7
First Order Partial Differential Equations

"The profound study of nature is the most fertile source of mathematical discoveries." - Joseph Fourier (1768-1830)

7.1 Introduction

We begin our study of partial differential equations with *first order partial differential equations.* Before doing so, we need to define a few terms.

Recall (see the appendix on differential equations) that an n-th order ordinary differential equation is an equation for an unknown function $y(x)$ that expresses a relationship between the unknown function and its first n derivatives. One could write this generally as

n-th order ordinary differential equation

$$F(y^{(n)}(x), y^{(n-1)}(x), \ldots, y'(x), y(x), x) = 0. \tag{7.1}$$

Here $y^{(n)}(x)$ represents the nth derivative of $y(x)$. Furthermore, and initial value problem consists of the differential equation plus the values of the first $n-1$ derivatives at a particular value of the independent variable, say x_0:

Initial value problem.

$$y^{(n-1)}(x_0) = y_{n-1}, \quad y^{(n-2)}(x_0) = y_{n-2}, \quad \ldots, \quad y(x_0) = y_0. \tag{7.2}$$

If conditions are instead provided at more than one value of the independent variable, then we have a boundary value problem. .

If the unknown function is a function of several variables, then the derivatives are partial derivatives and the resulting equation is a partial differential equation. Thus, if $u = u(x, y, \ldots)$, a general partial differential equation might take the form

$$F\left(x, y, \ldots, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \ldots, \frac{\partial^2 u}{\partial x^2}, \ldots\right) = 0. \tag{7.3}$$

Since the notation can get cumbersome, there are different ways to write the partial derivatives. First order derivatives could be written as

$$\frac{\partial u}{\partial x}, u_x, \partial_x u, D_x u.$$

Second order partial derivatives could be written in the forms

$$\frac{\partial^2 u}{\partial x^2}, u_{xx}, \partial_{xx} u, D_x^2 u.$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}, u_{xy}, \partial_{xy} u, D_y D_x u.$$

Note, we are assuming that $u(x, y, \ldots)$ has continuous partial derivatives. Then, according to Clairaut's Theorem (Alexis Claude Clairaut, 1713-1765) , mixed partial derivatives are the same.

Examples of some of the partial differential equation treated in this book are shown in Table 1.1. However, being that the highest order derivatives in these equation are of second order, these are second order partial differential equations. In this chapter we will focus on first order partial differential equations. Examples are given by

$$\begin{aligned} u_t + u_x &= 0. \\ u_t + uu_x &= 0. \\ u_t + uu_x &= u. \\ 3u_x - 2u_y + u &= x. \end{aligned}$$

For function of two variables, which the above are examples, a general first order partial differential equation for $u = u(x, y)$ is given as

$$F(x, y, u, u_x, u_y) = 0, \quad (x, y) \in D \subset R^2. \tag{7.4}$$

This equation is too general. So, restrictions can be placed on the form, leading to a classification of first order equations. A linear first order partial differential equation is of the form

Linear first order partial differential equation.

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y). \tag{7.5}$$

Note that all of the coefficients are independent of u and its derivatives and each term in linear in u, u_x, or u_y.

We can relax the conditions on the coefficients a bit. Namely, we could assume that the equation is linear only in u_x and u_y. This gives the quasilinear first order partial differential equation in the form

Quasilinear first order partial differential equation.

$$a(x, y, u)u_x + b(x, y, u)u_y = f(x, y, u). \tag{7.6}$$

Note that the u-term was absorbed by $f(x, y, u)$.

In between these two forms we have the semilinear first order partial differential equation in the form

Semilinear first order partial differential equation.

$$a(x, y)u_x + b(x, y)u_y = f(x, y, u). \tag{7.7}$$

Here the left side of the equation is linear in u, u_x and u_y. However, the right hand side can be nonlinear in u.

For the most part, we will introduce the Method of Characteristics for solving quasilinear equations. But, let us first consider the simpler case of linear first order constant coefficient partial differential equations.

7.2 Linear Constant Coefficient Equations

Let's consider the linear first order constant coefficient partial differential equation

$$au_x + bu_y + cu = f(x, y), \tag{7.8}$$

for a, b, and c constants with $a^2 + b^2 > 0$. We will consider how such equations might be solved. We do this by considering two cases, $b = 0$ and $b \neq 0$.

For the first case, $b = 0$, we have the equation

$$au_x + cu = f.$$

We can view this as a first order linear (ordinary) differential equation with y a parameter. Recall that the solution of such equations can be obtained using an integrating factor. [See the discussion after Equation (A.7).] First rewrite the equation as

$$u_x + \frac{c}{a}u = \frac{f}{a}.$$

Introducing the integrating factor

$$\mu(x) = \exp\left(\int^x \frac{c}{a}\,d\xi\right) = e^{\frac{c}{a}x},$$

the differential equation can be written as

$$(\mu u)_x = \frac{f}{a}\mu.$$

Integrating this equation and solving for $u(x, y)$, we have

$$\begin{aligned}
\mu(x)u(x,y) &= \frac{1}{a}\int f(\xi,y)\mu(\xi)\,d\xi + g(y) \\
e^{\frac{c}{a}x}u(x,y) &= \frac{1}{a}\int f(\xi,y)e^{\frac{c}{a}\xi}\,d\xi + g(y) \\
u(x,y) &= \frac{1}{a}\int f(\xi,y)e^{\frac{c}{a}(\xi-x)}\,d\xi + g(y)e^{-\frac{c}{a}x}.
\end{aligned} \tag{7.9}$$

Here $g(y)$ is an arbitrary function of y.

For the second case, $b \neq 0$, we have to solve the equation

$$au_x + bu_y + cu = f.$$

It would help if we could find a transformation which would eliminate one of the derivative terms reducing this problem to the previous case. That is what we will do.

We first note that

$$\begin{aligned}
au_x + bu_y &= (a\mathbf{i} + b\mathbf{j}) \cdot (u_x\mathbf{i} + u_y\mathbf{j}) \\
&= (a\mathbf{i} + b\mathbf{j}) \cdot \nabla u.
\end{aligned} \tag{7.10}$$

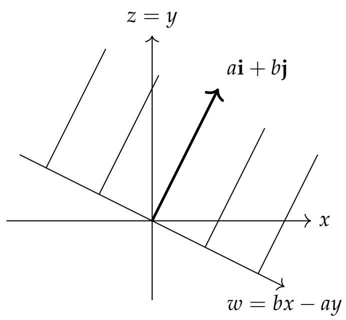


Figure 7.1: Coordinate systems for transforming $au_x + bu_y + cu = f$ into $bv_z + cv = f$ using the transformation $w = bx - ay$ and $z = y$.

Recall from multivariable calculus that the last term is nothing but a directional derivative of $u(x,y)$ in the direction $a\mathbf{i} + b\mathbf{j}$. [Actually, it is proportional to the directional derivative if $a\mathbf{i} + b\mathbf{j}$ is not a unit vector.]

Therefore, we seek to write the partial differential equation as involving a derivative in the direction $a\mathbf{i} + b\mathbf{j}$ but not in a directional orthogonal to this. In Figure 7.1 we depict a new set of coordinates in which the w direction is orthogonal to $a\mathbf{i} + b\mathbf{j}$.

We consider the transformation

$$\begin{aligned} w &= bx - ay, \\ z &= y. \end{aligned} \tag{7.11}$$

We first note that this transformation is invertible,

$$\begin{aligned} x &= \frac{1}{b}(w + az), \\ y &= z. \end{aligned} \tag{7.12}$$

Next we consider how the derivative terms transform. Let $u(x,y) = v(w,z)$. Then, we have

$$\begin{aligned} au_x + bu_y &= a\frac{\partial}{\partial x}v(w,z) + b\frac{\partial}{\partial y}v(w,z), \\ &= a\left[\frac{\partial v}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right] \\ &\quad +b\left[\frac{\partial v}{\partial w}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right] \\ &= a[bv_w + 0 \cdot v_z] + b[-av_w + v_z] \\ &= bv_z. \end{aligned} \tag{7.13}$$

Therefore, the partial differential equation becomes

$$bv_z + cv = f\left(\frac{1}{b}(w + az), z\right).$$

This is now in the same form as in the first case and can be solved using an integrating factor.

Example 7.1. Find the general solution of the equation $3u_x - 2u_y + u = x$.

First, we transform the equation into new coordinates.

$$w = bx - ay = -2x - 3y,$$

and $z = y$. The,

$$\begin{aligned} u_x - 2u_y &= 3[-2v_w + 0 \cdot v_z] - 2[-3v_w + v_z] \\ &= -2v_z. \end{aligned} \tag{7.14}$$

The new partial differential equation for $v(w,z)$ is

$$-2\frac{\partial v}{\partial z} + v = x = -\frac{1}{2}(w + 3z).$$

Rewriting this equation,

$$\frac{\partial v}{\partial z} - \frac{1}{2}v = \frac{1}{4}(w+3z),$$

we identify the integrating factor

$$\mu(z) = \exp\left[-\int^z \frac{1}{2}\,d\zeta\right] = e^{-z/2}.$$

Using this integrating factor, we can solve the differential equation for $v(w,z)$.

$$\begin{aligned}
\frac{\partial}{\partial z}\left(e^{-z/2}v\right) &= \frac{1}{4}(w+3z)e^{-z/2}, \\
e^{-z/2}v(w,z) &= \frac{1}{4}\int^z (w+3\zeta)e^{-\zeta/2}\,d\zeta \\
&= -\frac{1}{2}(w+6+3z)e^{-z/2} + c(w) \\
v(w,z) &= -\frac{1}{2}(w+6+3z) + c(w)e^{z/2} \\
u(x,y) &= x - 3 + c(-2x-3y)e^{y/2}.
\end{aligned} \tag{7.15}$$

7.3 *Quasilinear Equations: The Method of Characteristics*

7.3.1 *Geometric Interpretation*

We consider the quasilinear partial differential equation in two independent variables,

$$a(x,y,u)u_x + b(x,y,u)u_y - c(x,y,u) = 0. \tag{7.16}$$

Let $u = u(x,y)$ be a solution of this equation. Then,

$$f(x,y,u) = u(x,y) - u = 0$$

describes the solution surface, or integral surface,

Integral surface.

