Chapter 6

Sturm-Liouville Eigenvalue Problems

"Read Euler, read Euler. He is the master of us all."

"Nature laughs at the difficulties of integration." - Pierre-Simon Marquis de Laplace (1749-1827)

6.1 Introduction

In physics many problems arise in the form of boundary value problems involving second order ordinary differential equations. For example, when solving the wave equation and the heat equation in three dimensions using the Method of Separation of Variables, separating out the time dependence leads to a three dimensional boundary value problem in both cases. Further separation of variables leads to a set of boundary value problems involving second order ordinary differential equations.

In general, we might obtain equations of the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \tag{6.1}$$

for $y = y(x)$ subject to boundary conditions. We can write such an equation in operator form by defining the differential operator

$$L = a_2(x)D^2 + a_1(x)D + a_0(x),$$

where $D = d/dx$. Then, Equation (6.1) takes the form

$$Ly = f.$$

Recall that we had solved such nonhomogeneous differential equations in Chapter 1 and a review is in Section **??**. In this section we will show that these equations can be solved using eigenfunction expansions. Namely, we seek solutions to the eigenvalue problem

$$L\phi = \lambda\phi$$

with homogeneous boundary conditions on ϕ and then seek a solution of the nonhomogeneous problem, $Ly = f$, as an expansion over these eigenfunctions. Formally, we let

$$y(x) = \sum_{n=1}^{\infty} c_n\phi_n(x).$$

However, we are not guaranteed a nice set of eigenfunctions. We need an appropriate set to form a basis in the function space. Also, it would be nice to have orthogonality so that we can easily solve for the expansion coefficients.

It turns out that any linear second order differential operator can be turned into an operator that possesses just the right properties (self-adjointedness) to carry out this procedure. The resulting operator is referred to as a Sturm-Liouville operator. We will highlight some of the properties of these operators and see how they are used in applications.

The Sturm-Liouville operator.

We define the Sturm-Liouville operator as

$$\mathcal{L} = \frac{d}{dx}p(x)\frac{d}{dx} + q(x). \tag{6.2}$$

The Sturm-Liouville eigenvalue problem.

The Sturm-Liouville eigenvalue problem is given by the differential equation

$$\mathcal{L}y = -\lambda\sigma(x)y,$$

or

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y + \lambda\sigma(x)y = 0, \tag{6.3}$$

for $x \in (a,b)$, $y = y(x)$, plus boundary conditions. The functions $p(x)$, $p'(x)$, $q(x)$ and $\sigma(x)$ are assumed to be continuous on (a,b) and $p(x) > 0, \sigma(x) > 0$ on $[a,b]$. If the interval is finite and these assumptions on the coefficients are true on $[a,b]$, then the problem is said to be a regular Sturm-Liouville problem. Otherwise, it is called a singular Sturm-Liouville problem.

We also need to impose the set of homogeneous boundary conditions

$$\begin{aligned} \alpha_1 y(a) + \beta_1 y'(a) &= 0, \\ \alpha_2 y(b) + \beta_2 y'(b) &= 0. \end{aligned} \tag{6.4}$$

The α's and β's are constants. For different values, one has special types of boundary conditions. For $\beta_i = 0$, we have what are called Dirichlet boundary conditions. Namely, $y(a) = 0$ and $y(b) = 0$. For $\alpha_i = 0$, we have Neumann boundary conditions. In this case, $y'(a) = 0$ and $y'(b) = 0$. In terms of the heat equation example, Dirichlet conditions correspond to maintaining a fixed temperature at the ends of the rod. The Neumann boundary conditions would correspond to no heat flow across the ends, or insulating conditions, as there would be no temperature gradient at those points. The more general boundary conditions allow for partially insulated boundaries.

Types of boundary conditions:
Dirichlet boundary conditions - the solution takes fixed values on the boundary. These are named after Gustav Lejeune Dirichlet (1805-1859).

Neumann boundary conditions - the derivative of the solution takes fixed values on the boundary. These are named after Carl Neumann (1832-1925).

Robin boundary conditions, $ay(0) + by'(0) = 0$, named after Victor Gustave Robin (1855–1897).

Another type of boundary condition that is often encountered is the periodic boundary condition. Consider the heated rod that has been bent to form a circle. Then the two end points are physically the same. So, we would expect that the temperature and the temperature gradient should agree at those points. For this case we write $y(a) = y(b)$ and $y'(a) = y'(b)$. Boundary value problems using these conditions have to be handled differently than the above homogeneous conditions. These conditions leads to different types of eigenfunctions and eigenvalues.

As previously mentioned, equations of the form (6.1) occur often. We now show that any second order linear operator can be put into the form of the Sturm-Liouville operator. In particular, equation (6.1) can be put into the form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = F(x). \tag{6.5}$$

Differential equations of Sturm-Liouville form.

The proof of this is straight forward as we soon show. Let's first consider the equation (6.1) for the case that $a_1(x) = a_2'(x)$. Then, we can write the equation in a form in which the first two terms combine,

$$\begin{aligned} f(x) &= a_2(x)y'' + a_1(x)y' + a_0(x)y \\ &= (a_2(x)y')' + a_0(x)y. \end{aligned} \tag{6.6}$$

The resulting equation is now in Sturm-Liouville form. We just identify $p(x) = a_2(x)$ and $q(x) = a_0(x)$.

Not all second order differential equations are as simple to convert. Consider the differential equation

$$x^2y'' + xy' + 2y = 0.$$

In this case $a_2(x) = x^2$ and $a_2'(x) = 2x \neq a_1(x)$. So, this does not fall into this case. However, we can change the operator in this equation, $x^2D + xD$, to a Sturm-Liouville operator, $Dp(x)D$ for a $p(x)$ that depends on the coefficients x^2 and x.

In the Sturm Liouville operator the derivative terms are gathered together into one exact derivative, $Dp(x)D$. This is similar to what we saw for ODE [see Section **??**] when we solved linear first order equations. In that case we sought an integrating factor. We can do the same thing here. We seek a multiplicative function $\mu(x)$ that we can multiply through (6.1) so that it can be written in Sturm-Liouville form.

