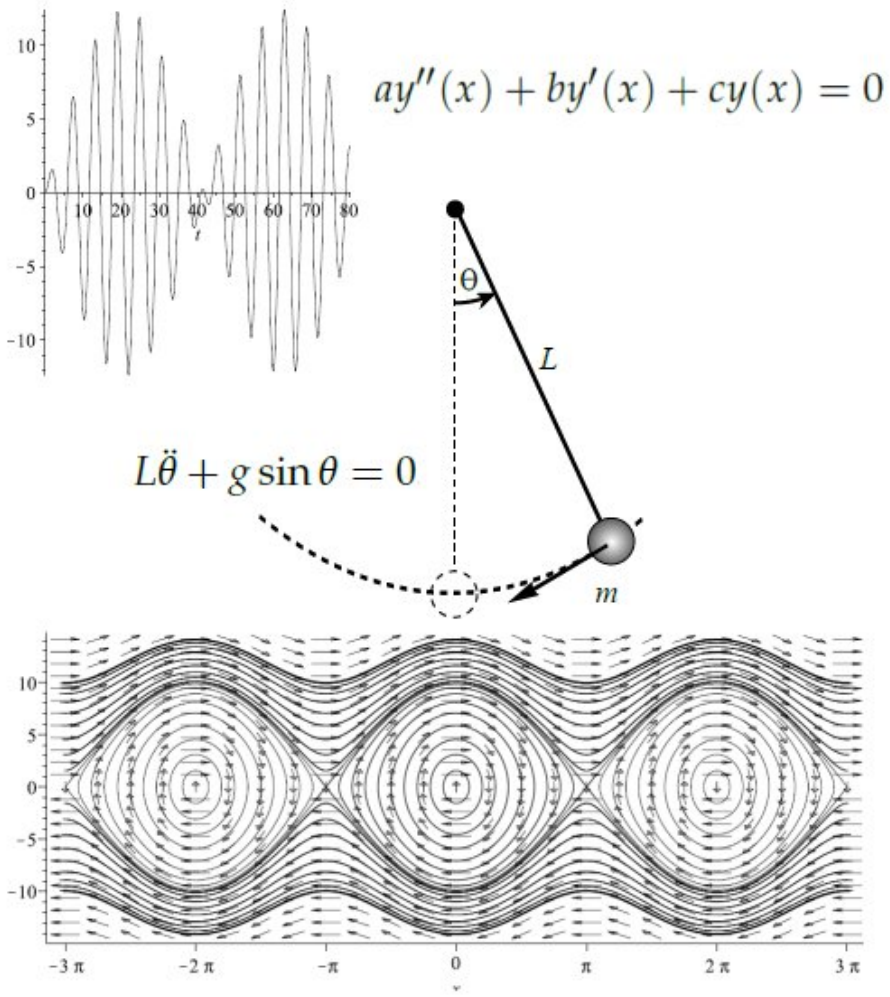
RUSSELL L. HERMAN

A SECOND COURSE IN ORDINARY DIFFERENTIAL EQUATIONS: DYNAMICAL SYSTEMS AND BOUNDARY VALUE PROBLEMS



R. L. HERMAN - VERSION DATE: APRIL 25, 2025

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A SECOND COURSE IN ORDINARY DIFFERENTIAL EQUATIONS: DYNAMICAL SYSTEMS AND BOUNDARY VALUE PROBLEMS by Russell Herman. Versions of these notes have been posted online since Spring 2005.
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April 2025

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Dedicated to those students who have endured previous versions of my notes and to current students who take the time to read this revised edition.

Chapter 1

Introduction

In order to solve a differential equation, you look at it till a solution occurs to you.
- George Pólya (1887-1985)

These are notes for a second course in differential equations originally taught in the Spring semester of 2005 at the University of North Carolina Wilmington to upper level and first year graduate students and later updated in Fall 2007, Fall 2008, and Fall 2024. It is assumed that you have had an introductory course in differential equations. However, we will begin this chapter with a review of some of the material from your first course in differential equations and then give an overview of the material we are about to cover.

Typically an introductory course in differential equations introduces students to analytical solutions of first order differential equations which are separable, first order linear differential equations, and sometimes to some other special types of equations. Students then explore the theory of second order differential equations generally restricted to the study of exact solutions of constant coefficient linear differential equations or even equations of the Cauchy-Euler type. These are later followed by the study of special techniques, such as power series methods or Laplace transform methods. If time permits, ones explores a few special functions, such as Legendre polynomials and Bessel functions, while using power series methods for solving differential equations.

More recently, variations on this inventory of topics have been introduced through the early introduction of systems of differential equations, qualitative studies of these systems and a more intense use of technology for understanding the behavior of solutions of differential equations. This is typically done at the expense of not covering power series methods, special functions, or Laplace transforms. In either case, the types of problems solved are initial value problems in which the differential equation to be solved is accompanied by a set of initial conditions.

In this course we will assume some exposure to the overlap of these two approaches. We will first give a quick review of the solution of separable and linear first order equations. Then we will review second order linear differential equations and Cauchy-Euler equations. This will then be fol-

lowed by an overview of some of the topics covered. As with any course in differential equations, we will emphasize analytical, graphical and (sometimes) approximate solutions of differential equations. Throughout we will present applications from physics, chemistry and biology.

1.1 Review of the First Course

In this section we review a few of the solution techniques encountered in a first course in differential equations. We will not review the basic theory except in possible references as reminders as to what we are doing.

We first recall that an *n-th order ordinary differential equation* is an equation for an unknown function $y(x)$ that expresses a relationship between the unknown function and its first n derivatives. One could write this generally as

$$F(y^{(n)}(x), y^{(n-1)}(x), \ldots, y'(x), y(x), x) = 0. \tag{1.1}$$

Here $y^{(n)}(x)$ represents the nth derivative of $y(x)$.

An *initial value problem* consists of the differential equation plus the values of the first $n-1$ derivatives at a particular value of the independent variable, say x_0:

$$y^{(n-1)}(x_0) = y_{n-1}, \quad y^{(n-2)}(x_0) = y_{n-2}, \quad \ldots, \quad y(x_0) = y_0. \tag{1.2}$$

A *linear nth order differential equation* takes the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \ldots + a_1(x)y'(x) + a_0(x)y(x)) = f(x). \tag{1.3}$$

If $f(x) \equiv 0$, then the equation is said to be *homogeneous*, otherwise it is *nonhomogeneous*.

1.1.1 First Order Differential Equations

Typically, the first differential equations encountered are first order equations. A *first order differential equation* takes the form

$$F(y', y, x) = 0. \tag{1.4}$$

There are two general forms for which one can formally obtain a solution. The first is the separable case and the second is a first order equation. We indicate that we can formally obtain solutions, as one can display the needed integration that leads to a solution. However, the resulting integrals are not always reducible to elementary functions nor does one obtain explicit solutions when the integrals are doable.

Separable equations.

A first order equation is separable if it can be written the form

$$\frac{dy}{dx} = f(x)g(y). \tag{1.5}$$

Special cases result when either $f(x) = 1$ or $g(y) = 1$. In the first case the equation is said to be *autonomous*.

The *general solution* to equation (1.5) is obtained in terms of two integrals:

$$\int \frac{dy}{g(y)} = \int f(x)\,dx + C, \tag{1.6}$$

where C is an integration constant. This yields a *1-parameter family of solutions* to the differential equation corresponding to different values of C. If one can solve (1.6) for $y(x)$, then one obtains an explicit solution. Otherwise, one has a family of implicit solutions. If an initial condition is given as well, then one might be able to find a member of the family that satisfies this condition, which is often called a *particular solution*.

Example 1.1. $y' = 2xy$, $y(0) = 2$.

Applying (1.6), one has

$$\int \frac{dy}{y} = \int 2x\,dx + C.$$

Integrating yields

$$\ln|y| = x^2 + C.$$

Exponentiating, one obtains the general solution,

$$y(x) = \pm e^{x^2+C} = Ae^{x^2}.$$

Here we have defined $A = \pm e^C$. Since C is an arbitrary constant, A is an arbitrary constant. Several solutions in this 1-parameter family are shown in Figure 1.1.

Next, one seeks a particular solution satisfying the initial condition. For $y(0) = 2$, one finds that $A = 2$. So, the particular solution satisfying the initial conditions is $y(x) = 2e^{x^2}$.

Example 1.2. $yy' = -x$.

Following the same procedure as in the last example, one obtains:

$$\int y\,dy = -\int x\,dx + C \Rightarrow y^2 = -x^2 + A, \quad \text{where} \quad A = 2C.$$

Thus, we obtain an implicit solution. Writing the solution as $x^2 + y^2 = A$, we see that this is a family of circles for $A > 0$ and the origin for $A = 0$. Plots of some solutions in this family are shown in Figure 1.2.

Linear first order equations.

The second type of first order equation encountered is the *linear first order differential equation* in the form

$$y'(x) + p(x)y(x) = q(x). \tag{1.7}$$

Integrating factor.

In this case one seeks an *integrating factor*, $\mu(x)$, which is a function that one can multiply through the equation making the left side a perfect derivative. Thus, obtaining,

$$\frac{d}{dx}\left[\mu(x)y(x)\right] = \mu(x)q(x). \tag{1.8}$$

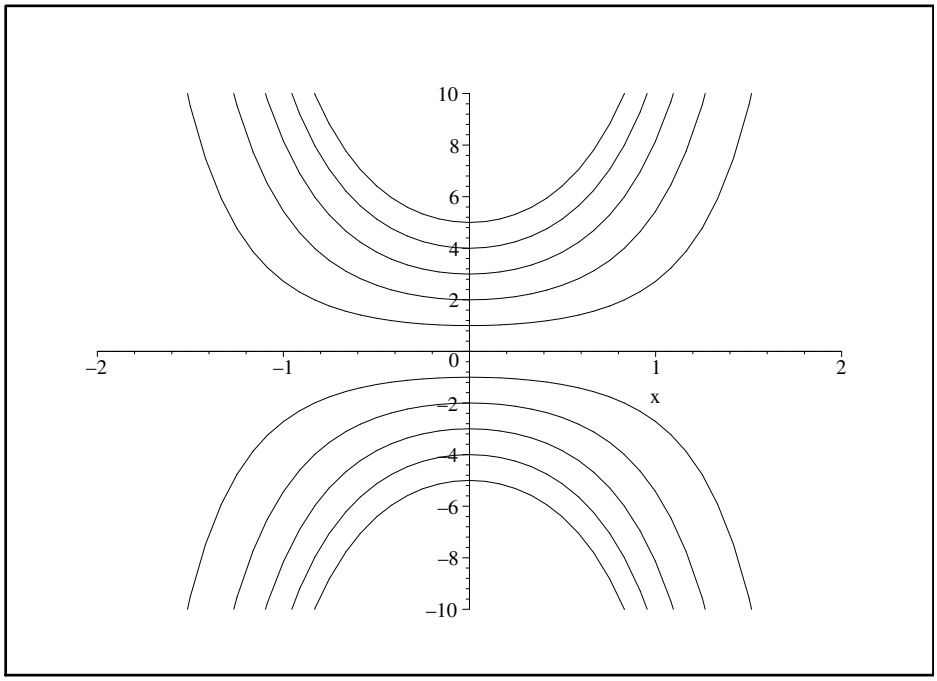


Figure 1.1: Plots of solutions from the 1-parameter family of solutions of Example 1.1 for several initial conditions.

The integrating factor that works is $\mu(x) = \exp(\int^x p(\xi)\,d\xi)$. One can show this by expanding the derivative in Equation (1.8),

$$\mu(x)y'(x) + \mu'(x)y(x) = \mu(x)q(x), \tag{1.9}$$

and comparing this equation to the one obtained from multiplying (1.7) by $\mu(x)$:

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \mu(x)q(x). \tag{1.10}$$

Note that these last two equations would be the same if

$$\frac{d\mu(x)}{dx} = \mu(x)p(x).$$

This is a separable first order equation whose solution is the above given form for the integrating factor,

$$\mu(x) = \exp\left(\int^x p(\xi)\,d\xi\right). \tag{1.11}$$

Equation (1.8) is easily integrated to obtain

$$y(x) = \frac{1}{\mu(x)}\left[\int^x \mu(\xi)q(\xi)\,d\xi + C\right]. \tag{1.12}$$

Example 1.3. $xy' + y = x, \quad x > 0, y(1) = 0.$

One first notes that this is a linear first order differential equation. Solving for y', one can see that the original equation is not separable. However, it is not in the standard form. So, we first rewrite the

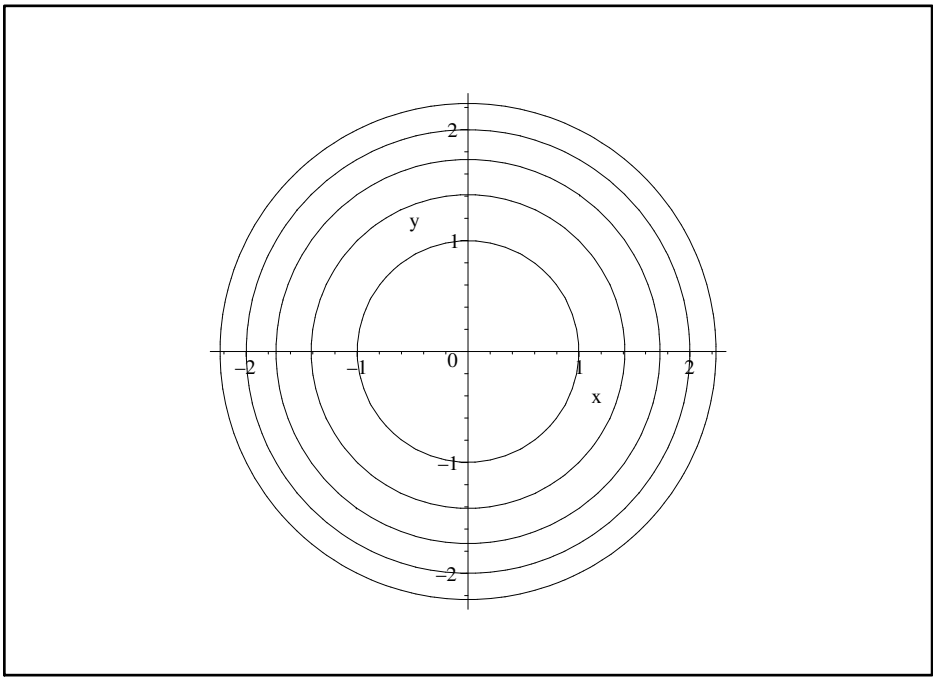


Figure 1.2: Plots of solutions of Example 1.2 for several initial conditions.

equation as

$$\frac{dy}{dx} + \frac{1}{x}y = 1. \tag{1.13}$$

Noting that $p(x) = \frac{1}{x}$, we determine the integrating factor

$$\mu(x) = \exp\left[\int^x \frac{d\xi}{\xi}\right] = e^{\ln x} = x.$$

Multiplying equation (1.13) by $\mu(x) = x$, we actually get back the original equation! In this case we have found that $xy' + y$ must have been the derivative of something to start. In fact, $(xy)' = xy' + x$. Therefore, equation (1.8) becomes

$$(xy)' = x.$$

Integrating one obtains

$$xy = \frac{1}{2}x^2 + C,$$

or

$$y(x) = \frac{1}{2}x + \frac{C}{x}.$$

Inserting the initial condition into this solution, we have $0 = \frac{1}{2} + C$. Therefore, $C = -\frac{1}{2}$. Thus, the solution of the initial value problem is $y(x) = \frac{1}{2}(x - \frac{1}{x})$.

Example 1.4. $(\sin x)y' + (\cos x)y = x^2 \sin x$.

Actually, this problem is easy if you realize that

$$\frac{d}{dx}((\sin x)y) = (\sin x)y' + (\cos x)y.$$

But, we will go through the process of finding the integrating factor for practice.

First, rewrite the original differential equation in standard form:

$$y' + (\cot x)y = x^2.$$

Then, compute the integrating factor as

$$\mu(x) = \exp\left(\int^x \cot\xi \, d\xi\right) = e^{-\ln(\sin x)} = \frac{1}{\sin x}.$$

Using the integrating factor, the original equation becomes

$$\frac{d}{dx}\left((\sin x)y\right) = x^2.$$

Integrating, we have

$$y \sin x = \frac{1}{3}x^3 + C.$$

So, the solution is

$$y = \left(\frac{1}{3}x^3 + C\right)\csc x.$$

There are other first order equations that one can solve for closed form solutions. However, many equations are not solvable, or one is simply interested in the behavior of solutions. In such cases one turns to direction fields. We will return to a discussion of the qualitative behavior of differential equations later in the course.

1.1.2 *Second Order Linear Differential Equations*

SECOND ORDER DIFFERENTIAL EQUATIONS ARE TYPICALLY harder than first order. In most cases students are only exposed to second order linear differential equations. A general form for a *second order linear differential equation* is given by

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{1.14}$$

One can rewrite this equation using operator terminology.[1] Namely, one first defines the differential operator $L = a(x)D^2 + b(x)D + c(x)$, where $D = \frac{d}{dx}$. Then equation (1.14) becomes

$$Ly = f. \tag{1.15}$$

The solutions of linear differential equations are found by making use of the linearity of L. Namely, we consider the *vector space* [2] consisting of real-valued functions over some domain. Let f and g be vectors in this function space. L is a *linear operator* if for two vectors f and g and scalar a, we have that

[1] We note that Leibniz introduced $\frac{dy}{dx}$ and Newton used the dot notation, $\dot{y}$. About a century later Lagrange introduced y' and Arbogast introduced the operator notation D.

[2] We assume that the reader has been introduced to concepts in linear algebra. Later in the text we will recall the definition of a vector space and see that linear algebra is in the background of the study of many concepts in the solution of differential equations.

a. $L(f+g) = Lf + Lg$

b. $L(af) = aLf$.

One typically solves (1.14) by finding the general solution of the homogeneous problem,

$$Ly_h = 0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p = f.$$

Then the general solution of (1.14) is simply given as $y = y_h + y_p$. This is true because of the linearity of L. Namely,

$$\begin{aligned} Ly &= L(y_h + y_p) \\ &= Ly_h + Ly_p \\ &= 0 + f = f. \end{aligned} \tag{1.16}$$

There are methods for finding a particular solution of a differential equation. These range from pure guessing to the Method of Undetermined Coefficients, or by making use of the Method of Variation of Parameters. We will review some of these methods later.

Determining solutions to the homogeneous problem, $Ly_h = 0$, is not always easy. However, others have studied a variety of second order linear equations and have saved us the trouble for some of the differential equations that often appear in applications.

Again, linearity is useful in producing the general solution of a homogeneous linear differential equation. If y_1 and y_2 are solutions of the homogeneous equation, then the *linear combination* $y = c_1y_1 + c_2y_2$ is also a solution of the homogeneous equation. In fact, if y_1 and y_2 are *linearly independent*,[3] then $y = c_1y_1 + c_2y_2$ is the general solution of the homogeneous problem. As you may recall, linear independence is established if the Wronskian of the solutions in not zero. In this case, we have

$$W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0. \tag{1.17}$$

[3] Recall, a set of functions $\{y_i(x)\}_{i=1}^n$ is a linearly independent set if and only if

$$c_1y(1(x) + \ldots + c_ny_n(x) = 0$$

implies $c_i = 0$, for $i = 1, \ldots, n$.

1.1.3 Constant Coefficient Equations

The simplest and most seen second order differential equations are those with constant coefficients. The general form for a homogeneous constant coefficient second order linear differential equation is given as

$$ay''(x) + by'(x) + cy(x) = 0, \tag{1.18}$$

where a, b, and c are constants.

Solutions to (1.18) are obtained by making a guess of $y(x) = e^{rx}$. Inserting this guess into (1.18) leads to the *characteristic equation*

Characteristic equation.

$$ar^2 + br + c = 0. \tag{1.19}$$

The roots of this equation in turn lead to three types of solution depending upon the nature of the roots as shown below.

Example 1.5. $y'' - y' - 6y = 0\ y(0) = 2, y'(0) = 0.$

The characteristic equation for this problem is $r^2 - r - 6 = 0$. The roots of this equation are found as $r = -2, 3$. Therefore, the general solution can be quickly written down:

$$y(x) = c_1 e^{-2x} + c_2 e^{3x}.$$

Note that there are two arbitrary constants in the general solution. Therefore, one needs two pieces of information to find a particular solution. Of course, we have the needed information in the form of the initial conditions.

One also needs to evaluate the first derivative

$$y'(x) = -2c_1 e^{-2x} + 3c_2 e^{3x}$$

in order to attempt to satisfy the initial conditions. Evaluating y and y' at $x = 0$ yields

$$\begin{aligned} 2 &= c_1 + c_2 \\ 0 &= -2c_1 + 3c_2 \end{aligned} \tag{1.20}$$

These two equations in two unknowns can readily be solved to give $c_1 = 6/5$ and $c_2 = 4/5$. Therefore, the solution of the initial value problem is obtained as $y(x) = \frac{6}{5}e^{-2x} + \frac{4}{5}e^{3x}$.

Classification of Roots of the Characteristic Equation for Second Order Constant Coefficient ODEs

1. **Real, distinct roots** r_1, r_2. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.

2. **Real, equal roots** $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the *Method of Reduction of Order*. This gives the second solution as xe^{rx}. Therefore, the general solution is found as $y(x) = (c_1 + c_2 x)e^{rx}$. [This is covered in the appendix to this chapter.]

3. **Complex conjugate roots** $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. Making use of Euler's identity, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, these complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$. [This is covered in the appendix to this chapter.]

Example 1.6. $y'' + 6y' + 9y = 0$.

In this example we have $r^2 + 6r + 9 = 0$. There is only one root, $r = -3$. Again, the solution is easily obtained as $y(x) = (c_1 + c_2 x)e^{-3x}$.

Example 1.7. $y'' + 4y = 0$.

The characteristic equation in this case is $r^2 + 4 = 0$. The roots are pure imaginary roots, $r = \pm 2i$ and the general solution consists purely of sinusoidal functions: $y(x) = c_1 \cos(2x) + c_2 \sin(2x)$.

Example 1.8. $y'' + 2y' + 4y = 0$.

The characteristic equation in this case is $r^2 + 2r + 4 = 0$. The roots are complex, $r = -1 \pm \sqrt{3}i$ and the general solution can be written as

$$y(x) = \left[c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)\right] e^{-x}.$$

One of the most important applications of the equations in the last two examples is in the study of oscillations. Typical systems are a mass on a spring, or a simple pendulum. For a mass m on a spring with spring constant $k > 0$, one has from Hooke's law that the position as a function of time, $x(t)$, satisfies the equation

$$mx'' + kx = 0.$$

This constant coefficient equation has pure imaginary roots ($\alpha = 0$) and the solutions are pure sines and cosines. Such motion is called *simple harmonic motion*.

Adding a damping term and periodic forcing complicates the dynamics, but is nonetheless solvable. The next example shows a forced harmonic oscillator.

Example 1.9. $y'' + 4y = \sin x$.

This is an example of a nonhomogeneous problem. The homogeneous problem was actually solved in Example 1.7. According to the theory, we need only seek a particular solution to the nonhomogeneous problem and add it to the solution of the last example to get the general solution.

The particular solution can be obtained by purely guessing, making an educated guess, or using the Method of Variation of Parameters. We will not review all of these techniques at this time. Due to the simple form of the driving term, we will make an intelligent guess of $y_p(x) = A \sin x$ and determine what A needs to be. Recall, this is the *Method of Undetermined Coefficients* which we review in the next section. Inserting our guess in the equation gives $(-A + 4A) \sin x = \sin x$. So, we see that $A = 1/3$ works. The general solution of the nonhomogeneous problem is therefore $y(x) = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{3} \sin x$.

1.1.4 *Method of Undetermined Coefficients*

TO DATE, WE ONLY KNOW HOW TO SOLVE constant coefficient, homogeneous equations. How does one solve a nonhomogeneous equation like

that in Equation (1.14),

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{1.21}$$

Recall, that one solves this equation by finding the general solution of the homogeneous problem,

$$Ly_h = 0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p = f.$$

General solution of the nonhomogeneous problem, $y = y_h + y_p$.

Then the general solution of (1.14) is simply given as $y = y_h + y_p$. So, how do we find the particular solution?

You could guess a solution, but that is not usually possible without a little bit of experience. So we need some other methods. There are two main methods. In the first case, the Method of Undetermined Coefficients, one makes an intelligent guess based on the form of $f(x)$. In the second method, one can systematically develop the particular solution. We will come back to this method the Method of Variation of Parameters, later in the book.

Let's solve a simple differential equation highlighting how we can handle nonhomogeneous equations.

Example 1.10. Consider the equation

$$y'' + 2y' - 3y = 4. \tag{1.22}$$

The first step is to determine the solution of the homogeneous equation. Thus, we solve

$$y_h'' + 2y_h' - 3y_h = 0. \tag{1.23}$$

The characteristic equation is $r^2 + 2r - 3 = 0$. The roots are $r = 1, -3$. So, we can immediately write the solution

$$y_h(x) = c_1 e^x + c_2 e^{-3x}.$$

The second step is to find a particular solution of (1.22). What possible function can we insert into this equation such that only a 4 remains? If we try something proportional to x, then we are left with a linear function after inserting x and its derivatives. Perhaps a constant function you might think. $y = 4$ does not work. But, we could try an arbitrary constant, $y = A$.

Let's see. Inserting $y = A$ into (1.22), we obtain

$$-3A = 4.$$

Ah ha! We see that we can choose $A = -\frac{4}{3}$ and this works. So, we have a particular solution, $y_p(x) = -\frac{4}{3}$. This step is done.

Combining our two solutions, we have the general solution to the original nonhomogeneous equation (1.22). Namely,

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}.$$

Insert this solution into the equation and verify that it is indeed a solution. If we had been given initial conditions, we could now use them to determine our arbitrary constants.

What if we had a different source term? Consider the equation

$$y'' + 2y' - 3y = 4x. \tag{1.24}$$

The only thing that would change is our particular solution. So, we need a guess.

We know a constant function does not work by the last example. So, let's try $y_p = Ax$. Inserting this function into Equation (**??**), we obtain

$$2A - 3Ax = 4x.$$

Picking $A = -4/3$ would get rid of the x terms, but will not cancel everything. We still have a constant left. So, we need something more general.

Let's try a linear function, $y_p(x) = Ax + B$. Then we get after substitution into (1.24)

$$2A - 3(Ax + B) = 4x.$$

Equating the coefficients of the different powers of x on both sides, we find a system of equations for the undetermined coefficients:

$$\begin{aligned} 2A - 3B &= 0 \\ -3A &= 4. \end{aligned} \tag{1.25}$$

These are easily solved to obtain

$$\begin{aligned} A &= -\frac{4}{3} \\ B &= \frac{2}{3}A = -\frac{8}{9}. \end{aligned} \tag{1.26}$$

So, our particular solution is

$$y_p(x) = -\frac{4}{3}x - \frac{8}{9}.$$

This gives the general solution to the nonhomogeneous problem as

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}x - \frac{8}{9}.$$

There are general forms that you can guess based upon the form of the driving term, $f(x)$. Some examples are given in Table 1.1. More general applications are covered in a standard text on differential equations. However,

the procedure is simple. Given $f(x)$ in a particular form, you make an appropriate guess up to some unknown parameters, or coefficients. Inserting the guess leads to a system of equations for the unknown coefficients. Solve the system and you have your solution. This solution is then added to the general solution of the homogeneous differential equation.

Table 1.1: Educated guesses given non-homogeneous $f(x)$.

$f(x)$	Guess
$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$	$A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$
ae^{bx}	Ae^{bx}
$a\cos\omega x + b\sin\omega x$	$A\cos\omega x + B\sin\omega x$

Example 1.11. As a final example, let's consider the equation

$$y'' + 2y' - 3y = 2e^{-3x}. \tag{1.27}$$

According to the above, we would guess a solution of the form $y_p = Ae^{-3x}$. Inserting our guess, we find

$$0 = 2e^{-3x}.$$

Oops! The coefficient, A, disappeared! We cannot solve for it. What went wrong?

The answer lies in the general solution of the homogeneous problem. Note that e^x and e^{-3x} are solutions to the homogeneous problem. So, a multiple of e^{-3x} will not get us anywhere. It turns out that there is one further modification of the method. If our driving term contains terms that are solutions of the homogeneous problem, then we need to make a guess consisting of the smallest possible power of x times the function which is no longer a solution of the homogeneous problem. Namely, we guess $y_p(x) = Axe^{-3x}$. We compute the derivative of our guess, $y_p' = A(1-3x)e^{-3x}$ and $y_p'' = A(9x-6)e^{-3x}$. Inserting these into the equation, we obtain

$$[(9x-6) + 2(1-3x) - 3x]Ae^{-3x} = 2e^{-3x},$$

or

$$-4A = 2.$$

So, $A = -1/2$ and $y_p(x) = -\frac{1}{2}xe^{-3x}$.

Modified Method of Undetermined Coefficients

In general, if any term in the guess $y_p(x)$ is a solution of the homogeneous equation, then multiply the guess by x^k, where k is the smallest positive integer such that no term in $x^k y_p(x)$ is a solution of the homogeneous problem.

1.1.5 Cauchy-Euler Equations

ANOTHER CLASS OF SOLVABLE LINEAR DIFFERENTIAL EQUATIONS that is of interest are the Cauchy-Euler type of equations. These are given by

$$ax^2y''(x) + bxy'(x) + cy(x) = 0. \tag{1.28}$$

Note that in such equations the power of x in each of the coefficients matches the order of the derivative in that term. These equations are solved in a manner similar to the constant coefficient equations.

One begins by making the guess $y(x) = x^r$. Inserting this function and its derivatives,

$$y'(x) = rx^{r-1}, \qquad y''(x) = r(r-1)x^{r-2},$$

into Equation (1.28), we have

$$\left[ar(r-1) + br + c\right]x^r = 0.$$

Since this has to be true for all x in the problem domain, we obtain the characteristic equation

Characteristic equation for the Cauchy-Euler Equation.

$$ar(r-1) + br + c = 0. \tag{1.29}$$

Just like the constant coefficient differential equation, we have a quadratic equation and the nature of the roots again leads to three classes of solutions. These are shown below. Some of the details are provided in the next section.

Classification of Roots of the Characteristic Equation for Cauchy-Euler Differential Equations

1. Real, distinct roots r_1, r_2. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1x^{r_1} + c_2x^{r_2}$.

2. Real, equal roots $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the Method of Reduction of Order. This gives the second solution as $x^r \ln|x|$. Therefore, the general solution is found as $y(x) = (c_1 + c_2 \ln|x|)x^r$.

3. Complex conjugate roots $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. These complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $x^\alpha \cos(\beta \ln|x|)$ and $x^\alpha \sin(\beta \ln|x|)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = x^\alpha(c_1 \cos(\beta \ln|x|) + c_2 \sin(\beta \ln|x|))$.

Example 1.12. $x^2y'' + 5xy' + 12y = 0$

As with the constant coefficient equations, we begin by writing down the characteristic equation. Doing a simple computation,

$$\begin{aligned} 0 &= r(r-1) + 5r + 12 \\ &= r^2 + 4r + 12 \\ &= (r+2)^2 + 8, \\ -8 &= (r+2)^2, \end{aligned} \tag{1.30}$$

one determines the roots are $r = -2 \pm 2\sqrt{2}i$. Therefore, the general solution is $y(x) = \left[c_1 \cos(2\sqrt{2}\ln|x|) + c_2 \sin(2\sqrt{2}\ln|x|)\right] x^{-2}$

Example 1.13. $t^2y'' + 3ty' + y = 0, \quad y(1) = 0, y'(1) = 1.$

For this example the characteristic equation takes the form

$$r(r-1) + 3r + 1 = 0,$$

or

$$r^2 + 2r + 1 = 0.$$

There is only one real root, $r = -1$. Therefore, the general solution is

$$y(t) = (c_1 + c_2 \ln|t|)t^{-1}.$$

However, this problem is an initial value problem. At $t = 1$ we know the values of y and y'. Using the general solution, we first have that

$$0 = y(1) = c_1.$$

Thus, we have so far that $y(t) = c_2 \ln|t|t^{-1}$. Now, using the second condition and

$$y'(t) = c_2(1 - \ln|t|)t^{-2},$$

we have

$$1 = y(1) = c_2.$$

Therefore, the solution of the initial value problem is $y(t) = \ln|t|t^{-1}$.

Nonhomogeneous Cauchy-Euler Equations

We can also solve some nonhomogeneous Cauchy-Euler equations using the Method of Undetermined Coefficients. We will demonstrate this with a couple of examples.

Example 1.14. Find the solution of $x^2y'' - xy' - 3y = 2x^2$.

First we find the solution of the homogeneous equation. The characteristic equation is $r^2 - 2r - 3 = 0$. So, the roots are $r = -1, 3$ and the solution is $y_h(x) = c_1x^{-1} + c_2x^3$.

We next need a particular solution. Let's guess $y_p(x) = Ax^2$. Inserting the guess into the nonhomogeneous differential equation, we have

$$\begin{aligned} 2x^2 &= x^2y'' - xy' - 3y = 2x^2 \\ &= 2Ax^2 - 2Ax^2 - 3Ax^2 \\ &= -3Ax^2. \end{aligned} \tag{1.31}$$

So, $A = -2/3$. Therefore, the general solution of the problem is

$$y(x) = c_1 x^{-1} + c_2 x^3 - \frac{2}{3}x^2.$$

Example 1.15. Find the solution of $x^2y'' - xy' - 3y = 2x^3$.

In this case the nonhomogeneous term is a solution of the homogeneous problem, which we solved in the last example. So, we will need a modification of the method. We have a problem of the form

$$ax^2y'' + bxy' + cy = dx^r,$$

where r is a solution of $ar(r-1) + br + c = 0$. Let's guess a solution of the form $y = Ax^r \ln x$. Then one finds that the differential equation reduces to $Ax^r(2ar - a + b) = dx^r$. [You should verify this for yourself.]

With this in mind, we can now solve the problem at hand. Let $y_p = Ax^3 \ln x$. Inserting into the equation, we obtain $4Ax^3 = 2x^3$, or $A = 1/2$. The general solution of the problem can now be written as

$$y(x) = c_1 x^{-1} + c_2 x^3 + \frac{1}{2}x^3 \ln x.$$

1.2 *Overview of the Course*

FOR THE MOST PART, YOUR FIRST COURSE IN DIFFERENTIAL EQUATIONS was about solving initial value problems. When second order equations did not fall into the above cases, then you might have learned how to obtain approximate solutions using power series methods, or even finding new functions from these methods. In this course we will explore two broad topics: systems of differential equations and boundary value problems.

We will see that there are interesting initial value problems when studying systems of differential equations. In fact, many of the second order equations that you have seen in the past can be written as a system of two first order equations. For example, the equation for simple harmonic motion,

$$x'' + \omega^2 x = 0,$$

can be written as the system

$$\begin{aligned} x' &= y \\ y' &= -\omega^2 x \end{aligned}.$$

Just note that $x'' = y' = -\omega^2 x$. Of course, one can generalize this to systems with more complicated right hand sides. The behavior of such systems can be fairly interesting and these systems result from a variety of physical models.

In the second part of the course we will explore boundary value problems. Often these problems evolve from the study of partial differential equations. Such examples stem from vibrating strings, temperature distributions, bending beams, etc. Boundary conditions are conditions that are

imposed at more than one point, while for initial value problems the conditions are specified at one point. For example, we could take the oscillation equation above and ask when solutions of the equation would satisfy the conditions $x(0) = 0$ and $x(1) = 0$. The general solution, as we have determined earlier, is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t.$$

Requiring $x(0) = 0$, we find that $c_1 = 0$, leaving $x(t) = c_2 \sin \omega t$. Also imposing that $0 = x(1) = c_2 \sin \omega$, we are forced to make $\omega = n\pi$, for $n = 1, 2, \ldots$. (Making $c_2 = 0$ would not give a nonzero solution of the problem.) Thus, there are an infinite number of solutions possible, if we have the freedom to choose our ω. In the second half of the course we will investigate techniques for solving boundary value problems and look at several applications, including seeing the connections with partial differential equations and Fourier series.

1.3 *Appendix: Reduction of Order and Complex Roots*

IN THIS SECTION WE PROVIDE SOME OF THE DETAILS leading to the general forms for the constant coefficient and Cauchy-Euler differential equations. In the first subsection we review how the Method of Reduction of Order is used to obtain the second linearly independent solutions for the case of one repeated root. In the second subsection we review how the complex solutions can be used to produce two linearly independent real solutions.

Method of Reduction of Order

FIRST WE CONSIDER CONSTANT COEFFICIENT EQUATIONS. In the case when there is a repeated real root, one has only one independent solution, $y_1(x) = e^{rx}$. The question is how does one obtain the second solution? Since the solutions are independent, we must have that the ratio $y_2(x)/y_1(x)$ is not a constant. So, we guess the form $y_2(x) = v(x)y_1(x) = v(x)e^{rx}$. For constant coefficient second order equations, we can write the equation as

$$(D - r)^2 y = 0,$$

where $D = \frac{d}{dx}$.

We now insert $y_2(x)$ into this equation. First we compute

$$(D - r)ve^{rx} = v'e^{rx}.$$

Then,

$$(D - r)^2 ve^{rx} = (D - r)v'e^{rx} = v''e^{rx}.$$

So, if $y_2(x)$ is to be a solution to the differential equation, $(D - r)^2 y_2 = 0$, then $v''(x)e^{rx} = 0$ for all x. So, $v''(x) = 0$, which implies that

$$v(x) = ax + b.$$

So,

$$y_2(x) = (ax + b)e^{rx}.$$

Without loss of generality, we can take $b = 0$ and $a = 1$ to obtain the second linearly independent solution, $y_2(x) = xe^{rx}$.

Deriving the solution for Case 2 for the Cauchy-Euler equations is messier, but works in the same way. First note that for the real root, $r = r_1$, the characteristic equation has to factor as $(r - r_1)^2 = 0$. Expanding, we have

$$r^2 - 2r_1 r + r_1^2 = 0.$$

The general characteristic equation is

$$ar(r - 1) + br + c = 0.$$

Rewriting this, we have

$$r^2 + (\frac{b}{a} - 1)r + \frac{c}{a} = 0.$$

Comparing equations, we find

$$\frac{b}{a} = 1 - 2r_1, \qquad \frac{c}{a} = r_1^2.$$

So, the general Cauchy-Euler equation in this case takes the form

$$x^2 y'' + (1 - 2r_1)xy' + r_1^2 y = 0.$$

Now we seek the second linearly independent solution in the form $y_2(x) = v(x)x^{r_1}$. We first list this function and its derivatives,

$$\begin{aligned} y_2(x) &= vx^{r_1}, \\ y_2'(x) &= (xv' + r_1 v)x^{r_1 - 1}, \\ y_2''(x) &= (x^2 v'' + 2r_1 xv' + r_1(r_1 - 1)v)x^{r_1 - 2}. \end{aligned} \tag{1.32}$$

Inserting these forms into the differential equation, we have

$$\begin{aligned} 0 &= x^2 y'' + (1 - 2r_1)xy' + r_1^2 y \\ &= (xv'' + v')x^{r_1 + 1}. \end{aligned} \tag{1.33}$$

Thus, we need to solve the equation

$$xv'' + v' = 0,$$

or

$$\frac{v''}{v'} = -\frac{1}{x}.$$

Integrating, we have

$$\ln|v'| = -\ln|x| + C.$$

Exponentiating, we have one last differential equation to solve,

$$v' = \frac{A}{x}.$$

Thus,

$$v(x) = A \ln|x| + k.$$

So, we have found that the second linearly independent equation can be written as

$$y_2(x) = x^{r_1} \ln|x|.$$

Complex Roots

WHEN ONE HAS COMPLEX ROOTS IN THE SOLUTION of constant coefficient equations, one needs to look at the solutions

$$y_{1,2}(x) = e^{(\alpha \pm i\beta)x}.$$

We make use of Euler's formula

$$e^{i\beta x} = \cos\beta x + i \sin\beta x. \tag{1.34}$$

Then the linear combination of $y_1(x)$ and $y_2(x)$ becomes

$$\begin{aligned} Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} &= e^{\alpha x}\left[Ae^{i\beta x} + Be^{-i\beta x}\right] \\ &= e^{\alpha x}\left[(A+B)\cos\beta x + i(A-B)\sin\beta x\right] \\ &\equiv e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x). \end{aligned} \tag{1.35}$$

Thus, we see that we have a linear combination of two real, linearly independent solutions, $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$.

When dealing with the Cauchy-Euler equations, we have solutions of the form $y(x) = x^{\alpha+i\beta}$. The key to obtaining real solutions is to first recall that

$$x^y = e^{\ln x^y} = e^{y \ln x}.$$

Thus, a power can be written as an exponential and the solution can be written as

$$y(x) = x^{\alpha+i\beta} = x^\alpha e^{i\beta \ln x}, \quad x > 0.$$

We can now find two real, linearly independent solutions, $x^\alpha \cos(\beta \ln|x|)$ and $x^\alpha \sin(\beta \ln|x|)$ following the same steps as above for the constant coefficient case.

1.4 Applications

DIFFERENTIAL EQUATIONS AROSE DUE TO THE NEED TO SOLVE PROBLEMS. In a first course you might have seen some typical applications of first and second order differential equations. For example first order equuations arise in the study of population dynamics and logistic growth, Newton's law of cooling, free fall with drag, or mixture problems. Some historically interesting problems such as finding orthogonal families of curves, pursuit curves, and tractional motion were taken up by some of the top mathematicians of the eighteenth century.

Second order differential equations appear in applications of simple harmonic motion (SHM or SHO in some texts)such as a mass on a spring, pendula (the plural of pendulum), and simple circuits. Some of these problems can be further generalized into problems involving systems of differential equations. So, we will describe some of these applications as well as some interesting first order problems which can also be found in *A First Course in Differential Equations for Scientists and Engineers* by the author.

1.4.1 Mass-Spring Systems

We begin with the case of a single block on a spring as shown in Figure 1.3. The net force in this case is the restoring force of the spring given by Hooke's Law,

$$F_s = -kx,$$

where $k > 0$ is the spring constant. Here x is the elongation, or displacement of the spring from equilibrium. When the displacement is positive, the spring force is negative and when the displacement is negative the spring force is positive. We have depicted a horizontal system sitting on a frictionless surface. A similar model can be provided for vertically oriented springs. However, you need to account for gravity to determine the location of equilibrium. Otherwise, the oscillatory motion about equilibrium is modeled the same.

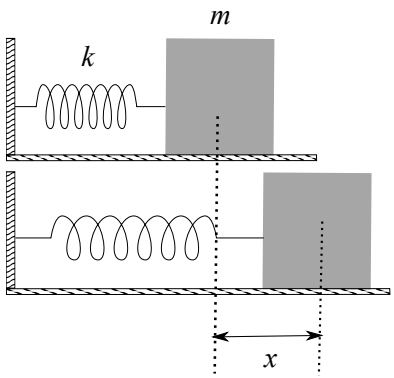


Figure 1.3: Spring-Mass system.

From Newton's Second Law, $F = m\ddot{x}$, we obtain the equation for the motion of the mass on the spring:

$$m\ddot{x} + kx = 0. \tag{1.36}$$

Dividing by the mass, this equation can be written in the form

$$\ddot{x} + \omega^2 x = 0, \tag{1.37}$$

where

$$\omega = \sqrt{\frac{k}{m}}.$$

This is the generic differential equation for simple harmonic motion. Two solutions of this equation are given by

$$\begin{aligned} x(t) &= A\cos\omega t, \\ x(t) &= A\sin\omega t, \end{aligned} \tag{1.38}$$

where ω is the angular frequency, measured in rad/s, and A is called the amplitude of the oscillation. .

The angular frequency is related to the frequency by

$$\omega = 2\pi f,$$

where f is measured in cycles per second, or Hertz. Furthermore, this is related to the period of oscillation, the time it takes the mass to go through one cycle:

$$T = 1/f.$$

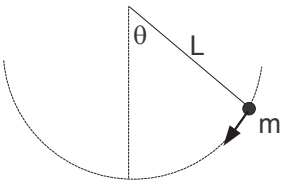


Figure 1.4: A simple pendulum consists of a point mass m attached to a string of length L. It is released from an angle θ_0.

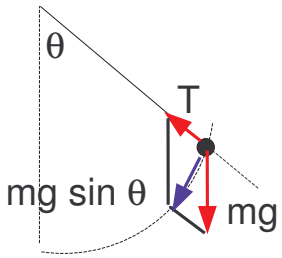


Figure 1.5: There are two forces acting on the mass, the weight mg and the tension T. The net force is found to be $F = mg\sin\theta$.

1.4.2 *The Simple Pendulum*

THE SIMPLE PENDULUM consists of a point mass m hanging on a string of length L from some support. [See Figure 1.4.] One pulls the mass back to some starting angle, θ_0, and releases it. The goal is to find the angular position as a function of time.

There are a couple of possible derivations. We could either use Newton's Second Law of Motion, $F = ma$, or its rotational analogue in terms of torque, $\tau = I\alpha$. We will use the former only to limit the amount of physics background needed.

There are two forces acting on the point mass. The first is gravity. This points downward and has a magnitude of mg, where g is the standard symbol for the acceleration due to gravity. The other force is the tension in the string. In Figure 1.5 these forces and their sum are shown. The magnitude of the sum is easily found as $F = mg\sin\theta$ using the addition of these two vectors.

Now, Newton's Second Law of Motion tells us that the net force is the mass times the acceleration. So, we can write

$$m\ddot{x} = -mg\sin\theta.$$

Next, we need to relate x and θ. x is the distance traveled, which is the length of the arc traced out by the point mass. The arclength is related to the angle, provided the angle is measure in radians. Namely, $x = r\theta$ for $r = L$. Thus, we can write

$$mL\ddot{\theta} = -mg\sin\theta.$$

Linear and nonlinear pendulum equation.

The equation for a compound pendulum takes a similar form. We start with the rotational form of Newton's second law $\tau = I\alpha$. Noting that the torque due to gravity acts at the center of mass position ℓ, the torque is given by $\tau = -mg\ell\sin\theta$. Since $\alpha = \ddot{\theta}$, we have $I\ddot{\theta} = -mg\ell\sin\theta$. Then, for small angles $\ddot{\theta} + \omega^2\theta = 0$, where $\omega = \frac{mg\ell}{I}$. For a simple pendulum, we let $\ell = L$ and $I = mL^2$, and obtain $\omega = \sqrt{g/L}$.

Canceling the masses, this then gives us the nonlinear pendulum equation

$$L\ddot{\theta} + g\sin\theta = 0. \tag{1.39}$$

We note that this equation is of the same form as the mass-spring system. We define $\omega = \sqrt{g/L}$ and obtain the equation for simple harmonic motion,

$$\ddot{\theta} + \omega^2\theta = 0.$$

There are several variations of Equation (1.39) which will be used in this text. The first one is the linear pendulum. This is obtained by making a small angle approximation. For small angles we know that $\sin\theta \approx \theta$. Under this approximation (1.39) becomes

$$L\ddot{\theta} + g\theta = 0. \tag{1.40}$$

1.4.3 *LRC Circuits*

ANOTHER TYPICAL PROBLEM OFTEN ENCOUNTERED in a first year physics class is that of an LRC series circuit. This circuit is pictured in Figure 1.6.

The resistor is a circuit element satisfying Ohm's Law. The capacitor is a device that stores electrical energy and an inductor, or coil, store magnetic energy.

The physics for this problem stems from Kirchoff's Rules for circuits. Namely, the sum of the drops in electric potential are set equal to the rises in electric potential. The potential drops across each circuit element are given by

1. Resistor: $V = IR$.
2. Capacitor: $V = \frac{q}{C}$.
3. Inductor: $V = L\frac{dI}{dt}$.

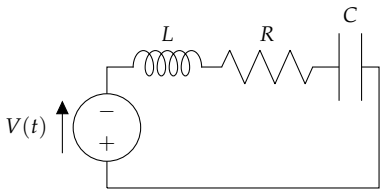


Figure 1.6: Series LRC Circuit.

Furthermore, we need to define the current as $I = \dfrac{dq}{dt}$. where q is the charge in the circuit. Adding these potential drops, we set them equal to the voltage supplied by the voltage source, $V(t)$. Thus, we obtain

$$IR + \frac{q}{C} + L\frac{dI}{dt} = V(t).$$

Since both q and I are unknown, we can replace the current by its expression in terms of the charge to obtain

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = V(t).$$

This is a second order equation for $q(t)$.

More complicated circuits are possible by looking at parallel connections, or other combinations, of resistors, capacitors and inductors. This will result in several equations for each loop in the circuit, leading to larger systems of differential equations. An example of another circuit setup is shown in Figure 1.7. This is not a problem that can be covered in the first year physics course. One can set up a system of second order equations and proceed to solve them. We will see how to solve such problems in the next chapter.

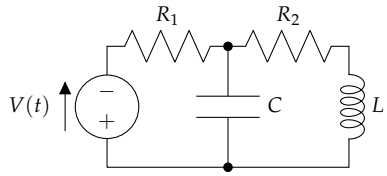


Figure 1.7: Parallel LRC Circuit.

1.4.4 *Orthogonal Trajectories of Curves**

There are many problems from geometry which have lead to the study of differential equations. One such problem is the construction of orthogonal trajectories. Give a a family of curves, $y_1(x;a)$, we seek another family of curves $y_2(x;c)$ such that the second family of curves are perpendicular the to given family. This means that the tangents of two intersecting curves at the point of intersection are perpendicular to each other. The slopes of the tangent lines are given by the derivatives $y_1'(x)$ and $y_2'(x)$. We recall from elementary geometry that the slopes of two perpendicular lines are related by

$$y_2'(x) = -\frac{1}{y_1'(x)}.$$

Example 1.16. Find a family of orthogonal trajectories to the family of parabolae $y_1(x;a) = ax^2$.

We note that the new collection of curves has to satisfy the equation

$$y_2'(x) = -\frac{1}{y_1'(x)} = -\frac{1}{2ax}.$$

Before solving for $y_2(x)$, we need to eliminate the parameter a. From the give function, we have that $a = \frac{y}{x^2}$. Inserting this into the equation for y_2', we have

$$y'(x) = -\frac{1}{2ax} = -\frac{x}{2y}.$$

Thus, to find $y_2(x)$, we have to solve the differential equation

$$2yy' + x = 0.$$

Noting that $(y^2)' = 2yy'$ and $(\frac{1}{2}x^2)' = x,$, this (exact) equation can be written as

$$\frac{d}{dx}\left(y^2 + \frac{1}{2}x^2\right) = 0.$$

Integrating, we find the family of solutions,

$$y^2 + \frac{1}{2}x^2 = k.$$

In Figure 1.8 we plot both families of orthogonal curves.

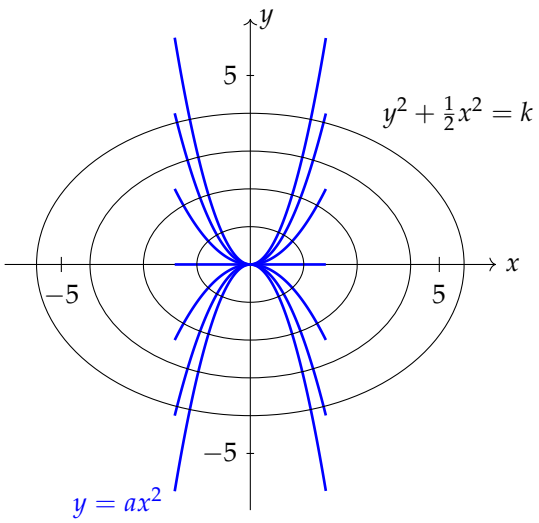


Figure 1.8: Plot of orthogonal families of curves, $y = ax^2$ and $y^2 + \frac{1}{2}x^2 = k$.

1.4.5 *Pursuit Curves**

ANOTHER APPLICATION THAT IS INTERESTING IS TO FIND the path that a body traces out as it moves towards a fixed point or another moving body. Such curses are know as pursuit curves. These could model aircraft or submarines following targets, or predators following prey. We demonstrate this with an example.

Example 1.17. A hawk at point (x, y) sees a sparrow traveling at speed v along a straight line. The hawk flies towards the sparrow at constant speed w but always in a direction along line of sight between their positions. If the hawk starts out at the point $(a, 0)$ at $t = 0$, when the sparrow is at $(0, 0)$, then what is the path the hawk needs to follow? Will the hawk catch the sparrow? The situation is shown in Figure 1.9. We pick the path of the sparrow to be along the $y-$axis. Therefore, the sparrow is at position $(0, vt)$.

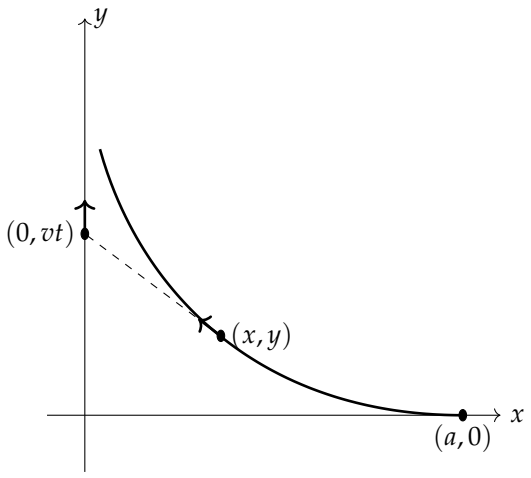


Figure 1.9: A hawk at point (x, y) sees a sparrow at point $(0, vt)$ and always follows the straight line between these points.

First we need the equation of the line of sight between the points (x, y) and $(0, vt)$. Considering that the slope of the line is the same as the slope of the tangent to the path, $y = y(x)$, we have

$$y' = \frac{y - vt}{x}.$$

The hawk is moving at a constant speed, w. Since the speed is related to the time through the distance the hawk travels. we need to find the arclength of the path between $(a, 0)$ and (x, y). This is given by

$$L = \int ds = \int_x^a \sqrt{1 + [y'(x)]^2}\, dx.$$

The distance is related to the speed, w, and the time, t, by $L = wt$.

Eliminating the time using $y' = \frac{y-vt}{x}$, we have

$$\int_x^a \sqrt{1 + [y'(x)]^2}\, dx = \frac{w}{v}(y - xy').$$

Furthermore, we can differentiate this result with respect to x to get rid of the integral,

$$\sqrt{1 + [y'(x)]^2} = \frac{w}{v} xy''.$$

Even though this is a second order differential equation for $y(x)$, it is a first order separable equation in the speed function $z(x) = y'(x)$. Namely,

$$\frac{w}{v} xz' = \sqrt{1 + z^2}.$$

Separating variables, we find

$$\frac{w}{v}\int \frac{dz}{\sqrt{1+z^2}} = \int \frac{dx}{x}.$$

The integrals can be computed using standard methods from calculus. We can easily integrate the right hand side,

$$\int \frac{dx}{x} = \ln|x| + c_1.$$

The left hand side takes a little extra work, or looking the value up in Tables or using a CAS package. Recall a trigonometric substitution is in order. [See the Appendix.] We let $z = \tan\theta$. Then $dz = \sec^2\theta\, d\theta$. The methods proceeds as follows:

$$\begin{aligned}
\int \frac{dz}{\sqrt{1+z^2}} &= \int \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\, d\theta \\
&= \int \sec\theta\, d\theta \\
&= \ln(\tan\theta + \sec\theta) + c_2 \\
&= \ln(z + \sqrt{1+z^2}) + c_2. \qquad (1.41)
\end{aligned}$$

Putting these together, we have for $x > 0$,

$$\ln(z + \sqrt{1+z^2}) = \frac{v}{w}\ln x + C.$$

Using the initial condition $z = y' = 0$ and $x = a$ at $t = 0$,

$$0 = \frac{v}{w}\ln a + C,$$

or $C = -\frac{v}{w}\ln a$.

Using this value for c, we find

$$\begin{aligned}
\ln(z + \sqrt{1+z^2}) &= \frac{v}{w}\ln x - \frac{v}{w}\ln a \\
\ln(z + \sqrt{1+z^2}) &= \frac{v}{w}\ln\frac{x}{a} \\
\ln(z + \sqrt{1+z^2}) &= \ln\left(\frac{x}{a}\right)^{\frac{v}{w}} \\
z + \sqrt{1+z^2} &= \left(\frac{x}{a}\right)^{\frac{v}{w}}. \qquad (1.42)
\end{aligned}$$

We can solve for $z = y'$, to find

$$y' = \frac{1}{2}\left[\left(\frac{x}{a}\right)^{\frac{v}{w}} - \left(\frac{x}{a}\right)^{-\frac{v}{w}}\right]$$

Integrating,

$$y(x) = \frac{a}{2}\left[\frac{\left(\frac{x}{a}\right)^{1+\frac{v}{w}}}{1+\frac{v}{w}} - \frac{\left(\frac{x}{a}\right)^{1-\frac{v}{w}}}{1-\frac{v}{w}}\right] + k.$$

The integration constant, k, can be found knowing $y(a) = 0$. This gives

$$\begin{aligned} 0 &= \frac{a}{2}\left[\frac{1}{1+\frac{v}{w}} - \frac{1}{1-\frac{v}{w}}\right] + k \\ k &= \frac{a}{2}\left[\frac{1}{1-\frac{v}{w}} - \frac{1}{1+\frac{v}{w}}\right] \\ &= \frac{avw}{w^2 - v^2}. \end{aligned} \tag{1.43}$$

The full solution for the path is given by

$$y(x) = \frac{a}{2}\left[\frac{\left(\frac{x}{a}\right)^{1+\frac{v}{w}}}{1+\frac{v}{w}} - \frac{\left(\frac{x}{a}\right)^{1-\frac{v}{w}}}{1-\frac{v}{w}}\right] + \frac{avw}{w^2 - v^2}.$$

Can the hawk catch the sparrow? This would happen if there is a time when $y(0) = vt$. Inserting $x = 0$ into the solution, we have $y(0) = \frac{avw}{w^2-v^2} = vt$. This is possible if $w > v$.

A related problem is one that was posed in 1676 by the French physician Claude Perrault (1613-1688), who was the brother of Charles Perrault, who published stories like *Cinderella* and *Little Red Riding Hood.* Claude Perrault was at a meeting in Paris and placed his watch in the middle of the table and pulled the end of the watch chain along the edge of the table. He demonstrated that when the end of the watch chain followed a straight line perpendicular to the starting point, the watch would be dragged along a curve. He asked. "What is the shape of the curve traced by the watch?" This curve was first studied by Christiaan Huygens (1629-1695) in 1692, who gave it the name tractrix.

Such a problem is an inverse tangent problem. One of the first such problems was originally posed in a letter from Florimond de Beaune (1601 - 1652) to Marin Mersenne (1588 - 1648) in 1638. Instead of seeking the tangent to a known curve, he asked for the curve which had a specific property. This led to being one of the first differential equations which in modern day notation is given by

$$\frac{dy}{dx} = \frac{x-y}{a}, \tag{1.44}$$

for some constant a. It is interesting to note that this was discussed in Leibniz's first publication in calculus in 1684, *Nova methodus pro maximis et minimis itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illis calculi genus.* After the work carried out by Bernoulli, Leibniz, Newton, and Euler on solution methods for differential equations, we now can easily solve this problem to obtain the curve $y = x + a\left(e^{-x/a} - 1\right)$.

The Perrault's tractrix problem is another inverse tangent problem. In Figure 1.10 is given the setup for the tractrix problem. Perrault's problem can be stated that one seeks the shape of a curve whose tangent (segment) has a constant length, a. From the diagram, we have the differential equation

$$\frac{dy}{dx} = -\frac{y}{\sqrt{a^2 - y^2}}. \tag{1.45}$$

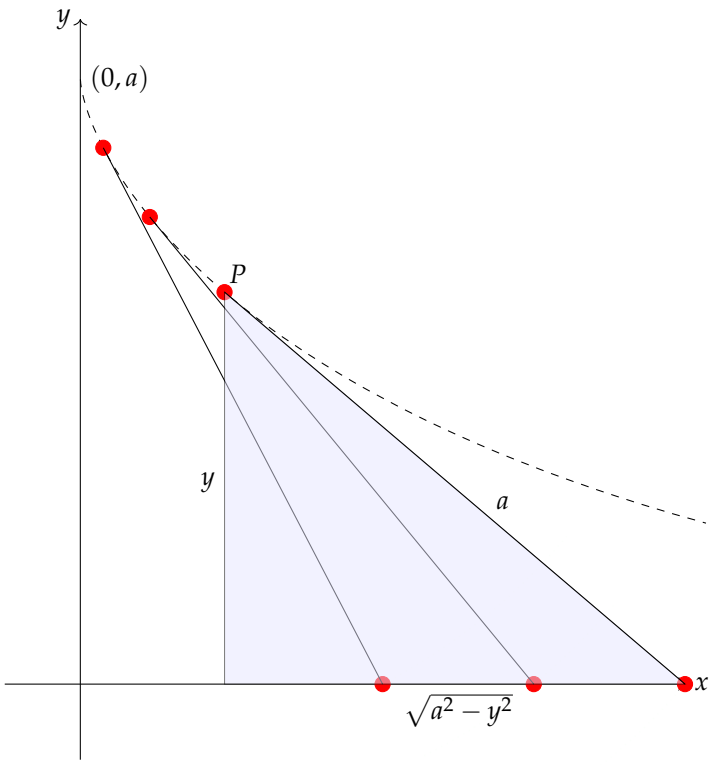


Figure 1.10: The tractrix problem: the end of the watch chain follows a straight line along the x-axis perpendicular to the starting point and the watch would be dragged along a curve coinded by Huygens as a tractrix.

This problem can be solved using several methods. Since it is separable, we have merely to evaluate an integral,

$$x + c = -\int \frac{\sqrt{a^2 - y^2}}{y}\, dy.$$

Due to the square root, nowadays one might think to use a trigonometric substitution. However, we could also use a hyperbolic trigonometric function substitution. Considering the argument of the square root is $a^2 - y^2$, we could let $y = a\,\text{sech}\, u$. Then,

$$a^2 - y^2 = a^2(1 - \text{sech}^2\, u) = a^2 \tanh^2 u$$

and $dy = -a\,\text{sech}\, u \tanh u\, du$ So, the integral becomes,

$$\begin{aligned} -\int \frac{\sqrt{a^2 - y^2}}{y}\, dy &= \int \frac{a \tanh u}{a\,\text{sech}\, u}\, a\,\text{sech}\, u \tanh u\, du \\ &= a \int \tanh^2 u\, du \\ &= a \int (1 - \text{sech}^2\, u)\, du \\ &= a(\tanh u - u) = x + C \end{aligned} \tag{1.46}$$

Therefore, we have a parametric solution

$$x(u) = a(u - \tanh u) + C, \quad y(u) = a\,\text{sech}\, u.$$

Requiring $(x(0), y(0)) = (0, a)$, then $C = 0$.

Finally, one can show that the evolute (the curve formed by the centers of curvature) of a tractrix is a catenary, which is the shape a hanging chain takes under its own weight. Also, the tractrix played a role in non-Euclidean geometry since rotating a tractrix around its asymptote generates a surface called a pseudosphere.

1.5 Other First Order Equations*

There are several nonlinear first order equations whose solution can be obtained using special techniques. We conclude this chapter by looking at a few of these equations named after famous mathematicians of the 17-18th century inspired by various applications.

1.5.1 Bernoulli Equation*

We begin with the Bernoulli equation, named after Jacob Bernoulli (1655-1705). The Bernoulli equation is of the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \quad n \neq 0, 1.$$

Note that when $n = 0, 1$ the equation is linear and can be solved using an integrating factor. The key to solving this equation is using the transformation $z(x) = \frac{1}{y^{n-1}(x)}$ to make the equation for $z(x)$ linear. We demonstrate the procedure using an example.

The Bernoulli's were a family of Swiss mathematicians spanning three generations. It all started with Jacob Bernoulli (1654-1705) and his brother Johann Bernoulli (1667-1748). Jacob had a son, Nicolaus Bernoulli (1687-1759) and Johann (1667-1748) had three sons, Nicolaus Bernoulli II (1695-1726), Daniel Bernoulli (1700-1872), and Johann Bernoulli II (1710-1790). The last generation consisted of Johann II's sons, Johann Bernoulli III (1747-1807) and Jacob Bernoulli II (1759-1789). Johann, Jacob and Daniel Bernoulli were the most famous of the Bernoulli's. Jacob studied with Leibniz, Johann studied under his older brother and later taught Leonhard Euler (1707-1783) and Daniel Bernoulli, who is known for his work in hydrodynamics. See Figure 1.11.

Example 1.18. Solve the Bernoulli equation $xy' + y = y^2 \ln x$ for $x > 0$.