We recall from multivariable, or vector, calculus that the normal to the integral surface is given by the gradient function,

$$\nabla f = (u_x, u_y, -1).$$

Now consider the vector of coefficients, $\mathbf{v} = (a,b,c)$ and the dot product with the gradient above:

$$\mathbf{v}\cdot\nabla f = au_x + bu_y - c.$$

This is the left hand side of the partial differential equation. Therefore, for the solution surface we have

$$\mathbf{v}\cdot\nabla f = 0,$$

or $\mathbf{v}$ is perpendicular to ∇f. Since ∇f is normal to the surface, $\mathbf{v} = (a,b,c)$ is tangent to the surface. Geometrically, $\mathbf{v}$ defines a direction field, called the characteristic field. These are shown in Figure 7.2.

The characteristic field.

7.3.2 Characteristics

WE SEEK THE FORMS OF THE CHARACTERISTIC CURVES such as the one shown in Figure 7.2. Recall that one can parametrize space curves,

$$\mathbf{c}(t) = (x(t), y(t), u(t)), \quad t \in [t_1, t_2].$$

The tangent to the curve is then

$$\mathbf{v}(t) = \frac{d\mathbf{c}(t)}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{du}{dt}\right).$$

However, in the last section we saw that $\mathbf{v}(t) = (a, b, c)$ for the partial differential equation $a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) = 0$. This gives the parametric form of the characteristic curves as

$$\frac{dx}{dt} = a, \frac{dy}{dt} = b, \frac{du}{dt} = c. \tag{7.17}$$

Another form of these equations is found by relating the differentials, dx, dy, du, to the coefficients in the differential equation. Since $x = x(t)$ and $y = y(t)$, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b}{a}.$$

Similarly, we can show that

$$\frac{du}{dx} = \frac{c}{a}, \quad \frac{du}{dy} = \frac{c}{b}.$$

All of these relations can be summarized in the form

$$dt = \frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}. \tag{7.18}$$

How do we use these characteristics to solve quasilinear partial differential equations? Consider the next example.

Example 7.2. Find the general solution: $u_x + u_y - u = 0$.

We first identify $a = 1$, $b = 1$, and $c = u$. The relations between the differentials is

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{u}.$$

We can pair the differentials in three ways:

$$\frac{dy}{dx} = 1, \quad \frac{du}{dx} = u, \quad \frac{du}{dy} = u.$$

Only two of these relations are independent. We focus on the first pair.

The first equation gives the characteristic curves in the xy-plane. This equation is easily solved to give

$$y = x + c_1.$$

The second equation can be solved to give $u = c_2 e^x$.

The goal is to find the general solution to the differential equation. Since $u = u(x,y)$, the integration "constant" is not really a constant, but is constant with respect to x. It is in fact an arbitrary constant function. In fact, we could view it as a function of c_1, the constant of integration in the first equation. Thus, we let $c_2 = G(c_1)$ for G and arbitrary function. Since $c_1 = y - x$, we can write the general solution of the differential equation as

$$u(x,y) = G(y-x)e^x.$$

Example 7.3. Solve the advection equation, $u_t + cu_x = 0$, for c a constant, and $u = u(x,t)$, $|x| < \infty$, $t > 0$.

The characteristic equations are

$$d\tau = \frac{dt}{1} = \frac{dx}{c} = \frac{du}{0} \tag{7.19}$$

and the parametric equations are given by

$$\frac{dx}{d\tau} = c, \quad \frac{du}{d\tau} = 0. \tag{7.20}$$

These equations imply that

- $u =$ const. $= c_1$.
- $x = ct +$ const. $= ct + c_2$.

Traveling waves.

As before, we can write c_1 as an arbitrary function of c_2. However, before doing so, let's replace c_1 with the variable ξ and then we have that

$$\xi = x - ct, \quad u(x,t) = f(\xi) = f(x-ct)$$

where f is an arbitrary function. Furthermore, we see that $u(x,t) = f(x-ct)$ indicates that the solution is a wave moving in one direction in the shape of the initial function, $f(x)$. This is known as a traveling wave. A typical traveling wave is shown in Figure 7.3.

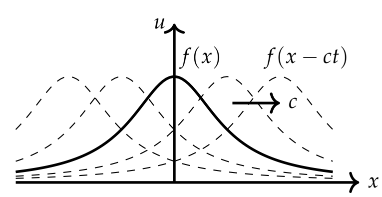


Figure 7.3: Depiction of a traveling wave. $u(x,t) = f(x)$ at $t = 0$ travels without changing shape.

Note that since $u = u(x,t)$, we have

$$\begin{aligned} 0 &= u_t + cu_x \\ &= \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} \\ &= \frac{du(x(t),t}{dt}. \end{aligned} \tag{7.21}$$

This implies that $u(x,t) =$ constant along the characteristics, $\frac{dx}{dt} = c$.

As with ordinary differential equations, the general solution provides an infinite number of solutions of the differential equation. If we want to pick out a particular solution, we need to specify some side conditions. We investigate this by way of examples.

Side conditions.

Example 7.4. Find solutions of $u_x + u_y - u = 0$ subject to $u(x,0) = 1$.

We found the general solution to the partial differential equation as $u(x,y) = G(y-x)e^x$. The side condition tells us that $u = 1$ along $y = 0$. This requires

$$1 = u(x,0) = G(-x)e^x.$$

Thus, $G(-x) = e^{-x}$. Replacing x with $-z$, we find

$$G(z) = e^z.$$

Thus, the side condition has allowed for the determination of the arbitrary function $G(y-x)$. Inserting this function, we have

$$u(x,y) = G(y-x)e^x = e^{y-x}e^x = e^y.$$

Side conditions could be placed on other curves. For the general line, $y = mx + d$, we have $u(x, mx+d) = g(x)$ and for $x = d$, $u(d,y) = g(y)$. As we will see, it is possible that a given side condition may not yield a solution. We will see that conditions have to be given on non-characteristic curves in order to be useful.

Example 7.5. Find solutions of $3u_x - 2u_y + u = x$ for a) $u(x,x) = x$ and b) $u(x,y) = 0$ on $3y + 2x = 1$.

Before applying the side condition, we find the general solution of the partial differential equation. Rewriting the differential equation in standard form, we have

$$3u_x - 2u_y = x = u.$$

The characteristic equations are

$$\frac{dx}{3} = \frac{dy}{-2} = \frac{du}{x-u}. \tag{7.22}$$

These equations imply that

- $-2dx = 3dy$

  This implies that the characteristic curves (lines) are $2x + 3y = c_1$.

- $\frac{du}{dx} = \frac{1}{3}(x-u)$.

  This is a linear first order differential equation, $\frac{du}{dx} + \frac{1}{3}u = \frac{1}{3}x$. It can be solved using the integrating factor,

$$\mu(x) = \exp\left(\frac{1}{3}\int^x d\xi\right) = e^{x/3}.$$

$$\begin{aligned}
\frac{d}{dx}\left(ue^{x/3}\right) &= \frac{1}{3}xe^{x/3} \\
ue^{x/3} &= \frac{1}{3}\int^x \xi e^{\xi/3}\,d\xi + c_2 \\
&= (x-3)e^{x/3} + c_2 \\
u(x,y) &= x - 3 + c_2e^{-x/3}.
\end{aligned} \tag{7.23}$$

As before, we write c_2 as an arbitrary function of $c_1 = 2x + 3y$. This gives the general solution

$$u(x, y) = x - 3 + G(2x + 3y)e^{-x/3}.$$

Note that this is the same answer that we had found in Example 7.1

Now we can look at any side conditions and use them to determine particular solutions by picking out specific G's.

a $u(x, x) = x$

This states that $u = x$ along the line $y = x$. Inserting this condition into the general solution, we have

$$x = x - 3 + G(5x)e^{-x/3},$$

or

$$G(5x) = 3e^{x/3}.$$

Letting $z = 5x$,

$$G(z) = 3e^{z/15}.$$

The particular solution satisfying this side condition is

$$\begin{aligned} u(x, y) &= x - 3 + G(2x + 3y)e^{-x/3} \\ &= x - 3 + 3e^{(2x+3y)/15}e^{-x/3} \\ &= x - 3 + 3e^{(y-x)/5}. \end{aligned} \tag{7.24}$$

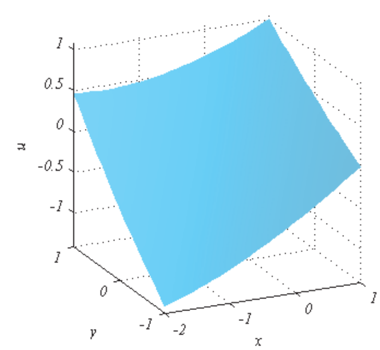


Figure 7.4: Integral surface found in Example 7.5.

This surface is shown in Figure 7.5.

In Figure 7.5 we superimpose the values of $u(x, y)$ along the characteristic curves. The characteristic curves are the red lines and the images of these curves are the black lines. The side condition is indicated with the blue curve drawn along the surface.

