Chapter 7

Special Functions

"You have no idea, how much poetry there is in the calculation of a table of logarithms!" - Carl Friedrich Gauss (1777-1855)

In this chapter we will look at some additional functions which arise often in physical applications and are eigenfunctions for some Sturm-Liouville boundary value problem. We begin with a collection of special functions, called the classical orthogonal polynomials. These include such polynomial functions as the Legendre polynomials, the Hermite polynomials, the Tchebycheff[1] and the Gegenbauer polynomials. Also, Bessel functions occur quite often. We will spend more time exploring the Legendre and Bessel functions. These functions are typically found as solutions of differential equations using power series methods in a first course in differential equations.

[1] In the 1955 letter Tchebycheff or Chebyshev? one finds an interesting discussion on the correct spelling of Pafnuty Lvovich Chebyshev's (1821-1894) last name. Some of the variants have been Tchebichef, Tschebyscheff, Tshebysheff, Chebyshev, Chebychev, Chebysheff, Chebyshov Tchebycheff, Tchebysheff, Tchebychev, Tchebycheff, Tschebyschev or Tschebyscheff. In Russian, the name is Пафну́тий Льво́вич Чебышёв. According to the Soviet Embassy letter, Tchebycheff is the French transliteration and Chebyshev is the English transliteration. In that 1955 letter, a writer notes that the AIP at the time had adopted the French version. Currently, it seems the French have accepted Tchebychev and the AMS uses Chebyshev. The correct transliteration may be Čebyšëv.

7.1 Classical Orthogonal Polynomials

There are other basis functions that can be used to develop series representations of functions. In this section we introduce the classical orthogonal polynomials. We begin by noting that the sequence of functions $\{1, x, x^2, \ldots\}$ is a basis of linearly independent functions. In fact, by the Stone-Weierstraß Approximation Theorem[2] this set is a basis of $L^2_\sigma(a, b)$, the space of square integrable functions over the interval $[a, b]$ relative to weight $\sigma(x)$. However, we will show that the sequence of functions $\{1, x, x^2, \ldots\}$ does not provide an orthogonal basis for these spaces. We will then proceed to find an appropriate orthogonal basis of functions.

[2] **Stone-Weierstraß Approximation Theorem** Suppose f is a continuous function defined on the interval $[a, b]$. For every $\epsilon > 0$, there exists a polynomial function $P(x)$ such that for all $x \in [a, b]$, we have $|f(x) - P(x)| < \epsilon$. Therefore, every continuous function defined on $[a, b]$ can be uniformly approximated as closely as we wish by a polynomial function.

We are familiar with being able to expand functions over a basis such as $\{1, x, x^2, \ldots\}$, since these expansions are just Maclaurin series representations of the functions about $x = 0$,

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^n.$$

However, this basis is not an orthogonal basis of functions. One can easily see this by integrating the product of two even, or two odd, basis functions with $\sigma(x) = 1$ and (a, b)=$(-1, 1)$. For example,

$$\int_{-1}^{1} x^0 x^2\, dx = \frac{2}{3}.$$

The Gram-Schmidt Orthogonalization Process.

Since we have found that orthogonal bases have been useful in determining the coefficients for expansions of given functions, we might ask, "Given a set of linearly independent basis vectors, can one find an orthogonal basis of the given space?" The answer is yes. We recall from introductory linear algebra, which mostly covers finite dimensional vector spaces, that there is a method for carrying this out called the Gram-Schmidt Orthogonalization Process. We will review this process for finite dimensional vectors and then generalize to function spaces.

Let's assume that we have three vectors that span the usual three dimensional space, $\mathbf{R}^3$, given by $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ and shown in Figure 7.1. We seek an orthogonal basis $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, beginning one vector at a time.

First we take one of the original basis vectors, say $\mathbf{a}_1$, and define

$$\mathbf{e}_1 = \mathbf{a}_1.$$

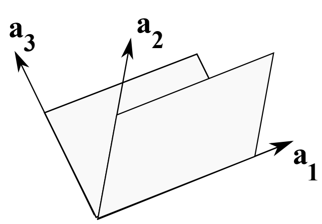


Figure 7.1: The basis $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, of $\mathbf{R}^3$.

It is sometimes useful to normalize these basis vectors, denoting such a normalized vector with a "hat":

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_1}{e_1},$$

where $e_1 = \sqrt{\mathbf{e}_1 \cdot \mathbf{e}_1}$.

Next, we want to determine an $\mathbf{e}_2$ that is orthogonal to $\mathbf{e}_1$. We take another element of the original basis, $\mathbf{a}_2$. In Figure 7.2 we show the orientation of the vectors. Note that the desired orthogonal vector is $\mathbf{e}_2$. We can now write $\mathbf{a}_2$ as the sum of $\mathbf{e}_2$ and the projection of $\mathbf{a}_2$ on $\mathbf{e}_1$. Denoting this projection by $\mathbf{pr}_1\mathbf{a}_2$, we then have

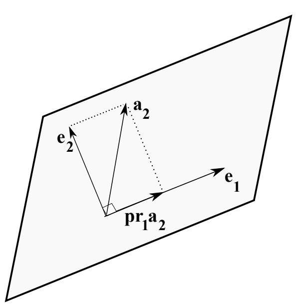


Figure 7.2: A plot of the vectors $\mathbf{e}_1$, $\mathbf{a}_2$, and $\mathbf{e}_2$ needed to find the projection of $\mathbf{a}_2$, on $\mathbf{e}_1$.

$$\mathbf{e}_2 = \mathbf{a}_2 - \mathbf{pr}_1\mathbf{a}_2. \tag{7.1}$$

Recall the projection of one vector onto another from your vector calculus class.

$$\mathbf{pr}_1\mathbf{a}_2 = \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1. \tag{7.2}$$

This is easily proven by writing the projection as a vector of length $a_2 \cos\theta$ in direction $\hat{\mathbf{e}}_1$, where θ is the angle between $\mathbf{e}_1$ and $\mathbf{a}_2$. Using the definition of the dot product, $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$, the projection formula follows.

Combining Equations (7.1)-(7.2), we find that

$$\mathbf{e}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1. \tag{7.3}$$

It is a simple matter to verify that $\mathbf{e}_2$ is orthogonal to $\mathbf{e}_1$:

$$\begin{aligned} \mathbf{e}_2 \cdot \mathbf{e}_1 &= \mathbf{a}_2 \cdot \mathbf{e}_1 - \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1 \cdot \mathbf{e}_1 \\ &= \mathbf{a}_2 \cdot \mathbf{e}_1 - \mathbf{a}_2 \cdot \mathbf{e}_1 = 0. \end{aligned} \tag{7.4}$$

Next, we seek a third vector $\mathbf{e}_3$ that is orthogonal to both $\mathbf{e}_1$ and $\mathbf{e}_2$. Pictorially, we can write the given vector $\mathbf{a}_3$ as a combination of vector projections along $\mathbf{e}_1$ and $\mathbf{e}_2$ with the new vector. This is shown in Figure 7.3. Thus, we can see that

$$\mathbf{e}_3 = \mathbf{a}_3 - \frac{\mathbf{a}_3 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1 - \frac{\mathbf{a}_3 \cdot \mathbf{e}_2}{e_2^2}\mathbf{e}_2. \tag{7.5}$$

Again, it is a simple matter to compute the scalar products with $\mathbf{e}_1$ and $\mathbf{e}_2$ to verify orthogonality.

The Gram-Schmidt Orthogonalization for a vector basis.

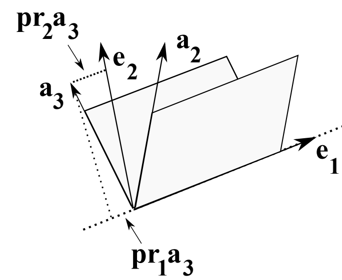


Figure 7.3: A plot of vectors for determining $\mathbf{e}_3$.

We can easily generalize this procedure to the N-dimensional case. Let $\mathbf{a}_n$, $n = 1, ..., N$ be a set of linearly independent vectors in $\mathbf{R}^N$. Then, an orthogonal basis can be found by setting $\mathbf{e}_1 = \mathbf{a}_1$ and defining

$$\mathbf{e}_n = \mathbf{a}_n - \sum_{j=1}^{n-1} \frac{\mathbf{a}_n \cdot \mathbf{e}_j}{e_j^2}\mathbf{e}_j, \quad n = 2, 3, \ldots, N. \tag{7.6}$$

Now, we can generalize this idea to (real) function spaces. Let $f_n(x)$, $n \in N_0 = \{0, 1, 2, \ldots\}$, be a linearly independent sequence of continuous functions defined for $x \in [a, b]$. Then, an orthogonal basis of functions, $\phi_n(x)$, $n \in N_0$ can be found and is given by

$$\phi_0(x) = f_0(x)$$

and

The Gram-Schmidt Orthogonalization for a basis of functions

$$\phi_n(x) = f_n(x) - \sum_{j=0}^{n-1} \frac{\langle f_n, \phi_j \rangle}{\|\phi_j\|^2} \phi_j(x), \quad n = 1, 2, \ldots. \tag{7.7}$$

Here we are using inner products relative to weight $\sigma(x)$,

$$\langle f, g \rangle = \int_a^b f(x)g(x)\sigma(x)\, dx. \tag{7.8}$$

Note the similarity between the orthogonal basis in (7.7) and the expression for the finite dimensional case in Equation (7.6).

Example 7.1. Apply the Gram-Schmidt Orthogonalization process to the set $f_n(x) = x^n$, $n \in N_0$, when $x \in (-1, 1)$ and $\sigma(x) = 1$.

First, we have $\phi_0(x) = f_0(x) = 1$. Note that

$$\int_{-1}^{1} \phi_0^2(x)\, dx = 2.$$

We could use this result to fix the normalization of the new basis, but we will hold off doing that for now.

Now, we compute the second basis element:

$$\begin{aligned} \phi_1(x) &= f_1(x) - \frac{\langle f_1, \phi_0 \rangle}{\|\phi_0\|^2} \phi_0(x) \\ &= x - \frac{\langle x, 1 \rangle}{\|1\|^2} 1 = x, \end{aligned} \tag{7.9}$$

since $\langle x, 1 \rangle$ is the integral of an odd function over a symmetric interval.

For $\phi_2(x)$, we have

$$\begin{aligned}
\phi_2(x) &= f_2(x) - \frac{\langle f_2, \phi_0 \rangle}{\|\phi_0\|^2}\,\phi_0(x) - \frac{\langle f_2, \phi_1 \rangle}{\|\phi_1\|^2}\,\phi_1(x) \\
&= x^2 - \frac{\langle x^2, 1 \rangle}{\|1\|^2}\,1 - \frac{\langle x^2, x \rangle}{\|x\|^2}\,x \\
&= x^2 - \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} dx} \\
&= x^2 - \frac{1}{3}. \qquad (7.10)
\end{aligned}$$

So far, we have the orthogonal set $\{1, x, x^2 - \frac{1}{3}\}$. If one chooses to normalize these by forcing $\phi_n(1) = 1$, then one obtains the classical Legendre polynomials, $P_n(x)$. Thus,

$$P_2(x) = \frac{1}{2}(3x^2 - 1).$$

Note that this normalization is different than the usual one. In fact, we see the $P_2(x)$ does not have a unit norm,

$$\|P_2\|^2 = \int_{-1}^{1} P_2^2(x)\,dx = \frac{2}{5}.$$

The set of Legendre[3] polynomials is just one set of classical orthogonal polynomials that can be obtained in this way. Many of these special functions had originally appeared as solutions of important boundary value problems in physics. They all have similar properties and we will just elaborate some of these for the Legendre functions in the next section. Others in this group are shown in Table 7.1.

[3] Adrien-Marie Legendre (1752-1833) was a French mathematician who made many contributions to analysis and algebra.

Table 7.1: Common classical orthogonal polynomials with the interval and weight function used to define them.

Polynomial	Symbol	Interval	$\sigma(x)$
Hermite	$H_n(x)$	$(-\infty, \infty)$	e^{-x^2}
Laguerre	$L_n^\alpha(x)$	$[0, \infty)$	e^{-x}
Legendre	$P_n(x)$	(-1,1)	1
Gegenbauer	$C_n^\lambda(x)$	(-1,1)	$(1 - x^2)^{\lambda - 1/2}$
Tchebycheff of the 1st kind	$T_n(x)$	(-1,1)	$(1 - x^2)^{-1/2}$
Tchebycheff of the 2nd kind	$U_n(x)$	(-1,1)	$(1 - x^2)^{-1/2}$
Jacobi	$P_n^{(\nu,\mu)}(x)$	(-1,1)	$(1 - x)^\nu (1 - x)^\mu$

For reference, we also note the differential equations satisfied by these functions as shown in Table 7.2.

