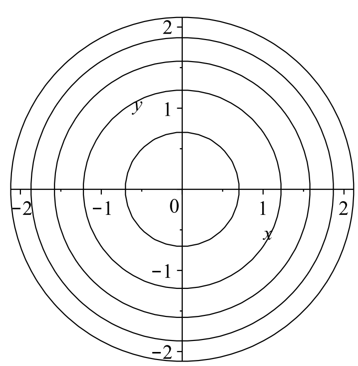

Figure A.2: Plots of solutions of Example A.2 for several initial conditions.

Example A.2. $yy' = -x$. Following the same procedure as in the last example, one obtains:

$$\int y\,dy = -\int x\,dx + C \Rightarrow y^2 = -x^2 + A, \quad \text{where} \quad A = 2C.$$

Thus, we obtain an implicit solution. Writing the solution as $x^2 + y^2 = A$, we see that this is a family of circles for $A > 0$ and the origin for $A = 0$. Plots of some solutions in this family are shown in Figure A.2.

A.1.2 Linear First Order Equations

THE SECOND TYPE OF FIRST ORDER EQUATION encountered is the linear first order differential equation in the standard form

$$y'(x) + p(x)y(x) = q(x). \tag{A.7}$$

In this case one seeks an integrating factor, $\mu(x)$, which is a function that one can multiply through the equation making the left side a perfect derivative. Thus, obtaining,

$$\frac{d}{dx}[\mu(x)y(x)] = \mu(x)q(x). \tag{A.8}$$

The integrating factor that works is $\mu(x) = \exp(\int^x p(\xi)\,d\xi)$. One can derive $\mu(x)$ by expanding the derivative in Equation (A.8),

$$\mu(x)y'(x) + \mu'(x)y(x) = \mu(x)q(x), \tag{A.9}$$

and comparing this equation to the one obtained from multiplying (A.7) by $\mu(x)$:

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \mu(x)q(x). \tag{A.10}$$

Note that these last two equations would be the same if the second terms were the same. Thus, we will require that

$$\frac{d\mu(x)}{dx} = \mu(x)p(x).$$

This is a separable first order equation for $\mu(x)$ whose solution is the integrating factor:

Integrating factor.

$$\mu(x) = \exp\left(\int^x p(\xi)\,d\xi\right). \tag{A.11}$$

Equation (A.8) is now easily integrated to obtain the general solution to the linear first order differential equation:

$$y(x) = \frac{1}{\mu(x)}\left[\int^x \mu(\xi)q(\xi)\,d\xi + C\right]. \tag{A.12}$$

Example A.3. $xy' + y = x, \quad x > 0,\ y(1) = 0.$

One first notes that this is a linear first order differential equation. Solving for y', one can see that the equation is not separable. Furthermore, it is not in the standard form (A.7). So, we first rewrite the equation as

$$\frac{dy}{dx} + \frac{1}{x}y = 1. \tag{A.13}$$

Noting that $p(x) = \frac{1}{x}$, we determine the integrating factor

$$\mu(x) = \exp\left[\int^x \frac{d\xi}{\xi}\right] = e^{\ln x} = x.$$

Multiplying equation (A.13) by $\mu(x) = x$, we actually get back the original equation! In this case we have found that $xy' + y$ must have been the derivative of something to start. In fact, $(xy)' = xy' + x$. Therefore, the differential equation becomes

$$(xy)' = x.$$

Integrating, one obtains

$$xy = \frac{1}{2}x^2 + C,$$

or

$$y(x) = \frac{1}{2}x + \frac{C}{x}.$$

Inserting the initial condition into this solution, we have $0 = \frac{1}{2} + C$. Therefore, $C = -\frac{1}{2}$. Thus, the solution of the initial value problem is

$$y(x) = \frac{1}{2}(x - \frac{1}{x}).$$

We can verify that this is the solution. Since $y' = \frac{1}{2} + \frac{1}{2x^2}$, we have

$$xy' + y = \frac{1}{2}x + \frac{1}{2x} + \frac{1}{2}\left(x - \frac{1}{x}\right) = x.$$

Also, $y(1) = \frac{1}{2}(1 - 1) = 0$.

Example A.4. $(\sin x)y' + (\cos x)y = x^2$.

Actually, this problem is easy if you realize that the left hand side is a perfect derivative. Namely,

$$\frac{d}{dx}((\sin x)y) = (\sin x)y' + (\cos x)y.$$

But, we will go through the process of finding the integrating factor for practice.

First, we rewrite the original differential equation in standard form. We divide the equation by $\sin x$ to obtain

$$y' + (\cot x)y = x^2 \csc x.$$

Then, we compute the integrating factor as

$$\mu(x) = \exp\left(\int^x \cot \xi \, d\xi\right) = e^{\ln(\sin x)} = \sin x.$$

Using the integrating factor, the standard form equation becomes

$$\frac{d}{dx}\left((\sin x)y\right) = x^2.$$

Integrating, we have

$$y \sin x = \frac{1}{3}x^3 + C.$$

So, the solution is

$$y(x) = \left(\frac{1}{3}x^3 + C\right) \csc x.$$

A.2 Second Order Linear Differential Equations

SECOND ORDER DIFFERENTIAL EQUATIONS ARE TYPICALLY HARDER than first order. In most cases students are only exposed to second order linear differential equations. A general form for a *second order linear differential equation* is given by

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{A.14}$$

One can rewrite this equation using operator terminology. Namely, one first defines the differential operator $L = a(x)D^2 + b(x)D + c(x)$, where $D = \frac{d}{dx}$. Then equation (A.14) becomes

$$Ly = f. \tag{A.15}$$

The solutions of linear differential equations are found by making use of the linearity of L. Namely, we consider the *vector space*[1] consisting of real-valued functions over some domain. Let f and g be vectors in this function space. L is a *linear operator* if for two vectors f and g and scalar a, we have that

a. $L(f+g) = Lf + Lg$
b. $L(af) = aLf$.

[1] We assume that the reader has been introduced to concepts in linear algebra. Later in the text we will recall the definition of a vector space and see that linear algebra is in the background of the study of many concepts in the solution of differential equations.

One typically solves (A.14) by finding the general solution of the homogeneous problem,

$$Ly_h = 0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p = f.$$

Then, the general solution of (A.14) is simply given as $y = y_h + y_p$. This is true because of the linearity of L. Namely,

$$\begin{aligned} Ly &= L(y_h + y_p) \\ &= Ly_h + Ly_p \\ &= 0 + f = f. \end{aligned} \tag{A.16}$$

There are methods for finding a particular solution of a nonhomogeneous differential equation. These methods range from pure guessing, the Method of Undetermined Coefficients, the Method of Variation of Parameters, or Green's functions. We will review these methods later in the chapter.