We first divide out the $a_2(x)$, giving

$$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = \frac{f(x)}{a_2(x)}.$$

Next, we multiply this differential equation by and integrating factor, μ,

$$\mu(x)y'' + \mu(x)\frac{a_1(x)}{a_2(x)}y' + \mu(x)\frac{a_0(x)}{a_2(x)}y = \mu(x)\frac{f(x)}{a_2(x)}.$$

The first two terms can now be combined into an exact derivative $(\mu y')'$ if the second coefficient is $\mu'(x)$. Therefore, $\mu(x)$ satisfies a first order, separable differential equation:

$$\frac{d\mu}{dx} = \mu(x)\frac{a_1(x)}{a_2(x)}.$$

This is formally solved to give the sought integrating factor[1]

$$\mu(x) = e^{\int \frac{a_1(x)}{a_2(x)}\,dx}.$$

[1] Actually, we should write this as

$$\mu(x) = e^{\int^x \frac{a_1(\xi)}{a_2(\xi)}\,d\xi}.$$

This distinguishes the variable of integration ξ from the evaluation at x.

Thus, the original equation can be multiplied by factor

$$\frac{\mu(x)}{a_2(x)} = \frac{1}{a_2(x)} e^{\int \frac{a_1(x)}{a_2(x)}\,dx}$$

to turn it into Sturm-Liouville form.

In summary,

Equation (6.1),

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x), \tag{6.7}$$

can be put into the Sturm-Liouville form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = F(x), \tag{6.8}$$

where

$$\begin{aligned} p(x) &= e^{\int \frac{a_1(x)}{a_2(x)}\,dx}, \\ q(x) &= p(x)\frac{a_0(x)}{a_2(x)}, \\ F(x) &= p(x)\frac{f(x)}{a_2(x)}. \end{aligned} \tag{6.9}$$

Example 6.1. Convert $x^2y'' + xy' + 2y = 0$ into Sturm-Liouville form.

We multiply this equation by

$$\frac{\mu(x)}{a_2(x)} = \frac{1}{x^2} e^{\int \frac{dx}{x}} = \frac{1}{x},$$

to put the equation in Sturm-Liouville form:

$$\begin{aligned} 0 &= xy'' + y' + \frac{2}{x}y \\ &= (xy')' + \frac{2}{x}y. \end{aligned} \tag{6.10}$$

Conversion of a linear second order differential equation to Sturm Liouville form.

We could also proceed as follows. Divide by x^2, giving

$$y'' + \frac{1}{x}y' + \frac{2}{x}y = 0.$$

Now find

$$\mu(x) = \int^x \frac{dx}{x} = \frac{1}{x}.$$

Multiply the original equation by $\mu(x)$ and rewrite the equation.

6.2 *Properties of Sturm-Liouville Eigenvalue Problems*

THERE ARE SEVERAL PROPERTIES THAT CAN BE PROVEN for the (regular) Sturm-Liouville eigenvalue problem in (6.3). However, we will not prove them all here. We will merely list some of the important facts and focus on a few of the properties.

Real, countable eigenvalues.

1. The eigenvalues are real, countable, ordered and there is a smallest eigenvalue. Thus, we can write them as $\lambda_1 < \lambda_2 < \ldots$. However, there is no largest eigenvalue and $n \to \infty$, $\lambda_n \to \infty$.

Oscillatory eigenfunctions.

2. For each eigenvalue λ_n there exists an eigenfunction ϕ_n with $n-1$ zeros on (a, b).

3. Eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the weight function, $\sigma(x)$. Defining the inner product of $f(x)$ and $g(x)$ as

$$\langle f, g \rangle = \int_a^b f(x)g(x)\sigma(x)\,dx, \tag{6.11}$$

then the orthogonality of the eigenfunctions can be written in the form

Orthogonality of eigenfunctions.

$$\langle \phi_n, \phi_m \rangle = \langle \phi_n, \phi_n \rangle \delta_{nm}, \quad n, m = 1, 2, \ldots. \tag{6.12}$$

4. The set of eigenfunctions is complete; i.e., any piecewise smooth function can be represented by a generalized Fourier series expansion of the eigenfunctions,

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x),$$

where

$$c_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

Actually, one needs $f(x) \in L^2_\sigma(a, b)$, the set of square integrable functions over $[a, b]$ with weight function $\sigma(x)$. By square integrable, we mean that $\langle f, f \rangle < \infty$. One can show that such a space is isomorphic to a Hilbert space, a complete inner product space. Hilbert spaces play a special role in quantum mechanics.

Complete basis of eigenfunctions.

5. The eigenvalues satisfy the Rayleigh quotient

$$\lambda_n = \frac{-p\phi_n \frac{d\phi_n}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{d\phi_n}{dx}\right)^2 - q\phi_n^2 \right] dx}{\langle \phi_n, \phi_n \rangle}.$$

The Rayleigh quotient is named after Lord Rayleigh, John William Strutt, 3rd Baron Raleigh (1842-1919).

This is verified by multiplying the eigenvalue problem

$$\mathcal{L}\phi_n = -\lambda_n \sigma(x)\phi_n$$

by ϕ_n and integrating. Solving this result for λ_n, we obtain the Rayleigh quotient. The Rayleigh quotient is useful for getting estimates of eigenvalues and proving some of the other properties.

Example 6.2. Verify some of these properties for the eigenvalue problem

$$y'' = -\lambda y, \quad y(0) = y(\pi) = 0.$$

This is a problem we had seen many times. The eigenfunctions for this eigenvalue problem are $\phi_n(x) = \sin nx$, with eigenvalues $\lambda_n = n^2$ for $n = 1, 2, \ldots$. These satisfy the properties listed above.

First of all, the eigenvalues are real, countable and ordered, $1 < 4 < 9 < \ldots$. There is no largest eigenvalue and there is a first one.

The eigenfunctions corresponding to each eigenvalue have $n - 1$ zeros on $(0, \pi)$. This is demonstrated for several eigenfunctions in Figure 6.1.

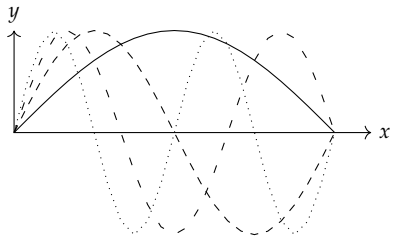


Figure 6.1: Plot of the eigenfunctions $\phi_n(x) = \sin nx$ for $n = 1, 2, 3, 4$.