The values of $u(x, y)$ are found from the side condition as follows. For $x = \xi$ on the blue curve, we know that $y = \xi$ and $u(\xi, \xi) = \xi$. Now, the characteristic lines are given by $2x + 3y = c_1$. The constant c_1 is found on the blue curve from the point of intersection with one of the black characteristic lines. For $x = y = \xi$, we have $c_1 = 5\xi$. Then, the equation of the characteristic line, which is red in Figure 7.5, is given by $y = \frac{1}{3}(5\xi - 2x)$.

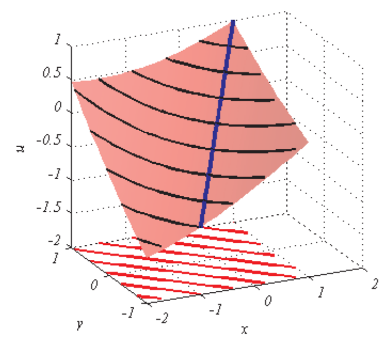


Figure 7.5: Integral surface with side condition and characteristics for Example 7.5.

Along these lines we need to find $u(x, y) = x - 3 + c_2 e^{-x/3}$. First we have to find c_2. We have on the blue curve, that

$$\begin{aligned} \xi &= u(\xi, \xi) \\ &= \xi - 3 + c_2 e^{-\xi/3}. \end{aligned} \tag{7.25}$$

Therefore, $c_2 = 3e^{\xi/3}$. Inserting this result into the expression for the solution, we have

$$u(x, y) = x - 3 + e^{(\xi - x)/3}.$$

So, for each ξ, one can draw a family of spacecurves

$$\left(x, \frac{1}{3}(5\xi - 2x), x - 3 + e^{(\xi - x)/3}\right)$$

yielding the integral surface.

b $u(x, y) = 0$ on $3y + 2x = 1$.
For this condition, we have

$$0 = x - 3 + G(1)e^{-x/3}.$$

We note that G is not a function in this expression. We only have one value for G. So, we cannot solve for $G(x)$. Geometrically, this side condition corresponds to one of the black curves in Figure 7.5.

7.4 Applications

7.4.1 Conservation Laws

There are many applications of quasilinear equations, especially in fluid dynamics. The advection equation is one such example and generalizations of this example to nonlinear equations leads to some interesting problems. These equations fall into a category of equations called conservation laws. We will first discuss one-dimensional (in space) conservations laws and then look at simple examples of nonlinear conservation laws.

Conservation laws are useful in modeling several systems. They can be boiled down to determining the rate of change of some stuff, $Q(t)$, in a region, $a \le x \le b$, as depicted in Figure 7.6. The simples model is to think of fluid flowing in one dimension, such as water flowing in a stream. Or, it could be the transport of mass, such as a pollutant. One could think of traffic flow down a straight road.

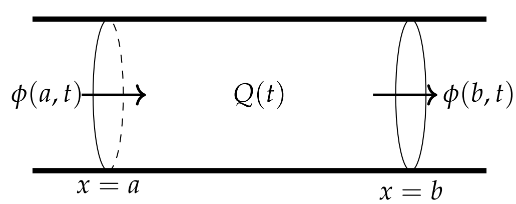


Figure 7.6: The rate of change of Q between $x = a$ and $x = b$ depends on the rates of flow through each end.

This is an example of a typical mixing problem. The rate of change of $Q(t)$ is given as

$$\text{the rate of change of } Q = \text{Rate in} - \text{Rate Out} + \text{ source term.}$$

Here the "Rate in" is how much is flowing into the region in Figure 7.6 from the $x = a$ boundary. Similarly, the "Rate out" is how much is flowing into the region from the $x = b$ boundary. [Of course, this could be the other way, but we can imagine for now that q is flowing from left to right.] We can describe this flow in terms of the flux, $\phi(x,t)$ over the ends of the region. On the left side we have a gain of $\phi(a,t)$ and on the right side of the region there is a loss of $\phi(b,t)$.

The source term would be some other means of adding or removing Q from the region. In terms of fluid flow, there could be a source of fluid inside the region such as a faucet adding more water. Or, there could be a drain letting water escape. We can denote this by the total source over the interval, $\int_a^b f(x,t)\,dx$. Here $f(x,t)$ is the source density.

In summary, the rate of change of $Q(x,t)$ can be written as

$$\frac{dQ}{dt} = \phi(a,t) - \phi(b,t) + \int_a^b f(x,y)\,dx.$$

We can write this in a slightly different form by noting that $\phi(a,t) - \phi(b,t)$ can be viewed as the evaluation of antiderivatives in the Fundamental Theorem of Calculus. Namely, we can recall that

$$\int_a^b \frac{\partial\phi(x,t)}{\partial x}\,dx = \phi(b,t) - \phi(a,t).$$

The difference is not exactly in the order that we desire, but it is easy to see that

Integral form of conservation law.

$$\frac{dQ}{dt} = -\int_a^b \frac{\partial\phi(x,t)}{\partial x}\,dx + \int_a^b f(x,t)\,dx. \tag{7.26}$$

This is the integral form of the conservation law.

We can rewrite the conservation law in differential form. First, we introduce the density function, $u(x,t)$, so that the total amount of stuff at a given time is

$$Q(t) = \int_a^b u(x,t)\,dx.$$

Introducing this form into the integral conservation law, we have

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = -\int_a^b \frac{\partial\phi}{\partial x}\,dx + \int_a^b f(x,t)\,dx. \tag{7.27}$$

Assuming that a and b are fixed in time and that the integrand is continuous, we can bring the time derivative inside the integrand and collect the three terms into one to find

$$\int_a^b \left(u_t(x,t) + \phi_x(x,t) - f(x,t)\right)\,dx = 0, \quad \forall x \in [a,b].$$

We cannot simply set the integrant to zero just because the integral vanishes. However, if this result holds for every region $[a,b]$, then we can conclude the integrand vanishes. So, under that assumption, we have the local conservation law,

Differential form of conservation law.

$$u_t(x,t) + \phi_x(x,t) = f(x,t). \tag{7.28}$$

This partial differential equation is actually an equation in terms of two unknown functions, assuming we know something about the source function. We would like to have a single unknown function. So, we need some additional information. This added information comes from the constitutive relation, a function relating the flux to the density function. Namely, we will assume that we can find the relationship $\phi = \phi(u)$. If so, then we can write

$$\frac{\partial \phi}{\partial x} = \frac{d\phi}{du}\frac{\partial u}{\partial x},$$

or $\phi_x = \phi'(u)u_x$.

Example 7.6. Inviscid Burgers' Equation Find the equation satisfied by $u(x,t)$ for $\phi(u) = \frac{1}{2}u^2$ and $f(x,t) \equiv 0$.

For this flux function we have $\phi_x = \phi'(u)u_x = uu_x$. The resulting equation is then $u_t + uu_x = 0$. This is the inviscid Burgers' equation. We will later discuss Burgers' equation.

Example 7.7. Traffic Flow

This is a simple model of one-dimensional traffic flow. Let $u(x,t)$ be the density of cars. Assume that there is no source term. For example, there is no way for a car to disappear from the flow by turning off the road or falling into a sinkhole. Also, there is no source of additional cars.

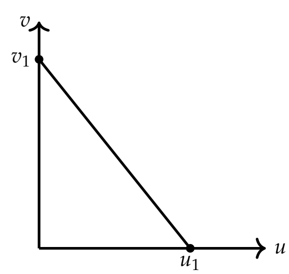


Figure 7.7: Car velocity as a function of car density.

Let $\phi(x,t)$ denote the number of cars per hour passing position x at time t. Note that the units are given by cars/mi times mi/hr. Thus, we can write the flux as $\phi = uv$, where v is the velocity of the carts at position x and time t.

In order to continue we need to assume a relationship between the car velocity and the car density. Let's assume the simplest form, a linear relationship. The more dense the traffic, we expect the speeds to slow down. So, a function similar to that in Figure 7.7 is in order. This is a straight line between the two intercepts $(0, v_1)$ and $(u_1, 0)$. It is easy to determine the equation of this line. Namely the relationship is given as

$$v = v_1 - \frac{v_1}{u_1}u.$$

This gives the flux as

$$\phi = uv = v_1\left(u - \frac{u^2}{u_1}\right).$$

We can now write the equation for the car density,

$$\begin{aligned} 0 &= u_t + \phi' u_x \\ &= u_t + v_1\left(1 - \frac{2u}{u_1}\right)u_x. \end{aligned} \tag{7.29}$$

7.4.2 *Nonlinear Advection Equations*

IN THIS SECTION WE CONSIDER EQUATIONS OF THE FORM $u_t + c(u)u_x = 0$. When $c(u)$ is a constant function, we have the advection equation. In the last two examples we have seen cases in which $c(u)$ is not a constant function.

We will apply the method of characteristics to these equations. First, we will recall how the method works for the advection equation.

The advection equation is given by $u_t + cu_x = 0$. The characteristic equations are given by

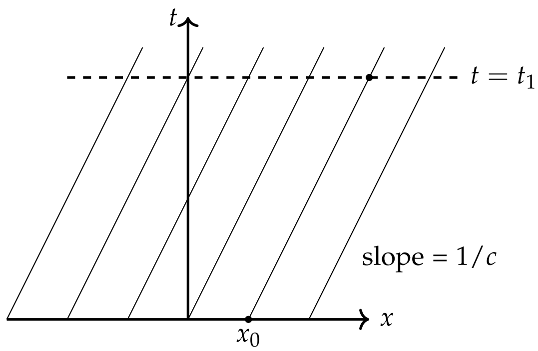
$$\frac{dx}{dt} = c, \quad \frac{du}{dt} = 0.$$

These are easily solved to give the result that

$$u(x,t) = \text{constant along the lines } x = ct + x_0,$$

where x_0 is an arbitrary constant.