Determining solutions to the homogeneous problem, $Ly_h = 0$, is not always easy. However, many now famous mathematicians and physicists have studied a variety of second order linear equations and they have saved us the trouble of finding solutions to the differential equations that often appear in applications. We will encounter many of these in the following chapters. We will first begin with some simple homogeneous linear differential equations.

Linearity is also useful in producing the general solution of a homogeneous linear differential equation. If y_1 and y_2 are solutions of the homogeneous equation, then the *linear combination* $y = c_1y_1 + c_2y_2$ is also a solution of the homogeneous equation. In fact, if y_1 and y_2 are *linearly independent*,[2] then $y = c_1y_1 + c_2y_2$ is the general solution of the homogeneous problem.

[2] A set of functions $\{y_i(x)\}_{i=1}^n$ is a linearly independent set if and only if

$$c_1y_1(x) + \ldots + c_ny_n(x) = 0$$

implies $c_i = 0$, for $i = 1, \ldots, n$.

For $n = 2$, $c_1y_1(x) + c_2y_2(x) = 0$. If y_1 and y_2 are linearly dependent, then the coefficients are not zero and $y_2(x) = -\frac{c_1}{c_2}y_1(x)$ and is a multiple of $y_1(x)$.

Linear independence can also be established by looking at the Wronskian of the solutions. For a second order differential equation the Wronskian is defined as

$$W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x). \quad (A.17)$$

The solutions are linearly independent if the Wronskian is not zero.

A.2.1 Constant Coefficient Equations

THE SIMPLEST SECOND ORDER DIFFERENTIAL EQUATIONS are those with constant coefficients. The general form for a homogeneous constant coefficient second order linear differential equation is given as

$$ay''(x) + by'(x) + cy(x) = 0, \quad (A.18)$$

where a, b, and c are constants.

Solutions to (A.18) are obtained by making a guess of $y(x) = e^{rx}$. Inserting this guess into (A.18) leads to the characteristic equation

$$ar^2 + br + c = 0. \quad (A.19)$$

The characteristic equation for $ay'' + by' + cy = 0$ is $ar^2 + br + c = 0$. Solutions of this quadratic equation lead to solutions of the differential equation.

Namely, we compute the derivatives of $y(x) = e^{rx}$, to get $y(x) = re^{rx}$, and $y(x) = r^2e^{rx}$. Inserting into (A.18), we have

$$0 = ay''(x) + by'(x) + cy(x) = (ar^2 + br + c)e^{rx}.$$

Since the exponential is never zero, we find that $ar^2 + br + c = 0$.

Two real, distinct roots, r_1 and r_2, give solutions of the form

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x}.$$

The roots of this equation, r_1, r_2, in turn lead to three types of solutions depending upon the nature of the roots. In general, we have two linearly independent solutions, $y_1(x) = e^{r_1x}$ and $y_2(x) = e^{r_2x}$, and the general solution is given by a linear combination of these solutions,

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x}.$$

For two real distinct roots, we are done. However, when the roots are real, but equal, or complex conjugate roots, we need to do a little more work to obtain usable solutions.

Example A.5. $y'' - y' - 6y = 0$ $y(0) = 2, y'(0) = 0$.

The characteristic equation for this problem is $r^2 - r - 6 = 0$. The roots of this equation are found as $r = -2, 3$. Therefore, the general solution can be quickly written down:

$$y(x) = c_1e^{-2x} + c_2e^{3x}.$$

Note that there are two arbitrary constants in the general solution. Therefore, one needs two pieces of information to find a particular solution. Of course, we have the needed information in the form of the initial conditions.

One also needs to evaluate the first derivative

$$y'(x) = -2c_1e^{-2x} + 3c_2e^{3x}$$

in order to attempt to satisfy the initial conditions. Evaluating y and y' at $x = 0$ yields

$$\begin{aligned} 2 &= c_1 + c_2 \\ 0 &= -2c_1 + 3c_2 \end{aligned} \tag{A.20}$$

These two equations in two unknowns can readily be solved to give $c_1 = 6/5$ and $c_2 = 4/5$. Therefore, the solution of the initial value problem is obtained as $y(x) = \frac{6}{5}e^{-2x} + \frac{4}{5}e^{3x}$.

Repeated roots, $r_1 = r_2 = r$, give solutions of the form

$$y(x) = (c_1 + c_2x)e^{rx}.$$

In the case when there is a repeated real root, one has only one solution, $y_1(x) = e^{rx}$. The question is how does one obtain the second linearly independent solution? Since the solutions should be independent, we must have that the ratio $y_2(x)/y_1(x)$ is not a constant. So, we guess the form $y_2(x) = v(x)y_1(x) = v(x)e^{rx}$. (This process is called the Method of Reduction of Order.)

For constant coefficient second order equations, we can write the equation as

$$(D - r)^2y = 0,$$

where $D = \frac{d}{dx}$. We now insert $y_2(x) = v(x)e^{rx}$ into this equation. First we compute

$$(D - r)ve^{rx} = v'e^{rx}.$$

Then,

$$0 = (D - r)^2ve^{rx} = (D - r)v'e^{rx} = v''e^{rx}.$$

So, if $y_2(x)$ is to be a solution to the differential equation, then $v''(x)e^{rx} = 0$ for all x. So, $v''(x) = 0$, which implies that

$$v(x) = ax + b.$$

So,

$$y_2(x) = (ax + b)e^{rx}.$$

Without loss of generality, we can take $b = 0$ and $a = 1$ to obtain the second linearly independent solution, $y_2(x) = xe^{rx}$. The general solution is then

$$y(x) = c_1e^{rx} + c_2xe^{rx}.$$

Example A.6. $y'' + 6y' + 9y = 0$.

In this example we have $r^2 + 6r + 9 = 0$. There is only one root, $r = -3$. From the above discussion, we easily find the solution $y(x) = (c_1 + c_2x)e^{-3x}$.

When one has complex roots in the solution of constant coefficient equations, one needs to look at the solutions

$$y_{1,2}(x) = e^{(\alpha \pm i\beta)x}.$$

We make use of Euler's formula (See Chapter 6 for more on complex variables)

$$e^{i\beta x} = \cos\beta x + i\sin\beta x. \tag{A.21}$$

Then, the linear combination of $y_1(x)$ and $y_2(x)$ becomes

$$\begin{aligned} Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} &= e^{\alpha x}\left[Ae^{i\beta x} + Be^{-i\beta x}\right] \\ &= e^{\alpha x}\left[(A+B)\cos\beta x + i(A-B)\sin\beta x\right] \\ &\equiv e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x). \end{aligned} \tag{A.22}$$

Thus, we see that we have a linear combination of two real, linearly independent solutions, $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$.