In this example $p(x) = 1$, $q(x) = \ln x$, and $n = 2$. Therefore, we let $z = \frac{1}{y}$. Then,

$$z' = -\frac{1}{y^2}y' = z^2y'.$$

Inserting $z = y^{-1}$ and $z' = z^2y'$ into the differential equation, we have

$$\begin{aligned} xy' + y &= y^2 \ln x \\ -x\frac{z'}{z^2} + \frac{1}{z} &= \frac{\ln x}{z^2} \\ -xz' + z &= \ln x \\ z' - \frac{1}{x}z &= -\frac{\ln x}{x}. \end{aligned} \tag{1.47}$$

Thus, the resulting equation is a linear first order differential equation. It can be solved using the integrating factor,

$$\mu(x) = \exp\left(-\int \frac{dx}{x}\right) = \frac{1}{x}.$$

Multiplying the differential equation by the integrating factor, we have

$$\left(\frac{z}{x}\right)' = \frac{\ln x}{x^2}.$$

Integrating, we obtain

$$\begin{aligned} \frac{z}{x} &= -\int \frac{\ln x}{x^2} + C \\ &= \frac{\ln x}{x} + \int \frac{dx}{x^2} + C \\ &= \frac{\ln x}{x} + \frac{1}{x} + C. \end{aligned} \tag{1.48}$$

Multiplying by x, we have $z = \ln x + 1 + Cx$. Since $z = y^{-1}$, the general solution to the problem is

$$y = \frac{1}{\ln x + 1 + Cx}.$$

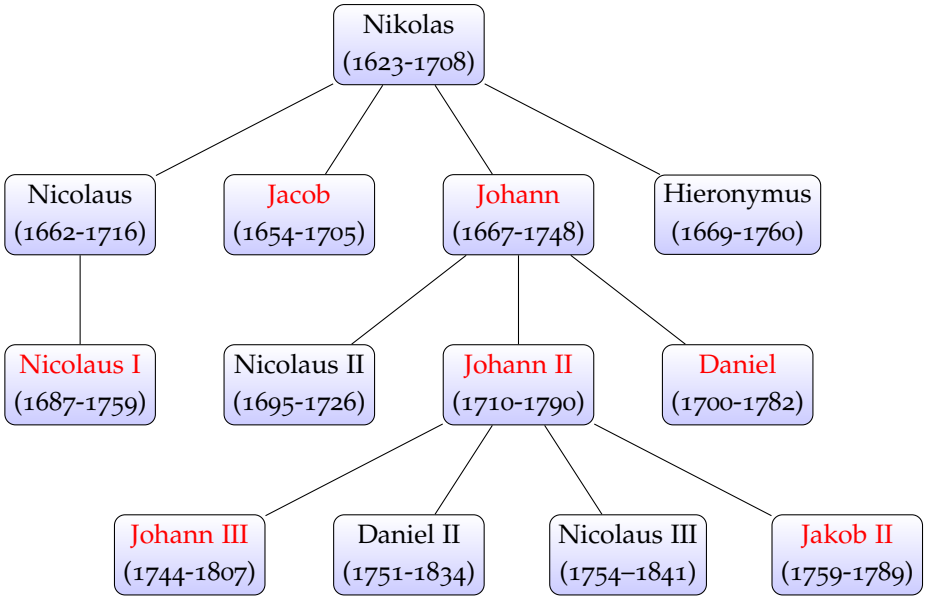


Figure 1.11: The Bernoulli's were a family of Swiss mathematicians spanning at least three generations starting with the brothers Jacob Bernoulli (1654-1705) and Johann Bernoulli (1667-1748).

1.5.2 *Lagrange and Clairaut Equations**

ALEXIS CLAUDE CLAIRAUT (1713-1765) SOLVED the differential equation

$$y = xy' + g(y').$$

This is a special case of the family of Lagrange equations,

$$y = xf(y') + g(y'),$$

named after Joseph Louis Lagrange (1736-1813). These equations also have solutions called singular solutions. Singular solution are solutions for which there is a failure of uniqueness to the initial value problem at every point on the curve. A singular solution is often one that is tangent to every solution in a family of solutions.

First, we consider solving the more general Lagrange equation. Let $p = y'$ in the Lagrange equation, giving

$$y = xf(p) + g(p). \tag{1.49}$$

Next, we differentiate with respect to x to find

$$y' = p = f(p) + xf'(p)p' + g'(p)p'.$$

Lagrange equations, $y = xf(y') + g(y')$.

Here we used the Chain Rule. For example,

$$\frac{dg(p)}{dx} = \frac{dg}{dp}\frac{dp}{dx}.$$

Solving for p', we have

$$\frac{dp}{dx} = \frac{p - f(p)}{xf'(p) + g'(p)}. \tag{1.50}$$

We have introduced $p = p(x)$, viewed as a function of x. Let's assume that we can invert this function to find $x = x(p)$. Then, from introductory calculus, we know that the derivatives of a function and its inverse are related,

$$\frac{dx}{dp} = \frac{1}{\frac{dp}{dx}}.$$

Applying this to Equation (1.50), we have

$$\begin{aligned} \frac{dx}{dp} &= \frac{xf'(p) + g'(p)}{p - f(p)} \\ x' - \frac{f'(p)}{p - f(p)}x &= \frac{g'(p)}{p - f(p)}, \end{aligned} \tag{1.51}$$

assuming that $p - f(p) \neq 0$.

As can be seen, we have transformed the Lagrange equation into a first order linear differential equation (1.51) for $x(p)$. Using methods from earlier in the chapter, we can in principle obtain a family of solutions

$$x = F(p, C),$$

where C is an arbitrary integration constant. Using Equation (1.49), one might be able to eliminate p in Equation (1.51) to obtain a family of solutions of the Lagrange equation in the form

$$\varphi(x, y, C) = 0.$$

If it is not possible to eliminate p from Equations (1.49) and (1.51), then one could report the family of solutions as a parametric family of solutions with p the parameter. So, the parametric solutions would take the form

$$\begin{aligned} x &= F(p, C), \\ y &= F(p, C)f(p) + g(p). \end{aligned} \tag{1.52}$$

We had also assumed the $p - f(p) \neq 0$. However, there might also be solutions of Lagrange's equation for which $p - f(p) = 0$. Such solutions are called singular solutions.

Singular solutions are possible for Lagrange equations.

Example 1.19. Solve the Lagrange equation $y = 2xy' - y'^2$.

We will start with Equation (1.51). Noting that $f(p) = 2p$, $g(p) = -p^2$, we have

$$\begin{aligned} x' - \frac{f'(p)}{p - f(p)}x &= \frac{g'(p)}{p - f(p)} \\ x' - \frac{2}{p - 2p}x &= \frac{-2p}{p - 2p} \\ x' + \frac{2}{p}x &= 2. \end{aligned} \tag{1.53}$$

This first order linear differential equation can be solved using an integrating factor. Namely,

$$\mu(p) = \exp\left(\int \frac{2}{p}\, dp\right) = e^{2\ln p} = p^2.$$

Multiplying the differential equation by the integrating factor, we have

$$\frac{d}{dp}\left(xp^2\right) = 2p^2.$$

Integrating,

$$xp^2 = \frac{2}{3}p^3 + C.$$

This gives the general solution

$$x(p) = \frac{2}{3}p + \frac{C}{p^2}.$$

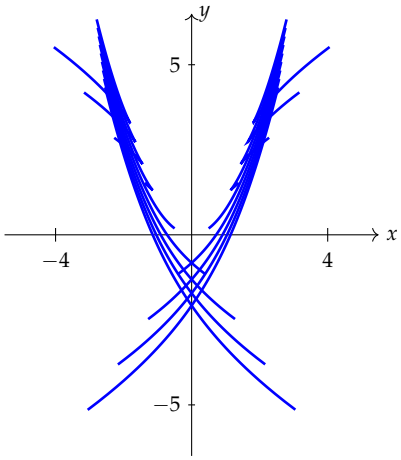


Figure 1.12: Family of solutions of the Lagrange equation $y = 2xy' - y'^2$.

Replacing $y' = p$ in the original differential equation, we have $y = 2xp - p^2$. The family of solutions is then given by the parametric equations

$$\begin{aligned} x &= \frac{2}{3}p + \frac{C}{p^2}, \\ y &= 2\left(\frac{2}{3}p + \frac{C}{p^2}\right)p - p^2 \\ &= \frac{1}{3}p^2 + \frac{2C}{p}. \end{aligned} \tag{1.54}$$

The plots of these solutions is shown in Figure 1.12.

We also need to check for a singular solution. We solve the equation $p - f(p) = 0$, or $p = 0$. This gives the solution $y(x) = (2xp - p^2)_{p=0} = 0$.

Clairaut equations, $y = xy' + g(y')$.

The Clairaut differential equation is given by

$$y = xy' + g(y').$$

Letting $p = y'$, we have

$$y = xp + g(p).$$

This is the Lagrange equation with $f(p) = p$. Differentiating with respect to x,

$$p = p + xp' + g'(p)p'.$$

Rearranging, we find

$$x = -g'(p)$$

So, we have the parametric solution

$$\begin{aligned} x &= -g'(p), \\ y &= -pg'(p) + g(p). \end{aligned} \tag{1.55}$$

For the case that $y' = C$, it can be seen that $y = Cx + g(C)$ is a general solution solution.

Example 1.20. Find the solutions of $y = xy' - y'^2$.

As noted, there is a family of straight line solutions $y = Cx - C^2$, since $g(p) = -p^2$. There might also by a parametric solution not contained n this family. It would be given by the set of equations

$$\begin{aligned} x &= -g'(p) = 2p, \\ y &= -pg'(p) + g(p) = 2p^2 - p^2 = p^2. \end{aligned} \tag{1.56}$$

Eliminating p, we have the parabolic curve $y = x^2/4$.

In Figure 1.13 we plot these solutions. The family of straight line solutions are shown in blue. The limiting curve traced out, much like string figures one might create, is the parametric curve.

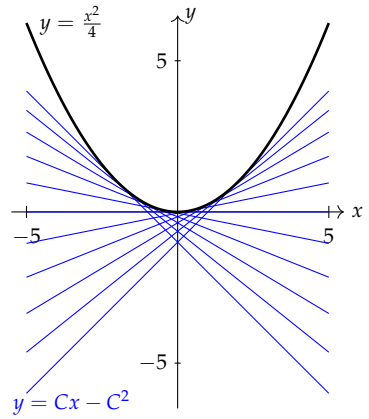


Figure 1.13: Plot of solutions to the Clairaut equation $y = xy' - y'^2$. The straight line solutions are a family of curves whose limit is the parametric slution.

1.5.3 *Riccati Equation**

Jacopo Francesco Riccati (1676-1754) studied curves with some specified curvature. He proposed an equation of the form

$$y' + a(x)y^2 + b(x)y + c(x) = 0$$

around 1720. He communicated this to the Bernoulli's. It was Daniel Bernoulli who had actually solved this equation. As noted by Ranjan Roy (2011), Riccati had published his equation in 1722 with a note that D. Bernoulli giving the solution in terms of an anagram. Furthermore, when $a \equiv 0$, the Riccati equation reduces to a Bernoulli equation.

In Section 3.2.1, we will show that the Ricatti equation can be transformed into a second order linear differential equation. However, there are special cases in which we can get our hands on the solutions. For example, if a, b, and c are constants, then the differential equation can be integrated directly. We have

$$\frac{dy}{dx} = -(ay^2 + by + c).$$

This equation is separable and we obtain

$$x - C = -\int \frac{dy}{ay^2 + by + c}.$$

When a differential equation is left in this form, it is said to be solved by quadrature when the resulting integral in principle can be computed in terms of elementary functions.[4]

[4] By elementary functions we mean well known functions like polynomials, trigonometric, hyperbolic, and some not so well know to undergraduates, such as Jacobi or Weierstrass elliptic functions.

If a particular solution is known, then one can obtain a solution to the Riccati equation. Let the known solution be $y_1(x)$ and assume that the general solution takes the form $y(x) = y_1(x) + z(x)$ for some unknown function $z(x)$. Substituting this form into the differential equation, we can show that $v(x) = 1/z(x)$ satisfies a first order linear differential equation.

Inserting $y = y_1 + z$ into the general Riccati equation, we have

$$\begin{aligned} 0 &= \frac{dy}{dx} + a(x)y^2 + b(x)y + c \\ &= \frac{dz}{dx} + az^2 + 2azy_1 + bz + \\ &\quad + \frac{dy_1}{dx} + ay_1^2 + by_1 + c \\ &= \frac{dz}{dx} + a(x)[2y_1z + z^2] + b(x)z \\ -a(x)z^2 &= \frac{dz}{dx} + [2a(x)y_1 + b(x)]z. \end{aligned} \tag{1.57}$$

The last equation is a Bernoulli equation with $n = 2$. So, we can make it a linear equation with the substitution $z = \frac{1}{v}$, $z' = -\frac{z'}{v^2}$. Then, we obtain a differential equation for $v(x)$. It is given by

$$v' - (2a(x)y_1(x) + b(x))v = a(x).$$

Example 1.21. Find the general solution of the Riccati equation, $y' - y^2 + 2e^x y - e^{2x} - e^x = 0$, using the particular solution $y_1(x) = e^x$.

We let the sought solution take the form $y(x) = z(x) + e^x$. Then, the equation for $z(x)$ is found as

$$\frac{dz}{dx} = z^2.$$

This equation is simple enough to integrate directly to obtain $z = \frac{1}{C-x}$. Then, the solution to the problem becomes

$$y(x) = \frac{1}{C - x} + e^x.$$

Problems

1. Find all of the solutions of the first order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $\frac{dy}{dx} = \frac{e^x}{2y}$.

b. $\frac{dy}{dt} = y^2(1 + t^2)$, $y(0) = 1$.

c. $\frac{dy}{dx} = \frac{\sqrt{1 - y^2}}{x}$.

d. $xy' = y(1-2y), \quad y(1) = 2.$

e. $y' - (\sin x)y = \sin x.$

f. $xy' - 2y = x^2, y(1) = 1.$

g. $\dfrac{ds}{dt} + 2s = st^2, \quad , s(0) = 1.$

h. $x' - 2x = te^{2t}.$

i. $\dfrac{dy}{dx} + y = \sin x, \, y(0) = 0.$

j. $\dfrac{dy}{dx} - \dfrac{3}{x}y = x^3, \, y(1) = 4.$

2. Consider the differential equation

$$\frac{dy}{dx} = \frac{x}{y} - \frac{x}{1+y}.$$

a. Find the 1-parameter family of solutions (general solution) of this equation.

b. Find the solution of this equation satisfying the initial condition $y(0) = 1$. Is this a member of the 1-parameter family?

3. Identify the type of differential equation. Find the general solution and plot several particular solutions. Also, find the singular solution if one exists.

a. $y = xy' + \frac{1}{y'}.$

b. $y = 2xy' + \ln y'.$

c. $y' + 2xy = 2xy^2.$

d. $y' + 2xy = y^2 e^{x^2}.$

4. The initial value problem

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2}, \quad y(1) = 1$$

does not fall into the class of problems considered in our review. However, if one substitutes $y(x) = xz(x)$ into the differential equation, one obtains an equation for $z(x)$ which can be solved. Use this substitution to solve the initial value problem for $y(x)$.

5. Find all of the solutions of the second order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $y'' - 9y' + 20y = 0.$

b. $y'' - 3y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1.$

c. $8y'' + 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0.$

d. $x'' - x' - 6x = 0$ for $x = x(t)$.

6. Prove that $y_1(x) = \sinh x$ and $y_2(x) = 3\sinh x - 2\cosh x$ are linearly independent solutions of $y'' - y = 0$. Write $y_3(x) = \cosh x$ as a linear combination of y_1 and y_2.

7. Find all of the solutions of the second order differential equations for $x > 0$.. When an initial condition is given, find the particular solution satisfying that condition.

a. $x^2y'' + 3xy' + 2y = 0$.

b. $x^2y'' - 3xy' + 3y = 0$.

c. $x^2y'' + 5xy' + 4y = 0$.

d. $x^2y'' - 2xy' + 3y = 0$.

e. $x^2y'' + 3xy' - 3y = x^2$.

8. Consider the nonhomogeneous differential equation $x'' - 3x' + 2x = 6e^{3t}$.

a. Find the general solution of the homogenous equation.

b. Find a particular solution using the Method of Undetermined Coefficients by guessing $x_p(t) = Ae^{3t}$.

c. Use your answers in the previous parts to write down the general solution for this problem.

9. Find the general solution of the given equation by the method given.

a. $y'' - 3y' + 2y = 10$. Method of Undetermined Coefficients.

b. $y'' + y' = 3x^2$. Method of Variation of Parameters.

10. Use the Method of Variation of Parameters to determine the general solution for the following problems.

a. $y'' + y = \tan x$.

b. $y'' - 4y' + 4y = 6xe^{2x}$.

11. Instead of assuming that $c_1'y_1 + c_2'y_2 = 0$ in the derivation of the solution using Variation of Parameters, assume that $c_1'y_1 + c_2'y_2 = h(x)$ for an arbitrary function $h(x)$ and show that one gets the same particular solution.

12. Find the general solution of each differential equation. When an initial condition is given, find the particular solution satisfying that condition.

a. $y'' - 3y' + 2y = 20e^{-2x}, \quad y(0) = 0, \quad y'(0) = 6$.

b. $y'' + y = 2\sin 3x$.

c. $y'' + y = 1 + 2\cos x$.

d. $x^2y'' - 2xy' + 2y = 3x^2 - x, \quad x > 0$.

13. Verify that the given function is a solution and use Reduction of Order to find a second linearly independent solution.

a. $x^2y'' - 2xy' - 4y = 0, \quad y_1(x) = x^4$.

b. $xy'' - y' + 4x^3y = 0, \quad y_1(x) = \sin(x^2)$.

14. A certain model of the motion of a tossed whiffle ball is given by

$$mx'' + cx' + mg = 0, \quad x(0) = 0, \quad x'(0) = v_0.$$

Here m is the mass of the ball, g=9.8 m/s^2 is the acceleration due to gravity and c is a measure of the damping. Since there is no x term, we can write this as a first order equation for the velocity $v(t) = x'(t)$:

$$mv' + cv + mg = 0.$$

a. Find the general solution for the velocity $v(t)$ of the linear first order differential equation above.

b. Use the solution of part a to find the general solution for the position $x(t)$.

c. Find an expression to determine how long it takes for the ball to reach it's maximum height?

d. Assume that $c/m = 10\ \text{s}^{-1}$. For $v_0 = 5, 10, 15, 20$ m/s, plot the solution, $x(t)$, versus the time.

e. From your plots and the expression in part c, determine the rise time. Do these answers agree?

f. What can you say about the time it takes for the ball to fall as compared to the rise time?

Chapter 2

Linear Systems of Differential Equations

"Do not worry too much about your difficulties in mathematics, I can assure you that mine are still greater." - Albert Einstein (1879-1955)

2.1 Coupled Systems

In this chapter we will begin our study of systems of differential equations. After defining first order systems, we will look at constant coefficient systems and the behavior of solutions for these systems. Also, most of the discussion will focus on planar, or two dimensional, systems. For such systems we will be able to look at a variety of graphical representations of the family of solutions and discuss the qualitative features of systems we can solve in preparation for the study of systems whose solutions cannot be found in an algebraic form. However, we first turn to some simple physical problems.

There are many problems in physics that can result in systems of equations. This is because the most basic law of physics is given by Newton's Second Law, which states that if a body experiences a net force, it will accelerate. Thus,

$$\sum \mathbf{F} = m\mathbf{a}.$$

Since $\mathbf{a} = \ddot{\mathbf{x}}$ we have a system of second order differential equations in general for three dimensional problems, or one second order differential equation for one dimensional problems for a single mass.

We have already seen reminded in the last chapter of the simple problem of a mass on a spring. This is shown in Figure 2.1. The mass slides on a frictionless surface and reacts to the restoring force of the spring attached to a wall. The restoring force of the spring given by Hooke's Law,

$$F_s = -kx,$$

where $k > 0$ is the spring constant and x is the elongation of the spring. When the spring elongation is positive, the spring force is negative and when the spring elongation is negative the spring force is positive. The

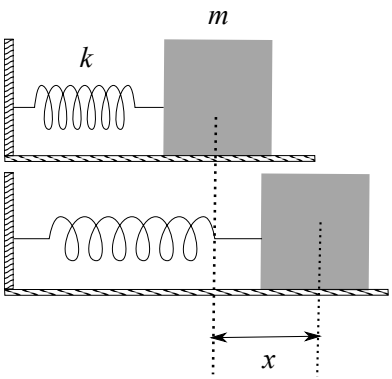


Figure 2.1: Spring-Mass system.

equation for simple harmonic motion for the mass-spring system is found from Newton's Second Law as

$$m\ddot{x} + kx = 0.$$

This second order equation, constant coefficient equation is easily solved using the methods in the previous chapter. However, it can also be written as a system of two first order equations in terms of the unknown position and velocity. We first set $y = \dot{x}$. Noting that $\ddot{x} = \dot{y}$, we rewrite the second order equation in terms of x and $\dot{y}$. Thus, we have

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{k}{m}x. \end{aligned} \tag{2.1}$$

One can look at more complicated spring-mass systems. Consider two blocks attached with two springs as in Figure 2.2. In this case we apply Newton's second law for each block. We will designate the elongations of each spring from equilibrium as x_1 and x_2. These are shown in Figure 2.2.

For mass m_1, the forces acting on it are due to each spring. The first spring with spring constant k_1 provides a force on m_1 of $-k_1x_1$. The second spring is stretched, or compressed, based upon the relative locations of the two masses. So, the second spring will exert a force on m_1 of $k_2(x_2 - x_1)$.

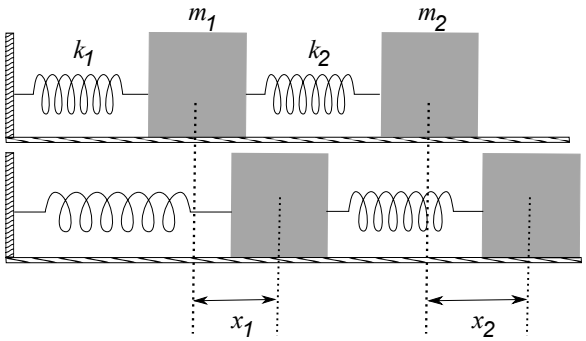


Figure 2.2: System of two masses and two springs.

Similarly, the only force acting directly on mass m_2 is provided by the restoring force from spring 2. So, that force is given by $-k_2(x_2 - x_1)$. The reader should think about the signs in each case.

Putting this all together, we apply Newton's Second Law to both masses. We obtain the two equations

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 + k_2(x_2 - x_1) \\ m_2\ddot{x}_2 &= -k_2(x_2 - x_1). \end{aligned} \tag{2.2}$$

Thus, we see that we have a coupled system of two second order differential equations. Each equation depends on the unknowns x_1 and x_2.

One can rewrite this system of two second order equations as a system of four first order equations by letting $x_3 = \dot{x}_1$ and $x_4 = \dot{x}_2$. This leads to

the system

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -\frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}(x_2 - x_1) \\
\dot{x}_4 &= -\frac{k_2}{m_2}(x_2 - x_1).
\end{aligned} \tag{2.3}$$

As we will see in the next chapter, this system can be written more compactly in matrix form:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{2.4}$$

We can solve this system of first order equations using matrix methods. However, we will first need to recall a few things from linear algebra. This will be done in the next chapter. For now, we will return to simpler systems and explore the behavior of typical solutions in planar systems.

2.2 *Planar Systems*

2.2.1 *Introduction*

We now consider examples of solving a coupled system of first order differential equations in the plane. We will focus on the theory of linear systems with constant coefficients. Understanding these simple systems will help in the study of nonlinear systems, which contain much more interesting behaviors, such as the onset of chaos. In the next chapter we will return to these systems and describe a matrix approach to obtaining the solutions.

A general form for first order systems in the plane is given by a system of two equations for unknowns $x(t)$ and $y(t)$:

$$\begin{aligned}
x'(t) &= P(x, y, t) \\
y'(t) &= Q(x, y, t).
\end{aligned} \tag{2.5}$$

An autonomous system is one in which there is no explicit time dependence:

Autonomous systems.

$$\begin{aligned}
x'(t) &= P(x, y) \\
y'(t) &= Q(x, y).
\end{aligned} \tag{2.6}$$

Otherwise the system is called nonautonomous.

A *linear system* takes the form

$$\begin{aligned}
x' &= a(t)x + b(t)y + e(t) \\
y' &= c(t)x + d(t)y + f(t).
\end{aligned} \tag{2.7}$$

A homogeneous linear system results when $e(t) = 0$ and $f(t) = 0$.

A linear, constant coefficient system of first order differential equations is given by

$$\begin{aligned} x' &= ax + by + e \\ y' &= cx + dy + f. \end{aligned} \tag{2.8}$$

A linear, homogeneous system of constant coefficient first order differential equations in the plane.

We will focus on linear, homogeneous systems of constant coefficient first order differential equations:

$$\boxed{\begin{aligned} x' &= ax + by \\ y' &= cx + dy. \end{aligned} \tag{2.9}}$$

As we will see later, such systems can result by a simple translation of the unknown functions. These equations are said to be coupled if either $b \neq 0$ or $c \neq 0$.

We begin by noting that the system (2.9) can be rewritten as a second order constant coefficient linear differential equation, which we already know how to solve. We differentiate the first equation in system (2.9) and systematically replace occurrences of y and y', since we also know from the first equation that $y = \frac{1}{b}(x' - ax)$. Thus, we have

$$\begin{aligned} x'' &= ax' + by' \\ &= ax' + b(cx + dy) \\ &= ax' + bcx + d(x' - ax). \end{aligned} \tag{2.10}$$

Rewriting the last line, we have

$$\boxed{x'' - (a + d)x' + (ad - bc)x = 0.} \tag{2.11}$$

This is a linear, homogeneous, constant coefficient ordinary differential equation. We know that we can solve this by first looking at the roots of the characteristic equation

$$r^2 - (a + d)r + ad - bc = 0 \tag{2.12}$$

and writing down the appropriate general solution for $x(t)$. Then we can find $y(t)$ using Equation (2.9):

$$y = \frac{1}{b}(x' - ax).$$

We now demonstrate this for a specific example.

Example 2.1. Consider the system of differential equations

$$\begin{aligned} x' &= -x + 6y \\ y' &= x - 2y. \end{aligned} \tag{2.13}$$

Carrying out the above outlined steps, we have that $x'' + 3x' - 4x = 0$. This can be shown as follows:

$$\begin{aligned} x'' &= -x' + 6y' \\ &= -x' + 6(x - 2y) \\ &= -x' + 6x - 12\left(\frac{x' + x}{6}\right) \\ &= -3x' + 4x \end{aligned} \tag{2.14}$$

The resulting differential equation has a characteristic equation of $r^2 + 3r - 4 = 0$. The roots of this equation are $r = 1, -4$. Therefore, $x(t) = c_1 e^t + c_2 e^{-4t}$. But, we still need $y(t)$. From the first equation of the system we have

$$y(t) = \frac{1}{6}(x' + x) = \frac{1}{6}(2c_1 e^t - 3c_2 e^{-4t}).$$

Thus, the solution to the system is

$$\begin{aligned} x(t) &= c_1 e^t + c_2 e^{-4t}, \\ y(t) &= \tfrac{1}{3} c_1 e^t - \tfrac{1}{2} c_2 e^{-4t}. \end{aligned} \tag{2.15}$$

Sometimes one needs initial conditions. For these systems we would specify conditions like $x(0) = x_0$ and $y(0) = y_0$. These would allow the determination of the arbitrary constants as before.

Solving systems with initial conditions.

Example 2.2. Solve

$$\begin{aligned} x' &= -x + 6y \\ y' &= x - 2y. \end{aligned} \tag{2.16}$$

given $x(0) = 2$, $y(0) = 0$.

We already have the general solution of this system in (2.15). Inserting the initial conditions, we have

$$\begin{aligned} 2 &= c_1 + c_2, \\ 0 &= \tfrac{1}{3} c_1 - \tfrac{1}{2} c_2. \end{aligned} \tag{2.17}$$

Solving for c_1 and c_2 gives $c_1 = 6/5$ and $c_2 = 4/5$. Therefore, the solution of the initial value problem is

$$\begin{aligned} x(t) &= \tfrac{2}{5}\left(3e^t + 2e^{-4t}\right), \\ y(t) &= \tfrac{2}{5}\left(e^t - e^{-4t}\right). \end{aligned} \tag{2.18}$$

2.2.2 *Equilibrium Solutions and Nearby Behaviors*

In studying systems of differential equations, it is often useful to study the behavior of solutions without obtaining an algebraic form for the solution. This is done by exploring equilibrium solutions and solutions

nearby equilibrium solutions. Such techniques will be seen to be useful later in studying nonlinear systems.

We begin this section by studying equilibrium solutions of system (2.8). For equilibrium solutions the system does not change in time. Therefore, equilibrium solutions satisfy the equations $x' = 0$ and $y' = 0$. Of course, this can only happen for constant solutions. Let x_0 and y_0 be the (constant) equilibrium solutions. Then, x_0 and y_0 must satisfy the system

Equilibrium solutions.

$$\begin{aligned} 0 &= ax_0 + by_0 + e, \\ 0 &= cx_0 + dy_0 + f. \end{aligned} \tag{2.19}$$

This is a linear system of nonhomogeneous algebraic equations. One only has a unique solution when the determinant of the system is not zero, i.e., $ad - bc \neq 0$. Using Cramer's (determinant) Rule for solving such systems, we have

$$x_0 = -\frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \qquad y_0 = -\frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}. \tag{2.20}$$

If the system is homogeneous, $e = f = 0$, then we have that the origin is the equilibrium solution; i.e., $(x_0, y_0) = (0, 0)$. Often we will have this case since one can always make a change of coordinates from (x, y) to (u, v) by $u = x - x_0$ and $v = y - y_0$. Then, $u_0 = v_0 = 0$.

Next we are interested in the behavior of solutions near the equilibrium solutions. Later this behavior will be useful in analyzing more complicated nonlinear systems. We will look at some simple systems that are readily solved.

Example 2.3. Stable Node (sink)

Consider the system

$$\begin{aligned} x' &= -2x \\ y' &= -y. \end{aligned} \tag{2.21}$$

This is a simple uncoupled system. Each equation is simply solved to give

$$x(t) = c_1 e^{-2t} \text{ and } y(t) = c_2 e^{-t}.$$

In this case we see that all solutions tend towards the equilibrium point, $(0, 0)$. This will be called a *stable node*, or a *sink*.

Before looking at other types of solutions, we will explore the stable node in the above example. There are several methods of looking at the behavior of solutions. We can look at solution plots of the dependent versus the independent variables, or we can look in the xy-plane at the parametric curves $(x(t), y(t))$.

Solution Plots: One can plot each solution as a function of t given a set of initial conditions. Examples are shown in Figure 2.3 for several initial

conditions. Note that the solutions decay for large t. Special cases result for various initial conditions. Note that for $t = 0$, $x(0) = c_1$ and $y(0) = c_2$. (Of course, one can provide initial conditions at any $t = t_0$. It is generally easier to pick $t = 0$ in our general explanations.) If we pick an initial condition with c_1=o, then $x(t) = 0$ for all t. One obtains similar results when setting $y(0) = 0$.

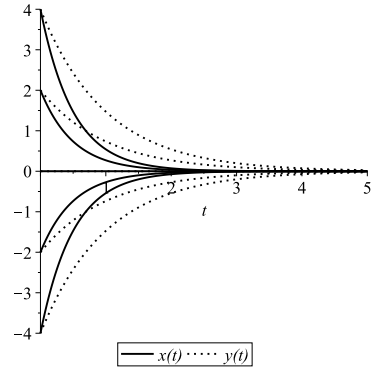


Figure 2.3: Plots of solutions of Example 2.3 for several initial conditions.

Phase Portrait: There are other types of plots which can provide additional information about the solutions even if we cannot find the exact solutions as we can for these simple examples. In particular, one can consider the solutions $x(t)$ and $y(t)$ as the coordinates along a parameterized path, or curve, in the plane: $\mathbf{r} = (x(t), y(t))$ Such curves are called trajectories or orbits. The xy-plane is called the phase plane and a collection of such orbits gives a phase portrait for the family of solutions of the given system.

One method for determining the equations of the orbits in the phase plane is to eliminate the parameter t between the known solutions to get a relationship between x and y. Since the solutions are known for the last example, we can do this, since the solutions are known. In particular, we have

$$x = c_1 e^{-2t} = c_1 \left(\frac{y}{c_2}\right)^2 \equiv Ay^2.$$

Another way to obtain information about the orbits comes from noting that the slopes of the orbits in the xy-plane are given by dy/dx. For autonomous systems, we can write this slope just in terms of x and y. This leads to a first order differential equation, which possibly could be solved analytically or numerically.

First we will obtain the orbits for Example 2.3 by solving the corresponding slope equation. Recall that for trajectories defined parametrically by $x = x(t)$ and $y = y(t)$, we have from the Chain Rule for $y = y(x(t))$ that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

Therefore,

The Slope of a parametric curve.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}. \tag{2.22}$$

For the system in (2.21) we use Equation (2.22) to obtain the equation for the slope at a point on the orbit:

$$\frac{dy}{dx} = \frac{y}{2x}.$$

The general solution of this first order differential equation is found using separation of variables as $x = Ay^2$ for A an arbitrary constant. Plots of these solutions in the phase plane are given in Figure 2.4. [Note that this is the same form for the orbits that we had obtained above by eliminating t from the solution of the system.]

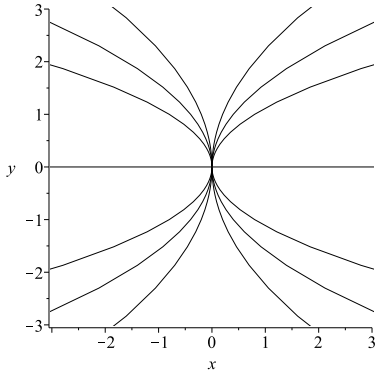


Figure 2.4: Orbits for Example 2.3.

Once one has solutions to differential equations, we often are interested in the long time behavior of the solutions. Given a particular initial condition

(x_0, y_0), how does the solution behave as time increases? For orbits near an equilibrium solution, do the solutions tend towards, or away from, the equilibrium point? The answer is obvious when one has the exact solutions $x(t)$ and $y(t)$. However, this is not always the case.

Let's consider the above example for initial conditions in the first quadrant of the phase plane. For a point in the first quadrant we have that

$$dx/dt = -2x < 0,$$

meaning that as $t \to \infty$, $x(t)$ get more negative. Similarly,

$$dy/dt = -y < 0,$$

indicating that $y(t)$ is also getting smaller for this problem. Thus, these orbits tend towards the origin as $t \to \infty$. This qualitative information was obtained without relying on the known solutions to the problem.

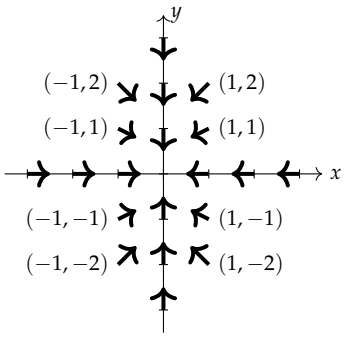


Figure 2.5: Sketch of tangent vectors using Example 2.3.

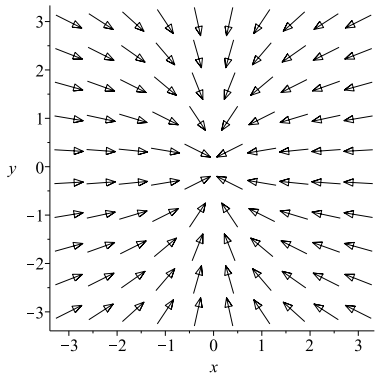


Figure 2.6: Direction field for Example 2.3.

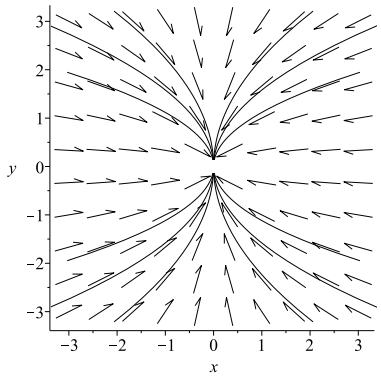


Figure 2.7: Phase portrait for Example 2.3. This is a stable node, or sink

Direction Fields: Another way to determine the behavior of the solutions of the system of differential equations is to draw the direction field. A direction field is a vector field in which one plots arrows in the direction of tangents to the orbits at selected points in the plane. This is done because the slopes of the tangent lines are given by dy/dx. For the general system (2.9), the slope is

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by}.$$

This is a first order differential equation which can be solved as we show in the following examples.

Example 2.4. Draw the direction field for Example 2.3.

We can use software to draw direction fields. However, one can sketch these fields by hand. We have that the slope of the tangent at this point is given by

$$\frac{dy}{dx} = \frac{-y}{-2x} = \frac{y}{2x}.$$

For each point in the plane one draws a piece of tangent line with this slope. In Figure 2.5 we show a few of these. For $(x, y) = (1, 1)$ the slope is $dy/dx = 1/2$. So, we draw an arrow with slope $1/2$ at this point. From system (2.21), we have that x' and y' are both negative at this point. Therefore, the vector points down and to the left.

We can do this for several points, as shown in Figure 2.5. Sometimes one can quickly sketch vectors with the same slope. For this example, when $y = 0$, the slope is zero and when $x = 0$ the slope is infinite. So, several vectors can be provided. Such vectors are tangent to curves known as *isoclines* in which $\frac{dy}{dx}$ =constant.

It is often difficult to provide an accurate sketch of a direction field. Computer software can be used to provide a better rendition. For Example 2.3 the direction field is shown in Figure 2.6. Looking at this direction field, one can begin to "see" the orbits by following the tangent vectors.

Of course, one can superimpose the orbits on the direction field. This is shown in Figure 2.7. Are these the patterns you saw in Figure 2.6?

In this example we see all orbits "flow" towards the origin, or equilibrium point. Again, this is an example of what is called a *stable node* or a *sink*. (Imagine what happens to the water in a sink when the drain is unplugged.)

This is another uncoupled system. The solutions are again simply gotten by integration. We have that $x(t) = c_1 e^{-t}$ and $y(t) = c_2 e^{t}$. Here we have that x decays as t gets large and y increases as t gets large. In particular, if one picks initial conditions with $c_2 = 0$, then orbits follow the x-axis towards the origin. For initial points with $c_1 = 0$, orbits originating on the y-axis will flow away from the origin. Of course, in these cases the origin is an equilibrium point and once at equilibrium, one remains there.

In fact, there is only one line on which to pick initial conditions such that the orbit leads towards the equilibrium point. No matter how small c_2 is, sooner, or later, the exponential growth term will dominate the solution. One can see this behavior in Figure 2.8.

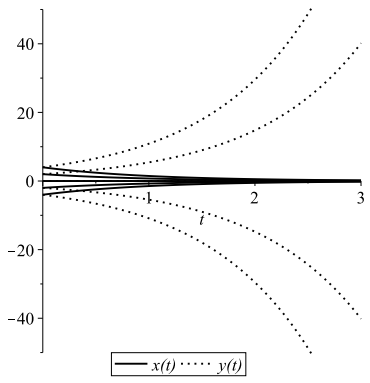


Figure 2.8: Plots of solutions of Example 2.5 for several initial conditions.

Example 2.5. Saddle Consider the system

$$\begin{aligned} x' &= -x \\ y' &= y. \end{aligned} \tag{2.23}$$

Similar to the first example, we can look at plots of solutions orbits in the phase plane. These are given by Figures 2.8-2.9. The orbits can be obtained from the system as

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{y}{x}.$$

The solution is $y = \frac{A}{x}$. For different values of $A \neq 0$ we obtain a family of hyperbolae. These are the same curves one might obtain for the level curves of a surface known as a saddle surface, $z = xy$. Thus, this type of equilibrium point is classified as a saddle point. From the phase portrait we can verify that there are many orbits that lead away from the origin (equilibrium point), but there is one line of initial conditions that leads to the origin and that is the x-axis. In this case, the line of initial conditions is given by the x-axis.

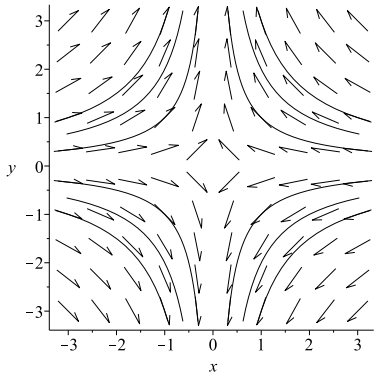


Figure 2.9: Phase portrait for Example 2.5. This is a saddle.

Example 2.6. Unstable Node (source)

$$\begin{aligned} x' &= 2x \\ y' &= y. \end{aligned} \tag{2.24}$$

This example is similar to Example 2.3. The solutions are obtained by replacing t with $-t$. The solutions, orbits, and direction fields can be seen in Figures 2.10-2.11. This is once again a node, but all orbits lead away from the equilibrium point. It is called an *unstable node* or a *source*.

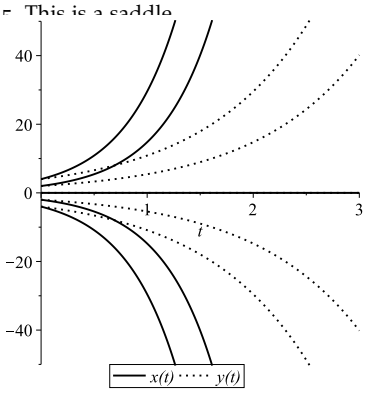


Figure 2.10: Plots of solutions of Example 2.6 for several initial conditions.

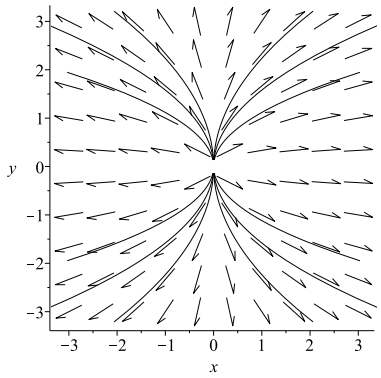


Figure 2.11: Phase portrait for Example 2.6, an unstable node or source.

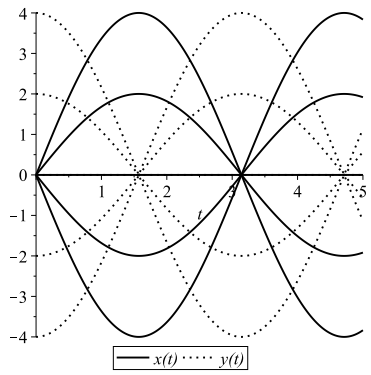


Figure 2.12: Plots of solutions of Example 2.7 for several initial conditions.

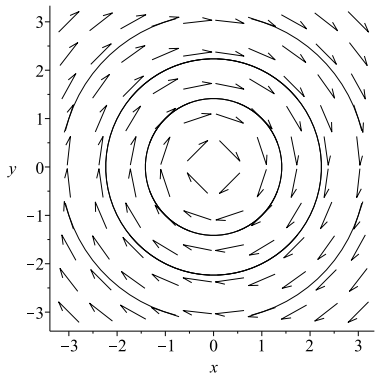


Figure 2.13: Phase portrait for Example 2.7, a center.

Example 2.7. Center

$$\begin{aligned} x' &= y \\ y' &= -x. \end{aligned} \tag{2.25}$$

This system is a simple, coupled system. Neither equation can be solved without some information about the other unknown function. However, we can differentiate the first equation and use the second equation to obtain

$$x'' + x = 0.$$

We recognize this equation as one that appears in the study of simple harmonic motion. The solutions are pure sinusoidal oscillations:

$$x(t) = c_1 \cos t + c_2 \sin t, \quad y(t) = -c_1 \sin t + c_2 \cos t.$$

In the phase plane the trajectories can be determined either by looking at the direction field, or solving the first order equation

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Performing a separation of variables and integrating, we find that

$$x^2 + y^2 = C.$$

Thus, we have a family of circles for $C > 0$. (Can you prove this using the general solution?) Looking at the results graphically in Figures 2.12-2.13 confirms this result. This type of point is called a center.

Example 2.8. Focus (spiral)

$$\begin{aligned} x' &= \alpha x + y \\ y' &= -x. \end{aligned} \tag{2.26}$$

In this example, we will see an additional set of behaviors of equilibrium points in planar systems. We have added one term, αx, to the system in Example 2.7. We will consider the effects for two specific values of the parameter: $\alpha = 0.1, -0.2$. The resulting behaviors are shown in the Figures 2.15-2.18. We see orbits that look like spirals. These orbits are stable and unstable spirals (or foci, the plural of focus.)

We can understand these behaviors by once again relating the system of first order differential equations to a second order differential equation. Using the usual method for obtaining a second order equation form a system, we find that $x(t)$ satisfies the differential equation

$$x'' - \alpha x' + x = 0.$$

We recall from our first course that this is a form of damped simple harmonic motion. The characteristic equation is $r^2 - \alpha r + 1 = 0$. The solution of this quadratic equation is

$$r = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}.$$

There are five special cases to consider as shown in the below classification.

Classification of Solutions of $x'' - \alpha x' + x = 0$

1. $\alpha = -2$. There is one real solution. This case is called *critical damping* since the solution $r = -1$ leads to exponential decay. The solution is $x(t) = (c_1 + c_2 t)e^{-t}$.
2. $\alpha < -2$. There are two real, negative solutions, $r = -\mu, -\nu$, $\mu, \nu > 0$. The solution is $x(t) = c_1 e^{-\mu t} + c_2 e^{-\nu t}$. In this case we have what is called *overdamped* motion. There are no oscillations
3. $-2 < \alpha < 0$. There are two complex conjugate solutions $r = \alpha/2 \pm i\beta$ with real part less than zero and $\beta = \frac{\sqrt{4-\alpha^2}}{2}$. The solution is $x(t) = (c_1 \cos \beta t + c_2 \sin \beta t)e^{\alpha t/2}$. Since $\alpha < 0$, this consists of a decaying exponential times oscillations. This is often called an *underdamped oscillation*.
4. $\alpha = 0$. This leads to *simple harmonic motion*.
5. $0 < \alpha < 2$. This is similar to the underdamped case, except $\alpha > 0$. The solutions are growing oscillations.
6. $\alpha = 2$. There is one real solution. The solution is $x(t) = (c_1 + c_2 t)e^{t}$. It leads to unbounded growth in time.
7. For $\alpha > 2$. There are two real, positive solutions $r = \mu, \nu > 0$. The solution is $x(t) = c_1 e^{\mu t} + c_2 e^{\nu t}$, which grows in time.

For $\alpha < 0$ the solutions are losing energy, so the solutions can oscillate with a diminishing amplitude. (See Figure 2.14.) For $\alpha > 0$, there is a growth in the amplitude, which is not typical. (See Figure 2.15.) Of course, there can be overdamped motion if the magnitude of α is too large.

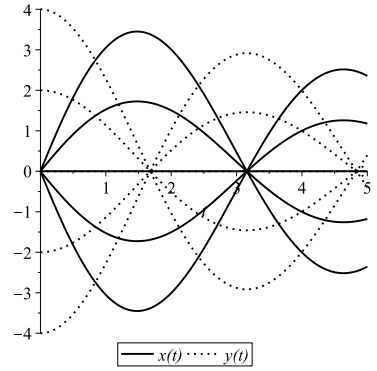


Figure 2.14: Plots of solutions of Example 2.8 for several initial conditions, $\alpha = -0.2$.

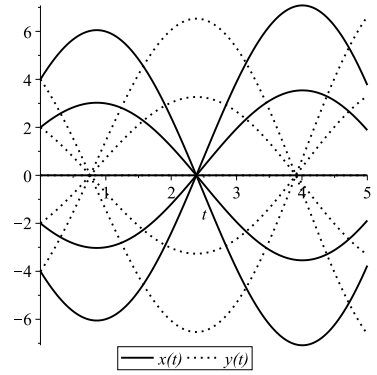


Figure 2.15: Plots of solutions of Example 2.8 for several initial conditions, $\alpha = 0.1$.

Example 2.9. Degenerate Node For this example, we will write out the solutions. It is a coupled system for which only the second equation is coupled.

$$\begin{aligned} x' &= -x \\ y' &= -2x - y. \end{aligned} \tag{2.27}$$

There are two possible approaches:

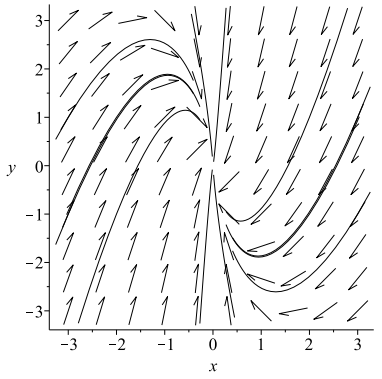


Figure 2.16: Phase portrait for 2.9. This is a degenerate node.

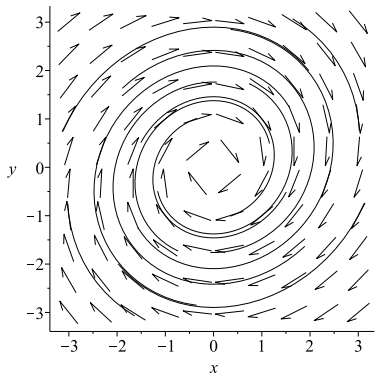


Figure 2.17: Phase portrait for Example 2.8 with $\alpha = -0.2$. This is a stable focus, or spiral.

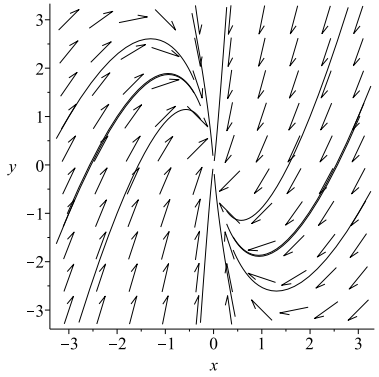


Figure 2.18: Phase portrait for Example 2.9. This is a degenerate node.

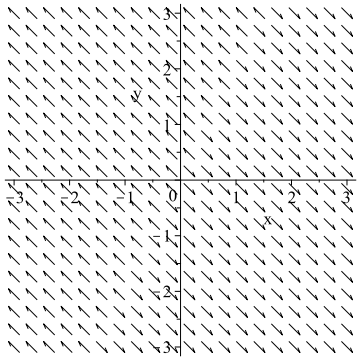


Figure 2.19: Plots of direction field of Example 2.10.

a. We could solve the first equation to find $x(t) = c_1 e^{-t}$. Inserting this solution into the second equation, we have

$$y' + y = -2c_1 e^{-t}.$$

This is a relatively simple linear first order equation for $y = y(t)$. The integrating factor is $\mu = e^t$. The solution is found as $y(t) = (c_2 - 2c_1 t)e^{-t}$.

b. Another method would be to proceed to rewrite this as a second order equation. Computing x'' does not get us very far. So, we look at

$$\begin{aligned} y'' &= -2x' - y' \\ &= 2x - y' \\ &= -2y' - y. \end{aligned} \tag{2.28}$$

Therefore, y satisfies

$$y'' + 2y' + y = 0.$$

The characteristic equation has one real root, $r = -1$. So, we write

$$y(t) = (k_1 + k_2 t)e^{-t}.$$

This is a stable *degenerate node.* Combining this with the solution $x(t) = c_1 e^{-t}$, we can show that $y(t) = (c_2 - 2c_1 t)e^{-t}$ as before.

In Figure 2.16 we see several orbits in this system. It differs from the stable node show in Figure 2.4 in that there is only one direction along which the orbits approach the origin instead of two. If one picks $c_1 = 0$, then $x(t) = 0$ and $y(t) = c_2 e^{-t}$. This leads to orbits running along the y-axis as seen in the figure.

Example 2.10. A Line of Equilibria, Zero Root

$$\begin{aligned} x' &= 2x - y \\ y' &= -2x + y. \end{aligned} \tag{2.29}$$

In this last example, we have a coupled set of equations. We rewrite it as a second order differential equation:

$$\begin{aligned} x'' &= 2x' - y' \\ &= 2x' - (-2x + y) \\ &= 2x' + 2x + (x' - 2x) = 3x'. \end{aligned} \tag{2.30}$$

So, the second order equation is

$$x'' - 3x' = 0$$

and the characteristic equation is $0 = r(r - 3)$. This gives the general solution as

$$x(t) = c_1 + c_2 e^{3t}$$

and thus

$$y = 2x - x' = 2(c_1 + c_2 e^3 t) - (3c_2 e^{3t}) = 2c_1 - c_2 e^{3t}.$$

In Figure 2.19 we show the direction field. The constant slope field seen in this example is confirmed by a simple computation:

$$\frac{dy}{dx} = \frac{-2x + y}{2x - y} = -1.$$

Furthermore, looking at initial conditions with $y = 2x$, we have at $t = 0$,

$$2c_1 - c_2 = 2(c_1 + c_2) \quad \Rightarrow \quad c_2 = 0.$$

Therefore, points on this line remain on this line forever, $(x, y) = (c_1, 2c_1)$. This line of fixed points is called a line of equilibria.

2.2.3 *Polar Representation of Spirals*

IN THE EXAMPLES WITH A CENTER OR A SPIRAL, one might be able to write the solutions in polar coordinates. Recall that a point in the plane can be described by either Cartesian (x, y) or polar (r, θ) coordinates. Given the polar form, one can find the Cartesian components using

$$x = r\cos\theta \text{ and } y = r\sin\theta.$$

Given the Cartesian coordinates, one can find the polar coordinates using

$$r^2 = x^2 + y^2 \text{ and } \tan\theta = \frac{y}{x}. \tag{2.31}$$

Since x and y are functions of t, then naturally we can think of r and θ as functions of t. Converting a system of equations in the plane for x' and y' to polar form requires knowing r' and θ'. So, we first find expressions for r' and θ' in terms of x' and y'.

Differentiating the first equation in (2.31) gives

$$rr' = xx' + yy'.$$

Inserting the expressions for x' and y' from system 2.9, we have

$$rr' = x(ax + by) + y(cx + dy).$$

In some cases this may be written entirely in terms of r's. Similarly, we have that

$$\theta' = \frac{xy' - yx'}{r^2},$$

which the reader can prove for homework.

In summary, when converting first order equations from rectangular to polar form, one needs the relations below.

Derivatives of Polar Variables

$$\begin{aligned} r' &= \frac{xx' + yy'}{r}, \\ \theta' &= \frac{xy' - yx'}{r^2}. \end{aligned} \tag{2.32}$$

Example 2.11. Rewrite the following system in polar form and solve the resulting system.

$$\begin{aligned} x' &= ax + by \\ y' &= -bx + ay. \end{aligned} \tag{2.33}$$

We first compute r' and θ':

$$rr' = xx' + yy' = x(ax + by) + y(-bx + ay) = ar^2.$$

$$r^2\theta' = xy' - yx' = x(-bx + ay) - y(ax + by) = -br^2.$$

This leads to simpler system

$$\begin{aligned} r' &= ar \\ \theta' &= -b. \end{aligned} \tag{2.34}$$

This system is uncoupled. The second equation in this system indicates that we traverse the orbit at a constant rate in the clockwise direction. Solving these equations, we have that $r(t) = r_0 e^{at}$, $\theta(t) = \theta_0 - bt$. Eliminating t between these solutions, we finally find the polar equation of the orbits:

$$r = r_0 e^{-a(\theta - \theta_0)t/b}.$$

If you graph this for $a \neq 0$, you will get stable or unstable spirals.

Example 2.12. Consider the specific system

$$\begin{aligned} x' &= -y + x \\ y' &= x + y. \end{aligned} \tag{2.35}$$

In order to convert this system into polar form, we compute

$$rr' = xx' + yy' = x(-y + x) + y(x + y) = r^2.$$

$$r^2\theta' = -xy' - yx' = x(x + y) - y(-y + x) = r^2.$$

This leads to simpler system

$$\begin{aligned} r' &= r \\ \theta' &= 1. \end{aligned} \tag{2.36}$$

Solving these equations yields

$$r(t) = r_0 e^t, \quad \theta(t) = t + \theta_0.$$

Eliminating t from this solution gives the orbits in the phase plane, $r(\theta) = r_0 e^{\theta - \theta_0}$.

A more complicated example arises for a nonlinear system of differential equations. Consider the following example.

Example 2.13.

$$\begin{aligned} x' &= -y + x(1 - x^2 - y^2) \\ y' &= x + y(1 - x^2 - y^2). \end{aligned} \tag{2.37}$$

Transforming to polar coordinates, one can show that in order to convert this system into polar form, we compute

$$r' = r(1 - r^2), \quad \theta' = 1.$$

This uncoupled system can be solved and this is left to the reader.

2.3 *Applications*

In this section we will describe some simple applications leading to systems of differential equations which can be solved using the methods in this chapter. These systems are left for homework problems and the as the start of further explorations for student projects.

2.3.1 *Mass-Spring Systems*

The first examples that we had seen involved masses on springs. Recall that for a simple mass on a spring we studied simple harmonic motion, which is governed by the equation

$$m\ddot{x} + kx = 0.$$

This second order equation can be written as two first order equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{k}{m}x, \end{aligned} \tag{2.38}$$

or

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 x, \end{aligned} \tag{2.39}$$

where $\omega^2 = \frac{k}{m}$. The coefficient matrix for this system is

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}.$$

We also looked at the system of two masses and two springs as shown in Figure 2.20. The equations governing the motion of the masses is

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1 x_1 + k_2(x_2 - x_1) \\ m_2\ddot{x}_2 &= -k_2(x_2 - x_1). \end{aligned} \tag{2.40}$$

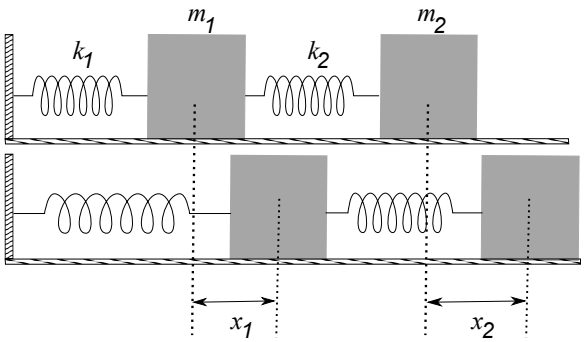


Figure 2.20: System of two masses and two springs.

We can rewrite this system as four first order equations

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -\frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}(x_2 - x_1) \\ \dot{x}_4 &= -\frac{k_2}{m_2}(x_2 - x_1). \end{aligned} \tag{2.41}$$

The coefficient matrix for this system is

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0 \end{pmatrix}.$$

We can study this system for specific values of the constants using the methods covered in the last sections.

Writing the spring-block system as a second order vector system.

Actually, one can also put the system (2.40) in the matrix form

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -(k_1+k_2) & k_2 \\ k_2 & -k_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{2.42}$$

This system can then be written compactly as

$$M\ddot{\mathbf{x}} = -K\mathbf{x}, \tag{2.43}$$

where

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad K = \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}.$$

This system can be solved by guessing a form for the solution. We could guess

$$\mathbf{x} = \mathbf{a}e^{i\omega t}$$

or

$$\mathbf{x} = \begin{pmatrix} a_1\cos(\omega t - \delta_1) \\ a_2\cos(\omega t - \delta_2) \end{pmatrix},$$

where δ_i are phase shifts determined from initial conditions.

Inserting $\mathbf{x} = \mathbf{a}e^{i\omega t}$ into the system gives

$$(K - \omega^2 M)\mathbf{a} = \mathbf{0}.$$

This is a homogeneous system. It is a generalized eigenvalue problem for eigenvalues ω^2 and eigenvectors $\mathbf{a}$. We solve this in a similar way to the standard matrix eigenvalue problems. The eigenvalue equation is found as

$$\det(K - \omega^2 M) = 0.$$

Once the eigenvalues are found, then one determines the eigenvectors and constructs the solution.

Example 2.14. Let $m_1 = m_2 = m$ and $k_1 = k_2 = k$. Then, we have to solve the system

$$\omega^2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

The eigenvalue equation is given by

$$\begin{aligned} 0 &= \begin{vmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{vmatrix} \\ &= (2k - m\omega^2)(k - m\omega^2) - k^2 \\ &= m^2\omega^4 - 3km\omega^2 + k^2. \end{aligned} \tag{2.44}$$

Solving this quadratic equation for ω^2, we have

$$\omega^2 = \frac{3 \pm 1}{2}\frac{k}{m}.$$

For positive values of ω, one can show that

$$\omega = \frac{1}{2}\left(\pm 1 + \sqrt{5}\right)\sqrt{\frac{k}{m}}.$$

The eigenvectors can be found for each eigenvalue by solving the homogeneous system

$$\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0.$$

The eigenvectors are given by

$$\mathbf{a}_1 = \begin{pmatrix} -\frac{\sqrt{5}+1}{2} \\ 1 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} \frac{\sqrt{5}-1}{2} \\ 1 \end{pmatrix}.$$

We are now ready to construct the real solutions to the problem. Similar to solving two first order systems with complex roots, we take the real and imaginary parts and take a linear combination of the solutions. In this problem there are four terms, giving the solution in the form

$$\mathbf{x}(t) = c_1\mathbf{a_1}cos\omega_1 t + c_2\mathbf{a_1}sin\omega_1 t + c_3\mathbf{a_2}cos\omega_2 t + c_4\mathbf{a_2}sin\omega_2 t,$$

where the ω's are the eigenvalues and the $\mathbf{a}$'s are the corresponding eigenvectors. The constants are determined from the initial conditions, $\mathbf{x}(0) = \mathbf{x}_0$ and $\dot{\mathbf{x}}(0) = \mathbf{v}_0$.

2.3.2 *Circuits**

In the last chapter we investigated simple series LRC circuits. More complicated circuits are possible by looking at parallel connections, or other combinations, of resistors, capacitors and inductors. This results in several equations for each loop in the circuit, leading to larger systems of differential equations. An example of another circuit setup is shown in Figure 2.21. This is not a problem that can be covered in the first year physics course.

There are two loops, indicated in Figure 2.22 as traversed clockwise. For each loop we need to apply Kirchoff's Loop Rule. There are three oriented currents, labeled I_i, $i = 1, 2, 3$. Corresponding to each current is a changing charge, q_i such that

$$I_i = \frac{dq_i}{dt}, \quad i = 1, 2, 3.$$

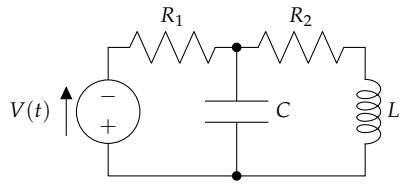


Figure 2.21: A circuit with two loops containing several different circuit elements.

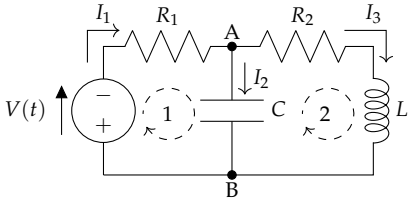


Figure 2.22: The previous parallel circuit with the directions indicated for traversing the loops in Kirchoff's Laws.

We have for loop one

$$I_1 R_1 + \frac{q_2}{C} = V(t) \tag{2.45}$$

and for loop two

$$I_3 R_2 + L\frac{dI_3}{dt} = \frac{q_2}{C}. \tag{2.46}$$

There are three unknown functions for the charge. Once we know the charge functions, differentiation will yield the three currents. However, we only have two equations. We need a third equation. This equation is found from Kirchoff's Point (Junction) Rule.

Consider the points A and B in Figure 2.22. Any charge (current) entering these junctions must be the same as the total charge (current) leaving the junctions. For point A we have

$$I_1 = I_2 + I_3, \tag{2.47}$$

or

$$\dot{q}_1 = \dot{q}_2 + \dot{q}_3. \tag{2.48}$$

Equations (2.45), (2.46), and (2.48) form a coupled system of differential equations for this problem. There are both first and second order derivatives involved. We can write the whole system in terms of charges as

$$\begin{aligned} R_1\dot{q}_1 + \frac{q_2}{C} &= V(t) \\ R_2\dot{q}_3 + L\ddot{q}_3 &= \frac{q_2}{C} \\ \dot{q}_1 &= \dot{q}_2 + \dot{q}_3. \end{aligned} \tag{2.49}$$

The question is whether, or not, we can write this as a system of first order differential equations. Since there is only one second order derivative, we can introduce the new variable $q_4 = \dot{q}_3$. The first equation can be solved for $\dot{q}_1$. The third equation can be solved for $\dot{q}_2$ with appropriate substitutions

for the other terms. $\dot{q}_3$ is gotten from the definition of q_4 and the second equation can be solved for $\ddot{q}_3$ and substitutions made to obtain the system

$$\begin{aligned}
\dot{q}_1 &= \frac{V}{R_1} - \frac{q_2}{R_1 C} \\
\dot{q}_2 &= \frac{V}{R_1} - \frac{q_2}{R_1 C} - q_4 \\
\dot{q}_3 &= q_4 \\
\dot{q}_4 &= \frac{q_2}{LC} - \frac{R_2}{L} q_4.
\end{aligned}$$

So, we have a nonhomogeneous first order system of differential equations.

2.3.3 *Mixture Problems*

There are many types of mixture problems. Such problems are standard in a first course on differential equations as examples of first order differential equations. Typically these examples consist of a tank of brine, water containing a specific amount of salt with pure water entering and the mixture leaving, or the flow of a pollutant into, or out of, a lake. We first saw such problems in Chapter 1.

In general one has a rate of flow of some concentration of mixture entering a region and a mixture leaving the region. The goal is to determine how much stuff is in the region at a given time. This is governed by the equation

Rate of change of substance = Rate In − Rate Out.

This can be generalized to the case of two interconnected tanks. We will provide an example, but first we review the single tank problem from Chapter 1.

Example 2.15. Single Tank Problem

A 50 gallon tank of pure water has a brine mixture with concentration of 2 pounds per gallon entering at the rate of 5 gallons per minute. [See Figure 2.23.] At the same time the well-mixed contents drain out at the rate of 5 gallons per minute. Find the amount of salt in the tank at time t. In all such problems one assumes that the solution is well mixed at each instant of time.

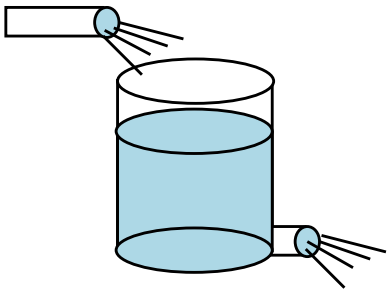


Figure 2.23: A typical mixing problem.