We also know from Chapter 5 that the set $\{\sin nx\}_{n=1}^{\infty}$ is an orthogonal set of basis functions of length

$$\|\phi_n\| = \sqrt{\frac{\pi}{2}}.$$

Thus, the Rayleigh quotient can be computed using $p(x) = 1$, $q(x) = 0$, and the eigenfunctions. It is given by

$$\begin{aligned} R &= \frac{-\phi_n\phi_n'\Big|_0^\pi + \int_0^\pi (\phi_n')^2\, dx}{\frac{\pi}{2}} \\ &= \frac{2}{\pi}\int_0^\pi \left(-n^2\cos nx\right)^2 dx = n^2. \end{aligned} \tag{6.13}$$

Therefore, knowing the eigenfunction, the Rayleigh quotient returns the eigenvalues as expected.[2]

[2] Here the length, or norm, of ϕ is defined by
$$\|\phi\|^2 = \langle\phi,\phi\rangle.$$

Example 6.3. 6.9 We seek the eigenfunctions of the operator found in Example 6.1. Namely, we want to solve the eigenvalue problem

$$\mathcal{L}y = (xy')' + \frac{2}{x}y = -\lambda\sigma y \tag{6.14}$$

subject to a set of homogeneous boundary conditions. Let's use the boundary conditions

$$y'(1) = 0, \quad y'(2) = 0.$$

[Note that we do not know $\sigma(x)$ yet, but will choose an appropriate function to obtain solutions. Why didn't we give a condition at $x = 0$?]

Expanding the derivative, we have

$$xy'' + y' + \frac{2}{x}y = -\lambda\sigma y.$$

Multiply through by x to obtain

$$x^2y'' + xy' + (2 + \lambda x\sigma)\, y = 0.$$

Notice that if we choose $\sigma(x) = x^{-1}$, then this equation can be made a Cauchy-Euler type equation. [see Section **??**.] Thus, we have

$$x^2y'' + xy' + (\lambda + 2)\, y = 0.$$

The characteristic equation, found assuming solutions of the form $y(x) = x^r$, is

$$r^2 + \lambda + 2 = 0.$$

For oscillatory solutions, we need $\lambda + 2 > 0$. Thus, the general solution is

$$y(x) = c_1\cos(\sqrt{\lambda+2}\ln|x|) + c_2\sin(\sqrt{\lambda+2}\ln|x|). \tag{6.15}$$

See the Appendix for a review of Cauchy-Euler equations.

Next we apply the boundary conditions. $y'(1) = 0$ forces $c_2 = 0$. This leaves

$$y(x) = c_1 \cos(\sqrt{\lambda + 2} \ln x).$$

The second condition, $y'(2) = 0$, yields

$$\sin(\sqrt{\lambda + 2} \ln 2) = 0.$$

This will give nontrivial solutions when

$$\sqrt{\lambda + 2} \ln 2 = n\pi, \quad n = 0, 1, 2, 3 \ldots.$$

In summary, the eigenfunctions for this eigenvalue problem are

$$y_n(x) = \cos\left(\frac{n\pi}{\ln 2} \ln x\right), \quad 1 \leq x \leq 2$$

and the eigenvalues are $\lambda_n = \left(\frac{n\pi}{\ln 2}\right)^2 - 2$ for $n = 0, 1, 2, \ldots$.

Note: We include the $n = 0$ case because $y(x) =$ constant is a solution of the $\lambda = -2$ case. More specifically, in this case the characteristic equation reduces to $r^2 = 0$. Thus, the general solution of this Cauchy-Euler equation is

$$y(x) = c_1 + c_2 \ln |x|.$$

Setting $y'(1) = 0$, forces $c_2 = 0$. $y'(2)$ automatically vanishes, leaving the solution in this case as $y(x) = c_1$.

We note that some of the properties listed in the beginning of the section hold for this example. The eigenvalues are seen to be real, countable and ordered. There is a least one, $\lambda_0 = -2$. Next, one can find the zeros of each eigenfunction on [1,2]. Then the argument of the cosine, $\frac{n\pi}{\ln 2} \ln x$, takes values 0 to $n\pi$ for $x \in [1, 2]$. The cosine function has $n - 1$ roots on this interval.

Orthogonality can be checked as well. We set up the integral and use the substitution $y = \pi \ln x / \ln 2$. This gives the inner product 6.11 as

$$\begin{aligned} \langle y_n, y_m \rangle &= \int_1^2 \cos\left(\frac{n\pi}{\ln 2} \ln x\right) \cos\left(\frac{m\pi}{\ln 2} \ln x\right) \frac{dx}{x} \\ &= \frac{\ln 2}{\pi} \int_0^\pi \cos ny \cos my \, dy \\ &= \frac{\ln 2}{2} \delta_{n,m}. \end{aligned} \tag{6.16}$$

6.2.1 *Adjoint Operators*

In the study of the spectral theory of matrices, one learns about the adjoint of the matrix, $A^\dagger$, and the role that self-adjoint, or Hermitian, matrices play in diagonalization. Also, one needs the concept of adjoint to discuss the existence of solutions to the matrix problem $\mathbf{y} = A\mathbf{x}$. In the same

spirit, one is interested in the existence of solutions of the operator equation $Lu = f$ and solutions of the corresponding eigenvalue problem. The study of linear operators on a Hilbert space is a generalization of what the reader had seen in a linear algebra course.

Just as one can find a basis of eigenvectors and diagonalize Hermitian, or self-adjoint, matrices (or, real symmetric in the case of real matrices), we will see that the Sturm-Liouville operator is self-adjoint. In this section we will define the domain of an operator and introduce the notion of adjoint operators. In the last section we discuss the role the adjoint plays in the existence of solutions to the operator equation $Lu = f$.

Adjoint of an operator.

We begin by defining the adjoint of an operator. The adjoint, $L^\dagger$, of operator L satisfies

$$\langle u, Lv\rangle = \langle L^\dagger u, v\rangle$$

for all v in the domain of L and u in the domain of $L^\dagger$. Here the domain of a differential operator L is the set of all $u \in L^2_\sigma(a,b)$ satisfying a given set of homogeneous boundary conditions. This is best understood through example.

Example 6.4. Find the adjoint of $L = a_2(x)D^2 + a_1(x)D + a_0(x)$ for $D = d/dx$.

In order to find the adjoint, we place the operator inside an integral. Consider the inner product

$$\langle u, Lv\rangle = \int_a^b u(a_2v'' + a_1v' + a_0v)\,dx.$$

We have to move the operator L from v and determine what operator is acting on u in order to formally preserve the inner product. For a simple operator like $L = \frac{d}{dx}$, this is easily done using integration by parts. For the given operator, we will need to apply several integrations by parts to the individual terms. We consider each derivative term in the integrand separately.