The characteristic lines are shown in Figure 7.8. We note that $u(x,t) = u(x_0,0) = f(x_0)$. So, if we know u initially, we can determine what u is at a later time.



$$u(x_0 + ct_1, t_1) = u(x_0, 0)$$

Figure 7.8: The characteristics lines the xt-plane.

In Figure 7.8 we see that the value of $u(x_0,)$ at $t = 0$ and $x = x_0$ propagates along the characteristic to a point at time $t = t_1$. From $x - ct = x_0$, we can solve for x in terms of t_1 and find that $u(x_0 + ct_1, t_1) = u(x_0, 0)$.

Plots of solutions $u(x,t)$ versus x for specific times give traveling waves as shown in Figure 7.3. In Figure 7.9 we show how each wave profile for different times are constructed for a given initial condition.

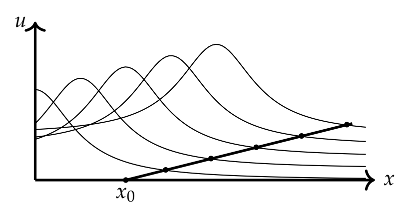


Figure 7.9: For each $x = x_0$ at $t = 0$, $u(x_0 + ct, t) = u(x_0, 0)$.

The nonlinear advection equation is given by $u_t + c(u)u_x = 0$, $|x| < \infty$. Let $u(x,0) = u_0(x)$ be the initial profile. The characteristic equations are given by

$$\frac{dx}{dt} = c(u), \quad \frac{du}{dt} = 0.$$

These are solved to give the result that

$$u(x,t) = \text{constant},$$

along the characteristic curves $x'(t) = c(u)$. The lines passing though $u(x_0,) = u_0(x_0)$ have slope $1/c(u_0(x_0))$.

Example 7.8. Solve $u_t + uu_x = 0$, $u(x,0) = e^{-x^2}$.

For this problem $u =$ constant along

$$\frac{dx}{dt} = u.$$

Since u is constant, this equation can be integrated to yield $x = u(x_0, 0)t + x_0$. Inserting the initial condition, $x = e^{-x_0^2}t + x_0$. Therefore, the solution is

$$u(x,t) = e^{-x_0^2} \text{ along } x = e^{-x_0^2}t + x_0.$$

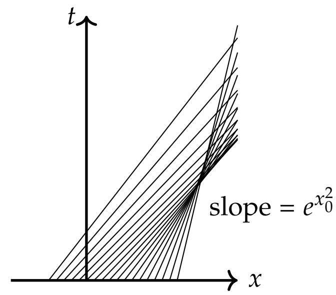


Figure 7.10: The characteristics lines the xt-plane for the nonlinear advection equation.

In Figure 7.10 the characteristics a shown. In this case we see that the characteristics intersect. In Figure charlines3 we look more specifically at the intersection of the characteristic lines for $x_0 = 0$ and $x_0 = 1$. These are approximately the first lines to intersect; i.e., there are (almost) no intersections at earlier times. At the intersection point the function $u(x,t)$ appears to take on more than one value. For the case shown, the solution wants to take the values $u = 0$ and $u = 1$.

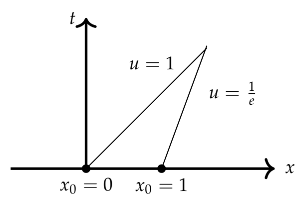


Figure 7.11: The characteristics lines for $x_0 = 0, 1$ in the xt-plane for the nonlinear advection equation.

In Figure 7.12 we see the development of the solution. This is found using a parametric plot of the points $(x_0 + te^{-x_0^2}, e^{-x_0^2})$ for different times. The initial profile propagates to the right with the higher points traveling faster than the lower points since $x'(t) = u > 0$. Around $t = 1.0$ the wave breaks and becomes multivalued. The time at which the function becomes multivalued is called the breaking time.

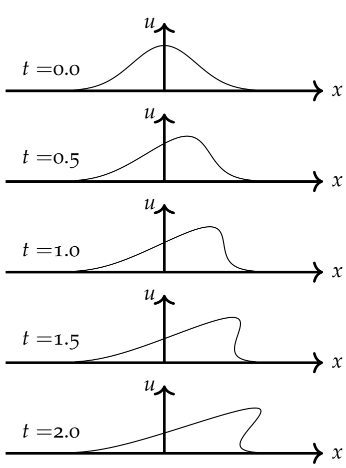


Figure 7.12: The development of a gradient catastrophe in Example 7.8 leading to a multivalued function.

7.4.3 *The Breaking Time*

In the last example we saw that for nonlinear wave speeds a gradient catastrophe might occur. The first time at which a catastrophe occurs is called the breaking time. We will determine the breaking time for the nonlinear advection equation, $u_t + c(u)u_x = 0$. For the characteristic corresponding to $x_0 = \xi$, the wavespeed is given by

$$F(\xi) = c(u_0(\xi))$$

and the characteristic line is given by

$$x = \xi + tF(\xi).$$

The value of the wave function along this characteristic is

$u_0(\xi) = u(\xi, 0)$.

$$\begin{aligned} u(x,t) &= u(\xi + tF(\xi), t) \\ &= . \end{aligned} \tag{7.30}$$

Therefore, the solution is

$$u(x,t) = u_0(\xi) \text{ along } x = \xi + tF(\xi).$$

This means that

$$u_x = u_0'(\xi)\xi_x \quad \textit{and} \quad u_t = u_0'(\xi)\xi_t.$$

We can determine ξ_x and ξ_t using the characteristic line

$$\xi = x - tF(\xi).$$

Then, we have

$$\begin{aligned} \xi_x &= 1 - tF'(\xi)\xi_x \\ &= \frac{1}{1 + tF'(\xi)}. \\ \xi_t &= \frac{\partial}{\partial t}(x - tF(\xi)) \\ &= -F(\xi) - tF'(\xi)\xi_t \\ &= \frac{-F(\xi)}{1 + tF'(\xi)}. \end{aligned} \tag{7.31}$$

Note that ξ_x and ξ_t are undefined if the denominator in both expressions vanishes, $1 + tF'(\xi) = 0$, or at time

$$t = -\frac{1}{F'(\xi)}.$$

The minimum time for this to happen in the breaking time,

The breaking time.

$$t_b = \min\left\{-\frac{1}{F'(\xi)}\right\}. \tag{7.32}$$

Example 7.9. Find the breaking time for $u_t + uu_x = 0$, $u(x,0) = e^{-x^2}$.

Since $c(u) = u$, we have

$$F(\xi) = c(u_0(\xi)) = e^{-\xi^2}$$

and

$$F'(\xi) = -2\xi e^{-\xi^2}.$$

This gives

$$t = \frac{1}{2\xi e^{-\xi^2}}.$$

We need to find the minimum time. Thus, we set the derivative equal to zero and solve for ξ.

$$\begin{aligned} 0 &= \frac{d}{d\xi}\left(\frac{e^{\xi^2}}{2\xi}\right) \\ &= \left(2 - \frac{1}{\xi^2}\right)\frac{e^{\xi^2}}{2}. \end{aligned} \tag{7.33}$$

Thus, the minimum occurs for $2 - \frac{1}{\xi^2} = 0$, or $\xi = 1/\sqrt{2}$. This gives

$$t_b = t\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\frac{2}{\sqrt{2}e^{-1/2}}} = \sqrt{\frac{e}{2}} \approx 1.16. \tag{7.34}$$

7.4.4 *Shock Waves*

Solutions of nonlinear advection equations can become multivalued due to a gradient catastrophe. Namely, the derivatives u_t and u_x become undefined. We would like to extend solutions past the catastrophe. However, this leads to the possibility of discontinuous solutions. Such solutions which may not be differentiable or continuous in the domain are known as weak solutions. In particular, consider the initial value problem

Weak solutions.

$$u_t + \phi_x = 0, \quad x \in R, \quad t > 0, \quad u(x,0) = u_0(x).$$

Then, $u(x,t)$ is a weak solution of this problem if

$$\int_0^\infty \int_{-\infty}^\infty [uv_t + \phi v_x]\, dxdt + \int_{-\infty}^\infty u_0(x)v(x,0)\, dx = 0$$

for all smooth functions $v \in C^\infty(R \times [0,\infty))$ with compact support, i.e., $v \equiv= 0$ outside some compact subset of the domain.

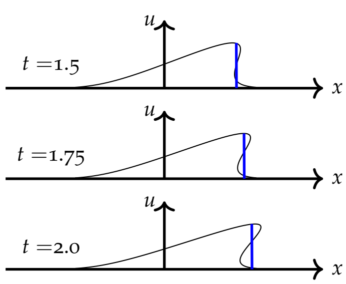


Figure 7.13: The shock solution after the breaking time.

Effectively, the weak solution that evolves will be a piecewise smooth function with a discontinuity, the shock wave, that propagates with shock speed. It can be shown that the form of the shock will be the discontinuity shown in Figure 7.13 such that the areas cut from the solutions will cancel leaving the total area under the solution constant. [See G. B. Whitham's *Linear and Nonlinear Waves*, 1973.] We will consider the discontinuity as shown in Figure 7.14.

We can find the equation for the shock path by using the integral form of the conservation law,

$$\frac{d}{dt}\int_a^b u(x,t)\, dx = \phi(a,t) - \phi(b,t).$$

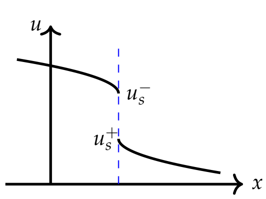


Figure 7.14: Depiction of the jump discontinuity at the shock position.