Let $x(t)$ be the amount of salt at time t. Then the rate at which the salt in the tank increases is due to the amount of salt entering the tank less that leaving the tank. To figure out these rates, one notes that dx/dt has units of pounds per minute. The amount of salt entering per minute is given by the product of the entering concentration times the rate at which the brine enters. This gives the correct units:

$$\left(2\frac{\text{pounds}}{\text{gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right) = 10\frac{\text{pounds}}{\text{min}}.$$

Similarly, one can determine the rate out as

$$\left(\frac{x \text{ pounds}}{50 \text{ gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right) = \frac{x}{10}\frac{\text{pounds}}{\text{min}}.$$

Thus, we have

$$\frac{dx}{dt} = 10 - \frac{x}{10}.$$

This equation is easily solved using the methods for first order equations.

Example 2.16. Double Tank Problem

One has two tanks connected together, labeled tank X and tank Y, as shown in Figure 2.24.

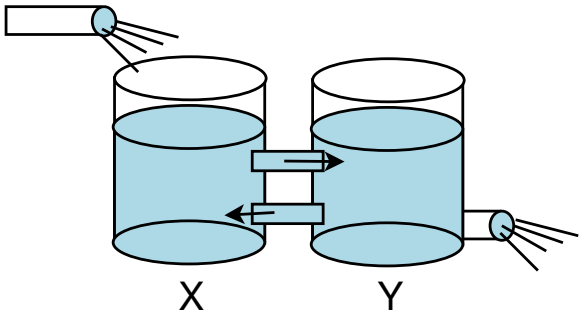


Figure 2.24: The two tank problem.

Let tank X initially have 100 gallons of brine made with 100 pounds of salt. Tank Y initially has 100 gallons of pure water. Pure water is pumped into tank X at a rate of 2.0 gallons per minute. Some of the mixture of brine and pure water flows into tank Y at 3 gallons per minute. To keep the tank levels the same, one gallon of the Y mixture flows back into tank X at a rate of one gallon per minute and 2.0 gallons per minute drains out. Find the amount of salt at any given time in the tanks. What happens over a long period of time?

In this problem we set up two equations. Let $x(t)$ be the amount of salt in tank X and $y(t)$ the amount of salt in tank Y. Again, we carefully look at the rates into and out of each tank in order to set up the system of differential equations. We obtain the system

$$\begin{aligned} \frac{dx}{dt} &= \frac{y}{100} - \frac{3x}{100} \\ \frac{dy}{dt} &= \frac{3x}{100} - \frac{3y}{100}. \end{aligned} \tag{2.50}$$

This is a linear, homogenous constant coefficient system of two first order equations, which we know how to solve. The matrix form of the system is given by

$$\dot{\mathbf{x}} = \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{3}{100} & -\frac{3}{100} \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 100 \\ 0 \end{pmatrix}.$$

The eigenvalues for the problem are given by $\lambda = -3 \pm \sqrt{3}$ and the eigenvectors are

$$\begin{pmatrix} 1 \\ \pm\sqrt{3} \end{pmatrix}.$$

Since the eigenvalues are real and distinct, the general solution is easily written down:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} e^{(-3+\sqrt{3})t} + c_2 \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} e^{(-3-\sqrt{3})t}.$$

Finally, we need to satisfy the initial conditions. So,

$$\mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \end{pmatrix},$$

or

$$c_1 + c_2 = 100, \quad (c_1 - c_2)\sqrt{3} = 0.$$

So, $c_2 = c_1 = 50$. The final solution is

$$\mathbf{x}(t) = 50 \left(\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} e^{(-3+\sqrt{3})t} + \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} e^{(-3-\sqrt{3})t} \right),$$

or

$$\begin{aligned} x(t) &= 50 \left(e^{(-3+\sqrt{3})t} + e^{(-3-\sqrt{3})t} \right) \\ y(t) &= 50\sqrt{3} \left(e^{(-3+\sqrt{3})t} - e^{(-3-\sqrt{3})t} \right). \end{aligned} \tag{2.51}$$

2.3.4 *Chemical Kinetics**

THERE ARE MANY PROBLEMS IN THE CHEMISTRY of chemical reactions which lead to systems of differential equations. The simplest reaction is when a chemical A turns into chemical B. This happens at a certain rate, $k > 0$. This reaction can be represented by the chemical formula

$$\boxed{A \xrightarrow[k]{} B.}$$

In this case we have that the rates of change of the concentrations of A, $[A]$, and B, $[B]$, are given by

$$\begin{aligned} \frac{d[A]}{dt} &= -k[A] \\ \frac{d[B]}{dt} &= k[A] \end{aligned} \tag{2.52}$$

The chemical reactions used in these examples are first order reactions. Second order reactions have rates proportional to the square of the concentration.

Think about this as it is a key to understanding the next reactions.

A more complicated reaction is given by

$$\boxed{A \xrightarrow[k_1]{} B \xrightarrow[k_2]{} C.}$$

Here there are three concentrations and two rates of change. The system of equations governing the reaction is

$$\begin{aligned} \frac{d[A]}{dt} &= -k_1[A], \\ \frac{d[B]}{dt} &= k_1[A] - k_2[B], \\ \frac{d[C]}{dt} &= k_2[B]. \end{aligned} \tag{2.53}$$

The more complication rate of change is when [B] increases from [A] changing to [B] and decrease when [B] changes to [C]. Thus, there are two terms in the rate of change equation for concentration [B].

One can further consider reactions in which a reverse reaction is possible. Thus, a further generalization occurs for the reaction

$$\boxed{A \underset{k_1}{\overset{k_3}{\rightleftarrows}} B \xrightarrow[k_2]{} C.}$$

The reverse reaction rates contribute to the reaction equations for [A] and [B]. The resulting system of equations is

$$\begin{aligned} \frac{d[A]}{dt} &= -k_1[A] + k_3[B], \\ \frac{d[B]}{dt} &= k_1[A] - k_2[B] - k_3[B], \\ \frac{d[C]}{dt} &= k_2[B]. \end{aligned} \tag{2.54}$$

Nonlinear chemical reactions will be discussed in the next chapter.

2.3.5 *Predator Prey Models**

ANOTHER COMMON POPULATION MODEL is that describing the coexistence of species. For example, we could consider a population of rabbits and foxes. Left to themselves, rabbits would tend to multiply, thus

$$\frac{dR}{dt} = aR,$$

with $a > 0$. In such a model the rabbit population would grow exponentially. Similarly, a population of foxes would decay without the rabbits to feed on. So, we have that

$$\frac{dF}{dt} = -bF$$

for $b > 0$.

Now, if we put these populations together on a deserted island, they would interact. The more foxes, the rabbit population would decrease. However, the more rabbits, the foxes would have plenty to eat and the population would thrive. Thus, we could model the competing populations as

$$\begin{aligned} \frac{dR}{dt} &= aR - cF, \\ \frac{dF}{dt} &= -bF + dR, \end{aligned} \tag{2.55}$$

where all of the constants are positive numbers. Studying this coupled system would lead to a study of the dynamics of these populations. The nonlinear version of this system, the Lotka-Volterra model, will be discussed in the next chapter.

2.3.6 *Love Affairs**

THE NEXT APPLICATION IS ONE THAT WAS INTRODUCED in 1988 by Strogatz as a cute system involving relationships.[1] One considers what happens to the affections that two people have for each other over time. Let R denote the affection that Romeo has for Juliet and J be the affection that Juliet has for Romeo. Positive values indicate love and negative values indicate dislike.

[1] Steven H. Strogatz introduced this problem as an interesting example of systems of differential equations in *Mathematics Magazine,* Vol. 61, No. 1 (Feb. 1988) p 35. He also describes it in his book *Nonlinear Dynamics and Chaos (1994).*

One possible model is given by

$$\begin{aligned} \frac{dR}{dt} &= bJ \\ \frac{dJ}{dt} &= cR \end{aligned} \tag{2.56}$$

with $b > 0$ and $c < 0$. In this case Romeo loves Juliet the more she likes him. But Juliet backs away when she finds his love for her increasing.

A typical system relating the combined changes in affection can be modeled as

$$\begin{aligned} \frac{dR}{dt} &= aR + bJ \\ \frac{dJ}{dt} &= cR + dJ. \end{aligned} \tag{2.57}$$

Several scenarios are possible for various choices of the constants. For example, if $a > 0$ and $b > 0$, Romeo gets more and more excited by Juliet's love for him. If $c > 0$ and $d < 0$, Juliet is being cautious about her relationship with Romeo. For specific values of the parameters and initial conditions, one can explore this match of an overly zealous lover with a cautious lover.

2.3.7 *Epidemics**

ANOTHER INTERESTING AREA OF APPLICATION of differential equation is in predicting the spread of disease. Typically, one has a population of susceptible people or animals. Several infected individuals are introduced into the population and one is interested in how the infection spreads and if the number of infected people drastically increases or dies off. Such models are typically nonlinear and we will look at what is called the SIR model in the next chapter. In this section we will model a simple linear model.

Let us break the population into three classes. First, we let $S(t)$ represent the healthy people, who are susceptible to infection. Let $I(t)$ be the number of infected people. Of these infected people, some will die from the infection and others could recover. We will consider the case that initially there is one infected person and the rest, say N, are healthy. Can we predict how many deaths have occurred by time t?

We model this problem using the compartmental analysis we had seen for mixing problems. The total rate of change of any population would be

due to those entering the group less those leaving the group. For example, the number of healthy people decreases due infection and can increase when some of the infected group recovers. Let's assume that a) the rate of infection is proportional to the number of healthy people, aS, and b) the number who recover is proportional to the number of infected people, rI. Thus, the rate of change of healthy people is found as

$$\frac{dS}{dt} = -aS + rI.$$

Let the number of deaths be $D(t)$. Then, the death rate could be taken to be proportional to the number of infected people. So,

$$\frac{dD}{dt} = dI$$

Finally, the rate of change of infected people is due to healthy people getting infected and the infected people who either recover or die. Using the corresponding terms in the other equations, we can write the rate of change of infected people as

$$\frac{dI}{dt} = aS - rI - dI.$$

This linear system of differential equations can be written in matrix form.

$$\frac{d}{dt}\begin{pmatrix} S \\ I \\ D \end{pmatrix} = \begin{pmatrix} -a & r & 0 \\ a & -d-r & 0 \\ 0 & d & 0 \end{pmatrix}\begin{pmatrix} S \\ I \\ D \end{pmatrix}. \tag{2.58}$$

The reader can find the solutions of this system and determine if this is a realistic model.

2.4 First Order Matrix Differential Equations

2.4.1 Matrix Formulation

WE HAVE INVESTIGATED SEVERAL LINEAR SYSTEMS in the plane and in the next chapter we will use some of these ideas to investigate nonlinear systems. We need a deeper insight into the solutions of planar systems. So, in this section we will recast the first order linear systems into matrix form. This will lead to a better understanding of first order systems and allow for extensions to higher dimensions and the solution of nonhomogeneous equations later in this chapter.

We start with the usual homogeneous system in Equation (2.9). Let the unknowns be represented by the vector

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Then we have that

$$\mathbf{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \equiv A\mathbf{x}.$$

Here we have introduced the *coefficient matrix* A. This is a first order vector differential equation,

$$\mathbf{x}' = A\mathbf{x}.$$

Formerly, we can write the solution as

$$\mathbf{x} = \mathbf{x}_0 e^{At}.$$

You can verify that this is a solution by simply differentiating,

$$\frac{d\mathbf{x}}{dt} = \mathbf{x}_0 \frac{d}{dt}\left(e^{At}\right) = A\mathbf{x}_0 e^{At} = A\mathbf{x}.$$

2.4.2 *Exponentiating a Matrix*

HOWEVER, THERE REMAINS THE QUESTION, "What does it mean to exponentiate a matrix?" The exponential of a matrix is defined using the Maclaurin series expansion

$$e^x = \sum_{k=0}^{\infty} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

We define

$$e^A = \sum_{k=0}^{\infty} \frac{1}{n!} A^n = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots. \tag{2.60}$$

In general it is difficult to sum this series, but it is doable for some simple examples.

The *exponential of a matrix* is defined using the Maclaurin series expansion

$$e^x = \sum_{k=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

So, we define

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots. \tag{2.59}$$

In general, it is difficult computing e^A unless A is diagonal.

Example 2.17. Evaluate e^{tA} for $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

$$\begin{aligned}
e^{tA} &= I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots. \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^2 + \frac{t^3}{3!}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^3 + \cdots \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} + \frac{t^3}{3!}\begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} + \cdots \\
&= \begin{pmatrix} 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} \cdots & 0 \\ 0 & 1 + 2t + \frac{2t^2}{2!} + \frac{8t^3}{3!} \cdots \end{pmatrix} \\
&= \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}
\end{aligned} \tag{2.61}$$

Example 2.18. Evaluate e^{tA} for $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We first note that

$$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Therefore,

$$A^n = \begin{cases} A, & n \text{ odd}, \\ I, & n \text{ even}. \end{cases}$$

Then, we have

$$\begin{aligned} e^{tA} &= I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots. \\ &= I + tA + \frac{t^2}{2!}I + \frac{t^3}{3!}A + \cdots. \\ &= \begin{pmatrix} 1 + \frac{t^2}{2!} + \frac{t^4}{4!} \cdots & t + \frac{t^3}{3!} + \frac{t^5}{5!} \cdots \\ t + \frac{t^3}{3!} + \frac{t^5}{5!} \cdots & 1 + \frac{t^2}{2!} + \frac{t^4}{4!} \cdots \end{pmatrix} \\ &= \begin{pmatrix} \cosh & \sinh t \\ \sinh t & \cosh t \end{pmatrix}. \end{aligned} \tag{2.62}$$

Since summing these infinite series might be difficult, we will now investigate the solutions of planar systems to see if we can find other approaches for solving linear systems using matrix methods. We begin by recalling the solution to the problem in Example (2.16). We obtained the solution to this system as

$$\begin{aligned} x(t) &= c_1 e^t + c_2 e^{-4t}, \\ y(t) &= \frac{1}{3}c_1 e^t - \frac{1}{2}c_2 e^{-4t}. \end{aligned} \tag{2.63}$$

This can be rewritten using matrix operations. Namely, we first write the solution in vector form.

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\ &= \begin{pmatrix} c_1 e^t + c_2 e^{-4t} \\ \frac{1}{3}c_1 e^t - \frac{1}{2}c_2 e^{-4t} \end{pmatrix} \\ &= \begin{pmatrix} c_1 e^t \\ \frac{1}{3}c_1 e^t \end{pmatrix} + \begin{pmatrix} c_2 e^{-4t} \\ -\frac{1}{2}c_2 e^{-4t} \end{pmatrix} \\ &= c_1 \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} e^{-4t}. \end{aligned} \tag{2.64}$$

We see that our solution is in the form of a linear combination of vectors of the form

$$\mathbf{x} = \mathbf{v}e^{\lambda t}$$

with $\mathbf{v}$ a constant vector and λ a constant number. This is similar to how we began to find solutions to second order constant coefficient equations. So, for the general problem (2.4.1) we insert this guess. Thus,

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \Rightarrow \\ \lambda \mathbf{v}e^{\lambda t} &= A\mathbf{v}e^{\lambda t}. \end{aligned} \tag{2.65}$$

For this to be true for all t, we have that

$$A\mathbf{v} = \lambda \mathbf{v}. \tag{2.66}$$

This is an eigenvalue problem. A is a 2×2 matrix for our problem, but could easily be generalized to a system of n first order differential equations. We will confine our remarks for now to planar systems. However, we need to recall how to solve eigenvalue problems and then see how solutions of eigenvalue problems can be used to obtain solutions to our systems of differential equations..

2.4.3 *Eigenvalue Problems*

We seek *nontrivial solutions* to the eigenvalue problem

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{2.67}$$

We note that $\mathbf{v} = \mathbf{0}$ is an obvious solution. Furthermore, it does not lead to anything useful. So, it is called a *trivial solution*. Typically, we are given the matrix A and have to determine the *eigenvalues*, λ, and the associated *eigenvectors*, $\mathbf{v}$, satisfying the above eigenvalue problem. Later in the course we will explore other types of eigenvalue problems.

For now we begin to solve the eigenvalue problem for $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Inserting this into Equation (2.67), we obtain the homogeneous algebraic system

$$\begin{aligned} (a-\lambda)v_1 + bv_2 &= 0, \\ cv_1 + (d-\lambda)v_2 &= 0. \end{aligned} \tag{2.68}$$

The solution of such a system would be unique if the determinant of the system is not zero. However, this would give the trivial solution $v_1 = 0$, $v_2 = 0$. To get a nontrivial solution, we need to force the determinant to be zero. This yields the eigenvalue equation

$$0 = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc.$$

This is a quadratic equation for the eigenvalues that would lead to nontrivial solutions. If we expand the right side of the equation, we find that

$$\lambda^2 - (a+d)\lambda + ad - bc = 0.$$

This is the same equation as the characteristic equation (2.12) for the general constant coefficient differential equation considered in the first chapter. Thus, the eigenvalues correspond to the solutions of the characteristic polynomial for the system.

Once we find the eigenvalues, then there are possibly an infinite number solutions to the algebraic system. We will see this in the examples.

So, the process is to

a) Write the coefficient matrix;

b) Find the eigenvalues from the equation $\det(A - \lambda I) = 0$; and,

c) Find the eigenvectors by solving the linear system $(A - \lambda I)\mathbf{v} = 0$ for each λ.

2.5 Solving Constant Coefficient Systems in 2D

Before proceeding to examples, we first indicate the types of solutions that could result from the solution of a homogeneous, constant coefficient system of first order differential equations.

We begin with the linear system of differential equations in matrix form.

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x} = A\mathbf{x}. \tag{2.69}$$

The type of behavior depends upon the eigenvalues of matrix A. The procedure is to determine the eigenvalues and eigenvectors and use them to construct the general solution.

If we have an initial condition, $\mathbf{x}(t_0) = \mathbf{x}_0$, we can determine the two arbitrary constants in the general solution in order to obtain the particular solution. Thus, if $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are two linearly independent solutions[2], then the general solution is given as

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t).$$

[2] Recall that linear independence means $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = \mathbf{0}$ if and only if $c_1, c_2 = 0$. The reader should derive the condition on the $\mathbf{x}_i$ for linear independence.

Then, setting $t = 0$, we get two linear equations for c_1 and c_2:

$$c_1\mathbf{x}_1(0) + c_2\mathbf{x}_2(0) = \mathbf{x}_0.$$

The major work is in finding the linearly independent solutions. This depends upon the different types of eigenvalues that one obtains from solving the eigenvalue equation, $\det(A - \lambda I) = 0$. The nature of these roots indicate the form of the general solution. In Table 2.1 we summarize the classification of solutions in terms of the eigenvalues of the coefficient matrix. We first make some general remarks about the plausibility of these solutions and then provide examples in the following section to clarify the matrix methods for our two dimensional systems.

The construction of the general solution in Case I is straight forward. However, the other two cases need a little explanation.

Let's consider Case III. Note that since the original system of equations does not have any i's, then we would expect real solutions. So, we look at the real and imaginary parts of the complex solution. We have that the complex solution satisfies the equation

$$\frac{d}{dt}\left[Re(\mathbf{y}(t)) + iIm(\mathbf{y}(t))\right] = A[Re(\mathbf{y}(t)) + iIm(\mathbf{y}(t))].$$

Differentiating the sum and splitting the real and imaginary parts of the equation, gives

$$\frac{d}{dt}Re(\mathbf{y}(t)) + i\frac{d}{dt}Im(\mathbf{y}(t)) = A[Re(\mathbf{y}(t))] + iA[Im(\mathbf{y}(t))].$$

Setting the real and imaginary parts equal, we have

$$\frac{d}{dt}Re(\mathbf{y}(t)) = A[Re(\mathbf{y}(t))],$$

and

$$\frac{d}{dt} Im(\mathbf{y}(t)) = A[Im(\mathbf{y}(t))].$$

Therefore, the real and imaginary parts each are linearly independent solutions of the system and the general solution can be written as a linear combination of these expressions.

Classification of the Solutions for Two Linear First Order Differential Equations

1. **Case I: Two real, distinct roots.**

   Solve the eigenvalue problem $A\mathbf{v} = \lambda\mathbf{v}$ for each eigenvalue obtaining two eigenvectors $\mathbf{v}_1, \mathbf{v}_2$. Then write the general solution as a linear combination $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$

2. **Case II: One Repeated Root**

   Solve the eigenvalue problem $A\mathbf{v} = \lambda\mathbf{v}$ for one eigenvalue λ, obtaining the first eigenvector $\mathbf{v}_1$. One then needs a second linearly independent solution. This is obtained by solving the nonhomogeneous problem $A\mathbf{v}_2 - \lambda\mathbf{v}_2 = \mathbf{v}_1$ for $\mathbf{v}_2$.

   The general solution is then given by $\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1)$.

3. **Case III: Two complex conjugate roots.**

   Solve the eigenvalue problem $A\mathbf{x} = \lambda\mathbf{x}$ for one eigenvalue, $\lambda = \alpha + i\beta$, obtaining one eigenvector $\mathbf{v}$. Note that this eigenvector may have complex entries. Thus, one can write the vector $\mathbf{y}(t) = e^{\lambda t}\mathbf{v} = e^{\alpha t}(\cos\beta t + i\sin\beta t)\mathbf{v}$. Now, construct two linearly independent solutions to the problem using the real and imaginary parts of $\mathbf{y}(t) : \mathbf{y}_1(t) = Re(\mathbf{y}(t))$ and $\mathbf{y}_2(t) = Im(\mathbf{y}(t))$. Then the general solution can be written as $\mathbf{x}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t)$.

Table 2.1: Solutions Types for Planar Systems with Constant Coefficients

We now turn to Case II. Writing the system of first order equations as a second order equation for $x(t)$ with the sole solution of the characteristic equation, $\lambda = \frac{1}{2}(a+d)$, we have that the general solution takes the form

$$x(t) = (c_1 + c_2 t)e^{\lambda t}.$$

This suggests that the second linearly independent solution involves a term of the form $\mathbf{v}te^{\lambda t}$. It turns out that the guess that works is

$$\mathbf{x} = te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2.$$

Inserting this guess into the system $\mathbf{x}' = A\mathbf{x}$ yields

$$\begin{aligned} (te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2)' &= A\left[te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2\right]. \\ e^{\lambda t}\mathbf{v}_1 + \lambda te^{\lambda t}\mathbf{v}_1 + \lambda e^{\lambda t}\mathbf{v}_2 &= \lambda te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}A\mathbf{v}_2. \\ e^{\lambda t}\left(\mathbf{v}_1 + \lambda\mathbf{v}_2\right) &= e^{\lambda t}A\mathbf{v}_2. \end{aligned} \tag{2.70}$$

Noting this is true for all t, we find that

$$\mathbf{v}_1 + \lambda \mathbf{v}_2 = A\mathbf{v}_2. \tag{2.71}$$

Therefore,

$$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1.$$

We know everything except for $\mathbf{v}_2$. So, we just solve for it and obtain the second linearly independent solution.

2.6 Examples of the Matrix Method

Here we will give some examples for typical systems for the three cases mentioned in the last section.

Example 2.19. $A = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$.

Eigenvalues: We first determine the eigenvalues.

$$0 = \begin{vmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix} \tag{2.72}$$

Therefore,

$$\begin{aligned} 0 &= (4-\lambda)(3-\lambda) - 6 \\ 0 &= \lambda^2 - 7\lambda + 6 \\ 0 &= (\lambda - 1)(\lambda - 6) \end{aligned} \tag{2.73}$$

The eigenvalues are then $\lambda = 1, 6$. This is an example of Case I.

Eigenvectors: Next we determine the eigenvectors associated with each of these eigenvalues. We have to solve the system $A\mathbf{v} = \lambda\mathbf{v}$ in each case.

Case $\lambda = 1$.

$$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.74}$$

$$\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.75}$$

This gives $3v_1 + 2v_2 = 0$. One possible solution yields an eigenvector of

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Case $\lambda = 6$.

$$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 6 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.76}$$

$$\begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.77}$$

For this case we need to solve $-2v_1 + 2v_2 = 0$. This yields

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

General Solution: We can now construct the general solution.

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 \\ &= c_1 e^t \begin{pmatrix} 2 \\ -3 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2c_1 e^t + c_2 e^{6t} \\ -3c_1 e^t + c_2 e^{6t} \end{pmatrix}. \end{aligned} \tag{2.78}$$

Example 2.20. $A = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$.

Eigenvalues: Again, one solves the eigenvalue equation.

$$0 = \begin{vmatrix} 3-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} \tag{2.79}$$

Therefore,

$$\begin{aligned} 0 &= (3-\lambda)(-1-\lambda) + 5 \\ 0 &= \lambda^2 - 2\lambda + 2 \\ \lambda &= \frac{-(-2) \pm \sqrt{4 - 4(1)(2)}}{2} = 1 \pm i. \end{aligned} \tag{2.80}$$

The eigenvalues are then $\lambda = 1+i, 1-i$. This is an example of Case III.

Eigenvectors: In order to find the general solution, we need only find the eigenvector associated with $1+i$.

$$\begin{aligned} \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= (1+i) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{2.81}$$

We need to solve $(2-i)v_1 - 5v_2 = 0$. Thus,

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}. \tag{2.82}$$

Complex Solution: In order to get the two real linearly independent solutions, we need to compute the real and imaginary parts of $\mathbf{v}e^{\lambda t}$.

$$e^{\lambda t} \begin{pmatrix} 2+i \\ 1 \end{pmatrix} = e^{(1+i)t} \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$$
\begin{aligned}
&= e^t(\cos t + i\sin t)\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\
&= e^t\begin{pmatrix} (2+i)(\cos t + i\sin t) \\ \cos t + i\sin t \end{pmatrix} \\
&= e^t\begin{pmatrix} (2\cos t - \sin t) + i(\cos t + 2\sin t) \\ \cos t + i\sin t \end{pmatrix} \\
&= e^t\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + ie^t\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}.
\end{aligned}
$$

General Solution: Now we can construct the general solution.

$$
\begin{aligned}
\mathbf{x}(t) &= c_1e^t\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2e^t\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} \\
&= e^t\begin{pmatrix} c_1(2\cos t - \sin t) + c_2(\cos t + 2\sin t) \\ c_1\cos t + c_2\sin t \end{pmatrix}. \qquad (2.83)
\end{aligned}
$$

Note: This can be rewritten as

$$
\mathbf{x}(t) = e^t\cos t\begin{pmatrix} 2c_1 + c_2 \\ c_1 \end{pmatrix} + e^t\sin t\begin{pmatrix} 2c_2 - c_1 \\ c_2 \end{pmatrix}.
$$

Example 2.21. $A = \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}$.

Eigenvalues:

$$
0 = \begin{vmatrix} 7-\lambda & -1 \\ 9 & 1-\lambda \end{vmatrix} \qquad (2.84)
$$

Therefore,

$$
\begin{aligned}
0 &= (7-\lambda)(1-\lambda) + 9 \\
0 &= \lambda^2 - 8\lambda + 16 \\
0 &= (\lambda - 4)^2. \qquad (2.85)
\end{aligned}
$$

There is only one real eigenvalue, $\lambda = 4$. This is an example of Case II.

Eigenvectors: In this case we first solve for $\mathbf{v}_1$ and then get the second linearly independent vector.

$$
\begin{aligned}
\begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= 4\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (2.86)
\end{aligned}
$$

Therefore, we have

$$
3v_1 - v_2 = 0, \quad \Rightarrow \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.
$$

Second Linearly Independent Solution:
Now we need to solve $A\mathbf{v}_2 - \lambda\mathbf{v}_2 = \mathbf{v}_1$.

$$\begin{aligned}\begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - 4\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix}. \end{aligned} \tag{2.87}$$

Expanding the matrix product, we obtain the system of equations

$$\begin{aligned} 3u_1 - u_2 &= 1 \\ 9u_1 - 3u_2 &= 3. \end{aligned} \tag{2.88}$$

The solution of this system is $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

General Solution: We construct the general solution as

$$\begin{aligned} \mathbf{y}(t) &= c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1). \\ &= c_1 e^{4t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{4t}\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right] \\ &= e^{4t}\begin{pmatrix} c_1 + c_2(1+t) \\ 3c_1 + c_2(2+3t) \end{pmatrix}. \end{aligned} \tag{2.89}$$

2.7 *Planar Systems - Summary*

The reader should have noted by now that there is a connection between the behavior of the solutions obtained in Section 2.2.2 and the eigenvalues found from the coefficient matrices in the previous examples. In Table 2.2 we summarize some of these cases.

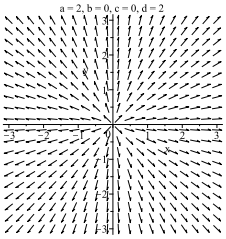
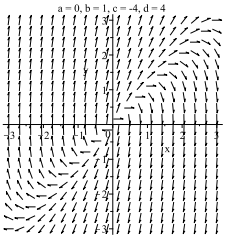
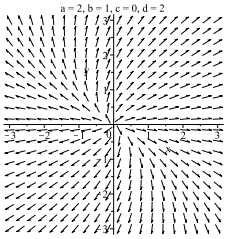
Type	Eigenvalues	Stability
Node	Real λ, same signs	$\lambda < 0$, stable $\lambda > 0$, unstable
Saddle	Real λ opposite signs	Mostly Unstable
Center	λ pure imaginary	—
Focus/Spiral	Complex λ, $\mathrm{Re}(\lambda) \neq 0$	$\mathrm{Re}(\lambda) < 0$, stable $\mathrm{Re}(\lambda) > 0$, unstable
Degenerate Node	Repeated roots,	$\lambda > 0$, stable
Lines of Equilibria	One zero eigenvalue	$\lambda < 0$, stable

Table 2.2: List of typical behaviors in planar systems.

The connection, as we have seen, is that the characteristic equation for the associated second order differential equation is the same as the eigenvalue equation of the coefficient matrix for the linear system. However, one should be a little careful in cases in which the coefficient matrix in not diagonalizable. In Table 2.3 are three examples of systems with repeated roots. The reader should look at these systems and look at the commonalities and

differences in these systems and their solutions. In these cases one has unstable nodes, though they are degenerate in that there is only one accessible eigenvector.

Table 2.3: Three examples of systems with a repeated root of $\lambda = 2$.

System 1	System 2	System 3
		
$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}$	$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} \mathbf{x}$	$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \mathbf{x}$

Another way to look at the classification of these solution is to use the determinant and trace of the coefficient matrix. Recall that the determinant and trace of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are given by $\det A = ad - bc$ and $\text{tr}A = a + d$.

We note that the general eigenvalue equation,

$$\lambda^2 - (a+d)\lambda + ad - bc = 0,$$

can be written as

$$\lambda^2 - (\text{tr}A)\lambda + \det A = 0. \tag{2.90}$$

Therefore, the eigenvalues are found from the quadratic formula as

$$\lambda_{1,2} = \frac{\text{tr}A \pm \sqrt{(\text{tr}A)^2 - 4\det A}}{2}. \tag{2.91}$$

The solution behavior then depends on the sign of discriminant,

$$(\text{tr}A)^2 - 4\det A.$$

If we consider a plot of where the discriminant vanishes, then we could plot

$$(\text{tr}A)^2 = 4\det A$$

in the $\det A \text{tr}A)$-plane. This is a parabolic cure as shown by the dashed line in Figure 2.25. The region inside the parabola have a negative discriminant, leading to complex roots. In these cases we have oscillatory solutions. If $\text{tr}A = 0$, then one has centers. If $\text{tr}A < 0$, the solutions are stable spirals; otherwise, they are unstable spirals. If the discriminant is positive, then the roots are real, leading to nodes or saddles in the regions indicated.

2.8 Theory of Homogeneous Constant Coefficient Systems

There is a general theory for solving homogeneous, constant coefficient systems of first order differential equations. We begin by once again recalling

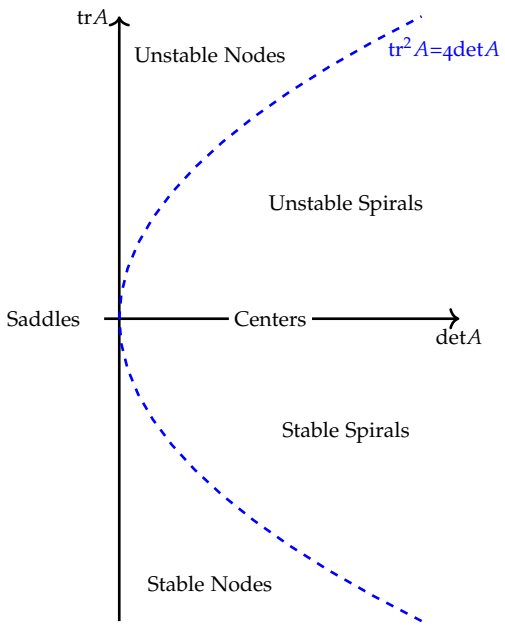


Figure 2.25: Solution Classification for Planar Systems.

the specific problem (2.16). We obtained the solution to this system as

$$\begin{aligned} x(t) &= c_1 e^t + c_2 e^{-4t}, \\ y(t) &= \frac{1}{3} c_1 e^t - \frac{1}{2} c_2 e^{-4t}. \end{aligned} \tag{2.92}$$

This time we rewrite the solution as

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} c_1 e^t + c_2 e^{-4t} \\ \frac{1}{3} c_1 e^t - \frac{1}{2} c_2 e^{-4t} \end{pmatrix} \\ &= \begin{pmatrix} e^t & e^{-4t} \\ \frac{1}{3} e^t & -\frac{1}{2} e^{-4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &\equiv \Phi(t)\mathbf{C}. \end{aligned} \tag{2.93}$$

Thus, we can write the general solution as a 2×2 matrix Φ times an arbitrary constant vector. The matrix Φ consists of two columns that are linearly independent solutions of the original system. This matrix is an example of what we will define as the *Fundamental Matrix* of solutions of the system. So, determining the Fundamental Matrix will allow us to find the general solution of the system upon multiplication by a constant matrix. In fact, we will see that it will also lead to a simple representation of the solution of the initial value problem for our system. We will outline the general theory.

Consider the homogeneous, constant coefficient system of first order differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n, \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n, \end{aligned}$$

$$\begin{aligned} &\vdots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n. \end{aligned} \tag{2.94}$$

As we have seen, this can be written in the matrix form $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Now, consider m vector solutions of this system: $\phi_1(t)$, $\phi_2(t)$,... $\phi_m(t)$. These solutions are said to be *linearly independent* on some domain if

$$c_1\phi_1(t) + c_2\phi_2(t) + \ldots + c_m\phi_m(t) = 0$$

for all t in the domain implies that $c_1 = c_2 = \ldots = c_m = 0$.

Let $\phi_1(t), \phi_2(t), \ldots \phi_n(t)$ be a set of n linearly independent set of solutions of our system, called a *fundamental set of solutions*. We construct a matrix from these solutions using these solutions as the column of that matrix. We define this matrix to be the *fundamental matrix solution*. This matrix takes the form

$$\Phi = \begin{pmatrix} \phi_1 & \ldots & \phi_n \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{pmatrix}.$$

What do we mean by a "matrix" solution? We have assumed that each ϕ_k is a solution of our system. Therefore, we have that $\phi_k' = A\phi_k$, for $k = 1, \ldots, n$. We say that Φ is a matrix solution because we can show that Φ also satisfies the matrix formulation of the system of differential equations. We can show this using the properties of matrices.

$$\begin{aligned} \frac{d}{dt}\Phi &= \begin{pmatrix} \phi_1' & \ldots & \phi_n' \end{pmatrix} \\ &= \begin{pmatrix} A\phi_1 & \ldots & A\phi_n \end{pmatrix} \\ &= A\begin{pmatrix} \phi_1 & \ldots & \phi_n \end{pmatrix} \\ &= A\Phi. \end{aligned} \tag{2.95}$$

Given a set of vector solutions of the system, when are they linearly independent? We consider a matrix solution $\Omega(t)$ of the system in which we have n vector solutions. Then, we define the *Wronskian* of $\Omega(t)$ to be

$$W = \det \Omega(t).$$

If $W(t) \neq 0$, then $\Omega(t)$ is a fundamental matrix solution.

Before continuing, we list the fundamental matrix solutions for the set of examples in the last section. (Refer to the solutions from those examples.) Furthermore, note that the fundamental matrix solutions are not unique as one can multiply any column by a nonzero constant and still have a fundamental matrix solution.

Example 2.19 $A = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$.

$$\Phi(t) = \begin{pmatrix} 2e^t & e^{6t} \\ -3e^t & e^{6t} \end{pmatrix}.$$

We should note in this case that the Wronskian is found as

$$\begin{aligned} W &= \det \Phi(t) \\ &= \begin{vmatrix} 2e^t & e^{6t} \\ -3e^t & e^{6t} \end{vmatrix} \\ &= 5e^{7t} \neq 0. \end{aligned} \tag{2.96}$$

Example 2.20 $A = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$.

$$\Phi(t) = \begin{pmatrix} e^t(2\cos t - \sin t) & e^t(\cos t + 2\sin t) \\ e^t \cos t & e^t \sin t \end{pmatrix}.$$

Example 2.21 $A = \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}$.

$$\Phi(t) = \begin{pmatrix} e^{4t} & e^{4t}(1+t) \\ 3e^{4t} & e^{4t}(2+3t) \end{pmatrix}.$$

So far we have only determined the general solution. This is done by the following steps:

Procedure for Determining the General Solution

1. Solve the eigenvalue problem $(A - \lambda I)\mathbf{v} = 0$.
2. Construct vector solutions from $\mathbf{v}e^{\lambda t}$. The method depends if one has real or complex conjugate eigenvalues.
3. Form the fundamental solution matrix $\Phi(t)$ from the vector solution.
4. The general solution is given by $\mathbf{x}(t) = \Phi(t)\mathbf{C}$ for $\mathbf{C}$ an arbitrary constant vector.

We are now ready to solve the initial value problem:

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

Starting with the general solution, we have that

$$\mathbf{x}_0 = \mathbf{x}(t_0) = \Phi(t_0)\mathbf{C}.$$

As usual, we need to solve for the c_k's. Using matrix methods, this is now easy. Since the Wronskian is not zero, then we can invert Φ at any value of t. So, we have

$$\mathbf{C} = \Phi^{-1}(t_0)\mathbf{x}_0.$$

Putting $\mathbf{C}$ back into the general solution, we obtain the solution to the initial value problem:

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0.$$

You can easily verify that this is a solution of the system and satisfies the initial condition at $t = t_0$.

The matrix combination $\Phi(t)\Phi^{-1}(t_0)$ is useful. So, we will define the resulting product to be the *principal matrix solution*, denoting it by

$$\Psi(t) = \Phi(t)\Phi^{-1}(t_0).$$

Thus, the solution of the initial value problem is $\mathbf{x}(t) = \Psi(t)\mathbf{x}_0$. Furthermore, we note that $\Psi(t)$ is a solution to the matrix initial value problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(t_0) = I,$$

where I is the $n \times n$ identity matrix.

Matrix Solution of the Homogeneous Problem

In summary, the matrix solution of

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

is given by

$$\mathbf{x}(t) = \Psi(t)\mathbf{x}_0 = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0,$$

where $\Phi(t)$ is the fundamental matrix solution and $\Psi(t)$ is the principal matrix solution.

Example 2.22. Let's consider the matrix initial value problem

$$\begin{aligned} x' &= 5x + 3y \\ y' &= -6x - 4y, \end{aligned} \tag{2.97}$$

satisfying $x(0) = 1$, $y(0) = 2$. Find the solution of this problem.

We first note that the coefficient matrix is

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}.$$

The eigenvalue equation is easily found from

$$\begin{aligned} 0 &= -(5-\lambda)(4+\lambda) + 18 \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda - 2)(\lambda + 1). \end{aligned} \tag{2.98}$$

So, the eigenvalues are $\lambda = -1, 2$. The corresponding eigenvectors are found to be

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Now we construct the fundamental matrix solution. The columns are obtained using the eigenvectors and the exponentials, $e^{\lambda t}$:

$$\phi_1(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}, \qquad \phi_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

So, the fundamental matrix solution is

$$\Phi(t) = \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix}.$$

The general solution to our problem is then

$$\mathbf{x}(t) = \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix} \mathbf{C}$$

for $\mathbf{C}$ is an arbitrary constant vector.

In order to find the particular solution of the initial value problem, we need the principal matrix solution. We first evaluate $\Phi(0)$, then we invert it:

$$\Phi(0) = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \quad \Rightarrow \quad \Phi^{-1}(0) = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}.$$

The particular solution is then

$$\begin{aligned} \mathbf{x}(t) &= \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix} \begin{pmatrix} -3 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -3e^{-t} + 4e^{2t} \\ 6e^{-t} - 4e^{2t} \end{pmatrix} \end{aligned} \tag{2.99}$$

Thus, $x(t) = -3e^{-t} + 4e^{2t}$ and $y(t) = 6e^{-t} - 4e^{2t}$.

2.9 *Nonhomogeneous Systems*

Before leaving the theory of systems of linear, constant coefficient systems, we will discuss nonhomogeneous systems. We would like to solve systems of the form

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t). \tag{2.100}$$

We will assume that we have found the fundamental matrix solution of the homogeneous equation. Furthermore, we will assume that $A(t)$ and $\mathbf{f}(t)$ are continuous on some common domain.

As with second order equations, we can look for solutions that are a sum of the general solution to the homogeneous problem plus a particular solution of the nonhomogeneous problem. Namely, we can write the general solution as

$$\mathbf{x}(t) = \Phi(t)\mathbf{C} + \mathbf{x}_p(t),$$

where $\mathbf{C}$ is an arbitrary constant vector, $\Phi(t)$ is the fundamental matrix solution of $\mathbf{x}' = A(t)\mathbf{x}$, and

$$\mathbf{x}_p' = A(t)\mathbf{x}_p + \mathbf{f}(t).$$

Such a representation is easily verified.

We need to find the particular solution, $\mathbf{x}_p(t)$. We can do this by applying *The Method of Variation of Parameters for Systems*. We consider a solution in the form of the solution of the homogeneous problem, but replace the constant vector by unknown parameter functions. Namely, we assume that

$$\mathbf{x}_p(t) = \Phi(t)\mathbf{c}(t).$$

Differentiating, we have that

$$\mathbf{x}_p' = \Phi'\mathbf{c} + \Phi\mathbf{c}' = A\Phi\mathbf{c} + \Phi\mathbf{c}',$$

or

$$\mathbf{x}_p' - A\mathbf{x}_p = \Phi\mathbf{c}'.$$

But the left side is $\mathbf{f}$. So, we have that,

$$\Phi\mathbf{c}' = \mathbf{f},$$

or, since Φ is invertible (why?),

$$\mathbf{c}' = \Phi^{-1}\mathbf{f}.$$

In principle, this can be integrated to give $\mathbf{c}$. Therefore, the particular solution can be written as

$$\mathbf{x}_p(t) = \Phi(t)\int^t \Phi^{-1}(s)\mathbf{f}(s)\,ds. \tag{2.101}$$

This is the *variation of parameters formula.*

The general solution of Equation (2.100) has been found as

$$\boxed{\mathbf{x}(t) = \Phi(t)\mathbf{C} + \Phi(t)\int^t \Phi^{-1}(s)\mathbf{f}(s)\,ds.} \tag{2.102}$$

We can use the general solution to find the particular solution of an initial value problem consisting of Equation (2.100) and the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$. This condition is satisfied for a solution of the form

$$\mathbf{x}(t) = \Phi(t)\mathbf{C} + \Phi(t)\int_{t_0}^t \Phi^{-1}(s)\mathbf{f}(s)\,ds \tag{2.103}$$

provided

$$\mathbf{x}_0 = \mathbf{x}(t_0) = \Phi(t_0)\mathbf{C}.$$

This can be solved for $\mathbf{C}$ as in the last section. Inserting the solution back into the general solution (2.103), we have

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)\,ds \tag{2.104}$$

This solution can be written a little neater in terms of the principal matrix solution, $\Psi(t) = \Phi(t)\Phi^{-1}(t_0)$:

$$\boxed{\mathbf{x}(t) = \Psi(t)\mathbf{x}_0 + \Psi(t)\int_{t_0}^{t} \Psi^{-1}(s)\mathbf{f}(s)\,ds} \tag{2.105}$$

Finally, one further simplification occurs when A is a constant matrix, which are the only types of problems we have solved in this chapter. In this case, we have that $\Psi^{-1}(t) = \Psi(-t)$. So, computing $\Psi^{-1}(t)$ is relatively easy.

Example 2.23. $x'' + x = 2\cos t$, $x(0) = 4$, $x'(0) = 0$. This example can be solved using the Method of Undetermined Coefficients. However, we will use the matrix method described in this section.

First, we write the problem in matrix form. The system can be written as

$$\begin{aligned} x' &= y \\ y' &= -x + 2\cos t. \end{aligned} \tag{2.106}$$

Thus, we have a nonhomogeneous system of the form

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 2\cos t \end{pmatrix}.$$

Next we need the fundamental matrix of solutions of the homogeneous problem. We have that

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda = \pm i$. An eigenvector associated with $\lambda = i$ is easily found as $\begin{pmatrix} 1 \\ i \end{pmatrix}$. This leads to a complex solution

$$\begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} = \begin{pmatrix} \cos t + i\sin t \\ i\cos t - \sin t \end{pmatrix}.$$

From this solution we can construct the fundamental solution matrix

$$\Phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

So, the general solution to the homogeneous problem is

$$\mathbf{x}_h = \Phi(t)\mathbf{C} = \begin{pmatrix} c_1\cos t + c_2\sin t \\ -c_1\sin t + c_2\cos t \end{pmatrix}.$$

Next we seek a particular solution to the nonhomogeneous problem. From Equation (2.103) we see that we need $\Phi^{-1}(s)\mathbf{f}(s)$. Thus, we have

$$\begin{aligned}\Phi^{-1}(s)\mathbf{f}(s) &= \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} 0 \\ 2\cos s \end{pmatrix} \\ &= \begin{pmatrix} -2\sin s \cos s \\ 2\cos^2 s \end{pmatrix}. \end{aligned} \tag{2.107}$$

We now compute

$$\begin{aligned}\Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)\,ds &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \int_{t_0}^{t} \begin{pmatrix} -2\sin s \cos s \\ 2\cos^2 s \end{pmatrix} ds \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} -\sin^2 t \\ t + \frac{1}{2}\sin(2t) \end{pmatrix} \\ &= \begin{pmatrix} t\sin t \\ \sin t + t\cos t \end{pmatrix}. \end{aligned} \tag{2.108}$$

therefore, the general solution is

$$\mathbf{x} = \begin{pmatrix} c_1\cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix} + \begin{pmatrix} t\sin t \\ \sin t + t\cos t \end{pmatrix}.$$

The solution to the initial value problem is

$$\mathbf{x} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} t\sin t \\ \sin t + t\cos t \end{pmatrix},$$

or

$$\mathbf{x} = \begin{pmatrix} 4\cos t + t\sin t \\ -3\sin t + t\cos t \end{pmatrix}.$$

2.10 *Appendix: More on Exponentiation of Matrices*

We have seen that in general that exponentiating a matrix using the Maclaurin series (2.60) can be difficult. If the matrix is diagonal, then the exponentiation is simple since powers of a diagonal matrix involves the powers of the diagonal elements. In other words, if

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \lambda_n \end{pmatrix},$$

then

$$e^A = \begin{pmatrix} e^{\lambda_1} & 0 & \cdots & \cdots & 0 \\ 0 & e^{\lambda_2} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & e^{\lambda_n} \end{pmatrix}.$$

If A is not diagonal, then if it is diagonalizable, then there is an invertible matrix P such that $P^{-1}AP = D$ is a diagonal matrix. Then, we have

$$\begin{aligned} e^A &= e^{PDP^{-1}} \\ &= \sum_{k=0}^{\infty} \frac{1}{n!}(PDP^{-1})^n \\ &= I + PDP^{-1} + \frac{PDP^{-1}PDP^{-1}}{2!} + \frac{PDP^{-1}PDP^{-1}PDP^{-1}}{3!} + \cdots \\ &= I + PDP^{-1} + \frac{PD^2P^{-1}}{2!} + \frac{PD^3P^{-1}}{3!} + \cdots \\ &= P\left(I + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \cdots\right)P^{-1} \\ &= Pe^DP^{-1}. \end{aligned} \tag{2.109}$$

Therefore, we have a prescription for exponentiating a diagonalizable matrix.

But when are matrices diagonalizable? We learn from a linear algebra course that real symmetric matrices are diagonalizable.[3] A real symmetric matrix A is one which is a matrix of real numbers such that A equals its transpose, $A^T = A$. Such matrices satisfy several properties, which we will not prove here.

- The eigenvalues of a real symmetric matrix are real.
- The eigenvectors of a real symmetric matrix corresponding to distinct eigenvalues are orthogonal.
- Every real symmetric $n \times n$ matrix posses a complete set of orthonormal vectors.
- For every real symmetric $n \times n$ matrix A there is an $n \times n$ real orthogonal matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix.

A real matrix P is orthogonal if $P^{-1} = P^T$. Therefore $P^TP = I$. Such orthogonal matrices have column (and row) vectors which are orthogonal. Thus, one can use the orthonormal eigenvectors of A to form the matrix P. Namely, we let $A\mathbf{v}_k = \lambda_k\mathbf{v}_k$ and $P = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Then,

$$\begin{aligned} P^{-1}AP &= (\mathbf{v}_1, \ldots, \mathbf{v}_n)^{-1}A(\mathbf{v}_1, \ldots, \mathbf{v}_n), \\ &= (\mathbf{v}_1, \ldots, \mathbf{v}_n)^T(\lambda_1\mathbf{v}_1, \ldots, \lambda_n\mathbf{v}_n), \\ &= \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{pmatrix} \begin{pmatrix} \lambda_1\mathbf{v}_1 & \ldots & \lambda_n\mathbf{v}_n \end{pmatrix} \end{aligned}$$

[3] This is a simple example of a more general theory where real symmetric matrices are replaced by complex Hermitian matrices which are self-adjoint and orthogonal matrices are replaced with unitary matrices. As matrices are representations of operators, the more general theory of self-adjoint operators are discussed later.

$$= \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \lambda_n \end{pmatrix}. \tag{2.110}$$

Thus, we have shown that $P^{-1}AP$ is a diagonal matrix and this is called a similarity transformation.

Similarity transformation.

Example 2.24. Consider the matrix $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$. Find e^{tA} using the diagonalization of A.

First, we solve the eigenvalue problem. The eigenvalue equation is given by $0 = |A - \lambda I| = (1-\lambda)^2 - 9$. So, $\lambda = 4, -2$.

Next, we seek the eigenvectors. Letting $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)^T$, we have

$$(1-\lambda)\mathbf{v}_1 + 3\mathbf{v}_2 = 0.$$

For $\lambda = 4$, we need to solve $(-3)\mathbf{v}_1 + 3\mathbf{v}_2 = 0$, or $\mathbf{v}_1 = \mathbf{v}_2$. For $\lambda = -2$, we need to solve $(3)\mathbf{v}_1 + 3\mathbf{v}_2 = 0$, or $\mathbf{v}_1 = -\mathbf{v}_2$. Therefore, the eigenvalues and normalized eigenvectors are given by

$$\lambda = 4, \quad \mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \lambda = -2, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

This gives[4]

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

[4] Since P is orthogonal, it is a rotation matrix. In two dimensions a rotation of coordinate axes about the origin by angle θ is given by

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Therefore, in the example $\theta = -\frac{\pi}{4}$.

Then, we have

$$\begin{aligned} P^{-1}AP &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 4 & -2 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 8 & 0 \\ 0 & -4 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}. \end{aligned} \tag{2.111}$$

Finally, we can exponentiate tA as

$$\begin{aligned} e^{tA} &= e^{tPDP^{-1}} = Pe^{tD}P^{-1} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{4t} & e^{4t} \\ -e^{-2t} & e^{-2t} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} e^{4t}+e^{-2t} & e^{4t}-e^{-2t} \\ e^{4t}-e^{-2t} & e^{4t}+e^{-2t} \end{pmatrix}. \end{aligned} \tag{2.112}$$

As a word of caution, we note that real symmetric matrices can be diagonalized. what about real nonsymmetric matrices? A real matrix can be transformed into a diagonal matrix through a similarity transformation as long as it has a full set of linearly independent eigenvectors. We need only look at Example 2.22. In that case

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}.$$

The eigenvalues were found as $\lambda = -1, 2$ and the corresponding eigenvectors were

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This gives

$$P = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}.$$

Then, we have

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & -2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}. \end{aligned} \tag{2.113}$$

So, we have a similarity transformation, but note that P is not an orthogonal matrix in this case.

We can use this diagonalization to compute e^{tA} as we show in the next example.

Example 2.25. Find e^{tA} for $A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$.

$$\begin{aligned} e^{tA} &= e^{tPDP^{-1}} = Pe^{tD}P^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} -e^{-t} & -e^{-t} \\ 2e^{2t} & e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} -e^{-t} + 2e^{2t} & -e^{-t} + 2e^{2t} \\ 2e^{-t} - 2e^{2t} & 2e^{-t} - e^{2t} \end{pmatrix}. \end{aligned} \tag{2.114}$$

From Example 2.22, the fundamental matrix solution is given as

$$\Phi(t) = \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix}$$

and

$$\Phi^{-1}(0) = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}.$$

So, the principal matrix solution is found as

$$\begin{aligned}\Phi(t) &= \Phi(t)\Phi^{-1}(0) \\ &= \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -e^{-t} + 2e^{2t} & -e^{-t} + 2e^{2t} \\ 2e^{-t} - 2e^{2t} & 2e^{-t} - e^{2t} \end{pmatrix}, \end{aligned} \tag{2.115}$$

which is the same as our evaluation of e^{tA} in the previous example.

There are other methods for exponentiating a matrix. These involve more sophisticated matrix methods and rely on the Cayley-Hamilton Theorem.

The Cayley-Hamilton Theorem Matrix A satisfies its own characteristic (eigenvalue) equation. This means that any power of A, and therefore e^A, can be expressed as a polynomial of A of degree $n-1$. We can also use this to evaluate A^{-1}. We demonstrate this by example.

Example 2.26. First, we note that

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} 7 & 3 \\ -6 & -2 \end{pmatrix}.$$

Inserting these into the characteristic equation [see Example 2.22], we have

$$A^2 - A - 2I = \begin{pmatrix} 7 & 3 \\ -6 & -2 \end{pmatrix} - \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0.$$

Now, we rearrange the equation as $A(A-I) = 2I$ and find that $A^{-1} = \frac{1}{2}(A-I)$. Thus,

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 4 & 3 \\ -6 & -5 \end{pmatrix} = \begin{pmatrix} 2 & \frac{3}{2} \\ -3 & -\frac{5}{2} \end{pmatrix}.$$

This agrees with the usual method for computing A^{-1}.

We can use the Cayley Hamilton Theorem to evaluate functions of A. Let $f(A)$ be a matrix polynomial of A of degree n Also, let $d(\lambda) = \det(A - \lambda I)$. If the characteristic equation for A is $d(\lambda) = 0$, then we know by the Cayley-Hamilton Theorem that $d(A) = 0$. Given polynomials $f(x)$ and $d(x)$, we know that we can write

$$f(x) = d(x)q(x) + r(x),$$

where $r(x)$ is a polynomial of degree $n-1$. So, we also have $f(A) = d(A)q(A) + r(A) = r(A)$. Therefore, $f(A)$ is a polynomial in A of degree $n-1$. So, we can write $f(A) = \alpha_{n-1}A^{n-1} + \alpha_{n-1}A^{n-1} + \ldots \alpha_0 I$.

Example 2.27. Let $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$. Find A^{10}.

The eigenvalues are $\lambda = 5, 2$. So, for $f(A) = A^{10}$, then $f(\lambda) = \lambda^{10}$ and $r(A)$ is a polynomial of degree 1. So, we have

$$A^{10} = r(A) = \alpha_1 A + \alpha_0 I.$$

Evaluating $f(\lambda_i) = r(\lambda_i)$, we obtain two equations for the two unknowns,

$$\begin{aligned} 5^{10} &= 5\alpha_1 + \alpha_0 \\ 2^{10} &= 2\alpha_1 + \alpha_0. \end{aligned} \tag{2.116}$$

Solving, we find

$$\alpha_1 = 3254867, \quad \alpha_0 = -6508710.$$

Then,

$$A^{10} = \begin{pmatrix} 6510758 & 3254867 \\ 6509734 & 3255891 \end{pmatrix}.$$

These methods can be generalized to evaluate arbitrary functions. As an example, we can exponentiate a matrix

Example 2.28. Find e^{tA} for $A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$.

This is Example 2.25. However, in this case we let

$$e^{tA} = \alpha_1 A + \alpha_0 I.$$

The eigenvalues are $\lambda = -1, 2$. So, for $f(A) = e^{tA}$, then $f(\lambda) = e^{t\lambda}$. We have $r(A) = \alpha_1 A + \alpha_0 I$ and $r(\lambda) = \alpha_1 \lambda + \alpha_0$. Substituting the eigenvalues, we obtain

$$\begin{aligned} e^{-t} &= -\alpha_1 + \alpha_0 \\ e^{2t} &= 2\alpha_1 + \alpha_0. \end{aligned} \tag{2.117}$$

Solving, we have

$$\alpha_0 = \frac{1}{3}\left(2e^{-t} + e^{2t}\right), \alpha_1 = \frac{1}{3}\left(e^{2t} - e^{-t}\right).$$

Then,

$$e^{tA} = \alpha_1 A + \alpha_0 I = \begin{pmatrix} -e^{-t} + 2e^{2t} & -e^{-t} + 2e^{2t} \\ 2e^{-t} - 2e^{2t} & 2e^{-t} - e^{2t} \end{pmatrix}.$$

Again, we arrive at the same result.

The Putzer Algorithm. Suppose that A is an $n \times n$ matrix with eigenvalues λ_k, $k = 1, \ldots, n$, not necessarily distinct. Then,

$$e^{tA} = \sum_{j=0}^{n-1} r_{j+1}(t) P_j$$

where

$$P_j = \prod_{k=1}^{j} (A - \lambda_k I)$$

and the $r_j(t)$'s are the components of the solutions of

$$\mathbf{p}'(t) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 1 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 1 & \lambda_n \end{pmatrix} \mathbf{p}(t)$$

satisfying $\mathbf{p}(0) = (1, 0, \ldots, 0)^T$.

Example 2.29. Let's try a different example. Consider $A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$. Find e^{tA} using the Putzer Algorithm.

Here $n = 2$ and the eigenvalues are $\lambda = -1, -3$. First we find the $P_j's$.

$$P_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_1 = A + I = \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix}.$$

Next, we solve the tridiagonal system

$$\begin{pmatrix} r_1'(t) \\ r_2'(t) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} r_1(t) \\ r_2(t) \end{pmatrix}$$

with $r_1(0) = 1, \quad r_2(0) = 0$. Expanding into a coupled system of differential equations,

$$\begin{aligned} r_1' &= -r_1, \quad r_1(0) = 1, \\ r_2' &= r_1 - 3r_2, \quad r_2(0) = 0. \end{aligned} \tag{2.118}$$

The first equation is readily solved to find $r_1(t) = e^{-t}$. The second equation becomes

$$r_2' + 3r_2 = e^{-t}, \quad r_2(0) = 0.$$

This is solved using and integrating factor. The result is $r_2(t) = \frac{1}{2}\left(e^{-t}e^{-3t}\right)$.

Now that we have the pieces for the algorithm, we can construct the answer.

$$\begin{aligned} e^{tA} &= \sum_{j=0}^{n-1} r_{j+1}(t) P_j \\ &= r_1(t)P_0 + r_2(t)P_1 \\ &= e^{-t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\left(e^{-t}e^{-3t}\right)\begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} \\ &= \begin{pmatrix} 2e^{-t} - e^{-3t} & -e^{-t}2e^{-3t} \\ 2e^{-t} - 2e^{-3t} & -e^{-t} + 2e^{-3t} \end{pmatrix}. \end{aligned} \tag{2.119}$$

Problems

1. Consider the system

$$\begin{aligned} x' &= -4x - y \\ y' &= x - 2y. \end{aligned}$$

a. Determine the second order differential equation satisfied by $x(t)$.

b. Solve the differential equation for $x(t)$.

c. Using this solution, find $y(t)$.

d. Verify your solutions for $x(t)$ and $y(t)$.

e. Find a particular solution to the system given the initial conditions $x(0) = 1$ and $y(0) = 0$.

2. Consider the following systems. Determine the families of orbits for each system and sketch several orbits in the phase plane and classify them by their type (stable node, etc.)

a.

$$\begin{aligned} x' &= 3x \\ y' &= -2y. \end{aligned}$$

b.

$$\begin{aligned} x' &= -y \\ y' &= -5x. \end{aligned}$$

c.

$$\begin{aligned} x' &= 2y \\ y' &= -3x. \end{aligned}$$

d.

$$\begin{aligned} x' &= x - y \\ y' &= y. \end{aligned}$$

e.

$$\begin{aligned} x' &= 2x + 3y \\ y' &= -3x + 2y. \end{aligned}$$

3. Use the transformations relating polar and Cartesian coordinates to prove that

$$\frac{d\theta}{dt} = \frac{1}{r^2}\left[x\frac{dy}{dt} - y\frac{dx}{dt}\right].$$

4. Consider the system of equations in Example 2.13.

a. Derive the polar form of the system.

b. Solve the radial equation, $r' = r(1 - r^2)$, for the initial values $r(0) = 0, 0.5, 1.0, 2.0$.

c. Based upon these solutions, plot and describe the behavior of all solutions to the original system in Cartesian coordinates.

5. Consider the following systems. For each system determine the coefficient matrix. When possible, solve the eigenvalue problem for each matrix and use the eigenvalues and eigenfunctions to provide solutions to the given systems. Finally, in the common cases which you investigated in Problem 2, make comparisons with your previous answers, such as what type of eigenvalues correspond to stable nodes.

a.

$$\begin{aligned} x' &= 3x - y \\ y' &= 2x - 2y. \end{aligned}$$

b.

$$\begin{aligned} x' &= -y \\ y' &= -5x. \end{aligned}$$

c.

$$\begin{aligned} x' &= x - y \\ y' &= y. \end{aligned}$$

d.

$$\begin{aligned} x' &= 2x + 3y \\ y' &= -3x + 2y. \end{aligned}$$

e.

$$\begin{aligned} x' &= -4x - y \\ y' &= x - 2y. \end{aligned}$$

f.

$$\begin{aligned} x' &= x - y \\ y' &= x + y. \end{aligned}$$

6. For the given matrix, evaluate e^{tA}, using the definition

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = I + tA + \frac{t^2}{2} A^2 + \frac{t^3}{3!} A^3 + \ldots,$$

and simplifying.

a. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

b. $A = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$.

c. $A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$.

d. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

e. $A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

f. $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

7. Find the fundamental matrix solution for the system $\mathbf{x}' = A\mathbf{x}$ where matrix A is given. If an initial condition is provided, find the solution of the initial value problem using the principal matrix.

a. $A = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$.

b. $A = \begin{pmatrix} 12 & -15 \\ 4 & -4 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

c. $A = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}$.

d. $A = \begin{pmatrix} 4 & -13 \\ 2 & -6 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

e. $A = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$.

f. $A = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}$.

g. $A = \begin{pmatrix} 8 & -5 \\ 16 & 8 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

h. $A = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}$.

i. $A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$.

8. Solve the following initial value problems using Equation (2.105), the solution of a nonhomogeneous system using the principal matrix solution.

a. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}, \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

b. $\mathbf{x}' = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ e^t \end{pmatrix}, \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

c. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

9. Add a third spring connected to mass two in the coupled system shown in Figure 2.2 to a wall on the far right. Assume that the masses are the same and the springs are the same.

a. Model this system with a set of first order differential equations.

b. If the masses are all 2.0 kg and the spring constants are all 10.0 N/m, then find the general solution for the system.

c. Move mass one to the left (of equilibrium) 10.0 cm and mass two to the right 5.0 cm. Let them go. find the solution and plot it as a function of time. Where is each mass at 5.0 seconds?

d. Model this initial value problem with a set of two second order differential equations. Set up the system in the form $M\ddot{\mathbf{x}} = -K\mathbf{x}$ and solve using the values in part b.

10. In Example 2.14 we investigated a couple mass-spring system as a pair of second order differential equations.

a. In that problem we used $\sqrt{\frac{3\pm\sqrt{5}}{2}} = \frac{\sqrt{5}\pm 1}{2}$. Prove this result.

b. Rewrite the system as a system of four first order equations.

c. Find the eigenvalues and eigenfunctions for the system of equations in part b to arrive at the solution found in Example 2.14.

d. Let $k = 5.00$ N/m and $m = 0.250$ kg. Assume that the masses are initially at rest and plot the positions as a function of time if initially i) $x_1(0) = x_2(0) = 10.0$ cm and i) $x_1(0) = -x_2(0) = 10.0$ cm. Describe the resulting motion.

11. Consider the series circuit in Figure 1.6 with $L = 1.00$ H, $R = 1.00 \times 10^2$ Ω, $C = 1.00 \times 10^{-4}$ F, and $V_0 = 1.00 \times 10^3$ V.

a. Set up the problem as a system of two first order differential equations for the charge and the current.

b. Suppose that no charge is present and no current is flowing at time $t = 0$ when V_0 is applied. Find the current and the charge on the capacitor as functions of time.

c. Plot your solutions and describe how the system behaves over time.