For the a_1v' term, we integrate by parts to find

$$\int_a^b u(x)a_1(x)v'(x)\,dx = a_1(x)u(x)v(x)\Big|_a^b - \int_a^b (u(x)a_1(x))'v(x)\,dx. \quad (6.17)$$

Now, we consider the a_2v'' term. In this case it will take two integrations by parts:

$$\begin{aligned}\int_a^b u(x)a_2(x)v''(x)\,dx &= a_2(x)u(x)v'(x)\Big|_a^b - \int_a^b (u(x)a_2(x))'v(x)'\,dx \\ &= \left[a_2(x)u(x)v'(x) - (a_2(x)u(x))'v(x)\right]\Big|_a^b \\ &\quad + \int_a^b (u(x)a_2(x))''v(x)\,dx. \quad (6.18)\end{aligned}$$

Combining these results, we obtain

$$\langle u, Lv\rangle = \int_a^b u(a_2v'' + a_1v' + a_0v)\,dx$$

$$= \left[a_1(x)u(x)v(x) + a_2(x)u(x)v'(x) - (a_2(x)u(x))'v(x)\right]\Big|_a^b + \int_a^b \left[(a_2u)'' - (a_1u)' + a_0u\right] v\,dx. \quad (6.19)$$

Before inserting the boundary conditions for v, one has to determine boundary conditions for u such that

$$\left[a_1(x)u(x)v(x) + a_2(x)u(x)v'(x) - (a_2(x)u(x))'v(x)\right]\Big|_a^b = 0.$$

This leaves

$$\langle u, Lv\rangle = \int_a^b \left[(a_2u)'' - (a_1u)' + a_0u\right] v\,dx \equiv \langle L^\dagger u, v\rangle.$$

Therefore,

$$L^\dagger = \frac{d^2}{dx^2}a_2(x) - \frac{d}{dx}a_1(x) + a_0(x). \quad (6.20)$$

Self-adjoint operators.

When $L^\dagger = L$, the operator is called formally self-adjoint, or Hermitian. When the domain of L is the same as the domain of $L^\dagger$, the term self-adjoint is used. As the domain is important in establishing self-adjointness, we need to do a complete example in which the domain of the adjoint is found.

Example 6.5. Determine $L^\dagger$ and its domain for operator $Lu = \frac{du}{dx}$ where u satisfies the boundary conditions $u(0) = 2u(1)$ on $[0, 1]$.

We need to find the adjoint operator satisfying $\langle v, Lu\rangle = \langle L^\dagger v, u\rangle$. Therefore, we rewrite the integral

$$\langle v, Lu\rangle = \int_0^1 v\frac{du}{dx}\,dx = uv\big|_0^1 - \int_0^1 u\frac{dv}{dx}\,dx = \langle L^\dagger v, u\rangle.$$

From this we have the adjoint problem consisting of an adjoint operator and the associated boundary condition (or, domain of $L^\dagger$.):

1. $L^\dagger = -\frac{d}{dx}$.
2. $uv\Big|_0^1 = 0 \Rightarrow 0 = u(1)[v(1) - 2v(0)] \Rightarrow v(1) = 2v(0)$.

6.2.2 *Lagrange's and Green's Identities*

Before turning to the proofs that the eigenvalues of a Sturm-Liouville problem are real and the associated eigenfunctions orthogonal, we will first need to introduce two important identities. For the Sturm-Liouville operator,

$$\mathcal{L} = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q,$$

we have the two identities:

Lagrange's Identity:	$u\mathcal{L}v - v\mathcal{L}u$	$= [p(uv' - vu')]'$.
Green's Identity:	$\int_a^b (u\mathcal{L}v - v\mathcal{L}u)\,dx$	$= [p(uv' - vu')]\big\|_a^b$.

The proof of Lagrange's identity follows by a simple manipulations of the operator:

$$\begin{aligned} u\mathcal{L}v - v\mathcal{L}u &= u\left[\frac{d}{dx}\left(p\frac{dv}{dx}\right) + qv\right] - v\left[\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu\right] \\ &= u\frac{d}{dx}\left(p\frac{dv}{dx}\right) - v\frac{d}{dx}\left(p\frac{du}{dx}\right) \\ &= u\frac{d}{dx}\left(p\frac{dv}{dx}\right) + p\frac{du}{dx}\frac{dv}{dx} - v\frac{d}{dx}\left(p\frac{du}{dx}\right) - p\frac{du}{dx}\frac{dv}{dx} \\ &= \frac{d}{dx}\left[pu\frac{dv}{dx} - pv\frac{du}{dx}\right]. \end{aligned} \tag{6.21}$$

Green's identity is simply proven by integrating Lagrange's identity.

6.2.3 *Orthogonality and Reality*

We are now ready to prove that the eigenvalues of a Sturm-Liouville problem are real and the corresponding eigenfunctions are orthogonal. These are easily established using Green's identity, which in turn is a statement about the Sturm-Liouville operator being self-adjoint.

Example 6.6. The eigenvalues of the Sturm-Liouville problem (6.3) are real.

Let $\phi_n(x)$ be a solution of the eigenvalue problem associated with λ_n:

$$\mathcal{L}\phi_n = -\lambda_n \sigma \phi_n.$$

We want to show that the eigenvalues are real. Namely, we show that $\overline{\lambda}_n = \lambda_n$, where the bar means complex conjugate. So, we also consider the complex conjugate of this eigenvalue problem,

$$\mathcal{L}\overline{\phi}_n = -\overline{\lambda}_n \sigma \overline{\phi}_n.$$

Now, multiply the first equation by $\overline{\phi}_n$, the second equation by ϕ_n, and then subtract the results. We obtain

$$\overline{\phi}_n \mathcal{L}\phi_n - \phi_n \mathcal{L}\overline{\phi}_n = (\overline{\lambda}_n - \lambda_n)\sigma \phi_n \overline{\phi}_n.$$

Integrating both sides of this equation, we have

$$\int_a^b \left(\overline{\phi}_n \mathcal{L}\phi_n - \phi_n \mathcal{L}\overline{\phi}_n\right)\, dx = (\overline{\lambda}_n - \lambda_n)\int_a^b \sigma \phi_n \overline{\phi}_n \, dx.$$

We apply Green's identity to the left hand side to find

$$[p(\overline{\phi}_n \phi_n' - \phi_n \overline{\phi}_n')]|_a^b = (\overline{\lambda}_n - \lambda_n)\int_a^b \sigma \phi_n \overline{\phi}_n \, dx.$$

Using the homogeneous boundary conditions (6.4) for a self-adjoint operator, the left side vanishes. This leaves

$$0 = (\overline{\lambda}_n - \lambda_n)\int_a^b \sigma \left|\phi_n\right|^2 \, dx.$$

The integral is positive, so we must have $\overline{\lambda}_n = \lambda_n$. Therefore, the eigenvalues are real.