7.2 *Fourier-Legendre Series*

In an earlier chapter we saw how useful Fourier series expansions were for solving the heat equation. When studying partial differential equa-

Polynomial	Differential Equation
Hermite	$y'' - 2xy' + 2ny = 0$
Laguerre	$xy'' + (\alpha + 1 - x)y' + ny = 0$
Legendre	$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$
Gegenbauer	$(1 - x^2)y'' - (2n+3)xy' + \lambda y = 0$
Tchebycheff of the 1st kind	$(1 - x^2)y'' - xy' + n^2 y = 0$
Jacobi	$(1 - x^2)y'' + (\nu - \mu + (\mu + \nu + 2)x)y'$
	$+n(n + 1 + \mu + \nu)y = 0$

Table 7.2: Differential equations satisfied by some of the common classical orthogonal polynomials.

tions in higher dimensions, one finds that problems with spherical symmetry may lead to the series representations in terms of a basis of Legendre polynomials. For example, we could consider the steady state temperature distribution inside a hemispherical igloo, which takes the form

$$\phi(r, \theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

in spherical coordinates. Evaluating this function at the surface $r = a$ as $\phi(a, \theta) = f(\theta)$, leads to a Fourier-Legendre series expansion of function f :

$$f(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta),$$

where $c_n = A_n a^n$

In this section we would like to explore Fourier-Legendre series expansions of functions $f(x)$ defined on $(-1, 1)$:

$$f(x) \sim \sum_{n=0}^{\infty} c_n P_n(x). \tag{7.11}$$

As with Fourier trigonometric series, we can determine the expansion coefficients by multiplying both sides of Equation (7.11) by $P_m(x)$ and integrating for $x \in [-1, 1]$. Orthogonality gives the usual form for the generalized Fourier coefficients,

$$c_n = \frac{\langle f, P_n \rangle}{\|P_n\|^2}, n = 0, 1, \ldots.$$

We will later show that

$$\|P_n\|^2 = \frac{2}{2n+1}.$$

Therefore, the Fourier-Legendre coefficients are

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx. \tag{7.12}$$

7.2.1 *Properties of Legendre Polynomials*

We can do examples of Fourier-Legendre expansions given just a few facts about Legendre polynomials. The first property that the Legendre polynomials have is the Rodrigues formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n, \quad n \in N_0. \tag{7.13}$$

The Rodrigues Formula is credited to Benjamin Olinde Rodrigues (1795-1851) who discovered it in 1816 according to Hermite in 1865 and named by Heinrich Heine in 1878. The formula was also independently discovered by Sir James Ivory (1824) and Carl Gustav Jacobi (1827). Similar formulae give the other classical orthogonal polynomials.

From the Rodrigues formula, one can show that $P_n(x)$ is an nth degree polynomial. Also, for n odd, the polynomial is an odd function and for n even, the polynomial is an even function.

Example 7.2. Determine $P_2(x)$ from Rodrigues formula:

$$\begin{aligned} P_2(x) &= \frac{1}{2^2 2!} \frac{d^2}{dx^2}(x^2-1)^2 \\ &= \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1) \\ &= \frac{1}{8}\frac{d}{dx}(4x^3-4x) \\ &= \frac{1}{8}(12x^2-4) \\ &= \frac{1}{2}(3x^2-1). \end{aligned} \tag{7.14}$$

Note that we get the same result as we found in the last section using orthogonalization.

Table 7.3: Tabular computation of the Legendre polynomials using the Rodrigues formula.

n	$(x^2-1)^n$	$\frac{d^n}{dx^n}(x^2-1)^n$	$\frac{1}{2^n n!}$	$P_n(x)$
0	1	1	1	1
1	x^2-1	$2x$	$\frac{1}{2}$	x
2	x^4-2x^2+1	$12x^2-4$	$\frac{1}{8}$	$\frac{1}{2}(3x^2-1)$
3	$x^6-3x^4+3x^2-1$	$120x^3-72x$	$\frac{1}{48}$	$\frac{1}{2}(5x^3-3x)$

The first several Legendre polynomials are computed using the Rodrigues formula in Table 7.3. In Figure 7.4 we show plots of these Legendre polynomials.

All of the classical orthogonal polynomials satisfy a three term recursion formula (or, recurrence relation or formula). In the case of the Legendre polynomials, we have

The Three Term Recursion Formula. This is also referred to as Bonnet's recursion formula. Named after Pierre Ossian Bonnet (1819-1892), who published the derivation in Bonnet [1852]. It was also in this paper that he developed the Fourier-Legendre series expansion. He designated them X_n.

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad n = 1, 2, \ldots. \tag{7.15}$$

This can also be rewritten by replacing n with $n-1$ as

$$(2n-1)xP_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x), \quad n = 1, 2, \ldots. \tag{7.16}$$

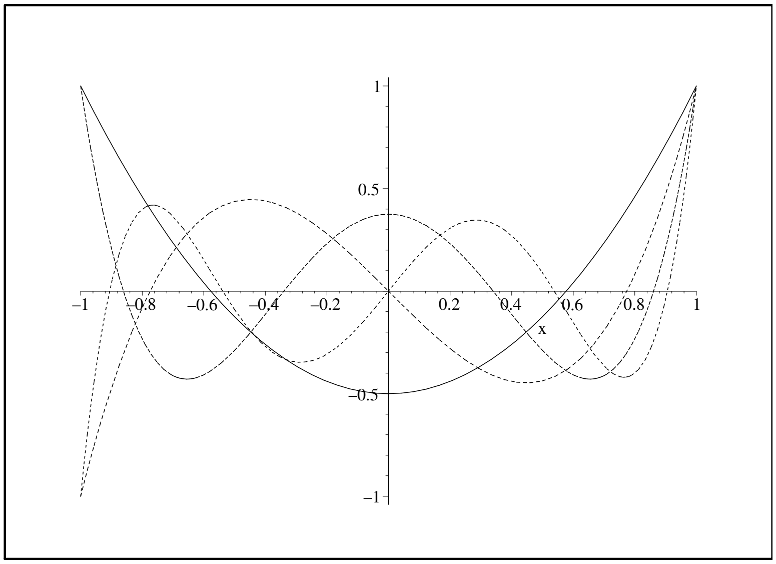


Figure 7.4: Plots of the Legendre polynomials $P_2(x)$, $P_3(x)$, $P_4(x)$, and $P_5(x)$.

Example 7.3. Use the recursion formula to find $P_2(x)$ and $P_3(x)$, given that $P_0(x) = 1$ and $P_1(x) = x$.

We first begin by inserting $n = 1$ into Equation (7.15):

$$2P_2(x) = 3xP_1(x) - P_0(x) = 3x^2 - 1.$$

So, $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

For $n = 2$, we have

$$\begin{aligned} 3P_3(x) &= 5xP_2(x) - 2P_1(x) \\ &= \frac{5}{2}x(3x^2 - 1) - 2x \\ &= \frac{1}{2}(15x^3 - 9x). \end{aligned} \tag{7.17}$$

This gives $P_3(x) = \frac{1}{2}(5x^3 - 3x)$. These expressions agree with the earlier results.

We will prove the three term recursion relation in two ways. The first proof we mention is using the orthogonality of the Legendre polynomials and is provided in Appendix 7.7. A more algebraic proof relies on the generating function for Legendre polynomials.

7.2.2 *The Generating Function for Legendre Polynomials*

A SECOND PROOF OF THE THREE TERM RECURSION FORMULA can be obtained from the generating function of the Legendre polynomials. Many special functions have such generating functions. In this case it is given by

$$g(x,t) = \frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \quad |x| \le 1, |t| < 1. \tag{7.18}$$

This generating function occurs often in applications. In particular, it arises in potential theory, such as electromagnetic or gravitational potentials. These potential functions are $\frac{1}{r}$ type functions.

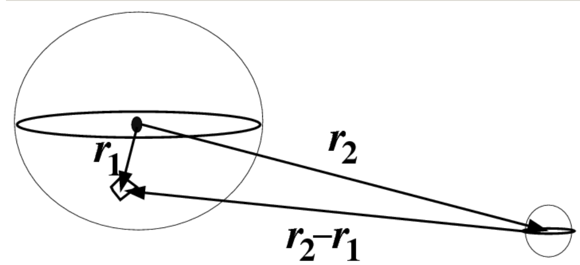


Figure 7.5: The position vectors used to describe the tidal force on the Earth due to the moon.

Legendre said that Pierre-Simon Laplace (1749-1827) introduced the potential function and Legendre provided the expansion. The polynomials were named in 1875 by Isaac Todhunter (1820-1884) as Legendre coefficients. Laplace and Legendre had written memoirs which came out in 1783 and 1785, respectively. But Legendre's work was published several years after it was written.

For example, the gravitational potential between the Earth and the moon is proportional to the reciprocal of the magnitude of the difference between their positions relative to some coordinate system. An even better example, would be to place the origin at the center of the Earth and consider the forces on the non-pointlike Earth due to the moon. Consider a piece of the Earth at position $\mathbf{r}_1$ and the moon at position $\mathbf{r}_2$ as shown in Figure 7.5. The tidal potential Φ is proportional to

$$\Phi \propto \frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|} = \frac{1}{\sqrt{(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{r}_2 - \mathbf{r}_1)}} = \frac{1}{\sqrt{r_1^2 - 2r_1r_2\cos\theta + r_2^2}},$$

where θ is the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$.

Typically, one of the position vectors is much larger than the other. Let's assume that $r_1 \ll r_2$. Then, one can write

$$\Phi \propto \frac{1}{\sqrt{r_1^2 - 2r_1r_2\cos\theta + r_2^2}} = \frac{1}{r_2}\frac{1}{\sqrt{1 - 2\frac{r_1}{r_2}\cos\theta + \left(\frac{r_1}{r_2}\right)^2}}.$$

Now, define $x = \cos\theta$ and $t = \frac{r_1}{r_2}$. We then have that the tidal potential is proportional to the generating function for the Legendre polynomials! So, we can write the tidal potential as

$$\Phi \propto \frac{1}{r_2}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r_1}{r_2}\right)^n.$$

The first term in the expansion, $\frac{1}{r_2}$, is the gravitational potential that gives the usual force between the Earth and the moon. [Recall that the gravitational potential for mass m at distance r from M is given by $\Phi = -\frac{GMm}{r}$ and that the force is the gradient of the potential, $\mathbf{F} = -\nabla\Phi \propto \nabla\left(\frac{1}{r}\right)$.] The next terms will give expressions for the tidal effects.

Now that we have some idea as to where this generating function might have originated, we can proceed to use it. First of all, the generating function can be used to obtain special values of the Legendre polynomials.

Example 7.4. Evaluate $P_n(0)$ using the generating function.

$P_n(0)$ is found by considering $g(0,t)$. Setting $x = 0$ in Equation (7.18), we have

$$\begin{aligned} g(0,t) &= \frac{1}{\sqrt{1+t^2}} \\ &= \sum_{n=0}^{\infty} P_n(0)t^n \\ &= P_0(0) + P_1(0)t + P_2(0)t^2 + P_3(0)t^3 + \ldots . \end{aligned} \tag{7.19}$$

We can use the binomial expansion to find the final answer. Namely, we have

$$\frac{1}{\sqrt{1+t^2}} = 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 + \ldots .$$

Comparing these expansions, we have the $P_n(0) = 0$ for n odd and for even integers one can show (see Problem 10) that[4]

$$P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}, \tag{7.20}$$

where $n!!$ is the *double factorial*,

$$n!! = \begin{cases} n(n-2)\ldots(3)1, & n > 0, \text{odd}, \\ n(n-2)\ldots(4)2, & n > 0, \text{even}, \\ 1 & n = 0, -1 \end{cases} .$$

[4] This example can be finished by first proving that

$$(2n)!! = 2^n n!$$

and

$$(2n-1)!! = \frac{(2n)!}{(2n)!!} = \frac{(2n)!}{2^n n!}.$$

Example 7.5. Evaluate $P_n(-1)$.

This is a simpler problem. In this case we have

$$g(-1,t) = \frac{1}{\sqrt{1+2t+t^2}} = \frac{1}{1+t} = 1 - t + t^2 - t^3 + \ldots .$$

Therefore, $P_n(-1) = (-1)^n$.

Example 7.6. Prove the three term recursion formula,

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0, \quad k = 1, 2, \ldots,$$

using the generating function.

Proof of the three term recursion formula using the generating function.

We can also use the generating function to find recurrence relations. To prove the three term recursion (7.15) that we introduced above, then we need only differentiate the generating function with respect to t in Equation (7.18) and rearrange the result. First note that

$$\frac{\partial g}{\partial t} = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \frac{x-t}{1-2xt+t^2} g(x,t).$$

Combining this with

$$\frac{\partial g}{\partial t} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1},$$

we have

$$(x-t)g(x,t) = (1-2xt+t^2)\sum_{n=0}^{\infty} nP_n(x)t^{n-1}.$$

Inserting the series expression for $g(x,t)$ and distributing the sum on the right side, we obtain

$$(x-t)\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} nP_n(x)t^{n-1} - \sum_{n=0}^{\infty} 2nxP_n(x)t^n + \sum_{n=0}^{\infty} nP_n(x)t^{n+1}.$$

Multiplying out the $x-t$ factor and rearranging, leads to three separate sums:

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} - \sum_{n=0}^{\infty}(2n+1)xP_n(x)t^n + \sum_{n=0}^{\infty}(n+1)P_n(x)t^{n+1} = 0. \quad (7.21)$$

Each term contains powers of t that we would like to combine into a single sum. This is done by reindexing. For the first sum, we could use the new index $k = n-1$. Then, the first sum can be written

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} = \sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k.$$

Using different indices is just another way of writing out the terms. Note that

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} = 0 + P_1(x) + 2P_2(x)t + 3P_3(x)t^2 + \ldots$$

and

$$\sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k = 0 + P_1(x) + 2P_2(x)t + 3P_3(x)t^2 + \ldots$$

actually give the same sum. The indices are sometimes referred to as *dummy indices* because they do not show up in the expanded expression and can be replaced with another letter.

If we want to do so, we could now replace all of the k's with n's. However, we will leave the k's in the first term and now reindex the next sums in Equation (7.21). The second sum just needs the replacement $n = k$ and the last sum we reindex using $k = n+1$. Therefore, Equation (7.21) becomes

$$\sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k - \sum_{k=0}^{\infty}(2k+1)xP_k(x)t^k + \sum_{k=1}^{\infty} kP_{k-1}(x)t^k = 0. \quad (7.22)$$

We can now combine all of the terms, noting the $k=-1$ term is automatically zero and the $k=0$ terms give

$$P_1(x) - xP_0(x) = 0. \quad (7.23)$$

Of course, we know this already. So, that leaves the $k>0$ terms:

$$\sum_{k=1}^{\infty}\left[(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x)\right]t^k = 0. \quad (7.24)$$

Since this is true for all t, the coefficients of the t^k's are zero, or

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0, \quad k = 1, 2, \ldots.$$

While this is the standard form for the three term recurrence relation, the earlier form is obtained by setting $k = n-1$.