12. Consider the series circuit in Figure 2.21 with $L = 1.00$ H, $R_1 = R_2 = 1.00 \times 10^2\ \Omega$, $C = 1.00 \times 10^{-4}$ F, and $V_0 = 1.00 \times 10^3$ V.

a. Set up the problem as a system of first order differential equations for the charges and the currents in each loop.

b. Suppose that no charge is present and no current is flowing at time $t = 0$ when V_0 is applied. Find the current and the charge on the capacitor as functions of time.

c. Plot your solutions and describe how the system behaves over time.

13. Initially a 100 gallon tank is filled with pure water. At time $t = 0$ water with a half a pound of salt per two gallons is added to the container at the rate of 3 gallons per minute, and the well-stirred mixture is drained from the container at the same rate.

a. Find the number of pounds of salt in the container as a function of time.

b. How many minutes does it take for the concentration to reach 2 pounds per gallon?

c. What does the concentration in the container approach for large values of time? Does this agree with your intuition?

14. You make two quarts of salsa for a party. The recipe calls for five teaspoons of lime juice per quart, but you had accidentally put in five tablespoons per quart. You decide to feed your guests the salsa anyway. Assume that the guests take a quarter cup of salsa per minute and that you replace what was taken with chopped tomatoes and onions without any lime juice. [1 quart = 4 cups and 1 Tb = 3 tsp.]

a. Write down the differential equation and initial condition for the amount of lime juice as a function of time in this mixture-type problem.

b. Solve this initial value problem.

c. How long will it take to get the salsa back to the recipe's suggested concentration?

15. Consider the chemical reaction leading to the system in (2.54). Let the rate constants be $k_1 = 0.20\ \text{ms}^{-1}$, $k_2 = 0.05\ \text{ms}^{-1}$, and $k_3 = 0.10\ \text{ms}^{-1}$. What do the eigenvalues of the coefficient matrix say about the behavior of the system? Find the solution of the system assuming $[A](0) = A_0 = 1.0$ μmol, $[B](0) = 0$, and $[C](0) = 0$. Plot the solutions for $t = 0.0$ to 50.0 ms and describe what is happening over this time.

16. Find and classify any equilibrium points in the Romeo and Juliet problem for the following cases. Solve the systems and describe their affections as a function of time.

a. $a = 0$, $b = 2$, $c = -1$, $d = 0$, $R(0) = 1$, $J(0) = 1$.

b. $a = 0$, $b = 2$, $c = 1$, $d = 0$, $R(0) = 1$, $J(0) = 1$.

c. $a = -1$, $b = 2$, $c = -1$, $d = 0$, $R(0) = 1$, $J(0) = 1$.

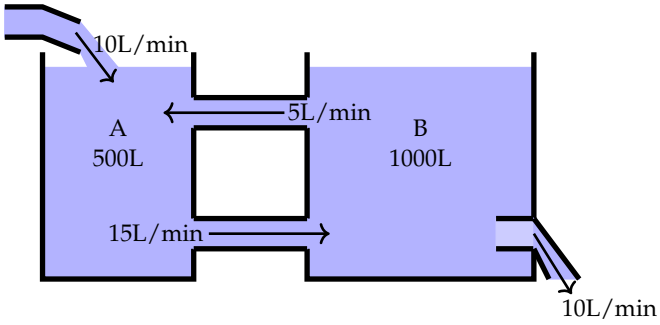


Figure 2.26: Figure for Problem 17.

17. Two tanks contain a mixture of water and alcohol with tank A containing 500 L and tank B 1000L. Initially, the concentration of alcohol in Tank A is 0% and that of tank B is 80%. Solution leaves tank A into B at a rate of 15 liter/min and the solution in tank B returns to A at a rate of 5 L/min while well mixed solution also leaves the system at 10 liter/min through an outlet. A mixture of water and alcohol enters tank A at the rate of 10 liter/min with the concentration of 10% through an inlet. What will be the concentration of the alcohol of the solution in each tank after 10 mins?

18. Consider the tank system in Problem 17. Add a third tank (C) to tank B with a volume of 300 L. Connect C with 8 L/min from tank B and 2 L/min flow back. Let 10 L/min flow out of the system. If the initial concentration is 10% in each tank and a mixture of water and alcohol enters tank A at the rate of 10 liter/min with the concentration of 20% through an inlet, what will be the concentration of the alcohol in each of the tanks after an hour?

19. Consider the epidemic model leading to the system in (2.58). Choose the constants as $a = 2.0$ days^{-1}, $d = 3.0$ days^{-1}, and $r = 1.0$ days^{-1}. What are the eigenvalues of the coefficient matrix? Find the solution of the system assuming an initial population of $1,000$ and one infected individual. Plot the solutions for $t = 0.0$ to 5.0 days and describe what is happening over this time. Is this model realistic?

Chapter 3

Nonlinear Systems

"The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful." - Jules Henri Poincaré (1854-1912)

3.1 Introduction

Some of the most interesting phenomena in the world are modeled by nonlinear systems. These systems can be modeled by differential equations when time is considered as a continuous variable or difference equations when time is treated in discrete steps. Applications involving differential equations can be found in many physical systems such as planetary systems, weather prediction, electrical circuits, and kinetics. Even in some simple dynamical systems a combination of damping and a driving force can lead to chaotic behavior. Namely, small changes in initial conditions could lead to very different outcomes. In this chapter we will explore a few different nonlinear systems and introduce some of the tools needed to investigate them. These tools are based on some of the material in Chapters 2 and 3 for linear systems of differential equations.

Nonlinear differential equations are either integrable, but difficult to solve, or they are not integrable and can only be solved numerically. We will see that we can sometimes approximate the solutions of nonlinear systems with linear systems in small regions of phase space and determine the qualitative behavior of the system without knowledge of the exact solution.

Nonlinear problems occur naturally. We will see problems from many of the same fields we explored in Section 2.3. We will concentrate mainly on continuous dynamical systems. We will begin with a simple population model and look at the behavior of equilibrium solutions of first order autonomous differential equations. We will then look at nonlinear systems in the plane, such as the nonlinear pendulum and other nonlinear oscillations. We will conclude by discussing a few other interesting physical examples stressing some of the key ideas of nonlinear dynamics.

3.2 *The Logistic Equation*

IN THIS SECTION WE WILL EXPLORE a simple nonlinear population model. Typically, we want to model the growth of a given population, $y(t)$, and the differential equation governing the growth behavior of this population is developed in a manner similar to that used previously for mixing problems. Namely, we note that the rate of change of the population is given by an equation of the form

$$\frac{dy}{dt} = \text{Rate In} \ - \ \text{Rate Out}.$$

The *Rate In* could be due to the number of births per unit time and the *Rate Out* by the number of deaths per unit time. While there are other potential contributions to these rates we will consider the birth and death rates in the simplest examples.

A simple population model can be obtained if one assumes that these rates are linear in the population. Thus, we assume that the

$$\text{Rate In} \ = by \text{ and the } \ \text{Rate Out} \ = my.$$

Here we have denoted the birth rate as b and the mortality rate as m. This gives the rate of change of population as

$$\frac{dy}{dt} = by - my. \tag{3.1}$$

Generally, these rates could depend on the time. In the case that they are both constant rates, we can define $k = b - m$ and obtain the familiar exponential model of population growth:

Malthusian population growth.

$$\frac{dy}{dt} = ky.$$

This is easily solved and one obtains exponential growth $(k > 0)$ or decay $(k < 0)$. This Malthusian growth model has been named after Thomas Robert Malthus (1766-1834), a clergyman who used this model to warn of the impending doom of the human race if its reproductive practices continued.

When populations get large enough, there is competition for resources, such as space and food, which can lead to a higher mortality rate. Thus, the mortality rate may be a function of the population size, $m = m(y)$. The simplest model would be a linear dependence, $m = \tilde{m} + cy$. Then, the previous exponential model takes the form

$$\frac{dy}{dt} = ky - cy^2, \tag{3.2}$$

where $k = b - \tilde{m}$. This is known as the *logistic model* of population growth. Typically, c is small and the added nonlinear term does not really kick in until the population gets large enough.

The logistic model was first published in 1838 by Pierre François Verhulst (1804-1849) in the form

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right),$$

where N is the population at time t, r is the growth rate, and K is what is called the carrying capacity. Note that in our model $r = k = Kc$.

Example 3.1. Show that Equation (3.2) can be written in the form

$$z' = kz(1-z)$$

which has only one parameter.

We carry this out be rescaling the population, $y(t) = \alpha z(t)$, where α is to be determined. Inserting this transformation, we have

$$\begin{aligned} y' &= ky - cy^2 \\ \alpha z' &= \alpha kz - c\alpha^2 z^2, \end{aligned}$$

or

$$z' = kz\left(1 - \alpha\frac{c}{k}z\right).$$

Thus, we obtain the result, $z' = kz(1-z)$, if we pick $\alpha = \frac{k}{c}$.

Before we obtain the exact solution, it is instructive to study the qualitative behavior of the solutions without actually writing down any explicit solutions. Such methods are useful for more difficult nonlinear equations as we will see later in this chapter.

We will demonstrate this analysis with a simple logistic equation example. We will first look for constant solutions, called equilibrium solutions, satisfying $y'(t) = 0$. Then, we will look at the behavior of solutions near the equilibrium solutions, or fixed points, and determine the stability of the equilibrium solutions. In the next section we will extend these ideas to other first order differential equations.

Example 3.2. Find and classify the equilibrium solutions of the logistic equation,

$$\frac{dy}{dt} = y - y^2. \tag{3.3}$$

First, we determine the equilibrium, or constant, solutions given by $y' = 0$. For this case, we have $y - y^2 = 0$. So, the equilibrium solutions are $y = 0$ and $y = 1$.

These solutions divide the ty-plane into three regions, $y < 0$, $0 < y < 1$, and $y > 1$. Solutions that originate in one of these regions at $t = t_0$ will remain in that region for all $t > t_0$ since solutions of this differential equation cannot intersect.

Next, we determine the behavior of solutions in the three regions. Noting that $y'(t)$ gives the slope of any solution in the plane, then we find that the solutions are monotonic in each region. Namely, in regions where $y'(t) > 0$, we have monotonically increasing functions and in regions where $y'(t) < 0$, we have monotonically decreasing functions. We determine the sign of $y'(t)$ from the right-hand side of the differential equation.

For example, in this problem $y - y^2 > 0$ only for the middle region and $y - y^2 < 0$ for the other two regions. Thus, the slope is positive in the middle region, giving a rising solution as shown in Figure 3.1. Note that this solution does not cross the equilibrium solutions. Similar statements can be made about the solutions in the other regions.

Note: If two solutions of the differential equation intersect then they have common values y_1 at time t_1. Using this information, we could set up an initial value problem for which the initial condition is $y(t_1) = y_1$. Since the two different solutions intersect at this point in the phase plane, we would have an initial value problem with two different solutions. This would violate the uniqueness theorem for initial value problems.

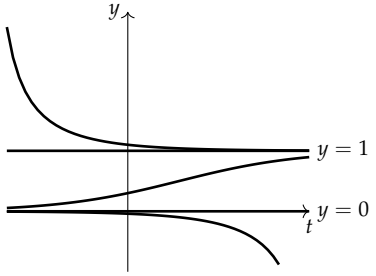


Figure 3.1: Representative solution behavior for $y' = y - y^2$.

Stable and unstable equilibria.

We further note that the solutions on either side of the equilibrium solution $y = 1$ tend to approach this equilibrium solution for large values of t. In fact, no matter how far these solutions are from $y = 1$, as long as $y(t) > 0$, the solutions will eventually approach this equilibrium solution as $t \to \infty$. We then say that the equilibrium solution, $y = 1$, is a *stable equilibrium*.

Similarly, we note that the solutions on either side of the equilibrium solution $y = 0$ tend away from $y = 0$ for large values of t. No matter how close a solution is to $y = 0$ at some given time, eventually these solutions will diverge as $t \to \infty$. We say that such equilibrium solutions are *unstable equilibria*.

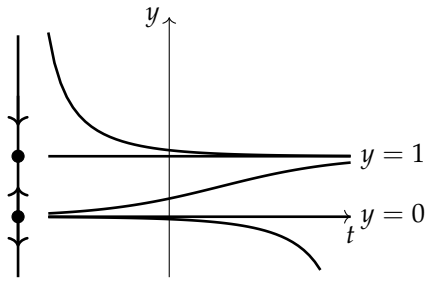


Figure 3.2: Representative solution behavior and the phase line for $y' = y - y^2$.

Phase lines.

If we are only interested in the behavior of the equilibrium solutions, we could just display a *phase line*. In Figure 3.2 we place a vertical line to the right of the ty-plane plot. On this line we first place dots at the corresponding equilibrium solutions and label the solutions. These points divide the phase line into three intervals.

In each interval we then place arrows pointing upward or downward indicating solutions with positive or negative slopes, respectively. For example, for the interval $y > 1$ there is a downward pointing arrow indicating that the slope is negative in that region.

Looking at the resulting phase line we can determine if a given equilibrium is stable (arrows pointing towards the point) or unstable (arrows pointing away from the point). In Figure 3.3 we draw the final phase line by itself. We see that $y = 1$ is a stable equilibrium point and $y = 0$ is an unstable equilibrium point.



Figure 3.3: Phase line for $y' = y - y^2$.

3.2.1 *The Riccati Equation**

We have seen that one does not need an explicit solution of the logistic equation (3.2) in order to study the behavior of its solutions. However, the logistic equation is an example of a nonlinear first order equation that is solvable. It is also an example of a general Riccati equation, a first order differential equation quadratic in the unknown function.

The Riccati equation is named after the Italian mathematician Jacopo Francesco Riccati (1676-1754). When $a(t) = 0$, the equation becomes a Bernoulli equation.

The general form of the *Riccati equation* is

$$\frac{dy}{dt} = a(t) + b(t)y + c(t)y^2. \tag{3.4}$$

As long as $c(t) \neq 0$, this equation can be reduced to a second order linear differential equation through the transformation

$$y(t) = -\frac{1}{c(t)}\frac{x'(t)}{x(t)}.$$

We will demonstrate the use of this transformation in obtaining the solution of the logistic equation.

Example 3.3. Solve the logistic equation

$$\frac{dy}{dt} = ky - cy^2 \tag{3.5}$$

using the transformation

$$y = \frac{1}{c}\frac{x'}{x}.$$

differentiating this transformation with respect to t, we obtain

$$\begin{aligned} \frac{dy}{dt} &= \frac{1}{c}\left[\frac{x''}{x} - \left(\frac{x'}{x}\right)^2\right] \\ &= \frac{1}{c}\left[\frac{x''}{x} - (cy)^2\right] \\ &= \frac{1}{c}\frac{x''}{x} - cy^2. \end{aligned} \tag{3.6}$$

Inserting this result into the logistic equation (3.5), we have

$$\frac{1}{c}\frac{x''}{x} - cy^2 = k\frac{1}{c}\left(\frac{x'}{x}\right) - cy^2.$$

Simplifying, we see that the logistic equation has been reduced to a second order linear, differential equation,

$$x'' = kx'.$$

This equation is readily solved. One integration gives

$$x'(t) = Be^{kt}.$$

A second integration gives

$$x(t) = A + Be^{kt},$$

where A and B are two arbitrary constants.

Inserting this result into the Riccati transformation, we obtain

$$y(t) = \frac{1}{c}\frac{x'}{x} = \frac{kBe^{kt}}{c(A + Be^{kt})}.$$

It appears that we have two arbitrary constants. However, we started out with a first order differential equation and so we expect only one arbitrary constant. We can resolve this dilemma by dividing[1] the numerator and denominator by Be^{kt} and defining $C = \frac{A}{B}$. Then, we have the solution

$$y(t) = \frac{k/c}{1 + Ce^{-kt}}, \tag{3.7}$$

showing that there really is only one arbitrary constant in the solution.

[1] This general solution holds for $B \neq 0$. If $B = 0$, then we have $x(t) = A$ and, thus, $y(t)$ is the constant equilibrium solution.

Plots of the solution (3.7) of the logistic equation for different initial conditions gives the solutions seen in the last section. In particular, setting all of the constants to unity, we have the sigmoid function,

$$y(t) = \frac{1}{1+e^{-t}}.$$

This is the signature *S*-shaped curve of the logistic model as shown in Figure 3.4. We should note that this is not the only way to obtain the solution to the logistic equation, though this approach has provided us with an introduction to Riccati equations. A more direct approach would be to use separation of variables on the logistic equation, which is Problem 1.

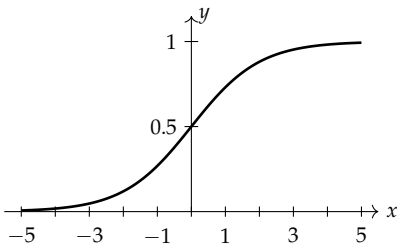


Figure 3.4: Plot of the sigmoid function.

3.3 Autonomous First Order Equations

In this section we will study the stability of nonlinear first order autonomous equations. We will then extend this study in the next section to looking at families of first order equations which are connected through a parameter.

Recall that a first order autonomous equation is given in the form

$$\frac{dy}{dt} = f(y). \tag{3.8}$$

We will assume that f and $\frac{\partial f}{\partial y}$ are continuous functions of y, so that we know that solutions of initial value problems exist and are unique.

A solution $y(t)$ of Equation (3.8) is called an *equilibrium solution*, or a *fixed point* solution, if it is a constant solution satisfying $y'(t) = 0$. Such solutions are the roots of the right-hand side of the differential equation, $f(y) = 0$.

Example 3.4. Find the equilibrium solutions of $y' = 1 - y^2$.

The equilibrium solutions are the roots of $f(y) = 1 - y^2 = 0$. The equilibria are found to be $y = \pm 1$.

Once we have determined the equilibrium solutions, we would like to classify them. Are they stable or unstable? As we had seen previously, we are interested in the behavior of solutions near the equilibria. This classification can be determined using a linearization of the given equation. This will provide an analytic criteria to establish the stability of equilibrium solutions without geometrically drawing the phase lines as we had done previously.

Linearization of first order equations.

Let y^* be an equilibrium solution of Equation (3.8). Then, any solution can be written in the form

$$y(t) = y^* + \xi(t),$$

where $\xi(t)$ measures how far the solution is from the equilibrium at any given time.

Inserting this form into Equation (3.8), we have

$$\frac{d\xi}{dt} = f(y^* + \xi).$$

We now consider small $\xi(t)$ in order to study solutions near the equilibrium solution. For such solutions, we can expand $f(y)$ about the equilibrium solution,

$$f(y^* + \xi) = f(y^*) + f'(y^*)\xi + \frac{1}{2!}f''(y^*)\xi^2 + \cdots.$$

Since y^* is an equilibrium solution, $f(y^*) = 0$, the first term in the Taylor series vanishes. If the first derivative does not vanish, then for solutions close to equilibrium, we can neglect higher order terms in the expansion. Then, $\xi(t)$ approximately satisfies the differential equation

$$\frac{d\xi}{dt} = f'(y^*)\xi. \tag{3.9}$$

This is called a linearization of the original nonlinear equation about the equilibrium point. This equation has exponential solutions for $f'(y^*) \neq 0$,

$$\xi(t) = \xi_0 e^{f'(y^*)t}.$$

Now we see how the stability criteria arise. If $f'(y^*) > 0$, $\xi(t)$ grows in time. Therefore, nearby solutions stray from the equilibrium solution for large times. On the other hand, if $f'(y^*) < 0$, $\xi(t)$ decays in time and nearby solutions approach the equilibrium solution for large t. Thus, we have the results:

The stability criteria for equilibrium solutions of a first order differential equation.

$$\begin{array}{ll} f'(y^*) < 0, & y^* \text{ is stable.} \\ f'(y^*) > 0, & y^* \text{ is unstable.} \end{array} \tag{3.10}$$

Example 3.5. Determine the stability of the equilibrium solutions of $y' = 1 - y^2$.

In the last example we found the equilibrium solutions, $y^* = \pm 1$. The stability criteria require computing

$$f'(y^*) = -2y^*.$$

For this problem we have $f'(\pm 1) = \mp 2$. Therefore, $y^* = 1$ is a stable equilibrium and $y^* = -1$ is an unstable equilibrium.

Example 3.6. Find and classify the equilibria for the logistic equation $y' = y - y^2$.

We had already investigated this problem using phase lines. There are two equilibria, $y = 0$ and $y = 1$.

We next apply the stability criteria. Noting that $f'(y) = 1 - 2y$, the first equilibrium solution gives $f'(0) = 1$. So, $y = 0$ is an unstable equilibrium. Since $f'(1) = -1 < 0$, we see that $y = 1$ is a stable equilibrium. These results are the same as we hade determined earlier using phase lines.

3.4 *Bifurcations for First Order Equations*

We now consider families of first order autonomous differential equations of the form

$$\frac{dy}{dt} = f(y; \mu).$$

Here μ is a parameter that we can change and then observe the resulting behaviors of the solutions of the differential equation. When a small change in the parameter leads to changes in the behavior of the solution, then the system is said to undergo a *bifurcation*. The value of the parameter, μ, at which the bifurcation occurs is called a *bifurcation point*.

Bifurcations and bifurcation points.

We will consider several generic examples, leading to special classes of bifurcations of first order autonomous differential equations. We will study the stability of equilibrium solutions using both phase lines and the stability criteria developed in the last section

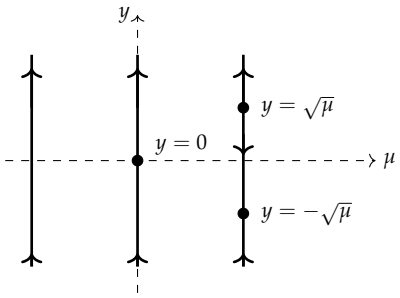


Figure 3.5: Phase lines for $y' = y^2 - \mu$. On the right $\mu > 0$ and on the left $\mu < 0$.

Example 3.7. $y' = y^2 - \mu$.

First note that equilibrium solutions occur for $y^2 = \mu$. In this problem, there are three cases to consider.

1. $\mu > 0$.

   In this case there are two real solutions of $y^2 = \mu$, $y = \pm\sqrt{\mu}$. Note that $y^2 - \mu < 0$ for $|y| < \sqrt{\mu}$. So, we have the right phase line in Figure 3.5.

2. $\mu = 0$.

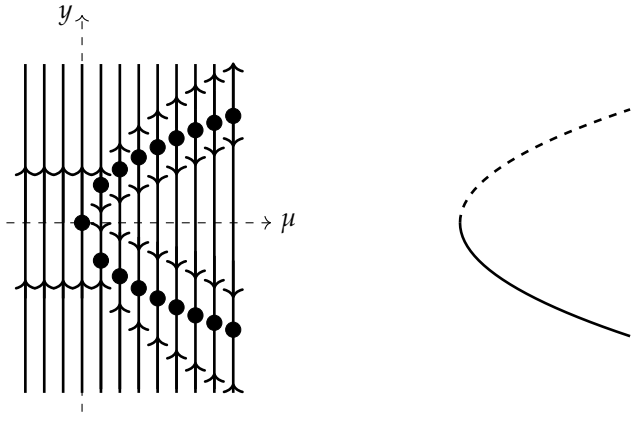
   There is only one equilibrium point at $y = 0$. The equation becomes $y' = y^2$. It is obvious that the right side of this equation is never negative. So, the phase line, which is shown as the middle line in Figure 3.5, has upward pointing arrows.

3. $\mu < 0$.

   In this case there are no equilibrium solutions. Since $y^2 - \mu > 0$, the slopes for all solutions are positive as indicated by the last phase line in Figure 3.5.

We can also confirm the behaviors of the equilibrium points by noting that $f'(y) = 2y$. Then, $f'(\pm\sqrt{\mu}) = \pm 2\sqrt{\mu}$ for $\mu \geq 0$. Therefore, the equilibria $y = +\sqrt{\mu}$ are unstable equilibria for $\mu > 0$. Similarly, the equilibria $y = -\sqrt{\mu}$ are stable equilibria for $\mu > 0$.

Figure 3.6: (a) The typical phase lines for $y' = y^2 - \mu$. (b) Bifurcation diagram for $y' = y^2 - \mu$. This is an example of a saddle-node bifurcation.



We can combine these results for the phase lines into one diagram known as a bifurcation diagram. We will plot the equilibrium solutions and their phase lines $y = \pm\sqrt{\mu}$ in the μy-plane. We begin by lining up the phase lines for various μ's. These are shown on the left side of Figure 3.6. Note the pattern of equilibrium points lies on the parabolic curve $y^2 = \mu$. The upper branch of this curve is a collection of unstable equilibria and the bottom is a stable branch. So, we can dispose of the phase lines and just keep the equilibria. However, we will draw the unstable branch as a dashed line and the stable branch as a solid line.

The bifurcation diagram is displayed on the right side of Figure 3.6. This type of bifurcation is called a *saddle-node bifurcation*. The point $\mu = 0$ at which the behavior changes is the *bifurcation point. As μ changes from negative to positive values, the system goes from having no equilibria to having one stable and one unstable equilibrium point.*

Example 3.8. $y' = y^2 - \mu y$.

Writing this equation in factored form, $y' = y(y - \mu)$, we see that there are two equilibrium points, $y = 0$ and $y = \mu$. The behavior of the solutions depends upon the sign of $y' = y(y - \mu)$. This leads to four cases with the indicated signs of the derivative. The regions indicating the signs of y' are shown in Figure 3.7.

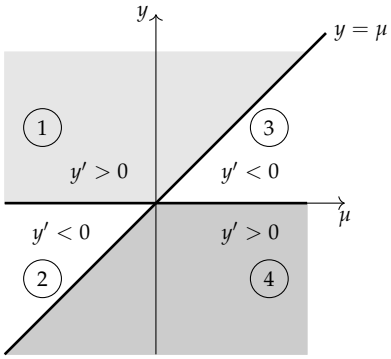


Figure 3.7: The regions indicating the different signs of the derivative for $y' = y^2 - \mu y$.

1. $y > 0, y - \mu > 0 \Rightarrow y' > 0$.
2. $y < 0, y - \mu > 0 \Rightarrow y' < 0$.
3. $y > 0, y - \mu < 0 \Rightarrow y' < 0$.
4. $y < 0, y - \mu < 0 \Rightarrow y' > 0$.

The corresponding phase lines and superimposed bifurcation diagram are shown in figure 3.8. The bifurcation diagram is on the right side of Figure 3.8 and this type of bifurcation is called a *transcritical bifurcation*.

Again, the stability can be determined from the derivative $f'(y) = 2y - \mu$ evaluated at $y = 0, \mu$. From $f'(0) = -\mu$, we see that $y = 0$ is stable for $\mu > 0$ and unstable for $\mu < 0$. Similarly, $f'(\mu) = \mu$ implies that $y = \mu$ is unstable for $\mu > 0$ and stable for $\mu < 0$. These results are consistent with the phase line plots.

Example 3.9. $y' = y^3 - \mu y$.

For this last example, we find from $y^3 - \mu y = y(y^2 - \mu) = 0$ that there are two cases.

1. $\mu < 0$. In this case there is only one equilibrium point at $y = 0$. For positive values of y we have that $y' > 0$ and for negative values of y we have that $y' < 0$. Therefore, this is an unstable equilibrium point.
2. $\mu > 0$. Here we have three equilibria, $y = 0, \pm\sqrt{\mu}$. A careful investigation shows that $y = 0$ is a stable equilibrium point and that the other two equilibria are unstable.

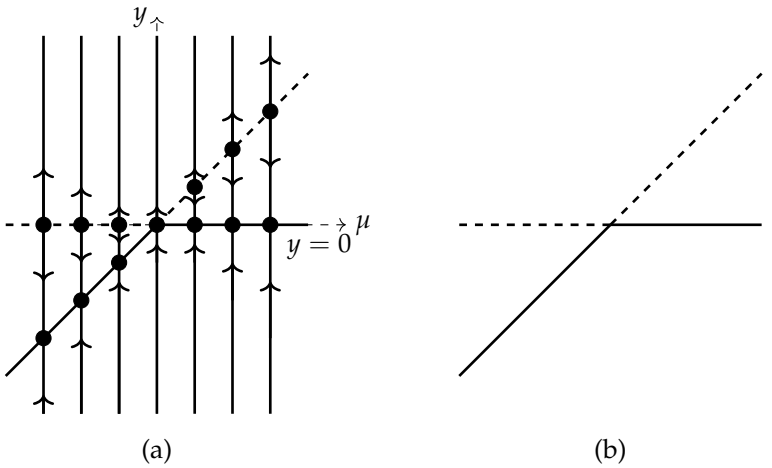


Figure 3.8: (a) Collection of phase lines for $y' = y^2 - \mu y$. (b) Bifurcation diagram for $y' = y^2 - \mu y$. This is an example of a transcritical bifurcation.

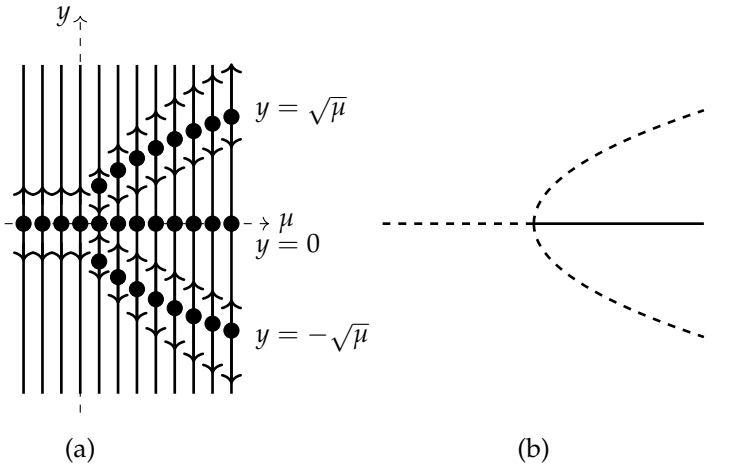


Figure 3.9: (a) The phase lines for $y' = y^3 - \mu y$. The left one corresponds to $\mu < 0$ and the right phase line is for $\mu > 0$. (b)Bifurcation diagram for $y' = y^3 - \mu y$. This is an example of a pitchfork bifurcation.

In Figure 3.9 we show the phase lines for these two cases. The corresponding bifurcation diagram is then sketched on the right side of Figure 3.9. For obvious reasons this has been labeled a *pitchfork bifurcation*.

When two of the prongs of the pitchfork are unstable branches, the bifurcation is called a subcritical pitchfork bifurcation. When two prongs are stable branches, the bifurcation is a supercritical pitchfork bifurcation.

Since $f'(y) = 3y^2 - \mu$, the stability analysis gives that $f'(0) = -\mu$. So, $y = 0$ is stable for $\mu > 0$ and unstable for $\mu < 0$. For $\mu > 0$, we have that $f'(\pm\sqrt{\mu}) = 2\mu$. Therefore, $y = \pm\sqrt{\mu}$, $\mu > 0$, is unstable. Thus, we have a subcritical pitchfork bifurcation.

3.5 *The Stability of Fixed Points in Nonlinear Systems*

We next investigate the stability of the equilibrium solutions of the nonlinear pendulum which we first encountered in Section 1.4.2. Along the way we will develop some basic methods for studying the stability of equilibria in nonlinear systems in general.

Recall that the derivation of the pendulum equation was based upon a simple point mass m hanging on a string of length L from some support as shown in Figure 3.10. One pulls the mass back to some starting angle, θ_0, and releases it. The goal is to find the angular position as a function of time, $\theta(t)$.

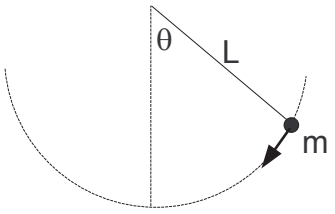


Figure 3.10: A simple pendulum consists of a point mass m attached to a string of length L. It is released from an angle θ_0.

In Chapter 2 we derived the nonlinear pendulum equation,

$$L\ddot{\theta} + g\sin\theta = 0. \tag{3.11}$$

There are several variations of Equation (3.11) which we have used in this text. The first one is the linear pendulum, which was obtained using a small angle approximation,

$$L\ddot{\theta} + g\theta = 0. \tag{3.12}$$

We also made the system more realistic by adding damping and forcing. A variety of these oscillation problems are summarized in the table below.

Equations for Pendulum Motion

1. Nonlinear Pendulum: $L\ddot{\theta} + g\sin\theta = 0$.
2. Damped Nonlinear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\sin\theta = 0$.
3. Linear Pendulum: $L\ddot{\theta} + g\theta = 0$.
4. Damped Linear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\theta = 0$.
5. Forced Damped Nonlinear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\sin\theta = F\cos\omega t$.
6. Forced Damped Linear Pendulum: $L\ddot{\theta} + b\dot{\theta} + g\theta = F\cos\omega t$.

There are two simple systems that we will consider, the damped linear pendulum, in the form

$$x'' + bx' + \omega^2 x = 0$$

and the the damped nonlinear pendulum, in the form

$$x'' + bx' + \omega^2\sin x = 0.$$

These are second order differential equations and can be cast as a system of two first order differential equations using the methods of Chapter 6.

The linear equation can be written as

$$\begin{aligned} x' &= y, \\ y' &= -by - \omega^2 x. \end{aligned} \tag{3.13}$$

This system has only one equilibrium solution, $x = 0$, $y = 0$.

The damped nonlinear pendulum takes the form

$$\begin{aligned} x' &= y, \\ y' &= -by - \omega^2\sin x. \end{aligned} \tag{3.14}$$

This system also has the equilibrium solution $x = 0$, $y = 0$. However, there are actually an infinite number of solutions. The equilibria are determined from

$$y = 0 \text{ and } -by - \omega^2 \sin x = 0. \tag{3.15}$$

These equations imply that $y = 0$ and $\sin x = 0$. There are an infinite number of solutions to the latter equation: $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. So, this system has an infinite number of equilibria, $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$.

The next step is to determine the stability of the equilibrium solutions these systems. This can be accomplished just as we had done for first order equations. To do this we need a more general theory for nonlinear systems. So, we will develop the needed machinery.

We begin with the $n-$dimensional system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathrm{R}^n. \tag{3.16}$$

Linear stability analysis of systems.

Here $\mathbf{f}: \mathrm{R}^n \to \mathrm{R}^n$ is a mapping from R^n to R^n. We define the equilibrium solutions, or fixed points, of this system as the points $\mathbf{x}^*$ satisfying $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$.

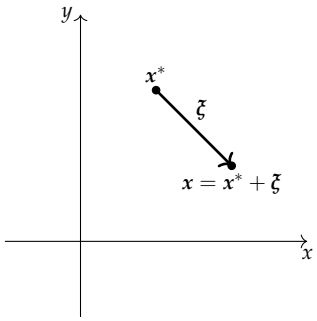


Figure 3.11: A general point in the plane, which is near the fixed point, in the form $\mathbf{x} = \mathbf{x}^* + \boldsymbol{\xi}$,

The stability in the neighborhood of equilibria will now be determined. We are interested in what happens to solutions of the system with initial conditions starting near a fixed point. We will represent a general point in the plane, which is near the fixed point, in the form $\mathbf{x} = \mathbf{x}^* + \boldsymbol{\xi}$. We note that the length of $\boldsymbol{\xi}$ gives an indication of how close we are to the fixed point. So, we consider that initially, $|\boldsymbol{\xi}| \ll 1$.

As the system evolves, $\boldsymbol{\xi}$ will change. The change of $\boldsymbol{\xi}$ in time is in turn governed by a system of equations. We can approximate this evolution as follows. First, we note that

$$\mathbf{x}' = \boldsymbol{\xi}'.$$

Next, we have that

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}^* + \boldsymbol{\xi}).$$

We can expand the right side about the fixed point using a multidimensional version of Taylor's Theorem. Thus, we have that

$$\mathbf{f}(\mathbf{x}^* + \boldsymbol{\xi}) = \mathbf{f}(\mathbf{x}^*) + D\mathbf{f}(\mathbf{x}^*)\boldsymbol{\xi} + O(|\boldsymbol{\xi}|^2).$$

The Jacobian matrix.

Here $D\mathbf{f}(\mathbf{x})$ is the *Jacobian matrix*, defined as

$$D\mathbf{f}(\mathbf{x}^*) \equiv \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

Noting that $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$, we then have that system (3.16) becomes

$$\xi' \approx D\mathbf{f}(\mathbf{x}^*)\boldsymbol{\xi}. \tag{3.17}$$

It is this equation which describes the behavior of the system near the fixed point. As with first order equations, we say that system (3.16) has been linearized or that Equation (3.17) is the linearization of system (3.16).

Linearization of the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$.

The stability of the equilibrium point of the nonlinear system is now reduced to analyzing the behavior of the linearized system given by Equation (3.17). We can use the methods from the last two chapters to investigate the eigenvalues of the Jacobian matrix evaluated at each equilibrium point. We will demonstrate this procedure with several examples.

Example 3.10. Determine the equilibrium points and their stability for the system

$$\begin{aligned} x' &= -2x - 3xy, \\ y' &= 3y - y^2. \end{aligned} \tag{3.18}$$

We first determine the fixed points. Setting the right-hand side equal to zero and factoring, we have

$$\begin{aligned} -x(2+3y) &= 0, \\ y(3-y) &= 0. \end{aligned} \tag{3.19}$$

From the second equation, we see that either $y = 0$ or $y = 3$. The first equation then gives $x = 0$ in either case. So, there are two fixed points: $(0,0)$ and $(0,3)$.

Next, we linearize the system of differential equations about each fixed point. First, we note that the Jacobian matrix is given by

$$D\mathbf{f}(x,y) = \begin{pmatrix} -2-3y & -3x \\ 0 & 3-2y \end{pmatrix}. \tag{3.20}$$

1. Case I Equilibrium point $(0,0)$.

   In this case we find that

   $$D\mathbf{f}(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}. \tag{3.21}$$

   Therefore, the linearized equation becomes

   $$\xi' = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} \xi. \tag{3.22}$$

   This is equivalently written out as the system

   $$\begin{aligned} \xi_1' &= -2\xi_1, \\ \xi_2' &= 3\xi_2. \end{aligned} \tag{3.23}$$

   This is the linearized system about the origin. Note the similarity with the original system.

   We should emphasize that the linearized equations are constant coefficient equations and we can use matrix methods to determine the nature of the equilibrium point. The eigenvalues of this system are obviously $\lambda = -2, 3$. Therefore, we have that the origin is a saddle point.

2. Case II Equilibrium point $(0, 3)$.

   Again we evaluate the Jacobian matrix at the equilibrium point and look at its eigenvalues to determine the type of fixed point. The Jacobian matrix for this case becomes

$$D\mathbf{f}(0,3) = \begin{pmatrix} -11 & 0 \\ 0 & -3 \end{pmatrix}. \tag{3.24}$$

   The eigenvalues are $\lambda = -11, -3$. So, this fixed point is a stable node.

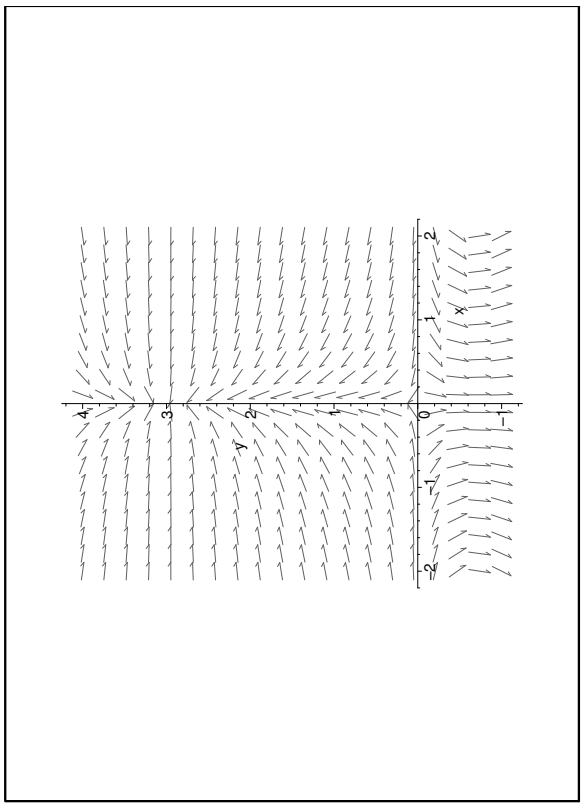


Figure 3.12: Phase plane for the system $x' = -2x - 3xy$, $y' = 3y - y^2$.

This analysis has given us a saddle and a stable node. We know what the behavior is like near each fixed point, but we have to resort to other means to say anything about the behavior far from these points. The phase portrait for this system is given in Figure 3.12. You should be able to locate the saddle point and the node in the figure. Notice how solutions behave in regions far from these points.

We can expect to be able to perform a linearization under general conditions. These are given in the *Hartman-Großman Theorem*:

Theorem 3.1. *A continuous map exists between the linear and nonlinear systems when $D\mathbf{f}(\mathbf{x}^*)$ does not have any eigenvalues with zero real part.*

Generally, there are several types of behavior that one can see in nonlinear systems. One can see sinks or sources, hyperbolic (saddle) points, elliptic points (centers) or foci. We have defined some of these for planar

systems. In general, if at least two eigenvalues have real parts with opposite signs, then the fixed point is a *hyperbolic point*. If the real part of a nonzero eigenvalue is zero, then we have a center, or *elliptic point.*

For linear systems in the plane, this classification was done in Chapter 6. The Jacobian matrix evaluated at the equilibrium points is simply the 2×2 coefficient matrix we had called A.

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.25}$$

Here we are using $J = D\mathbf{f}(\mathbf{x}^*)$.

The eigenvalue equation is given by

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

However, $a + d$ is the trace, $\text{tr}(J)$ and $\det(J) = ad - bc$. Therefore, we can write the eigenvalue equation as

$$\lambda^2 - \text{tr}(J)\lambda + \det(J) = 0.$$

The solution of this equation is found using the quadratic formula,

$$\lambda = \frac{1}{2}\left[-\text{tr}(J) \pm \sqrt{\text{tr}^2(J) - 4\det(J)}\right].$$

We had seen in previous chapter that equilibrium points in planar systems can be classified as nodes, saddles, centers, or spirals (foci). The type of behavior can be determined from solutions of the eigenvalue equation. Since the nature of the eigenvalues depends on the trace and determinant of the Jacobian matrix at the equilibrium point, we can relate the types of equilibria to points in the det-tr plane. This is shown in Figure 3.13, which is similar to Figure 2.25.

In Figure 3.13 the parabola $\text{tr}^2(J) = 4\det(J)$ divides the det-tr plane. Points on this curve give a vanishing discriminant in the computation of the eigenvalues. In these cases one finds repeated roots, or eigenvalues. Along this curve one can find stable and unstable degenerate nodes. Also along this line are stable and unstable proper nodes, called star nodes. These arise from systems of the form $x' = ax$, $y' = ay$.

In the case that $\det(J) < 0$, we have that the discriminant

$$\Delta \equiv \text{tr}^2(J) - 4\det(J)$$

is positive. Not only that, $\Delta > \text{tr}^2(J)$. Thus, we obtain two real and distinct eigenvalues with opposite signs. These lead to saddle points.

In the case that $\det(J) > 0$, we can have either $\Delta > 0$ or $\Delta < 0$. The discriminant is negative for points inside the parabolic curve. It is in this region that one finds centers and spirals, corresponding to complex eigenvalues. When $\text{tr}(J) > 0$, there are unstable spirals. There are stable spirals when $\text{tr}(J) < 0$. For the case that $\text{tr}(J) = 0$, the eigenvalues are pure imaginary, giving centers.

There are several other types of behavior depicted in the figure, but we will now turn to studying a few of examples.

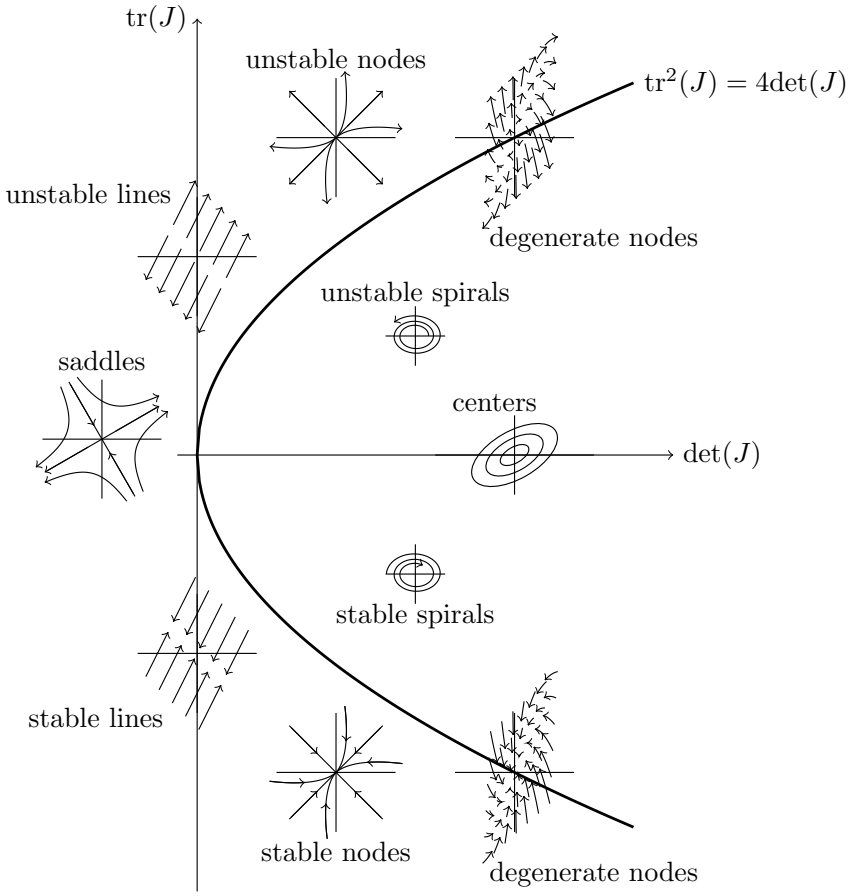


Figure 3.13: Diagram indicating the behavior of equilibrium points in the det – tr plane. The parabolic curve

$$\mathrm{tr}^2(J) = 4\mathrm{det}(J)$$

indicates where the discriminant vanishes.

Example 3.11. Find and classify all of the equilibrium solutions of the nonlinear system

$$\begin{aligned} x' &= 2x - y + 2xy + 3(x^2 - y^2), \\ y' &= x - 3y + xy - 3(x^2 - y^2). \end{aligned} \tag{3.26}$$

In Figure 3.14 we show the direction field for this system. Try to locate and classify the equilibrium points visually. After the stability analysis, you should return to this figure and determine if you identified the equilibrium points correctly.

We will first determine the equilibrium points. Setting the right-hand side of each differential equation to zero, we have

$$\begin{aligned} 2x - y + 2xy + 3(x^2 - y^2) &= 0, \\ x - 3y + xy - 3(x^2 - y^2) &= 0. \end{aligned} \tag{3.27}$$

This system of algebraic equations can be solved exactly. Adding the equations, we have

$$3x - 4y + 3xy = 0.$$

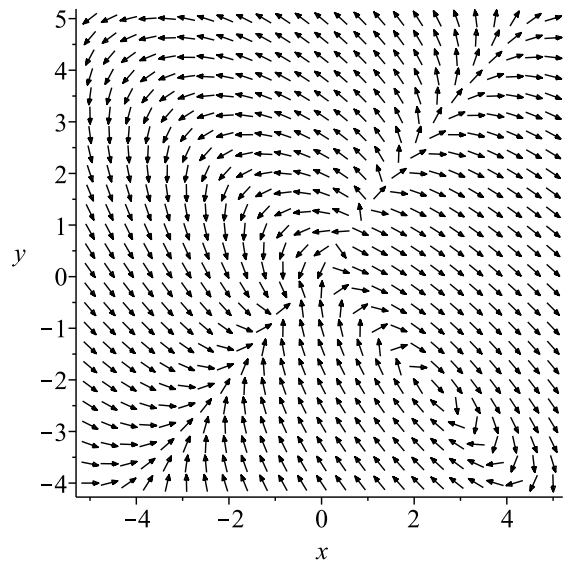


Figure 3.14: Phase plane for the system
$x' = 2x - y + 2xy + 3(x^2 - y^2)$,
$y' = x - 3y + xy - 3(x^2 - y^2)$.

Solving for x,

$$x = \frac{4y}{3(1+y)},$$

and substituting the result for x into the first algebraic equation, we find an equation for y :

$$\frac{y(1-y)(9y^2+22y+5)}{3(1+y)^2} = 0.$$

The solutions to this equation are

$$y = 0, 1, -\frac{11}{9} \pm \frac{2}{9}\sqrt{19}.$$

The corresponding values for x are

$$x = 0, \frac{2}{3}, 1 \mp \frac{\sqrt{19}}{3}.$$

Now that we have located the equilibria, we can classify them. The Jacobian matrix is given by

$$D\mathbf{f}(x,y) = \begin{pmatrix} 6x+2y+2 & 2x-6y-1 \\ -6x+y+1 & x+6y-3 \end{pmatrix}. \tag{3.28}$$

Now, we evaluate the Jacobian at each equilibrium point and find the eigenvalues.

1. Case I. Equilibrium point $(0,0)$.
   In this case we find that

$$D\mathbf{f}(0,0) = \begin{pmatrix} -2 & -1 \\ 1 & -3 \end{pmatrix}. \tag{3.29}$$

The eigenvalues of this matrix are $\lambda = -\frac{1}{2} \pm \frac{\sqrt{21}}{2}$. Therefore, the origin is a saddle point.

2. Case II. Equilibrium point $(\frac{2}{3}, 1)$.

   Again we evaluate the Jacobian matrix at the equilibrium point and look at its eigenvalues to determine the type of fixed point. The Jacobian matrix for this case becomes

$$D\mathbf{f}\left(\frac{2}{3}, 1\right) = \begin{pmatrix} 8 & -\frac{17}{3} \\ -2 & \frac{11}{3} \end{pmatrix}. \tag{3.30}$$

   The eigenvalues are $\lambda = \frac{35}{6} \pm \frac{\sqrt{577}}{6} \approx 9.84, 1.83$. This fixed point is an unstable node.

3. Case III. Equilibrium point $(1 \mp \frac{\sqrt{19}}{3}, -\frac{11}{9} \pm \frac{2}{9}\sqrt{19})$.

   The Jacobian matrix for this case becomes

$$D\mathbf{f}\left(1 \mp \frac{\sqrt{19}}{3}, -\frac{11}{9} \pm \frac{2}{9}\sqrt{19}\right) = \begin{pmatrix} \frac{50}{9} \mp \frac{14}{9}\sqrt{19} & \frac{25}{3} \mp 2\sqrt{19} \\ -\frac{56}{9} \pm \frac{20}{9}\sqrt{19} & -\frac{28}{3} \pm \sqrt{19} \end{pmatrix}. \tag{3.31}$$

   There are two equilibrium points under this case. The first is given by

$$(1 - \frac{\sqrt{19}}{3}, -\frac{11}{9} + \frac{2}{9}\sqrt{19}) \approx (0.453, -0.254).$$

   The eigenvalues for this point are

$$\lambda = -\frac{17}{9} - \frac{5}{18}\sqrt{19} \pm \frac{1}{18}\sqrt{3868\sqrt{19} - 16153}.$$

   These are approximately -4.58 and -1.62 So, this equilibrium point is a stable node.

   The other equilibrium is $(1 + \frac{\sqrt{19}}{3}, -\frac{11}{9} - \frac{2}{9}\sqrt{19}) \approx (2.45, -2.19)$. The corresponding eigenvalues are complex with negative real parts,

$$\lambda = -\frac{17}{9} + \frac{5}{18}\sqrt{19} \pm \frac{i}{18}\sqrt{16153 + 3868\sqrt{19}},$$

   or $\lambda \approx -0.678 \pm 10.1i$. This point is a stable spiral.

Plots of the phase plane are given in Figures 3.12 and 3.14. The reader can look at the direction field and verify these results for the behavior of equilibrium solutions. A zoomed in view is shown in Figure 3.15 with several orbits indicated.

Example 3.12. Damped Nonlinear Pendulum Equilibria

We are now ready to establish the behavior of the fixed points of the damped nonlinear pendulum system in Equation (3.14). Recall that the system for the damped nonlinear pendulum was given by

$$\begin{aligned} x' &= y, \\ y' &= -by - \omega^2 \sin x. \end{aligned} \tag{3.32}$$

For a damped system, we will need $b > 0$. We had found that there are an infinite number of equilibrium points at $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$.

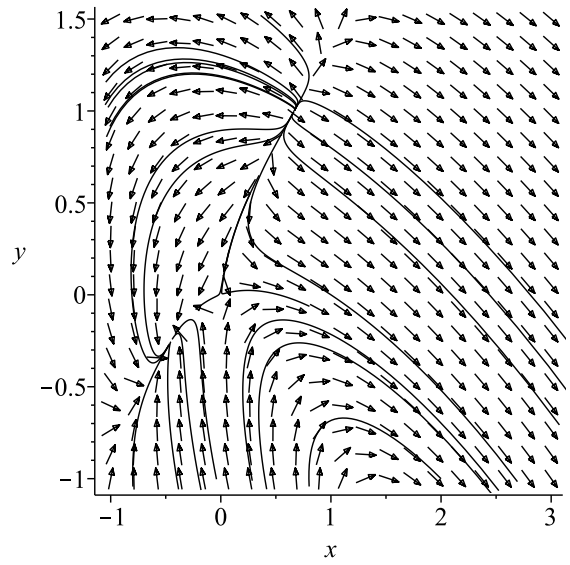


Figure 3.15: A closer look at the phase plane for the system

$$x' = 2x - y + 2xy + 3(x^2 - y^2),$$

$$y' = x - 3y + xy - 3(x^2 - y^2)$$

with a few trajectories shown.

The Jacobian matrix for this systems is

$$D\mathbf{f}(x,y) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x & -b \end{pmatrix}. \tag{3.33}$$

Evaluating this matrix at the fixed points, we find that

$$D\mathbf{f}(n\pi,0) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos n\pi & -b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (-1)^{n+1}\omega^2 & -b \end{pmatrix}. \tag{3.34}$$

The eigenvalue equation is given by

$$\lambda^2 + b\lambda + (-1)^n \omega^2 = 0.$$

There are two cases to consider: n even and n odd. For the first case, we find the eigenvalues

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4\omega^2}}{2}.$$

For $b^2 < 4\omega^2$, we have two complex conjugate roots with a negative real part. Thus, we have stable foci for even n values. If there is no damping, then we obtain centers ($\lambda = \pm i\omega$).

In the second case, n odd, we find

$$\lambda = \frac{-b \pm \sqrt{b^2 + 4\omega^2}}{2}.$$

Since $b^2 + 4\omega^2 > b^2$, these roots will be real with opposite signs. Thus, we have hyperbolic points, or saddles. If there is no damping, the eigenvalues reduce to $\lambda = \pm\omega$.

In Figure (3.16) we show the phase plane for the undamped nonlinear pendulum with $\omega = 1.25$. We see that we have a mixture of centers

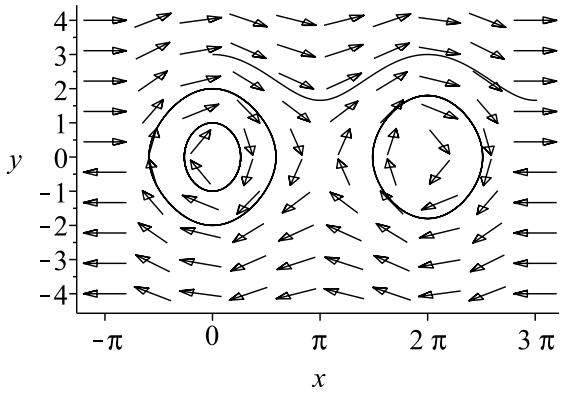


Figure 3.16: Phase plane for the undamped nonlinear pendulum. Solution curves are shown for initial conditions $(x_0, y_0) = (0,3), (0,2), (0,1), (5,1)$.

and saddles. There are orbits for which there is periodic motion. In the case that $\theta = \pi$ we have an inverted pendulum. This is an unstable position and this is reflected in the presence of saddle points, especially if the pendulum is constructed using a massless rod.

There are also unbounded orbits, going through all possible angles. These correspond to the mass spinning around the pivot in one direction forever due to initially having large enough energies.

We have indicated in the figure solution curves with the initial conditions $(x_0, y_0) = (0,3), (0,2), (0,1), (5,1)$. These show the various types of motions that we have described.

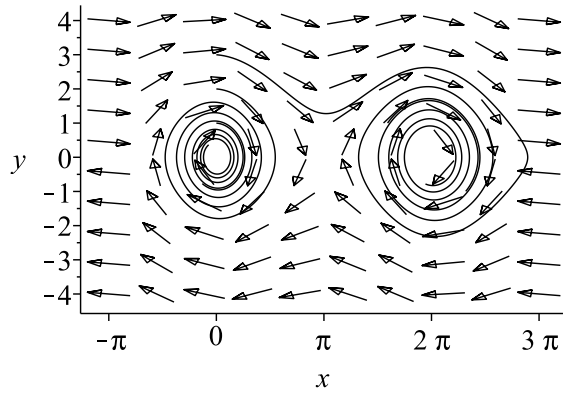


Figure 3.17: Phase plane for the damped nonlinear pendulum. Solution curves are shown for initial conditions $(x_0, y_0) = (0,3), (0,2), (0,1), (5,1)$.

When there is damping, we see that we can have a variety of other behaviors as seen in Figure (3.17). In this example we have set $b = 0.08$ and $\omega = 1.25$. We see that energy loss results in the mass settling around one of the stable fixed points. This leads to an understanding as to why there are an infinite number of equilibria, even though physically the mass traces out a bound set of Cartesian points. We have indicated in the Figure (3.17) solution curves with the initial conditions $(x_0, y_0) = (0,3), (0,2), (0,1), (5,1)$.

In Figure 3.18 we show a region of the phase plane which corresponds to oscillations about $x = 0$. For small angles the pendulum oscillates following somewhat elliptical orbits. As the angles get larger, due to greater initial

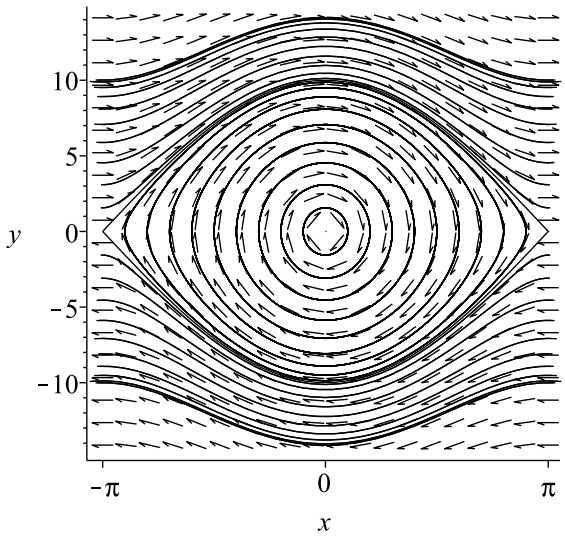


Figure 3.18: Several orbits in the phase plane for the undamped nonlinear pendulum with $\omega = 5.0$. The orbits surround a center at $(0,0)$. At the edges there are saddle points, $(\pm\pi, 0)$.

energies, these orbits begin to change from ellipses to other periodic orbits. There is a limiting orbit, beyond which one has unbounded motion. The limiting orbit connects the saddle points on either side of the center. The curve is called a separatrix and being that these trajectories connect two saddles, they are often referred to as heteroclinic orbits.

Heteroclinc orbits and separatrices.

In Figures 3.19-3.19 we show more orbits, including both bound and unbound motion beyond the interval $x \in [-\pi, \pi]$. For both plots we have chosen $\omega = 5$ and the same set of initial conditions, $x(0) = \pi k/10$, $k = -20, \ldots, 20$. for $y(0) = 0, \pm 10$. The time interval is taken for $t \in [-3, 3]$. The only difference is that in the damped case we have $b = 0.5$. In these plots one can see what happens to the heteroclinic orbits and nearby unbounded orbits under damping.

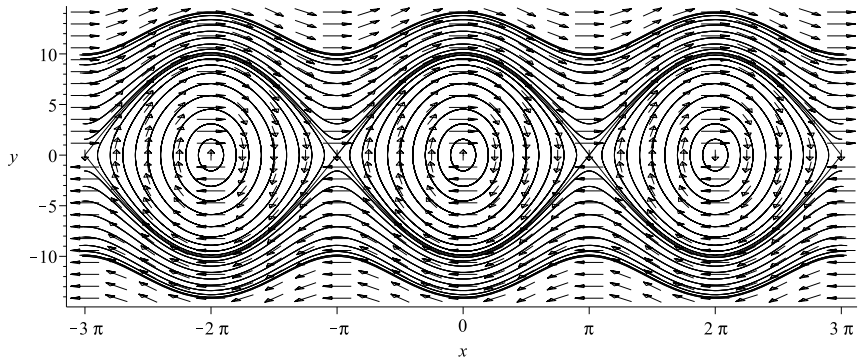


Figure 3.19: Several orbits in the phase plane for the undamped nonlinear pendulum with $\omega = 5.0$.

Before leaving this problem, we should note that the orbits in the phase plane for the undamped nonlinear pendulum can be obtained graphically. Recall from Equation (3.70), the total mechanical energy for the nonlinear

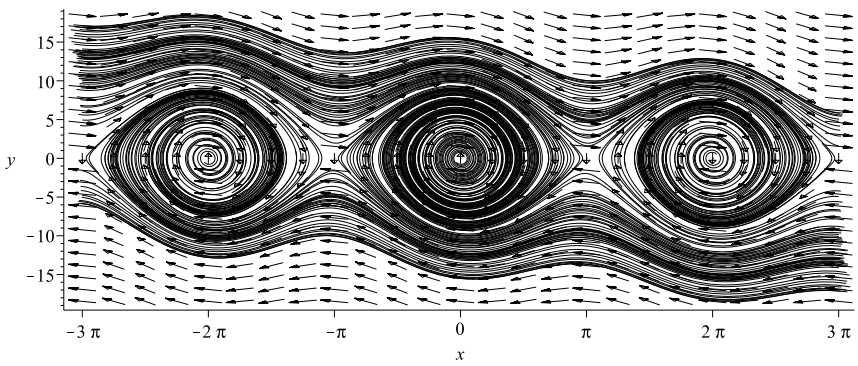


Figure 3.20: Several orbits in the phase plane for the damped nonlinear pendulum with $\omega = 5.0$ and $b = 0.5$.

pendulum is

$$E = \frac{1}{2}mL^2\dot{\theta}^2 + mgL(1 - \cos\theta).$$

From this equation we obtained Equation (3.71),

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = -\omega^2\cos\theta_0.$$

Letting $y = \dot{\theta}$, $x = \theta$, and defining $z = -\omega^2\cos\theta_0$, this equation can be written as

$$\frac{1}{2}y^2 - \omega^2\cos x = z. \tag{3.35}$$

For each energy (z), this gives a constant energy curve. Plotting the family of energy curves we obtain the phase portrait shown in Figure 3.21.

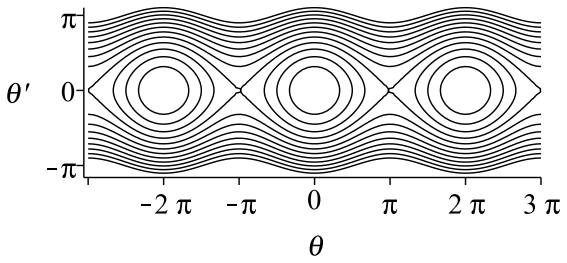


Figure 3.21: A family of energy curves in the phase plane for $\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = z$. Here we took $\omega = 1.0$ and $z \in [-5, 15]$.

3.6 Nonlinear Population Models

We have already encountered several models of population dynamics in this and previous chapters. Of course, one could dream up several other examples. While such models might seem far from applications in physics, it turns out that these models lead to systems od differential equations which also appear in physical systems such as the coupling of waves in lasers, in plasma physics, and in chemical reactions.

Two well-known nonlinear population models are the predator-prey and competing species models. In the predator-prey model, one typically has one species, the predator, feeding on the other, the prey. We will look at the standard Lotka-Volterra model in this section. The competing species model looks similar, except there are a few sign changes, since one species is not feeding on the other. Also, we can build in logistic terms into our model. We will save this latter type of model for the homework.

The Lotka-Volterra model is named after Alfred James Lotka (1880-1949) and Vito Volterra (1860-1940).

The Lotka-Volterra model of population dynamics.

The Lotka-Volterra model takes the form

$$\begin{aligned} \dot{x} &= ax - bxy, \\ \dot{y} &= -dy + cxy, \end{aligned} \tag{3.36}$$

where a, b, c, and d are positive constants. In this model, we can think of x as the population of rabbits (prey) and y is the population of foxes (predators). Choosing all constants to be positive, we can describe the terms.

- ax: When left alone, the rabbit population will grow. Thus a is the natural growth rate without predators.
- $-dy$: When there are no rabbits, the fox population should decay. Thus, the coefficient needs to be negative.
- $-bxy$: We add a nonlinear term corresponding to the depletion of the rabbits when the foxes are around.
- cxy: The more rabbits there are, the more food for the foxes. So, we add a nonlinear term giving rise to an increase in fox population.

Example 3.13. Determine the equilibrium points and their stability for the Lotka-Volterra system.

The analysis of the Lotka-Volterra model begins with determining the fixed points. So, we have from Equation (3.36)

$$\begin{aligned} x(a - by) &= 0, \\ y(-d + cx) &= 0. \end{aligned} \tag{3.37}$$

Therefore, the origin, $(0, 0)$, and $(\frac{d}{c}, \frac{a}{b})$ are the fixed points.