Complex roots, $r = \alpha \pm i\beta$, give solutions of the form

$$y(x) = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x).$$

Example A.7. $y'' + 4y = 0$.

The characteristic equation in this case is $r^2 + 4 = 0$. The roots are pure imaginary roots, $r = \pm 2i$, and the general solution consists purely of sinusoidal functions, $y(x) = c_1\cos(2x) + c_2\sin(2x)$, since $\alpha = 0$ and $\beta = 2$.

Example A.8. $y'' + 2y' + 4y = 0$.

The characteristic equation in this case is $r^2 + 2r + 4 = 0$. The roots are complex, $r = -1 \pm \sqrt{3}i$ and the general solution can be written as

$$y(x) = \left[c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x)\right]e^{-x}.$$

Example A.9. $y'' + 4y = \sin x$.

This is an example of a nonhomogeneous problem. The homogeneous problem was actually solved in Example A.7. According to the theory, we need only seek a particular solution to the nonhomogeneous problem and add it to the solution of the last example to get the general solution.

The particular solution can be obtained by purely guessing, making an educated guess, or using the Method of Variation of Parameters. We will not review all of these techniques at this time. Due to the simple form of the driving term, we will make an intelligent guess of $y_p(x) = A\sin x$ and determine what A needs to be. Inserting this guess into the differential equation gives $(-A + 4A)\sin x = \sin x$. So, we see that $A = 1/3$ works. The general solution of the nonhomogeneous problem is therefore $y(x) = c_1\cos(2x) + c_2\sin(2x) + \frac{1}{3}\sin x$.

The three cases for constant coefficient linear second order differential equations are summarized below.

Classification of Roots of the Characteristic Equation for Second Order Constant Coefficient ODEs

1. **Real, distinct roots** r_1, r_2. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.
2. **Real, equal roots** $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the *Method of Reduction of Order*. This gives the second solution as xe^{rx}. Therefore, the general solution is found as $y(x) = (c_1 + c_2 x)e^{rx}$.
3. **Complex conjugate roots** $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. Making use of Euler's identity, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, these complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$.

A.3 Forced Systems

Many problems can be modeled by nonhomogeneous second order equations. Thus, we want to find solutions of equations of the form

$$Ly(x) = a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{A.23}$$

As noted in Section A.2, one solves this equation by finding the general solution of the homogeneous problem,

$$Ly_h = 0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p = f.$$

Then, the general solution of (A.14) is simply given as $y = y_h + y_p$.

So far, we only know how to solve constant coefficient, homogeneous equations. So, by adding a nonhomogeneous term to such equations we will need to find the particular solution to the nonhomogeneous equation.

We could guess a solution, but that is not usually possible without a little bit of experience. So, we need some other methods. There are two main methods. In the first case, the Method of Undetermined Coefficients, one makes an intelligent guess based on the form of $f(x)$. In the second method, one can systematically developed the particular solution. We will come back to the Method of Variation of Parameters and we will also introduce the powerful machinery of Green's functions later in this section.

A.3.1 Method of Undetermined Coefficients

Let's solve a simple differential equation highlighting how we can handle nonhomogeneous equations.

Example A.10. Consider the equation

$$y'' + 2y' - 3y = 4. \tag{A.24}$$

The first step is to determine the solution of the homogeneous equation. Thus, we solve

$$y_h'' + 2y_h' - 3y_h = 0. \tag{A.25}$$

The characteristic equation is $r^2 + 2r - 3 = 0$. The roots are $r = 1, -3$. So, we can immediately write the solution

$$y_h(x) = c_1 e^x + c_2 e^{-3x}.$$

The second step is to find a particular solution of (A.24). What possible function can we insert into this equation such that only a 4 remains? If we try something proportional to x, then we are left with a linear function after inserting x and its derivatives. Perhaps a constant function you might think. $y = 4$ does not work. But, we could try an arbitrary constant, $y = A$.

Let's see. Inserting $y = A$ into (A.24), we obtain

$$-3A = 4.$$

Ah ha! We see that we can choose $A = -\frac{4}{3}$ and this works. So, we have a particular solution, $y_p(x) = -\frac{4}{3}$. This step is done.

Combining the two solutions, we have the general solution to the original nonhomogeneous equation (A.24). Namely,

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}.$$

Insert this solution into the equation and verify that it is indeed a solution. If we had been given initial conditions, we could now use them to determine the arbitrary constants.

Example A.11. What if we had a different source term? Consider the equation

$$y'' + 2y' - 3y = 4x. \tag{A.26}$$

The only thing that would change is the particular solution. So, we need a guess.

We know a constant function does not work by the last example. So, let's try $y_p = Ax$. Inserting this function into Equation (A.26), we obtain

$$2A - 3Ax = 4x.$$

Picking $A = -4/3$ would get rid of the x terms, but will not cancel everything. We still have a constant left. So, we need something more general.

Let's try a linear function, $y_p(x) = Ax + B$. Then we get after substitution into (A.26)

$$2A - 3(Ax + B) = 4x.$$

Equating the coefficients of the different powers of x on both sides, we find a system of equations for the undetermined coefficients:

$$\begin{aligned} 2A - 3B &= 0 \\ -3A &= 4. \end{aligned} \tag{A.27}$$

These are easily solved to obtain

$$\begin{aligned} A &= -\frac{4}{3} \\ B &= \frac{2}{3}A = -\frac{8}{9}. \end{aligned} \tag{A.28}$$

So, the particular solution is

$$y_p(x) = -\frac{4}{3}x - \frac{8}{9}.$$

This gives the general solution to the nonhomogeneous problem as

$$y(x) = y_h(x) + y_p(x) = c_1e^x + c_2e^{-3x} - \frac{4}{3}x - \frac{8}{9}.$$

There are general forms that you can guess based upon the form of the driving term, $f(x)$. Some examples are given in Table A.1. More general applications are covered in a standard text on differential equations. However, the procedure is simple. Given $f(x)$ in a particular form, you make an appropriate guess up to some unknown parameters, or coefficients. Inserting the guess leads to a system of equations for the unknown coefficients. Solve the system and you have the solution. This solution is then added to the general solution of the homogeneous differential equation.

$f(x)$	Guess
$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$	$A_nx^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0$
ae^{bx}	Ae^{bx}
$a\cos\omega x + b\sin\omega x$	$A\cos\omega x + B\sin\omega x$

Table A.1: Forms used in the Method of Undetermined Coefficients.

Example A.12. Solve

$$y'' + 2y' - 3y = 2e^{-3x}. \tag{A.29}$$

According to the above, we would guess a solution of the form $y_p = Ae^{-3x}$. Inserting our guess, we find

$$0 = 2e^{-3x}.$$

Oops! The coefficient, A, disappeared! We cannot solve for it. What went wrong?