Recall that one can differentiate under the integral if $u(x,t)$ and $u_t(x,t)$ are continuous in x and t in an appropriate subset of the domain. In particular, we will integrate over the interval $[a,b]$ as shown in Figure 7.15. The domains on either side of shock path are denoted as R^+ and R^- and the limits of $x(t)$ and $u(x,t)$ as one approaches from the left of the shock are denoted by $x_s^-(t)$ and $u^- = u(x_s^-, t)$. Similarly, the limits of $x(t)$ and $u(x,t)$ as one approaches from the right of the shock are denoted by $x_s^+(t)$ and $u^+ = u(x_s^+, t)$.

We need to be careful in differentiating under the integral,

$$\begin{aligned} \frac{d}{dt}\int_a^b u(x,t)\,dx &= \frac{d}{dt}\left[\int_a^{x_s^-(t)} u(x,t)\,dx + \int_{x_s^+(t)}^b u(x,t)\,dx\right] \\ &= \int_a^{x_s^-(t)} u_t(x,t)\,dx + \int_{x_s^+(t)}^b u_t(x,t)\,dx \\ &\quad + u(x_s^-,t)\frac{dx_s^-}{dt} - u(x_s^+,t)\frac{dx_s^+}{dt} \\ &= \phi(a,t) - \phi(b,t). \end{aligned} \tag{7.35}$$

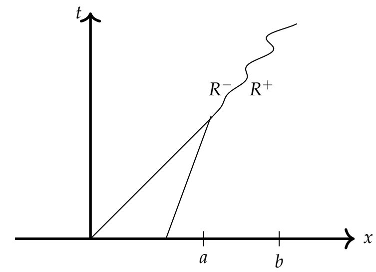


Figure 7.15: Domains on either side of shock path are denoted as R^+ and R^-.

Taking the limits $a \to x_s^-$ and $b \to x_s^+$, we have that

$$\left(u(x_s^-,t) - u(x_s^+,t)\right)\frac{dx_s}{dt} = \phi(x_s^-,t) - \phi(x_s^+,t).$$

Adopting the notation

$$[f] = f(x_s^+) - f(x_s^-),$$

we arrive at the Rankine-Hugonoit jump condition

The Rankine-Hugonoit jump condition.

$$\frac{dx_s}{dt} = \frac{[\phi]}{[u]}. \tag{7.36}$$

This gives the equation for the shock path as will be shown in the next example.

Example 7.10. Consider the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 1, & x \le 0, \\ 0, & x > 0. \end{cases}$$

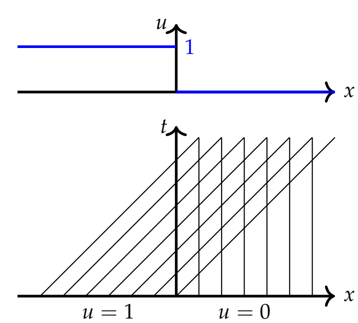


Figure 7.16: Initial condition and characteristics for Example 7.10.

The characteristics for this partial differential equation are familiar by now. The initial condition and characteristics are shown in Figure 7.16. From $x'(t) = u$, there are two possibilities. If $u = 0$, then we have a constant. If $u = 1$ along the characteristics, the we have straight lines of slope one. Therefore, the characteristics are given by

$$x(t) = \begin{cases} x_0, & x > 0, \\ t + x_0, & x < 0. \end{cases}$$

As seen in Figure 7.16 the characteristics intersect immediately at $t = 0$. The shock path is found from the Rankine-Hugonoit jump condition. We first note that $\phi(u) = \frac{1}{2}u^2$, since $\phi_x = uu_x$. Then, we have

$$\begin{aligned} \frac{dx_s}{dt} &= \frac{[\phi]}{[u]} \\ &= \frac{\frac{1}{2}u^{+2} - \frac{1}{2}u^{-2}}{u^+ - u^-} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{(u^+ + u^-)(u^+ - u^-)}{u^+ - u^-} \\
&= \frac{1}{2}(u^+ + u^-) \\
&= \frac{1}{2}(0+1) = \frac{1}{2}. \qquad (7.37)
\end{aligned}
$$

Now we need only solve the ordinary differential equation $x_s'(t) = \frac{1}{2}$ with initial condition $x_s(0) = 0$. This gives $x_s(t) = \frac{t}{2}$. This line separates the characteristics on the left and right side of the shock solution. The solution is given by

$$
u(x,t) = \begin{cases} 1, & x \le t/2, \\ 0, & x > t/2. \end{cases}
$$

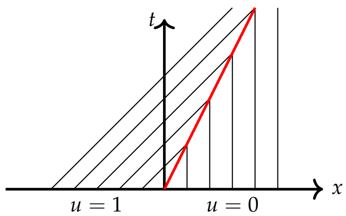


Figure 7.17: The characteristic lines end at the shock path (in red). On the left $u = 1$ and on the right $u = 0$.

In Figure 7.17 we show the characteristic lines ending at the shock path (in red) with $u = 0$ and on the right and $u = 1$ on the left of the shock path. This is consistent with the solution. One just sees the initial step function moving to the right with speed 1/2 without changing shape.

7.4.5 *Rarefaction Waves*

Shocks are not the only type of solutions encountered when the velocity is a function of u. There may sometimes be regions where the characteristic lines do not appear. A simple example is the following.

Example 7.11. Draw the characteristics for the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$
u(x,0) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0. \end{cases}
$$

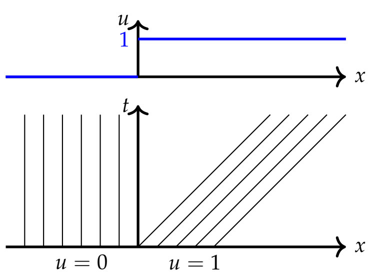


Figure 7.18: Initial condition and characteristics for Example 7.14.

In this case the solution is zero for negative values of x and positive for positive values of x as shown in Figure 7.18. Since the wavespeed is given by u, the $u = 1$ initial values have the waves on the right moving to the right and the values on the left stay fixed. This leads to the characteristics in Figure 7.18 showing a region in the xt-plane that has no characteristics. In this section we will discover how to fill in the missing characteristics and, thus, the details about the solution between the $u = 0$ and $u = 1$ values.

As motivation, we consider a smoothed out version of this problem.

Example 7.12. Draw the characteristics for the initial condition

$$
u(x,0) = \begin{cases} 0, & x \le -\epsilon, \\ \frac{x+\epsilon}{2\epsilon}, & |x| \le \epsilon, \\ 1, & x > \epsilon. \end{cases}
$$

The function is shown in the top graph in Figure 7.19. The leftmost and rightmost characteristics are the same as the previous example.

The only new part is determining the equations of the characteristics for $|x| \leq \epsilon$. These are found using the method of characteristics as

$$x = \xi + u_0(\xi)t, \quad u_0(\xi) = \frac{\xi + \epsilon}{2\epsilon}t.$$

These characteristics are drawn in Figure 7.19 in red. Note that these lines take on slopes varying from infinite slope to slope one, corresponding to speeds going from zero to one.

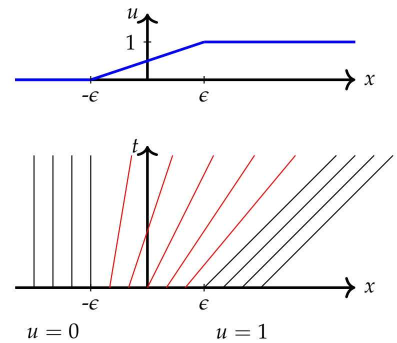


Figure 7.19: The function and characteristics for the smoothed step function.

Characteristics for rarefaction, or expansion, waves are fan-like characteristics.

Comparing the last two examples, we see that as ϵ approaches zero, the last example converges to the previous example. The characteristics in the region where there were none become a "fan". We can see this as follows. Since $|\xi| < \epsilon$ for the fan region, as ϵ gets small, so does this interval. Let's scale ξ as $\xi = \sigma\epsilon$, $\sigma \in [-1, 1]$. Then,

$$x = \sigma\epsilon + u_0(\sigma\epsilon)t, \quad u_0(\sigma\epsilon) = \frac{\sigma\epsilon + \epsilon}{2\epsilon}t = \frac{1}{2}(\sigma + 1)t.$$

For each $\sigma \in [-1, 1]$ there is a characteristic. Letting $\epsilon \to 0$, we have

$$x = ct, \quad c = \frac{1}{2}(\sigma + 1)t.$$

Thus, we have a family of straight characteristic lines in the xt-plane passing through $(0, 0)$ of the form $x = ct$ for c varying from $c = 0$ to $c = 1$. These are shown as the red lines in Figure 7.20.

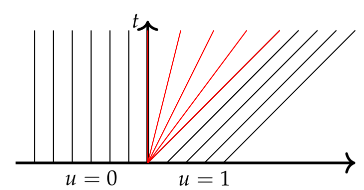


Figure 7.20: The characteristics for Example 7.14 showing the "fan" characteristics.

The fan characteristics can be written as $x/t =$ constant. So, we can seek to determine these characteristics analytically and in a straight forward manner by seeking solutions of the form $u(x,t) = g(\frac{x}{t})$.

Example 7.13. Determine solutions of the form $u(x,t) = g(\frac{x}{t})$ to $u_t + uu_x = 0$.