Next, we determine their stability, by linearization about the fixed points. We can use the Jacobian matrix, or we could just expand the right-hand side of each equation in (3.36) about the equilibrium points as shown in he next example. The Jacobian matrix for this system is

$$Df(x, y) = \begin{pmatrix} a - by & -bx \\ cy & -d + cx \end{pmatrix}.$$

Evaluating at each fixed point, we have

$$Df(0, 0) = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix}, \tag{3.38}$$

$$Df\left(\frac{d}{c}, \frac{a}{b}\right) = \begin{pmatrix} 0 & -\frac{bd}{c} \\ \frac{ac}{b} & 0 \end{pmatrix}. \tag{3.39}$$

The eigenvalues of (3.38) are $\lambda = a, -d$. So, the origin is a saddle point.

The eigenvalues of (3.39) satisfy $\lambda^2 + ad = 0$. So, the other point is a center. In Figure 3.22 we show a sample direction field for the Lotka-Volterra system.

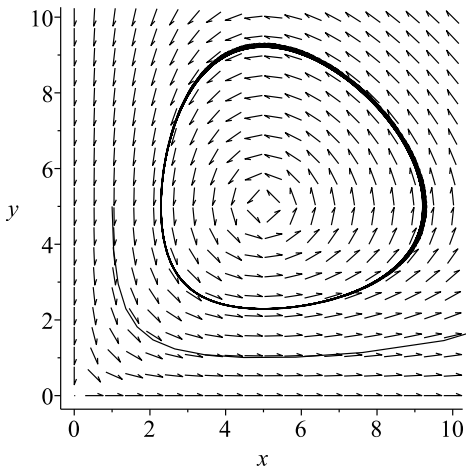


Figure 3.22: Phase plane for the Lotka-Volterra system given by $\dot{x} = x - 0.2xy$, $\dot{y} = -y + 0.2xy$. Solution curves are shown for initial conditions $(x_0, y_0) = (8,3), (1,5)$.

Another way to carry out the linearization of the system of differential equations is to expand the equations about the fixed points. For fixed points (x^*, y^*), we let

$$(x, y) = (x^* + u, y^* + v).$$

Direct linearization of a system is carried out by introducing $\mathbf{x} = \mathbf{x}^* + \boldsymbol{\xi}$, or $(x, y) = (x^* + u, y^* + v)$ into the system and dropping nonlinear terms in u and v.

Inserting this translation of the origin into the equations of the system, and dropping nonlinear terms in u and v, results in the linearized system. This method is equivalent to analyzing the Jacobian matrix for each fixed point.

Example 3.14. Expand the Lotka-Volterra system about the equilibrium points.

For the origin $(0,0)$ the linearization about the origin amounts to simply dropping the nonlinear terms. In this case we have

$$\begin{aligned} \dot{u} &= au, \\ \dot{v} &= -dv. \end{aligned} \tag{3.40}$$

The coefficient matrix for this system is the same as $Df(0,0)$.

For the second fixed point, we let

$$(x, y) = \left(\frac{d}{c} + u, \frac{a}{b} + v\right).$$

Inserting this transformation into the system gives

$$\begin{aligned} \dot{u} &= a\left(\frac{d}{c} + u\right) - b\left(\frac{d}{c} + u\right)\left(\frac{a}{b} + v\right), \\ \dot{v} &= -d\left(\frac{a}{b} + v\right) + c\left(\frac{d}{c} + u\right)\left(\frac{a}{b} + v\right). \end{aligned} \tag{3.41}$$

Expanding, we obtain

$$\begin{aligned}\dot{u} &= \frac{ad}{c} + au - b\left(\frac{ad}{bc} + \frac{d}{c}v + \frac{a}{b}u + uv\right),\\ \dot{v} &= -\frac{ad}{b} - dv + c\left(\frac{ad}{bc} + \frac{d}{c}v + \frac{a}{b}u + uv\right).\end{aligned} \tag{3.42}$$

In both equations the constant terms cancel and linearization is simply getting rid of the uv terms. This leaves the linearized system

$$\begin{aligned}\dot{u} &= au - b\left(\frac{d}{c}v + \frac{a}{b}u\right),\\ \dot{v} &= -dv + c\left(+\frac{d}{c}v + \frac{a}{b}u\right),\end{aligned} \tag{3.43}$$

or

$$\begin{aligned}\dot{u} &= -\frac{bd}{c}v,\\ \dot{v} &= \frac{ac}{b}u.\end{aligned} \tag{3.44}$$

The coefficient matrix for this linearized system is the same as $Df\left(\frac{d}{c}, \frac{a}{b}\right)$. In fact, for nearby orbits, they are almost circular orbits. From this linearized system, we have $\ddot{u} + adu = 0$.

We can take $u = A\cos(\sqrt{ad}t + \phi)$, where A and ϕ can be determined from the initial conditions. Then,

$$\begin{aligned}v &= -\frac{c}{bd}\dot{u}\\ &= \frac{c}{bd}A\sqrt{ad}\sin(\sqrt{ad}t + \phi)\\ &= \frac{c}{b}\sqrt{\frac{a}{d}}A\sin(\sqrt{ad}t + \phi).\end{aligned} \tag{3.45}$$

Therefore, the solutions near the center are given by

$$(x, y) = \left(\frac{d}{c} + A\cos(\sqrt{ad}t + \phi), \frac{a}{b} + \frac{c}{b}\sqrt{\frac{a}{d}}A\sin(\sqrt{ad}t + \phi)\right).$$

For $a = d = 1$, $b = c = 0.2$, and initial values of $(x_0, y_0) = (5.5, 5)$, these solutions become

$$x(t) = 5.0 + 0.5\cos t, \quad y(t) = 5.0 + 0.5\sin t.$$

Plots of these solutions are shown in Figure (3.23).

It is also possible to find a first integral of the Lotka-Volterra system whose level curves give the phase portrait of the system. As we had done in Chapter 2, we can write

$$\begin{aligned}\frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}}\\ &= \frac{-dy + cxy}{ax - bxy}\\ &= \frac{y(-d + cx)}{x(a - by)}.\end{aligned} \tag{3.46}$$

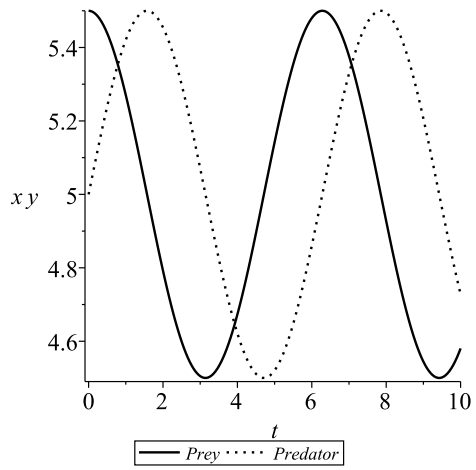


Figure 3.23: The linearized solutions of Lotka-Volterra system $\dot{x} = x - 0.2xy$, $\dot{y} = -y + 0.2xy$ for the initial conditions $(x_0, y_0) = (5.5, 5)$.

This is an equation of the form seen in Problem 2.**??**. This equation is now a separable differential equation. The solution this differential equation is given in implicit form as

$$a \ln y + d \ln x - cx - by = C,$$

The first integral of the Lotka-Volterra system.

where C is an arbitrary constant. This expression is known as the first integral of the Lotka-Volterra system. These level curves are shown in Figure 3.24.

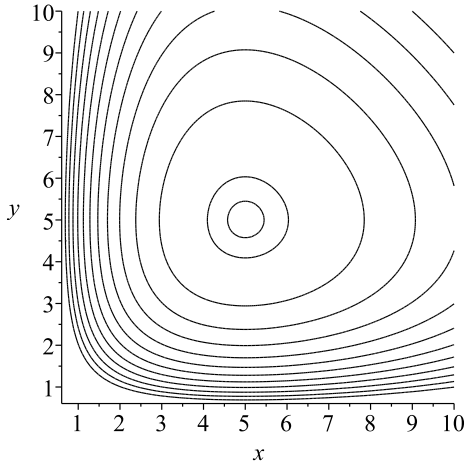


Figure 3.24: Phase plane for the Lotka-Volterra system given by $\dot{x} = x - 0.2xy$, $\dot{y} = -y + 0.2xy$ based upon the first integral of the system.

3.7 *Limit Cycles*

So far we have just been concerned with equilibrium solutions and their behavior. However, asymptotically stable fixed points are not the only attractors. There are other types of solutions, known as limit cycles, towards

which a solution may tend. In this section we will look at some examples of these periodic solutions.

Such solutions are common in nature. Rayleigh investigated the problem

$$x'' + c\left(\frac{1}{3}(x')^2 - 1\right)x' + x = 0 \tag{3.47}$$

in the study of the vibrations of a violin string. Balthasar van der Pol (1889-1959) studied an electrical circuit, modeling this behavior. Others have looked into biological systems, such as neural systems, chemical reactions, such as Michaelis-Menten kinetics, and other chemical systems leading to chemical oscillations. One of the most important models in the historical study of dynamical systems is that of planetary motion and investigating the stability of planetary orbits. As is well known, these orbits are periodic.

Limit cycles are isolated periodic solutions towards which neighboring states might tend when stable. A key example exhibiting a limit cycle is given in the next example.

Example 3.15. Find the limit cycle in the system

$$\begin{aligned} x' &= \mu x - y - x(x^2 + y^2) \\ y' &= x + \mu y - y(x^2 + y^2). \end{aligned} \tag{3.48}$$

It is clear that the origin is a fixed point. The Jacobian matrix is given as

$$Df(0,0) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}. \tag{3.49}$$

The eigenvalues are found to be $\lambda = \mu \pm i$. For $\mu = 0$ we have a center. For $\mu < 0$ we have a stable spiral and for $\mu > 0$ we have an unstable spiral. However, this spiral does not wander off to infinity. We see in Figure 3.25 that the equilibrium point is a spiral. However, in Figure 3.26 it is clear that the solution does not spiral out to infinity. It is bounded by a circle.

One can actually find the radius of this circle. This requires rewriting the system in polar form. Recall from Chapter 2 that we can change derivatives of Cartesian coordinates to derivatives of polar coordinates by using the relations

$$rr' = xx' + yy', \tag{3.50}$$

$$r^2\theta' = xy' - yx'. \tag{3.51}$$

Inserting the system (3.48) into these expressions, we have

$$rr' = \mu r^2 - r^4, \qquad r^2\theta' = r^2.$$

This leads to the system

$$\begin{aligned} r' &= \mu r - r^3, \\ \theta' &= 1. \end{aligned} \tag{3.52}$$

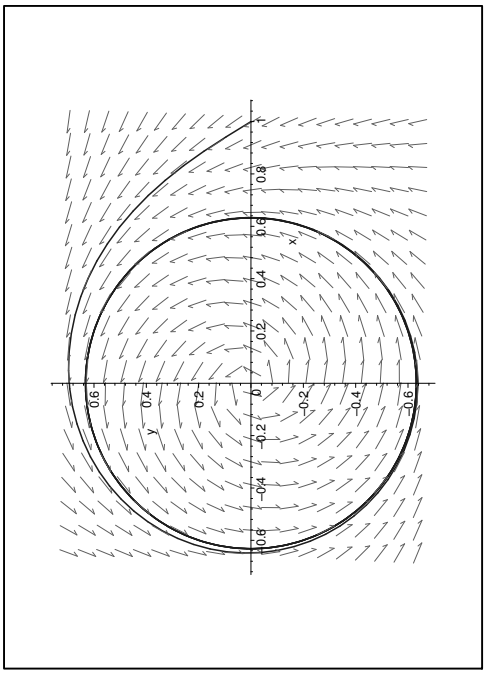


Figure 3.25: Phase plane for system (3.48) with $\mu = 0.4$.

Of course, for a circle the radius is constant, $r = \text{const}$. Therefore, in order to find the limit cycle, we need to look at the equilibrium solutions of Equation (3.52). This amounts to finding the constant solutions of $\mu r - r^3 = 0$. The equilibrium solutions are $r = 0, \pm\sqrt{\mu}$. The limit cycle corresponds to the positive radius solution, $r = \sqrt{\mu}$.

In Figures 3.25-3.26 we take $\mu = 0.4$. In this case we expect a circle with $r = \sqrt{0.4} \approx 0.63$. From the θ equation, we have that $\theta' > 0$. This means that we follow the limit cycle in a counterclockwise direction as time increases.

Limit cycles are not always circles. In Figures 3.27-3.28 we show the behavior of the Rayleigh system (3.47) for $c = 0.4$ and $c = 2.0$. In this case we see that solutions tend towards a noncircular limit cycle in a clockwise direction.

A slight change of the Rayleigh system leads to the van der Pol equation:

$$x'' + c(x^2 - 1)x' + x = 0 \tag{3.53}$$

The van der Pol system.

The limit cycle for $c = 2.0$ is shown in Figure 3.29.

Can one determine ahead of time if a given nonlinear system will have a limit cycle? In order to answer this question, we will introduce some definitions.

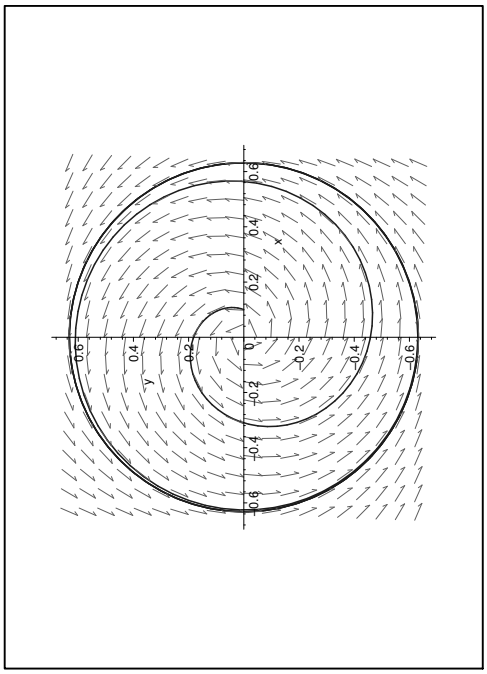


Figure 3.26: Phase plane for system (3.48) with $\mu = 0.4$ showing that the inner spiral is bounded by a limit cycle.

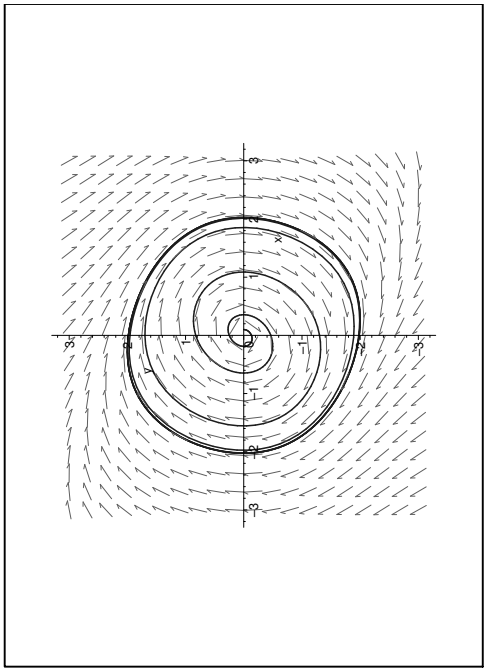


Figure 3.27: Phase plane for the Rayleigh system (3.47) with $c = 0.4$.

Figure 3.28: Phase plane for the van der Pol system (3.53) with $c = 2.0$.

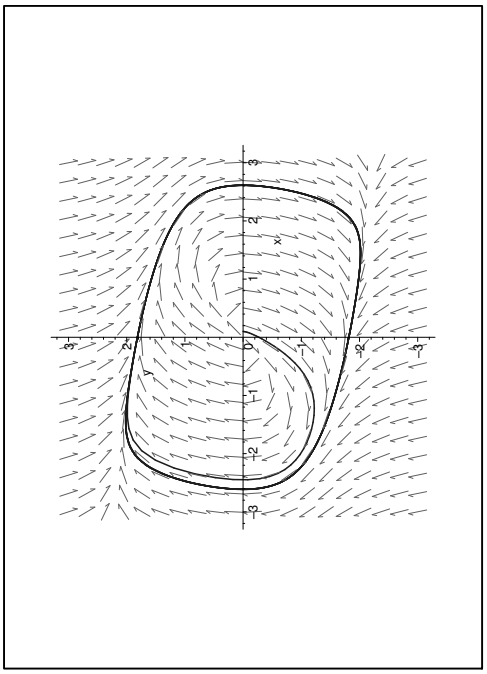
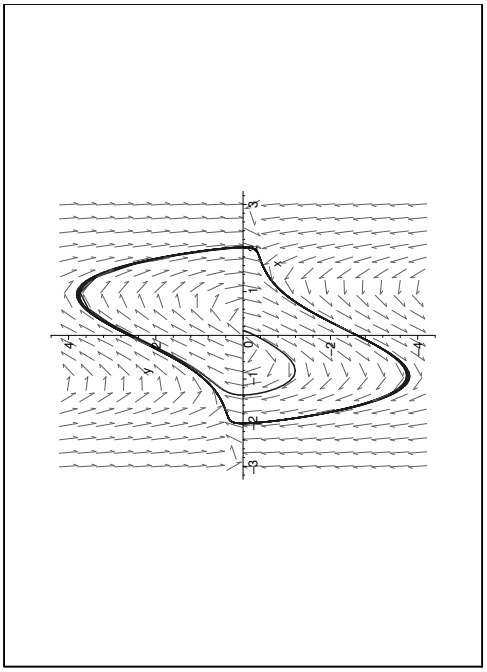


Figure 3.29: Phase plane for the van der Pol system (3.53) with $c = 0.4$.



We first describe different trajectories and families of trajectories. A *flow* on R^2 is a function ϕ that satisfies the following

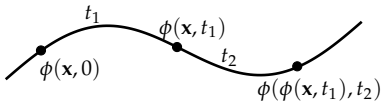


Figure 3.30: A sketch depicting the idea of trajectory, or orbit, passing through $\mathbf{x}$.

1. $\phi(\mathbf{x}, t)$ is continuous in both arguments.

2. $\phi(\mathbf{x}, 0) = \mathbf{x}$ for all $\mathbf{x} \in R^2$.

3. $\phi(\phi(\mathbf{x}, t_1), t_2) = \phi(\mathbf{x}, t_1 + t_2)$.

Flows.

The *orbit*, or *trajectory*, through $\mathbf{x}$ is defined as $\gamma = \{\phi(\mathbf{x}, t) | t \in I\}$. In Figure 3.30 we demonstrate these properties. For $t = 0$, $\phi(\mathbf{x}, 0) = \mathbf{x}$. Increasing t, one follows the trajectory until one reaches the point $\phi(\mathbf{x}, t_1)$. Continuing t_2 further, one is then at $\phi(\phi(\mathbf{x}, t_1), t_2)$. By the third property, this is the same as going from $\mathbf{x}$ to $\phi(\mathbf{x}, t_1 + t_2)$ for $t = t_1 + t_2$.

Orbits and trajectories.

Having defined the orbits, we need to define the asymptotic behavior of the orbit for both positive and negative large times. We define the *positive semiorbit* through $\mathbf{x}$ as $\gamma^+ = \{\phi(\mathbf{x}, t) | t > 0\}$. The *negative semiorbit* through $\mathbf{x}$ is defined as $\gamma^- = \{\phi(\mathbf{x}, t) | t < 0\}$. Thus, we have $\gamma = \gamma^+ \cup \gamma^-$.

Limit sets and limit points.

The *positive limit set*, or *ω-limit set*, of point $\mathbf{x}$ is defined as

$$\Lambda^+ = \{\mathbf{y} | \text{ there exists a sequence of } t_n \to \infty \text{ such that } \phi(\mathbf{x}, t_n) \to \mathbf{y}\}.$$

The $\mathbf{y}$'s are referred to as *ω-limit points*. This is shown in Figure 3.31.

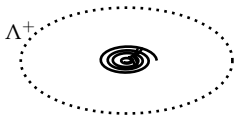


Figure 3.31: A sketch depicting an ω-limit set. Note that the orbits tend towards the set as t increases.

Similarly, we define the *negative limit set*, or the *alpha-limit set*, of point $\mathbf{x}$ is defined as

$$\Lambda^- = \{\mathbf{y} | \text{ there exists a sequences of } t_n \to -\infty \text{ such that } \phi(\mathbf{x}, t_n) \to \mathbf{y}\}$$

and the corresponding $\mathbf{y}$'s are *α-limit points*. This is shown in Figure 3.32.

Cycles and periodic orbits.

There are several types of orbits that a system might possess. A *cycle* or *periodic orbit* is any closed orbit which is not an equilibrium point. A periodic orbit is stable if for every neighborhood of the orbit such that all nearby orbits stay inside the neighborhood. Otherwise, it is unstable. The orbit is asymptotically stable if all nearby orbits converge to the periodic orbit.

A limit cycle is a cycle which is the α or ω-limit set of some trajectory other than the limit cycle. A limit cycle Γ is stable if $\Lambda^+ = \Gamma$ for all $\mathbf{x}$ in some neighborhood of Γ. A limit cycle Γ is unstable if $\Lambda^- = \Gamma$ for all $\mathbf{x}$ in some neighborhood of Γ. Finally, a limit cycle is semistable if it is attracting on one side and repelling on the other side. In the previous examples, we saw limit cycles that were stable. Figures 3.31 and 3.32 depict stable and unstable limit cycles, respectively.



Figure 3.32: A sketch depicting an α-limit set. Note that the orbits tend away from the set as t increases.

We now state a theorem which describes the type of orbits we might find in our system.

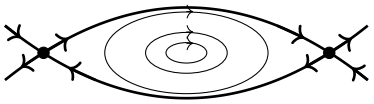


Figure 3.33: A heteroclinic orbit connecting two critical points.

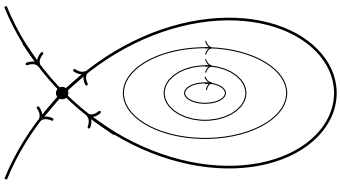


Figure 3.34: A homoclinic orbit returning to the point it left.

Theorem 3.2. Poincaré-Bendixon Theorem *Let γ^+ be contained in a bounded region in which there are finitely many critical points. Then Λ^+ is either*

1. *a single critical point;*
2. *a single closed orbit;*
3. *a set of critical points joined by heteroclinic orbits. [Compare Figures 3.33 and 3.34.]*

We are interested in determining when limit cycles may, or may not, exist. A consequence of the Poincaré-Bendixon Theorem is given by the following corollary.

Corollary Let D be a bounded closed set containing no critical points and suppose that $\gamma^+ \subset D$. Then there exists a limit cycle contained in D.

More specific criteria allow us to determine if there is a limit cycle in a given region. These are given by Dulac's Criteria and Bendixon's Criteria.

Dulac's Criteria Consider the autonomous planar system

$$x' = f(x,y), \quad y' = g(x,y)$$

and a continuously differentiable function ψ defined on an annular region D contained in some open set. If

$$\frac{\partial}{\partial x}(\psi f) + \frac{\partial}{\partial y}(\psi g)$$

does not change sign in D, then there is at most one limit cycle contained entirely in D.

Bendixon's Criteria Consider the autonomous planar system

$$x' = f(x,y), \quad y' = g(x,y)$$

defined on a simply connected domain D such that

$$\frac{\partial}{\partial x}(\psi f) + \frac{\partial}{\partial y}(\psi g) \neq 0$$

in D. Then, there are no limit cycles entirely in D.

Proof. These are easily proved using Green's Theorem in the Plane. (See your calculus text.) We prove Bendixon's Criteria. Let $\mathbf{f} = (f,g)$. Assume that Γ is a closed orbit lying in D. Let S be the interior of Γ. Then

$$\begin{aligned} \int_S \nabla \cdot \mathbf{f}\, dxdy &= \oint_\Gamma (f\, dy - g\, dx) \\ &= \int_0^T (f\dot{y} - g\dot{x})dt \\ &= \int_0^T (fg - gf)dt = 0. \end{aligned} \tag{3.54}$$

So, if $\nabla \cdot \mathbf{f}$ is not identically zero and does not change sign in S, then from the continuity of $\nabla \cdot \mathbf{f}$ in S we have that the right side above is either positive or negative. Thus, we have a contradiction and there is no closed orbit lying in D □

Example 3.16. Consider the earlier example in (3.48) with $\mu = 1$.

$$\begin{aligned} x' &= x - y - x(x^2 + y^2) \\ y' &= x + y - y(x^2 + y^2). \end{aligned} \tag{3.55}$$

We already know that a limit cycle exists at $x^2 + y^2 = 1$. A simple computation gives that

$$\nabla \cdot \mathbf{f} = 2 - 4x^2 - 4y^2.$$

For an arbitrary annulus $a < x^2 + y^2 < b$, we have

$$2 - 4b < \nabla \cdot \mathbf{f} < 2 - 4a.$$

For $a = 3/4$ and $b = 5/4$, $-3 < \nabla \cdot \mathbf{f} < -1$. Thus, $\nabla \cdot \mathbf{f} < 0$ in the annulus $3/4 < x^2 + y^2 < 5/4$. Therefore, by Dulac's Criteria there is at most one limit cycle in this annulus.

Example 3.17. Consider the system

$$\begin{aligned} x' &= y \\ y' &= -ax - by + cx^2 + dy^2. \end{aligned} \tag{3.56}$$

Let $\psi(x, y) = e^{-2dx}$. Then,

$$\frac{\partial}{\partial x}(\psi y) + \frac{\partial}{\partial y}(\psi(-ax - by + cx^2 + dy^2)) = -be^{-2dx} \neq 0.$$

We conclude by Bendixon's Criteria that there are no limit cycles for this system.

3.8 Nonautonomous Nonlinear Systems

IN THIS SECTION WE DISCUSS NONAUTONOMOUS SYSTEMS. Recall that an autonomous system is one in which there is no explicit time dependence. A simple example is the forced nonlinear pendulum given by the nonhomogeneous equation

$$\ddot{x} + \omega^2 \sin x = f(t). \tag{3.57}$$

We can set this up as a system of two first order equations:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 \sin x + f(t). \end{aligned} \tag{3.58}$$

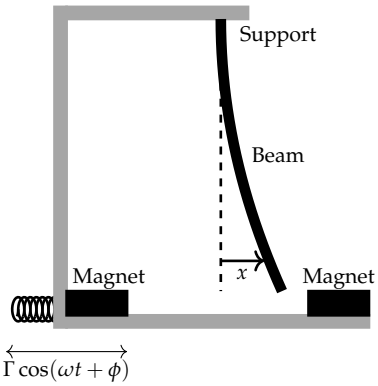


Figure 3.35: One model of the Duffing equation describes a periodically forced beam which interacts with two magnets.

This system is not in a form for which we could use the earlier methods. Namely, it is a nonautonomous system. However, we introduce a new variable $z(t) = t$ and turn it into an autonomous system in one more dimension. The new system takes the form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 \sin x + f(z). \\ \dot{z} &= 1. \end{aligned} \tag{3.59}$$

The system is now a three dimensional autonomous, possibly nonlinear, system and can be explored using methods from Chapters 2 and 3.

A more interesting model is provided by the Duffing Equation. This equation, named after Georg Wilhelm Christian Caspar Duffing (1861-1944), models hard spring and soft spring oscillations. It also models a periodically forced beam as shown in Figure 3.35. It is of interest because it is a simple system which exhibits chaotic dynamics and will motivate us towards using new visualization methods for nonautonomous systems.

The most general form of Duffing's equation is given by the damped, forced system

$$\ddot{x} + k\dot{x} + (\beta x^3 \pm \omega_0^2 x) = \Gamma \cos(\omega t + \phi). \tag{3.60}$$

This equation models hard spring, $(\beta > 0)$, and soft spring, $(\beta < 0)$, oscillations. However, we will use the simpler version of the Duffing equation:

$$\ddot{x} + k\dot{x} + x^3 - x = \Gamma \cos \omega t. \tag{3.61}$$

An equation of this form can be obtained by setting $\phi = 0$ and rescaling x and t in the original equation. We will explore the behavior of the system as we vary the remaining parameters. In Figures 3.36-3.39 we show some typical solution plots superimposed on the direction field.

The undamped, unforced Duffing equation.

We start with the undamped ($k = 0$) and unforced ($\Gamma = 0$) Duffing equation,

$$\ddot{x} + x^3 - x = 0.$$

We can write this second order equation as the autonomous system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x(1 - x^2). \end{aligned} \tag{3.62}$$

We see that there are three equilibrium points at $(0, 0)$ and $(\pm 1, 0)$. In Figure 3.36 we plot several orbits for $k = 0$, and $\Gamma = 0$. We see that the three equilibrium points consist of two centers and a saddle.

The unforced Duffing equation.

We now turn on the damping. The system becomes

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -ky + x(1 - x^2). \end{aligned} \tag{3.63}$$

In Figures 3.37 and 3.38 we show what happens when $k = 0.1$. These plots are reminiscent of the plots for the nonlinear pendulum; however, there are fewer equilibria. Note that the centers become stable spirals for $k > 0$.

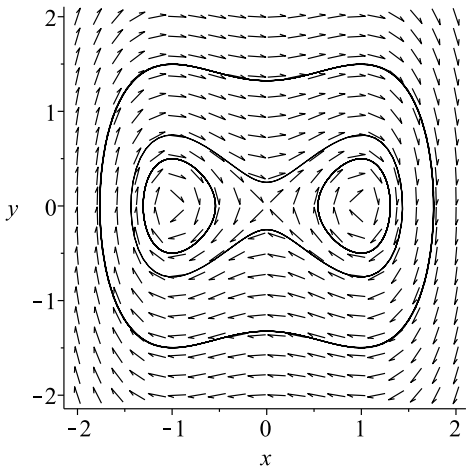


Figure 3.36: Phase plane for the undamped, unforced Duffing equation ($k = 0, \Gamma = 0$).

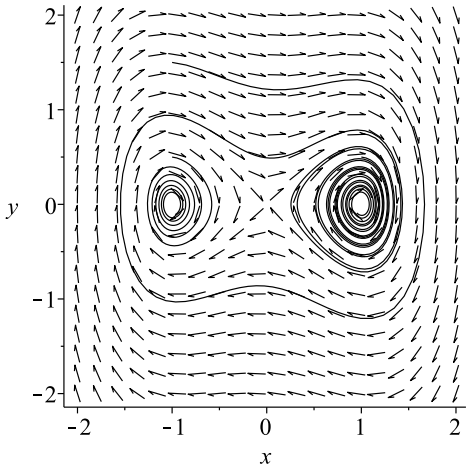


Figure 3.37: Phase plane for the unforced Duffing equation with $k = 0.1$ and $\Gamma = 0$.

Next we turn on the forcing to obtain a damped, forced Duffing equation. The system is now nonautonomous.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x(1-x^2) + \Gamma \cos \omega t. \end{aligned} \tag{3.64}$$

The damped, forced Duffing equation.

In Figure 3.39 we only show one orbit with $k = 0.1$, $\Gamma = 0.5$, and $\omega = 1.25$. The solution intersects itself and look a bit messy. We can imagine what we would get if we added any more orbits. For completeness, we show in Figure 3.40 an example with four different orbits.

In cases for which one has periodic orbits such as the Duffing equation, Poincaré introduced the notion of *surfaces of section*. One embeds the orbit in a higher dimensional space so that there are no self intersections, like we saw in Figures 3.39 and 3.40. In Figure 3.42 we show an example where a simple orbit is shown as it periodically pierces a given surface.

In order to simplify the resulting pictures, one only plots the points at which the orbit pierces the surface as sketched in Figure 3.41. In practice, there is a natural frequency, such as ω in the forced Duffing equation. Then,

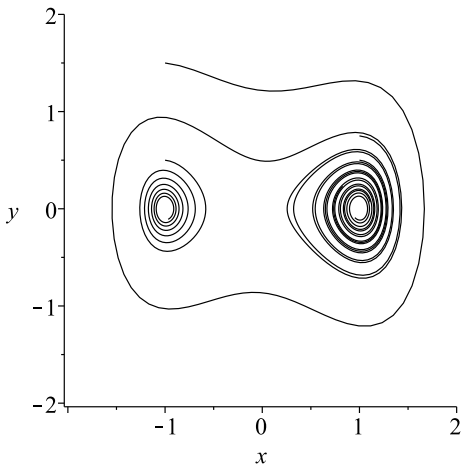


Figure 3.38: Display of two orbits for the unforced Duffing equation with $k = 0.1$ and $\Gamma = 0$.

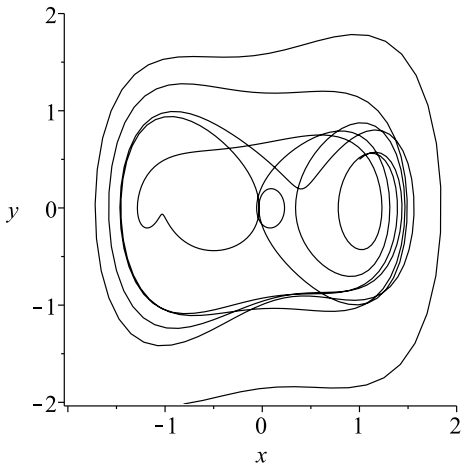


Figure 3.39: Phase plane for the Duffing equation with $k = 0.1$, $\Gamma = 0.5$, and $\omega = 1.25$. In this case we show only one orbit which was generated from the initial condition $(x_0 = 1.0, \quad y_0 = 0.5)$.

one plots points at times that are multiples of the period, $T = \frac{2\pi}{\omega}$. In Figure 3.43 we show what the plot for one orbit would look like for the damped, unforced Duffing equation.

The more interesting case, is when there is forcing and damping. In this case the surface of section plot is given in Figure 3.44. While this is not as busy as the solution plot in Figure 3.39, it still provides some interesting behavior. What one finds is what is called a strange attractor. Plotting many orbits, we find that after a long time, all of the orbits are attracted to a small region in the plane, much like a stable node attracts nearby orbits. However, this set consists of more than one point. Also, the flow on the attractor is chaotic in nature. Thus, points wander in an irregular way throughout the attractor. This is one of the interesting topics in chaos theory and this whole theory of dynamical systems has only been touched in this text leaving the reader to wander of into further depth into this fascinating field.

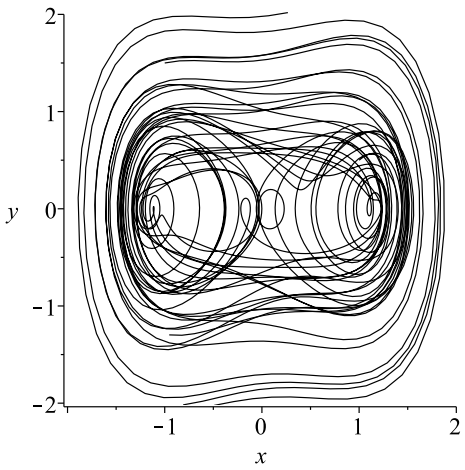


Figure 3.40: Phase plane for the Duffing equation with $k = 0.1$, $\Gamma = 0.5$, and $\omega = 1.25$. In this case four initial conditions were used to generate four orbits.

The surface of section plots at the end of the last section were obtained using code from S. Lynch's book *Dynamical Systems with Applications Using Maple*. For reference, the plots in Figures 3.36 and 3.37 were generated in Maple using the following commands:

```
> with(DEtools):
> Gamma:=0.5:omega:=1.25:k:=0.1:
> DEplot([diff(x(t),t)=y(t), diff(y(t),t)=x(t)-k*y(t)-(x(t))^3
  + Gamma*cos(omega*t)], [x(t),y(t)],t=0..500,[[x(0)=1,y(0)=0.5],
  [x(0)=-1,y(0)=0.5], [x(0)=1,y(0)=0.75], [x(0)=-1,y(0)=1.5]],
  x=-2..2,y=-2..2, stepsize=0.1, linecolor=blue, thickness=1,
  color=black);
```

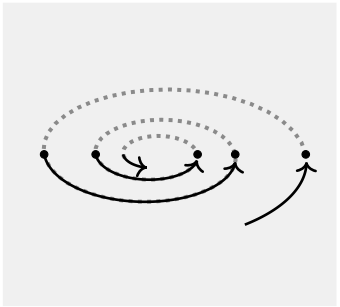


Figure 3.41: As an orbit pierces the surface of section, one plots the point of intersection in that plane to produce the surface of section plot.

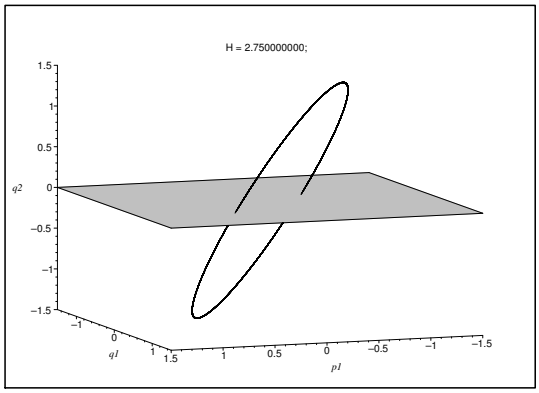


Figure 3.42: Poincaré's surface of section. One notes each time the orbit pierces the surface.

Figure 3.43: Poincaré's surface of section plot for the damped, unforced Duffing equation.

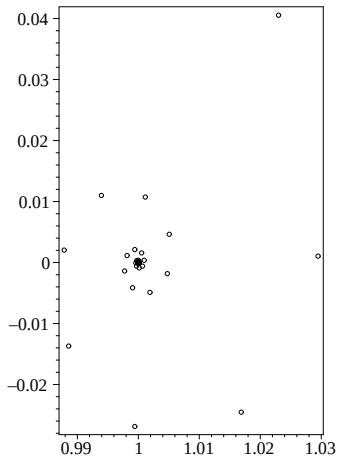
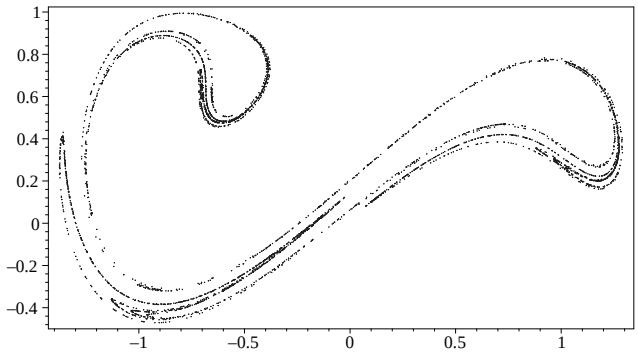


Figure 3.44: Poincaré's surface of section plot for the damped, forced Duffing equation. This leads to what is known as a strange attractor.



3.9 The Period of the Nonlinear Pendulum*

RECALL THAT THE PERIOD OF THE SIMPLE PENDULUM is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \tag{3.65}$$

for

$$\omega \equiv \sqrt{\frac{g}{L}}. \tag{3.66}$$

This was based upon the solving the linear pendulum equation (3.12). This equation was derived assuming a small angle approximation. How good is this approximation? What is meant by a *small* angle?

We recall that the Taylor series approximation of $\sin\theta$ about $\theta = 0$:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots. \tag{3.67}$$

One can obtain a bound on the error when truncating this series to one term after taking a numerical analysis course. But we can just simply plot the relative error, which is defined as

$$\text{Relative Error } = \left|\frac{\sin\theta - \theta}{\sin\theta}\right|.$$

A plot of the relative error is given in Figure 3.45. Thus for $\theta \approx 0.4$ radians (or, 23^o) we have that the relative error is about 2.6%.

Relative error in $\sin\theta$ approximation.

We would like to do better than this. So, we now turn to the nonlinear pendulum equation (3.11) in the simpler form

$$\ddot{\theta} + \omega^2 \sin\theta = 0. \tag{3.68}$$

Solution of nonlinear pendulum equation.

We next employ a technique that is useful for equations of the form

$$\ddot{\theta} + F(\theta) = 0$$

when it is easy to integrate the function $F(\theta)$. Namely, we note that

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 + \int^{\theta(t)} F(\phi)\, d\phi\right] = (\ddot{\theta} + F(\theta))\dot{\theta}.$$

For the nonlinear pendulum problem, we multiply Equation (3.68) by $\dot{\theta}$,

$$\ddot{\theta}\dot{\theta} + \omega^2 \sin\theta\dot{\theta} = 0$$

and note that the left side of this equation is a perfect derivative. Thus,

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta\right] = 0.$$

Therefore, the quantity in the brackets is a constant. So, we can write

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = c. \tag{3.69}$$

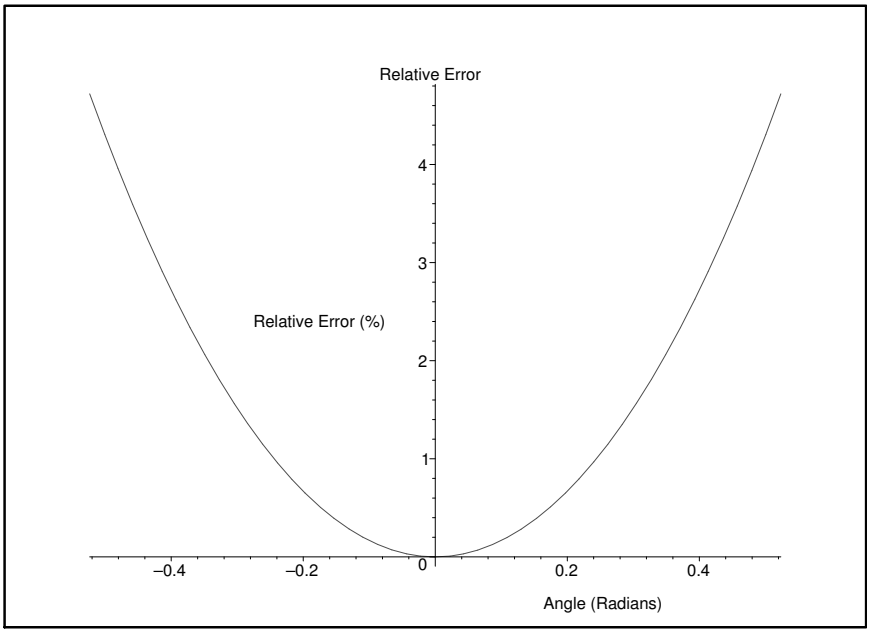


Figure 3.45: The relative error in percent when approximating $\sin\theta$ by θ.

Solving for $\dot{\theta}$, we obtain

$$\frac{d\theta}{dt} = \sqrt{2(c + \omega^2 \cos\theta)}.$$

This equation is a separable first order equation and we can rearrange and integrate the terms to find that

$$t = \int dt = \int \frac{d\theta}{\sqrt{2(c + \omega^2 \cos\theta)}}.$$

Of course, we need to be able to do the integral. When one finds a solution in this implicit form, one says that the problem has been solved by quadratures. Namely, the solution is given in terms of some integral.

In fact, the above integral can be transformed into what is known as an elliptic integral of the first kind. We will rewrite this result and then use it to obtain an approximation to the period of oscillation of the nonlinear pendulum, leading to corrections to the linear result found earlier.

We will first rewrite the constant found in (3.69). This requires a little physics. The swinging of a mass on a string, assuming no energy loss at the pivot point, is a conservative process. Namely, the total mechanical energy is conserved. Thus, the total of the kinetic and gravitational potential energies is a constant. The kinetic energy of the mass on the string is given as

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\dot{\theta}^2.$$

The potential energy is the gravitational potential energy. If we set the potential energy to zero at the bottom of the swing, then the potential energy is $U = mgh$, where h is the height that the mass is from the bottom of the

swing. A little trigonometry gives that $h = L(1 - \cos\theta)$. So,

$$U = mgL(1 - \cos\theta).$$

Total mechanical energy for the nonlinear pendulum.

So, the total mechanical energy is

$$E = \frac{1}{2}mL^2\dot{\theta}^2 + mgL(1 - \cos\theta). \tag{3.70}$$

We note that a little rearranging shows that we can relate this result to Equation (3.69). Dividing by m and L^2 and using the definition of $\omega^2 = g/L$, we have

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = \frac{1}{mL^2}E - \omega^2.$$

Therefore, we have determined the integration constant in terms of the total mechanical energy,

$$c = \frac{1}{mL^2}E - \omega^2.$$

We can use Equation (3.70) to get a value for the total energy. At the top of the swing the mass is not moving, if only for a moment. Thus, the kinetic energy is zero and the total mechanical energy is pure potential energy. Letting θ_0 denote the angle at the highest angular position, we have that

$$E = mgL(1 - \cos\theta_0) = mL^2\omega^2(1 - \cos\theta_0).$$

Therefore, we have found that

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = -\omega^2\cos\theta_0. \tag{3.71}$$

We can solve for $\dot{\theta}$ and integrate the differential equation to obtain

$$t = \int dt = \int \frac{d\theta}{\omega\sqrt{2(\cos\theta - \cos\theta_0)}}.$$

Using the half angle formula,

$$\sin^2\frac{\theta}{2} = \frac{1}{2}(1 - \cos\theta),$$

we can rewrite the argument in the radical as

$$\cos\theta - \cos\theta_0 = 2\left[\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right].$$

Noting that a motion from $\theta = 0$ to $\theta = \theta_0$ is a quarter of a cycle, we have that

$$T = \frac{2}{\omega}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}}. \tag{3.72}$$

This result can now be transformed into an elliptic integral.[2] We define

$$z = \frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}}$$

[2] Elliptic integrals were first studied by Leonhard Euler and Giulio Carlo de' Toschi di Fagnano (1682-1766) , who studied the lengths of curves such as the ellipse and the lemniscate,

$$(x^2 + y^2)^2 = x^2 - y^2.$$

and

$$k = \sin\frac{\theta_0}{2}.$$

Then, Equation (3.72) becomes

$$T = \frac{4}{\omega}\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}. \tag{3.73}$$

This is done by noting that $dz = \frac{1}{2k}\cos\frac{\theta}{2}\,d\theta = \frac{1}{2k}(1-k^2z^2)^{1/2}\,d\theta$ and that $\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2} = k^2(1-z^2)$. The integral in this result is called the complete elliptic integral of the first kind.

We note that the incomplete elliptic integral of the first kind is defined as

$$F(\phi,k) \equiv \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\sin\phi} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

The complete and incomplete elliptic integrals of the first kind.

Then, the complete elliptic integral of the first kind is given by $K(k) = F(\frac{\pi}{2},k)$, or

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

Therefore, the period of the nonlinear pendulum is given by

$$T = \frac{4}{\omega}K\left(\sin\frac{\theta_0}{2}\right). \tag{3.74}$$

There are table of values for elliptic integrals. However, one can use a computer algebra system to compute values of such integrals. We will look for small angle approximations.

For small angles ($\theta_0 \ll \frac{\pi}{2}$), we have that k is small. So, we can develop a series expansion for the period, T, for small k. This is simply done by using the binomial expansion,

$$(1-k^2z^2)^{-1/2} = 1 + \frac{1}{2}k^2z^2 + \frac{3}{8}k^2z^4 + O((kz)^6)$$

Inserting this expansion into the integrand for the complete elliptic integral and integrating term by term, we find that

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \ldots\right]. \tag{3.75}$$

The first term of the expansion gives the well known period of the simple pendulum for small angles. The next terms in the expression give further corrections to the linear result which are useful for larger amplitudes of oscillation. In Figure 3.46 we show the relative errors incurred when keeping the k^2 (quadratic) and k^4 (quartic) terms as compared to the exact value of the period.

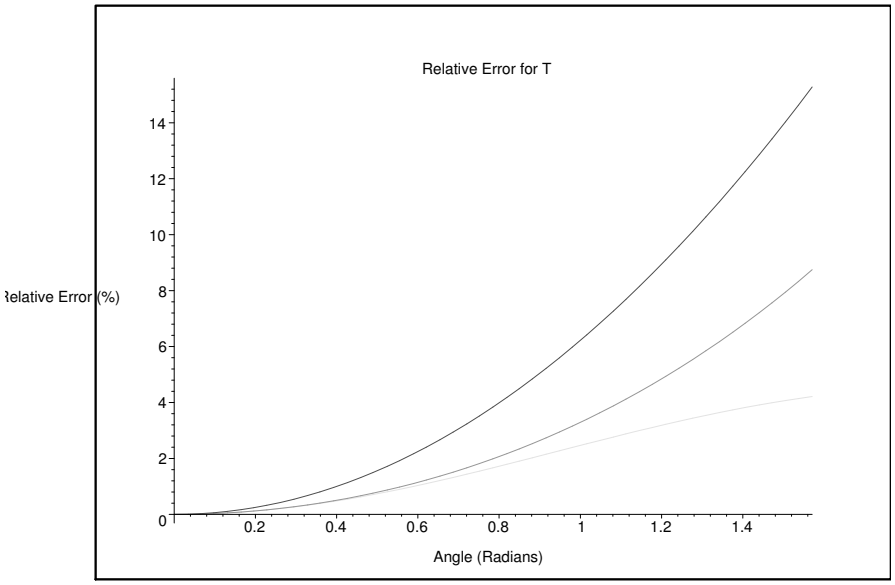


Figure 3.46: The relative error in percent when approximating the exact period of a nonlinear pendulum with one (solid), two (dashed), or three (dotted) terms in Equation (3.75).

3.10 *Exact Solutions Using Elliptic Functions**

THE SOLUTION IN EQUATION (3.73) OF THE NONLINEAR PENDULUM EQUATION led to the introduction of elliptic integrals. The incomplete elliptic integral of the first kind is defined as

$$F(\phi,k) \equiv \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\sin\phi} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}. \tag{3.76}$$

The complete integral of the first kind is given by $K(k) = F(\frac{\pi}{2},k)$, or

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{1} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

Elliptic integrals of the second kind are defined as

$$E(\phi,k) = \int_0^{\phi} \sqrt{1-k^2\sin^2\theta}\,d\theta = \int_0^{\sin\phi} \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,dt \tag{3.77}$$

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\,d\theta = \int_0^{1} \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,dt \tag{3.78}$$

Recall, a first integration of the nonlinear pendulum equation from Equation (3.70),

$$\left(\frac{d\theta}{dt}\right)^2 - \omega^2\cos\theta = -\omega^2\cos\theta_0.$$

or

$$\left(\frac{d\theta}{dt}\right)^2 = 2\omega^2\left[\sin^2\frac{\theta}{2} - \sin^2\frac{\theta_0}{2}\right].$$

Letting

$$kz = \sin\frac{\theta}{2} \text{ and } k = \sin\frac{\theta_0}{2},$$

the differential equation becomes

$$\frac{dz}{d\tau} = \pm\omega\sqrt{1-z^2}\sqrt{1-k^2z^2}.$$

Applying separation of variables, we find

$$\begin{aligned} \pm\omega(t-t_0) &= \frac{1}{\omega}\int_1^z \frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}} && (3.79)\\ &= \int_0^1 \frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}} - \int_0^z \frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}} && (3.80)\\ &= K(k) - F(\sin^{-1}(k^{-1}\sin\theta), k). && (3.81) \end{aligned}$$

The solution, $\theta(t)$, is then found by solving for z and using $kz = \sin\frac{\theta}{2}$ to solve for θ. This requires that we know how to invert the elliptic integral, $F(z,k)$.

Elliptic functions result from the inversion of elliptic integrals. Consider

$$\begin{aligned} u(\sin\phi, k) = F(\phi, k) &= \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}. && (3.82)\\ &= \int_0^{\sin\phi} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. && (3.83) \end{aligned}$$

Note:$F(\phi, 0) = \phi$ and $F(\phi, 1) = \ln(\sec\phi + \tan\phi)$. In these cases F is obviously monotone increasing and thus there must be an inverse.

The inverse of Equation (3.76) is defined as $\phi = F^{-1}(u,k) = \text{am}(u,k)$, where $u = \sin\phi$. The function $\text{am}(u,k)$ is called the Jacobi amplitude function and k is the elliptic modulus. [In some references and software like MATLAB packages, $m = k^2$ is used as the parameter.] Three of the Jacobi elliptic functions, shown in Figure 3.47, can be defined in terms of the amplitude function by

$$\begin{aligned} \text{sn}(u,k) &= \sin\text{am}(u,k) = \sin\phi,\\ \text{cn}(u,k) &= \cos\text{am}(u,k) = \cos\phi, \end{aligned}$$

Jacobi elliptic functions.

and the delta amplitude

$$\text{dn}(u,k) = \sqrt{1-k^2\sin^2\phi}.$$

They are related through the identities

$$\text{cn}^2(u,k) + \text{sn}^2(u,k) = 1, \quad (3.84)$$

$$\text{dn}^2(u,k) + k^2\,\text{sn}^2(u,k) = 1. \quad (3.85)$$

The elliptic functions can be extended to the complex plane. In this case the functions are doubly periodic. However, we will not need to consider this in the current text.

Also, we see that these functions are periodic. The period is given in terms of the complete elliptic integral of the first kind, $K(k)$. Consider

$$\begin{aligned} F(\phi + 2\pi, k) &= \int_0^{\phi+2\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.\\ &= \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} + \int_\phi^{\phi+2\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \end{aligned}$$

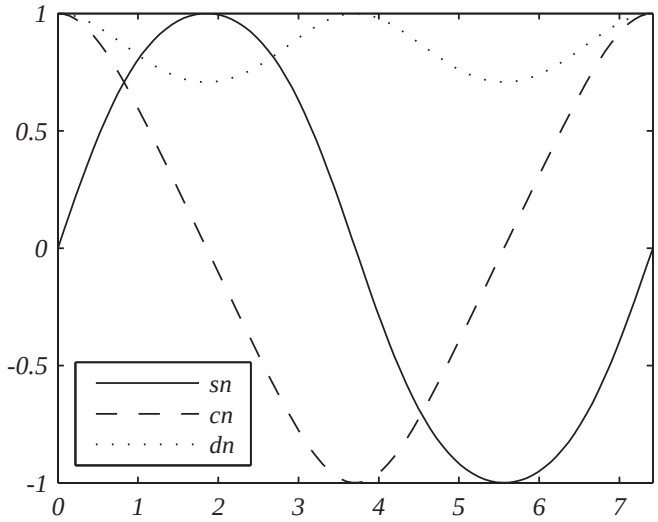


Figure 3.47: Plots of the Jacobi elliptic functions $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$, and $\mathrm{dn}(u,k)$ for $m = k^2 = 0.5$. Here $K(k) = 1.8541$.

$$\begin{aligned} &= F(\phi,k) + \int_0^{2\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \\ &= F(\phi,k) + 4K(k). \end{aligned} \tag{3.86}$$

Since $F(\phi + 2\pi, k) = u + 4K$, we have

$$\mathrm{sn}(u+4K) = \sin(\mathrm{am}(u+4K)) = \sin(\mathrm{am}(u)+2\pi) = \sin\mathrm{am}(u) = \mathrm{sn}\,u.$$

In general, we have

$$\mathrm{sn}(u+2K,k) = -\mathrm{sn}(u,k) \tag{3.87}$$
$$\mathrm{cn}(u+2K,k) = -\mathrm{cn}(u,k) \tag{3.88}$$
$$\mathrm{dn}(u+2K,k) = \mathrm{dn}(u,k). \tag{3.89}$$

The plots of $\mathrm{sn}(u)$, $\mathrm{cn}(u)$, and $dn(u)$, are shown in Figures 3.48-3.50.

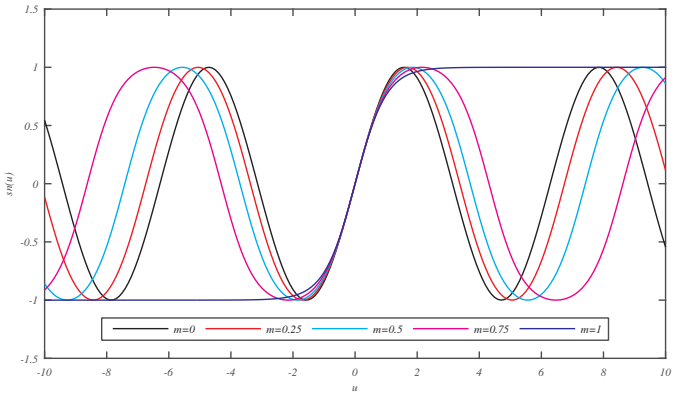


Figure 3.48: Plots of $\mathrm{sn}(u,k)$ for $m = 0, 0.25, 0.50, 0.75, 1.00$.

Namely,

$$\mathrm{sn}(u+K,k) = \frac{\mathrm{cn}\,u}{\mathrm{dn}\,u}, \quad \mathrm{sn}(u+2K,k) = -\mathrm{sn}\,u,$$

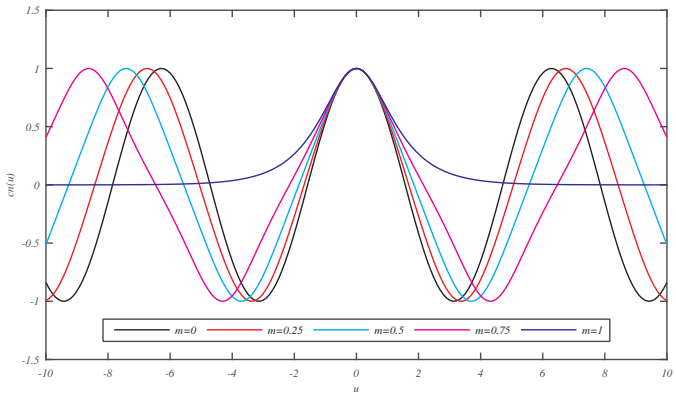


Figure 3.49: Plots of $\mathrm{cn}(u,k)$ for $m = 0, 0.25, 0.50, 0.75, 1.00$.

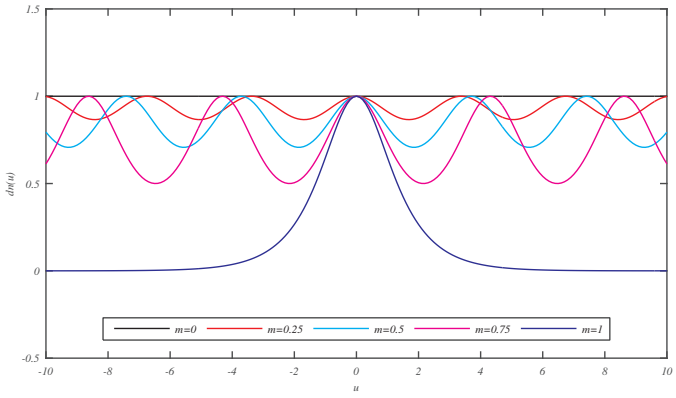


Figure 3.50: Plots of $\mathrm{dn}(u,k)$ for $m = 0, 0.25, 0.50, 0.75, 1.00$.

$$\mathrm{cn}(u+K,k) = -\sqrt{1-k^2}\,\frac{\mathrm{sn}\,u}{\mathrm{dn}\,u}, \quad \mathrm{dn}(u+2K,k) = -\,\mathrm{cn}\,u,$$

$$\mathrm{dn}(u+K,k) = \frac{\sqrt{1-k^2}}{\mathrm{dn}\,u}, \quad \mathrm{dn}(u+2K,k) = \mathrm{dn}\,u.$$

Therefore, dn and cn have a period of $4K$ and dn has a period of $2K$.

Special values found in Figure 3.47 are seen as

$$\mathrm{sn}(K,k) = 1,$$

$$\mathrm{cn}(K,k) = 0,$$

$$\mathrm{dn}(K,k) = \sqrt{1-k^2} = k',$$

where k' is called the complementary modulus.

Important to this section are the derivatives of these elliptic functions,

$$\frac{\partial}{\partial u}\,\mathrm{sn}(u,k) = \mathrm{cn}(u,k)\,\mathrm{dn}(u,k),$$

$$\frac{\partial}{\partial u}\,\mathrm{cn}(u,k) = -\,\mathrm{sn}(u,k)\,\mathrm{dn}(u,k),$$

$$\frac{\partial}{\partial u}\,\mathrm{dn}(u,k) = -k^2\,\mathrm{sn}(u,k)\,\mathrm{cn}(u,k),$$

and the amplitude function

$$\frac{\partial}{\partial u}\,\mathrm{am}(u,k) = \mathrm{dn}(u,k).$$

Sometimes the Jacobi elliptic functions are displayed without reference to the elliptic modulus, such as $\text{sn}(u) = \text{sn}(u,k)$. When k is understood, we can do the same.

Example 3.18. Show that $\text{sn}(u)$ satisfies the differential equation

$$y'' + (1+k^2)y = 2k^2y^3.$$

From the above derivatives, we have that

$$\begin{aligned} \frac{d^2}{du^2}\,\text{sn}(u) &= \frac{d}{du}(\text{cn}(u)\,\text{dn}(u)) \\ &= -\,\text{sn}(u)\,\text{dn}^2(u) - k^2\,\text{sn}(u)\,\text{cn}^2(u). \end{aligned} \tag{3.90}$$

Letting $y(u) = \text{sn}(u)$ and using the identities (3.84)-(3.85), we have that

$$y'' = -y(1-k^2y^2) - k^2y(1-y^2) = -(1+k^2)y + 2k^2y^3.$$

This is the desired result.

Example 3.19. Show that $\theta(t) = 2\sin^{-1}(k\,\text{sn}\,t)$ is a solution of the equation $\ddot{\theta} + \sin\theta = 0$.

Differentiating $\theta(t) = 2\sin^{-1}(k\,\text{sn}\,t)$, we have

$$\begin{aligned} \frac{d^2}{dt^2}\left(2\sin^{-1}(k\,\text{sn}\,t)\right) &= \frac{d}{dt}\left(2\frac{k\,\text{cn}\,t\,\text{dn}\,t}{\sqrt{1-k^2\,\text{sn}^2\,t}}\right) \\ &= \frac{d}{dt}\left(2k\,\text{cn}\,t\right) \\ &= -2k\,\text{sn}\,t\,\text{dn}\,t. \end{aligned} \tag{3.91}$$

However, we can evaluate $\sin\theta$ for a range of θ. Thus, we have

$$\begin{aligned} \sin\theta &= \sin(2\sin^{-1}(k\,\text{sn}\,t)) \\ &= 2\sin(\sin^{-1}(k\,\text{sn}\,t))\cos(\sin^{-1}(k\,\text{sn}\,t)) \\ &= 2k\,\text{sn}\,t\sqrt{1-k^2\,\text{sn}^2\,t} \\ &= 2k\,\text{sn}\,t\,\text{dn}\,t. \end{aligned} \tag{3.92}$$

Comparing these results, we have shown that $\ddot{\theta} + \sin\theta = 0$.

The solution to the last example can be used to obtain the exact solution to the nonlinear pendulum problem, $\ddot{\theta} + \omega^2\sin\theta = 0$, $\theta(0) = \theta_0$, $\dot{\theta}(0) = 0$. The general solution is given by $\theta(t) = 2\sin^{-1}(k\,\text{sn}(\omega t + \phi))$ where ϕ has to be determined from the initial conditions. We note that

$$\begin{aligned} \frac{d\,\text{sn}(u+K)}{du} &= \text{cn}(u+K)\,\text{dn}(u+K) \\ &= \left(-\sqrt{1-k^2}\frac{\text{sn}\,u}{\text{dn}\,u}\right)\left(\frac{\sqrt{1-k^2}}{\text{dn}\,u}\right) \\ &= -(1-k^2)\frac{\text{sn}\,u}{\text{dn}^2\,u}. \end{aligned} \tag{3.93}$$

Evaluating at $u = 0$, we have $\text{sn}'(K) = 0$.

Therefore, if we pick $\phi = K$, then $\dot{\theta}(0) = 0$ and the solution is

$$\theta(t) = 2\sin^{-1}(k\,\mathrm{sn}(\omega t + K)).$$

Furthermore, the other initial value is found to be

$$\theta(0) = 2\sin^{-1}(k\,\mathrm{sn}\,K) = 2\sin^{-1}k.$$

Thus, $k = \sin\frac{\theta_0}{2}$, as we had seen in the earlier derivation of the elliptic integral solution. The solution is given as

$$\theta(t) = 2\sin^{-1}(\sin\frac{\theta_0}{2}\,\mathrm{sn}(\omega t + K)).$$

In Figures 3.51-3.52 we show comparisons of the exact solutions of the linear and nonlinear pendulum problems for $L = 1.0$ m and initial angles $\theta_0 = 10^o$ and $\theta_0 = 50^o$.

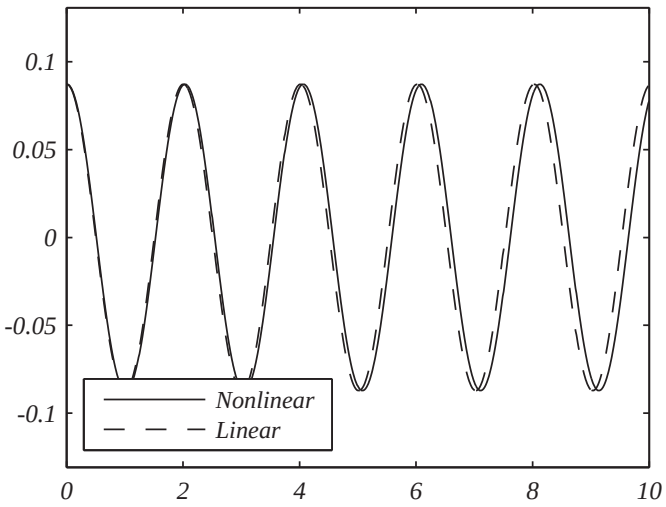


Figure 3.51: Comparison of exact solutions of the linear and nonlinear pendulum problems for $L = 1.0$ m and $\theta_0 = 10^o$.