The answer lies in the general solution of the homogeneous problem. Note that e^x and e^{-3x} are solutions to the homogeneous problem. So, a multiple of e^{-3x} will not get us anywhere. It turns out that there is one further modification of the method. If the driving term contains terms that are solutions of the homogeneous problem, then we need to make a guess consisting of the smallest possible power of x times the function which is no longer a solution of the homogeneous problem. Namely, we guess $y_p(x) = Axe^{-3x}$ and differentiate this guess to obtain the derivatives $y_p' = A(1-3x)e^{-3x}$ and $y_p'' = A(9x-6)e^{-3x}$.

Inserting these derivatives into the differential equation, we obtain

$$[(9x-6)+2(1-3x)-3x]Ae^{-3x} = 2e^{-3x}.$$

Comparing coefficients, we have

$$-4A = 2.$$

So, $A = -1/2$ and $y_p(x) = -\frac{1}{2}xe^{-3x}$. Thus, the solution to the problem is

$$y(x) = \left(2 - \frac{1}{2}x\right)e^{-3x}.$$

Modified Method of Undetermined Coefficients

In general, if any term in the guess $y_p(x)$ is a solution of the homogeneous equation, then multiply the guess by x^k, where k is the smallest positive integer such that no term in $x^k y_p(x)$ is a solution of the homogeneous problem.

A.3.2 Periodically Forced Oscillations

A SPECIAL TYPE OF FORCING is periodic forcing. Realistic oscillations will dampen and eventually stop if left unattended. For example, mechanical clocks are driven by compound or torsional pendula and electric oscillators are often designed with the need to continue for long periods of time. However, they are not perpetual motion machines and will need a periodic injection of energy. This can be done systematically by adding periodic forcing. Another simple example is the motion of a child on a swing in the park. This simple damped pendulum system will naturally slow down to equilibrium (stopped) if left alone. However, if the child pumps energy into the swing at the right time, or if an adult pushes the child at the right time, then the amplitude of the swing can be increased.

There are other systems, such as airplane wings and long bridge spans, in which external driving forces might cause damage to the system. A well know example is the wind induced collapse of the Tacoma Narrows Bridge due to strong winds. Of course, if one is not careful, the child in the

The Tacoma Narrows Bridge opened in Washington State (U.S.) in mid 1940. However, in November of the same year the winds excited a transverse mode of vibration, which eventually (in a few hours) lead to large amplitude oscillations and then collapse.

last example might get too much energy pumped into the system causing a similar failure of the desired motion.

While there are many types of forced systems, and some fairly complicated, we can easily get to the basic characteristics of forced oscillations by modifying the mass-spring system by adding an external, time-dependent, driving force. Such as system satisfies the equation

$$m\ddot{x} + b\dot{(x)} + kx = F(t), \tag{A.30}$$

where m is the mass, b is the damping constant, k is the spring constant, and $F(t)$ is the driving force. If $F(t)$ is of simple form, then we can employ the Method of Undetermined Coefficients. Since the systems we have considered so far are similar, one could easily apply the following to pendula or circuits.

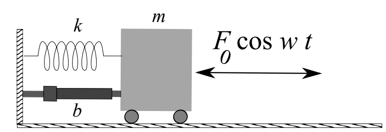

Figure A.3: An external driving force is added to the spring-mass-damper system.

As the damping term only complicates the solution, we will consider the simpler case of undamped motion and assume that $b = 0$. Furthermore, we will introduce a sinusoidal driving force, $F(t) = F_0 \cos \omega t$ in order to study periodic forcing. This leads to the simple periodically driven mass on a spring system

$$m\ddot{x} + kx = F_0 \cos \omega t. \tag{A.31}$$

In order to find the general solution, we first obtain the solution to the homogeneous problem,

$$x_h = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t,$$

where $\omega_0 = \sqrt{\frac{k}{m}}$. Next, we seek a particular solution to the nonhomogeneous problem. We will apply the Method of Undetermined Coefficients.

A natural guess for the particular solution would be to use $x_p = A \cos \omega t + B\ sin \omega t$. However, recall that the guess should not be a solution of the homogeneous problem. Comparing x_p with x_h, this would hold if $\omega \neq \omega_0$. Otherwise, one would need to use the Modified Method of Undetermined Coefficients as described in the last section. So, we have two cases to consider.

Dividing through by the mass, we solve the simple driven system,

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t.$$

Example A.13. Solve $\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$, for $\omega \neq \omega_0$.

In this case we continue with the guess $x_p = A \cos \omega t + B\ sin \omega t$. Since there is no damping term, one quickly finds that $B = 0$. Inserting $x_p = A \cos \omega t$ into the differential equation, we find that

$$\left(-\omega^2 + \omega_0^2\right) A \cos \omega t = \frac{F_0}{m} \cos \omega t.$$

Solving for A, we obtain

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)}.$$

The general solution for this case is thus,

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t. \tag{A.32}$$

Example A.14. Solve $\ddot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t$.

In this case, we need to employ the Modified Method of Undetermined Coefficients. So, we make the guess $x_p = t\left(A\cos\omega_0 t + B\, sin\omega_0 t\right)$. Since there is no damping term, one finds that $A = 0$. Inserting the guess in to the differential equation, we find that

$$B = \frac{F_0}{2m\omega_0},$$

or the general solution is

$$x(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \frac{F_0}{2m\omega}t\sin\omega t. \tag{A.33}$$

The general solution to the problem is thus

$$x(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \begin{cases} \frac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega t, & \omega \neq \omega_0, \\ \frac{F_0}{2m\omega_0}t\sin\omega_0 t, & \omega = \omega_0. \end{cases} \tag{A.34}$$

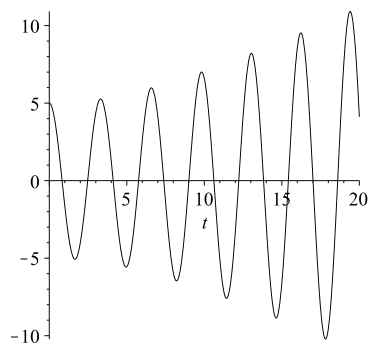

Figure A.4: Plot of

$$x(t) = 5\cos 2t + \frac{1}{2}t\sin 2t,$$

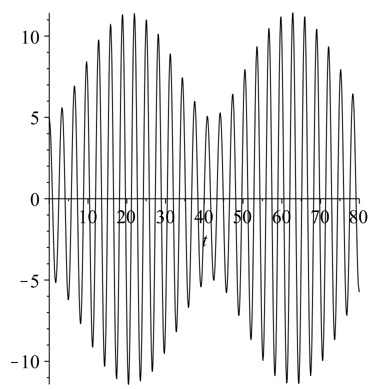

Figure A.5: Plot of

$$x(t) = \frac{1}{249}\left(2045\cos 2t - 800\cos\frac{43}{20}t\right),$$

a solution of $\ddot{x} + 4x = 2\cos 2.15t$.