Example 6.7. The eigenfunctions corresponding to different eigenvalues of the Sturm-Liouville problem (6.3) are orthogonal.

This is proven similar to the last example. Let $\phi_n(x)$ be a solution of the eigenvalue problem associated with λ_n,

$$\mathcal{L}\phi_n = -\lambda_n \sigma \phi_n,$$

and let $\phi_m(x)$ be a solution of the eigenvalue problem associated with $\lambda_m \neq \lambda_n$,

$$\mathcal{L}\phi_m = -\lambda_m \sigma \phi_m,$$

Now, multiply the first equation by ϕ_m and the second equation by ϕ_n. Subtracting these results, we obtain

$$\phi_m \mathcal{L}\phi_n - \phi_n \mathcal{L}\phi_m = (\lambda_m - \lambda_n)\sigma \phi_n \phi_m$$

Integrating both sides of the equation, using Green's identity with homogeneous boundary conditions, we obtain

$$0 = (\lambda_m - \lambda_n) \int_a^b \sigma \phi_n \phi_m \, dx.$$

Since the eigenvalues are distinct, we can divide by $\lambda_m - \lambda_n$, leaving the desired result,

$$\int_a^b \sigma \phi_n \phi_m \, dx = 0.$$

Therefore, the eigenfunctions are orthogonal with respect to the weight function $\sigma(x)$.

6.2.4 *The Rayleigh Quotient*

The Rayleigh quotient is useful for getting estimates of eigenvalues and proving some of the other properties associated with Sturm-Liouville eigenvalue problems. The Rayleigh quotient is general and finds applications for both matrix eigenvalue problems as well as self-adjoint operators. For a Hermitian matrix M the Rayleigh quotient is given by

$$R(\mathbf{v}) = \frac{\langle \mathbf{v}, M\mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

One can show that the critical values of the Rayleigh quotient, as a function of $\mathbf{v}$, are the eigenvectors of M and the values of R at these critical values are the corresponding eigenvectors. In particular, minimizing $R(\mathbf{v})$ over the vector space will give the lowest eigenvalue. This leads to the Rayleigh-Ritz method for computing the lowest eigenvalues when the eigenvectors are not known.

This definition can easily be extended to Sturm-Liouville operators,

$$R(\phi_n) = \frac{\langle \phi_n \mathcal{L}\phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

We begin by multiplying the eigenvalue problem,

$$\mathcal{L}\phi_n = -\lambda_n \sigma(x)\phi_n,$$

by ϕ_n and integrating. This gives

$$\int_a^b \left[\phi_n \frac{d}{dx}\left(p\frac{d\phi_n}{dx}\right) + q\phi_n^2\right]\,dx = -\lambda_n \int_a^b \phi_n^2 \sigma\,dx.$$

One can solve the last equation for λ to find

$$\lambda_n = \frac{-\int_a^b \left[\phi_n \frac{d}{dx}\left(p\frac{d\phi_n}{dx}\right) + q\phi_n^2\right]\,dx}{\int_a^b \phi_n^2 \sigma\,dx} = R(\phi_n).$$

It appears that we have solved for the eigenvalues and have not needed the machinery used in studying boundary value problems. However, we really cannot evaluate this expression when we do not know the eigenfunctions, $\phi_n(x)$. Nevertheless, we will see what we can determine from the Rayleigh quotient.

One can rewrite this result by performing an integration by parts on the first term in the numerator. Namely, pick $u = \phi_n$ and $dv = \frac{d}{dx}\left(p\frac{d\phi_n}{dx}\right)\,dx$ for the standard integration by parts formula. Then, we have

$$\int_a^b \phi_n \frac{d}{dx}\left(p\frac{d\phi_n}{dx}\right)\,dx = p\phi_n\frac{d\phi_n}{dx}\Big|_a^b - \int_a^b \left[p\left(\frac{d\phi_n}{dx}\right)^2 - q\phi_n^2\right]\,dx.$$

Inserting the new formula into the expression for λ, leads to the Rayleigh Quotient

$$\lambda_n = \frac{-p\phi_n\frac{d\phi_n}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{d\phi_n}{dx}\right)^2 - q\phi_n^2\right]\,dx}{\int_a^b \phi_n^2\sigma\,dx}. \tag{6.22}$$

In many applications the sign of the eigenvalue is important. As we had seen in the solution of the heat equation, $T' + k\lambda T = 0$. Since we expect the heat energy to diffuse, the solutions should decay in time. Thus, we would expect $\lambda > 0$. In studying the wave equation, one expects vibrations and these are only possible with the correct sign of the eigenvalue (positive again). Thus, in order to have nonnegative eigenvalues, we see from (6.22) that

a. $q(x) \leq 0$, and

b. $-p\phi_n\frac{d\phi_n}{dx}\Big|_a^b \geq 0.$

Furthermore, if λ is a zero eigenvalue, then $q(x) \equiv 0$ and $\alpha_1 = \alpha_2 = 0$ in the homogeneous boundary conditions. This can be seen by setting the numerator equal to zero. Then, $q(x) = 0$ and $\phi_n'(x) = 0$. The second of these conditions inserted into the boundary conditions forces the restriction on the type of boundary conditions.

One of the properties of Sturm-Liouville eigenvalue problems with homogeneous boundary conditions is that the eigenvalues are ordered, $\lambda_1 <$

$\lambda_2 < \ldots$. Thus, there is a smallest eigenvalue. It turns out that for any continuous function, $y(x)$,

$$\lambda_1 = \min_{y(x)} \frac{-py\frac{dy}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{dy}{dx}\right)^2 - qy^2\right] dx}{\int_a^b y^2\sigma\, dx} \tag{6.23}$$

and this minimum is obtained when $y(x) = \phi_1(x)$. This result can be used to get estimates of the minimum eigenvalue by using trial functions which are continuous and satisfy the boundary conditions, but do not necessarily satisfy the differential equation.