Inserting this guess into the differential equation, we have

$$\begin{aligned} 0 &= u_t + uu_x \\ &= \frac{1}{t}g'\left(g - \frac{x}{t}\right). \end{aligned} \tag{7.38}$$

Seek rarefaction fan waves using $u(x,t) = g(\frac{x}{t})$.

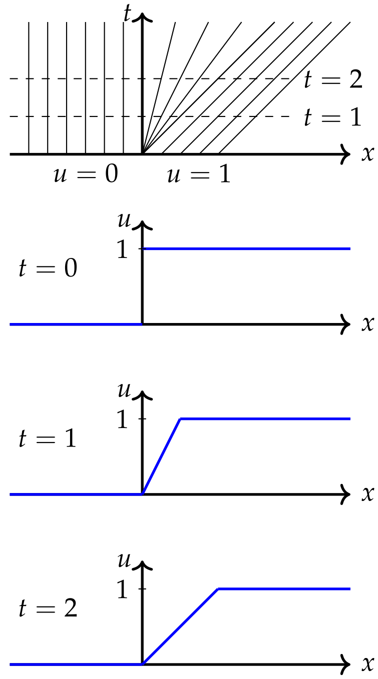
Thus, either $g' = 0$ or $g = \frac{x}{t}$. The first case will not work since this gives constant solutions. The second solution is exactly what we had obtained before. Recall that solutions along characteristics give $u(x,t) = \frac{x}{t} =$ constant. The characteristics and solutions for $t = 0, 1, 2$ are shown in Figure rarefactionfig4. At a specific time one can draw a line (dashed lines in figure) and follow the characteristics back to the $t = 0$ values, $u(\xi, 0)$ in order to construct $u(x,t)$.

As a last example, let's investigate a nonlinear model which possesses both shock and rarefaction waves.

Example 7.14. Solve the initial value problem $u_t + u^2u_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 0, & x \leq 0, \\ 1, & 0 < x < 2, \\ 0, & x \geq 2. \end{cases}$$

Figure 7.21: The characteristics and solutions for $t = 0, 1, 2$ for Example 7.14



The method of characteristics gives

$$\frac{dx}{dt} = u^2, \quad \frac{du}{dt} = 0.$$

Therefore,

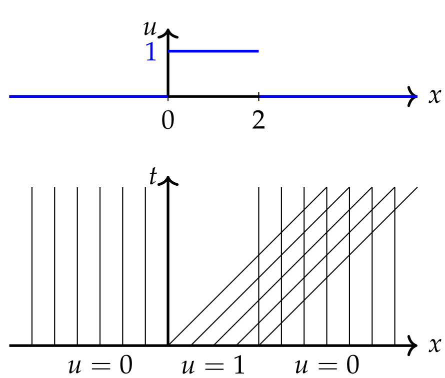
$$u(x,t) = u_0(\xi) = \text{ const. along the lines } x(t) = u_0^2(\xi)t + \xi.$$

There are three values of $u_0(\xi)$,

$$u_0(\xi) = \begin{cases} 0, & \xi \leq 0, \\ 1, & 0 < \xi < 2, \\ 0, & \xi \geq 2. \end{cases}$$

In Figure 7.22 we see that there is a rarefaction and a gradient catastrophe.

Figure 7.22: In this example there occurs a rarefaction and a gradient catastrophe.



In order to fill in the fan characteristics, we need to find solutions $u(x,t) = g(x/t)$. Inserting this guess into the differential equation, we have

$$\begin{aligned} 0 &= u_t + u^2 u_x \\ &= \frac{1}{t} g' \left(g^2 - \frac{x}{t} \right). \end{aligned} \tag{7.39}$$

Thus, either $g' = 0$ or $g^2 = \frac{x}{t}$. The first case will not work since this gives constant solutions. The second solution gives

$$g\left(\frac{x}{t}\right) = \sqrt{\frac{x}{t}}.$$

. Therefore, along the fan characteristics the solutions are $u(x,t) = \sqrt{\frac{x}{t}} =$ constant. These fan characteristics are added in Figure 7.23.

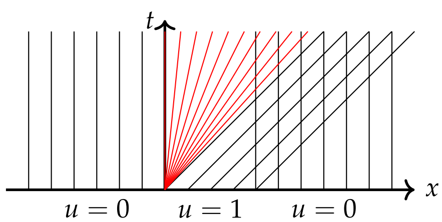


Figure 7.23: The fan characteristics are added to the other characteristic lines.

Next, we turn to the shock path. We see that the first intersection occurs at the point $(x,t) = (2,0)$. The Rankine-Hugonoit condition gives

$$\begin{aligned} \frac{dx_s}{dt} &= \frac{[\phi]}{[u]} \\ &= \frac{\frac{1}{3}u^{+3} - \frac{1}{3}u^{-3}}{u^+ - u^-} \\ &= \frac{1}{3}\frac{(u^+ - u^-)(u^{+2} + u^+u^- + u^{-2})}{u^+ - u^-} \\ &= \frac{1}{3}(u^{+2} + u^+u^- + u^{-2}) \\ &= \frac{1}{3}(0 + 0 + 1) = \frac{1}{3}. \end{aligned} \tag{7.40}$$

Thus, the shock path is given by $x_s'(t) = \frac{1}{3}$ with initial condition $x_s(0) = 2$. This gives $x_s(t) = \frac{t}{3} + 2$. In Figure 7.24 the shock path is shown in red with the fan characteristics and vertical lines meeting the path. Note that the fan lines and vertical lines cross the shock path. This leads to a change in the shock path.

The new path is found using the Rankine-Hugonoit condition with $u^+ = 0$ and $u^- = \sqrt{\frac{x}{t}}$. Thus,

$$\frac{dx_s}{dt} = \frac{[\phi]}{[u]}$$

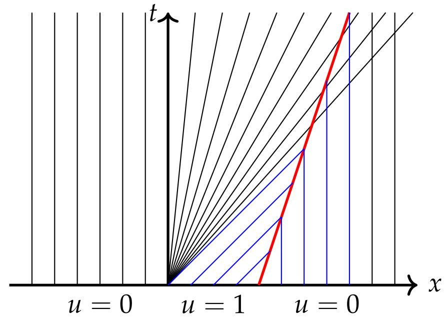


Figure 7.24: The shock path is shown in red with the fan characteristics and vertical lines meeting the path.

$$
\begin{aligned}
&= \frac{\frac{1}{3}u^{+3} - \frac{1}{3}u^{-3}}{u^+ - u^-} \\
&= \frac{1}{3}\frac{(u^+ - u^-)(u^{+2} + u^+u^- + u^{-2})}{u^+ - u^-} \\
&= \frac{1}{3}(u^{+2} + u^+u^- + u^{-2}) \\
&= \frac{1}{3}(0 + 0 + \sqrt{\frac{x_s}{t}}) = \frac{1}{3}\frac{x_s}{t}. \qquad (7.41)
\end{aligned}
$$

We need to solve the initial value problem

$$\frac{dx_s}{dt} = \frac{1}{3}\frac{x_s}{t}, \quad x_s(3) = 3.$$

This can be done using separation of variables. Namely,

$$\int \frac{dx_s}{x_s} = \frac{1}{3}\int \frac{dt}{t}.$$

This gives the solution

$$\ln x_s = \frac{1}{3}\ln t + c \quad \Rightarrow \quad x_s = At^{1/3}.$$

Since the second shock solution starts at the point $(3, 3)$, we can determine $A = 3^{2/3}$. This gives the shock path as

$$x_s(t) = 3^{2/3}t^{1/3}.$$

In Figure 7.25 we show this shock path and the other characteristics ending on the path.

It is interesting to construct the solution at different times based on the characteristics. For a given time, t, one draws a horizontal line in the xt-plane and reads off the values of $u(x,t)$ using the values at $t = 0$ and the rarefaction solutions. This is shown in Figure 7.26. The right discontinuity in the initial profile continues as a shock front until $t = 3$. At that time the back rarefaction wave has caught up to the shock. After $t = 3$, the shock propagates forward slightly slower and the height of the shock begins to decrease. Due to the fact that the partial differential equation is a conservation law, the area under the shock remains constant as it stretches and decays in amplitude.

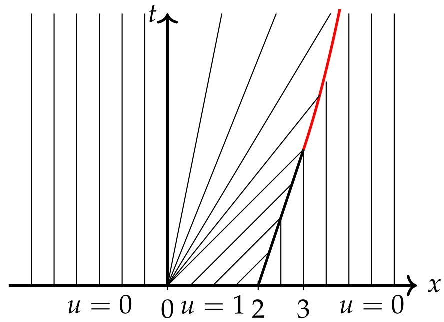


Figure 7.25: The second shock path is shown in red with the characteristics shown in all regions.

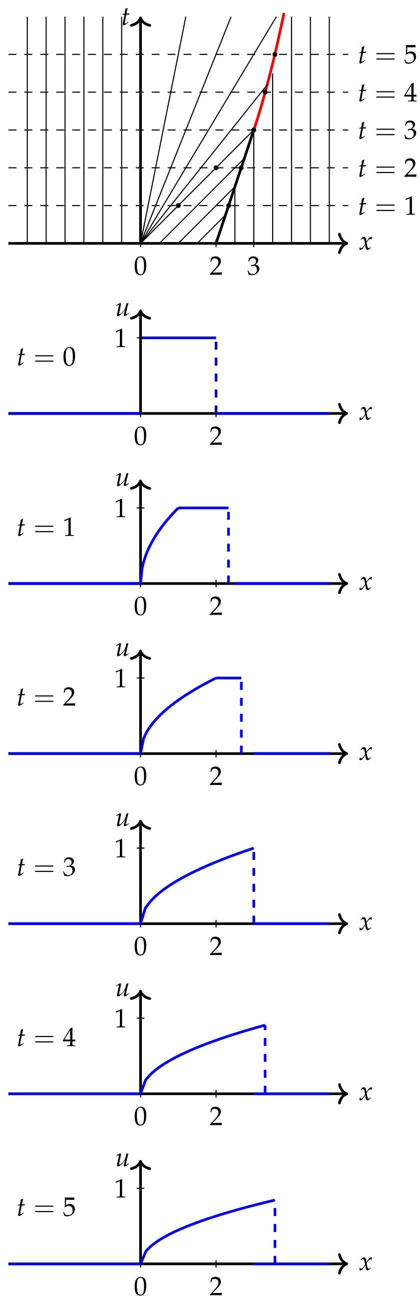


Figure 7.26: Solutions for the shock-rarefaction example.