Problems

1. Solve the general logistic problem,

$$\frac{dy}{dt} = ky - cy^2, \quad y(0) = y_0 \tag{3.94}$$

using separation of variables.

2. Find the equilibrium solutions and determine their stability for the following systems. For each case draw representative solutions and phase lines.

a. $y' = y^2 - 6y - 16$.

b. $y' = \cos y$.

c. $y' = y(y-2)(y+3)$.

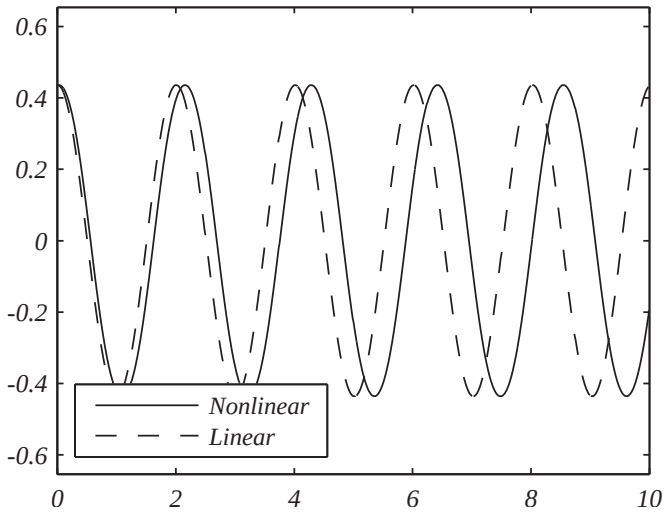


Figure 3.52: Comparison of the exact solutions of the linear and nonlinear pendulum problems for $L = 1.0$ m and $\theta_0 = 50^o$.

d. $y' = y^2(y+1)(y-4)$.

3. For $y' = y - y^2$, find the general solution corresponding to $y(0) = y_0$. Provide specific solutions for the following initial conditions and sketch them: a. $y(0) = 0.25$, b. $y(0) = 1.5$, and c. $y(0) = -0.5$.

4. For each problem determine equilibrium points, bifurcation points and construct a bifurcation diagram. Discuss the different behaviors in each system.

a. $y' = y - \mu y^2$

b. $y' = y(\mu - y)(\mu - 2y)$

c. $x' = \mu - x^3$

d. $x' = x - \frac{\mu x}{1+x^2}$

5. Consider the family of differential equations $x' = x^3 + \delta x^2 - \mu x$.

a. Sketch a bifurcation diagram in the $x\mu$-plane for $\delta = 0$.

b. Sketch a bifurcation diagram in the $x\mu$-plane for $\delta > 0$.

Hint: Pick a few values of δ and μ in order to get a feel for how this system behaves.

6. System 3.52 can be solved exactly. Integrate the r-equation using separation of variables. For initial conditions a) $r(0) = 0.25$, $\theta(0) = 0$, and b) $r(0) = 1.5$, $\theta(0) = 0$, and $\mu = 1.0$, find and plot the solutions in the xy-plane showing the approach to a limit cycle.

7. Consider the system

$$\begin{aligned} x' &= -y + x\left[\mu - x^2 - y^2\right], \\ y' &= x + y\left[\mu - x^2 - y^2\right]. \end{aligned}$$

Rewrite this system in polar form. Look at the behavior of the r equation and construct a bifurcation diagram in μr space. What might this diagram look like in the three dimensional μxy space? (Think about the symmetry in this problem.) This leads to what is called a *Hopf bifurcation.*

8. Find the fixed points of the following systems. Linearize the system about each fixed point and determine the nature and stability in the neighborhood of each fixed point, when possible. Verify your findings by plotting phase portraits using a computer.

a.

$$\begin{aligned} x' &= x(100 - x - 2y), \\ y' &= y(150 - x - 6y). \end{aligned}$$

b.

$$\begin{aligned} x' &= x + x^3, \\ y' &= y + y^3. \end{aligned}$$

c.

$$\begin{aligned} x' &= x - x^2 + xy, \\ y' &= 2y - xy - 6y^2. \end{aligned}$$

d.

$$\begin{aligned} x' &= -2xy, \\ y' &= -x + y + xy - y^3. \end{aligned}$$

9. Plot phase portraits for the Lienard system

$$\begin{aligned} x' &= y - \mu(x^3 - x) \\ y' &= -x. \end{aligned}$$

for a small and a not so small value of μ. Describe what happens as one varies μ.

10. Consider the period of a nonlinear pendulum. Let the length be $L = 1.0$ m and $g = 9.8$ m/s^2. Sketch T vs the initial angle θ_0 and compare the linear and nonlinear values for the period. For what angles can you use the linear approximation confidently?

11. Another population model is one in which species compete for resources, such as a limited food supply. Such a model is given by

$$\begin{aligned} x' &= ax - bx^2 - cxy, \\ y' &= dy - ey^2 - fxy. \end{aligned}$$

In this case, assume that all constants are positive.

a Describe the effects/purpose of each terms.

b Find the fixed points of the model.

c Linearize the system about each fixed point and determine the stability.

d From the above, describe the types of solution behavior you might expect, in terms of the model.

12. Consider a model of a food chain of three species. Assume that each population on its own can be modeled by logistic growth. Let the species be labeled by $x(t)$, $y(t)$, and $z(t)$. Assume that population x is at the bottom of the chain. That population will be depleted by population y. Population y is sustained by x's, but eaten by z's. A simple, but scaled, model for this system can be given by the system

$$\begin{aligned} x' &= x(1-x) - xy \\ y' &= y(1-y) + xy - yz \\ z' &= z(1-z) + yz. \end{aligned}$$

a. Find the equilibrium points of the system.

b. Find the Jacobian matrix for the system and evaluate it at the equilibrium points.

c. Find the eigenvalues and eigenvectors.

d. Describe the solution behavior near each equilibrium point.

e. Which of these equilibria are important in the study of the population model and describe the interactions of the species in the neighborhood of these point(s).

13. Derive the first integral of the Lotka-Volterra system, $a \ln y + d \ln x - cx - by = C$.

14. Show that the system $x' = x - y - x^3$, $y' = x + y - y^3$, has a unique limit cycle by picking an appropriate $\psi(x, y)$ in Dulac's Criteria.

15. The Lorenz model is a simple model for atmospheric convection developed by Edward Lorenz in 1963. The system is given by the three equations

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y - x) \\ \frac{dy}{dt} &= x(\rho - z) - y \\ \frac{dz}{dt} &= xy - \beta z. \end{aligned}$$

a. Find the equilibrium points of the system.

b. Find the Jacobian matrix for the system and evaluate it at the equilibrium points.

c. Determine any bifurcation points and describe what happens near the bifurcation point(s). Consider $\sigma = 10$, $\beta = 8/3$, and vary ρ.

d. This system is know to exhibit chaotic behavior. Lorenz found a so-called strange attractor for parameter values $\sigma = 10$, $\beta = 8/3$, and $\rho = 28$. Using a computer, locate this strange attractor.

16. The Michaelis-Menten kinetics reaction is given by

$$\boxed{E + S \underset{k_1}{\overset{k_3}{\leftrightarrows}} ES \xrightarrow[k_2]{} E + P.}$$

The resulting system of equations for the chemical concentrations is

$$\begin{aligned} \frac{d[S]}{dt} &= -k_1[E][S] + k_3[ES], \\ \frac{d[E]}{dt} &= -k_1[E][S] + (k_2 + k_3)[ES], \\ \frac{d[ES]}{dt} &= k_1[E][S] - (k_2 + k_3)[ES], \\ \frac{d[P]}{dt} &= k_2[ES]. \end{aligned} \tag{3.95}$$

In chemical kinetics one seeks to determine the rate of product formation ($v = d[P]/dt = k_3[ES]$). Assuming that $[ES]$ is a constant, find v as a function of $[S]$ and the total enzyme concentration $[E_T] = [E] + [ES]$. As a nonlinear dynamical system, what are the equilibrium points?

17. In Equation (2.58) we saw a linear version of an epidemic model. The commonly used nonlinear SIR model is given by

$$\begin{aligned} \frac{dS}{dt} &= -\beta SI \\ \frac{dI}{dt} &= \beta SI - \gamma I \\ \frac{dR}{dt} &= \gamma I, \end{aligned} \tag{3.96}$$

where S is the number of susceptible individuals, I is the number of infected individuals, and R are the number who have been removed from the the other groups, either by recovering or dying.

a. Let $N = S + I + R$ be the total population. Prove that $N =$ constant. Thus, one need only solve the first two equations and find $R = N - S - I$ afterwards.

b. Find and classify the equilibria. Describe the equilibria in terms of the population behavior.

c. Let $\beta = 0.05$ and $\gamma = 0.2$. Assume that in a population of 100 there is one infected person. Numerically solve the system of equations for $S(t)$ and $I(t)$ and describe the solution being careful to determine the units of population and the constants.

d. The equations can be modified by adding constant birth and death rates. Assuming these are te same, one would have a new system.

$$\frac{dS}{dt} = -\beta SI + \mu(N - S)$$

$$\begin{aligned} \frac{dI}{dt} &= \beta SI - \gamma I - \mu I \\ \frac{dR}{dt} &= \gamma I - \mu R. \end{aligned} \tag{3.97}$$

How does this affect any equilibrium solutions?

e. Again, let $\beta = 0.05$ and $\gamma = 0.2$. Let $\mu = 0.1$ For a population of 100 with one infected person numerically solve the system of equations for $S(t)$ and $I(t)$ and describe the solution being careful to determine the units of population and the constants.

18. An undamped, unforced Duffing equation, $\ddot{x} + \omega^2 x + \epsilon x^3 = 0$, can be solved exactly in terms of elliptic functions. Using the results of Exercise 3.18, determine the solution of this equation and determine if there are any restrictions on the parameters.

19. Determine the circumference of an ellipse in terms of an elliptic integral.

20. Evaluate the following in terms of elliptic integrals and compute the values to four decimal places.

a. $\int_0^{\pi/4} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$.

b. $\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{4}\sin^2\theta}}$.

c. $\int_0^2 \frac{dx}{\sqrt{(9-x^2)(4-x^2)}}$.

d. $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}}$.

e. $\int_1^\infty \frac{dx}{\sqrt{x^4-1}}$.

Chapter 4

Boundary Value Problems

"Once you learn the concept of a differential equation, you see differential equations all over, no matter what you do." - Gian-Carlo Rota (1932-1999)

4.1 Introduction

Until this point we have solved initial value problems. For an initial value problem one has to solve a differential equation subject to conditions on the unknown function and its derivatives at one value of the independent variable. For example, for $x = x(t)$ we could have the initial value problem

$$x'' + x = 2, \quad x(0) = 1, \quad x'(0) = 0. \tag{4.1}$$

In the next chapters we will study boundary value problems and various tools for solving such problems. In this chapter we will motivate our interest in boundary value problems by looking into solving the one-dimensional heat equation, which is a partial differential equation. for the rest of the section, we will use this solution to show that in the background of our solution of boundary value problems is a structure based upon linear algebra and analysis leading to the study of inner product spaces. Though technically, we should be lead to Hilbert spaces, which are complete inner product spaces.

For an initial value problem one has to solve a differential equation subject to conditions on the unknown function or its derivatives at more than one value of the independent variable. As an example, we have a slight modification of the above problem: Find the solution $x = x(t)$ for $0 \leq t \leq 1$ that satisfies the problem

$$x'' + x = 2, \quad x(0) = 1, \quad x(1) = 0. \tag{4.2}$$

Typically, initial value problems involve time dependent functions and boundary value problems are spatial. So, with an initial value problem one knows how a system evolves in terms of the differential equation and the state of the system at some fixed time. Then one seeks to determine the state of the system at a later time.

For boundary values problems, one knows how each point responds to its neighbors, but there are conditions that have to be satisfied at the endpoints. An example would be a horizontal beam supported at the ends, like a bridge. The shape of the beam under the influence of gravity, or other forces, would lead to a differential equation and the boundary conditions at the beam ends would affect the solution of the problem. There are also a variety of other types of boundary conditions. In the case of a beam, one end could be fixed and the other end could be free to move. We will explore the effects of different boundary value conditions in our discussions and exercises.

Let's solve the above boundary value problem. As with initial value problems, we need to find the general solution and then apply any conditions that we may have. This is a nonhomogeneous differential equation, so we have that the solution is a sum of a solution of the homogeneous equation and a particular solution of the nonhomogeneous equation, $x(t) = x_h(t) + x_p(t)$. The solution of $x'' + x = 0$ is easily found as

$$x_h(t) = c_1 \cos t + c_2 \sin t.$$

The particular solution is easily found using the Method of Undetermined Coefficients,

$$x_p(t) = 2.$$

Thus, the general solution is

$$x(t) = 2 + c_1 \cos t + c_2 \sin t.$$

We now apply the boundary conditions and see if there are values of c_1 and c_2 that yield a solution to our problem. The first condition, $x(0) = 0$, gives

$$0 = 2 + c_1.$$

Thus, $c_1 = -2$. Using this value for c_1, the second condition, $x(1) = 1$, gives

$$0 = 2 - 2\cos 1 + c_2 \sin 1.$$

This yields

$$c_2 = \frac{2(\cos 1 - 1)}{\sin 1}.$$

We have found that there is a solution to the boundary value problem and it is given by

$$x(t) = 2\left(1 - \cos t \frac{(\cos 1 - 1)}{\sin 1} \sin t\right).$$

Boundary value problems arise in many physical systems, just as many of the initial values problems we have seen. We will see in the next section that boundary value problems for ordinary differential equations often appear in the solution of partial differential equations.

4.2 Partial Differential Equations

In this section we will introduce some generic partial differential equations and see how the discussion of such equations leads naturally to the study of boundary value problems for ordinary differential equations. However, we will not derive the particular equations, leaving that to courses in differential equations, mathematical physics, etc.

For ordinary differential equations, the unknown functions are functions of a single variable, e.g., $y = y(x)$. Partial differential equations are equations involving an unknown function of several variables, such as $u = u(x,y)$, $u = u(x,y)$, $u = u(x,y,z,t)$, and its (partial) derivatives. Therefore, the derivatives are partial derivatives. We will use the standard notations $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, etc.

There are a few standard equations that one encounters. These can be studied in one to three dimensions and are all linear differential equations. A list is provided in Table 4.1. Here we have introduced the Laplacian operator, $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$. Depending on the types of boundary conditions imposed and on the geometry of the system (rectangular, cylindrical, spherical, etc.), one encounters many interesting boundary value problems for ordinary differential equations.

Name	2 Vars	3 D
Heat Equation	$u_t = ku_{xx}$	$u_t = k\nabla^2 u$
Wave Equation	$u_{tt} = c^2 u_{xx}$	$u_{tt} = c^2 \nabla^2 u$
Laplace's Equation	$u_{xx} + u_{yy} = 0$	$\nabla^2 u = 0$
Poisson's Equation	$u_{xx} + u_{yy} = F(x,y)$	$\nabla^2 u = F(x,y,z)$
Schrödinger's Equation	$iu_t = u_{xx} + F(x,t)u$	$iu_t = \nabla^2 u + F(x,y,z,t)u$

Table 4.1: List of generic partial differential equations.

Let's look at the heat equation in one dimension. This could describe the heat conduction in a thin insulated rod of length L. It could also describe the diffusion of pollutant in a long narrow stream, or the flow of traffic down a road. In problems involving diffusion processes, one instead calls this equation the diffusion equation.

A typical initial-boundary value problem for the heat equation would be that initially one has a temperature distribution $u(x,0) = f(x)$. Placing the bar in an ice bath and assuming the heat flow is only through the ends of the bar, one has the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$. Of course, we are dealing with Celsius temperatures and we assume there is plenty of ice to keep that temperature fixed at each end for all time. So, the problem one would need to solve is given as

1D Heat Equation

$$\begin{array}{lcc} \text{PDE} & u_t = ku_{xx} & 0 < t, \quad 0 \le x \le L \\ \text{IC} & u(x,0) = f(x) & 0 < x < L \\ \text{BC} & u(0,t) = 0 & t > 0 \\ & u(L,t) = 0 & t > 0 \end{array} \tag{4.3}$$

Here, k is the heat conduction constant and is determined using properties of the bar.

Another problem that will come up in later discussions is that of the vibrating string. A string of length L is stretched out horizontally with both ends fixed. Think of a violin string or a guitar string. Then the string is plucked, giving the string an initial profile. Let $u(x,t)$ be the vertical displacement of the string at position x and time t. The motion of the string is governed by the one dimensional wave equation. The initial-boundary value problem for this problem is given as

1D Wave Equation

$$\begin{array}{lcc} \text{PDE} & u_{tt} = c^2u_{xx} & 0 < t, \quad 0 \le x \le L \\ \text{IC} & u(x,0) = f(x) & 0 < x < L \\ \text{BC} & u(0,t) = 0 & t > 0 \\ & u(L,t) = 0 & t > 0 \end{array} \tag{4.4}$$

In this problem c is the wave speed in the string. It depends on the mass per unit length of the string and the tension placed on the string.

4.2.1 *Solving the Heat Equation*

WE WOULD LIKE TO SEE HOW THE SOLUTION OF SUCH PROBLEMS involving partial differential equations will lead naturally to studying boundary value problems for ordinary differential equations. We will see this as we attempt the solution of the heat equation problem 4.3. We will employ a method typically used in studying linear partial differential equations, called *the method of separation of variables*.

We assume that u can be written as a product of single variable functions of each independent variable,

$$u(x,t) = X(x)T(t).$$

Substituting this guess into the heat equation, we find that

$$XT' = kX''T.$$

Dividing both sides by k and $u = XT$, we then get

$$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X}.$$

We have separated the functions of time on one side and space on the other side. The only way that a function of t equals a function of x is if the functions are constant functions. Therefore, we set each function equal to a constant, λ :

$$\underbrace{\frac{1}{k}\frac{T'}{T}}_{\text{function of } t} = \underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{\lambda}_{\text{constant}}.$$

This leads to two equations:

$$T' = k\lambda T, \tag{4.5}$$

$$X'' = \lambda X. \tag{4.6}$$

These are ordinary differential equations. The general solutions to these equations are readily found as

$$T(t) = Ae^{k\lambda t}, \tag{4.7}$$

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}. \tag{4.8}$$

We need to be a little careful at this point. The aim is to force our product solutions to satisfy both the boundary conditions and initial conditions. Also, we should note that λ is arbitrary and may be positive, zero, or negative. We first look at how the boundary conditions on u lead to conditions on X.

The first condition is $u(0,t) = 0$. This implies that

$$X(0)T(t) = 0$$

for all t. The only way that this is true is if $X(0) = 0$. Similarly, $u(L,t) = 0$ implies that $X(L) = 0$. So, we have to solve the boundary value problem

$$X'' - \lambda X = 0, \quad X(0) = 0 = X(L). \tag{4.9}$$

We are seeking nonzero solutions, as $X \equiv 0$ is an obvious and uninteresting solution. We call such solutions *trivial solutions*.

There are three cases to consider, depending on the sign of λ.

I. $\underline{\lambda > 0}$

In this case we have the exponential solutions

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}. \tag{4.10}$$

For $X(0) = 0$, we have

$$0 = c_1 + c_2.$$

We will take $c_2 = -c_1$. Then, $X(x) = c_1(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}) = 2c_1 \sinh \sqrt{\lambda}x$. Applying the second condition, $X(L) = 0$ yields

$$c_1 \sinh \sqrt{\lambda}L = 0.$$

This will be true only if $c_1 = 0$, since $\lambda > 0$. Thus, the only solution in this case is $X(x) = 0$. This leads to a trivial solution, $u(x,t) = 0$.

II. $\underline{\lambda = 0}$

For this case it is easier to set λ to zero in the differential equation. So, $X'' = 0$. Integrating twice, one finds

$$X(x) = c_1 x + c_2.$$

Setting $x = 0$, we have $c_2 = 0$, leaving $X(x) = c_1 x$. Setting $x = L$, we find $c_1 L = 0$. So, $c_1 = 0$ and we are once again left with a trivial solution.

III. $\underline{\lambda < 0}$

In this case is would be simpler to write $\lambda = -\mu^2$. Then the differential equation is

$$X'' + \mu^2 X = 0.$$

The general solution is

$$X(x) = c_1 \cos \mu x + c_2 \sin \mu x.$$

At $x = 0$ we get $0 = c_1$. This leaves $X(x) = c_2 \sin \mu x$. At $x = L$, we find

$$0 = c_2 \sin \mu L.$$

So, either $c_2 = 0$ or $\sin \mu L = 0$. $c_2 = 0$ leads to a trivial solution again. But, there are cases when the sine is zero. Namely,

$$\mu L == n\pi, \quad n = 1, 2, \ldots.$$

Note that $n = 0$ is not included since this leads to a trivial solution. Also, negative values of n are redundant, since the sine function is an odd function.

In summary, we can find solutions to the boundary value problem (4.9) for particular values of λ. The solutions are

$$X_n(x) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

for

$$\lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$

Product solutions of the heat equation (4.3) satisfying the boundary conditions are therefore

$$\boxed{u_n(x,t) = b_n e^{k\lambda_n t} \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots,} \tag{4.11}$$

where b_n is an arbitrary constant. However, these do not necessarily satisfy the initial condition $u(x, 0) = f(x)$. What we do get is

$$u_n(x, 0) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots.$$

So, if our initial condition is in one of these forms, we can pick out the right n and we are done.

For other initial conditions, we have to do more work. Note, since the heat equation is linear, we can write a linear combination of our product solutions and obtain the *general solution* satisfying the given boundary conditions as

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \sin \frac{n\pi x}{L}. \tag{4.12}$$

The only thing to impose is the initial condition:

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

So, if we are given $f(x)$, can we find the constants b_n? If we can, then we will have the solution to the full initial-boundary value problem. This will be the subject of the next chapter. However, first we will look at the general form of our boundary value problem and relate what we have done to the theory of infinite dimensional vector spaces.

4.3 Connections to Linear Algebra

We have already seen in earlier chapters that ideas from linear algebra crop up in our studies of differential equations. Namely, we solved eigenvalue problems associated with our systems of differential equations in order to determine the local behavior of dynamical systems near fixed points. In our study of boundary value problems we will find more connections with the theory of vector spaces. However, we will find that our problems lie in the realm of infinite dimensional vector spaces. In this section we will begin to see these connections.

4.3.1 Eigenfunction Expansions for PDEs

In the last section we sought solutions of the heat equation. Let's formally write the heat equation in the form

$$\frac{1}{k}u_t = L[u], \tag{4.13}$$

where

$$L = \frac{\partial^2}{\partial x^2}.$$

L is another example of a linear differential operator. [See Section 1.1.2.] It is a differential operator because it involves derivative operators. We sometimes define $D_x = \frac{\partial}{\partial x}$, so that $L = D_x^2$. It is linear, because for functions $f(x)$ and $g(x)$ and constants α, β we have

$$L[\alpha f + \beta g] = \alpha L[f] + \beta L[g]$$

When solving the heat equation, using the method of separation of variables, we found an infinite number of product solutions $u_n(x,t) = T_n(t)X_n(x)$. We did this by solving the boundary value problem

$$L[X] = \lambda X, \quad X(0) = 0 = X(L). \tag{4.14}$$

Here we see that an operator acts on an unknown function and spits out an unknown constant times that unknown. Where have we done this before? This is the same form as $A\mathbf{v} = \lambda\mathbf{v}$. So, we see that Equation (4.14) is really an eigenvalue problem for the operator L and given boundary conditions. When we solved the heat equation in the last section, we found the *eigenvalues*

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2$$

and the *eigenfunctions*

$$X_n(x) = \sin\frac{n\pi x}{L}.$$

We used these to construct the general solution that is essentially a linear combination over the eigenfunctions,

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x).$$

Note that these eigenfunctions live in an infinite dimensional function space.

We would like to generalize this method to problems in which L comes from an assortment of linear differential operators. So, we consider the more general partial differential equation

$$u_t = L[u], \quad a \le x \le b, \quad t > 0,$$

satisfying the boundary conditions

$$B[u](a,t) = 0, \qquad B[u](b,t) = 0, \quad t > 0,$$

and initial condition

$$u(x,0) = f(x), \quad a \le x \le b.$$

The form of the allowed boundary conditions $B[u]$ will be taken up later. Also, we will later see specific examples and properties of linear differential operators that will allow for this procedure to work.

We assume product solutions of the form $u_n(x,t) = b_n(t)\phi_n(x)$, where the ϕ_n's are the eigenfunctions of the operator L,

$$L\phi_n = \lambda_n\phi_n, \quad n = 1, 2, \ldots, \tag{4.15}$$

satisfying the boundary conditions

$$B[\phi_n](a) = 0, \qquad B[\phi_n](b) = 0. \tag{4.16}$$

Inserting the general solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)$$

into the partial differential equation, we have

$$\begin{aligned} u_t &= L[u], \\ \frac{\partial}{\partial t}\sum_{n=1}^{\infty} b_n(t)\phi_n(x) &= L\left[\sum_{n=1}^{\infty} b_n(t)\phi_n(x)\right] \end{aligned} \tag{4.17}$$

On the left we differentiate term by term and on the right side we use the linearity of L:

$$\sum_{n=1}^{\infty} \frac{db_n(t)}{dt}\phi_n(x) = \sum_{n=1}^{\infty} b_n(t)L[\phi_n(x)] \tag{4.18}$$

Now, we make use of the result of applying L to the eigenfunction ϕ_n:

$$\sum_{n=1}^{\infty} \frac{db_n(t)}{dt}\phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\lambda_n\phi_n(x). \tag{4.19}$$

Infinite series cannot always be differentiated, so one must be careful. When we ignore such details for the time being, we say that we *formally* differentiate the series and formally apply the differential operator to the series. Such operations need to be justified later.

Comparing both sides, or using the linear independence of the eigenfunctions, we see that

$$\frac{db_n(t)}{dt} = \lambda_n b_n(t),$$

whose solution is

$$b_n(t) = b_n(0)e^{\lambda_n t}.$$

So, the general solution becomes

$$u(x,t) = \sum_{n=1}^{\infty} b_n(0)e^{\lambda_n t}\phi_n(x).$$

This solution satisfies, at least formally, the partial differential equation and satisfies the boundary conditions.

Finally, we need to determine the $b_n(0)$'s, which are so far arbitrary. We use the initial condition $u(x,0) = f(x)$ to find that

$$f(x) = \sum_{n=1}^{\infty} b_n(0)\phi_n(x).$$

So, given $f(x)$, we are left with the problem of extracting the coefficients $b_n(0)$ in an expansion of f in the eigenfunctions ϕ_n. We will see that this is related to Fourier series expansions, which we will take up in the next chapter.

4.3.2 *Eigenfunction Expansions for Nonhomogeneous ODEs*

Partial differential equations are not the only applications of the method of eigenfunction expansions, as seen in the last section. We can apply these method to nonhomogeneous two point boundary value problems for ordinary differential equations assuming that we can solve the associated eigenvalue problem.

Let's begin with the nonhomogeneous boundary value problem:

$$\begin{aligned} L[u] &= f(x), \quad a \le x \le b \\ B[u](a) &= 0, \quad B[u](b) = 0. \end{aligned} \tag{4.20}$$

We first solve the eigenvalue problem,

$$\begin{aligned} L[\phi] &= \lambda\phi, \quad a \le x \le b \\ B[\phi](a) &= 0, \quad B[\phi](b) = 0, \end{aligned} \tag{4.21}$$

and obtain a family of eigenfunctions, $\{\phi_n(x)\}_{n=1}^{\infty}$. Then we assume that $u(x)$ can be represented as a linear combination of these eigenfunctions:

$$u(x) = \sum_{n=1}^{\infty} b_n \phi_n(x).$$

Inserting this into the differential equation, we have

$$\begin{aligned} f(x) &= L[u] \\ &= L\left[\sum_{n=1}^{\infty} b_n \phi_n(x)\right] \\ &= \sum_{n=1}^{\infty} b_n L\left[\phi_n(x)\right] \\ &= \sum_{n=1}^{\infty} \lambda_n b_n \phi_n(x) \\ &\equiv \sum_{n=1}^{\infty} c_n \phi_n(x). \end{aligned} \tag{4.22}$$

Therefore, we have to find the expansion coefficients $c_n = \lambda_n b_n$ of the given $f(x)$ in a series expansion over the eigenfunctions. This is similar to what we had found for the heat equation problem and its generalization in the last section.

There are a lot of questions and details that have been glossed over in our formal derivations. Can we always find such eigenfunctions for a given operator? Do the infinite series expansions converge? Can we differentiate our expansions terms by term? Can one find expansions that converge to given functions like $f(x)$ above? We will begin to explore these questions in the case that the eigenfunctions are simple trigonometric functions like the $\phi_n(x) = \sin \frac{n\pi x}{L}$ in the solution of the heat equation.

4.3.3 *Linear Vector Spaces*

MUCH OF THE DISCUSSION AND TERMINOLOGY THAT WE WILL USE comes from the theory of vector spaces. Until now you may only have dealt with finite dimensional vector spaces in your classes. Even then, you might only be comfortable with two and three dimensions. We will review a little of what we know about finite dimensional spaces so that we can deal with the more general function spaces, which is where our eigenfunctions live.

The notion of a vector space is a generalization of our three dimensional vector spaces. In three dimensions, we have things called vectors, which are arrows of a specific length and pointing in a given direction. To each vector, we can associate a point in a three dimensional Cartesian system. We just attach the tail of the vector $\mathbf{v}$ to the origin and the head lands at (x, y, z). We then use unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ along the coordinate axes to write

$$\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Having defined vectors, we then learned how to add vectors and multiply vectors by numbers, or scalars. Under these operations, we expected to get back new vectors. Then we learned that there were two types of multiplication of vectors. We could multiply then to get a scalar or a vector. This lead to the dot and cross products, respectively. The dot product was useful for determining the length of a vector, the angle between two vectors, or if the vectors were orthogonal.

These notions were later generalized to spaces of more than three dimensions in your linear algebra class. The properties outlined roughly above need to be preserved. So, we have to start with a space of vectors and the operations between them. We also need a set of scalars, which generally come from some *field*. However, in our applications the field will either be the set of real numbers or the set of complex numbers.

Definition 4.1. A *vector space* V over a field F is a set that is closed under addition and scalar multiplication and satisfies the following conditions: For any $u, v, w \in V$ and $a, b \in F$

1. $u + v = v + u$.
2. $(u + v) + w = u + (v + w)$.
3. There exists a 0 such that $0 + v = v$.
4. There exists a $-v$ such that $v + (-v) = 0$.
5. $a(bv) = (ab)v$.
6. $(a + b)v = av + bv$.
7. $a(u + v) = au + bv$.
8. $1(v) = v$.

Now, for an n-dimensional vector space, we have the idea that any vector in the space can be represented as the sum over n linearly independent vectors. Recall that a linearly independent set of vectors $\{\mathbf{v}_j\}_{j=1}^n$ satisfies

$$\sum_{j=1}^{n} c_j \mathbf{v}_j = \mathbf{0} \quad \Leftrightarrow \quad c_j = 0.$$

This leads to the idea of a basis set. The standard basis in an n-dimensional vector space is a generalization of the standard basis in three dimensions (**i**, **j** and **k**). We define

$$\mathbf{e}_k = (0, \ldots, 0, \underbrace{1}_{k\text{th space}}, 0, \ldots, 0), \quad k = 1, \ldots, n. \tag{4.23}$$

Then, we can expand any $\mathbf{v} \in V$ as

$$\mathbf{v} = \sum_{k=1}^{n} v_k \mathbf{e}_k, \tag{4.24}$$

where the v_k's are called the components of the vector in this basis and one can write $\mathbf{v}$ as an n-tuple $(v_1, v_2, \ldots, v_n)$.

The only other thing we will need at this point is to generalize the dot product, or scalar product. Recall that there are two forms for the dot product in three dimensions. First, one has that

$$\mathbf{u} \cdot \mathbf{v} = uv \cos\theta, \tag{4.25}$$

where u and v denote the length of the vectors. The other form, is the component form:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{k=1}^{3} u_k v_k. \tag{4.26}$$

Of course, this form is easier to generalize. So, we define the *scalar product* between to n-dimensional vectors as

$$< \mathbf{u}, \mathbf{v} >= \sum_{k=1}^{n} u_k v_k. \tag{4.27}$$

Actually, there are a number of notations that are used in other texts. One can write the scalar product as $(\mathbf{u}, \mathbf{v})$ or even use the Dirac notation $< \mathbf{u}|\mathbf{v} >$ for applications in quantum mechanics.

While it does not always make sense to talk about angles between general vectors in higher dimensional vector spaces, there is one concept that is useful. It is that of orthogonality, which in three dimensions another way of say vectors are perpendicular to each other. So, we also say that vectors $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if and only if $< \mathbf{u}, \mathbf{v} >= 0$. If $\{\mathbf{a}_k\}_{k=1}^n$, is a set of basis vectors such that

$$< \mathbf{a}_j, \mathbf{a}_k >= 0, \quad k \neq j,$$

then it is called an *orthogonal basis*. If in addition each basis vector is a unit vector, then one has an *orthonormal basis*.

Let $\{\mathbf{a}_k\}_{k=1}^n$, be a set of basis vectors for vector space V. We know that any vector $\mathbf{v}$ can be represented in terms of this basis, $\mathbf{v} = \sum_{k=1}^n v_k \mathbf{a}_k$. If we know the basis and vector, can we find the components? The answer is, yes. We can use the scalar product of $\mathbf{v}$ with each basis element $\mathbf{a}_j$. So, we have for $j = 1, \ldots, n$

$$\begin{aligned} < \mathbf{a}_j, \mathbf{v} > &= < \mathbf{a}_j, \sum_{k=1}^{n} v_k \mathbf{a}_k > \\ &= \sum_{k=1}^{n} v_k < \mathbf{a}_j, \mathbf{a}_k > . \end{aligned} \tag{4.28}$$

Since we know the basis elements, we can easily compute the numbers

$$A_{jk} \equiv < \mathbf{a}_j, \mathbf{a}_k >$$

and

$$b_j \equiv < \mathbf{a}_j, \mathbf{v} > .$$

Therefore, the system (4.28) for the v_k's is a linear algebraic system, which takes the form $A\mathbf{v} = \mathbf{b}$. However, if the basis is orthogonal, then the matrix A is diagonal and the system is easily solvable. We have that

$$< \mathbf{a}_j, \mathbf{v} >= v_j < \mathbf{a}_j, \mathbf{a}_j >, \tag{4.29}$$

or

$$v_j = \frac{< \mathbf{a}_j, \mathbf{v} >}{< \mathbf{a}_j, \mathbf{a}_j >}. \tag{4.30}$$

In fact, if the basis is orthonormal, A is the identity matrix and the solution is simpler:

$$v_j =< \mathbf{a}_j, \mathbf{v} > . \tag{4.31}$$

We spent some time looking at this simple case of extracting the components of a vector in a finite dimensional space. The keys to doing this simply were to have a scalar product and an orthogonal basis set. These are the key ingredients that we will need in the infinite dimensional case. Recall that when we solved the heat equation, we had a function (vector) that we wanted to expand in a set of eigenfunctions (basis) and we needed to find the expansion coefficients (components). As you can see, we need to extend the concepts for finite dimensional spaces to their analogs in infinite dimensional spaces. Linear algebra will provide some of the backdrop for what is to follow: The study of many boundary value problems amounts to the solution of eigenvalue problems over infinite dimensional vector spaces (complete inner product spaces, the space of square integrable functions, or Hilbert spaces).

We will consider the space of functions of a certain type. They could be the space of continuous functions on [0,1], or the space of differentiably continuous functions, or the set of functions integrable from a to b. Later, we will specify the types of functions needed. We will further need to be able to add functions and multiply them by scalars. So, we can easily obtain a vector space of functions.

We will also need a scalar product defined on this space of functions. There are several types of scalar products, or inner products, that we can define. For a real vector space, we define

Definition 4.2. An inner product $<,>$ on a real vector space V is a mapping from $V \times V$ into R such that for $u, v, w \in V$ and $\alpha \in R$ one has

1. $< u + v, w >=< u, w > + < v, w > .$
2. $< \alpha v, w >= \alpha < v, w > .$
3. $< v, w >=< w, v > .$
4. $< v, v > \geq 0$ and $< v, v >= 0$ iff $v = 0$.

A real vector space equipped with the above inner product leads to a real inner product space. A more general definition with the third item replaced with $< v, w >= \overline{< w, v >}$ is needed for complex inner product spaces.

For the time being, we are dealing just with real valued functions. We need an inner product appropriate for such spaces. One such definition is the following. Let $f(x)$ and $g(x)$ be functions defined on $[a, b]$. Then, we define the *inner product*, if the integral exists, as

$$< f, g >= \int_a^b f(x)g(x)\,dx. \tag{4.32}$$

So far, we have functions spaces equipped with an inner product. Can we find a basis for the space? For an n-dimensional space we need n basis vectors. For an infinite dimensional space, how many will we need? How do we know when we have enough? We will think about those things later.

Let's assume that we have a basis of functions $\{\phi_n(x)\}_{n=1}^{\infty}$. Given a function $f(x)$, how can we go about finding the components of f in this basis? In other words, let

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

How do we find the c_n's? Does this remind you of the problem we had earlier?

Formally, we take the inner product of f with each ϕ_j, to find

$$\begin{aligned} < \phi_j, f > &= < \phi_j, \sum_{n=1}^{\infty} c_n \phi_n > \\ &= \sum_{n=1}^{\infty} c_n < \phi_j, \phi_n > . \end{aligned} \tag{4.33}$$

If our basis is an *orthogonal basis*, then we have

$$< \phi_j, \phi_n >= N_j \delta_{jn}, \tag{4.34}$$

where δ_{ij} is the Kronecker delta defined as

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases} \tag{4.35}$$

Thus, we have

$$\begin{aligned} < \phi_j, f > &= \sum_{n=1}^{\infty} c_n < \phi_j, \phi_n > \\ &= \sum_{n=1}^{\infty} c_n N_j \delta_{jn} \\ &= c_1 N_j \delta_{j1} + c_2 N_j \delta_{j2} + \ldots + c_j N_j \delta_{jj} + \ldots \\ &= c_j N_j. \end{aligned} \tag{4.36}$$

So, the expansion coefficient is

$$c_j = \frac{< \phi_j, f >}{N_j} = \frac{< \phi_j, f >}{< \phi_j, \phi_j >}.$$

We summarize this important result:

Generalized Basis Expansion

Let $f(x)$ be represented by an expansion over a basis of orthogonal functions, $\{\phi_n(x)\}_{n=1}^{\infty}$,

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

Then, the expansion coefficients are formally determined as

$$c_n = \frac{< \phi_n, f >}{< \phi_n, \phi_n >}.$$

In our preparation for later sections, let's determine if the set of functions $\phi_n(x) = \sin nx$ for $n = 1, 2, \ldots$ is orthogonal on the interval $[-\pi, \pi]$. We need to show that $< \phi_n, \phi_m >= 0$ for $n \neq m$. Thus, we have for $n \neq m$

$$\begin{aligned} < \phi_n, \phi_m > &= \int_{-\pi}^{\pi} \sin nx \sin mx \, dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} [\cos(n-m)x - \cos(n+m)x] \, dx \\ &= \frac{1}{2} \left[\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m} \right]_{-\pi}^{\pi} = 0. \end{aligned} \tag{4.37}$$

Here we have made use of a trigonometric identity for the product of two sines. We recall how this identity is derived. Recall the addition formulae for cosines:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

Adding, or subtracting, these equations gives

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B),$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B).$$

So, we have determined that the set $\phi_n(x) = \sin nx$ for $n = 1, 2, \ldots$ is an orthogonal set of functions on the interval $[= \pi, \pi]$. Just as with vectors in three dimensions, we can normalize our basis functions to arrive at an *orthonormal basis*, $< \phi_n, \phi_m >= \delta_{nm}$, $m, n = 1, 2, \ldots$. This is simply done by dividing by the *length* of the vector. Recall that the length of a vector was obtained as $v = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ In the same way, we define the *norm* of our functions by

$$\|f\| = \sqrt{< f, f >}.$$

Note, there are many types of norms, but this will be sufficient for us.

For the above basis of sine functions, we want to first compute the norm of each function. Then we would like to find a new basis from this one such that each basis eigenfunction has unit length and is therefore an orthonormal basis. We first compute

$$\begin{aligned} \|\phi_n\|^2 &= \int_{-\pi}^{\pi} \sin^2 nx\, dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} [1 - \cos 2nx]\, dx \\ &= \frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi} = \pi. \end{aligned} \tag{4.38}$$

We have found for our example that

$$< \phi_n, \phi_m >= \pi\delta_{nm} \tag{4.39}$$

and that $\|\phi_n\| = \sqrt{\pi}$. Defining $\psi_n(x) = \frac{1}{\sqrt{\pi}}\phi_n(x)$, we have *normalized* the ϕ_n's and have obtained an orthonormal basis of functions on $[-\pi, \pi]$.

Expansions of functions in trigonometric bases occur often and originally resulted from the study of partial differential equations. They have been named Fourier series and will be the topic of the next chapter.

Problems

1. Solve the following problems when possible. Note that there might not always exist a unique solution.

a. $x'' + x = 2, \quad x(0) = 0, \quad x'(1) = 0.$

b. $y'' + y = 1, \quad y(0) = 0, \quad y(\pi) = 0.$

c. $y'' + y = \sin 2x, \quad y(0) = 0, \quad y(\pi) = 0.$

d. $(x^2y')' + y = 0, \quad y(1) = 0, \quad y'(e) = 0.$

e. $y'' - 7y' + 12y = 4e^{2x}, \quad y(0) = 3, \quad y(1) = 5e^2.$

f. $x^2y'' + xy' - 4y = 0, \quad |y(0)| < \infty, \quad y'(1) = 6.$

2. Let $y = y(x)$ satisfy the simple differential equation $y'' + y = 0$. Determine the solutions which satisfy the following the boundary conditions:

a. $y(0) = y(\pi).$

b. $y(0) = y(\pi), \quad y'(0) = y'(\pi).$

c. $y(0) = y(2\pi), \quad y'(0) = y'(2\pi).$

3. Consider the boundary value problem

$$\frac{d^2y}{dx^2} + a^2y = \sin \pi x, \quad 0 < x < 1,$$

for $y(0) = 1, \quad y(1) = -2$, for all a. What are the solutions when $a = \pm\pi$?

4. Consider the boundary value problem for the deflection of a horizontal beam fixed at one end,

$$\frac{d^4y}{dx^4} = C, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(L) = 0, \quad y'''(L) = 0.$$

Solve this problem assuming that C is a constant.

5. The theory of horizontally embedded beams leads a the differential equation for the displacement $y(x)$ from horizontal. The fourth order equatrion is given as

$$EI\frac{d^4y}{dx^4} = w(x),$$

where E is Young's modulus, I is the moment of intertia of the beam cross-section, and $w(x)$ represents the load. If the load is just the weight of a uniform beam, then $w(x)$ is constant.

There are different boundary conditions that can be applied to the ends as seen in Figure 4.1.

- At an embedded end, y and y' vanish.
- At a simply supported end, y and y'' vanish.
- At a free end, y'' and y''' vanish.

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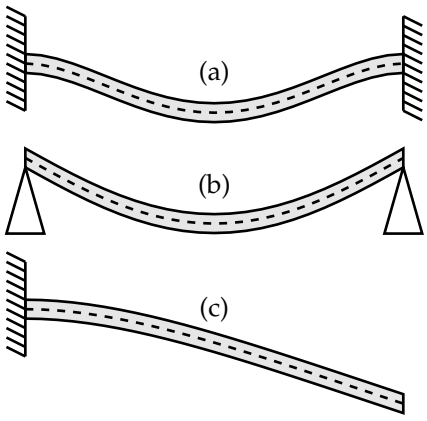


Figure 4.1: Deflections of a horizontal beam. (a) Both ends are embedded. (b) Both ends are simply supported. (c) One end is embedded and gthe other is a free end.

Find the indicated solutions in the case that $EI = 40$, $w(x) = 120$, and the length of the beam is 8.

a. The differential equation is easily integrated to obtain a general solution. Find this solution.

b. Consider the case in Figure 4.1(a) in which both ends are embedded. Plot the solution. Does this agree with the figure?

c. Consider the case in Figure 4.1(b) in which both ends are simply supported.

d. Consider the case in Figure 4.1(c) in which one end is embedded and the other is free. Plot the solution. Does this agree with the figure?

6. Find product solutions, $u(x,t) = b(t)\phi(x)$, to the heat equation satisfying the boundary conditions $u_x(0,t) = 0$ and $u(L,t) = 0$. Use these solutions to find a general solution of the heat equation satisfying these boundary conditions.

7. Consider the following boundary value problems. Determine the eigenvalues, λ, and eigenfunctions, $y(x)$ for each problem.

a. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = 0.$

b. $y'' - \lambda y = 0, \quad y(-\pi) = 0, \quad y'(\pi) = 0.$

c. $x^2y'' + xy' + \lambda y = 0, \quad y(1) = 0, \quad y(2) = 0.$

d. $(x^2y')' + \lambda y = 0, \quad y(1) = 0, \quad y'(e) = 0.$[1]

[1] In problem d you will not get exact eigenvalues. Show that you obtain a transcendental equation for the eigenvalues in the form $\tan z = 2z$. Find the first three eigenvalues numerically.

8. For the following sets of functions: i) show that each is orthogonal on the given interval, and ii) determine the corresponding orthonormal set.

a. $\{\sin 2nx\}, \quad n = 1, 2, 3, \ldots, \quad 0 \le x \le \pi.$

b. $\{\cos n\pi x\}, \quad n = 0, 1, 2, \ldots, \quad 0 \le x \le 2.$

c. $\{\sin \frac{n\pi x}{L}\}, \quad n = 1, 2, 3, \ldots, \quad x \in [-L, L].$

Chapter 5

Fourier Series

"The differential equations of the propagation of heat express the most general conditions, and reduce the physical questions to problems of pure analysis, and this is the proper object of theory." - Jean Baptiste Joseph Fourier (1768-1830)

5.1 Introduction

In this chapter we will look at trigonometric series. Previously, we saw that such series expansion occurred naturally in the solution of the heat equation and other boundary value problems. In the last chapter we saw that such functions could be viewed as a basis in an infinite dimensional vector space of functions. Given a function in that space, when will it have a representation as a trigonometric series? For what values of x will it converge? Finding such series is at the heart of Fourier, or spectral, analysis.

There are many applications using spectral analysis. At the root of these studies is the belief that many continuous waveforms are comprised of a number of harmonics. Such ideas stretch back to the Pythagorean study of the vibrations of strings, which lead to their view of a world of harmony. This idea was carried further by Johannes Kepler in his harmony of the spheres approach to planetary orbits. In the 1700's others worked on the superposition theory for vibrating waves on a stretched spring, starting with the wave equation and leading to the superposition of right and left traveling waves. This work was carried out by people such as John Wallis, Brook Taylor and Jean le Rond d'Alembert.

In 1742 d'Alembert solved the wave equation

$$c^2 \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0,$$

where y is the string height and c is the wave speed. However, his solution led himself and others, like Leonhard Euler and Daniel Bernoulli, to investigate what "functions" could be the solutions of this equation. In fact, this lead to a more rigorous approach to the study of analysis by first coming to grips with the concept of a function. For example, in 1749 Euler sought the solution for a plucked string in which case the initial condition $y(x, 0) = h(x)$ has a discontinuous derivative!

In 1753 Daniel Bernoulli viewed the solutions as a superposition of simple vibrations, or harmonics. Such superpositions amounted to looking at solutions of the form

$$y(x,t) = \sum_k a_k \sin\frac{k\pi x}{L}\cos\frac{k\pi ct}{L},$$

where the string extends over the interval $[0, L]$ with fixed ends at $x = 0$ and $x = L$. However, the initial conditions for such superpositions are

$$y(x,0) = \sum_k a_k \sin\frac{k\pi x}{L}.$$

It was determined that many functions could not be represented by a finite number of harmonics, even for the simply plucked string given by an initial condition of the form

$$y(x,0) = \begin{cases} cx, & 0 \le x \le L/2 \\ c(L-x), & L/2 \le x \le L \end{cases}.$$

Thus, the solution consists generally of an infinite series of trigonometric functions.

Such series expansions were also of importance in Joseph Fourier's solution of the heat equation. The use of such Fourier expansions became an important tool in the solution of linear partial differential equations, such as the wave equation and the heat equation. As seen in the last chapter, using the Method of Separation of Variables, allows higher dimensional problems to be reduced to several one dimensional boundary value problems. However, these studies lead to very important questions, which in turn opened the doors to whole fields of analysis. Some of the problems raised were

1. What functions can be represented as the sum of trigonometric functions?

2. How can a function with discontinuous derivatives be represented by a sum of smooth functions, such as the above sums?

3. Do such infinite sums of trigonometric functions a actually converge to the functions they represents?

Sums over sinusoidal functions naturally occur in music and in studying sound waves. A pure note can be represented as

$$y(t) = A\sin(2\pi ft),$$

where A is the amplitude, f is the frequency in hertz (Hz), and t is time in seconds. The amplitude is related to the volume, or intensity, of the sound. The larger the amplitude, the louder the sound. In Figure 5.1 we show plots of two such tones with $f = 2$ Hz in the top plot and $f = 5$ Hz in the bottom one.

Next, we consider what happens when we add several pure tones. After all, most of the sounds that we hear are in fact a combination of pure tones

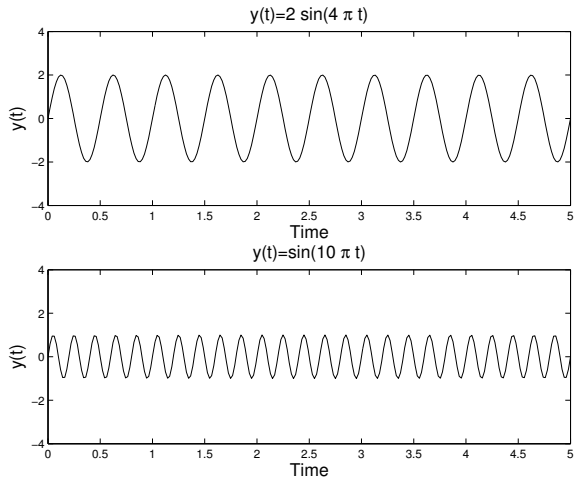


Figure 5.1: Plots of $y(t) = \sin(2\pi f t)$ on $[0,5]$ for $f = 2$ Hz and $f = 5$ Hz.

with different amplitudes and frequencies. In Figure 5.2 we see what happens when we add several sinusoids. Note that as one adds more and more tones with different characteristics, the resulting signal gets more complicated. However, we still have a function of time. In this chapter we will ask, "Given a function $f(t)$, can we find a set of sinusoidal functions whose sum converges to $f(t)$?"

Looking at the superpositions in Figure 5.2, we see that the sums yield functions that appear to be periodic. This is not to be unexpected. We recall that a periodic function is one in which the function values repeat over the domain of the function. The length of the smallest part of the domain which repeats is called the *period*. We can define this more precisely.

Definition 5.1. A function is said to be *periodic with period* T if $f(t+T) = f(t)$ for all t and the smallest such positive number T is called the *period.*

For example, we consider the functions used in Figure 5.2. We began with $y(t) = 2\sin(4\pi t)$. Recall from your first studies of trigonometric functions that one can determine the period by dividing the coefficient of t into 2π to get the period. In this case we have

$$T = \frac{2\pi}{4\pi} = \frac{1}{2}.$$

Looking at the top plot in Figure 5.1 we can verify this result. (You can count the full number of cycles in the graph and divide this into the total time to get a more accurate value of the period.)

In general, if $y(t) = A\sin(2\pi f t)$, the period is found as

$$T = \frac{2\pi}{2\pi f} = \frac{1}{f}.$$

Of course, this result makes sense, as the unit of frequency, the hertz, is also defined as s^{-1}, or cycles per second.

Returning to the superpositions in Figure 5.2, we have that $y(t) = \sin(10\pi t)$ has a period of 0.2 Hz and $y(t) = \sin(16\pi t)$ has a period of 0.125 Hz. The

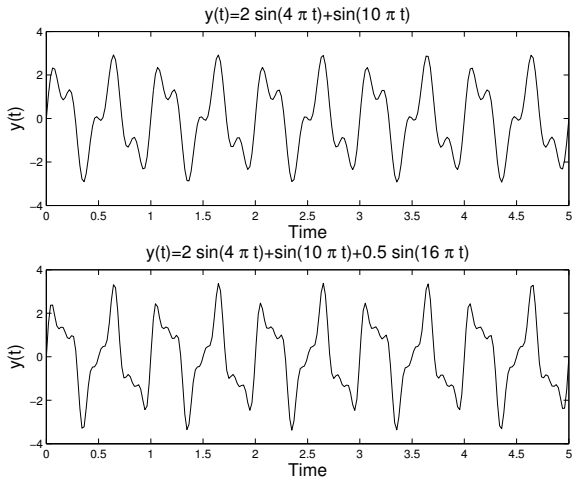


Figure 5.2: Superposition of several sinusoids. Top: Sum of signals with $f = 2$ Hz and $f = 5$ Hz. Bottom: Sum of signals with $f = 2$ Hz, $f = 5$ Hz, and and $f = 8$ Hz.

two superpositions retain the largest period of the signals added, which is 0.5 Hz.

Our goal will be to start with a function and then determine the amplitudes of the simple sinusoids needed to sum to that function. First of all, we will see that this might involve an infinite number of such terms. Thus, we will be studying an infinite series of sinusoidal functions.

Secondly, we will find that using just sine functions will not be enough either. This is because we can add sinusoidal functions that do not necessarily peak at the same time. We will consider two signals that originate at different times. This is similar to when your music teacher would make sections of the class sing a song like "Row, Row, Row your Boat" starting at slightly different times.

We can easily add shifted sine functions. In Figure 5.3 we show the functions $y(t) = 2\sin(4\pi t)$ and $y(t) = 2\sin(4\pi t + 7\pi/8)$ and their sum. Note that this shifted sine function can be written as $y(t) = 2\sin(4\pi(t + 7/32))$. Thus, this corresponds to a time shift of $-7/8$.

So, we should account for shifted sine functions in our general sum. Of course, we would then need to determine the unknown time shift as well as the amplitudes of the sinusoidal functions that make up our signal, $f(t)$. While this is one approach that some researchers use to analyze signals, there is a more common approach. This results from another reworking of the shifted function. Consider the general shifted function

$$y(t) = A\sin(2\pi ft + \phi).$$

Note that $2\pi ft + \phi$ is called the *phase* of our sine function and ϕ is called the *phase shift*. We can use our trigonometric identity for the sine of the sum of two angles to obtain

$$y(t) = A\sin(2\pi ft + \phi) = A\sin(\phi)\cos(2\pi ft) + A\cos(\phi)\sin(2\pi ft).$$

Defining $a = A\sin(\phi)$ and $b = A\cos(\phi)$, we can rewrite this as

$$y(t) = a\cos(2\pi ft) + b\sin(2\pi ft).$$

Thus, we see that our signal is a sum of sine and cosine functions with the same frequency and different amplitudes. If we can find a and b, then we can easily determine A and ϕ:

$$A = \sqrt{a^2 + b^2} \quad \tan\phi = \frac{b}{a}.$$

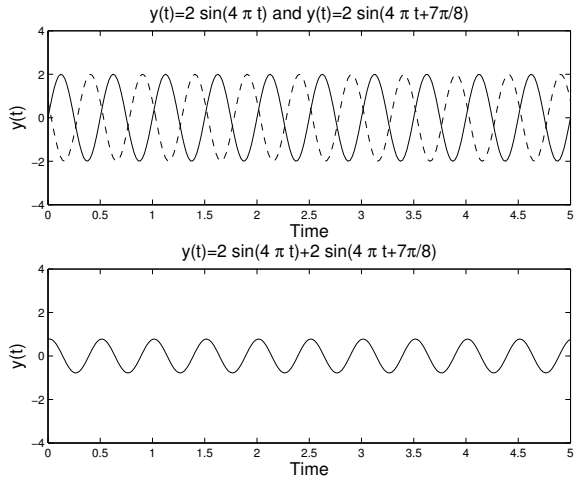


Figure 5.3: Plot of the functions $y(t) = 2\sin(4\pi t)$ and $y(t) = 2\sin(4\pi t + 7\pi/8)$ and their sum.

We are now in a position to state our goal in this chapter.

Goal

Given a signal $f(t)$, we would like to determine its frequency content by finding out what combinations of sines and cosines of varying frequencies and amplitudes will sum to the given function. This is called *Fourier Analysis.*

5.2 Fourier Trigonometric Series

As we have seen in the last section, we are interested in finding representations of functions in terms of sines and cosines. Given a function $f(x)$ we seek a representation in the form

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]. \tag{5.1}$$

Notice that we have opted to drop reference to the frequency form of the phase. This will lead to a simpler discussion for now and one can always make the transformation $nx = 2\pi f_n t$ when applying these ideas to applications.

The series representation in Equation (5.1) is called a *Fourier trigonometric series.* We will simply refer to this as a *Fourier series* for now. The set of constants a_0, a_n, b_n, $n = 1, 2, \ldots$ are called the *Fourier coefficients.* The constant term is chosen in this form to make later computations simpler, though

some other authors choose to write the constant term as a_0. Our goal is to find the Fourier series representation given $f(x)$. Having found the Fourier series representation, we will be interested in determining when the Fourier series converges and to what function it converges.

From our discussion in the last section, we see that the infinite series is periodic. The largest period of the terms comes from the $n = 1$ terms. The periods of $\cos x$ and $\sin x$ are $T = 2\pi$. Thus, the Fourier series has period 2π. This means that the series should be able to represent functions that are periodic of period 2π.

While this appears restrictive, we could also consider functions that are defined over one period. In Figure 5.4 we show a function defined on $[0, 2\pi]$. In the same figure, we show its periodic extension. These are just copies of the original function shifted by the period and glued together. The extension can now be represented by a Fourier series and restricting the Fourier series to $[0, 2\pi]$ will give a representation of the original function. Therefore, we will first consider Fourier series representations of functions defined on this interval. Note that we could just as easily considered functions defined on $[-\pi, \pi]$ or any interval of length 2π.

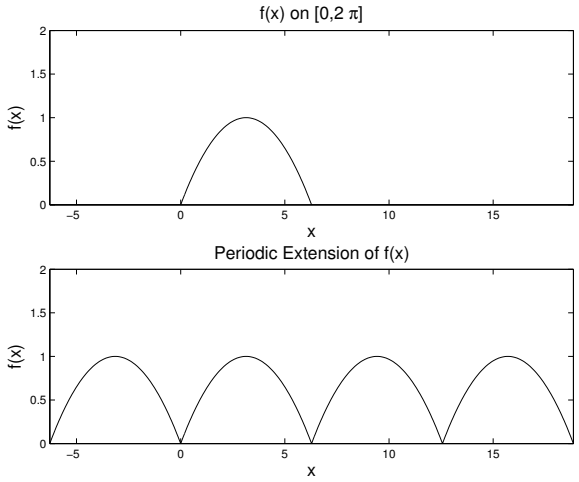


Figure 5.4: Plot of the functions $f(t)$ defined on $[0, 2\pi]$ and its periodic extension.

Fourier Coefficients

Theorem 5.1. *The Fourier series representation of $f(x)$ defined on $[0, 2\pi]$ when it exists, is given by (5.1) with Fourier coefficients*

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx, \quad n = 1, 2, \ldots. \end{aligned} \tag{5.2}$$

These expressions for the Fourier coefficients are obtained by considering special integrations of the Fourier series. We will look at the derivations of the a_n's. First we obtain a_0.

We begin by integrating the Fourier series term by term in Equation (5.1).

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} \frac{a_0}{2}\,dx + \int_0^{2\pi} \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]\,dx. \tag{5.3}$$

We assume that we can integrate the infinite sum term by term. Then we need to compute

$$\begin{aligned} \int_0^{2\pi} \frac{a_0}{2}\,dx &= \frac{a_0}{2}(2\pi) = \pi a_0, \\ \int_0^{2\pi} \cos nx\,dx &= \left[\frac{\sin nx}{n}\right]_0^{2\pi} = 0, \\ \int_0^{2\pi} \sin nx\,dx &= \left[\frac{-\cos nx}{n}\right]_0^{2\pi} = 0. \end{aligned} \tag{5.4}$$

From these results we see that only one term in the integrated sum does not vanish leaving

$$\int_0^{2\pi} f(x)\,dx = \pi a_0.$$

This confirms the value for a_0.

Next, we need to find a_n. We will multiply the Fourier series (5.1) by $\cos mx$ for some positive integer m. This is like multiplying by $\cos 2x$, $\cos 5x$, etc. We are multiplying by all possible $\cos mx$ functions for different integers m all at the same time. We will see that this will allow us to solve for the a_n's.

We find the integrated sum of the series times $\cos mx$ is given by

$$\begin{aligned} \int_0^{2\pi} f(x)\cos mx\,dx &= \int_0^{2\pi} \frac{a_0}{2}\cos mx\,dx \\ &+ \int_0^{2\pi} \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]\cos mx\,dx. \end{aligned} \tag{5.5}$$

Integrating term by term, the right side becomes

$$\frac{a_0}{2}\int_0^{2\pi} \cos mx\,dx + \sum_{n=1}^{\infty}\left[a_n \int_0^{2\pi} \cos nx \cos mx\,dx + b_n \int_0^{2\pi} \sin nx \cos mx\,dx\right]. \tag{5.6}$$

We have already established that $\int_0^{2\pi} \cos mx\,dx = 0$, which implies that the first term vanishes.

Next we need to compute integrals of products of sines and cosines. This requires that we make use of some trigonometric identities. While you have seen such integrals before in your calculus class, we will review how to carry out such integrals. For future reference, we list several useful identities, some of which we will prove along the way.

Useful Trigonometric Identities

$$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x \tag{5.7}$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \tag{5.8}$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \tag{5.9}$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \tag{5.10}$$

$$\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y)) \tag{5.11}$$

$$\cos x \cos y = \frac{1}{2}(\cos(x + y) + \cos(x - y)) \tag{5.12}$$

$$\sin x \cos y = \frac{1}{2}(\sin(x + y) + \sin(x - y)) \tag{5.13}$$

We first want to evaluate $\int_0^{2\pi} \cos nx \cos mx \, dx$. We do this by using the product identity. We had done this in the last chapter, but will repeat the derivation for the reader's benefit. Recall the addition formulae for cosines:

$$\cos(A + B) = \cos A \cos B - \sin A \sin B,$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

Adding these equations gives

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B).$$

We can use this identity with $A = mx$ and $B = nx$ to complete the integration. We have

$$\begin{aligned} \int_0^{2\pi} \cos nx \cos mx \, dx &= \frac{1}{2} \int_0^{2\pi} [\cos(m + n)x + \cos(m - n)x] \, dx \\ &= \frac{1}{2} \left[\frac{\sin(m + n)x}{m + n} + \frac{\sin(m - n)x}{m - n} \right]_0^{2\pi} \\ &= 0. \end{aligned} \tag{5.14}$$

There is one caveat when doing such integrals. What if one of the denominators $m \pm n$ vanishes? For our problem $m + n \neq 0$, since both m and n are positive integers. However, it is possible for $m = n$. This means that the vanishing of the integral can only happen when $m \neq n$. So, what can we do about the $m = n$ case? One way is to start from scratch with our integration. (Another way is to compute the limit as n approaches m in our result and use L'Hopital's Rule. Try it!)

So, for $n = m$ we have to compute $\int_0^{2\pi} \cos^2 mx \, dx$. This can also be handled using a trigonometric identity. Recall that

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta.)$$

Inserting this into the integral, we find

$$\int_0^{2\pi} \cos^2 mx \, dx = \frac{1}{2} \int_0^{2\pi} (1 + \cos^2 2mx) \, dx$$

$$\begin{aligned} &= \frac{1}{2}\left[x + \frac{1}{2m}\sin 2mx\right]_0^{2\pi} \\ &= \frac{1}{2}(2\pi) = \pi. \end{aligned} \tag{5.15}$$

To summarize, we have shown that

$$\int_0^{2\pi} \cos nx \cos mx\, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n. \end{cases} \tag{5.16}$$

This holds true for $m, n = 0, 1, \ldots$. [Why did we include $m, n = 0$?] When we have such a set of functions, they are said to be an orthogonal set over the integration interval.

Definition 5.2. A set of (real) functions $\{\phi_n(x)\}$ is said to be *orthogonal* on $[a, b]$ if $\int_a^b \phi_n(x)\phi_m(x)\, dx = 0$ when $n \neq m$. Furthermore, if we also have that $\int_a^b \phi_n^2(x)\, dx = 1$, these functions are called *orthonormal.*

The set of functions $\{\cos nx\}_{n=0}^{\infty}$ are orthogonal on $[0, 2\pi]$. Actually, they are orthogonal on any interval of length 2π. We can make them orthonormal by dividing each function by $\sqrt{\pi}$ as indicated by Equation (5.15).

The notion of orthogonality is actually a generalization of the orthogonality of vectors in finite dimensional vector spaces. The integral $\int_a^b f(x)f(x)\, dx$ is the generalization of the dot product, and is called the scalar product of $f(x)$ and $g(x)$, which are thought of as vectors in an infinite dimensional vector space spanned by a set of orthogonal functions. But that is another topic for later.