Example 6.8. We have already solved the eigenvalue problem $\phi'' + \lambda\phi = 0$, $\phi(0) = 0$, $\phi(1) = 0$. In this case, the lowest eigenvalue is $\lambda_1 = \pi^2$. We can pick a nice function satisfying the boundary conditions, say $y(x) = x - x^2$. Inserting this into Equation (6.23), we find

$$\lambda_1 \leq \frac{\int_0^1 (1-2x)^2\, dx}{\int_0^1 (x-x^2)^2\, dx} = 10.$$

Indeed, $10 \geq \pi^2$.

The only condition we placed on our trial function was that it satisfied the boundary conditions. Thus, this procedure would work with other functions obtaining bounds on the eigenvalue. With a good choice, one might get a better estimate of the true value.

6.3 The Eigenfunction Expansion Method

In this section we solve the nonhomogeneous problem $\mathcal{L}y = f$ using expansions over the basis of Sturm-Liouville eigenfunctions. We have seen that Sturm-Liouville eigenvalue problems have the requisite set of orthogonal eigenfunctions. In this section we will apply the eigenfunction expansion method to solve a particular nonhomogeneous boundary value problem.

Recall that one starts with a nonhomogeneous differential equation

$$\mathcal{L}y = f,$$

where $y(x)$ is to satisfy given homogeneous boundary conditions. The method makes use of the eigenfunctions satisfying the eigenvalue problem

$$\mathcal{L}\phi_n = -\lambda_n\sigma\phi_n$$

subject to the given boundary conditions. Then, one assumes that $y(x)$ can be written as an expansion in the eigenfunctions,

$$y(x) = \sum_{n=1}^{\infty} c_n\phi_n(x),$$

and inserts the expansion into the nonhomogeneous equation. This gives

$$f(x) = \mathcal{L}\left(\sum_{n=1}^{\infty} c_n \phi_n(x)\right) = -\sum_{n=1}^{\infty} c_n \lambda_n \sigma(x) \phi_n(x).$$

The expansion coefficients are then found by making use of the orthogonality of the eigenfunctions. Namely, we multiply the last equation by $\phi_m(x)$ and integrate. We obtain

$$\int_a^b f(x)\phi_m(x)\,dx = -\sum_{n=1}^{\infty} c_n \lambda_n \int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx.$$

Orthogonality yields

$$\int_a^b f(x)\phi_m(x)\,dx = -c_m \lambda_m \int_a^b \phi_m^2(x)\sigma(x)\,dx.$$

Solving for c_m, we have

$$c_m = -\frac{\int_a^b f(x)\phi_m(x)\,dx}{\lambda_m \int_a^b \phi_m^2(x)\sigma(x)\,dx}.$$

Example 6.9. As an example, we consider the solution of the boundary value problem

$$(xy')' + \frac{y}{x} = \frac{1}{x}, \quad x \in [1, e], \tag{6.24}$$

$$y(1) = 0 = y(e). \tag{6.25}$$

This equation is already in self-adjoint form. So, we know that the associated Sturm-Liouville eigenvalue problem has an orthogonal set of eigenfunctions. We first determine this set. Namely, we need to solve the eigenvalue problem

$$(x\phi')' + \frac{\phi}{x} = -\lambda\sigma\phi, \quad \phi(1) = 0 = \phi(e). \tag{6.26}$$

Rearranging the terms and multiplying by x, we have that

$$x^2\phi'' + x\phi' + (1 + \lambda\sigma x)\phi = 0.$$

This is almost an equation of Cauchy-Euler type similar to Example . Picking the weight function $\sigma(x) = \frac{1}{x}$, we have

$$x^2\phi'' + x\phi' + (1 + \lambda)\phi = 0.$$

This is easily solved. From Section **??**, the characteristic equation is

$$r^2 + (1 + \lambda) = 0.$$

One obtains nontrivial solutions [3] of the eigenvalue problem satisfying the boundary conditions when $\lambda > -1$. The solutions are

$$\phi_n(x) = A \sin(n\pi \ln x), \quad n = 1, 2, \ldots.$$

[3] The general solution is

$$\phi(x) = A \sin(\beta \ln x) + A \cos(\beta \ln x),$$

where $\beta = \sqrt{(1+\lambda)}$. The boundary conditions given that $B = 0$ and $\sin\beta = 0$. So, $\beta = n\pi$, $n = 1, 2, \ldots$.

where $\lambda_n = n^2\pi^2 - 1$.

It is often useful to normalize the eigenfunctions. This means that one chooses A so that the norm of each eigenfunction is one. Thus, we have

$$\begin{aligned} 1 &= \int_1^e \phi_n(x)^2 \sigma(x)\, dx \\ &= A^2 \int_1^e \sin(n\pi \ln x)\frac{1}{x}\, dx \\ &= A^2 \int_0^1 \sin(n\pi y)\, dy = \frac{1}{2}A^2. \end{aligned} \tag{6.27}$$

Thus, $A = \sqrt{2}$. Several of these eigenfunctions are show in Figure 6.2.

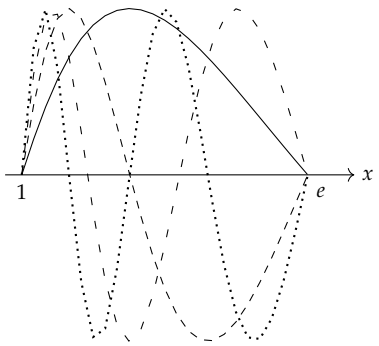


Figure 6.2: Plots of the first five eigenfunctions, $y(x) = \sqrt{2}\sin(n\pi \ln x)$.

We now turn towards solving the nonhomogeneous problem, $\mathcal{L}y = \frac{1}{x}$. We first expand the unknown solution in terms of the normalized eigenfunctions,

$$y(x) = \sum_{n=1}^{\infty} c_n \sqrt{2}\sin(n\pi \ln x).$$

Inserting this solution into the differential equation, we have

$$\frac{1}{x} = \mathcal{L}y = -\sum_{n=1}^{\infty} c_n \lambda_n \sqrt{2}\sin(n\pi \ln x)\frac{1}{x}.$$

Next, we make use of orthogonality. Multiplying both sides by the eigenfunction $\phi_m(x) = \sqrt{2}\sin(m\pi \ln x)$ and integrating, gives

$$\lambda_m c_m = \int_1^e \sqrt{2}\sin(m\pi \ln x)\frac{1}{x}\, dx = \frac{\sqrt{2}}{m\pi}[(-1)^m - 1].$$

Solving for c_m, we have

$$c_m = \frac{\sqrt{2}}{m\pi}\frac{[(-1)^m - 1]}{m^2\pi^2 - 1}.$$

Finally, we insert these coefficients into the expansion for $y(x)$. The solution is then

$$y(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\frac{[(-1)^n - 1]}{n^2\pi^2 - 1}\sin(n\pi \ln(x)).$$

We plot this solution in Figure 6.3. In Problem 14 you can directly solve this problem and compare your solution with the solution in Figure 6.3.