7.4.6 Traffic Flow

An interesting application is that of traffic flow. We had already derived the flux function. Let's investigate examples with varying initial conditions that lead to shock or rarefaction waves. As we had seen earlier in modeling traffic flow, we can consider the flux function

$$\phi = uv = v_1 \left(u - \frac{u^2}{u_1} \right),$$

which leads to the conservation law

$$u_t + v_1(1 - \frac{2u}{u_1})u_x = 0.$$

Here $u(x,t)$ represents the density of the traffic and u_1 is the maximum density and v_1 is the initial velocity.

First, consider the flow of traffic vas it approaches a red light as shown in Figure 7.27. The traffic that is stopped has reached the maximum density u_1. The incoming traffic has a lower density, u_0. For this red light problem, we consider the initial condition

$$u(x,0) = \begin{cases} u_0, & x < 0, \\ u_1, & x \geq 0. \end{cases}$$

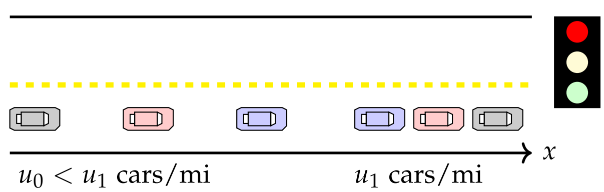


Figure 7.27: Cars approaching a red light.

The characteristics for this problem are given by

$$x = c(u(x_0,t))t + x_0,$$

where

$$c(u(x_0,t)) = v_1 \left(1 - \frac{2u(x_0,0)}{u_1} \right).$$

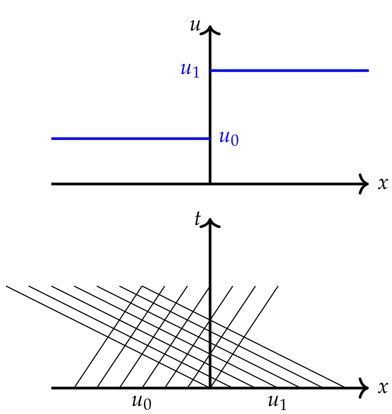


Figure 7.28: Initial condition and characteristics for the red light problem.

Since the initial condition is a piecewise-defined function, we need to consider two cases.

First, for $x \geq 0$, we have

$$c(u(x_0,t)) = c(u_1) = v_1 \left(1 - \frac{2u_1}{u_1} \right) = -v_1.$$

Therefore, the slopes of the characteristics, $x = -v_1 t + x_0$ are $-1/v_1$.

For $x_0 < 0$, we have

$$c(u(x_0,t)) = c(u_0) = v_1 \left(1 - \frac{2u_0}{u_1} \right).$$

So, the characteristics are $x = -v_1\left(1 - \frac{2u_0}{u_1}\right)t + x_0$.

In Figure 7.28 we plot the initial condition and the characteristics for $x < 0$ and $x > 0$. We see that there are crossing characteristics and the begin crossing at $t = 0$. Therefore, the breaking time is $t_b = 0$. We need to find the shock path satisfying $x_s(0) = 0$. The Rankine-Hugonoit conditions give

$$\begin{aligned}
\frac{dx_s}{dt} &= \frac{[\phi]}{[u]} \\
&= \frac{\frac{1}{2}u^{+2} - \frac{1}{2}u^{-2}}{u^+ - u^-} \\
&= \frac{1}{2}\frac{0 - v_1\frac{u_0^2}{u_1}}{u_1 - u_0} \\
&= -v_1\frac{u_0}{u_1}. \qquad (7.42)
\end{aligned}$$

Thus, the shock path is found as $x_s(t) = -v_1\frac{u_0}{u_1}t$.

In Figure 7.29 we show the shock path. In the top figure the red line shows the path. In the lower figure the characteristics are stopped on the shock path to give the complete picture of the characteristics. The picture was drawn with $v_1 = 2$ and $u_0/u_1 = 1/3$.

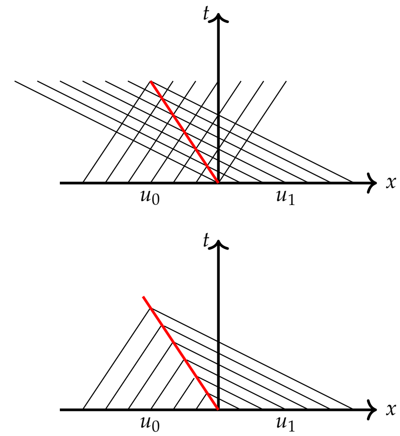


Figure 7.29: The addition of the shock path for the red light problem.

The next problem to consider is stopped traffic as the light turns green. The cars in Figure 7.30 begin to fan out when the traffic light turns green. In this model the initial condition is given by

$$u(x,0) = \begin{cases} u_1, & x \le 0, \\ 0, & x > 0. \end{cases}$$

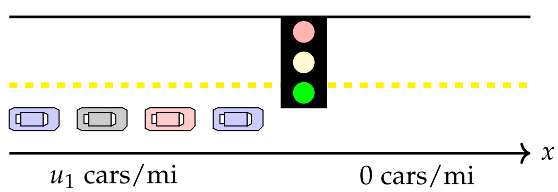


Figure 7.30: Cars begin to fan out when the traffic light turns green.

Again,

$$c(u(x_0,t)) = v_1\left(1 - \frac{2u(x_0,0)}{u_1}\right).$$

Inserting the initial values of u into this expression, we obtain constant speeds, $\pm v_1$. The resulting characteristics are given by

$$x(t) = \begin{cases} -v_1 t + x_0, & x \le 0, \\ v_1 t + x_0, & x > 0. \end{cases}$$

This leads to a rarefaction wave with the solution in the rarefaction region given by

$$u(x,t) = g(x/t) = \frac{1}{2}u_1\left(1 - \frac{1}{v_1}\frac{x}{t}\right).$$

The characteristics are shown in Figure 7.30. The full solution is then

$$u(x,t) = \begin{cases} u_1, & x \leq -v_1 t, \\ g(x/t), & |x| < v_1 t, \\ 0, & x > v_1 t. \end{cases}$$

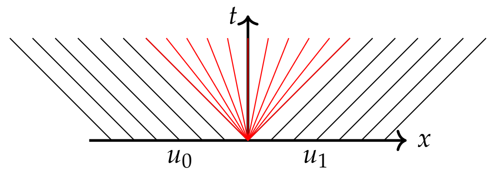


Figure 7.31: The characteristics for the green light problem.

7.5 General First Order PDEs

We have spent time solving quasilinear first order partial differential equations. We now turn to nonlinear first order equations of the form

$$F(x, y, u, u_x, u_y) = 0,$$

for $u = u(x, y)$.

If we introduce new variables, $p = u_x$ and $q = u_y$, then the differential equation takes the form

$$F(x, y, u, p, q) = 0.$$

Note that for $u(x,t)$ a function with continuous derivatives, we have

$$p_y = u_{xy} = u_{yx} = q_x.$$

We can view $F = 0$ as a surface in a five dimensional space. Since the arguments are functions of x and y, we have from the multivariable Chain Rule that

$$\begin{aligned} \frac{dF}{dx} &= F_x + F_u \frac{\partial u}{\partial x} + F_p \frac{\partial p}{\partial x} + F_q \frac{\partial q}{\partial x} \\ 0 &= F_x + pF_u + p_x F_p + p_y F_q. \end{aligned} \tag{7.43}$$

This can be rewritten as a quasilinear equation for $p(x, y)$:

$$F_p p_x + F_q p_x = -F_x - pF_u.$$

The characteristic equations are

$$\frac{dx}{F_p} = \frac{dy}{F_q} = -\frac{dp}{F_x + pF_u}.$$

Similarly, from $\frac{dF}{dy} = 0$ we have that

$$\frac{dx}{F_p} = \frac{dy}{F_q} = -\frac{dq}{F_y + qF_u}.$$

Furthermore, since $u = u(x, y)$,

$$\begin{aligned} du &= \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \\ &= pdx + qdy \\ &= pdx + q\frac{F_q}{F_p}dx \\ &= \left(p + q\frac{F_q}{F_p}\right). \end{aligned} \tag{7.44}$$

Therefore,

$$\frac{dx}{F_p} = \frac{du}{pF_p + qF_q}.$$

Combining these results we have the Charpit Equations

$$\boxed{\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q} = -\frac{dp}{F_x + pF_u} = -\frac{dq}{F_y + qF_u}.} \tag{7.45}$$

The Charpit equations. These were named after the French mathematician Paul Charpit Villecourt, who was probably the first to present the method in his thesis the year of his death, 1784. His work was further extended in 1797 by Lagrange and given a geometric explanation by Gaspard Monge (1746-1818) in 1808. This method is often called the Lagrange-Charpit method.

These equations can be used to find solutions of nonlinear first order partial differential equations as seen in the following examples.