There are other recursion relations which we list in the box below. Equation (7.25) was derived using the generating function. Differentiating it with respect to x, we find Equation (7.26). Equation (7.27) can be proven using the generating function by differentiating $g(x,t)$ with respect to x and rearranging the resulting infinite series just as in this last manipulation. This will be left as Problem 7. Combining this result with Equation (7.25), we can derive Equations (7.28)-(7.29). Adding and subtracting these equations yields Equations (7.30)-(7.31).

Recursion Formulae for Legendre Polynomials for $n = 1, 2, \ldots$.

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \tag{7.25}$$

$$(n+1)P'_{n+1}(x) = (2n+1)[P_n(x) + xP'_n(x)] - nP'_{n-1}(x) \tag{7.26}$$

$$P_n(x) = P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x) \tag{7.27}$$

$$P'_{n-1}(x) = xP'_n(x) - nP_n(x) \tag{7.28}$$

$$P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x) \tag{7.29}$$

$$P'_{n+1}(x) + P'_{n-1}(x) = 2xP'_n(x) + P_n(x). \tag{7.30}$$

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x). \tag{7.31}$$

$$(x^2-1)P'_n(x) = nxP_n(x) - nP_{n-1}(x) \tag{7.32}$$

Finally, Equation (7.32) can be obtained using Equations (7.28) and (7.29). Just multiply Equation (7.28) by x,

$$x^2P'_n(x) - nxP_n(x) = xP'_{n-1}(x).$$

Now use Equation (7.29), but first replace n with $n-1$ to eliminate the $xP'_{n-1}(x)$ term:

$$x^2P'_n(x) - nxP_n(x) = P'_n(x) - nP_{n-1}(x).$$

Rearranging gives the Equation (7.32).

Example 7.7. Use the generating function to prove

$$\|P_n\|^2 = \int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}.$$

The normalization constant.

Another use of the generating function is to obtain the normalization constant. This can be done by first squaring the generating function in order to get the products $P_n(x)P_m(x)$, and then integrating over x.

Squaring the generating function has to be done with care, as we need to make proper use of the dummy summation index. So, we first write

$$\begin{aligned} \frac{1}{1-2xt+t^2} &= \left[\sum_{n=0}^{\infty} P_n(x)t^n\right]^2 \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(x)P_m(x)t^{n+m}. \end{aligned} \tag{7.33}$$

Integrating from $x = -1$ to $x = 1$ and using the orthogonality of the Legendre polynomials, we have

$$\begin{aligned} \int_{-1}^{1} \frac{dx}{1-2xt+t^2} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t^{n+m} \int_{-1}^{1} P_n(x)P_m(x)\,dx \\ &= \sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} P_n^2(x)\,dx. \end{aligned} \tag{7.34}$$

However, one can show that[5]

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right).$$

Expanding this expression about $t = 0$, we obtain[6]

$$\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right) = \sum_{n=0}^{\infty} \frac{2}{2n+1}t^{2n}.$$

Comparing this result with Equation (7.34), we find that

$$\|P_n\|^2 = \int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}. \tag{7.35}$$

[5] You will need the integral

$$\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx) + C.$$

[6] You will need the series expansion

$$\begin{aligned} \ln(1+x) &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots. \end{aligned}$$

7.2.3 The Differential Equation for Legendre Polynomials

THE LEGENDRE POLYNOMIALS SATISFY a second order linear differential equation. This differential equation occurs naturally in the solution of initial-boundary value problems in three dimensions which possess some spherical symmetry. We will see this in the next chapter. There are two approaches we could take in showing that the Legendre polynomials satisfy a particular differential equation. Either we can write down the equations and attempt to solve it,[7] or we could use the above properties to obtain the equation. For now, we will seek the differential equation satisfied by $P_n(x)$ using the above recursion relations.

We begin by differentiating Equation (7.32) and using Equation (7.28) to simplify:

$$\begin{aligned} \frac{d}{dx}\left((x^2-1)P_n'(x)\right) &= nP_n(x) + nxP_n'(x) - nP_{n-1}'(x) \\ &= nP_n(x) + n^2P_n(x) \\ &= n(n+1)P_n(x). \end{aligned} \tag{7.36}$$

[7] The standard approach to solving Legendre's Equation (7.37) is to use power series. This is typically seen in an introductory differential equations course. We insert

$$y(x) = \sum_{k=0}^{\infty} c_k x^k, \quad x \in (-1,1)$$

into Equation (7.37) to find that the coefficient satisfy a recurrence relation,

$$c_{k+1} = \frac{(n-k)(n+k+1)}{(k+1)(k+2)}c_k, \quad k \geq 0.$$

When $k = n$, the series truncates to a polynomial.

Therefore, Legendre polynomials, or Legendre functions of the first kind, are solutions of the differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0. \tag{7.37}$$

Legendre's differential equation.

As this is a linear second order differential equation, we expect two linearly independent solutions. The second solution, called the Legendre function of the second kind, is given by $Q_n(x)$ and is not well behaved at $x = \pm 1$. For example,

$$Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x}.$$

We will not need these for physically interesting examples in this book.

A generalization of the Legendre equation is given by $(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0$. Solutions to this equation, $P_n^m(x)$ and $Q_n^m(x)$, are called the associated Legendre functions of the first and second kind.

Example 7.8. Find the series solution to the Legendre equation (7.37) for n a nonnegative integer.

We first note that $x = 0$ is an ordinary point.[8] Therefore, as we learn in a first course in differential equations, we can proceed to obtain solutions in the form of Maclaurin series expansions. Insert the series expansions

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} c_k x^k, \\ y'(x) &= \sum_{k=1}^{\infty} k c_k x^{k-1}, \\ y''(x) &= \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2}, \end{aligned} \tag{7.38}$$

into the differential equation to obtain

$$\begin{aligned} 0 &= (1-x^2)y'' - 2xy' + n(n+1)y \\ &= (1-x^2)\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - 2x\sum_{k=1}^{\infty} k c_k x^{k-1} + n(n+1)\sum_{k=0}^{\infty} c_k x^k \\ &= \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - \sum_{k=2}^{\infty} k(k-1)c_k x^k - \sum_{k=1}^{\infty} 2k c_k x^k \\ &\quad + \sum_{k=0}^{\infty} n(n+1)c_k x^k \\ &= \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} + \sum_{k=0}^{\infty}[n(n+1) - k(k+1)]c_k x^k. \end{aligned} \tag{7.39}$$

Re-indexing the first sum with $\ell = k-2$, we have

$$\begin{aligned} 0 &= \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} + \sum_{k=0}^{\infty}[n(n+1) - k(k+1)]c_k x^k \\ &= \sum_{\ell=0}^{\infty}(\ell+2)(\ell+1)c_{\ell+2}x^{\ell} + \sum_{\ell=0}^{\infty}[n(n+1) - \ell(\ell+1)]c_\ell x^\ell \\ &= 2c_2 + 6c_3 x + n(n+1)c_0 + n(n+1)c_1 x - 2c_1 x \\ &\quad + \sum_{\ell=2}^{\infty}\left((\ell+2)(\ell+1)c_{\ell+2} + [n(n+1) - \ell(\ell+1)]c_\ell\right)x^\ell. \end{aligned} \tag{7.40}$$

[8] For a second order differential equations in the form

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0,$$

where $P(x)$, $Q(x)$, and $R(x)$ are polynomials in x, x_0 is called an ordinary point if $P(x_0) \neq 0$. Otherwise, x_0 is called a singular point.

Matching terms, we have

$$\begin{aligned} \ell &= 0: \quad 2c_2 = -n(n+1)c_0. \\ \ell &= 1: \quad 6c_3 = [2-n(n+1)]c_1. \\ \ell &\geq 2: \quad (\ell+2)(\ell+1)c_{\ell+2} = [\ell(\ell+1)-n(n+1)]c_\ell. \end{aligned} \tag{7.41}$$

For $n = 0$, the first equation gives $c_2 = 0$ and the third equation gives $c_{2m} = 0$ for $m = 1, 2, 3, \ldots$. This leads to $y_1(x) = c_0$ as the solution for $n = 0$.

Similarly, for $n = 1$, the second equation gives $c_3 = 0$ and the third equation gives $c_{2m+1} = 0$ for $m = 1, 2, 3, \ldots$. Thus, $y_1(x) = c_1 x$ is a solution for $n = 1$.

In fact, for n any nonnegative integer the series truncates. For example, if $n = 2$, then these equations reduce to

$$\begin{aligned} \ell &= 0: \quad 2c_2 = -6c_0. \\ \ell &= 1: \quad 6c_3 = -4c_1. \\ \ell &\geq 2: \quad (\ell+2)(\ell+1)c_{\ell+2} = [\ell(\ell+1)-2(3)]c_\ell. \end{aligned} \tag{7.42}$$

For $\ell = 2$, we have $12c_4 = 0$. So, $c_6 = c_8 = \ldots = 0$. Also, we have $c_2 = -3c_0$. This gives

$$y(x) = c_0(1-3x^2) + (c_1 x + c_3 x^3 + c_5 x^5 + c_7 x^7 + \ldots).$$

Therefore, there is a polynomial solution of degree 2. The remaining coefficients are proportional to c_1, yielding the second linearly independent solution, which is not a polynomial.

For other nonnegative integer values of $n > 2$, we have

$$c_{\ell+2} = \frac{\ell(\ell+1)-n(n+1)}{(\ell+2)(\ell+1)} c_\ell, \quad \ell \geq 2.$$

When $\ell = n$, the right side of the equation vanishes, making the remaining coefficients vanish. Thus, we will be left with a polynomial of degree n. These are the Legendre polynomials, $P_n(x)$.

7.2.4 *Fourier-Legendre Series*

With these properties of Legendre functions we are now prepared to compute the expansion coefficients for the Fourier-Legendre series representation of a given function.

Example 7.9. Expand $f(x) = x^3$ in a Fourier-Legendre series.

We simply need to compute

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} x^3 P_n(x)\, dx. \tag{7.43}$$

We first note that

$$\int_{-1}^{1} x^m P_n(x)\, dx = 0 \quad \text{for } m > n.$$

As a result, we have that $c_n = 0$ for $n > 3$. We could just compute $\int_{-1}^{1} x^3 P_m(x)\,dx$ for $m = 0, 1, 2, \ldots$ outright by looking up Legendre polynomials. But, note that x^3 is an odd function. So, $c_0 = 0$ and $c_2 = 0$.

This leaves us with only two coefficients to compute. We refer to Table 7.3 and find that

$$c_1 = \frac{3}{2}\int_{-1}^{1} x^4\,dx = \frac{3}{5}$$

$$c_3 = \frac{7}{2}\int_{-1}^{1} x^3\left[\frac{1}{2}(5x^3 - 3x)\right]dx = \frac{2}{5}.$$

Thus,

$$x^3 = \frac{3}{5}P_1(x) + \frac{2}{5}P_3(x).$$

Of course, this is simple to check using Table 7.3:

$$\frac{3}{5}P_1(x) + \frac{2}{5}P_3(x) = \frac{3}{5}x + \frac{2}{5}\left[\frac{1}{2}(5x^3 - 3x)\right] = x^3.$$

We could have obtained this result without doing any integration. Write x^3 as a linear combination of $P_1(x)$ and $P_3(x)$:

$$\begin{aligned} x^3 &= c_1 x + \frac{1}{2}c_2(5x^3 - 3x) \\ &= (c_1 - \frac{3}{2}c_2)x + \frac{5}{2}c_2 x^3. \end{aligned} \tag{7.44}$$

Equating coefficients of like terms, we have that $c_2 = \dfrac{2}{5}$ and $c_1 = \dfrac{3}{2}c_2 = \dfrac{3}{5}$.

Example 7.10. Expand the Heaviside[9] function in a Fourier-Legendre series.

The Heaviside function is defined as

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \tag{7.45}$$

In this case, we cannot find the expansion coefficients without some integration. We have to compute

$$\begin{aligned} c_n &= \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx \\ &= \frac{2n+1}{2}\int_{0}^{1} P_n(x)\,dx. \end{aligned} \tag{7.46}$$

We can make use of identity (7.31),

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x), \quad n > 0. \tag{7.47}$$

We have for $n > 0$

$$c_n = \frac{1}{2}\int_{0}^{1}[P'_{n+1}(x) - P'_{n-1}(x)]\,dx = \frac{1}{2}[P_{n-1}(0) - P_{n+1}(0)].$$

[9] Oliver Heaviside (1850-1925) was an English mathematician, physicist and engineer who used complex analysis to study circuits and was a co-founder of vector analysis. The Heaviside function is also called the step function. The modern idea of the step function is credited to Bernhard Riemann (1826-1866) and was used by Fourier and Augustin-Louis Cauchy (1789-1857). However, Heaviside's usage in his operational calculus carried over to electrical engineering and Laplace transforms.