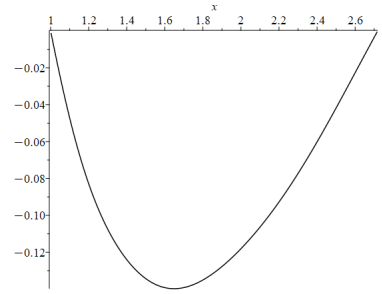


Figure 6.3: Plot of the solution in Example 6.9.

6.4 *Appendix: The Fredholm Alternative Theorem - Optional*

GIVEN THAT $Ly = f$, WHEN CAN ONE EXPECT to find a solution? Is it unique? These questions are answered by the Fredholm Alternative Theorem. This theorem occurs in many forms from a statement about solutions

to systems of algebraic equations to solutions of boundary value problems and integral equations. The theorem comes in two parts, thus the term "alternative". Either the equation has exactly one solution for all f, or the equation has many solutions for some f's and none for the rest.

The reader is familiar with the statements of the Fredholm Alternative for the solution of systems of algebraic equations. One seeks solutions of the system $Ax = b$ for A an $n \times m$ matrix. Defining the matrix adjoint, A^* through $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x, y, \in \mathcal{C}^n$, then either

Theorem 6.1. First Alternative

The equation $Ax = b$ has a solution if and only if $\langle b, v\rangle = 0$ for all v satisfying $A^*v = 0$.

or

Theorem 6.2. Second Alternative

A solution of $Ax = b$, if it exists, is unique if and only if $x = 0$ is the only solution of $Ax = 0$.

The second alternative is more familiar when given in the form: The solution of a nonhomogeneous system of n equations and n unknowns is unique if the only solution to the homogeneous problem is the zero solution. Or, equivalently, A is invertible, or has nonzero determinant.

Proof. We prove the second theorem first. Assume that $Ax = 0$ for $x \neq 0$ and $Ax_0 = b$. Then $A(x_0 + \alpha x) = b$ for all α. Therefore, the solution is not unique. Conversely, if there are two different solutions, x_1 and x_2, satisfying $Ax_1 = b$ and $Ax_2 = b$, then one has a nonzero solution $x = x_1 - x_2$ such that $Ax = A(x_1 - x_2) = 0$.

The proof of the first part of the first theorem is simple. Let $A^*v = 0$ and $Ax_0 = b$. Then we have

$$\langle b, v\rangle = \langle Ax_0, v\rangle = \langle x_0, A^*v\rangle = 0.$$

For the second part we assume that $\langle b, v\rangle = 0$ for all v such that $A^*v = 0$. Write b as the sum of a part that is in the range of A and a part that in the space orthogonal to the range of A, $b = b_R + b_O$. Then, $0 = \langle b_O, Ax\rangle = \langle A^*b, x\rangle$ for all x. Thus, $A^*b_O = 0$. Since $\langle b, v\rangle = 0$ for all v in the nullspace of A^*, then $\langle b, b_O\rangle = 0$.

Therefore, $\langle b, v\rangle = 0$ implies that

$$0 = \langle b, b_O\rangle = \langle b_R + b_O, b_O\rangle = \langle b_O, b_O\rangle.$$

This means that $b_O = 0$, giving $b = b_R$ is in the range of A. So, $Ax = b$ has a solution. □

Example 6.10. Determine the allowed forms of $\mathbf{b}$ for a solution of $A\mathbf{x} = \mathbf{b}$ to exist, where

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}.$$

First note that $A^* = \overline{A}^T$. This is seen by looking at

$$\begin{aligned} \langle A\mathbf{x}, \mathbf{y} \rangle &= \langle \mathbf{x}, A^*\mathbf{y} \rangle \\ \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_j \bar{y}_i &= \sum_{j=1}^{n} x_j \sum_{j=1}^{n} a_{ij} \bar{y}_i \\ &= \sum_{j=1}^{n} x_j \overline{\sum_{j=1}^{n} (\bar{a}^T)_{ji}\, y_i}. \end{aligned} \tag{6.28}$$

For this example,

$$A^* = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}.$$

We next solve $A^*\mathbf{v} = 0$. This means, $v_1 + 3v_2 = 0$. So, the nullspace of A^* is spanned by $\mathbf{v} = (3, -1)^T$. For a solution of $A\mathbf{x} = \mathbf{b}$ to exist, $\mathbf{b}$ would have to be orthogonal to $\mathbf{v}$. Therefore, a solution exists when

$$\mathbf{b} = \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

So, what does the Fredholm Alternative say about solutions of boundary value problems? We extend the Fredholm Alternative for linear operators. A more general statement would be

Theorem 6.3. *If L is a bounded linear operator on a Hilbert space, then $Ly = f$ has a solution if and only if $\langle f, v \rangle = 0$ for every v such that $L^\dagger v = 0$.*

The statement for boundary value problems is similar. However, we need to be careful to treat the boundary conditions in our statement. As we have seen, after several integrations by parts we have that

$$\langle \mathcal{L}u, v \rangle = S(u, v) + \langle u, \mathcal{L}^\dagger v \rangle,$$

where $S(u, v)$ involves the boundary conditions on u and v. Note that for nonhomogeneous boundary conditions, this term may no longer vanish.

Theorem 6.4. *The solution of the boundary value problem $\mathcal{L}u = f$ with boundary conditions $Bu = g$ exists if and only if*

$$\langle f, v \rangle - S(u, v) = 0$$

for all v satisfying $\mathcal{L}^\dagger v = 0$ and $B^\dagger v = 0$.

Example 6.11. Consider the problem

$$u'' + u = f(x), \quad u(0) - u(2\pi) = \alpha, u'(0) - u'(2\pi) = \beta.$$

Only certain values of α and β will lead to solutions. We first note that

$$L = L^\dagger = \frac{d^2}{dx^2} + 1.$$

Solutions of

$$L^\dagger v = 0, \quad v(0) - v(2\pi) = 0, v'(0) - v'(2\pi) = 0$$

are easily found to be linear combinations of $v = \sin x$ and $v = \cos x$.