Example 7.15. Find the general solutio of $u_x^2 + yu_y - u = 0$.

First, we introduce $u_x = p$ and $u_y = q$. Then,

$$F(x, y, u, p, q) = p^2 + qy - u = 0.$$

Next we identify

$$F_p = 2p, \quad F_q = y, \quad F_u = -1, \quad F_x = 0, \quad , F_y = q.$$

Then,

$$\begin{aligned} pF_p + qF_q &= 2p^2 + qy, \\ F_x + pF_u &= -p, \\ F_y + qF_u &= q - q = 0. \end{aligned}$$

The Charpit equations are then

$$\frac{dx}{2p} = \frac{dy}{y} = \frac{du}{2p^2 + qy} = \frac{dp}{p} = \frac{dq}{0}.$$

The first conclusion is that $q = c_1 =$ constant. So, from the partial differential equation we have $u = p^2 + c_1 y$.

Since $du = pdx + qdy = pdx + c_1 dy$, then

$$du - cdy = \sqrt{u - c_1 y}\,dx.$$

Therefore,

$$\begin{aligned} \int \frac{d(u - c_1 y)}{\sqrt{u - c_1 y}} &= \int dx \\ \int \frac{z}{\sqrt{z}} &= x + c_2 \\ 2\sqrt{u - c_1 y} &= x + c_2. \end{aligned} \tag{7.46}$$

Solving for u, we have

$$u(x,y) = \frac{1}{4}(x+c_2)^2 + c_1 y.$$

This example required a few tricks to implement the solution. Sometimes one needs to find parametric solutions. Also, if an initial condition is given, one needs to find the particular solution. In the next example we show how parametric solutions are found to the initial value problem.

Example 7.16. Solve the initial value problem $u_x^2 + u_y + u = 0$, $u(x,0) = x$.

We consider the parametric form of the Charpit equations,

$$dt = \frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q} = -\frac{dp}{F_x + pF_u} = -\frac{dq}{F_y + qF_u}. \tag{7.47}$$

This leads to the system of equations

$$\begin{aligned}
\frac{dx}{dt} &= F_p = 2p. \\
\frac{dy}{dt} &= F_q = 1. \\
\frac{du}{dt} &= pF_p + qF_q = 2p^2 + q. \\
\frac{dp}{dt} &= -(F_x + pF_u) = -p. \\
\frac{dq}{dt} &= -(F_y + qF_u) = -q.
\end{aligned}$$

The second, fourth, and fifth equations can be solved to obtain

$$\begin{aligned}
y &= t + c_1. \\
p &= c_2 e^{-t}. \\
q &= c_3 e^{-t}.
\end{aligned}$$

Inserting these results into the remaining equations, we have

$$\begin{aligned}
\frac{dx}{dt} &= 2c_2 e^{-t}. \\
\frac{du}{dt} &= 2c_2^2 e^{-2t} + c_3 e^{-t}.
\end{aligned}$$

These equations can be integrated to find Inserting these results into the remaining equations, we have

$$\begin{aligned}
x &= -2c_2 e^{-t} + c_4. \\
u &= -c_2^2 e^{-2t} - c_3 e^{-t} + c_5.
\end{aligned}$$

This is a parametric set of equations for $u(x,t)$. Since

$$e^{-t} = \frac{x - c_4}{-2c_2},$$

we have

$$
\begin{aligned}
u(x,y) &= -c_2^2 e^{-2t} - c_3 e^{-t} + c_5. \\
&= -c_2^2 \left(\frac{x-c_4}{-2c_2} \right)^2 - c_3 \left(\frac{x-c_4}{-2c_2} \right) + c_5 \\
&= \frac{1}{4}(x-c_4)^2 + \frac{c_3}{2c_2}(x-c_4). \qquad (7.48)
\end{aligned}
$$

We can use the initial conditions by first parametrizing the conditions. Let $x(s,0) = s$ and $y(s,0) = 0$, Then, $u(s,0) = s$. Since $u(x,0) = x$, $u_x(x,0) = 1$, or $p(s,0) = 1$.

From the partial differential equation, we have $p^2 + q + u = 0$. Therefore,

$$q(s,0) = -p^2(s,0) - u(s,0) = -(1+s).$$

These relations imply that

$$
\begin{aligned}
y(s,t)|_{t-0} &= 0 \Rightarrow c_1 = 0. \\
p(s,t)|_{t-0} &= 1 \Rightarrow c_2 = 1. \\
q(s,t)|_{t-0} &= -(1+s) = c_3.
\end{aligned}
$$

So,

$$
\begin{aligned}
y(s,t) &= t. \\
p(s,t) &= e^{-t}. \\
q(s,t) &= -(1+s)e^{-t}.
\end{aligned}
$$

The conditions on x and u give

$$
\begin{aligned}
x(s,t) &= (s+2) - 2e^{-t}, \\
u(s,t) &= (s+1)e^{-t} - e^{-2t}.
\end{aligned}
$$

7.6 Modern Nonlinear PDEs

THE STUDY OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS is a hot research topic. We will (eventually) describe some examples of important evolution equations and discuss their solutions in the last chapter.

Problems

1. Write the following equations in conservation law form, $u_t + \phi_x = 0$ by finding the flux function $\phi(u)$.

a. $u_t + cu_x = 0$.

b. $u_t + uu_x - \mu u_{xx} = 0$.

c. $u_t + 6uu_x + u_{xxx} = 0.$

d. $u_t + u^2 u_x + u_{xxx} = 0.$

2. Consider the Klein-Gordon equation, $u_{tt} - au_{xx} = bu$ for a and b constants. Find traveling wave solutions $u(x,t) = f(x - ct)$.

3. Find the general solution $u(x,y)$ to the following problems.

a. $u_x = 0.$

b. $yu_x - xu_y = 0.$

c. $2u_x + 3u_y = 1.$

d. $u_x + u_y = u.$

4. Solve the following problems.

a. $u_x + 2u_y = 0$, $u(x,0) = \sin x$.

b. $u_t + 4u_x = 0$, $u(x,0) = \frac{1}{1+x^2}$.

c. $yu_x - xu_y = 0$, $u(x,0) = x$.

d. $u_t + xtu_x = 0$, $u(x,0) = \sin x$.

e. $yu_x + xu_y = 0$, $u(0,y) = e^{-y^2}$.

f. $xu_t - 2xtu_x = 2tu$, $u(x,0) = x^2$.

g. $(y - u)u_x + (u - x)u_y = x - y$, $u = 0$ on $xy = 1$.

h. $yu_x + xu_y = xy$, $x, y > 0$, for $u(x,0) = e^{-x^2}, x > 0$ and $u(0,y) = e^{-y^2}, y > 0$.

5. Consider the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition $u(x,0) = \frac{1}{1+x^2}$.

a. Find and plot the characteristics.

b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.

c. Analytically determine the breaking time.

d. Plot solutions $u(x,t)$ at times before and after the breaking time.

6. Consider the problem $u_t + u^2 u_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition $u(x,0) = \frac{1}{1+x^2}$.

a. Find and plot the characteristics.

b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.

c. Analytically determine the breaking time.

d. Plot solutions $u(x,t)$ at times before and after the breaking time.

7. Consider the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 2, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

a. Find and plot the characteristics.

b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.

c. Analytically determine the breaking time.

d. Find the shock wave solution.

8. Consider the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 1, & x \leq 0, \\ 2, & x > 0. \end{cases}$$

a. Find and plot the characteristics.

b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.

c. Analytically determine the breaking time.

d. Find the shock wave solution.

9. Consider the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 0, & x \leq -1, \\ 2, & |x| < 1, \\ 1, & x > 1. \end{cases}$$

a. Find and plot the characteristics.

b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.

c. Analytically determine the breaking time.

d. Find the shock wave solution.

10. Solve the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 1, & x \leq 0, \\ 1 - \frac{x}{a}, & 0 < x < a, \\ 0, & x \geq a. \end{cases}$$

11. Solve the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 0, & x \leq 0, \\ \frac{x}{a}, & 0 < x < a, \\ 1, & x \geq a. \end{cases}$$

12. Consider the problem $u_t + u^2 u_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 2, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

a. Find and plot the characteristics.

b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.

c. Analytically determine the breaking time.

d. Find the shock wave solution.

13. Consider the problem $u_t + u^2 u_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 1, & x \le 0, \\ 2, & x > 0. \end{cases}$$

a. Find and plot the characteristics.

b. Find and plot the fan characteristics.

c. Write out the rarefaction wave solution for all regions of the xt-plane.

14. Solve the initial-value problem $u_t + uu_x = 0$ $|x| < \infty$, $t > 0$ satisfying

$$u(x,0) = \begin{cases} 1, & x \le 0, \\ 1 - x, & 0 \le x \le 1, \\ 0, & x \ge 1. \end{cases}$$

15. Consider the stopped traffic problem in a situation where the maximum car density is 200 cars per mile and the maximum speed is 50 miles per hour. Assume that the cars are arriving at 30 miles per hour. Find the solution of this problem and determine the rate at which the traffic is backing up. How does the answer change if the cars were arriving at 15 miles per hour.

16. Solve the following nonlinear equations where $p = u_x$ and $q = u_y$.

a. $p^2 + q^2 = 1$, $u(x,x) = x$.

b. $pq = u$, $u(0,y) = y^2$.

c. $p + q = pq$, $u(x,0) = x$.

d. $pq = u^2$

e. $p^2 + qy = u$.

17. Find the solution of $xp + qy - p^2 q - u = 0$ in parametric form for the initial conditions at $t = 0$:

$$x(t,s) = s, \quad y(t,s) = 2, \quad u(t,s) = s + 1$$

.