For $n = 0$, we have

$$c_0 = \frac{1}{2}\int_0^1 dx = \frac{1}{2}.$$

This leads to the expansion

$$f(x) \sim \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}[P_{n-1}(0) - P_{n+1}(0)]P_n(x).$$

We still need to evaluate the Fourier-Legendre coefficients

$$c_n = \frac{1}{2}[P_{n-1}(0) - P_{n+1}(0)].$$

Since $P_n(0) = 0$ for n odd, the c_n's vanish for n even. Letting $n = 2k-1$, we re-index the sum, obtaining

$$f(x) \sim \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{\infty}[P_{2k-2}(0) - P_{2k}(0)]P_{2k-1}(x).$$

We can compute the nonzero Fourier coefficients, $c_{2k-1} = \frac{1}{2}[P_{2k-2}(0) - P_{2k}(0)]$, using a result from Problem 10:

$$P_{2k}(0) = (-1)^k\frac{(2k-1)!!}{(2k)!!}. \tag{7.48}$$

Namely, we have

$$\begin{aligned} c_{2k-1} &= \frac{1}{2}[P_{2k-2}(0) - P_{2k}(0)] \\ &= \frac{1}{2}\left[(-1)^{k-1}\frac{(2k-3)!!}{(2k-2)!!} - (-1)^k\frac{(2k-1)!!}{(2k)!!}\right] \\ &= -\frac{1}{2}(-1)^k\frac{(2k-3)!!}{(2k-2)!!}\left[1 + \frac{2k-1}{2k}\right] \\ &= -\frac{1}{2}(-1)^k\frac{(2k-3)!!}{(2k-2)!!}\frac{4k-1}{2k}. \end{aligned} \tag{7.49}$$

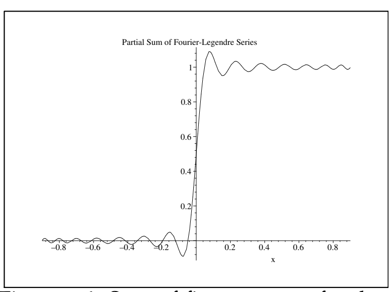


Figure 7.6: Sum of first 21 terms for the Fourier-Legendre series expansion of the Heaviside function.

Thus, the Fourier-Legendre series expansion for the Heaviside function is given by

$$f(x) \sim \frac{1}{2} - \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2n-3)!!}{(2n-2)!!}\frac{4n-1}{2n}P_{2n-1}(x). \tag{7.50}$$

The sum of the first 21 terms of this series are shown in Figure 7.6. We note the slow convergence to the Heaviside function. Also, we see that the Gibbs phenomenon is present due to the jump discontinuity at $x = 0$. [See Section 5.5.]

7.3 *Gamma Function*

The name and symbol for the Gamma function were first given by Legendre in 1811. However, the search for a generalization of the factorial extends back to the 1720's when Euler provided the first representation of the factorial as an infinite product, later to be modified by others like Gauss, Weierstrass, and Legendre.

A FUNCTION THAT OFTEN OCCURS IN THE STUDY OF SPECIAL FUNCTIONS is the Gamma function. We will need the Gamma function in the next section on Fourier-Bessel series.

For $x > 0$, we define the Gamma function as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt, \quad x > 0. \tag{7.51}$$

The Gamma function is a generalization of the factorial function and a plot is shown in Figure 7.7.

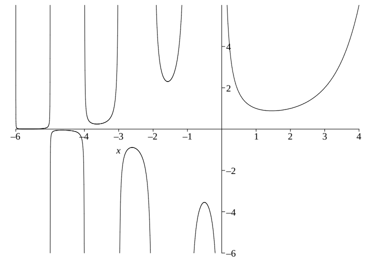


Figure 7.7: Plot of the Gamma function.

In fact, we have

$$\Gamma(1) = 1$$

and

$$\Gamma(x+1) = x\Gamma(x).$$

The reader can prove this identity by simply performing an integration by parts. (See Problem 14.) In particular, for integers $n \in Z^+$, we then have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-2) = n(n-1)\cdots 2\Gamma(1) = n!.$$

The Gamma function was a result of the problem of interpolating the factorial function. This originated with letters from Daniel Bernoulli and Leonhard Euler to Christian Goldbach (1690-1764). Bernoulli wrote October 6, 1729 with

$$x! = \lim_{n\to\infty} \left(n + 1 + \frac{x}{2}\right)^{x-1} \prod_{k=1}^{n} \frac{k+1}{k+x}.$$

He made a slight error computing $\frac{3}{2}!$.

October 13, 1729 Euler gave the nth approximation

$$\frac{1\cdot 2\cdot 3\cdots n(n+1)^m}{(1+m)(2+m)\cdots(n+m)}$$

and on January 8, 1730 Euler gave the integral representation

$$n! = \int_0^1 (-\ln x)^n \, dx.$$

See Problem 17.

We can also define the Gamma function for negative, non-integer values of x. We first note that by iteration on $n \in Z^+$, we have

$$\Gamma(x+n) = (x+n-1)\cdots(x+1)x\Gamma(x), \quad x+n > 0.$$

Solving for $\Gamma(x)$, we then find

$$\Gamma(x) = \frac{\Gamma(x+n)}{(x+n-1)\cdots(x+1)x}, \quad -n < x < 0$$

Note that the Gamma function is undefined at zero and the negative integers.

Example 7.11. We now prove that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

This is done by direct computation of the integral:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} \, dt.$$

Letting $t = z^2$, we have

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-z^2} \, dz.$$

Due to the symmetry of the integrand, we obtain the classic integral

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^\infty e^{-z^2} \, dz,$$

which can be performed using a standard trick. Consider the integral

$$I = \int_{-\infty}^\infty e^{-x^2} \, dx.$$

More generally, we have

$$\int_{-\infty}^\infty e^{-\beta y^2} \, dy = \sqrt{\frac{\pi}{\beta}}.$$

For $\beta = 1$ this is referred to as the Gauss error integral, or Gauss-Poisson integral. Gauss published this result in 1809, but it was Abraham de Moivre (1667-1754) who originally discovered these types of integrals in 1733. More general Gaussian integrals are reserved for the homework such as in Problems 2 and 19.

Then,

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy.$$

Note that we changed the integration variable. This will allow us to write this product of integrals as a double integral:

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dxdy.$$

This is an integral over the entire xy-plane. We can transform this Cartesian integration to an integration over polar coordinates. The integral becomes

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,rdrd\theta.$$

This is simple to integrate and we have $I^2 = \pi$. So, the final result is found by taking the square root of both sides:

$$\Gamma\left(\frac{1}{2}\right) = I = \sqrt{\pi}.$$

In Problem 10 the reader will prove the identity

$$\Gamma(n+\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}.$$

Another useful relation, which we only state, is

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

There are many other important relations, including infinite products, which we will not need at this point. The reader is encouraged to read about these elsewhere. In the meantime, we move on to the discussion of another important special function in physics and mathematics.

7.4 *Fourier-Bessel Series*

Bessel functions arise in many problems in physics possessing cylindrical symmetry such as the vibrations of circular drumheads and the radial modes in optical fibers. They also provide us with another orthogonal set of basis functions.

Bessel functions have a long history and were named after Friedrich Wilhelm Bessel (1784-1846).

The first occurrence of Bessel functions (zeroth order) was in the work of Daniel Bernoulli on heavy chains (1738). More general Bessel functions were studied by Leonhard Euler in 1781 and in his study of the vibrating membrane in 1764. Joseph Fourier found them in the study of heat conduction in solid cylinders and Siméon Poisson (1781-1840) in heat conduction of spheres (1823).

The history of Bessel functions does not just originate in the study of the wave and heat equations. These solutions originally came up in the study of the Kepler problem, describing planetary motion. According to G. N.

Watson in his *Treatise on Bessel Functions*, the formulation and solution of Kepler's Problem was discovered by Joseph-Louis Lagrange (1736-1813), in 1770. Namely, the problem was to express the radial coordinate and what is called the eccentric anomaly, E, as functions of time. Lagrange found expressions for the coefficients in the expansions of r and E in trigonometric functions of time. However, he only computed the first few coefficients. In 1816 Friedrich Wilhelm Bessel (1784-1846) had shown that the coefficients in the expansion for r could be given an integral representation. In 1824 he presented a thorough study of these functions, which are now called Bessel functions.

You might have seen Bessel functions in a course on differential equations as solutions of the differential equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0. \tag{7.52}$$

Solutions to this equation are obtained in the form of series expansions.[10] Namely, one seeks solutions of the form

$$y(x) = \sum_{j=0}^{\infty} a_j x^{j+n}$$

by determining the form the coefficients must take. We will leave this for a homework exercise and simply report the results.

[10] Since $x = 0$ is a regular singular point, we solve Bessel's equation using the Method of Frobenius. This differs from the series solution of Legendre's differential equation.

One solution of the differential equation is the *Bessel function of the first kind of order p*, given as

$$y(x) = J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}. \tag{7.53}$$

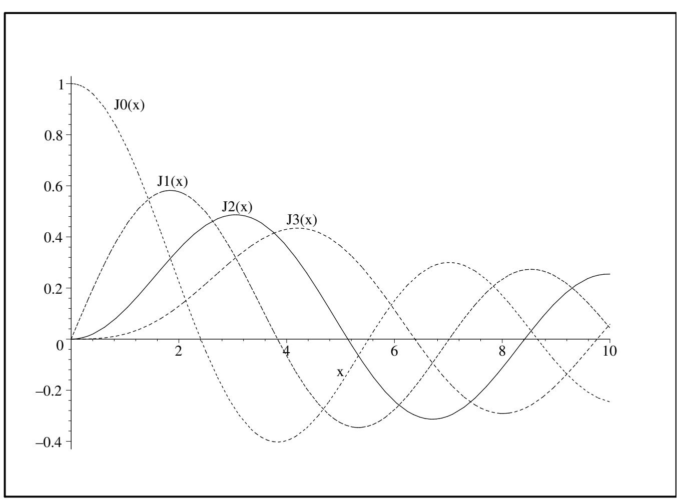


Figure 7.8: Plots of the Bessel functions $J_0(x)$, $J_1(x)$, $J_2(x)$, and $J_3(x)$.

In Figure 7.8 we display the first few Bessel functions of the first kind of integer order. Note that these functions can be described as decaying oscillatory functions.

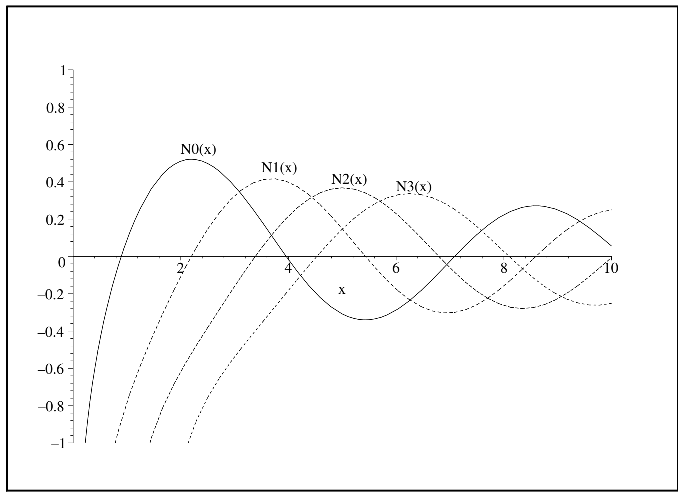
A second linearly independent solution is obtained for p not an integer as $J_{-p}(x)$. However, for p an integer, the $\Gamma(n+p+1)$ factor leads to evaluations of the Gamma function at zero, or negative integers, when p is negative. Thus, the above series is not defined in these cases.

Another method for obtaining a second linearly independent solution is through a linear combination of $J_p(x)$ and $J_{-p}(x)$ as

$$N_p(x) = Y_p(x) = \frac{\cos \pi p J_p(x) - J_{-p}(x)}{\sin \pi p}. \tag{7.54}$$

These functions are called the Neumann functions, or Bessel functions of the second kind of order p.

Figure 7.9: Plots of the Neumann functions $N_0(x)$, $N_1(x)$, $N_2(x)$, and $N_3(x)$.



In Figure 7.9 we display the first few Bessel functions of the second kind of integer order. Note that these functions are also decaying oscillatory functions. However, they are singular at $x = 0$.

In many applications one desires bounded solutions at $x = 0$. These functions do not satisfy this boundary condition. For example, we will later study one standard problem is to describe the oscillations of a circular drumhead. For this problem one solves the two dimensional wave equation using separation of variables in cylindrical coordinates. The radial equation leads to a Bessel equation. The Bessel function solutions describe the radial part of the solution and one does not expect a singular solution at the center of the drum. The amplitude of the oscillation must remain finite. Thus, only Bessel functions of the first kind can be used.

Bessel functions satisfy a variety of properties, which we will only list at this time for Bessel functions of the first kind. The reader will have the opportunity to prove these for homework.

Derivative Identities These identities follow directly from the manipulation of the series solution.

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x). \tag{7.55}$$

$$\frac{d}{dx}\left[x^{-p}J_p(x)\right] = -x^{-p}J_{p+1}(x). \tag{7.56}$$

Recursion Formulae The next identities follow from adding, or subtracting, the derivative identities.

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x}J_p(x). \tag{7.57}$$

$$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x). \tag{7.58}$$

Orthogonality As we will see in the next chapter, one can recast the Bessel equation into an eigenvalue problem whose solutions form an orthogonal basis of functions on $L_x^2(0,a)$. Using Sturm-Liouville theory, one can show that

$$\int_0^a xJ_p(j_{pn}\frac{x}{a})J_p(j_{pm}\frac{x}{a})\,dx = \frac{a^2}{2}\left[J_{p+1}(j_{pn})\right]^2\delta_{n,m}, \tag{7.59}$$

where j_{pn} is the nth root of $J_p(x)$, $J_p(j_{pn}) = 0$, $n = 1, 2, \ldots$. A list of some of these roots are provided in Table 7.4.

n	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
1	2.405	3.832	5.136	6.380	7.588	8.771
2	5.520	7.016	8.417	9.761	11.065	12.339
3	8.654	10.173	11.620	13.015	14.373	15.700
4	11.792	13.324	14.796	16.223	17.616	18.980
5	14.931	16.471	17.960	19.409	20.827	22.218
6	18.071	19.616	21.117	22.583	24.019	25.430
7	21.212	22.760	24.270	25.748	27.199	28.627
8	24.352	25.904	27.421	28.908	30.371	31.812
9	27.493	29.047	30.569	32.065	33.537	34.989

Table 7.4: The zeros of Bessel Functions, $J_m(j_{mn}) = 0$.