Returning to the evaluation of the integrals in equation (5.6), we still have to evaluate $\int_0^{2\pi} \sin nx \cos mx\, dx$. This can also be evaluated using trigonometric identities. In this case, we need an identity involving products of sines and cosines. Such products occur in the addition formulae for sine functions:

$$\sin(A + B) = \sin A \cos B + \sin B \cos A,$$

$$\sin(A - B) = \sin A \cos B - \sin B \cos A.$$

Adding these equations, we find that

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B.$$

Setting $A = nx$ and $B = mx$, we find that

$$\begin{aligned} \int_0^{2\pi} \sin nx \cos mx\, dx &= \frac{1}{2}\int_0^{2\pi} [\sin(n + m)x + \sin(n - m)x]\, dx \\ &= \frac{1}{2}\left[\frac{-\cos(n + m)x}{n + m} + \frac{-\cos(n - m)x}{n - m}\right]_0^{2\pi} \\ &= (-1 + 1) + (-1 + 1) = 0. \end{aligned} \tag{5.17}$$

For these integrals we also should be careful about setting $n = m$. In this special case, we have the integrals

$$\int_0^{2\pi} \sin mx \cos mx\, dx = \frac{1}{2}\int_0^{2\pi} \sin 2mx\, dx = \frac{1}{2}\left[\frac{-\cos 2mx}{2m}\right]_0^{2\pi} = 0.$$

Finally, we can finish our evaluation of (5.6). We have determined that all but one integral vanishes. In that case, $n = m$. This leaves us with

$$\int_0^{2\pi} f(x) \cos mx \, dx = a_m \pi.$$

Solving for a_m gives

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos mx \, dx.$$

Since this is true for all $m = 1, 2, \ldots$, we have proven this part of the theorem. The only part left is finding the b_n's This will be left as an exercise for the reader.

We now consider examples of finding Fourier coefficients for given functions. In all of these cases we define $f(x)$ on $[0, 2\pi]$.

Example 5.1. $f(x) = 3\cos 2x$, $x \in [0, 2\pi]$.

We first compute the integrals for the Fourier coefficients.

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} 3\cos 2x \, dx = 0. \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} 3\cos 2x \cos nx \, dx = 0, \quad n \neq 2. \\ a_2 &= \frac{1}{\pi} \int_0^{2\pi} 3\cos^2 2x \, dx = 3, \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} 3\cos 2x \sin nx \, dx = 0, \forall n. \end{aligned} \tag{5.18}$$

Therefore, we have that the only nonvanishing coefficient is $a_2 = 3$. So there is one term and $f(x) = 3\cos 2x$. Well, we should have know this before doing all of these integrals. So, if we have a function expressed simply in terms of sums of simple sines and cosines, then it should be easy to write down the Fourier coefficients without much work.

Example 5.2. $f(x) = \sin^2 x$, $x \in [0, 2\pi]$.

We could determine the Fourier coefficients by integrating as in the last example. However, it is easier to use trigonometric identities. We know that

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x.$$

There are no sine terms, so $b_n = 0$, $n = 1, 2, \ldots$. There is a constant term, implying $a_0/2 = 1/2$. So, $a_0 = 1$. There is a $\cos 2x$ term, corresponding to $n = 2$, so $a_2 = -\frac{1}{2}$. That leaves $a_n = 0$ for $n \neq 0, 2$.

Example 5.3. $f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$.

This example will take a little more work. We cannot bypass evaluating any integrals at this time. This function is discontinuous, so we will have to compute each integral by breaking up the integration into two integrals, one over $[0, \pi]$ and the other over $[\pi, 2\pi]$.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(-1)\,dx \\ &= \frac{1}{\pi}(\pi) + \frac{1}{\pi}(-2\pi+\pi) = 0. \end{aligned} \tag{5.19}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ &= \frac{1}{\pi}\left[\int_0^{\pi}\cos nx\,dx - \int_{\pi}^{2\pi}\cos nx\,dx\right] \\ &= \frac{1}{\pi}\left[\left(\frac{1}{n}\sin nx\right)_0^{\pi} - \left(\frac{1}{n}\sin nx\right)_{\pi}^{2\pi}\right] \\ &= 0. \end{aligned} \tag{5.20}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \\ &= \frac{1}{\pi}\left[\int_0^{\pi}\sin nx\,dx - \int_{\pi}^{2\pi}\sin nx\,dx\right] \\ &= \frac{1}{\pi}\left[\left(-\frac{1}{n}\cos nx\right)_0^{\pi} + \left(\frac{1}{n}\cos nx\right)_{\pi}^{2\pi}\right] \\ &= \frac{1}{\pi}\left[-\frac{1}{n}\cos n\pi + \frac{1}{n} + \frac{1}{n} - \frac{1}{n}\cos n\pi\right] \\ &= \frac{2}{n\pi}(1-\cos n\pi). \end{aligned} \tag{5.21}$$

We have found the Fourier coefficients for this function. Before inserting them into the Fourier series (5.1), we note that $\cos n\pi = (-1)^n$. Therefore,

$$1-\cos n\pi = \begin{cases} 0, & n \text{ even} \\ 2, & n \text{ odd}. \end{cases} \tag{5.22}$$

So, half of the b_n's are zero. While we could write the Fourier series representation as

$$f(x) \sim \frac{4}{\pi}\sum_{n=1,\ \text{odd}}^{\infty}\frac{1}{n}\sin nx,$$

we could let $n = 2k-1$ and write

$$f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)x}{2k-1},$$

But does this series converge? Does it converge to $f(x)$? We will discuss this question later in the chapter.

5.3 Fourier Series Over Other Intervals

In many applications we are interested in determining Fourier series representations of functions defined on intervals other than $[0, 2\pi]$. In this section we will determine the form of the series expansion and the Fourier coefficients in these cases.

The most general type of interval is given as $[a, b]$. However, this often is too general. More common intervals are of the form $[-\pi, \pi]$, $[0, L]$, or $[-L/2, L/2]$. The simplest generalization is to the interval $[0, L]$. Such intervals arise often in applications. For example, one can study vibrations of a one dimensional string of length L and set up the axes with the left end at $x = 0$ and the right end at $x = L$. Another problem would be to study the temperature distribution along a one dimensional rod of length L. Such problems lead to the original studies of Fourier series. As we will see later, symmetric intervals, $[-a, a]$, are also useful.

Given an interval $[0, L]$, we could apply a transformation to an interval of length 2π by simply rescaling our interval. Then we could apply this transformation to our Fourier series representation to obtain an equivalent one useful for functions defined on $[0, L]$.

We define $x \in [0, 2\pi]$ and $t \in [0, L]$. A linear transformation relating these intervals is simply $x = \frac{2\pi t}{L}$ as shown in Figure 5.5. So, $t = 0$ maps to $x = 0$ and $t = L$ maps to $x = 2\pi$. Furthermore, this transformation maps $f(x)$ to a new function $g(t) = f(x(t))$, which is defined on $[0, L]$. We will determine the Fourier series representation of this function using the representation for $f(x)$.

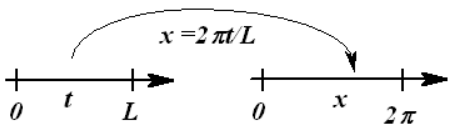


Figure 5.5: A sketch of the transformation between intervals $x \in [0, 2\pi]$ and $t \in [0, L]$.

Recall the form of the Fourier representation for $f(x)$ in Equation (5.1):

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]. \tag{5.23}$$

Inserting the transformation relating x and t, we have

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi t}{L} + b_n \sin \frac{2n\pi t}{L} \right]. \tag{5.24}$$

This gives the form of the series expansion for $g(t)$ with $t \in [0, L]$. But, we still need to determine the Fourier coefficients.

Recall, that

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx.$$

We need to make a substitution in the integral of $x = \frac{2\pi t}{L}$. We also will need to transform the differential, $dx = \frac{2\pi}{L}dt$. Thus, the resulting form for our coefficient is

$$a_n = \frac{2}{L}\int_0^L g(t)\cos\frac{2n\pi t}{L}\,dt. \tag{5.25}$$

Similarly, we find that

$$b_n = \frac{2}{L}\int_0^L g(t)\sin\frac{2n\pi t}{L}\,dt. \tag{5.26}$$

Fourier Series on $[0, L]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\frac{2n\pi x}{L} + b_n\sin\frac{2n\pi x}{L}\right]. \tag{5.27}$$

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}\,dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}\,dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{5.28}$$

Fourier Series on $[-\frac{L}{2}, \frac{L}{2}]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\frac{2n\pi x}{L} + b_n\sin\frac{2n\pi x}{L}\right]. \tag{5.29}$$

$$\begin{aligned} a_n &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\cos\frac{2n\pi x}{L}\,dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\sin\frac{2n\pi x}{L}\,dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{5.30}$$

Fourier Series on $[-\pi, \pi]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos nx + b_n\sin nx\right]. \tag{5.31}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{5.32}$$

Table 5.1: Special Fourier Series Representations on Different Intervals

We note first that when $L = 2\pi$ we get back the series representation that we first studied. Also, the period of $\cos\frac{2n\pi t}{L}$ is L/n, which means that the representation for $g(t)$ has a period of L.

At the end of this section we present the derivation of the Fourier series representation for a general interval for the interested reader. In Table 5.1 we summarize some commonly used Fourier series representations.

We will end our discussion for now with some special cases and an example for a function defined on $[-\pi, \pi]$.

Example 5.4. Let $f(x) = |x|$ on $[-\pi, \pi]$ We compute the coefficients, beginning as usual with a_0. We have

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} |x|\,dx = \pi \end{aligned} \tag{5.33}$$

At this point we need to remind the reader about the integration of even and odd functions.

1. **Even Functions:** In this evaluation we made use of the fact that the integrand is an even function. Recall that $f(x)$ is an *even function* if $f(-x) = f(x)$ for all x. One can recognize even functions as they are symmetric with respect to the y-axis as shown in Figure 5.6(A). If one integrates an even function over a symmetric interval, then one has that

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx. \tag{5.34}$$

One can prove this by splitting off the integration over negative values of x, using the substitution $x = -y$, and employing the evenness of $f(x)$. Thus,

$$\begin{aligned} \int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= -\int_{a}^{0} f(-y)\,dy + \int_{0}^{a} f(x)\,dx \\ &= \int_{0}^{a} f(y)\,dy + \int_{0}^{a} f(x)\,dx \\ &= 2\int_{0}^{a} f(x)\,dx. \end{aligned} \tag{5.35}$$

This can be visually verified by looking at Figure 5.6(A).

2. **Odd Functions:** A similar computation could be done for odd functions. $f(x)$ is an *odd function* if $f(-x) = -f(x)$ for all x. The graphs of such functions are symmetric with respect to the origin as shown in Figure 5.6(B). If one integrates an odd function over a symmetric interval, then one has that

$$\int_{-a}^{a} f(x)\,dx = 0. \tag{5.36}$$

We now continue with our computation of the Fourier coefficients for $f(x) = |x|$ on $[-\pi, \pi]$. We have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx. \tag{5.37}$$

Here we have made use of the fact that $|x|\cos nx$ is an even function. In order to compute the resulting integral, we need to use integration by parts,

$$\int_{a}^{b} u\,dv = uv\Big|_{a}^{b} - \int_{a}^{b} v\,du,$$

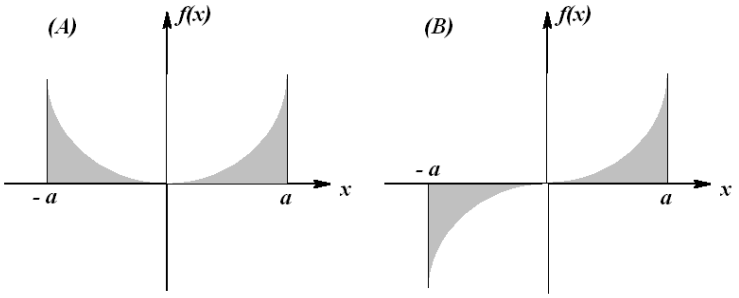


Figure 5.6: Examples of the areas under (A) even and (B) odd functions on symmetric intervals, $[-a, a]$.

by letting $u = x$ and $dv = \cos nx\,dx$. Thus, $du = dx$ and $v = \int dv = \frac{1}{n}\sin nx$. Continuing with the computation, we have

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi x\cos nx\,dx. \\ &= \frac{2}{\pi}\left[\frac{1}{n}x\sin nx\Big|_0^\pi - \frac{1}{n}\int_0^\pi \sin nx\,dx\right] \\ &= -\frac{2}{n\pi}\left[-\frac{1}{n}\cos nx\right]_0^\pi \\ &= -\frac{2}{\pi n^2}(1-(-1)^n). \end{aligned} \tag{5.38}$$

Here we have used the fact that $\cos n\pi = (-1)^n$ for any integer n. This lead to a factor $(1-(-1)^n)$. This factor can be simplified as

$$1-(-1)^n = \begin{cases} 2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}. \tag{5.39}$$

So, $a_n = 0$ for n even and $a_n = -\frac{4}{\pi n^2}$ for n odd.

Computing the b_n's is simpler. We note that we have to integrate $|x|\sin nx$ from $x = -\pi$ to π. The integrand is an odd function and this is a symmetric interval. So, the result is that $b_n = 0$ for all n.

Putting this all together, the Fourier series representation of $f(x) = |x|$ on $[-\pi, \pi]$ is given as

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,\text{ odd}}^{\infty}\frac{\cos nx}{n^2}. \tag{5.40}$$

While this is correct, we can rewrite the sum over only odd n by reindexing. We let $n = 2k-1$ for $k = 1, 2, 3, \ldots$. Then we only get the odd integers. The series can then be written as

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2}. \tag{5.41}$$

Throughout our discussion we have referred to such results as Fourier representations. We have not looked at the convergence of these series. Here is an example of an infinite series of functions. What does this series sum to? We show in Figure 5.7 the first few partial sums. They appear to be converging to $f(x) = |x|$ fairly quickly.

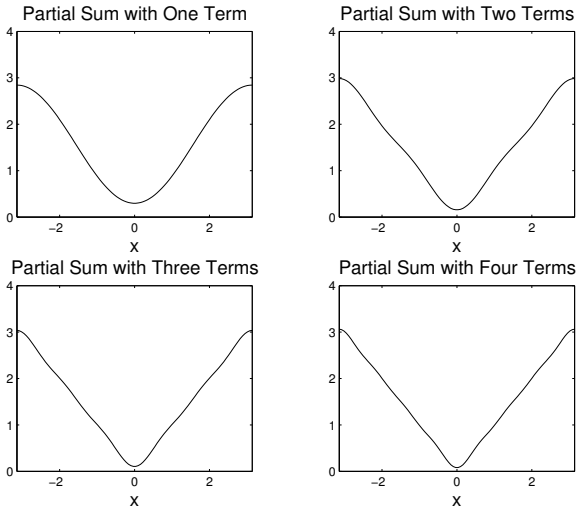


Figure 5.7: Plot of the first partial sums of the Fourier series representation for $f(x) = |x|$.

Even though $f(x)$ was defined on $[-\pi, \pi]$ we can still evaluate the Fourier series at values of x outside this interval. In Figure 5.8, we see that the representation agrees with $f(x)$ on the interval $[-\pi, \pi]$. Outside this interval we have a periodic extension of $f(x)$ with period 2π.

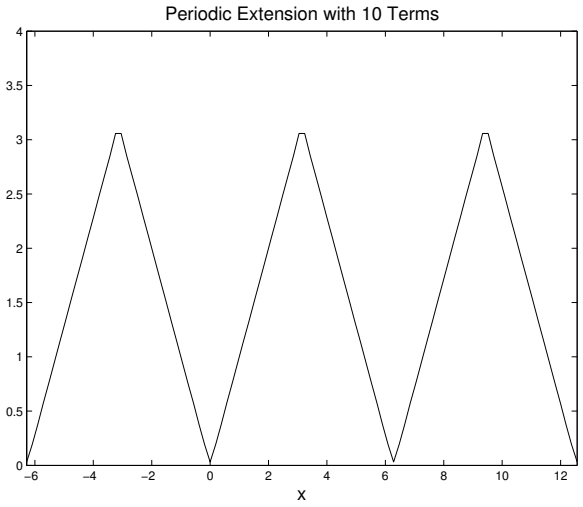


Figure 5.8: Plot of the first 10 terms of the Fourier series representation for $f(x) = |x|$ on the interval $[-2\pi, 4\pi]$.

Another example is the Fourier series representation of $f(x) = x$ on $[-\pi, \pi]$ as left for Problem 4. This is determined to be

$$f(x) \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx. \tag{5.42}$$

As seen in Figure 5.9 we again obtain the periodic extension of our function. In this case we needed many more terms. Also, the vertical parts of the first plot are nonexistent. In the second plot we only plot the points and not the typical connected points that most software packages plot as the default style.

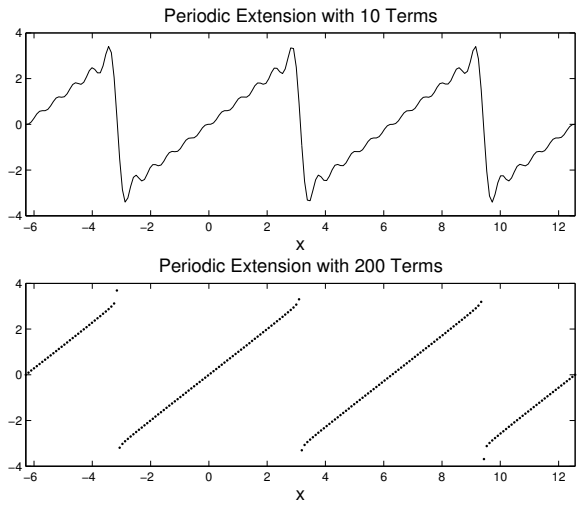


Figure 5.9: Plot of the first 10 terms and 200 terms of the Fourier series representation for $f(x) = x$ on the interval $[-2\pi, 4\pi]$.

Example 5.5. It is interesting to note that one can use Fourier series to obtain sums of some infinite series. For example, in the last example we found that

$$x \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

Now, what if we chose $x = \frac{\pi}{2}$? Then, we have

$$\frac{\pi}{2} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2} = 2\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right].$$

This gives a well known expression for π:

$$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right].$$

5.3.1 *Fourier Series on* $[a, b]$

A FOURIER SERIES REPRESENTATION IS ALSO POSSIBLE for a general interval, $t \in [a, b]$. As before, we just need to transform this interval to $[0, 2\pi]$. Let

$$x = 2\pi \frac{t - a}{b - a}.$$

Inserting this into the Fourier series (5.1) representation for $f(x)$ we obtain

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi(t-a)}{b-a} + b_n \sin \frac{2n\pi(t-a)}{b-a}\right]. \tag{5.43}$$

Well, this expansion is ugly. It is not like the last example, where the transformation was straightforward. If one were to apply the theory to applications, it might seem to make sense to just shift the data so that $a = 0$ and be done with any complicated expressions. However, mathematics students enjoy the challenge of developing such generalized expressions. So, let's see what is involved.

First, we apply the addition identities for trigonometric functions and rearrange the terms.

$$\begin{aligned} g(t) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi(t-a)}{b-a} + b_n \sin \frac{2n\pi(t-a)}{b-a} \right] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \left(\cos \frac{2n\pi t}{b-a} \cos \frac{2n\pi a}{b-a} + \sin \frac{2n\pi t}{b-a} \sin \frac{2n\pi a}{b-a} \right) \right. \\ &\quad \left. + b_n \left(\sin \frac{2n\pi t}{b-a} \cos \frac{2n\pi a}{b-a} - \cos \frac{2n\pi t}{b-a} \sin \frac{2n\pi a}{b-a} \right) \right] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\cos \frac{2n\pi t}{b-a} \left(a_n \cos \frac{2n\pi a}{b-a} - b_n \sin \frac{2n\pi a}{b-a} \right) \right. \\ &\quad \left. + \sin \frac{2n\pi t}{b-a} \left(a_n \sin \frac{2n\pi a}{b-a} + b_n \cos \frac{2n\pi a}{b-a} \right) \right]. \end{aligned} \tag{5.44}$$

Defining $A_0 = a_0$ and

$$\begin{aligned} A_n &\equiv a_n \cos \frac{2n\pi a}{b-a} - b_n \sin \frac{2n\pi a}{b-a} \\ B_n &\equiv a_n \sin \frac{2n\pi a}{b-a} + b_n \cos \frac{2n\pi a}{b-a}, \end{aligned} \tag{5.45}$$

we arrive at the more desirable form for the Fourier series representation of a function defined on the interval $[a, b]$.

$$g(t) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos \frac{2n\pi t}{b-a} + B_n \sin \frac{2n\pi t}{b-a} \right]. \tag{5.46}$$

We next need to find expressions for the Fourier coefficients. We insert the known expressions for a_n and b_n and rearrange. First, we note that under the transformation $x = 2\pi \frac{t-a}{b-a}$ we have

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\ &= \frac{2}{b-a} \int_a^b g(t) \cos \frac{2n\pi(t-a)}{b-a} \, dt, \end{aligned} \tag{5.47}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\ &= \frac{2}{b-a} \int_a^b g(t) \sin \frac{2n\pi(t-a)}{b-a} \, dt. \end{aligned} \tag{5.48}$$

Then, inserting these integrals in A_n, combining integrals and making use of the addition formula for the cosine of the sum of two angles, we obtain

$$\begin{aligned} A_n &\equiv a_n \cos \frac{2n\pi a}{b-a} - b_n \sin \frac{2n\pi a}{b-a} \\ &= \frac{2}{b-a} \int_a^b g(t) \left[\cos \frac{2n\pi(t-a)}{b-a} \cos \frac{2n\pi a}{b-a} - \sin \frac{2n\pi(t-a)}{b-a} \sin \frac{2n\pi a}{b-a} \right] dt \\ &= \frac{2}{b-a} \int_a^b g(t) \cos \frac{2n\pi t}{b-a} \, dt. \end{aligned} \tag{5.49}$$

A similar computation gives

$$B_n = \frac{2}{b-a}\int_a^b g(t)\sin\frac{2n\pi t}{b-a}\,dt. \tag{5.50}$$

Summarizing, we have shown that:

Theorem 5.2. *The Fourier series representation of $f(x)$ defined on $[a,b]$ when it exists, is given by*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\frac{2n\pi x}{b-a} + b_n\sin\frac{2n\pi x}{b-a}\right]. \tag{5.51}$$

with Fourier coefficients

$$\begin{aligned} a_n &= \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx. \quad n = 0,1,2,\ldots, \\ b_n &= \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx. \quad n = 1,2,\ldots. \end{aligned} \tag{5.52}$$

5.4 Sine and Cosine Series

IN THE LAST TWO EXAMPLES ($f(x) = |x|$ and $f(x) = x$ on $[-\pi,\pi]$) we have seen Fourier series representations that contain only sine or cosine terms. As we know, the sine functions are odd functions and thus sum to odd functions. Similarly, cosine functions sum to even functions. Such occurrences happen often in practice. Fourier representations involving just sines are called sine series and those involving just cosines (and the constant term) are called cosine series.

Another interesting result, based upon these examples, is that the original functions, $|x|$ and x agree on the interval $[0,\pi]$. Note from Figures 5.7-5.9 that their Fourier series representations do as well. Thus, more than one series can be used to represent functions defined on finite intervals. All they need to do is to agree with the function over that particular interval. Sometimes one of these series is more useful because it has additional properties needed in the given application.

We have made the following observations from the previous examples:

1. There are several trigonometric series representations for a function defined on a finite interval.

2. Odd functions on a symmetric interval are represented by sine series and even functions on a symmetric interval are represented by cosine series.

These two observations are related and are the subject of this section. We begin by defining a function $f(x)$ on interval $[0,L]$. We have seen that the Fourier series representation of this function appears to converge to a periodic extension of the function.

In Figure 5.10 we show a function defined on $[0,1]$. To the right is its periodic extension to the whole real axis. This representation has a period

of $L = 1$. The bottom left plot is obtained by first reflecting f about the y-axis to make it an even function and then graphing the periodic extension of this new function. Its period will be $2L = 2$. Finally, in the last plot we flip the function about each axis and graph the periodic extension of the new odd function. It will also have a period of $2L = 2$.

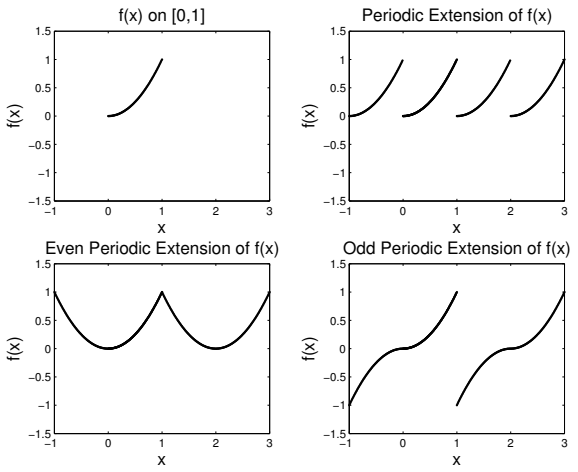


Figure 5.10: This is a sketch of a function and its various extensions. The original function $f(x)$ is defined on $[0, 1]$ and graphed in the upper left corner. To its right is the periodic extension, obtained by adding replicas. The two lower plots are obtained by first making the original function even or odd and then creating the periodic extensions of the new function.

In general, we obtain three different periodic representations. In order to distinguish these we will refer to them simply as the periodic, even and odd extensions. Now, starting with $f(x)$ defined on $[0, L]$, we would like to determine the Fourier series representations leading to these extensions. [For easy reference, the results are summarized in Table **??**] We have already seen that the periodic extension of $f(x)$ is obtained through the Fourier series representation in Equation (5.2).

Given $f(x)$ defined on $[0, L]$, the *even periodic extension* is obtained by simply computing the Fourier series representation for the even function

$$f_e(x) \equiv \begin{cases} f(x), & 0 < x < L, \\ f(-x) & -L < x < 0. \end{cases} \tag{5.59}$$

Since $f_e(x)$ is an even function on a symmetric interval $[-L, L]$, we expect that the resulting Fourier series will not contain sine terms. Therefore, the series expansion will be given by [Use the general case in (5.51) with $a = -L$ and $b = L$.]:

$$f_e(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{5.60}$$

with Fourier coefficients

$$a_n = \frac{1}{L} \int_{-L}^{L} f_e(x) \cos \frac{n\pi x}{L} \, dx. \quad n = 0, 1, 2, \ldots. \tag{5.61}$$

However, we can simplify this by noting that the integrand is even and the interval of integration can be replaced by $[0, L]$. On this interval $f_e(x) = f(x)$. So, we have the *Cosine Series Representation* of $f(x)$ for $x \in [0, L]$ is given as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{5.62}$$

Table 5.2: Fourier Cosine and Sine Series Representations on $[0, L]$

Fourier Series on $[0, L]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right]. \tag{5.53}$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{5.54}$$

Fourier Cosine Series on $[0, L]$

$$f(x) \sim a_0/2 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{5.55}$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots. \tag{5.56}$$

Fourier Sine Series on $[0, L]$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{5.57}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \tag{5.58}$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx. \quad n = 0, 1, 2, \ldots. \tag{5.63}$$

Similarly, given $f(x)$ defined on $[0, L]$, the *odd periodic extension* is obtained by simply computing the Fourier series representation for the odd function

$$f_o(x) \equiv \begin{cases} f(x), & 0 < x < L, \\ -f(-x) & -L < x < 0. \end{cases} \tag{5.64}$$

The resulting series expansion leads to defining the *Sine Series Representation* of $f(x)$ for $x \in [0, L]$ as

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}. \tag{5.65}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \quad n = 1, 2, \ldots. \tag{5.66}$$

Example 5.6. In Figure 5.10 we actually provided plots of the various extensions of the function $f(x) = x^2$ for $x \in [0, 1]$. Let's determine the representations of the periodic, even and odd extensions of this function.

For a change, we will use a CAS (Computer Algebra System) package to do the integrals. In this case we can use Maple. A general code for doing this for the periodic extension is shown in Table 5.3.

Table 5.3: Maple code for computing Fourier coefficients and plotting partial sums of the Fourier series.

```
> restart:
> L:=1:
> f:=x^2:
> assume(n,integer):
> a0:=2/L*int(f,x=0..L);
                                a0 := 2/3
> an:=2/L*int(f*cos(2*n*Pi*x/L),x=0..L);
                                      1
                              an := -------
                                     2   2
                                   n~  Pi
> bn:=2/L*int(f*sin(2*n*Pi*x/L),x=0..L);
                                       1
                              bn := - -----
                                     n~ Pi
> F:=a0/2+sum((1/(k*Pi)^2)*cos(2*k*Pi*x/L)
     -1/(k*Pi)*sin(2*k*Pi*x/L),k=1..50):
> plot(F,x=-1..3,title='Periodic Extension',
      titlefont=[TIMES,ROMAN,14],font=[TIMES,ROMAN,14]);
```

Example 5.7. Periodic Extension - Trigonometric Fourier Series

Using the above code, we have that $a_0 = \frac{2}{3}$ $a_n = \frac{1}{n^2\pi^2}$ and $b_n = -\frac{1}{n\pi}$. Thus, the resulting series is given as

$$f(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty} \left[\frac{1}{n^2\pi^2} \cos 2n\pi x - \frac{1}{n\pi} \sin 2n\pi x \right].$$

In Figure 5.11 we see the sum of the first 50 terms of this series. Generally, we see that the series seems to be converging to the periodic extension of f. There appear to be some problems with the convergence around integer values of x. We will later see that this is because of the discontinuities in the periodic extension and the resulting overshoot is referred to as the Gibbs phenomenon which is discussed in the appendix.

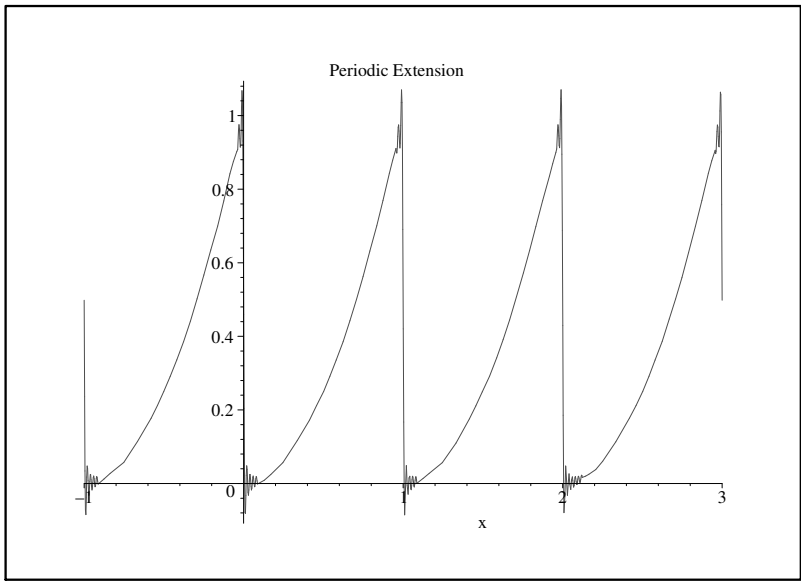


Figure 5.11: The periodic extension of $f(x) = x^2$ on $[0, 1]$.

Example 5.8. Even Periodic Extension - Cosine Series

In this case we compute $a_0 = \frac{2}{3}$ and $a_n = \frac{4(-1)^n}{n^2\pi^2}$. Therefore, we have

$$f(x) \sim \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x.$$

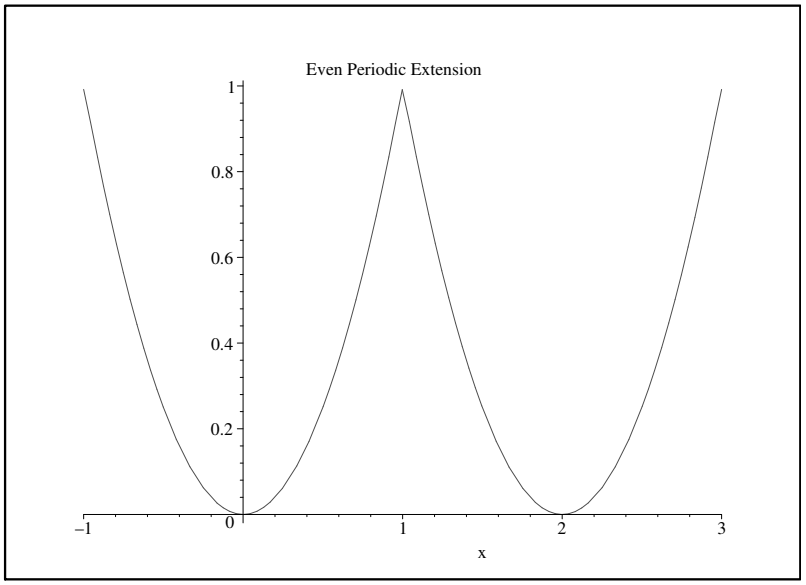
In Figure 5.12 we see the sum of the first 50 terms of this series. In this case the convergence seems to be much better than in the periodic extension case. We also see that it is converging to the even extension.

Example 5.9. Odd Periodic Extension - Sine Series

Finally, we look at the sine series for this function. We find that $b_n = -\frac{2}{n^3\pi^3}(n^2\pi^2(-1)^n - 2(-1)^n + 2)$. Therefore,

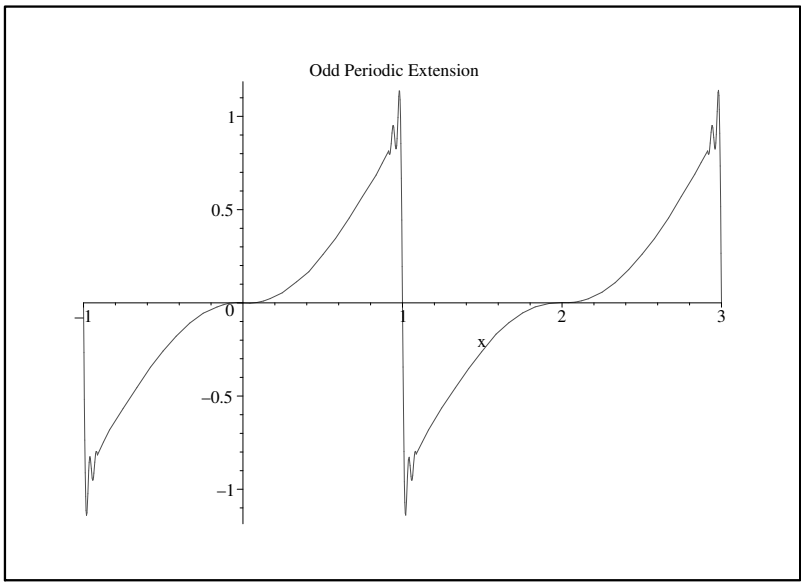
$$f(x) \sim -\frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3}(n^2\pi^2(-1)^n - 2(-1)^n + 2) \sin n\pi x.$$

Figure 5.12: The even periodic extension of $f(x) = x^2$ on $[0, 1]$.



Once again we see discontinuities in the extension as seen in Figure 5.13. However, we have verified that our sine series appears to be converging to the odd extension as we first sketched in Figure 5.10.

Figure 5.13: The odd periodic extension of $f(x) = x^2$ on $[0, 1]$.



5.5 *Appendix: The Gibbs Phenomenon*

We have seen that when there is a jump discontinuity in the periodic extension of our functions, whether the function originally had a discontinuity or developed one due to a mismatch in the values of the endpoints. This can be seen in Figures 5.9, 5.11 and 5.13. The Fourier series has

a difficult time converging at the point of discontinuity and these graphs of the Fourier series show a distinct overshoot which does not go away. This is called the *Gibbs phenomenon* and the amount of overshoot can be computed.

In one of our first examples, Example 5.3, we found the Fourier series representation of the piecewise defined function

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$$

to be

$$f(x) \sim \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}.$$

In Figure 5.14 we display the sum of the first ten terms. Note the wiggles, overshoots and under shoots near $x = 0, \pm\pi$. These are seen more when we plot the representation for $x \in [-3\pi, 3\pi]$, as shown in Figure 5.15. We note that the overshoots and undershoots occur at discontinuities in the periodic extension of $f(x)$. These occur whenever $f(x)$ has a discontinuity or if the values of $f(x)$ at the endpoints of the domain do not agree.

One might expect that we only need to add more terms. In Figure 5.16 we show the sum for twenty terms. Note the sum appears to converge better for points far from the discontinuities. But, the overshoots and undershoots are still present. In Figures 5.17 and 5.18 show magnified plots of the overshoot at $x = 0$ for $N = 100$ and $N = 500$, respectively. We see that the overshoot persists. The peak is at about the same height, but its location seems to be getting closer to the origin. We will show how one can estimate the size of the overshoot.

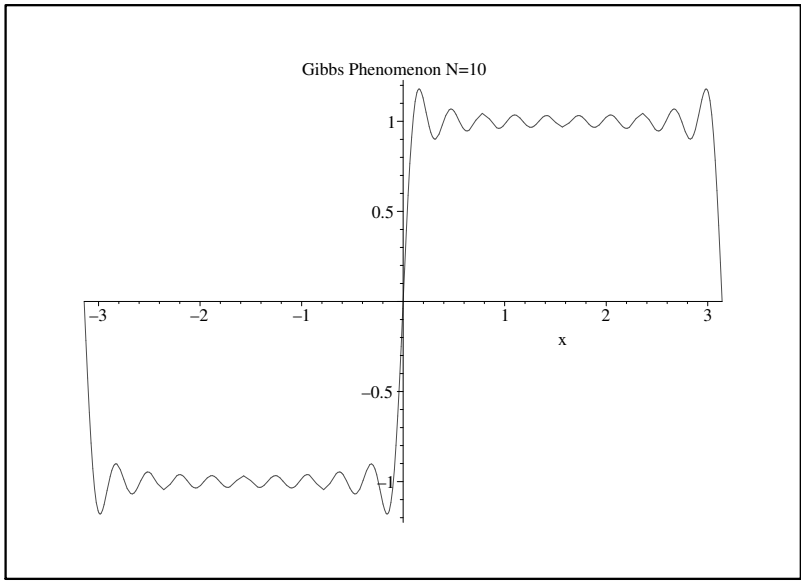


Figure 5.14: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 10$.

We can study the Gibbs phenomenon by looking at the partial sums of general Fourier trigonometric series for functions $f(x)$ defined on the interval $[-L, L]$. Writing out the partial sums, inserting the Fourier coefficients

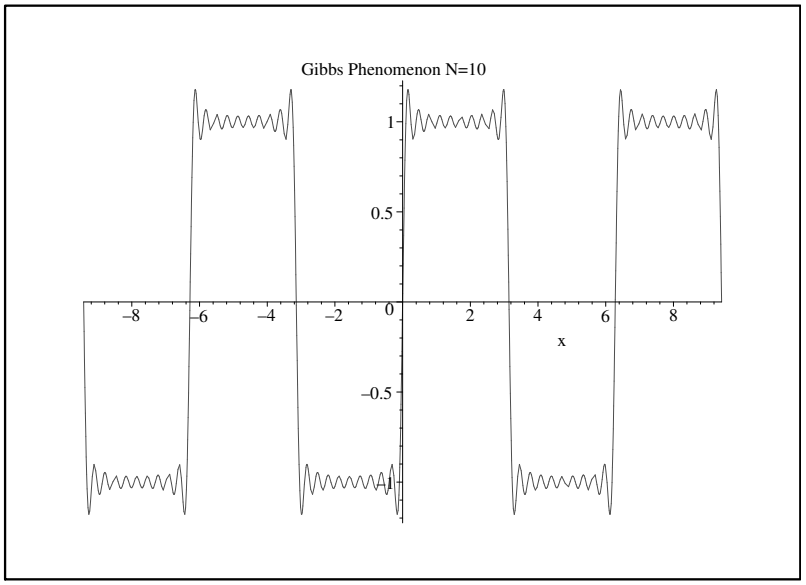


Figure 5.15: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 10$ plotted on $[-3\pi, 3\pi]$ displaying the periodicity.

and rearranging, we have

$$\begin{aligned}
S_N(x) &= a_0 + \sum_{n=1}^{N} \left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right] \\
&= \frac{1}{2L}\int_{-L}^{L} f(y)\,dy + \sum_{n=1}^{N} \left[\left(\frac{1}{L}\int_{-L}^{L} f(y)\cos\frac{n\pi y}{L}\,dy \right)\cos\frac{n\pi x}{L} \right. \\
&\quad \left. + \left(\frac{1}{L}\int_{-L}^{L} f(y)\sin\frac{n\pi y}{L}\,dy \right)\sin\frac{n\pi x}{L} \right] \\
&= \frac{1}{L}\int_{-L}^{L} \left\{ \frac{1}{2} + \sum_{n=1}^{N} \left(\cos\frac{n\pi y}{L}\cos\frac{n\pi x}{L} + \sin\frac{n\pi y}{L}\sin\frac{n\pi x}{L} \right) \right\} f(y)\,dy \\
&= \frac{1}{L}\int_{-L}^{L} \left\{ \frac{1}{2} + \sum_{n=1}^{N} \cos\frac{n\pi(y-x)}{L} \right\} f(y)\,dy \\
&\equiv \frac{1}{L}\int_{-L}^{L} D_N(y-x) f(y)\,dy. \qquad (5.67)
\end{aligned}$$

We have defined

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N} \cos\frac{n\pi x}{L},$$

which is called the *N-th Dirichlet Kernel*. We now prove

Proposition:

$$D_n(x) = \begin{cases} \frac{\sin((n+\frac{1}{2})\frac{\pi x}{L})}{2\sin\frac{\pi x}{2L}}, & \sin\frac{\pi x}{2L} \neq 0 \\ n + \frac{1}{2}, & \sin\frac{\pi x}{2L} = 0 \end{cases}.$$

Proof: Let $\theta = \frac{\pi x}{L}$ and multiply $D_n(x)$by $2\sin\frac{\theta}{2}$to obtain:

$$2\sin\frac{\theta}{2} D_n(x) = 2\sin\frac{\theta}{2}\left[\frac{1}{2} + \cos\theta + \cdots + \cos n\theta \right]$$

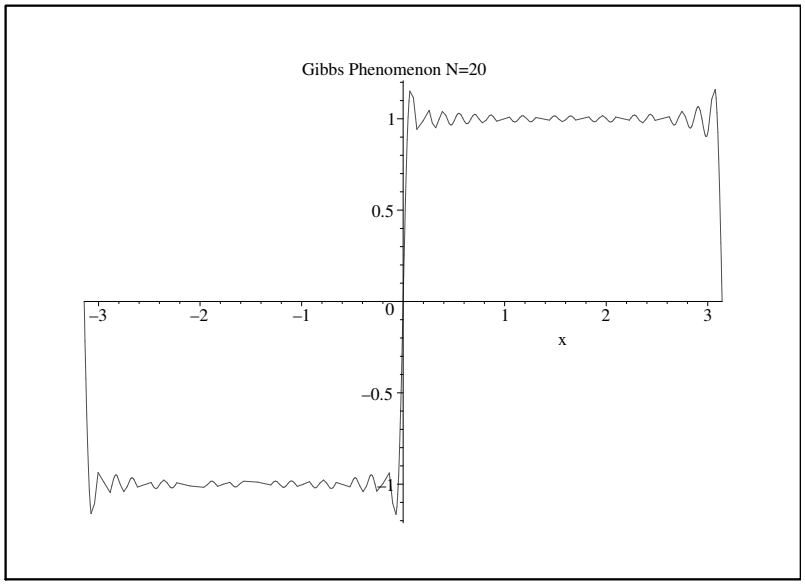


Figure 5.16: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 20$.

$$
\begin{aligned}
&= \sin\frac{\theta}{2} + 2\cos\theta\sin\frac{\theta}{2} + 2\cos 2\theta\sin\frac{\theta}{2} + \cdots + 2\cos n\theta\sin\frac{\theta}{2} \\
&= \sin\frac{\theta}{2} + \left(\sin\frac{3\theta}{2} - \sin\frac{\theta}{2}\right) + \left(\sin\frac{5\theta}{2} - \sin\frac{3\theta}{2}\right) + \cdots \\
&\quad + \left[\sin\left(n+\frac{1}{2}\right)\theta - \sin\left(n-\frac{1}{2}\right)\theta\right] \\
&= \sin\left(n+\frac{1}{2}\right)\theta. \qquad (5.68)
\end{aligned}
$$

Thus,

$$2\sin\frac{\theta}{2}D_n(x) = \sin\left(n+\frac{1}{2}\right)\theta,$$

or if $\sin\frac{\theta}{2} \neq 0$,

$$D_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}, \qquad \theta = \frac{\pi x}{L}.$$

If $\sin\frac{\theta}{2} = 0$,then one needs to apply L'Hospital's Rule:

$$
\begin{aligned}
\lim_{\theta\to 2m\pi}\frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} &= \lim_{\theta\to 2m\pi}\frac{(n+\frac{1}{2})\cos\left(n+\frac{1}{2}\right)\theta}{\cos\frac{\theta}{2}} \\
&= \frac{(n+\frac{1}{2})\cos\left(2mn\pi + m\pi\right)}{\cos m\pi} \\
&= n+\frac{1}{2}. \qquad (5.69)
\end{aligned}
$$

We further note that $D_N(x)$ is periodic with period $2L$ and is an even function.

So far, we have found that

$$S_N(x) = \frac{1}{L}\int_{-L}^{L} D_N(y-x)f(y)\,dy. \qquad (5.70)$$

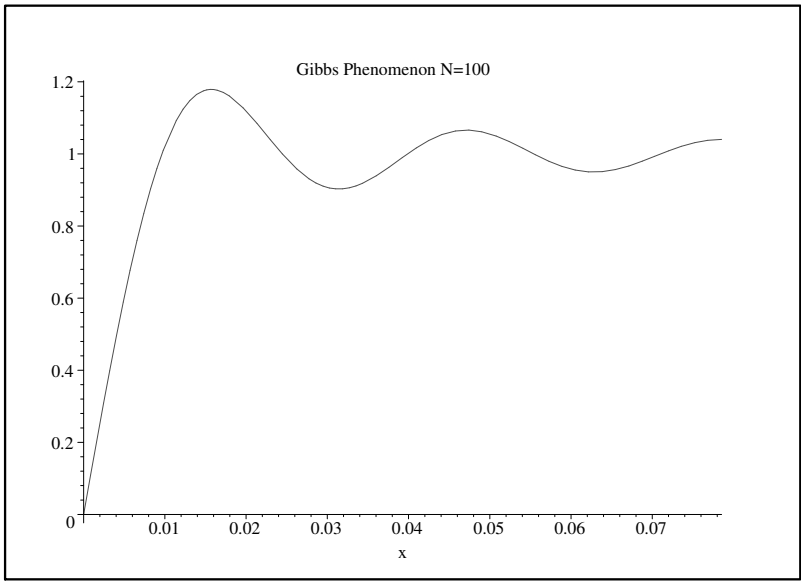


Figure 5.17: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 100$.

Now, make the substitution $\xi = y - x$. Then,

$$\begin{aligned} S_N(x) &= \frac{1}{L}\int_{-L-x}^{L-x} D_N(\xi) f(\xi + x)\, d\xi \\ &= \frac{1}{L}\int_{-L}^{L} D_N(\xi) f(\xi + x)\, d\xi. \end{aligned} \tag{5.71}$$

In the second integral we have made use of the fact that $f(x)$ and $D_N(x)$ are periodic with period $2L$ and shifted the interval back to $[-L, L]$.

Now split the integration and use the fact that $D_N(x)$ is an even function. Then,

$$\begin{aligned} S_N(x) &= \frac{1}{L}\int_{-L}^{0} D_N(\xi) f(\xi + x)\, d\xi + \frac{1}{L}\int_{0}^{L} D_N(\xi) f(\xi + x)\, d\xi \\ &= \frac{1}{L}\int_{0}^{L} \left[f(x - \xi) + f(\xi + x)\right] D_N(\xi)\, d\xi. \end{aligned} \tag{5.72}$$

We can use this result to study the Gibbs phenomenon whenever it occurs. In particular, we will only concentrate on our earlier example. Namely,

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$$

For this case, we have

$$S_N(x) = \frac{1}{\pi}\int_{0}^{\pi} \left[f(x - \xi) + f(\xi + x)\right] D_N(\xi)\, d\xi \tag{5.73}$$

for

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N} \cos nx.$$

Also, one can show that

$$f(x - \xi) + f(\xi + x) = \begin{cases} 2, & 0 \le \xi < x, \\ 0, & x \le \xi < \pi - x, \\ -2, & \pi - x \le \xi < \pi. \end{cases}$$

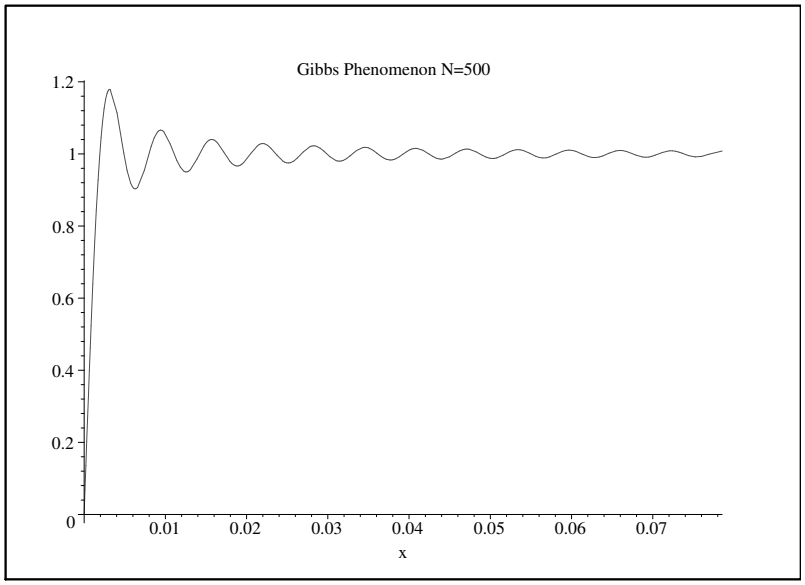


Figure 5.18: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 500$.

Thus, we have

$$\begin{aligned} S_N(x) &= \frac{2}{\pi}\int_0^x D_N(\xi)\,d\xi - \frac{2}{\pi}\int_{\pi-x}^{\pi} D_N(\xi)\,d\xi \\ &= \frac{2}{\pi}\int_0^x D_N(z)\,dz + \frac{2}{\pi}\int_0^x D_N(\pi - z)\,dz. \end{aligned} \tag{5.74}$$

Here we made the substitution $z = \pi - \xi$ in the second integral. The Dirichlet kernel in the proposition for $L = \pi$ is given by

$$D_N(x) = \frac{\sin(N + \frac{1}{2})x}{2\sin\frac{x}{2}}.$$

For N large, we have $N + \frac{1}{2} \approx N$, and for small x, we have $\sin\frac{x}{2} \approx \frac{x}{2}$. So, under these assumptions,

$$D_N(x) \approx \frac{\sin Nx}{x}.$$

Therefore,

$$S_N(x) \to \frac{2}{\pi}\int_0^x \frac{\sin N\xi}{\xi}\,d\xi.$$

If we want to determine the locations of the minima and maxima, where the undershoot and overshoot occur, then we apply the first derivative test for extrema to $S_N(x)$. Thus,

$$\frac{d}{dx}S_N(x) = \frac{2}{\pi}\frac{\sin Nx}{x} = 0.$$

The extrema occur for $Nx = m\pi$, $m = \pm 1, \pm 2, \ldots$. One can show that there is a maximum at $x = \pi/N$ and a minimum for $x = 2\pi/N$. The value for the overshoot can be computed as

$$S_N(\pi/N) = \frac{2}{\pi}\int_0^{\pi/N} \frac{\sin N\xi}{\xi}\,d\xi$$

$$
\begin{aligned}
&= \frac{2}{\pi}\int_0^{\pi}\frac{\sin t}{t}\,dt \\
&= \frac{2}{\pi}\mathrm{Si}(\pi) \\
&= 1.178979744\ldots. \qquad (5.75)
\end{aligned}
$$

Note that this value is independent of N and is given in terms of the sine integral,

$$\mathrm{Si}(x) \equiv \int_0^x \frac{\sin t}{t}\,dt.$$

5.6 Plotting Fourier Series

An example of Maple code was shown in Table 5.3 for finding Fourier coefficients and plotting Fourier series. If the Fourier coefficients are known, then the code is simpler to write:

```
plot((1/2)*Pi+2*(sum((cos(n*Pi)-1)*cos(n*x)/n^2, n = 1 .. 20))/Pi,
     x = -5*Pi .. 5*Pi)
```

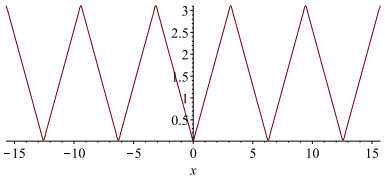


Figure 5.19: Nth Partial Sum in Maple.

However, if one does not have access to Maple, then one can use other software tools. For quick computations, one can be use online systems like WolframAlpha. For example, entering one of the following produces Fourier series plots.

```
plot sum_(k=1)^10(4/pi sin((2 k-1)*x)/(2 k-1)) x=0 to 2 pi
plot pi/2-sum_(k=1)^10(4/pi cos((2 k-1)*x)/(2 k-1)^2)
FourierSeries[abs(x), x, 10]
```

There are several other platforms like Sagemath with interactive Fourier Series `https://sage.brandoncurtis.com/fourier.html`, or at Desmos.com you can find several examples like `https://www.desmos.com/calculator/73hxmnkyvn` and `https://www.desmos.com/calculator/oowvp6lolf`. We now give some examples using MATLAB and Python.

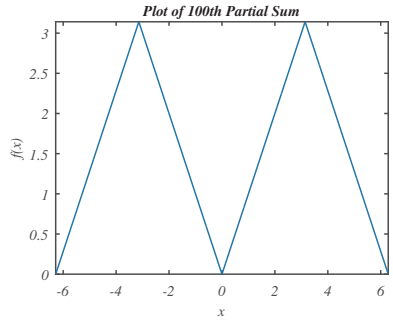


Figure 5.20: Nth Partial Sum done in MATLAB.

5.6.1 MATLAB and GNU Octave Files

If one has access to MATLAB's symbolic toolset, then one way that the Nth partial sum can be plotted is by saving the m-file below as **plotfs.m** and running it. It produces the plot in Figure 5.20.

```
% Plot the Nth partial sum of a Fourier series
% of f on the interval [a,b].

clear
syms n x;
N=100;
a=-2*pi;
b=2*pi;
f = pi/2+symsum((2/(pi*n^2))*((-1)^n-1)*cos(n*x),n,1,N);
```

```
ezplot(f,[a,b])
xlabel('x')
ylabel('f(x)')
title(['Plot of ', num2str(N),'th Partial Sum'] )
axis([a,b,0,pi])
```

GNU Octave is mostly compatable with MATLAB and m-files can often be run in GNU Octave. However, to run these symbolic examples, one has to download and install the symbolic package first.

```
pkg install symbolic-3.2.1.tar.gz
```

Afterwards it needs to be loaded when it is needed using `pkg load symbolic`.

A version of MATLAB that does not need symbolic computation can be run with a slight modification. We can call this **plotfso.m**.

```
N=10;
a=-2*pi;
b=2*pi;

t=linspace(a,b,100);
fs=pi/2;
for k=1:N
    fs=fs+(2/(pi*k^2))*((-1)^k-1)*cos(k*t);
end

plot(t,fs)
xlabel('x')
ylabel('f(x)')
title(['Plot of ', num2str(N),'th Partial Sum'] )
axis([a,b,0,pi])
```

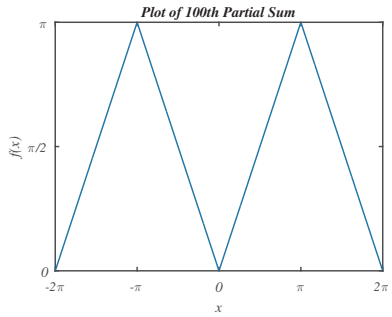


Figure 5.21: Nth Partial Sum showing labels in terms of π.

One can modify the plot labels to show increments of π or $\frac{\pi}{2}$ in MATLAB, as shown in Figure 5.21, by adding the following lines:

```
% Code to label axes with pi instead of the default
 xtick = [-2, -1, 0, 1, 2] * pi;
 xticklabel = { '-2 \pi', '-\pi', '0','\pi', '2\pi'};
 set (gca, 'xtick', xtick, 'xticklabel', xticklabel)
 ytick = [0 1/2 1]*pi;
 yticklabel = {'0','\pi/2','\pi'};
 set (gca, 'ytick', ytick, 'yticklabel', yticklabel)
```

The MATLAB code for **plotfso.m** will run in GNU Octave. A slight modification, as shown in the next section, will work in Python.

MATLAB now has a symbolic toolbox and one can do some computer algebra routines. The following code is used to find the Fourier coefficients for $f(x) = x^2$ on $x \in [0, 1]$.

```
syms n x
```

```
n = sym('n','integer');
f = x^2;
L = 1;
a0 = simplify(2/L*int(f,0,L))
an = simplify(2/L*int(f*cos(2*n*pi*x/L),0,L))
bn = simplify(2/L*int(f*sin(2*n*pi*x/L),0,L))
```

Running this code produces

```
>> FScoeff
a0 = 2/3
an = 1/(pi^2*n^2)
bn = -1/(pi*n)
```

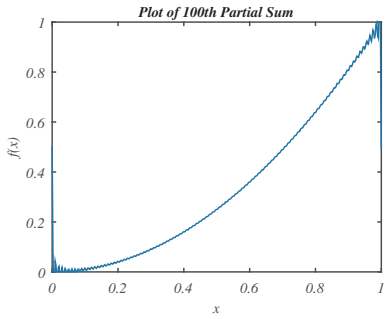


Figure 5.22: Nth Partial Sum for $f(x) = x^2$ on $x \in [0,1]$ as found in MATLAB using symbolic computation of the Fourier coefficients.

Adding the following lines,

```
N=100;
f = a0/2+symsum(an*cos(2*n*pi*x/L)+bn*sin(2*n*pi*x/L),n,1,N);
ezplot(f,[0,L])
xlabel('x')
ylabel('f(x)')
title(['Plot of ', num2str(N),'th Partial Sum'] )
axis([0,L,0,1])
```

we can plot the partial sums for the corresponding Fourier series. We show a plot for $N = 100$ in Figure 5.22. A similar result can be obtained with Python as shown in the next section.

5.6.2 *Python Scripts*

Another environment for plotting Fourier series is in Python. There are several libraries which come with the installation of Python for doing scientific computation. The first program shows how one can use the Sympy library to compute the Fourier coefficients symbolically.

```
from sympy import *

n = symbols('n', integer=True, positive=True)
k = symbols('k', integer=True, positive=True)
x = Symbol('x')

a = 0
b = 1
L=b-a

f = x**2

a_0 = (2/L) * integrate(f, (x, a, b))
a_n = (2/L) * integrate(f*cos(2*n*pi*x/L), (x,a,b))
```

```
b_n = (2/L) * integrate(f*sin(2*n*pi*x/L), (x,a,b))

print "a_0 =", simplify(a_0)
print "a_n =", simplify(a_n)
print "b_n =", simplify(b_n)
```

Once one has the Fourier coefficients, then one would like to plot the series. There are a number of ways to do this, using the several different libraries for plots. An example for using the explicit series is

```
from pylab import *

N=10;
a=-2*pi;
b=2*pi;

t=linspace(a,b,100);
fs=pi/2;
for k in xrange(1,N):
    fs=fs+(2/(pi*k**2))*((-1)**k-1)*cos(k*t);

plot(t,fs)
xlabel('x')
ylabel('f(x)')
title('Plot of Partial Sum' )
axis([a,b,0,pi])
```

Using Python entirely to compute the coefficients and plot the series could be done by direct integration in the partial sum:

```
from sympy import *
import numpy as np
import matplotlib.pyplot as plt

n = symbols('n', integer=True, positive=True)
x = Symbol('x')
t = Symbol('t')

L=1

FS = (1/L*integrate(t**2,(t,0,L))
       +sum(2/L*integrate(t**2*cos(2*n*pi*t/L),(t,0,L))
       *cos(2*n*pi*x/L)+2/L*integrate(t**2*sin(2*n*pi*t/L),(t,0,L))
       *sin(2*n*pi*x/L) for n in range(1, 20, 1)) )

evFS = lambdify(x, FS, modules=['numpy'])
tt = np.linspace(0, L, 100)
plt.plot(tt, evFS(tt), 'b', label='Fourier Series')
plt.show()
```

Problems

1. Write $y(t) = 3\cos 2t - 4\sin 2t$ in the form $y(t) = A\cos(2\pi f t + \phi)$.

2. Derive the coefficients b_n in Equation (5.2).

3. Consider the square wave function

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi. \end{cases}$$

Find the Fourier series representation of this function and plot the first 50 terms.

4. Find the Fourier Series of each function $f(x)$ of period 2π. For each series, plot the Nth partial sum,

$$S_N = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos nx + b_n \sin nx\right],$$

for $N = 5, 10, 50$ and describe the convergence (is it fast? what is it converging to, etc.)

a. $f(x) = x$, $|x| < \pi$.

b. $f(x) = \frac{x^2}{4}$, $|x| < \pi$.

c. $f(x) = \pi - |x|$, $|x| < \pi$.

d. $f(x) = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi, \\ -\frac{\pi}{2}, & \pi < x < 2\pi. \end{cases}$

e. $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$

5. Consider the function $f(x) = 4\sin^3 2x$

a. Derive an identity relating $\sin^3\theta$ in terms of $\sin\theta$ and $\sin 3\theta$ and express $f(x)$ in terms of simple sine functions.

b. Determine the Fourier coefficients of $f(x)$ in a Fourier series expansion on $[0, 2\pi]$ without computing any integrals!

6. Find the Fourier series of $f(x) = x$ on the given interval with the given period T. Plot the Nth partial sums and describe what you see.

a. $0 < x < 2$, $T = 2$.

b. $-2 < x < 2$, $T = 4$.

7. The result in problem 4b above gives a Fourier series representation of $\frac{x^2}{4}$. By picking the right value for x and a little arrangement of the series, show that [See Example 5.5.]

a.

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots.$$

b.
$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

8. Consider the function $f(x) = x$, $-\pi < x < \pi$.

a. Show that $x = 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}$.

b. Integrate the series in part a and show that
$$x^2 = \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos nx}{n^2}.$$

c. Find the Fourier cosine series of $f(x) = x^2$ on $[0, \pi]$ and compare it to the result in part b.

9. Sketch (by hand) the graphs of each of the following functions over four periods. Then sketch the extensions each of the functions as both an even and odd periodic function. Determine the corresponding Fourier sine and cosine series and verify the convergence to the desired function using Maple.

a. $f(x) = x^2$, $0 < x < 1$.

b. $f(x) = x(2 - x)$, $0 < x < 2$.

c. $f(x) = \begin{cases} 0, & 0 < x < 1, \\ 1, & 1 < x < 2. \end{cases}$

d. $f(x) = \begin{cases} \pi, & 0 < x < \pi, \\ 2\pi - x, & \pi < x < 2\pi. \end{cases}$

Chapter 6

Sturm-Liouville Eigenvalue Problems

"Read Euler, read Euler. He is the master of us all."

"Nature laughs at the difficulties of integration." - Pierre-Simon Marquis de Laplace (1749-1827)

6.1 Introduction

In physics many problems arise in the form of boundary value problems involving second order ordinary differential equations. For example, when solving the wave equation and the heat equation in three dimensions using the Method of Separation of Variables, separating out the time dependence leads to a three dimensional boundary value problem in both cases. Further separation of variables leads to a set of boundary value problems involving second order ordinary differential equations.

In general, we might obtain equations of the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \tag{6.1}$$

for $y = y(x)$ subject to boundary conditions. We can write such an equation in operator form by defining the differential operator

$$L = a_2(x)D^2 + a_1(x)D + a_0(x),$$

where $D = d/dx$. Then, Equation (6.1) takes the form

$$Ly = f.$$

Recall that we had solved such nonhomogeneous differential equations in Chapter 1 and a review is in Section **??**. In this section we will show that these equations can be solved using eigenfunction expansions. Namely, we seek solutions to the eigenvalue problem

$$L\phi = \lambda\phi$$

with homogeneous boundary conditions on ϕ and then seek a solution of the nonhomogeneous problem, $Ly = f$, as an expansion over these eigenfunctions. Formally, we let

$$y(x) = \sum_{n=1}^{\infty} c_n\phi_n(x).$$

However, we are not guaranteed a nice set of eigenfunctions. We need an appropriate set to form a basis in the function space. Also, it would be nice to have orthogonality so that we can easily solve for the expansion coefficients.

It turns out that any linear second order differential operator can be turned into an operator that possesses just the right properties (self-adjointedness) to carry out this procedure. The resulting operator is referred to as a Sturm-Liouville operator. We will highlight some of the properties of these operators and see how they are used in applications.

The Sturm-Liouville operator.

We define the Sturm-Liouville operator as

$$\mathcal{L} = \frac{d}{dx}p(x)\frac{d}{dx} + q(x). \tag{6.2}$$

The Sturm-Liouville eigenvalue problem.

The Sturm-Liouville eigenvalue problem is given by the differential equation

$$\mathcal{L}y = -\lambda\sigma(x)y,$$

or

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y + \lambda\sigma(x)y = 0, \tag{6.3}$$

for $x \in (a,b)$, $y = y(x)$, plus boundary conditions. The functions $p(x)$, $p'(x)$, $q(x)$ and $\sigma(x)$ are assumed to be continuous on (a,b) and $p(x) > 0, \sigma(x) > 0$ on $[a,b]$. If the interval is finite and these assumptions on the coefficients are true on $[a,b]$, then the problem is said to be a regular Sturm-Liouville problem. Otherwise, it is called a singular Sturm-Liouville problem.