Special cases of these solutions provide interesting physics, which can be explored by the reader in the homework. In the case that $\omega = \omega_0$, we see that the solution tends to grow as t gets large. This is what is called a resonance. Essentially, one is driving the system at its natural frequency. As the system is moving to the left, one pushes it to the left. If it is moving to the right, one is adding energy in that direction. This forces the amplitude of oscillation to continue to grow until the system breaks. An example of such an oscillation is shown in Figure A.4.

In the case that $\omega \neq \omega_0$, one can rewrite the solution in a simple form. Let's choose the initial conditions that $c_1 = -F_0/(m(\omega_0^2 - \omega^2))$, $c_2 = 0$. Then one has

$$x(t) = \frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\frac{(\omega_0-\omega)t}{2}\sin\frac{(\omega_0+\omega)t}{2}. \tag{A.35}$$

For values of ω near ω_0, one finds the solution consists of a rapid oscillation, due to the $\sin\frac{(\omega_0+\omega)t}{2}$ factor, with a slowly varying amplitude, $\frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\frac{(\omega_0-\omega)t}{2}$. The reader can investigate this solution.

This slow variation is called a beat and the beat frequency is given by $f = \frac{|\omega_0-\omega|}{4\pi}$. In Figure A.5 we see the high frequency oscillations are contained by the lower beat frequency, $f = \frac{0.15}{4\pi}$s. This corresponds to a period of $T = 1/f \approx 83.7$ Hz, which looks about right from the figure.

Example A.15. Solve $\ddot{x} + x = 2\cos\omega t$, $x(0) = 0$, $\dot{x}(0) = 0$, for $\omega = 1, 1.15$. For each case, we need the solution of the homogeneous problem,

$$x_h(t) = c_1\cos t + c_2\sin t.$$

The particular solution depends on the value of ω.

For $\omega = 1$, the driving term, $2\cos\omega t$, is a solution of the homogeneous problem. Thus, we assume

$$x_p(t) = At\cos t + Bt\sin t.$$

Inserting this into the differential equation, we find $A = 0$ and $B = 1$. So, the general solution is

$$x(t) = c_1 \cos t + c_2 \sin t + t \sin t.$$

Imposing the initial conditions, we find

$$x(t) = t \sin t.$$

This solution is shown in Figure A.6.

For $\omega = 1.15$, the driving term, $2 \cos \omega 1.15t$, is not a solution of the homogeneous problem. Thus, we assume

$$x_p(t) = A \cos 1.15t + B \sin 1.15t.$$

Inserting this into the differential equation, we find $A = -\frac{800}{129}$ and $B = 0$. So, the general solution is

$$x(t) = c_1 \cos t + c_2 \sin t - \frac{800}{129} \cos t.$$

Imposing the initial conditions, we find

$$x(t) = \frac{800}{129} \left(\cos t - \cos 1.15t \right).$$

This solution is shown in Figure A.7. The beat frequency in this case is the same as with Figure A.5.

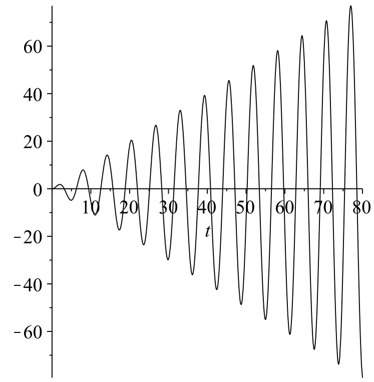

Figure A.6: Plot of

$$x(t) = t \sin 2t,$$

a solution of $\ddot{x} + x = 2 \cos t$.

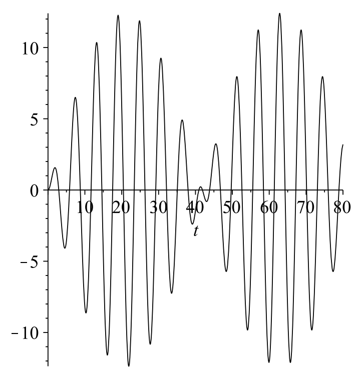

Figure A.7: Plot of

$$x(t) = \frac{800}{129} \left(\cos t - \cos \frac{23}{20} t \right),$$

a solution of $\ddot{x} + x = 2 \cos 1.15t$.

A.3.3 Method of Variation of Parameters

A MORE SYSTEMATIC WAY to find particular solutions is through the use of the Method of Variation of Parameters. The derivation is a little detailed and the solution is sometimes messy, but the application of the method is straight forward if you can do the required integrals. We will first derive the needed equations and then do some examples.

We begin with the nonhomogeneous equation. Let's assume it is of the standard form

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{A.36}$$

We know that the solution of the homogeneous equation can be written in terms of two linearly independent solutions, which we will call $y_1(x)$ and $y_2(x)$:

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x).$$

Replacing the constants with functions, then we no longer have a solution to the homogeneous equation. Is it possible that we could stumble across the right functions with which to replace the constants and somehow end up with $f(x)$ when inserted into the left side of the differential equation? It turns out that we can.

So, let's assume that the constants are replaced with two unknown functions, which we will call $c_1(x)$ and $c_2(x)$. This change of the parameters is where the name of the method derives. Thus, we are assuming that a particular solution takes the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x). \tag{A.37}$$

If this is to be a solution, then insertion into the differential equation should make the equation hold. To do this we will first need to compute some derivatives.

We assume the nonhomogeneous equation has a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

The first derivative is given by

$$y_p'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x) + c_1'(x)y_1(x) + c_2'(x)y_2(x). \tag{A.38}$$

Next we will need the second derivative. But, this will yield eight terms. So, we will first make a simplifying assumption. Let's assume that the last two terms add to zero:

$$c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0. \tag{A.39}$$

It turns out that we will get the same results in the end if we did not assume this. The important thing is that it works!