Generating Function

$$e^{x(t-\frac{1}{t})/2} = \sum_{n=-\infty}^{\infty} J_n(x)t^n, \quad x > 0, t \neq 0. \tag{7.60}$$

Integral Representation

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - n\theta)\,d\theta, \quad x > 0, n \in \mathrm{Z}. \tag{7.61}$$

Fourier-Bessel Series

Since the Bessel functions are an orthogonal set of functions of a Sturm-Liouville problem, we can expand square integrable functions in this basis. In fact, the Sturm-Liouville problem is given in the form

$$x^2y'' + xy' + (\lambda x^2 - p^2)y = 0, \quad x \in [0, a], \tag{7.62}$$

satisfying the boundary conditions: $y(x)$ is bounded at $x = 0$ and $y(a) = 0$. The solutions are then of the form $J_p(\sqrt{\lambda}x)$, as can be shown by making the substitution $t = \sqrt{\lambda}x$ in the differential equation. Namely, we let $y(x) = u(t)$ and note that

$$\frac{dy}{dx} = \frac{dt}{dx}\frac{du}{dt} = \sqrt{\lambda}\frac{du}{dt}.$$

Then,

$$t^2u'' + tu' + (t^2 - p^2)u = 0,$$

which has a solution $u(t) = J_p(t)$.

Using Sturm-Liouville theory, one can show that $J_p(j_{pn}\frac{x}{a})$ is a basis of eigenfunctions and the resulting *Fourier-Bessel series expansion* of $f(x)$ defined on $x \in [0, a]$ is

$$f(x) = \sum_{n=1}^{\infty} c_n J_p(j_{pn}\frac{x}{a}), \tag{7.63}$$

where the Fourier-Bessel coefficients are found using the orthogonality relation as

$$c_n = \frac{2}{a^2\left[J_{p+1}(j_{pn})\right]^2}\int_0^a x f(x) J_p(j_{pn}\frac{x}{a})\, dx. \tag{7.64}$$

In the study of boundary value problems in differential equations, Sturm-Liouville problems are a bountiful source of basis functions for the space of square integrable functions.

Example 7.12. Expand $f(x) = 1$ for $0 < x < 1$ in a Fourier-Bessel series of the form

$$f(x) = \sum_{n=1}^{\infty} c_n J_0(j_{0n}x)$$

.

We need only compute the Fourier-Bessel coefficients in Equation (7.64):

$$c_n = \frac{2}{[J_1(j_{0n})]^2}\int_0^1 x J_0(j_{0n}x)\, dx. \tag{7.65}$$

From the identity

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x). \tag{7.66}$$

we have

$$\begin{aligned}\int_0^1 x J_0(j_{0n}x)\, dx &= \frac{1}{j_{0n}^2}\int_0^{j_{0n}} y J_0(y)\, dy \\ &= \frac{1}{j_{0n}^2}\int_0^{j_{0n}} \frac{d}{dy}\left[yJ_1(y)\right]\, dy\end{aligned}$$

$$= \frac{1}{j_{0n}^2}\left[yJ_1(y)\right]_0^{j_{0n}}$$
$$= \frac{1}{j_{0n}}J_1(j_{0n}). \quad (7.67)$$

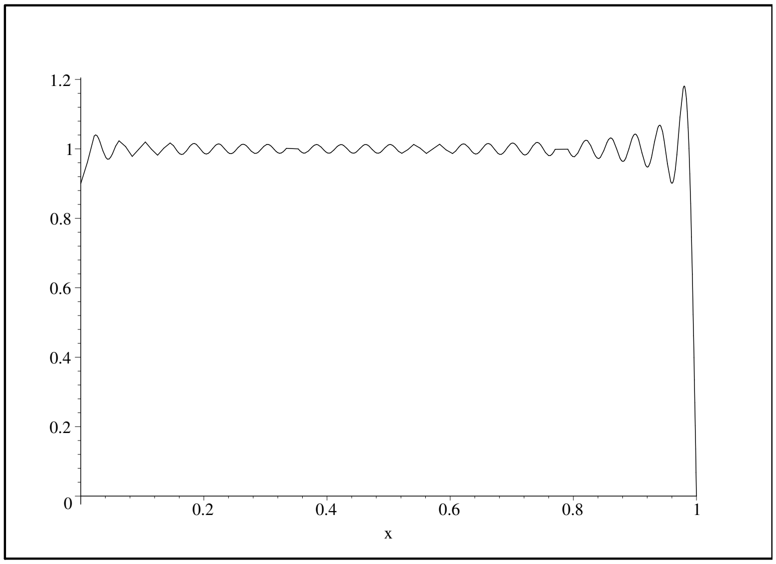


Figure 7.10: Plot of the first 50 terms of the Fourier-Bessel series in Equation (7.68) for $f(x) = 1$ on $0 < x < 1$.

As a result, the desired Fourier-Bessel expansion is given as

$$1 = 2\sum_{n=1}^{\infty} \frac{J_0(j_{0n}x)}{j_{0n}J_1(j_{0n})}, \quad 0 < x < 1. \quad (7.68)$$

In Figure 7.10 we show the partial sum for the first fifty terms of this series. Note once again the slow convergence due to the Gibbs phenomenon.

7.5 *Hypergeometric Functions*

Hypergeometric functions are probably the most useful, but least understood, class of functions. They typically do not make it into the undergraduate curriculum and seldom in graduate curriculum. Most functions that you know can be expressed using hypergeometric functions. There are many approaches to these functions and the literature can fill books. [11]

[11] See for example *Special Functions* by G. E. Andrews, R. Askey, and R. Roy, 1999, Cambridge University Press.

In 1812 Gauss published a study of the *hypergeometric series*

$$y(x) = 1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha(1+\alpha)(1+\beta)}{2!\gamma(1+\gamma)}x^2 + \frac{\alpha(1+\alpha)(2+\alpha)\beta(1+\beta)(2+\beta)}{3!\gamma(1+\gamma)(2+\gamma)}x^3 + \ldots. \quad (7.69)$$

Here α, β, γ, and x are real numbers. If one sets $\alpha = 1$ and $\beta = \gamma$, this series reduces to the familiar geometric series

$$y(x) = 1 + x + x^2 + x^3 + \ldots.$$

The hypergeometric series is actually a solution of the differential equation

$$x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]\, y' - \alpha\beta y = 0. \tag{7.70}$$

This equation was first introduced by Euler and latter studied extensively by Gauss, Kummer and Riemann. It is sometimes called Gauss' equation. Note that there is a symmetry in that α and β may be interchanged without changing the equation. The points $x = 0$ and $x = 1$ are regular singular points. Series solutions may be sought using the Frobenius method. It can be confirmed that the above hypergeometric series results.

A more compact form for the hypergeometric series may be obtained by introducing new notation. One typically introduces the *Pochhammer symbol*, $(\alpha)_n$, satisfying (i) $(\alpha)_0 = 1$ if $\alpha \neq 0$. and (ii) $(\alpha)_k = \alpha(1+\alpha)\ldots(k-1+\alpha)$, for $k = 1, 2, \ldots$.

Consider $(1)_n$. For $n = 0$, $(1)_0 = 1$. For $n > 0$,

$$(1)_n = 1(1+1)(2+1)\ldots[(n-1)+1].$$

This reduces to $(1)_n = n!$. In fact, one can show that

$$(k)_n = \frac{(n+k-1)!}{(k-1)!}$$

for k and n positive integers. In fact, one can extend this result to noninteger values for k by introducing the gamma function:

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.$$

We can now write the hypergeometric series in standard notation as

$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n} x^n.$$

Using this one can show that the general solution of Gauss' equation is

$$y(x) = A_2F_1(\alpha, \beta; \gamma; x) + B_2 x_2^{1-\gamma} F_1(1-\gamma+\alpha, 1-\gamma+\beta; 2-\gamma; x).$$

By carefully letting β approach ∞, one obtains what is called the *confluent hypergeometric function*. This in effect changes the nature of the differential equation. Gauss' equation has three regular singular points at $x = 0, 1, \infty$. One can transform Gauss' equation by letting $x = u/\beta$. This changes the regular singular points to $u = 0, \beta, \infty$. Letting $\beta \to \infty$, two of the singular points merge.

The new confluent hypergeometric function is then given as

$${}_1F_1(\alpha; \gamma; u) = \lim_{\beta\to\infty} {}_2F_1\left(\alpha, \beta; \gamma; \frac{u}{\beta}\right).$$

This function satisfies the differential equation

$$xy'' + (\gamma - x)y' - \alpha y = 0.$$

The purpose of this section is only to introduce the hypergeometric function. Many other special functions are related to the hypergeometric function after making some variable transformations. For example, the Legendre polynomials are given by

$$P_n(x) =_2 F_1(-n, n+1; 1; \frac{1-x}{2}).$$

In fact, one can also show that

$$\sin^{-1} x = x_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right).$$

The Bessel function $J_p(x)$ can be written in terms of confluent geometric functions as

$$J_p(x) = \frac{1}{\Gamma(p+1)}\left(\frac{z}{2}\right)^p e^{-iz}\,{}_1F_1\left(\frac{1}{2}+p, 1+2p; 2iz\right).$$

These are just a few connections of the powerful hypergeometric functions to some of the elementary functions that you know.

7.6 Appendix: The Binomial Expansion

In this section we had to recall the binomial expansion. This is simply the expansion of the expression $(a+b)^p$. We will investigate this expansion first for nonnegative integer powers p and then derive the expansion for other values of p.

Lets list some of the common expansions for nonnegative integer powers.

$$\begin{aligned} (a+b)^0 &= 1 \\ (a+b)^1 &= a+b \\ (a+b)^2 &= a^2+2ab+b^2 \\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\ (a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\ &\cdots \end{aligned} \tag{7.71}$$

We now look at the patterns of the terms in the expansions. First, we note that each term consists of a product of a power of a and a power of b. The powers of a are decreasing from n to 0 in the expansion of $(a+b)^n$. Similarly, the powers of b increase from 0 to n. The sums of the exponents in each term is n. So, we can write the $(k+1)$st term in the expansion as $a^{n-k}b^k$. For example, in the expansion of $(a+b)^{51}$ the 6th term is $a^{51-5}b^5 = a^{46}b^5$. However, we do not know the numerical coefficient in the expansion.

We now list the coefficients for the above expansions.

$$\begin{array}{rccccccccc} n=0: & & & & & 1 & & & & \\ n=1: & & & & 1 & & 1 & & & \\ n=2: & & & 1 & & 2 & & 1 & & \\ n=3: & & 1 & & 3 & & 3 & & 1 & \\ n=4: & 1 & & 4 & & 6 & & 4 & & 1 \end{array} \tag{7.72}$$

This pattern is the famous Pascal's triangle. There are many interesting features of this triangle. But we will first ask how each row can be generated.

We see that each row begins and ends with a one. Next the second term and next to last term has a coefficient of n. Next we note that consecutive pairs in each row can be added to obtain entries in the next row. For example, we have

$$\begin{array}{rccccccc} n=2: & & 1 & & 2 & & 1 & \\ & & \searrow & \swarrow & & \searrow & \swarrow & \\ n=3: & 1 & & 3 & & 3 & & 1 \end{array} \tag{7.73}$$

With this in mind, we can generate the next several rows of our triangle.

$$\begin{array}{rccccccccccccc} n=3: & & & & 1 & & 3 & & 3 & & 1 & & & \\ n=4: & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ n=5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ n=6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array} \tag{7.74}$$

Of course, it would take a while to compute each row up to the desired n. We need a simple expression for computing a specific coefficient. Consider the kth term in the expansion of $(a+b)^n$. Let $r = k-1$. Then this term is of the form $C_r^n a^{n-r} b^r$. We have seen the the coefficients satisfy

$$C_r^n = C_r^{n-1} + C_{r-1}^{n-1}.$$

Actually, the coefficients have been found to take a simple form.

$$C_r^n = \frac{n!}{(n-r)!r!} = \left(\begin{array}{c} n \\ r \end{array} \right).$$

This is nothing other than the combinatoric symbol for determining how to choose n things r at a time. In our case, this makes sense. We have to count the number of ways that we can arrange the products of r b's with $n-r$ a's. There are n slots to place the b's. For example, the $r=2$ case for $n=4$ involves the six products: *aabb*, *abab*, *abba*, *baab*, *baba*, and *bbaa*. Thus, it is natural to use this notation. The original problem that concerned Pascal was in gambling.

So, we have found that

$$(a+b)^n = \sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} a^{n-r} b^r. \tag{7.75}$$

What if $a \gg b$? Can we use this to get an approximation to $(a+b)^n$? If we neglect b then $(a+b)^n \simeq a^n$. How good of an approximation is this? This is where it would be nice to know the order of the next term in the expansion, which we could state using big O notation. In order to do this we first divide out a as

$$(a+b)^n = a^n (1 + \frac{b}{a})^n.$$

Now we have a small parameter, $\frac{b}{a}$. According to what we have seen above, we can use the binomial expansion to write

$$(1+\frac{b}{a})^n = \sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} \left(\frac{b}{a}\right)^r. \tag{7.76}$$

Thus, we have a finite sum of terms involving powers of $\frac{b}{a}$. Since $a \gg b$, most of these terms can be neglected. So, we can write

$$(1+\frac{b}{a})^n = 1 + n\frac{b}{a} + O\left(\left(\frac{b}{a}\right)^2\right).$$

note that we have used the observation that the second coefficient in the nth row of Pascal's triangle is n.