Next, one computes

$$\begin{aligned} S(u,v) &= \left[u'v - uv'\right]_0^{2\pi} \\ &= u'(2\pi)v(2\pi) - u(2\pi)v'(2\pi) - u'(0)v(0) + u(0)v'(0). \end{aligned} \tag{6.29}$$

For $v(x) = \sin x$, this yields

$$S(u, \sin x) = -u(2\pi) + u(0) = \alpha.$$

Similarly,

$$S(u, \cos x) = \beta.$$

Using $\langle f, v\rangle - S(u,v) = 0$, this leads to the conditions that we were seeking,

$$\int_0^{2\pi} f(x) \sin x \, dx = \alpha,$$

$$\int_0^{2\pi} f(x) \cos x \, dx = \beta.$$

Problems

1. Prove the if $u(x)$ and $v(x)$ satisfy the general homogeneous boundary conditions

$$\begin{aligned} \alpha_1 u(a) + \beta_1 u'(a) &= 0, \\ \alpha_2 u(b) + \beta_2 u'(b) &= 0 \end{aligned} \tag{6.30}$$

at $x = a$ and $x = b$, then

$$p(x)[u(x)v'(x) - v(x)u'(x)]_{x=a}^{x=b} = 0.$$

2. Prove Green's Identity $\int_a^b (u\mathcal{L}v - v\mathcal{L}u)\, dx = [p(uv' - vu')]\Big|_a^b$ for the general Sturm-Liouville operator $\mathcal{L}$.

3. Find the adjoint operator and its domain for $Lu = u'' + 4u' - 3u$, $u'(0) + 4u(0) = 0$, $u'(1) + 4u(1) = 0$.

4. Show that a Sturm-Liouville operator with periodic boundary conditions on $[a,b]$ is self-adjoint if and only if $p(a) = p(b)$. [Recall, periodic boundary conditions are given as $u(a) = u(b)$ and $u'(a) = u'(b)$.]

5. Rewrite the following in Sturm-Liouville form:

a. $y'' + 2y' = -\lambda y$.

b. $x^2y'' + xy' = -\lambda y$.

c. $(1 - x^2)y'' - xy' = -\lambda y, \quad |x| < 1$.

d. $xy'' + (1 - x)y' = -\lambda y, \quad x > 0$.

6. Consider the differential equation given by

$$x^2y'' + xy' + (\lambda x^2 - n^2)y = 0.$$

Rewrite this equation in self-adjoint form. Give the integral form for the orthogonality of the eigenfunctions.

7. The Hermite differential equation is given by $y'' - 2xy' + \lambda y = 0$. Rewrite this equation in self-adjoint form. From the Sturm-Liouville form obtained, verify that the differential operator is self adjoint on $(-\infty, \infty)$. Give the integral form for the orthogonality of the eigenfunctions.

8. Find the eigenvalues and eigenfunctions of the given Sturm-Liouville problems.

a. $y'' + \lambda y = 0$, $y'(0) = 0 = y'(\pi)$.

b. $y'' + \lambda y = 0$, $hy(0) - y'(0) = 0, y'(1) = 0, \quad h > 0$.

c. $y'' + 2y' + y + \lambda y = 0$, $y(0) = 0 = y(1)$.

d. $(xy')' + \frac{\lambda}{x}y = 0$, $y(1) = y(e^2) = 0$.

e. $x^2y'' + xy' = -\lambda y, \quad 1 < x < e^\pi, y(1) = 0, y(e^\pi) = 0$.

f. $(xy')' + \frac{\lambda}{x}y = 0$, $y(1) = y(e^2) = 0$.

g. $y'' + \lambda y = 0$, $2y(0) - y'(0) = 0$, $y'(1) = 0$. In this problem you will need a computer to find the first two eigenvalues.

9. Consider the Sturm-Liouville problem $y'' + \lambda y = 0$, $y(0) + y'(0) = 0$, $y(1) + 2y'(1) = 0$.

a. Can $\lambda = 0$?

b. Show that for $\lambda < 0$ there is only one solution. Determine this solution numerically to three decimal places.

c. Show that for $\lambda > 0$ there are an infinite number of eigenvalues. Numerically find the first three to three decimal places.

10. The eigenvalue problem $x^2y'' - \lambda xy' + \lambda y = 0$ with $y(1) = y(2) = 0$ is not a Sturm-Liouville eigenvalue problem. Show that none of the eigenvalues are real by solving this eigenvalue problem.

11. Consider the eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = y'(0), \quad y(1) = \alpha y'(1).$$

For what value of α is $\lambda = 0$ an eigenvalue? What is the corresponding eigenfunction?

12. In Example 6.8 we found a bound on the lowest eigenvalue for the given eigenvalue problem.

a. Verify the computation in the example.

b. Apply the method using

$$y(x) = \begin{cases} x, & 0 < x < \frac{1}{2} \\ 1 - x, & \frac{1}{2} < x < 1. \end{cases}$$

Is this an upper bound on λ_1

c. Use the Rayleigh quotient to obtain a good upper bound for the lowest eigenvalue of the eigenvalue problem: $\phi'' + (\lambda - x^2)\phi = 0$, $\phi(0) = 0$, $\phi'(1) = 0$.

13. Consider the eigenvalue problem, $y''(x) + (\lambda - x^2)y(x) = 0$, $y'(0) = 0$, $y(1) = 0$. Using the Rayleigh quotient, find a bound on the first eigenvalue.

14. One can rewrite the problem in Example 6.9 as

$$x^2y''(x) + xy'(x) + y(x) = 1, \quad y(1) = y(e) = 0.$$

Find the closed form solution and plot the solutions with the eigenfunction expansion in Example 6.9 to verify that the solutions are the same.

15. Use the method of eigenfunction expansions to solve the problems:

a. $y'' = x^2, \quad y(0) = y(1) = 0.$

b. $y'' + 4y = x^2, \quad y'(0) = y'(1) = 0.$

Note that these problems can also be solved directly. Find the solutions and plot these solutions with the eigenfunction expansions to verify that the solutions are the same.

16. Determine the solvability conditions for the nonhomogeneous boundary value problem: $u'' + 4u = f(x)$, $u(0) = \alpha$, $u'(\pi/4) = \beta$.