Under the assumption the first derivative simplifies to

$$y_p'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x). \tag{A.40}$$

The second derivative now only has four terms:

$$y_p'(x) = c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x). \tag{A.41}$$

Now that we have the derivatives, we can insert the guess into the differential equation. Thus, we have

$$\begin{aligned} f(x) \quad = \quad & a(x)\left[c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right] \\ & + b(x)\left[c_1(x)y_1'(x) + c_2(x)y_2'(x)\right] \\ & + c(x)\left[c_1(x)y_1(x) + c_2(x)y_2(x)\right]. \end{aligned} \tag{A.42}$$

Regrouping the terms, we obtain

$$\begin{aligned} f(x) \quad = \quad & c_1(x)\left[a(x)y_1''(x) + b(x)y_1'(x) + c(x)y_1(x)\right] \\ & + c_2(x)\left[a(x)y_2''(x) + b(x)y_2'(x) + c(x)y_2(x)\right] \\ & + a(x)\left[c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right]. \end{aligned} \tag{A.43}$$

Note that the first two rows vanish since y_1 and y_2 are solutions of the homogeneous problem. This leaves the equation

$$f(x) = a(x)\left[c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right],$$

which can be rearranged as

$$c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = \frac{f(x)}{a(x)}. \tag{A.44}$$

In summary, we have assumed a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

This is only possible if the unknown functions $c_1(x)$ and $c_2(x)$ satisfy the system of equations

$$\begin{aligned} c_1'(x)y_1(x) + c_2'(x)y_2(x) &= 0 \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) &= \frac{f(x)}{a(x)}. \end{aligned} \qquad \text{(A.45)}$$

In order to solve the differential equation $Ly = f$, we assume

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x),$$

for $Ly_{1,2} = 0$. Then, one need only solve a simple system of equations (A.45). System (A.45) can be solved as

$$c_1'(x) = -\frac{fy_2}{aW(y_1, y_2)},$$

$$c_1'(x) = \frac{fy_1}{aW(y_1, y_2)},$$

where $W(y_1, y_2) = y_1y_2' - y_1'y_2$ is the Wronskian. We use this solution in the next section.

It is standard to solve this system for the derivatives of the unknown functions and then present the integrated forms. However, one could just as easily start from this system and solve the system for each problem encountered.

Example A.16. Find the general solution of the nonhomogeneous problem: $y'' - y = e^{2x}$.

The general solution to the homogeneous problem $y_h'' - y_h = 0$ is

$$y_h(x) = c_1e^x + c_2e^{-x}.$$

In order to use the Method of Variation of Parameters, we seek a solution of the form

$$y_p(x) = c_1(x)e^x + c_2(x)e^{-x}.$$

We find the unknown functions by solving the system in (A.45), which in this case becomes

$$\begin{aligned} c_1'(x)e^x + c_2'(x)e^{-x} &= 0 \\ c_1'(x)e^x - c_2'(x)e^{-x} &= e^{2x}. \end{aligned} \qquad \text{(A.46)}$$

Adding these equations we find that

$$2c_1'e^x = e^{2x} \to c_1' = \frac{1}{2}e^x.$$

Solving for $c_1(x)$ we find

$$c_1(x) = \frac{1}{2}\int e^x\,dx = \frac{1}{2}e^x.$$

Subtracting the equations in the system yields

$$2c_2'e^{-x} = -e^{2x} \to c_2' = -\frac{1}{2}e^{3x}.$$

Thus,

$$c_2(x) = -\frac{1}{2}\int e^{3x}\,dx = -\frac{1}{6}e^{3x}.$$

The particular solution is found by inserting these results into y_p:

$$\begin{aligned} y_p(x) &= c_1(x)y_1(x) + c_2(x)y_2(x) \\ &= (\frac{1}{2}e^x)e^x + (-\frac{1}{6}e^{3x})e^{-x} \\ &= \frac{1}{3}e^{2x}. \end{aligned} \qquad \text{(A.47)}$$

Thus, we have the general solution of the nonhomogeneous problem as

$$y(x) = c_1 e^x + c_2 e^{-x} + \frac{1}{3} e^{2x}.$$

Example A.17. Now consider the problem: $y'' + 4y = \sin x$.

The solution to the homogeneous problem is

$$y_h(x) = c_1 \cos 2x + c_2 \sin 2x. \tag{A.48}$$

We now seek a particular solution of the form

$$y_h(x) = c_1(x) \cos 2x + c_2(x) \sin 2x.$$

We let $y_1(x) = \cos 2x$ and $y_2(x) = \sin 2x$, $a(x) = 1$, $f(x) = \sin x$ in system (A.45):

$$\begin{aligned} c_1'(x) \cos 2x + c_2'(x) \sin 2x &= 0 \\ -2c_1'(x) \sin 2x + 2c_2'(x) \cos 2x &= \sin x. \end{aligned} \tag{A.49}$$

Now, use your favorite method for solving a system of two equations and two unknowns. In this case, we can multiply the first equation by $2\sin 2x$ and the second equation by $\cos 2x$. Adding the resulting equations will eliminate the c_1' terms. Thus, we have

$$c_2'(x) = \frac{1}{2} \sin x \cos 2x = \frac{1}{2}(2\cos^2 x - 1) \sin x.$$

Inserting this into the first equation of the system, we have

$$c_1'(x) = -c_2'(x) \frac{\sin 2x}{\cos 2x} = -\frac{1}{2} \sin x \sin 2x = -\sin^2 x \cos x.$$

These can easily be solved:

$$c_2(x) = \frac{1}{2} \int (2\cos^2 x - 1) \sin x \, dx = \frac{1}{2}(\cos x - \frac{2}{3} \cos^3 x).$$

$$c_1(x) = -\int \sin^x \cos x \, dx = -\frac{1}{3} \sin^3 x.$$

The final step in getting the particular solution is to insert these functions into $y_p(x)$. This gives

$$\begin{aligned} y_p(x) &= c_1(x) y_1(x) + c_2(x) y_2(x) \\ &= (-\frac{1}{3} \sin^3 x) \cos 2x + (\frac{1}{2} \cos x - \frac{1}{3} \cos^3 x) \sin x \\ &= \frac{1}{3} \sin x. \end{aligned} \tag{A.50}$$

So, the general solution is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3} \sin x. \tag{A.51}$$

A.4 Cauchy-Euler Equations

ANOTHER CLASS OF SOLVABLE LINEAR DIFFERENTIAL EQUATIONS that is of interest are the Cauchy-Euler type of equations, also referred to in some books as Euler's equation. These are given by

$$ax^2y''(x) + bxy'(x) + cy(x) = 0. \tag{A.52}$$

Note that in such equations the power of x in each of the coefficients matches the order of the derivative in that term. These equations are solved in a manner similar to the constant coefficient equations.

One begins by making the guess $y(x) = x^r$. Inserting this function and its derivatives,

$$y'(x) = rx^{r-1}, \qquad y''(x) = r(r-1)x^{r-2},$$

into Equation (A.52), we have

$$[ar(r-1) + br + c]\, x^r = 0.$$

Since this has to be true for all x in the problem domain, we obtain the characteristic equation

$$ar(r-1) + br + c = 0. \tag{A.53}$$

The solutions of Cauchy-Euler equations can be found using the characteristic equation $ar(r-1) + br + c = 0$.

Just like the constant coefficient differential equation, we have a quadratic equation and the nature of the roots again leads to three classes of solutions. If there are two real, distinct roots, then the general solution takes the form $y(x) = c_1x^{r_1} + c_2x^{r_2}$.