Summarizing, this then gives

$$\begin{aligned} (a+b)^n &= a^n (1+\frac{b}{a})^n \\ &= a^n (1 + n\frac{b}{a} + O\left(\left(\frac{b}{a}\right)^2\right)) \\ &= a^n + na^n\frac{b}{a} + a^n O\left(\left(\frac{b}{a}\right)^2\right). \end{aligned} \tag{7.77}$$

Therefore, we can approximate $(a+b)^n \simeq a^n + nba^{n-1}$, with an error on the order of ba^{n-2}. Note that the order of the error does not include the constant factor from the expansion. We could also use the approximation that $(a+b)^n \simeq a^n$, but it is not as good because the error in this case is of the order ba^{n-1}.

We have seen that

$$\frac{1}{1-x} = 1 + x + x^2 + \dots.$$

But, $\frac{1}{1-x} = (1-x)^{-1}$. This is again a binomial to a power, but the power is not a nonnegative integer. It turns out that the coefficients of such a binomial expansion can be written similar to the form in Equation (7.75).

This example suggests that our sum may no longer be finite. So, for p a real number, we write

$$(1+x)^p = \sum_{r=0}^{\infty} \begin{pmatrix} p \\ r \end{pmatrix} x^r. \tag{7.78}$$

However, we quickly run into problems with this form. Consider the coefficient for $r = 1$ in an expansion of $(1+x)^{-1}$. This is given by

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{(-1)!}{(-1-1)!1!} = \frac{(-1)!}{(-2)!1!}.$$

But what is $(-1)!$? By definition, it is

$$(-1)! = (-1)(-2)(-3)\cdots.$$

This product does not seem to exist! But with a little care, we note that

$$\frac{(-1)!}{(-2)!} = \frac{(-1)(-2)!}{(-2)!} = -1.$$

So, we need to be careful not to interpret the combinatorial coefficient literally. There are better ways to write the general binomial expansion. We can write the general coefficient as

$$\begin{aligned} \begin{pmatrix} p \\ r \end{pmatrix} &= \frac{p!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)(p-r)!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)}{r!}. \end{aligned} \tag{7.79}$$

With this in mind we now state the theorem:

General Binomial Expansion The general binomial expansion for $(1+x)^p$ is a simple generalization of Equation (7.75). For p real, we have that

$$\begin{aligned} (1+x)^p &= \sum_{r=0}^{\infty} \frac{p(p-1)\cdots(p-r+1)}{r!} x^r \\ &= \sum_{r=0}^{\infty} \frac{\Gamma(p+1)}{r!\Gamma(p-r+1)} x^r. \end{aligned} \tag{7.80}$$

Often we need the first few terms for the case that $x \ll 1$:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2} x^2 + O(x^3). \tag{7.81}$$

7.7 *Appendix: Orthogonality Proof of Three Term Recursion Formula - Optional*

WE WILL PROVE THE THREE TERM RECURSION FORMULA using the orthogonality properties of Legendre polynomials and the following lemma.

The first proof of the three term recursion formula is based upon the nature of the Legendre polynomials as an orthogonal basis, while the second proof is derived using generating functions.

Lemma 7.1. *The leading coefficient of x^n in $P_n(x)$ is* $\frac{1}{2^n n!}\frac{(2n)!}{n!}$.

Proof. We can prove this using the Rodrigues formula. First, we focus on the leading coefficient of $(x^2-1)^n$, which is x^{2n}. The first derivative of x^{2n} is $2nx^{2n-1}$. The second derivative is $2n(2n-1)x^{2n-2}$. The jth derivative is

$$\frac{d^j x^{2n}}{dx^j} = [2n(2n-1)\dots(2n-j+1)]x^{2n-j}.$$

Thus, the nth derivative is given by

$$\frac{d^n x^{2n}}{dx^n} = [2n(2n-1)\dots(n+1)]x^n.$$

This proves that $P_n(x)$ has degree n. The leading coefficient of $P_n(x)$ can now be written as

$$\begin{aligned} \frac{[2n(2n-1)\dots(n+1)]}{2^n n!} &= \frac{[2n(2n-1)\dots(n+1)]}{2^n n!}\frac{n(n-1)\dots 1}{n(n-1)\dots 1} \\ &= \frac{1}{2^n n!}\frac{(2n)!}{n!}. \end{aligned} \tag{7.82}$$

□

Theorem 7.1. *Legendre polynomials satisfy the three term recursion formula*

$$(2n-1)xP_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x), \quad n = 1, 2, \dots. \tag{7.83}$$

Proof. In order to prove the three term recursion formula we consider the expression $(2n-1)xP_{n-1}(x) - nP_n(x)$. While each term is a polynomial of degree n, the leading order terms cancel. We need only look at the coefficient of the leading order term first expression. It is

$$\frac{2n-1}{2^{n-1}(n-1)!}\frac{(2n-2)!}{(n-1)!} = \frac{1}{2^{n-1}(n-1)!}\frac{(2n-1)!}{(n-1)!} = \frac{(2n-1)!}{2^{n-1}\left[(n-1)!\right]^2}.$$

The coefficient of the leading term for $nP_n(x)$ can be written as

$$n\frac{1}{2^n n!}\frac{(2n)!}{n!} = n\left(\frac{2n}{2n^2}\right)\left(\frac{1}{2^{n-1}(n-1)!}\right)\frac{(2n-1)!}{(n-1)!}\frac{(2n-1)!}{2^{n-1}\left[(n-1)!\right]^2}.$$

It is easy to see that the leading order terms in the expression $(2n-1)xP_{n-1}(x) - nP_n(x)$ cancel.

The next terms will be of degree $n-2$. This is because the P_n's are either even or odd functions, thus only containing even, or odd, powers of x. We conclude that

$$(2n-1)xP_{n-1}(x) - nP_n(x) = \text{ polynomial of degree } n-2.$$

Therefore, since the Legendre polynomials form a basis, we can write this polynomial as a linear combination of Legendre polynomials:

$$(2n-1)xP_{n-1}(x) - nP_n(x) = c_0P_0(x) + c_1P_1(x) + \dots + c_{n-2}P_{n-2}(x). \tag{7.84}$$

Multiplying Equation (7.84) by $P_m(x)$ for $m = 0, 1, \ldots, n-3$, integrating from -1 to 1, and using orthogonality, we obtain

$$0 = c_m \|P_m\|^2, \quad m = 0, 1, \ldots, n-3.$$

[Note: $\int_{-1}^{1} x^k P_n(x)\, dx = 0$ for $k \leq n-1$. Thus, $\int_{-1}^{1} x P_{n-1}(x) P_m(x)\, dx = 0$ for $m \leq n-3$.]

Thus, all of these c_m's are zero, leaving Equation (7.84) as

$$(2n-1)xP_{n-1}(x) - nP_n(x) = c_{n-2}P_{n-2}(x).$$

The final coefficient can be found by using the normalization condition, $P_n(1) = 1$. Thus, $c_{n-2} = (2n-1) - n = n-1$. ☐

7.8 Appendix: The Adomain Decomposition Method - Optional

The Adomian Decomposition Method (ADM) is an analytical technique designed to solve a wide variety of linear and nonlinear differential equations, integral equations, and partial differential equations. Developed by George Adomian (1922-1996) in the 1980s, this method provides a systematic approach to decomposing complex equations into simpler subproblems, allowing for iterative refinement of the solution. ADM is particularly advantageous for tackling nonlinear problems directly without the need for linearization or perturbative methods. In this section we will provide some examples solutions of linear differential equations using ADM in order to demonstrate this method.

The core idea of ADM involves expressing the solution of a given differential equation as an infinite series of unknown functions to be determined iteratively. If u denotes the solution, it is represented as:

$$u = \sum_{n=0}^{\infty} u_n, \tag{7.85}$$

where u_n are the components to be determined. For nonlinear PDEs, the method decomposes the nonlinear terms using Adomian polynomials. If $N(u)$ represents a nonlinear operator acting on u, it is expressed as

$$N(u) = \sum_{n=0}^{\infty} A_n, \tag{7.86}$$

where the A_n are the Adomian polynomials that depend on the components u_n.

The procedure begins by rewriting the differential equation in the form:

$$Lu + Ru + N(u) = g, \tag{7.87}$$

where L is a linear operator, R is a linear but potentially inhomogeneous operator, N is a nonlinear operator, and g is a source term.

Using the decomposed forms of u and $N(u)$, the equation becomes:

$$L\left(\sum_{n=0}^{\infty} u_n\right) + R\left(\sum_{n=0}^{\infty} u_n\right) + \sum_{n=0}^{\infty} A_n = g. \tag{7.88}$$

Equating terms of the same order, we derive a set of recursive relations for the components u_n. The zeroth component u_0 is obtained by solving:

$$u_0 = L^{-1}g + \text{initial/boundary values for} u, \tag{7.89}$$

and subsequent components are determined using:

$$u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n, \quad n \geq 0. \tag{7.90}$$

This is just a formal explanation of the process. We will give several examples implementing the process to see how it works in practice.

Example 7.13. Use the ADM to solve the first order (nonlinear) initial value problem

$$\frac{dy}{dt} = 1 + y^2, \quad y(0) = 0.$$

We identify the differential equation in operator form as

$$Ly + Ry = f(t), \tag{7.91}$$

where $L = \frac{d}{dt}$, $Ry = -y^2$, and $f(t) = 1$. Now, we prepare the problem for solution.

We write $Ly = 1 + y^2$. We want to solve for y. So, we apply L^{-1} to the equation,

$$L^{-1}Ly = L^{-1}(1) + L^{-1}(y^2).$$

But what is $L^{-1}Ly$? Since $L = \dfrac{d}{dt}$, then L^{-1} is just integration. We have

$$L^{-1}Ly = \int_0^t \frac{dy(\tau)}{d\tau}\, d\tau = y(t) - y(0). \tag{7.92}$$

[Note that we needed to distinguish between the integration variable τ and the independent variable t.] Also,

$$L^{-1}(1) = \int_0^t 1\, d\tau = t.$$

So, now the solution can be written

$$y(t) = y(0) + t + L^{-1}(y^2).$$

Next, we decompose the solution,

$$y(t) = \sum_{n=0}^{\infty} y_n(t) = y_0(t) + y_1(t) + y_2(t) + \ldots. \tag{7.93}$$

Using the initial condition, $y(0) = 0$, we have

$$y_0(t) + y_1(t) + y_2(t) + \ldots = t + L^{-1}[(y_0(t) + y_1(t) + \ldots)^2].$$

Next, we formulate a recursive process to solve for the components, $y_n(t)$. First, we let $y_0(t) = t$. [If $y(0) \neq 0$, we would have included it as well.]

We then expand y^2 and group the terms so that

$$\begin{aligned} \sum_{n=0}^{\infty} A_n &= (y_0(t) + y_1(t) + y_2(t) + \ldots)^2 \\ &= y_0^2 + 2y_0y_1 + y_1^2 + 2y_0y_2 + \ldots . \end{aligned} \tag{7.94}$$

The A_n's are chosen so that they only contain y_k's with $k \leq n$. Then, we can solve for the components recursively as

$$y_{k+1}(t) = L^{-1}(A_k), \quad k \geq 0. \tag{7.95}$$

The A_k's are referred to as the Adomian polynomials. In this case we have

$$\begin{aligned} A_0 &= y_0^2 \\ A_1 &= 2y_0y_1 \\ A_2 &= y_1^2 + 2y_0y_2, \text{ etc.} \end{aligned} \tag{7.96}$$

There would be different polynomials for a nonlinear term of higher order.

Since $y_0(t) = t$, we can begin to find other components:

$$\begin{aligned} y_1(t) &= \int_0^t y_0^2\, d\tau = \frac{t^3}{3} \\ y_2(t) &= \int_0^t 2y_0y_1\, d\tau = \frac{2t^5}{15} \\ y_3(t) &= \int_0^t (y_1^2 + 2y_0y_2)\, d\tau = \frac{17t^7}{315} \end{aligned} \tag{7.97}$$

Summing the components, we have

$$y(t) = t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{17t^7}{315} + \ldots . \tag{7.98}$$

This series can be truncated, giving an approximate solution, or it could be summed, if you are clever.

In this case, we could easily have integrated the separable equation.

$$t = \int_0^y \frac{dz}{1+z^2} = \tan^{-1} y,$$

or $y(t) = \tan t$. Looking up the series expansion for $\tan t$, we find agreement with the series found using the ADM.

Example 7.14. Forced Oscillator Problem

As the next example, we consider the second order differential equation for forced oscillations with no damping. It is given by

$$\frac{d^2u}{dt^2} + \omega_0^2 u = A\cos(\omega t), \tag{7.99}$$

where ω_0 is the natural frequency, and $f(t) = A\cos(\omega t)$ is the forcing function with amplitude A and forcing frequency ω.

We first rewrite the differential equation in operator form as

$$Lu + Ru = f(t), \tag{7.100}$$

where $L = \dfrac{d^2}{dt^2}$ and $Ru = \omega_0^2 u$.

Before proceeding, we need to be a little careful. We wish to solve for u. So, from $Lu = f(t) - Ru$, we obtain the solution by applying L^{-1} to both sides of the equation. This gives

$$L^{-1}Lu = L^{-1}(f(t) - Ru).$$

In the same way we inverted the first order derivative operator in the previous example, we need to be careful inverting the second derivative. We use τ and z for time integration variables to obtain

$$\begin{aligned}
L^{-1}Lu &= \int_0^t \int_0^z \frac{d^2u(\tau)}{d\tau^2}\, d\tau dz \\
&= \int_0^t \frac{du(\tau)}{d\tau}\Big|_{\tau=0}^{z} dz \\
&= \int_0^t \left[\frac{du(z)}{dz} - u'(0)\right] dz \\
&= u(t) - u(0) - tu'(0).
\end{aligned} \tag{7.101}$$

So, we now have the solution as

$$u(t) = u(0) + tu'(0) + L^{-1}(f(t) - Ru) \tag{7.102}$$

Then, we decompose the solution,

$$u(t) = \sum_{n=0}^{\infty} u_n(t). \tag{7.103}$$

In this case the forcing term does not need to be decomposed as it does not depend on u.

Finally, we finish setting up the problem by finding the needed recursive relations. The zeroth component is determined from using the source and initial values in Equation (7.102),

$$u_0 = u(0) + tu'(0) + L^{-1}f(t). \tag{7.104}$$

Then, subsequent components are found recursively using

$$u_{n+1} = -L^{-1}Ru_n = -\omega_0^2 L^{-1}u_n. \tag{7.105}$$

The solution is found through an iterative procedure starting with the solution for u_0. For the forcing $f(t) = A\cos(\omega t)$, we have

$$u_0 = u(0) + tu'(0) + L^{-1}(A\cos(\omega t)).$$

Since, $L = \dfrac{d^2}{dt^2}$, we carefully integrate twice with respect to t as

$$L^{-1}(\cdot) = \int_0^t \int_0^z (\cdot)\, d\tau dz.$$

This gives

$$\begin{aligned} L^{-1}(A\cos(\omega t)) &= \int_0^t \int_0^z A\cos(\omega\tau)\, d\tau dz \\ &= \frac{A}{\omega}\int_0^t \sin(\omega z)\, dz \\ &= -\frac{A}{\omega^2}\cos(\omega u)\Big|_{z=0}^t . \\ &= -\frac{A}{\omega^2}(\cos(\omega t) - 1). \end{aligned} \tag{7.106}$$

Therefore,

$$u_0(t) = u(0) + tu'(0) - \frac{A}{\omega^2}(\cos(\omega t) - 1). \tag{7.107}$$

Next, we solve for the components $u_n(t)$, which satisfy the equations

$$\begin{aligned} u_1(t) &= -\omega_0^2 L^{-1} u_0, \\ u_2(t) &= -\omega_0^2 L^{-1} u_1, \\ u_3(t) &= -\omega_0^2 L^{-1} u_2, \text{ etc.} \end{aligned} \tag{7.108}$$

Carrying out the computations for these three terms, one finds

$$\begin{aligned} u_1(t) &= \frac{\omega_0^2\left(u'(0)\omega^4 t^3 + 3u(0)\omega^4 t^2 + 3A\omega^2 t^2 + 6A\cos(\omega t) - 6A\right)}{6\omega^4} \\ u_2(t) &= \frac{\omega_0^4\left(-u'(0)\omega^6 t^5 - 5\omega^6 u(0)\, t^4 - 5A\omega^4 t^4 + 60A\omega^2 t^2\right)}{120\omega^6} \\ &\quad + A\frac{\cos(\omega t) - 1}{\omega^6} \\ u_3(t) &= \frac{\omega_0^6\left(u'(0)\omega^8 t^7 + 7\omega^8 u(0)\, t^6 + 7A\omega^6 t^6 - 210A\omega^4 t^4 + 2520A\omega^2 t^2\right)}{5040\omega^8} \\ &\quad + A\frac{\cos(\omega t) - 1}{\omega^8}. \end{aligned} \tag{7.109}$$

Summing these, we get the approximate solution

$$\begin{aligned} u(t) &= \left(1 - \frac{1}{2}\omega_0^2 t^2 + \frac{1}{24}\omega_0^4 t^4 - \frac{1}{720}\omega_0^6 t^6\right) u(0) \\ &\quad + \left(t - \frac{1}{6}\omega_0^2 t^3 + \frac{1}{120}\omega_0^4 t^5 - \frac{1}{5040}\omega_0^6 t^7)\right) u'(0) \end{aligned}$$

$$-\frac{A\cos(\omega t)}{\omega^2}-\frac{\omega_0^2 A\cos(\omega t)}{\omega^4}-\frac{\omega_0^4 A\cos(\omega t)}{\omega^6}-\frac{\omega_0^6 A\cos(\omega t)}{\omega^8}$$
$$+\frac{\omega_0^4 A\, t^4}{24\omega^2}-\frac{\omega_0^4 A\, t^2}{2\omega^4}+\frac{\omega_0^6 A}{\omega^8}+\frac{\omega_0^2 A}{\omega^4}-\frac{\omega_0^2 A\, t^2}{2\omega^2}-\frac{\omega_0^6 A\, t^6}{720\omega^2}$$
$$+\frac{\omega_0^6 A\, t^4}{24\omega^4}-\frac{\omega_0^6 A\, t^2}{2\omega^6}+\frac{\omega_0^4 A}{\omega^6}+\frac{A}{\omega^2}. \tag{7.110}$$

Of course, there is an easier way to solve this problem. The solution to the homogeneous problem is

$$u_h(t)=a\sin\omega_0 t+b\cos\omega_0 t.$$

The particular solution can be found using the Method of Undetermined Coefficients. We assume a form for the particular solution,

$$u_p(t)=B\cos\omega t.$$

[This is possible since there is no first order derivative term in the equation.] Inserting this guess, we find that $B=\dfrac{A}{\omega_0^2-\omega^2}$. So, the general solution is

$$u(t)=a\sin\omega_0 t+b\cos\omega_0 t+\frac{A}{\omega_0^2-\omega^2}\cos\omega t.$$

Finally, we need to impose the initial conditions.

$$u(0)=b+\frac{A}{\omega_0^2-\omega^2},\qquad u'(0)=a\omega_0.$$

Solving for the coefficients, gives

$$u(t)=\frac{u'(0)}{\omega_0}\sin\omega_0 t+\left(u(0)-\frac{A}{\omega_0^2-\omega^2}\right)\cos\omega_0 t+\frac{A}{\omega_0^2-\omega^2}\cos\omega t.$$

Do these solutions seem compatible? Is the ADM truncated series an approximation to the exact solution? We should recognize from the first two lines of Equation (7.110) as $u(0)\cos\omega_0 t$ and $\dfrac{u'(0)}{\omega_0}\sin\omega_0 t$, respectively. In order to obtain the third line, we consider an expansion for B. Namely,

$$\begin{aligned}\frac{A}{\omega_0^2-\omega^2} &= -\frac{A}{\omega^2}\frac{1}{1-\frac{\omega_0^2}{\omega^2}}\\ &= -\frac{A}{\omega^2}\left(1+\frac{\omega_0^2}{\omega^2}+\frac{\omega_0^4}{\omega^4}+\frac{\omega_0^6}{\omega^6}+\dots\right)\end{aligned} \tag{7.111}$$

This is what multiplies a $\cos\omega t$. All that is left to account for is the term

$$-\frac{A}{\omega_0^2-\omega^2}\cos\omega_0 t.$$

However, a series expansion of this expression about $\omega_0=0$ captures the remaining terms.

Example 7.15. Obtain solutions to the Airy Equation,

$$u''(x) = xu(x), \quad u(0) = A, u'(0) = B. \tag{7.112}$$

This equation takes the form $Lu = xu$, where $L = \frac{d^2}{dx^2}$. Applying L^{-1} to the differential equation gives the solution

$$u(x) = L^{-1}(xu) + \text{terms involving initial conditions.}$$

As before, we find the terms involving initial conditions from

$$L^{-1}Lu = \int_0^x \int_0^z u''(\xi)\, d\xi dz = u(x) - u(0) - xu'(0).$$

Using the initial conditions, the solution takes the form

$$u(x) = \sum_{n=0}^{\infty} u_n(x) = A + Bx + L^{-1}\left(x \sum_{n=0}^{\infty} u_n(x) \right).$$

Then, the recursive scheme takes the form

$$\begin{aligned} u_0(x) &= A + Bx, \\ u_{k+1}(x) &= L^{-1}(xu_k(x)). \end{aligned} \tag{7.113}$$

We compute $u_1(x)$ as

$$\begin{aligned} u_1(x) &= \int_0^x \int_0^z \xi u_0(\xi)\, d\xi dz \\ &= \int_0^x \left(\frac{1}{3} B\, z^3 + \frac{1}{2} A\, z^2 \right) dz \\ &= \frac{1}{12} B\, x^4 + \frac{1}{6} A\, x^3. \end{aligned} \tag{7.114}$$

Similarly, we find

$$\begin{aligned} u_2(x) &= \frac{1}{504} x^7 B + \frac{1}{180} x^6 A \\ u_3(x) &= \frac{1}{45360} x^{10} B + \frac{1}{12960} x^9 A. \end{aligned} \tag{7.115}$$

Adding the components, we obtain

$$\begin{aligned} u(x) &= A\left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \frac{x^9}{12960} + \dots\right) \\ &\quad + B\left(x + \frac{x^4}{12} + \frac{x^7}{504} + \frac{x^{10}}{45360} + \dots\right). \end{aligned} \tag{7.116}$$

Here we have the series solution that Airy and Stokes obtained using power series methods when trying to explain the superluminaries of a rainbow.

Example 7.16. Apply the Adomian Decomposition Method to the Mathieu Differential Equation.

The Mathieu equation is a linear second-order differential equation with periodic coefficients:

$$\frac{d^2y}{dx^2} + (a - 2q\cos(2x))y = 0,$$

where a and q are real parameters. Such equations arise in problems involving elliptical cylinder functions, parametric resonance in oscillating systems, and stability analyses in engineering.

We will apply the Adomian Decomposition Method to obtain an approximate analytical solution. In the case of the Mathieu equation, the term involving y is linear but contains a variable coefficient. So, we write the Mathieu equation in operator form as

$$Ly = -Ry,$$

where:

$$L = \frac{d^2}{dx^2}, \quad R = (a - 2q\cos(2x)).$$

We will also need the inverse operator L^{-1}, which is defined as the double integral

$$L^{-1}(f)(x) = \int_0^x \int_0^\xi f(\eta)\, d\eta\, d\xi,$$

assuming the initial conditions:

$$y(0) = y_0, \quad y'(0) = y_1.$$

We decompose the solution into an infinite series,

$$y(x) = \sum_{n=0}^{\infty} y_n(x),$$

and then apply L^{-1} to both sides of the Mathieu equation in operator form to obtain

$$y(x) = y_0 + y_1 x - L^{-1}\left[(a - 2q\cos(2x))y(x)\right].$$

Next we derive the recursive scheme. Assume that

$$y(x) = \sum_{n=0}^{\infty} y_n(x),$$

with

$$y_0(x) = y_0 + y_1 x, \quad y_{n+1}(x) = -L^{-1}\left[(a - 2q\cos(2x))y_n(x)\right].$$

Next, we match up terms on both sides of the equation.

The zeroth approximation is given by

$$y_0(x) = y_0 + y_1 x.$$

The first iteration is then

$$\begin{aligned} y_1(x) &= -L^{-1}\left[(a-2q\cos(2x))y_0(x)\right] = -L^{-1}\left[(a-2q\cos(2x))(y_0+y_1x)\right] \\ &= -L^{-1}\left[a(y_0+y_1x)-2q\cos(2x)(y_0+y_1x)\right]. \end{aligned} \tag{7.117}$$

Integratong term-by-term, we have

$$y_1(x) = -\int_0^x \int_0^\xi \left[a(y_0+y_1\eta)-2q\cos(2\eta)(y_0+y_1\eta)\right] d\eta\, d\xi.$$

This yields an explicit expression after computing the integrals.

To improve the accuracy we could carry out a second iteration.

$$y_2(x) = -L^{-1}\left[(a-2q\cos(2x))y_1(x)\right].$$

This term involves nested integrals and can be computed symbolically or numerically. Up to second order, the approximate solution is

$$y(x) \approx y_0(x) + y_1(x) + y_2(x).$$

The accuracy improves as more terms are added. ADM is particularly useful for handling periodic coefficients analytically to a reasonable approximation. Thus, the Adomian Decomposition Method allows for approximate analytical solutions of the Mathieu equation by systematically constructing a recursive series solution. While the Mathieu equation has known Floquet-type solutions, ADM offers an approach for approximating the behavior when exact solutions are intractable or when generalizations are introduced.

Problems

1. Consider the set of vectors $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$.

a. Use the Gram-Schmidt process to find an orthonormal basis for R^3 using this set in the given order.

b. What do you get if you do reverse the order of these vectors?

2. Use the Gram-Schmidt process to find the first four orthogonal polynomials satisfying the following:

a. Interval: $(-\infty,\infty)$ Weight Function: e^{-x^2}.

b. Interval: $(0,\infty)$ Weight Function: e^{-x}.

3. The Tchebysheff polynomials of the first kind are defined as $T_n(\cos\theta) = \cos(n\theta)$. Show that these functions are orthogonal on $[-1,1]$ with weight function, $(1-x^2)^{-1/2}$ as claimed in Table 7.1.

4. Let $T_n(x), n = 0,1,2,\ldots$ be solutions of the differential equation

$$(1-x^2)y'' - xy' + n^2y = 0.$$

a. Verify that $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$, satisfy the respective differential equations for $n = 0, 1, 2, 3$.

b. Show that, in general, $T_n(x) = \cos(n \cos^{-1} x), n = 1, 2, 3, \dots$ are polynomial solutions of degree n. Hint: Use de Moivre's formula

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta,$$

where $x = \cos\theta$.