For two real, distinct roots, the general solution takes the form

$$y(x) = c_1x^{r_1} + c_2x^{r_2}.$$

Example A.18. Find the general solution: $x^2y'' + 5xy' + 12y = 0$.

As with the constant coefficient equations, we begin by writing down the characteristic equation. Doing a simple computation,

$$\begin{aligned} 0 &= r(r-1) + 5r + 12 \\ &= r^2 + 4r + 12 \\ &= (r+2)^2 + 8, \\ -8 &= (r+2)^2, \end{aligned} \tag{A.54}$$

one determines the roots are $r = -2 \pm 2\sqrt{2}i$. Therefore, the general solution is $y(x) = \left[c_1 \cos(2\sqrt{2}\ln|x|) + c_2 \sin(2\sqrt{2}\ln|x|)\right] x^{-2}$

Deriving the solution for Case 2 for the Cauchy-Euler equations works in the same way as the second for constant coefficient equations, but it is a bit messier. First note that for the real root, $r = r_1$, the characteristic equation has to factor as $(r - r_1)^2 = 0$. Expanding, we have

$$r^2 - 2r_1r + r_1^2 = 0.$$

The general characteristic equation is

$$ar(r-1) + br + c = 0.$$

Dividing this equation by a and rewriting, we have

$$r^2 + (\frac{b}{a} - 1)r + \frac{c}{a} = 0.$$

Comparing equations, we find

$$\frac{b}{a} = 1 - 2r_1, \qquad \frac{c}{a} = r_1^2.$$

So, the Cauchy-Euler equation for this case can be written in the form

$$x^2y'' + (1 - 2r_1)xy' + r_1^2 y = 0.$$

Now we seek the second linearly independent solution in the form $y_2(x) = v(x)x^{r_1}$. We first list this function and its derivatives,

$$\begin{aligned} y_2(x) &= vx^{r_1}, \\ y_2'(x) &= (xv' + r_1 v)x^{r_1-1}, \\ y_2''(x) &= (x^2v'' + 2r_1xv' + r_1(r_1-1)v)x^{r_1-2}. \end{aligned} \tag{A.55}$$

Inserting these forms into the differential equation, we have

$$\begin{aligned} 0 &= x^2y'' + (1 - 2r_1)xy' + r_1^2 y \\ &= (xv'' + v')x^{r_1+1}. \end{aligned} \tag{A.56}$$

Thus, we need to solve the equation

$$xv'' + v' = 0,$$

or

$$\frac{v''}{v'} = -\frac{1}{x}.$$

Integrating, we have

$$\ln|v'| = -\ln|x| + C,$$

where $A = \pm e^C$ absorbs C and the signs from the absolute values. Exponentiating, we obtain one last differential equation to solve,

$$v' = \frac{A}{x}.$$

Thus,

$$v(x) = A\ln|x| + k.$$

For one root, $r_1 = r_2 = r$, the general solution is of the form

$$y(x) = (c_1 + c_2 \ln|x|)x^r.$$

So, we have found that the second linearly independent equation can be written as

$$y_2(x) = x^{r_1}\ln|x|.$$

Therefore, the general solution is found as $y(x) = (c_1 + c_2 \ln|x|)x^r$.

Example A.19. Solve the initial value problem: $t^2y'' + 3ty' + y = 0$, with the initial conditions $y(1) = 0$, $y'(1) = 1$.

For this example the characteristic equation takes the form

$$r(r-1) + 3r + 1 = 0,$$

or

$$r^2 + 2r + 1 = 0.$$

There is only one real root, $r = -1$. Therefore, the general solution is

$$y(t) = (c_1 + c_2 \ln|t|)t^{-1}.$$

However, this problem is an initial value problem. At $t = 1$ we know the values of y and y'. Using the general solution, we first have that

$$0 = y(1) = c_1.$$

Thus, we have so far that $y(t) = c_2 \ln|t|t^{-1}$. Now, using the second condition and

$$y'(t) = c_2(1 - \ln|t|)t^{-2},$$

we have

$$1 = y(1) = c_2.$$

Therefore, the solution of the initial value problem is $y(t) = \ln|t|t^{-1}$.

For complex conjugate roots, $r = \alpha \pm i\beta$, the general solution takes the form

$y(x) = x^\alpha(c_1 \cos(\beta \ln|x|) + c_2 \sin(\beta \ln|x|))$.

We now turn to the case of complex conjugate roots, $r = \alpha \pm i\beta$. When dealing with the Cauchy-Euler equations, we have solutions of the form $y(x) = x^{\alpha + i\beta}$. The key to obtaining real solutions is to first rewrite x^y :

$$x^y = e^{\ln x^y} = e^{y \ln x}.$$

Thus, a power can be written as an exponential and the solution can be written as

$$y(x) = x^{\alpha+i\beta} = x^\alpha e^{i\beta \ln x}, \quad x > 0.$$

Recalling that

$$e^{i\beta \ln x} = \cos(\beta \ln|x|) + i \sin(\beta \ln|x|),$$

we can now find two real, linearly independent solutions, $x^\alpha \cos(\beta \ln|x|)$ and $x^\alpha \sin(\beta \ln|x|)$ following the same steps as earlier for the constant coefficient case. This gives the general solution as

$$y(x) = x^\alpha(c_1 \cos(\beta \ln|x|) + c_2 \sin(\beta \ln|x|)).$$

Example A.20. Solve: $x^2y'' - xy' + 5y = 0$.

The characteristic equation takes the form

$$r(r-1) - r + 5 = 0,$$

or

$$r^2 - 2r + 5 = 0.$$

The roots of this equation are complex, $r_{1,2} = 1 \pm 2i$. Therefore, the general solution is $y(x) = x(c_1 \cos(2\ln|x|) + c_2 \sin(2\ln|x|))$.

The three cases are summarized in the table below.

Classification of Roots of the Characteristic Equation for Cauchy-Euler Differential Equations

1. Real, distinct roots r_1, r_2. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1 x^{r_1} + c_2 x^{r_2}$.

2. Real, equal roots $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the Method of Reduction of Order. This gives the second solution as $x^r \ln|x|$. Therefore, the general solution is found as $y(x) = (c_1 + c_2 \ln|x|)x^r$.

3. Complex conjugate roots $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. These complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $x^\alpha \cos(\beta \ln|x|)$ and $x^\alpha \sin(\beta \ln|x|)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = x^\alpha (c_1 \cos(\beta \ln|x|) + c_2 \sin(\beta \ln|x|))$.

Nonhomogeneous Cauchy-Euler Equations

We can also solve some nonhomogeneous Cauchy-Euler equations using the Method of Undetermined Coefficients or the Method of Variation of Parameters. We will demonstrate this with a couple of examples.