5. Do the following:

a. Show that the Tchebysheff polynomials, $T_n(x) = \cos(n \cos^{-1} x)$, for $n = 1, 2, 3, \dots$, satisfy a three term recursion formula,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

b. The generating function for the Tchebysheff polynomials is given by

$$g(x,t) = \frac{1 - tx}{1 - 2tx + t^2}.$$

Use the binomial expansion to show this is true for $n = 0, 1, 2, 3$. [See Problem 4.]

6. Find $P_4(x)$ using

a. The Rodrigues' Formula in Equation (7.13).

b. The three term recursion formula in Equation (7.15).

7. In Equations (7.25)-(7.32) we provide several identities for Legendre polynomials. Derive the results in Equations (7.26)-(7.32) as described in the text. Namely,

a. Differentiating Equation (7.25) with respect to x, derive Equation (7.26).

b. Derive Equation (7.27) by differentiating $g(x,t)$ with respect to x and rearranging the resulting infinite series.

c. Combining the last result with Equation (7.25), derive Equations (7.28)-(7.29).

d. Adding and subtracting Equations (7.28)-(7.29), obtain Equations (7.30)-(7.31).

e. Derive Equation (7.32) using some of the other identities.

8. Use the recursion relation (7.15) to evaluate $\int_{-1}^{1} xP_n(x)P_m(x)\,dx$, $n \le m$. Namely, insert $xP_n(x) = \dfrac{n+1}{(2n+1)}P_{n+1}(x) + \dfrac{n}{(2n+1)}P_{n-1}(x)$ in the integral and use the orthogonality of the Legendre polynomials to evaluate the integral.

9. Expand the following in a Fourier-Legendre series for $x \in (-1, 1)$.

a. $f(x) = x^2$.

b. $f(x) = 5x^4 + 2x^3 - x + 3.$

c. $f(x) = \begin{cases} -1, & -1 < x < 0, \\ 1, & 0 < x < 1. \end{cases}$

d. $f(x) = \begin{cases} x, & -1 < x < 0, \\ 0, & 0 < x < 1. \end{cases}$

10. In Maple one can type **simplify(LegendreP(2*n-2,0)-LegendreP(2*n,0));** to find a value for $P_{2n-2}(0) - P_{2n}(0)$. It gives the result in terms of Gamma functions. However, in Example 7.10 for Fourier-Legendre series, the value is given in terms of double factorials! So, we have

$$P_{2n-2}(0) - P_{2n}(0) = \frac{\sqrt{\pi}(4n-1)}{2\Gamma(n+1)\Gamma\left(\frac{3}{2}-n\right)} = (-1)^n \frac{(2n-3)!!}{(2n-2)!!}\frac{4n-1}{2n}.$$

You will verify that both results are the same by doing the following:

a. Prove that $P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}$ using the generating function and a binomial expansion.

b. Prove that $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$ using $\Gamma(x) = (x-1)\Gamma(x-1)$ and iteration.

c. Verify the result from Maple that $P_{2n-2}(0) - P_{2n}(0) = \frac{\sqrt{\pi}(4n-1)}{2\Gamma(n+1)\Gamma\left(\frac{3}{2}-n\right)}$.

d. Can either expression for $P_{2n-2}(0) - P_{2n}(0)$ be simplified further?

11. In 1838 the English Astronomer Royal George Biddell Airy developed a theory for rainbows to explain the appearance of supernumeraries, faint, pastel-colored arcs that appear inside the primary rainbow due to wave interference. This led to what are now called Airy integrals or functions,

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right) dt$$

These functions are solutions of the differential equation $\frac{d^2y}{dx^2} - xy = 0.$

a. Show that the Airy function is a solution of the differential equation.

b. Assume a power series solution for Airy's differential equation of the form $y(x) = \sum_{k=0}^\infty a_k x^k$. Find a recursion relation for the $a_k's$.

c. Show that the power series coefficients lead to the two linearly independent solutions [Compare this problem to Example 7.15.]

$$\begin{aligned} y_1(x) &= 1 + \sum_{n=1}^\infty \frac{1}{3^n n!\,[2\cdot 5\cdots(3n-1)]}x^{3n} \\ y_2(x) &= x + \sum_{n=1}^\infty \frac{1}{3^n n!\,[1\cdot 4\cdots(3n+1)]}x^{3n+1}. \end{aligned} \quad (7.118)$$

d. Plot the solutions $y_1(x)$ and $y_2(x)$ in the previous part of the problem focusing on $x \in [-10, 10]$.

12. Solve Hermite's differential equation $y'' - 2xy' + \lambda y = 0$ using a series solution about x_0. Find solutions for $\lambda = 0, 1, 2$.

13. The Hermite polynomials, $H_n(x)$, satisfy the following:

i. $< H_n, H_m >= \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \sqrt{\pi} 2^n n! \delta_{n,m}$.

ii. $H_n'(x) = 2nH_{n-1}(x)$.

iii. $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$.

iv. $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$.

Using these, show that

a. $H_n'' - 2xH_n' + 2nH_n = 0$. [Use properties ii. and iii.]

b. $\int_{-\infty}^{\infty} xe^{-x^2} H_n(x) H_m(x)\, dx = \sqrt{\pi} 2^{n-1} n! \left[\delta_{m,n-1} + 2(n+1)\delta_{m,n+1}\right]$. [Use properties i. and iii.]

c. $H_n(0) = \begin{cases} 0, & n \text{ odd}, \\ (-1)^m \frac{(2m)!}{m!}, & n = 2m. \end{cases}$ [Let $x = 0$ in iii. and iterate. Note from iv. that $H_0(x) = 1$ and $H_1(x) = 2x$.]

14. Use integration by parts to show $\Gamma(x+1) = x\Gamma(x)$.

15. Prove the double factorial identities:

$$(2n)!! = 2^n n!$$

and

$$(2n-1)!! = \frac{(2n)!}{2^n n!}.$$

16. In 1733 de Moivre added two supplements to his 1730 book after communicating with James Stirling, containing the first use of an approximation for large factorials. However, some say that the first appearance of the Stirling approximation appeared in a letter from Euler to Goldbach in 1744. We now know of the leading order approximation of the factorial for large N as

$$N! \approx \sqrt{2\pi} e^{-N} N^{N+\frac{1}{2}}.$$

Use this Stirling approximation to find an approximation for the following:

a. $\Gamma(n)$.

b. $N!!$

c. What do these approximation give for 100! and 100!!? How do they compare with the exact values? [Use a computer algebra system to answer this, do not do this by hand.]

17. Express the following as Gamma functions. Namely, noting the form $\Gamma(x+1) = \int_0^{\infty} t^x e^{-t}\, dt$ and using an appropriate substitution, each expression can be written in terms of a Gamma function.

a. $\int_0^{\infty} x^{2/3} e^{-x}\, dx$.

b. $\int_0^\infty x^5 e^{-x^2}\,dx$

c. $\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^n\,dx$

18. Show that

$$-\int_0^1 x^k \ln x\,dx = \frac{1}{(k+1)^2}, \quad k > -1$$

by first transforming the integral into a Gamma function.

19. Gaussian integrals are important in statistical mechanics and quantum mechanics. Show that

a. $\int_0^\infty x^{2n+1} e^{-ax^2}\,dx = \dfrac{n!}{2a^{n+1}}$.

b. $\int_0^\infty x^{2n} e^{-ax^2}\,dx = \dfrac{(2n-1)!!}{2^{n+1}a^n}\sqrt{\dfrac{\pi}{a}}$.

20. Prove that

$$\int_0^\infty e^{-x^4}\,dx = \Gamma\left(\frac{5}{4}\right) = \left(\frac{1}{4}\right)!.$$

21. The coefficients C_k^p in the binomial expansion for $(1+x)^p$ are given by

$$C_k^p = \frac{p(p-1)\cdots(p-k+1)}{k!}.$$

a. Write C_k^p in terms of Gamma functions.

b. For $p = 1/2$ use the properties of Gamma functions to write $C_k^{1/2}$ in terms of factorials.

c. Confirm your answer in part b by deriving the Maclaurin series expansion of $(1+x)^{1/2}$.

22. A solution Bessel's equation, $x^2y'' + xy' + (x^2 - n^2)y = 0$, , can be found using the guess $y(x) = \sum_{j=0}^\infty a_j x^{j+n}$. One obtains the recurrence relation $a_j = \frac{-1}{j(2n+j)}a_{j-2}$. Show that for $a_0 = (n!2^n)^{-1}$ we get the Bessel function of the first kind of order n from the even values $j = 2k$:

$$J_n(x) = \sum_{k=0}^\infty \frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k}.$$

23. Use the infinite series in the last problem to derive the derivative identities (7.66) and (7.56):

a. $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$.

b. $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$.

24. Prove the following identities based on those in the last problem.

a. $J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$.

b. $J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x)$.

25. Use the derivative identities of Bessel functions,(7.66)-(7.56), and integration by parts to show that

$$\int x^3 J_0(x)\,dx = x^3 J_1(x) - 2x^2 J_2(x) + C.$$

26. Use the generating function to find $J_n(0)$ and $J_n'(0)$.

27. Bessel functions $J_p(\lambda x)$ are solutions of $x^2y'' + xy' + (\lambda^2x^2 - p^2)y = 0$. Assume that $x \in (0,1)$ and that $J_p(\lambda) = 0$ and $J_p(0)$ is finite.

a. Show that this equation can be written in the form

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (\lambda^2 x - \frac{p^2}{x})y = 0.$$

This is the standard Sturm-Liouville form for Bessel's equation.

b. Prove that

$$\int_0^1 xJ_p(\lambda x)J_p(\mu x)\,dx = 0, \quad \lambda \neq \mu$$

by considering

$$\int_0^1 \left[J_p(\mu x)\frac{d}{dx}\left(x\frac{d}{dx}J_p(\lambda x)\right) - J_p(\lambda x)\frac{d}{dx}\left(x\frac{d}{dx}J_p(\mu x)\right)\right]dx.$$

Thus, the solutions corresponding to different eigenvalues (λ, μ) are orthogonal.

c. Prove that

$$\int_0^1 x\left[J_p(\lambda x)\right]^2\,dx = \frac{1}{2}J_{p+1}^2(\lambda) = \frac{1}{2}J_p'^2(\lambda).$$

[Hint: Compute the limit as $\mu \to \lambda$ of the previous result,

$$\int_0^1 x\left[J_p(\lambda x)\right]^2\,dx = \lim_{\mu\to\lambda}\int_0^1 xJ_p(\lambda x)J_p(\mu x)\,dx.$$

using l'Hôpital's Rule and the identities

$$\frac{dJ_n(z)}{dz} = \frac{n}{z}J_n(z) - J_{n+1}(z) = J_{n-1}(z) - \frac{n}{z}J_n(z).$$

]

28. We can rewrite Bessel functions, $J_\nu(x)$, in a form which will allow the order to be non-integer by using the gamma function. You will need the results from Problem 10b for $\Gamma\left(k+\frac{1}{2}\right)$.

a. Extend the series definition of the Bessel function of the first kind of order ν, $J_\nu(x)$, for $\nu \geq 0$ by writing the series solution for $y(x)$ in Problem 22 using the gamma function.

b. Extend the series to $J_{-\nu}(x)$, for $\nu \geq 0$. Discuss the resulting series and what happens when ν is a positive integer.

c. Use these results to obtain the closed form expressions

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x,$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x.$$

d. Use the results in part c with the recursion formula for Bessel functions to obtain a closed form for $J_{3/2}(x)$.

29. In this problem you will derive the expansion

$$x^2 = \frac{c^2}{2} + 4\sum_{j=2}^{\infty} \frac{J_0(\alpha_j x)}{\alpha_j^2 J_0(\alpha_j c)}, \quad 0 < x < c,$$

where the $\alpha_j's$ are the positive roots of $J_1(\alpha c) = 0$, by following the below steps.

a. List the first five values of α for $J_1(\alpha c) = 0$ using the Table 7.4 and Figure 7.8. [Note: Be careful determining α_1.]

b. Show that $\|J_0(\alpha_1 x)\|^2 = \frac{c^2}{2}$. Recall,

$$\|J_0(\alpha_j x)\|^2 = \int_0^c x J_0^2(\alpha_j x)\, dx.$$

c. Show that $\|J_0(\alpha_j x)\|^2 = \frac{c^2}{2}\left[J_0(\alpha_j c)\right]^2$, $j = 2, 3, \ldots$. (This is the most involved step.) First note from Problem 27 that $y(x) = J_0(\alpha_j x)$ is a solution of

$$x^2 y'' + xy' + \alpha_j^2 x^2 y = 0.$$

i. Verify the Sturm-Liouville form of this differential equation: $(xy')' = -\alpha_j^2 xy$.

ii. Multiply the equation in part i. by $y(x)$ and integrate from $x = 0$ to $x = c$ to obtain

$$\begin{aligned} \int_0^c (xy')' y\, dx &= -\alpha_j^2 \int_0^c xy^2\, dx \\ &= -\alpha_j^2 \int_0^c x J_0^2(\alpha_j x)\, dx. \end{aligned} \tag{7.119}$$

iii. Noting that $y(x) = J_0(\alpha_j x)$, integrate the left hand side by parts and use the following to simplify the resulting equation.

1. $J_0'(x) = -J_1(x)$ from Equation (7.56).
2. Equation (7.59).
3. $J_2(\alpha_j c) + J_0(\alpha_j c) = 0$ from Equation (7.57).

iv. Now you should have enough information to complete this part.

d. Use the results from parts b and c and Problem 16 to derive the expansion coefficients for

$$x^2 = \sum_{j=1}^{\infty} c_j J_0(\alpha_j x)$$

in order to obtain the desired expansion.