Example A.21. Find the solution of $x^2 y'' - xy' - 3y = 2x^2$.

First we find the solution of the homogeneous equation. The characteristic equation is $r^2 - 2r - 3 = 0$. So, the roots are $r = -1, 3$ and the solution is $y_h(x) = c_1 x^{-1} + c_2 x^3$.

We next need a particular solution. Let's guess $y_p(x) = Ax^2$. Inserting the guess into the nonhomogeneous differential equation, we have

$$\begin{aligned} 2x^2 &= x^2 y'' - xy' - 3y = 2x^2 \\ &= 2Ax^2 - 2Ax^2 - 3Ax^2 \\ &= -3Ax^2. \end{aligned} \tag{A.57}$$

So, $A = -2/3$. Therefore, the general solution of the problem is

$$y(x) = c_1 x^{-1} + c_2 x^3 - \frac{2}{3}x^2.$$

Example A.22. Find the solution of $x^2 y'' - xy' - 3y = 2x^3$.

In this case the nonhomogeneous term is a solution of the homogeneous problem, which we solved in the last example. So, we will need a modification of the method. We have a problem of the form

$$ax^2 y'' + bxy' + cy = dx^r,$$

where r is a solution of $ar(r-1)+br+c=0$. Let's guess a solution of the form $y = Ax^r \ln x$. Then one finds that the differential equation reduces to $Ax^r(2ar-a+b) = dx^r$. [You should verify this for yourself.]

With this in mind, we can now solve the problem at hand. Let $y_p = Ax^3 \ln x$. Inserting into the equation, we obtain $4Ax^3 = 2x^3$, or $A = 1/2$. The general solution of the problem can now be written as

$$y(x) = c_1 x^{-1} + c_2 x^3 + \frac{1}{2}x^3 \ln x.$$

Example A.23. Find the solution of $x^2y'' - xy' - 3y = 2x^3$ using Variation of Parameters.

As noted in the previous examples, the solution of the homogeneous problem has two linearly independent solutions, $y_1(x) = x^{-1}$ and $y_2(x) = x^3$. Assuming a particular solution of the form $y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$, we need to solve the system (A.45):

$$\begin{aligned} c_1'(x)x^{-1} + c_2'(x)x^3 &= 0 \\ -c_1'(x)x^{-2} + 3c_2'(x)x^2 &= \frac{2x^3}{x^2} = 2x. \end{aligned} \tag{A.58}$$

From the first equation of the system we have $c_1'(x) = -x^4c_2'(x)$. Substituting this into the second equation gives $c_2'(x) = \frac{1}{2x}$. So, $c_2(x) = \frac{1}{2}\ln|x|$ and, therefore, $c_1(x) = \frac{1}{8}x^4$. The particular solution is

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x) = \frac{1}{8}x^3 + \frac{1}{2}x^3\ln|x|.$$

Adding this to the homogeneous solution, we obtain the same solution as in the last example using the Method of Undetermined Coefficients. However, since $\frac{1}{8}x^3$ is a solution of the homogeneous problem, it can be absorbed into the first terms, leaving

$$y(x) = c_1 x^{-1} + c_2 x^3 + \frac{1}{2}x^3 \ln x.$$

Problems

1. Find all of the solutions of the first order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $\dfrac{dy}{dx} = \dfrac{e^x}{2y}$.

b. $\dfrac{dy}{dt} = y^2(1+t^2)$, $y(0) = 1$.

c. $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{x}$.

d. $xy' = y(1-2y), \quad y(1) = 2$.

e. $y' - (\sin x)y = \sin x$.

f. $xy' - 2y = x^2, y(1) = 1$.

g. $\dfrac{ds}{dt} + 2s = st^2, \quad , s(0) = 1.$

h. $x' - 2x = te^{2t}.$

i. $\dfrac{dy}{dx} + y = \sin x,\ y(0) = 0.$

j. $\dfrac{dy}{dx} - \dfrac{3}{x}y = x^3,\ y(1) = 4.$

2. Consider the differential equation

$$\frac{dy}{dx} = \frac{x}{y} - \frac{x}{1+y}.$$

a. Find the 1-parameter family of solutions (general solution) of this equation.

b. Find the solution of this equation satisfying the initial condition $y(0) = 1$. Is this a member of the 1-parameter family?

3. Identify the type of differential equation. Find the general solution and plot several particular solutions. Also, find the singular solution if one exists.

a. $y = xy' + \frac{1}{y'}.$

b. $y = 2xy' + \ln y'.$

c. $y' + 2xy = 2xy^2.$

d. $y' + 2xy = y^2 e^{x^2}.$

4. Find all of the solutions of the second order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $y'' - 9y' + 20y = 0.$

b. $y'' - 3y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1.$

c. $8y'' + 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0.$

d. $x'' - x' - 6x = 0$ for $x = x(t)$.

5. Verify that the given function is a solution and use Reduction of Order to find a second linearly independent solution.

a. $x^2 y'' - 2xy' - 4y = 0, \quad y_1(x) = x^4.$

b. $xy'' - y' + 4x^3 y = 0, \quad y_1(x) = \sin(x^2).$

6. Prove that $y_1(x) = \sinh x$ and $y_2(x) = 3\sinh x - 2\cosh x$ are linearly independent solutions of $y'' - y = 0$. Write $y_3(x) = \cosh x$ as a linear combination of y_1 and y_2.

7. Consider the nonhomogeneous differential equation $x'' - 3x' + 2x = 6e^{3t}$.

a. Find the general solution of the homogenous equation.

b. Find a particular solution using the Method of Undetermined Coefficients by guessing $x_p(t) = Ae^{3t}$.

c. Use your answers in the previous parts to write down the general solution for this problem.

8. Find the general solution of the given equation by the method given.

a. $y'' - 3y' + 2y = 10$. Method of Undetermined Coefficients.

b. $y'' + y' = 3x^2$. Variation of Parameters.

9. Use the Method of Variation of Parameters to determine the general solution for the following problems.

a. $y'' + y = \tan x$.

b. $y'' - 4y' + 4y = 6xe^{2x}$.

10. Instead of assuming that $c_1' y_1 + c_2' y_2 = 0$ in the derivation of the solution using Variation of Parameters, assume that $c_1' y_1 + c_2' y_2 = h(x)$ for an arbitrary function $h(x)$ and show that one gets the same particular solution.

11. Find all of the solutions of the second order differential equations for $x > 0$.. When an initial condition is given, find the particular solution satisfying that condition.

a. $x^2 y'' + 3xy' + 2y = 0$.

b. $x^2 y'' - 3xy' + 3y = 0$.

c. $x^2 y'' + 5xy' + 4y = 0$.

d. $x^2 y'' - 2xy' + 3y = 0$.

e. $x^2 y'' + 3xy' - 3y = x^2$.