We also need to impose the set of homogeneous boundary conditions

$$\begin{aligned} \alpha_1 y(a) + \beta_1 y'(a) &= 0, \\ \alpha_2 y(b) + \beta_2 y'(b) &= 0. \end{aligned} \tag{6.4}$$

The α's and β's are constants. For different values, one has special types of boundary conditions. For $\beta_i = 0$, we have what are called Dirichlet boundary conditions. Namely, $y(a) = 0$ and $y(b) = 0$. For $\alpha_i = 0$, we have Neumann boundary conditions. In this case, $y'(a) = 0$ and $y'(b) = 0$. In terms of the heat equation example, Dirichlet conditions correspond to maintaining a fixed temperature at the ends of the rod. The Neumann boundary conditions would correspond to no heat flow across the ends, or insulating conditions, as there would be no temperature gradient at those points. The more general boundary conditions allow for partially insulated boundaries.

Types of boundary conditions:
Dirichlet boundary conditions - the solution takes fixed values on the boundary. These are named after Gustav Lejeune Dirichlet (1805-1859).

Neumann boundary conditions - the derivative of the solution takes fixed values on the boundary. These are named after Carl Neumann (1832-1925).

Robin boundary conditions, $ay(0) + by'(0) = 0$, named after Victor Gustave Robin (1855–1897).

Another type of boundary condition that is often encountered is the periodic boundary condition. Consider the heated rod that has been bent to form a circle. Then the two end points are physically the same. So, we would expect that the temperature and the temperature gradient should agree at those points. For this case we write $y(a) = y(b)$ and $y'(a) = y'(b)$. Boundary value problems using these conditions have to be handled differently than the above homogeneous conditions. These conditions leads to different types of eigenfunctions and eigenvalues.

As previously mentioned, equations of the form (6.1) occur often. We now show that any second order linear operator can be put into the form of the Sturm-Liouville operator. In particular, equation (6.1) can be put into the form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = F(x). \tag{6.5}$$

Differential equations of Sturm-Liouville form.

The proof of this is straight forward as we soon show. Let's first consider the equation (6.1) for the case that $a_1(x) = a_2'(x)$. Then, we can write the equation in a form in which the first two terms combine,

$$\begin{aligned} f(x) &= a_2(x)y'' + a_1(x)y' + a_0(x)y \\ &= (a_2(x)y')' + a_0(x)y. \end{aligned} \tag{6.6}$$

The resulting equation is now in Sturm-Liouville form. We just identify $p(x) = a_2(x)$ and $q(x) = a_0(x)$.

Not all second order differential equations are as simple to convert. Consider the differential equation

$$x^2y'' + xy' + 2y = 0.$$

In this case $a_2(x) = x^2$ and $a_2'(x) = 2x \neq a_1(x)$. So, this does not fall into this case. However, we can change the operator in this equation, $x^2D + xD$, to a Sturm-Liouville operator, $Dp(x)D$ for a $p(x)$ that depends on the coefficients x^2 and x.

In the Sturm Liouville operator the derivative terms are gathered together into one exact derivative, $Dp(x)D$. This is similar to what we saw for ODE [see Section **??**] when we solved linear first order equations. In that case we sought an integrating factor. We can do the same thing here. We seek a multiplicative function $\mu(x)$ that we can multiply through (6.1) so that it can be written in Sturm-Liouville form.

We first divide out the $a_2(x)$, giving

$$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = \frac{f(x)}{a_2(x)}.$$

Next, we multiply this differential equation by and integrating factor, μ,

$$\mu(x)y'' + \mu(x)\frac{a_1(x)}{a_2(x)}y' + \mu(x)\frac{a_0(x)}{a_2(x)}y = \mu(x)\frac{f(x)}{a_2(x)}.$$

The first two terms can now be combined into an exact derivative $(\mu y')'$ if the second coefficient is $\mu'(x)$. Therefore, $\mu(x)$ satisfies a first order, separable differential equation:

$$\frac{d\mu}{dx} = \mu(x)\frac{a_1(x)}{a_2(x)}.$$

This is formally solved to give the sought integrating factor[1]

$$\mu(x) = e^{\int \frac{a_1(x)}{a_2(x)}\,dx}.$$

[1] Actually, we should write this as

$$\mu(x) = e^{\int^x \frac{a_1(\xi)}{a_2(\xi)}\,d\xi}.$$

This distinguishes the variable of integration ξ from the evaluation at x.

Thus, the original equation can be multiplied by factor

$$\frac{\mu(x)}{a_2(x)} = \frac{1}{a_2(x)} e^{\int \frac{a_1(x)}{a_2(x)}\,dx}$$

to turn it into Sturm-Liouville form.

In summary,

Equation (6.1),

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x), \tag{6.7}$$

can be put into the Sturm-Liouville form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = F(x), \tag{6.8}$$

where

$$\begin{aligned} p(x) &= e^{\int \frac{a_1(x)}{a_2(x)}\,dx}, \\ q(x) &= p(x)\frac{a_0(x)}{a_2(x)}, \\ F(x) &= p(x)\frac{f(x)}{a_2(x)}. \end{aligned} \tag{6.9}$$

Example 6.1. Convert $x^2y'' + xy' + 2y = 0$ into Sturm-Liouville form.

We multiply this equation by

$$\frac{\mu(x)}{a_2(x)} = \frac{1}{x^2} e^{\int \frac{dx}{x}} = \frac{1}{x},$$

to put the equation in Sturm-Liouville form:

$$\begin{aligned} 0 &= xy'' + y' + \frac{2}{x}y \\ &= (xy')' + \frac{2}{x}y. \end{aligned} \tag{6.10}$$

Conversion of a linear second order differential equation to Sturm Liouville form.

We could also proceed as follows. Divide by x^2, giving

$$y'' + \frac{1}{x}y' + \frac{2}{x}y = 0.$$

Now find

$$\mu(x) = \int^x \frac{dx}{x} = \frac{1}{x}.$$

Multiply the original equation by $\mu(x)$ and rewrite the equation.

6.2 *Properties of Sturm-Liouville Eigenvalue Problems*

THERE ARE SEVERAL PROPERTIES THAT CAN BE PROVEN for the (regular) Sturm-Liouville eigenvalue problem in (6.3). However, we will not prove them all here. We will merely list some of the important facts and focus on a few of the properties.

Real, countable eigenvalues.

1. The eigenvalues are real, countable, ordered and there is a smallest eigenvalue. Thus, we can write them as $\lambda_1 < \lambda_2 < \ldots$. However, there is no largest eigenvalue and $n \to \infty$, $\lambda_n \to \infty$.

Oscillatory eigenfunctions.

2. For each eigenvalue λ_n there exists an eigenfunction ϕ_n with $n-1$ zeros on (a, b).

3. Eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the weight function, $\sigma(x)$. Defining the inner product of $f(x)$ and $g(x)$ as

$$\langle f, g \rangle = \int_a^b f(x)g(x)\sigma(x)\,dx, \tag{6.11}$$

then the orthogonality of the eigenfunctions can be written in the form

Orthogonality of eigenfunctions.

$$\langle \phi_n, \phi_m \rangle = \langle \phi_n, \phi_n \rangle \delta_{nm}, \quad n, m = 1, 2, \ldots. \tag{6.12}$$

4. The set of eigenfunctions is complete; i.e., any piecewise smooth function can be represented by a generalized Fourier series expansion of the eigenfunctions,

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x),$$

where

$$c_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

Actually, one needs $f(x) \in L^2_\sigma(a, b)$, the set of square integrable functions over $[a, b]$ with weight function $\sigma(x)$. By square integrable, we mean that $\langle f, f \rangle < \infty$. One can show that such a space is isomorphic to a Hilbert space, a complete inner product space. Hilbert spaces play a special role in quantum mechanics.

Complete basis of eigenfunctions.

5. The eigenvalues satisfy the Rayleigh quotient

$$\lambda_n = \frac{-p\phi_n \frac{d\phi_n}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{d\phi_n}{dx}\right)^2 - q\phi_n^2 \right] dx}{\langle \phi_n, \phi_n \rangle}.$$

The Rayleigh quotient is named after Lord Rayleigh, John William Strutt, 3rd Baron Raleigh (1842-1919).

This is verified by multiplying the eigenvalue problem

$$\mathcal{L}\phi_n = -\lambda_n \sigma(x)\phi_n$$

by ϕ_n and integrating. Solving this result for λ_n, we obtain the Rayleigh quotient. The Rayleigh quotient is useful for getting estimates of eigenvalues and proving some of the other properties.

Example 6.2. Verify some of these properties for the eigenvalue problem

$$y'' = -\lambda y, \quad y(0) = y(\pi) = 0.$$

This is a problem we had seen many times. The eigenfunctions for this eigenvalue problem are $\phi_n(x) = \sin nx$, with eigenvalues $\lambda_n = n^2$ for $n = 1, 2, \ldots$. These satisfy the properties listed above.

First of all, the eigenvalues are real, countable and ordered, $1 < 4 < 9 < \ldots$. There is no largest eigenvalue and there is a first one.

The eigenfunctions corresponding to each eigenvalue have $n - 1$ zeros on $(0, \pi)$. This is demonstrated for several eigenfunctions in Figure 6.1.

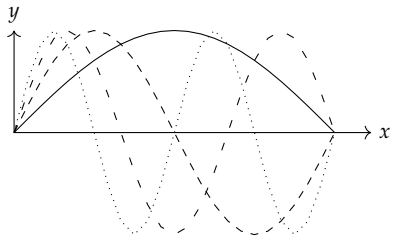


Figure 6.1: Plot of the eigenfunctions $\phi_n(x) = \sin nx$ for $n = 1, 2, 3, 4$.

We also know from Chapter 5 that the set $\{\sin nx\}_{n=1}^{\infty}$ is an orthogonal set of basis functions of length

$$\|\phi_n\| = \sqrt{\frac{\pi}{2}}.$$

Thus, the Rayleigh quotient can be computed using $p(x) = 1$, $q(x) = 0$, and the eigenfunctions. It is given by

$$\begin{aligned} R &= \frac{-\phi_n\phi_n'\Big|_0^\pi + \int_0^\pi (\phi_n')^2\, dx}{\frac{\pi}{2}} \\ &= \frac{2}{\pi}\int_0^\pi \left(-n^2\cos nx\right)^2 dx = n^2. \end{aligned} \tag{6.13}$$

Therefore, knowing the eigenfunction, the Rayleigh quotient returns the eigenvalues as expected.[2]

[2] Here the length, or norm, of ϕ is defined by
$$\|\phi\|^2 = \langle\phi,\phi\rangle.$$

Example 6.3. 6.9 We seek the eigenfunctions of the operator found in Example 6.1. Namely, we want to solve the eigenvalue problem

$$\mathcal{L}y = (xy')' + \frac{2}{x}y = -\lambda\sigma y \tag{6.14}$$

subject to a set of homogeneous boundary conditions. Let's use the boundary conditions

$$y'(1) = 0, \quad y'(2) = 0.$$

[Note that we do not know $\sigma(x)$ yet, but will choose an appropriate function to obtain solutions. Why didn't we give a condition at $x = 0$?]

Expanding the derivative, we have

$$xy'' + y' + \frac{2}{x}y = -\lambda\sigma y.$$

Multiply through by x to obtain

$$x^2y'' + xy' + (2 + \lambda x\sigma)\, y = 0.$$

Notice that if we choose $\sigma(x) = x^{-1}$, then this equation can be made a Cauchy-Euler type equation. [see Section **??**.] Thus, we have

$$x^2y'' + xy' + (\lambda + 2)\, y = 0.$$

The characteristic equation, found assuming solutions of the form $y(x) = x^r$, is

$$r^2 + \lambda + 2 = 0.$$

For oscillatory solutions, we need $\lambda + 2 > 0$. Thus, the general solution is

$$y(x) = c_1\cos(\sqrt{\lambda+2}\ln|x|) + c_2\sin(\sqrt{\lambda+2}\ln|x|). \tag{6.15}$$

See the Appendix for a review of Cauchy-Euler equations.

Next we apply the boundary conditions. $y'(1) = 0$ forces $c_2 = 0$. This leaves

$$y(x) = c_1 \cos(\sqrt{\lambda + 2} \ln x).$$

The second condition, $y'(2) = 0$, yields

$$\sin(\sqrt{\lambda + 2} \ln 2) = 0.$$

This will give nontrivial solutions when

$$\sqrt{\lambda + 2} \ln 2 = n\pi, \quad n = 0, 1, 2, 3 \ldots.$$

In summary, the eigenfunctions for this eigenvalue problem are

$$y_n(x) = \cos\left(\frac{n\pi}{\ln 2} \ln x\right), \quad 1 \leq x \leq 2$$

and the eigenvalues are $\lambda_n = \left(\frac{n\pi}{\ln 2}\right)^2 - 2$ for $n = 0, 1, 2, \ldots$.

Note: We include the $n = 0$ case because $y(x) =$ constant is a solution of the $\lambda = -2$ case. More specifically, in this case the characteristic equation reduces to $r^2 = 0$. Thus, the general solution of this Cauchy-Euler equation is

$$y(x) = c_1 + c_2 \ln |x|.$$

Setting $y'(1) = 0$, forces $c_2 = 0$. $y'(2)$ automatically vanishes, leaving the solution in this case as $y(x) = c_1$.

We note that some of the properties listed in the beginning of the section hold for this example. The eigenvalues are seen to be real, countable and ordered. There is a least one, $\lambda_0 = -2$. Next, one can find the zeros of each eigenfunction on [1,2]. Then the argument of the cosine, $\frac{n\pi}{\ln 2} \ln x$, takes values 0 to $n\pi$ for $x \in [1, 2]$. The cosine function has $n - 1$ roots on this interval.

Orthogonality can be checked as well. We set up the integral and use the substitution $y = \pi \ln x / \ln 2$. This gives the inner product 6.11 as

$$\begin{aligned} \langle y_n, y_m \rangle &= \int_1^2 \cos\left(\frac{n\pi}{\ln 2} \ln x\right) \cos\left(\frac{m\pi}{\ln 2} \ln x\right) \frac{dx}{x} \\ &= \frac{\ln 2}{\pi} \int_0^\pi \cos ny \cos my \, dy \\ &= \frac{\ln 2}{2} \delta_{n,m}. \end{aligned} \tag{6.16}$$

6.2.1 *Adjoint Operators*

In the study of the spectral theory of matrices, one learns about the adjoint of the matrix, $A^\dagger$, and the role that self-adjoint, or Hermitian, matrices play in diagonalization. Also, one needs the concept of adjoint to discuss the existence of solutions to the matrix problem $\mathbf{y} = A\mathbf{x}$. In the same

spirit, one is interested in the existence of solutions of the operator equation $Lu = f$ and solutions of the corresponding eigenvalue problem. The study of linear operators on a Hilbert space is a generalization of what the reader had seen in a linear algebra course.

Just as one can find a basis of eigenvectors and diagonalize Hermitian, or self-adjoint, matrices (or, real symmetric in the case of real matrices), we will see that the Sturm-Liouville operator is self-adjoint. In this section we will define the domain of an operator and introduce the notion of adjoint operators. In the last section we discuss the role the adjoint plays in the existence of solutions to the operator equation $Lu = f$.

Adjoint of an operator.

We begin by defining the adjoint of an operator. The adjoint, $L^\dagger$, of operator L satisfies

$$\langle u, Lv\rangle = \langle L^\dagger u, v\rangle$$

for all v in the domain of L and u in the domain of $L^\dagger$. Here the domain of a differential operator L is the set of all $u \in L^2_\sigma(a,b)$ satisfying a given set of homogeneous boundary conditions. This is best understood through example.

Example 6.4. Find the adjoint of $L = a_2(x)D^2 + a_1(x)D + a_0(x)$ for $D = d/dx$.

In order to find the adjoint, we place the operator inside an integral. Consider the inner product

$$\langle u, Lv\rangle = \int_a^b u(a_2v'' + a_1v' + a_0v)\,dx.$$

We have to move the operator L from v and determine what operator is acting on u in order to formally preserve the inner product. For a simple operator like $L = \frac{d}{dx}$, this is easily done using integration by parts. For the given operator, we will need to apply several integrations by parts to the individual terms. We consider each derivative term in the integrand separately.

For the a_1v' term, we integrate by parts to find

$$\int_a^b u(x)a_1(x)v'(x)\,dx = a_1(x)u(x)v(x)\Big|_a^b - \int_a^b (u(x)a_1(x))'v(x)\,dx. \quad (6.17)$$

Now, we consider the a_2v'' term. In this case it will take two integrations by parts:

$$\begin{aligned}\int_a^b u(x)a_2(x)v''(x)\,dx &= a_2(x)u(x)v'(x)\Big|_a^b - \int_a^b (u(x)a_2(x))'v(x)'\,dx \\ &= \left[a_2(x)u(x)v'(x) - (a_2(x)u(x))'v(x)\right]\Big|_a^b \\ &\quad + \int_a^b (u(x)a_2(x))''v(x)\,dx. \quad (6.18)\end{aligned}$$

Combining these results, we obtain

$$\langle u, Lv\rangle = \int_a^b u(a_2v'' + a_1v' + a_0v)\,dx$$

$$= \left[a_1(x)u(x)v(x) + a_2(x)u(x)v'(x) - (a_2(x)u(x))'v(x)\right]\Big|_a^b + \int_a^b \left[(a_2u)'' - (a_1u)' + a_0u\right] v\,dx. \quad (6.19)$$

Before inserting the boundary conditions for v, one has to determine boundary conditions for u such that

$$\left[a_1(x)u(x)v(x) + a_2(x)u(x)v'(x) - (a_2(x)u(x))'v(x)\right]\Big|_a^b = 0.$$

This leaves

$$\langle u, Lv\rangle = \int_a^b \left[(a_2u)'' - (a_1u)' + a_0u\right] v\,dx \equiv \langle L^\dagger u, v\rangle.$$

Therefore,

$$L^\dagger = \frac{d^2}{dx^2}a_2(x) - \frac{d}{dx}a_1(x) + a_0(x). \quad (6.20)$$

Self-adjoint operators.

When $L^\dagger = L$, the operator is called formally self-adjoint, or Hermitian. When the domain of L is the same as the domain of $L^\dagger$, the term self-adjoint is used. As the domain is important in establishing self-adjointness, we need to do a complete example in which the domain of the adjoint is found.

Example 6.5. Determine $L^\dagger$ and its domain for operator $Lu = \frac{du}{dx}$ where u satisfies the boundary conditions $u(0) = 2u(1)$ on $[0, 1]$.

We need to find the adjoint operator satisfying $\langle v, Lu\rangle = \langle L^\dagger v, u\rangle$. Therefore, we rewrite the integral

$$\langle v, Lu\rangle = \int_0^1 v\frac{du}{dx}\,dx = uv\big|_0^1 - \int_0^1 u\frac{dv}{dx}\,dx = \langle L^\dagger v, u\rangle.$$

From this we have the adjoint problem consisting of an adjoint operator and the associated boundary condition (or, domain of $L^\dagger$.):

1. $L^\dagger = -\frac{d}{dx}$.
2. $uv\Big|_0^1 = 0 \Rightarrow 0 = u(1)[v(1) - 2v(0)] \Rightarrow v(1) = 2v(0)$.

6.2.2 *Lagrange's and Green's Identities*

Before turning to the proofs that the eigenvalues of a Sturm-Liouville problem are real and the associated eigenfunctions orthogonal, we will first need to introduce two important identities. For the Sturm-Liouville operator,

$$\mathcal{L} = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q,$$

we have the two identities:

Lagrange's Identity:	$u\mathcal{L}v - v\mathcal{L}u$	$= [p(uv' - vu')]'$.
Green's Identity:	$\int_a^b (u\mathcal{L}v - v\mathcal{L}u)\,dx$	$= [p(uv' - vu')]\big\|_a^b$.

The proof of Lagrange's identity follows by a simple manipulations of the operator:

$$\begin{aligned} u\mathcal{L}v - v\mathcal{L}u &= u\left[\frac{d}{dx}\left(p\frac{dv}{dx}\right) + qv\right] - v\left[\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu\right] \\ &= u\frac{d}{dx}\left(p\frac{dv}{dx}\right) - v\frac{d}{dx}\left(p\frac{du}{dx}\right) \\ &= u\frac{d}{dx}\left(p\frac{dv}{dx}\right) + p\frac{du}{dx}\frac{dv}{dx} - v\frac{d}{dx}\left(p\frac{du}{dx}\right) - p\frac{du}{dx}\frac{dv}{dx} \\ &= \frac{d}{dx}\left[pu\frac{dv}{dx} - pv\frac{du}{dx}\right]. \end{aligned} \tag{6.21}$$

Green's identity is simply proven by integrating Lagrange's identity.

6.2.3 *Orthogonality and Reality*

We are now ready to prove that the eigenvalues of a Sturm-Liouville problem are real and the corresponding eigenfunctions are orthogonal. These are easily established using Green's identity, which in turn is a statement about the Sturm-Liouville operator being self-adjoint.

Example 6.6. The eigenvalues of the Sturm-Liouville problem (6.3) are real.

Let $\phi_n(x)$ be a solution of the eigenvalue problem associated with λ_n:

$$\mathcal{L}\phi_n = -\lambda_n \sigma \phi_n.$$

We want to show that the eigenvalues are real. Namely, we show that $\overline{\lambda}_n = \lambda_n$, where the bar means complex conjugate. So, we also consider the complex conjugate of this eigenvalue problem,

$$\mathcal{L}\overline{\phi}_n = -\overline{\lambda}_n \sigma \overline{\phi}_n.$$

Now, multiply the first equation by $\overline{\phi}_n$, the second equation by ϕ_n, and then subtract the results. We obtain

$$\overline{\phi}_n \mathcal{L}\phi_n - \phi_n \mathcal{L}\overline{\phi}_n = (\overline{\lambda}_n - \lambda_n)\sigma \phi_n \overline{\phi}_n.$$

Integrating both sides of this equation, we have

$$\int_a^b \left(\overline{\phi}_n \mathcal{L}\phi_n - \phi_n \mathcal{L}\overline{\phi}_n\right)\, dx = (\overline{\lambda}_n - \lambda_n)\int_a^b \sigma \phi_n \overline{\phi}_n \, dx.$$

We apply Green's identity to the left hand side to find

$$[p(\overline{\phi}_n \phi_n' - \phi_n \overline{\phi}_n')]|_a^b = (\overline{\lambda}_n - \lambda_n)\int_a^b \sigma \phi_n \overline{\phi}_n \, dx.$$

Using the homogeneous boundary conditions (6.4) for a self-adjoint operator, the left side vanishes. This leaves

$$0 = (\overline{\lambda}_n - \lambda_n)\int_a^b \sigma \left|\phi_n\right|^2 \, dx.$$

The integral is positive, so we must have $\overline{\lambda}_n = \lambda_n$. Therefore, the eigenvalues are real.

Example 6.7. The eigenfunctions corresponding to different eigenvalues of the Sturm-Liouville problem (6.3) are orthogonal.

This is proven similar to the last example. Let $\phi_n(x)$ be a solution of the eigenvalue problem associated with λ_n,

$$\mathcal{L}\phi_n = -\lambda_n \sigma \phi_n,$$

and let $\phi_m(x)$ be a solution of the eigenvalue problem associated with $\lambda_m \neq \lambda_n$,

$$\mathcal{L}\phi_m = -\lambda_m \sigma \phi_m,$$

Now, multiply the first equation by ϕ_m and the second equation by ϕ_n. Subtracting these results, we obtain

$$\phi_m \mathcal{L}\phi_n - \phi_n \mathcal{L}\phi_m = (\lambda_m - \lambda_n)\sigma \phi_n \phi_m$$

Integrating both sides of the equation, using Green's identity with homogeneous boundary conditions, we obtain

$$0 = (\lambda_m - \lambda_n) \int_a^b \sigma \phi_n \phi_m \, dx.$$

Since the eigenvalues are distinct, we can divide by $\lambda_m - \lambda_n$, leaving the desired result,

$$\int_a^b \sigma \phi_n \phi_m \, dx = 0.$$

Therefore, the eigenfunctions are orthogonal with respect to the weight function $\sigma(x)$.

6.2.4 *The Rayleigh Quotient*

The Rayleigh quotient is useful for getting estimates of eigenvalues and proving some of the other properties associated with Sturm-Liouville eigenvalue problems. The Rayleigh quotient is general and finds applications for both matrix eigenvalue problems as well as self-adjoint operators. For a Hermitian matrix M the Rayleigh quotient is given by

$$R(\mathbf{v}) = \frac{\langle \mathbf{v}, M\mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

One can show that the critical values of the Rayleigh quotient, as a function of $\mathbf{v}$, are the eigenvectors of M and the values of R at these critical values are the corresponding eigenvectors. In particular, minimizing $R(\mathbf{v})$ over the vector space will give the lowest eigenvalue. This leads to the Rayleigh-Ritz method for computing the lowest eigenvalues when the eigenvectors are not known.

This definition can easily be extended to Sturm-Liouville operators,

$$R(\phi_n) = \frac{\langle \phi_n \mathcal{L}\phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

We begin by multiplying the eigenvalue problem,

$$\mathcal{L}\phi_n = -\lambda_n \sigma(x)\phi_n,$$

by ϕ_n and integrating. This gives

$$\int_a^b \left[\phi_n \frac{d}{dx}\left(p\frac{d\phi_n}{dx}\right) + q\phi_n^2\right]\,dx = -\lambda_n \int_a^b \phi_n^2 \sigma\,dx.$$

One can solve the last equation for λ to find

$$\lambda_n = \frac{-\int_a^b \left[\phi_n \frac{d}{dx}\left(p\frac{d\phi_n}{dx}\right) + q\phi_n^2\right]\,dx}{\int_a^b \phi_n^2 \sigma\,dx} = R(\phi_n).$$

It appears that we have solved for the eigenvalues and have not needed the machinery used in studying boundary value problems. However, we really cannot evaluate this expression when we do not know the eigenfunctions, $\phi_n(x)$. Nevertheless, we will see what we can determine from the Rayleigh quotient.

One can rewrite this result by performing an integration by parts on the first term in the numerator. Namely, pick $u = \phi_n$ and $dv = \frac{d}{dx}\left(p\frac{d\phi_n}{dx}\right)\,dx$ for the standard integration by parts formula. Then, we have

$$\int_a^b \phi_n \frac{d}{dx}\left(p\frac{d\phi_n}{dx}\right)\,dx = p\phi_n\frac{d\phi_n}{dx}\Big|_a^b - \int_a^b \left[p\left(\frac{d\phi_n}{dx}\right)^2 - q\phi_n^2\right]\,dx.$$

Inserting the new formula into the expression for λ, leads to the Rayleigh Quotient

$$\lambda_n = \frac{-p\phi_n\frac{d\phi_n}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{d\phi_n}{dx}\right)^2 - q\phi_n^2\right]\,dx}{\int_a^b \phi_n^2\sigma\,dx}. \tag{6.22}$$

In many applications the sign of the eigenvalue is important. As we had seen in the solution of the heat equation, $T' + k\lambda T = 0$. Since we expect the heat energy to diffuse, the solutions should decay in time. Thus, we would expect $\lambda > 0$. In studying the wave equation, one expects vibrations and these are only possible with the correct sign of the eigenvalue (positive again). Thus, in order to have nonnegative eigenvalues, we see from (6.22) that

a. $q(x) \leq 0$, and

b. $-p\phi_n\frac{d\phi_n}{dx}\Big|_a^b \geq 0.$

Furthermore, if λ is a zero eigenvalue, then $q(x) \equiv 0$ and $\alpha_1 = \alpha_2 = 0$ in the homogeneous boundary conditions. This can be seen by setting the numerator equal to zero. Then, $q(x) = 0$ and $\phi_n'(x) = 0$. The second of these conditions inserted into the boundary conditions forces the restriction on the type of boundary conditions.

One of the properties of Sturm-Liouville eigenvalue problems with homogeneous boundary conditions is that the eigenvalues are ordered, $\lambda_1 <$

$\lambda_2 < \ldots$. Thus, there is a smallest eigenvalue. It turns out that for any continuous function, $y(x)$,

$$\lambda_1 = \min_{y(x)} \frac{-py\frac{dy}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{dy}{dx}\right)^2 - qy^2\right] dx}{\int_a^b y^2\sigma\, dx} \tag{6.23}$$

and this minimum is obtained when $y(x) = \phi_1(x)$. This result can be used to get estimates of the minimum eigenvalue by using trial functions which are continuous and satisfy the boundary conditions, but do not necessarily satisfy the differential equation.

Example 6.8. We have already solved the eigenvalue problem $\phi'' + \lambda\phi = 0$, $\phi(0) = 0$, $\phi(1) = 0$. In this case, the lowest eigenvalue is $\lambda_1 = \pi^2$. We can pick a nice function satisfying the boundary conditions, say $y(x) = x - x^2$. Inserting this into Equation (6.23), we find

$$\lambda_1 \leq \frac{\int_0^1 (1-2x)^2\, dx}{\int_0^1 (x-x^2)^2\, dx} = 10.$$

Indeed, $10 \geq \pi^2$.

The only condition we placed on our trial function was that it satisfied the boundary conditions. Thus, this procedure would work with other functions obtaining bounds on the eigenvalue. With a good choice, one might get a better estimate of the true value.

6.3 The Eigenfunction Expansion Method

In this section we solve the nonhomogeneous problem $\mathcal{L}y = f$ using expansions over the basis of Sturm-Liouville eigenfunctions. We have seen that Sturm-Liouville eigenvalue problems have the requisite set of orthogonal eigenfunctions. In this section we will apply the eigenfunction expansion method to solve a particular nonhomogeneous boundary value problem.

Recall that one starts with a nonhomogeneous differential equation

$$\mathcal{L}y = f,$$

where $y(x)$ is to satisfy given homogeneous boundary conditions. The method makes use of the eigenfunctions satisfying the eigenvalue problem

$$\mathcal{L}\phi_n = -\lambda_n\sigma\phi_n$$

subject to the given boundary conditions. Then, one assumes that $y(x)$ can be written as an expansion in the eigenfunctions,

$$y(x) = \sum_{n=1}^{\infty} c_n\phi_n(x),$$

and inserts the expansion into the nonhomogeneous equation. This gives

$$f(x) = \mathcal{L}\left(\sum_{n=1}^{\infty} c_n \phi_n(x)\right) = -\sum_{n=1}^{\infty} c_n \lambda_n \sigma(x) \phi_n(x).$$

The expansion coefficients are then found by making use of the orthogonality of the eigenfunctions. Namely, we multiply the last equation by $\phi_m(x)$ and integrate. We obtain

$$\int_a^b f(x)\phi_m(x)\,dx = -\sum_{n=1}^{\infty} c_n \lambda_n \int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx.$$

Orthogonality yields

$$\int_a^b f(x)\phi_m(x)\,dx = -c_m \lambda_m \int_a^b \phi_m^2(x)\sigma(x)\,dx.$$

Solving for c_m, we have

$$c_m = -\frac{\int_a^b f(x)\phi_m(x)\,dx}{\lambda_m \int_a^b \phi_m^2(x)\sigma(x)\,dx}.$$

Example 6.9. As an example, we consider the solution of the boundary value problem

$$(xy')' + \frac{y}{x} = \frac{1}{x}, \quad x \in [1, e], \tag{6.24}$$

$$y(1) = 0 = y(e). \tag{6.25}$$

This equation is already in self-adjoint form. So, we know that the associated Sturm-Liouville eigenvalue problem has an orthogonal set of eigenfunctions. We first determine this set. Namely, we need to solve the eigenvalue problem

$$(x\phi')' + \frac{\phi}{x} = -\lambda\sigma\phi, \quad \phi(1) = 0 = \phi(e). \tag{6.26}$$

Rearranging the terms and multiplying by x, we have that

$$x^2\phi'' + x\phi' + (1 + \lambda\sigma x)\phi = 0.$$

This is almost an equation of Cauchy-Euler type similar to Example . Picking the weight function $\sigma(x) = \frac{1}{x}$, we have

$$x^2\phi'' + x\phi' + (1 + \lambda)\phi = 0.$$

This is easily solved. From Section **??**, the characteristic equation is

$$r^2 + (1 + \lambda) = 0.$$

One obtains nontrivial solutions [3] of the eigenvalue problem satisfying the boundary conditions when $\lambda > -1$. The solutions are

$$\phi_n(x) = A \sin(n\pi \ln x), \quad n = 1, 2, \ldots.$$

[3] The general solution is

$$\phi(x) = A \sin(\beta \ln x) + A \cos(\beta \ln x),$$

where $\beta = \sqrt{(1+\lambda)}$. The boundary conditions given that $B = 0$ and $\sin\beta = 0$. So, $\beta = n\pi$, $n = 1, 2, \ldots$.

where $\lambda_n = n^2\pi^2 - 1$.

It is often useful to normalize the eigenfunctions. This means that one chooses A so that the norm of each eigenfunction is one. Thus, we have

$$\begin{aligned} 1 &= \int_1^e \phi_n(x)^2 \sigma(x)\, dx \\ &= A^2 \int_1^e \sin(n\pi \ln x)\frac{1}{x}\, dx \\ &= A^2 \int_0^1 \sin(n\pi y)\, dy = \frac{1}{2}A^2. \end{aligned} \tag{6.27}$$

Thus, $A = \sqrt{2}$. Several of these eigenfunctions are show in Figure 6.2.

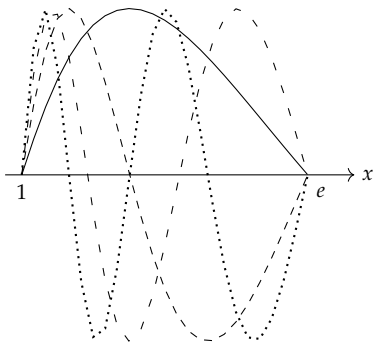


Figure 6.2: Plots of the first five eigenfunctions, $y(x) = \sqrt{2}\sin(n\pi \ln x)$.

We now turn towards solving the nonhomogeneous problem, $\mathcal{L}y = \frac{1}{x}$. We first expand the unknown solution in terms of the normalized eigenfunctions,

$$y(x) = \sum_{n=1}^{\infty} c_n \sqrt{2}\sin(n\pi \ln x).$$

Inserting this solution into the differential equation, we have

$$\frac{1}{x} = \mathcal{L}y = -\sum_{n=1}^{\infty} c_n \lambda_n \sqrt{2}\sin(n\pi \ln x)\frac{1}{x}.$$

Next, we make use of orthogonality. Multiplying both sides by the eigenfunction $\phi_m(x) = \sqrt{2}\sin(m\pi \ln x)$ and integrating, gives

$$\lambda_m c_m = \int_1^e \sqrt{2}\sin(m\pi \ln x)\frac{1}{x}\, dx = \frac{\sqrt{2}}{m\pi}[(-1)^m - 1].$$

Solving for c_m, we have

$$c_m = \frac{\sqrt{2}}{m\pi}\frac{[(-1)^m - 1]}{m^2\pi^2 - 1}.$$

Finally, we insert these coefficients into the expansion for $y(x)$. The solution is then

$$y(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\frac{[(-1)^n - 1]}{n^2\pi^2 - 1}\sin(n\pi \ln(x)).$$

We plot this solution in Figure 6.3. In Problem 14 you can directly solve this problem and compare your solution with the solution in Figure 6.3.

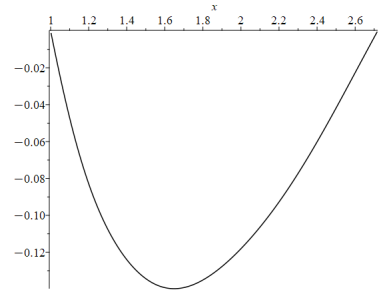


Figure 6.3: Plot of the solution in Example 6.9.

6.4 *Appendix: The Fredholm Alternative Theorem - Optional*

GIVEN THAT $Ly = f$, WHEN CAN ONE EXPECT to find a solution? Is it unique? These questions are answered by the Fredholm Alternative Theorem. This theorem occurs in many forms from a statement about solutions

to systems of algebraic equations to solutions of boundary value problems and integral equations. The theorem comes in two parts, thus the term "alternative". Either the equation has exactly one solution for all f, or the equation has many solutions for some f's and none for the rest.

The reader is familiar with the statements of the Fredholm Alternative for the solution of systems of algebraic equations. One seeks solutions of the system $Ax = b$ for A an $n \times m$ matrix. Defining the matrix adjoint, A^* through $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x, y, \in \mathcal{C}^n$, then either

Theorem 6.1. First Alternative

The equation $Ax = b$ has a solution if and only if $\langle b, v\rangle = 0$ for all v satisfying $A^*v = 0$.

or

Theorem 6.2. Second Alternative

A solution of $Ax = b$, if it exists, is unique if and only if $x = 0$ is the only solution of $Ax = 0$.

The second alternative is more familiar when given in the form: The solution of a nonhomogeneous system of n equations and n unknowns is unique if the only solution to the homogeneous problem is the zero solution. Or, equivalently, A is invertible, or has nonzero determinant.

Proof. We prove the second theorem first. Assume that $Ax = 0$ for $x \neq 0$ and $Ax_0 = b$. Then $A(x_0 + \alpha x) = b$ for all α. Therefore, the solution is not unique. Conversely, if there are two different solutions, x_1 and x_2, satisfying $Ax_1 = b$ and $Ax_2 = b$, then one has a nonzero solution $x = x_1 - x_2$ such that $Ax = A(x_1 - x_2) = 0$.

The proof of the first part of the first theorem is simple. Let $A^*v = 0$ and $Ax_0 = b$. Then we have

$$\langle b, v\rangle = \langle Ax_0, v\rangle = \langle x_0, A^*v\rangle = 0.$$

For the second part we assume that $\langle b, v\rangle = 0$ for all v such that $A^*v = 0$. Write b as the sum of a part that is in the range of A and a part that in the space orthogonal to the range of A, $b = b_R + b_O$. Then, $0 = \langle b_O, Ax\rangle = \langle A^*b, x\rangle$ for all x. Thus, $A^*b_O = 0$. Since $\langle b, v\rangle = 0$ for all v in the nullspace of A^*, then $\langle b, b_O\rangle = 0$.

Therefore, $\langle b, v\rangle = 0$ implies that

$$0 = \langle b, b_O\rangle = \langle b_R + b_O, b_O\rangle = \langle b_O, b_O\rangle.$$

This means that $b_O = 0$, giving $b = b_R$ is in the range of A. So, $Ax = b$ has a solution. □

Example 6.10. Determine the allowed forms of $\mathbf{b}$ for a solution of $A\mathbf{x} = \mathbf{b}$ to exist, where

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}.$$

First note that $A^* = \overline{A}^T$. This is seen by looking at

$$\begin{aligned} \langle A\mathbf{x}, \mathbf{y} \rangle &= \langle \mathbf{x}, A^*\mathbf{y} \rangle \\ \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_j \bar{y}_i &= \sum_{j=1}^{n} x_j \sum_{j=1}^{n} a_{ij} \bar{y}_i \\ &= \sum_{j=1}^{n} x_j \overline{\sum_{j=1}^{n} (\bar{a}^T)_{ji}\, y_i}. \end{aligned} \tag{6.28}$$

For this example,

$$A^* = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}.$$

We next solve $A^*\mathbf{v} = 0$. This means, $v_1 + 3v_2 = 0$. So, the nullspace of A^* is spanned by $\mathbf{v} = (3, -1)^T$. For a solution of $A\mathbf{x} = \mathbf{b}$ to exist, $\mathbf{b}$ would have to be orthogonal to $\mathbf{v}$. Therefore, a solution exists when

$$\mathbf{b} = \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

So, what does the Fredholm Alternative say about solutions of boundary value problems? We extend the Fredholm Alternative for linear operators. A more general statement would be

Theorem 6.3. *If L is a bounded linear operator on a Hilbert space, then $Ly = f$ has a solution if and only if $\langle f, v \rangle = 0$ for every v such that $L^\dagger v = 0$.*

The statement for boundary value problems is similar. However, we need to be careful to treat the boundary conditions in our statement. As we have seen, after several integrations by parts we have that

$$\langle \mathcal{L}u, v \rangle = S(u, v) + \langle u, \mathcal{L}^\dagger v \rangle,$$

where $S(u, v)$ involves the boundary conditions on u and v. Note that for nonhomogeneous boundary conditions, this term may no longer vanish.

Theorem 6.4. *The solution of the boundary value problem $\mathcal{L}u = f$ with boundary conditions $Bu = g$ exists if and only if*

$$\langle f, v \rangle - S(u, v) = 0$$

for all v satisfying $\mathcal{L}^\dagger v = 0$ and $B^\dagger v = 0$.

Example 6.11. Consider the problem

$$u'' + u = f(x), \quad u(0) - u(2\pi) = \alpha, u'(0) - u'(2\pi) = \beta.$$

Only certain values of α and β will lead to solutions. We first note that

$$L = L^\dagger = \frac{d^2}{dx^2} + 1.$$

Solutions of

$$L^\dagger v = 0, \quad v(0) - v(2\pi) = 0, v'(0) - v'(2\pi) = 0$$

are easily found to be linear combinations of $v = \sin x$ and $v = \cos x$.

Next, one computes

$$\begin{aligned} S(u,v) &= \left[u'v - uv'\right]_0^{2\pi} \\ &= u'(2\pi)v(2\pi) - u(2\pi)v'(2\pi) - u'(0)v(0) + u(0)v'(0). \end{aligned} \tag{6.29}$$

For $v(x) = \sin x$, this yields

$$S(u, \sin x) = -u(2\pi) + u(0) = \alpha.$$

Similarly,

$$S(u, \cos x) = \beta.$$

Using $\langle f, v\rangle - S(u,v) = 0$, this leads to the conditions that we were seeking,

$$\int_0^{2\pi} f(x) \sin x \, dx = \alpha,$$

$$\int_0^{2\pi} f(x) \cos x \, dx = \beta.$$

Problems

1. Prove the if $u(x)$ and $v(x)$ satisfy the general homogeneous boundary conditions

$$\begin{aligned} \alpha_1 u(a) + \beta_1 u'(a) &= 0, \\ \alpha_2 u(b) + \beta_2 u'(b) &= 0 \end{aligned} \tag{6.30}$$

at $x = a$ and $x = b$, then

$$p(x)[u(x)v'(x) - v(x)u'(x)]_{x=a}^{x=b} = 0.$$

2. Prove Green's Identity $\int_a^b (u\mathcal{L}v - v\mathcal{L}u)\, dx = [p(uv' - vu')]\Big|_a^b$ for the general Sturm-Liouville operator $\mathcal{L}$.

3. Find the adjoint operator and its domain for $Lu = u'' + 4u' - 3u$, $u'(0) + 4u(0) = 0$, $u'(1) + 4u(1) = 0$.

4. Show that a Sturm-Liouville operator with periodic boundary conditions on $[a,b]$ is self-adjoint if and only if $p(a) = p(b)$. [Recall, periodic boundary conditions are given as $u(a) = u(b)$ and $u'(a) = u'(b)$.]

5. Rewrite the following in Sturm-Liouville form:

a. $y'' + 2y' = -\lambda y$.

b. $x^2y'' + xy' = -\lambda y$.

c. $(1 - x^2)y'' - xy' = -\lambda y, \quad |x| < 1$.

d. $xy'' + (1 - x)y' = -\lambda y, \quad x > 0$.

6. Consider the differential equation given by

$$x^2y'' + xy' + (\lambda x^2 - n^2)y = 0.$$

Rewrite this equation in self-adjoint form. Give the integral form for the orthogonality of the eigenfunctions.

7. The Hermite differential equation is given by $y'' - 2xy' + \lambda y = 0$. Rewrite this equation in self-adjoint form. From the Sturm-Liouville form obtained, verify that the differential operator is self adjoint on $(-\infty, \infty)$. Give the integral form for the orthogonality of the eigenfunctions.

8. Find the eigenvalues and eigenfunctions of the given Sturm-Liouville problems.

a. $y'' + \lambda y = 0$, $y'(0) = 0 = y'(\pi)$.

b. $y'' + \lambda y = 0$, $hy(0) - y'(0) = 0, y'(1) = 0, \quad h > 0$.

c. $y'' + 2y' + y + \lambda y = 0$, $y(0) = 0 = y(1)$.

d. $(xy')' + \frac{\lambda}{x}y = 0$, $y(1) = y(e^2) = 0$.

e. $x^2y'' + xy' = -\lambda y, \quad 1 < x < e^\pi, y(1) = 0, y(e^\pi) = 0$.

f. $(xy')' + \frac{\lambda}{x}y = 0$, $y(1) = y(e^2) = 0$.

g. $y'' + \lambda y = 0$, $2y(0) - y'(0) = 0$, $y'(1) = 0$. In this problem you will need a computer to find the first two eigenvalues.

9. Consider the Sturm-Liouville problem $y'' + \lambda y = 0$, $y(0) + y'(0) = 0$, $y(1) + 2y'(1) = 0$.

a. Can $\lambda = 0$?

b. Show that for $\lambda < 0$ there is only one solution. Determine this solution numerically to three decimal places.

c. Show that for $\lambda > 0$ there are an infinite number of eigenvalues. Numerically find the first three to three decimal places.

10. The eigenvalue problem $x^2y'' - \lambda xy' + \lambda y = 0$ with $y(1) = y(2) = 0$ is not a Sturm-Liouville eigenvalue problem. Show that none of the eigenvalues are real by solving this eigenvalue problem.

11. Consider the eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = y'(0), \quad y(1) = \alpha y'(1).$$

For what value of α is $\lambda = 0$ an eigenvalue? What is the corresponding eigenfunction?

12. In Example 6.8 we found a bound on the lowest eigenvalue for the given eigenvalue problem.

a. Verify the computation in the example.

b. Apply the method using

$$y(x) = \begin{cases} x, & 0 < x < \frac{1}{2} \\ 1 - x, & \frac{1}{2} < x < 1. \end{cases}$$

Is this an upper bound on λ_1

c. Use the Rayleigh quotient to obtain a good upper bound for the lowest eigenvalue of the eigenvalue problem: $\phi'' + (\lambda - x^2)\phi = 0$, $\phi(0) = 0$, $\phi'(1) = 0$.

13. Consider the eigenvalue problem, $y''(x) + (\lambda - x^2)y(x) = 0$, $y'(0) = 0$, $y(1) = 0$. Using the Rayleigh quotient, find a bound on the first eigenvalue.

14. One can rewrite the problem in Example 6.9 as

$$x^2y''(x) + xy'(x) + y(x) = 1, \quad y(1) = y(e) = 0.$$

Find the closed form solution and plot the solutions with the eigenfunction expansion in Example 6.9 to verify that the solutions are the same.

15. Use the method of eigenfunction expansions to solve the problems:

a. $y'' = x^2, \quad y(0) = y(1) = 0.$

b. $y'' + 4y = x^2, \quad y'(0) = y'(1) = 0.$

Note that these problems can also be solved directly. Find the solutions and plot these solutions with the eigenfunction expansions to verify that the solutions are the same.

16. Determine the solvability conditions for the nonhomogeneous boundary value problem: $u'' + 4u = f(x)$, $u(0) = \alpha$, $u'(\pi/4) = \beta$.

Chapter 7

Special Functions

"You have no idea, how much poetry there is in the calculation of a table of logarithms!" - Carl Friedrich Gauss (1777-1855)

In this chapter we will look at some additional functions which arise often in physical applications and are eigenfunctions for some Sturm-Liouville boundary value problem. We begin with a collection of special functions, called the classical orthogonal polynomials. These include such polynomial functions as the Legendre polynomials, the Hermite polynomials, the Tchebycheff[1] and the Gegenbauer polynomials. Also, Bessel functions occur quite often. We will spend more time exploring the Legendre and Bessel functions. These functions are typically found as solutions of differential equations using power series methods in a first course in differential equations.

[1] In the 1955 letter Tchebycheff or Chebyshev? one finds an interesting discussion on the correct spelling of Pafnuty Lvovich Chebyshev's (1821-1894) last name. Some of the variants have been Tchebichef, Tschebyscheff, Tshebyscheff, Chebyshev, Chebychev, Chebyscheff, Chebyshov Tchebycheff, Tchebysheff, Tchebychev, Tchebycheff, Tschebyschev or Tschebyscheff. In Russian, the name is Пафну́тий Льво́вич Чебышёв. According to the Soviet Embassy letter, Tchebycheff is the French transliteration and Chebyshev is the English transliteration. In that 1955 letter, a writer notes that the AIP at the time had adopted the French version. Currently, it seems the French have accepted Tchebychev and the AMS uses Chebyshev. The correct transliteration may be Čebyšëv.

7.1 Classical Orthogonal Polynomials

There are other basis functions that can be used to develop series representations of functions. In this section we introduce the classical orthogonal polynomials. We begin by noting that the sequence of functions $\{1, x, x^2, \ldots\}$ is a basis of linearly independent functions. In fact, by the Stone-Weierstraß Approximation Theorem[2] this set is a basis of $L^2_\sigma(a, b)$, the space of square integrable functions over the interval $[a, b]$ relative to weight $\sigma(x)$. However, we will show that the sequence of functions $\{1, x, x^2, \ldots\}$ does not provide an orthogonal basis for these spaces. We will then proceed to find an appropriate orthogonal basis of functions.

[2] **Stone-Weierstraß Approximation Theorem** Suppose f is a continuous function defined on the interval $[a, b]$. For every $\epsilon > 0$, there exists a polynomial function $P(x)$ such that for all $x \in [a, b]$, we have $|f(x) - P(x)| < \epsilon$. Therefore, every continuous function defined on $[a, b]$ can be uniformly approximated as closely as we wish by a polynomial function.

We are familiar with being able to expand functions over a basis such as $\{1, x, x^2, \ldots\}$, since these expansions are just Maclaurin series representations of the functions about $x = 0$,

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^n.$$

However, this basis is not an orthogonal basis of functions. One can easily see this by integrating the product of two even, or two odd, basis functions

with $\sigma(x) = 1$ and (a, b)=$(-1, 1)$. For example,

$$\int_{-1}^{1} x^0 x^2 \, dx = \frac{2}{3}.$$

The Gram-Schmidt Orthogonalization Process.

Since we have found that orthogonal bases have been useful in determining the coefficients for expansions of given functions, we might ask, "Given a set of linearly independent basis vectors, can one find an orthogonal basis of the given space?" The answer is yes. We recall from introductory linear algebra, which mostly covers finite dimensional vector spaces, that there is a method for carrying this out called the Gram-Schmidt Orthogonalization Process. We will review this process for finite dimensional vectors and then generalize to function spaces.

Let's assume that we have three vectors that span the usual three dimensional space, $\mathbf{R}^3$, given by $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ and shown in Figure 7.1. We seek an orthogonal basis $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, beginning one vector at a time.

First we take one of the original basis vectors, say $\mathbf{a}_1$, and define

$$\mathbf{e}_1 = \mathbf{a}_1.$$

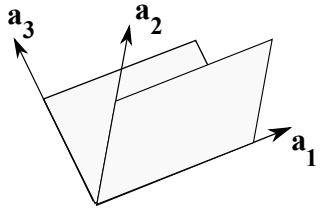


Figure 7.1: The basis $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, of $\mathbf{R}^3$.

It is sometimes useful to normalize these basis vectors, denoting such a normalized vector with a "hat":

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_1}{e_1},$$

where $e_1 = \sqrt{\mathbf{e}_1 \cdot \mathbf{e}_1}$.

Next, we want to determine an $\mathbf{e}_2$ that is orthogonal to $\mathbf{e}_1$. We take another element of the original basis, $\mathbf{a}_2$. In Figure 7.2 we show the orientation of the vectors. Note that the desired orthogonal vector is $\mathbf{e}_2$. We can now write $\mathbf{a}_2$ as the sum of $\mathbf{e}_2$ and the projection of $\mathbf{a}_2$ on $\mathbf{e}_1$. Denoting this projection by $\mathbf{pr}_1\mathbf{a}_2$, we then have

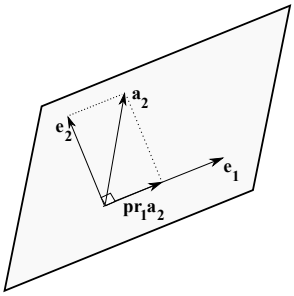


Figure 7.2: A plot of the vectors $\mathbf{e}_1$, $\mathbf{a}_2$, and $\mathbf{e}_2$ needed to find the projection of $\mathbf{a}_2$, on $\mathbf{e}_1$.

$$\mathbf{e}_2 = \mathbf{a}_2 - \mathbf{pr}_1\mathbf{a}_2. \tag{7.1}$$

Recall the projection of one vector onto another from your vector calculus class.

$$\mathbf{pr}_1\mathbf{a}_2 = \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1. \tag{7.2}$$

This is easily proven by writing the projection as a vector of length $a_2 \cos\theta$ in direction $\hat{\mathbf{e}}_1$, where θ is the angle between $\mathbf{e}_1$ and $\mathbf{a}_2$. Using the definition of the dot product, $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$, the projection formula follows.

Combining Equations (7.1)-(7.2), we find that

$$\mathbf{e}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1. \tag{7.3}$$

It is a simple matter to verify that $\mathbf{e}_2$ is orthogonal to $\mathbf{e}_1$:

$$\begin{aligned} \mathbf{e}_2 \cdot \mathbf{e}_1 &= \mathbf{a}_2 \cdot \mathbf{e}_1 - \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1 \cdot \mathbf{e}_1 \\ &= \mathbf{a}_2 \cdot \mathbf{e}_1 - \mathbf{a}_2 \cdot \mathbf{e}_1 = 0. \end{aligned} \tag{7.4}$$

Next, we seek a third vector $\mathbf{e}_3$ that is orthogonal to both $\mathbf{e}_1$ and $\mathbf{e}_2$. Pictorially, we can write the given vector $\mathbf{a}_3$ as a combination of vector projections along $\mathbf{e}_1$ and $\mathbf{e}_2$ with the new vector. This is shown in Figure 7.3. Thus, we can see that

$$\mathbf{e}_3 = \mathbf{a}_3 - \frac{\mathbf{a}_3 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1 - \frac{\mathbf{a}_3 \cdot \mathbf{e}_2}{e_2^2}\mathbf{e}_2. \tag{7.5}$$

Again, it is a simple matter to compute the scalar products with $\mathbf{e}_1$ and $\mathbf{e}_2$ to verify orthogonality.

The Gram-Schmidt Orthogonalization for a vector basis.

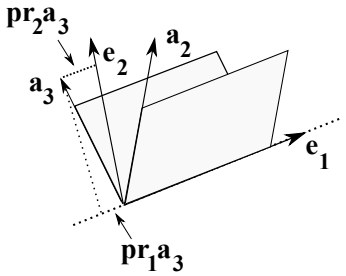


Figure 7.3: A plot of vectors for determining $\mathbf{e}_3$.

We can easily generalize this procedure to the N-dimensional case. Let $\mathbf{a}_n$, $n = 1, ..., N$ be a set of linearly independent vectors in $\mathbf{R}^N$. Then, an orthogonal basis can be found by setting $\mathbf{e}_1 = \mathbf{a}_1$ and defining

$$\mathbf{e}_n = \mathbf{a}_n - \sum_{j=1}^{n-1} \frac{\mathbf{a}_n \cdot \mathbf{e}_j}{e_j^2}\mathbf{e}_j, \quad n = 2, 3, \ldots, N. \tag{7.6}$$

Now, we can generalize this idea to (real) function spaces. Let $f_n(x)$, $n \in N_0 = \{0, 1, 2, \ldots\}$, be a linearly independent sequence of continuous functions defined for $x \in [a, b]$. Then, an orthogonal basis of functions, $\phi_n(x)$, $n \in N_0$ can be found and is given by

$$\phi_0(x) = f_0(x)$$

and

The Gram-Schmidt Orthogonalization for a basis of functions

$$\phi_n(x) = f_n(x) - \sum_{j=0}^{n-1} \frac{\langle f_n, \phi_j \rangle}{\|\phi_j\|^2}\phi_j(x), \quad n = 1, 2, \ldots. \tag{7.7}$$

Here we are using inner products relative to weight $\sigma(x)$,

$$\langle f, g \rangle = \int_a^b f(x)g(x)\sigma(x)\,dx. \tag{7.8}$$

Note the similarity between the orthogonal basis in (7.7) and the expression for the finite dimensional case in Equation (7.6).

Example 7.1. Apply the Gram-Schmidt Orthogonalization process to the set $f_n(x) = x^n$, $n \in N_0$, when $x \in (-1, 1)$ and $\sigma(x) = 1$.

First, we have $\phi_0(x) = f_0(x) = 1$. Note that

$$\int_{-1}^{1} \phi_0^2(x)\,dx = 2.$$

We could use this result to fix the normalization of the new basis, but we will hold off doing that for now.

Now, we compute the second basis element:

$$\begin{aligned} \phi_1(x) &= f_1(x) - \frac{\langle f_1, \phi_0 \rangle}{\|\phi_0\|^2}\phi_0(x) \\ &= x - \frac{\langle x, 1 \rangle}{\|1\|^2}1 = x, \end{aligned} \tag{7.9}$$

since $\langle x, 1 \rangle$ is the integral of an odd function over a symmetric interval.

For $\phi_2(x)$, we have

$$\begin{aligned}\phi_2(x) &= f_2(x) - \frac{\langle f_2, \phi_0\rangle}{\|\phi_0\|^2}\,\phi_0(x) - \frac{\langle f_2, \phi_1\rangle}{\|\phi_1\|^2}\,\phi_1(x)\\ &= x^2 - \frac{\langle x^2, 1\rangle}{\|1\|^2}\,1 - \frac{\langle x^2, x\rangle}{\|x\|^2}\,x\\ &= x^2 - \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} dx}\\ &= x^2 - \frac{1}{3}. \end{aligned} \tag{7.10}$$

So far, we have the orthogonal set $\{1, x, x^2 - \frac{1}{3}\}$. If one chooses to normalize these by forcing $\phi_n(1) = 1$, then one obtains the classical Legendre polynomials, $P_n(x)$. Thus,

$$P_2(x) = \frac{1}{2}(3x^2 - 1).$$

Note that this normalization is different than the usual one. In fact, we see the $P_2(x)$ does not have a unit norm,

$$\|P_2\|^2 = \int_{-1}^{1} P_2^2(x)\,dx = \frac{2}{5}.$$

[3] Adrien-Marie Legendre (1752-1833) was a French mathematician who made many contributions to analysis and algebra.

The set of Legendre[3] polynomials is just one set of classical orthogonal polynomials that can be obtained in this way. Many of these special functions had originally appeared as solutions of important boundary value problems in physics. They all have similar properties and we will just elaborate some of these for the Legendre functions in the next section. Others in this group are shown in Table 7.1.

Table 7.1: Common classical orthogonal polynomials with the interval and weight function used to define them.

Polynomial	Symbol	Interval	$\sigma(x)$
Hermite	$H_n(x)$	$(-\infty, \infty)$	e^{-x^2}
Laguerre	$L_n^{\alpha}(x)$	$[0, \infty)$	e^{-x}
Legendre	$P_n(x)$	(-1,1)	1
Gegenbauer	$C_n^{\lambda}(x)$	(-1,1)	$(1-x^2)^{\lambda-1/2}$
Tchebycheff of the 1st kind	$T_n(x)$	(-1,1)	$(1-x^2)^{-1/2}$
Tchebycheff of the 2nd kind	$U_n(x)$	(-1,1)	$(1-x^2)^{-1/2}$
Jacobi	$P_n^{(\nu,\mu)}(x)$	(-1,1)	$(1-x)^{\nu}(1-x)^{\mu}$

For reference, we also note the differential equations satisfied by these functions as shown in Table 7.2.

7.2 Fourier-Legendre Series

In an earlier chapter we saw how useful Fourier series expansions were for solving the heat equation. When studying partial differential equa-

Polynomial	Differential Equation
Hermite	$y'' - 2xy' + 2ny = 0$
Laguerre	$xy'' + (\alpha + 1 - x)y' + ny = 0$
Legendre	$(1-x^2)y'' - 2xy' + n(n+1)y = 0$
Gegenbauer	$(1-x^2)y'' - (2n+3)xy' + \lambda y = 0$
Tchebycheff of the 1st kind	$(1-x^2)y'' - xy' + n^2 y = 0$
Jacobi	$(1-x^2)y'' + (\nu - \mu + (\mu + \nu + 2)x)y'$
	$+n(n+1+\mu+\nu)y = 0$

Table 7.2: Differential equations satisfied by some of the common classical orthogonal polynomials.

tions in higher dimensions, one finds that problems with spherical symmetry may lead to the series representations in terms of a basis of Legendre polynomials. For example, we could consider the steady state temperature distribution inside a hemispherical igloo, which takes the form

$$\phi(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

in spherical coordinates. Evaluating this function at the surface $r = a$ as $\phi(a,\theta) = f(\theta)$, leads to a Fourier-Legendre series expansion of function f :

$$f(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta),$$

where $c_n = A_n a^n$

In this section we would like to explore Fourier-Legendre series expansions of functions $f(x)$ defined on $(-1,1)$:

$$f(x) \sim \sum_{n=0}^{\infty} c_n P_n(x). \tag{7.11}$$

As with Fourier trigonometric series, we can determine the expansion coefficients by multiplying both sides of Equation (7.11) by $P_m(x)$ and integrating for $x \in [-1,1]$. Orthogonality gives the usual form for the generalized Fourier coefficients,

$$c_n = \frac{\langle f, P_n \rangle}{\|P_n\|^2}, n = 0, 1, \ldots.$$

We will later show that

$$\|P_n\|^2 = \frac{2}{2n+1}.$$

Therefore, the Fourier-Legendre coefficients are

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx. \tag{7.12}$$

7.2.1 Properties of Legendre Polynomials

We can do examples of Fourier-Legendre expansions given just a few facts about Legendre polynomials. The first property that the Legendre polynomials have is the Rodrigues formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad n \in N_0. \tag{7.13}$$

The Rodrigues Formula is credited to Benjamin Olinde Rodrigues (1795-1851) who discovered it in 1816 according to Hermite in 1865 and named by Heinrich Heine in 1878. The formula was also independently discovered by Sir James Ivory (1824) and Carl Gustav Jacobi (1827). Similar formulae give the other classical orthogonal polynomials.

From the Rodrigues formula, one can show that $P_n(x)$ is an nth degree polynomial. Also, for n odd, the polynomial is an odd function and for n even, the polynomial is an even function.

Example 7.2. Determine $P_2(x)$ from Rodrigues formula:

$$\begin{aligned} P_2(x) &= \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 \\ &= \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1) \\ &= \frac{1}{8}\frac{d}{dx}(4x^3-4x) \\ &= \frac{1}{8}(12x^2-4) \\ &= \frac{1}{2}(3x^2-1). \end{aligned} \tag{7.14}$$

Note that we get the same result as we found in the last section using orthogonalization.

Table 7.3: Tabular computation of the Legendre polynomials using the Rodrigues formula.

n	$(x^2-1)^n$	$\frac{d^n}{dx^n}(x^2-1)^n$	$\frac{1}{2^n n!}$	$P_n(x)$
0	1	1	1	1
1	x^2-1	$2x$	$\frac{1}{2}$	x
2	x^4-2x^2+1	$12x^2-4$	$\frac{1}{8}$	$\frac{1}{2}(3x^2-1)$
3	$x^6-3x^4+3x^2-1$	$120x^3-72x$	$\frac{1}{48}$	$\frac{1}{2}(5x^3-3x)$

The first several Legendre polynomials are computed using the Rodrigues formula in Table 7.3. In Figure 7.4 we show plots of these Legendre polynomials.

All of the classical orthogonal polynomials satisfy a three term recursion formula (or, recurrence relation or formula). In the case of the Legendre polynomials, we have

The Three Term Recursion Formula. This is also referred to as Bonnet's recursion formula. Named after Pierre Ossian Bonnet (1819-1892), who published the derivation in Bonnet [1852]. It was also in this paper that he developed the Fourier-Legendre series expansion. He designated them X_n.

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad n = 1, 2, \ldots. \tag{7.15}$$

This can also be rewritten by replacing n with $n-1$ as

$$(2n-1)xP_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x), \quad n = 1, 2, \ldots. \tag{7.16}$$

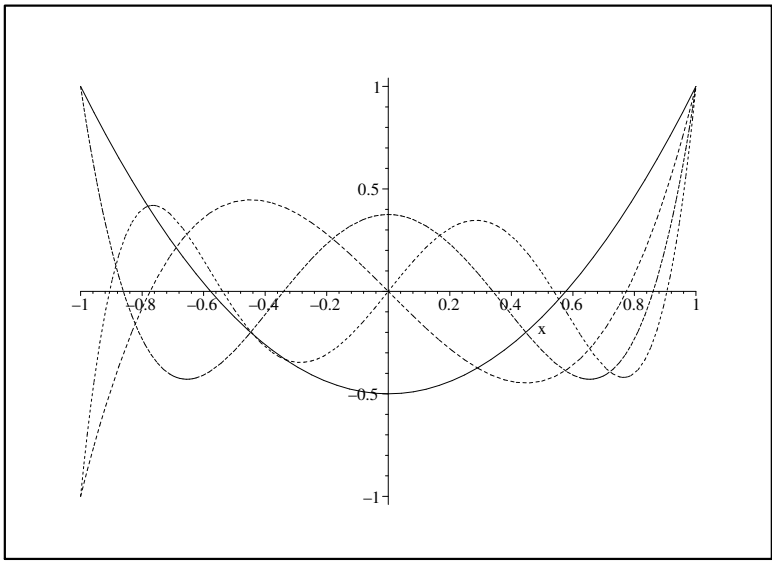


Figure 7.4: Plots of the Legendre polynomials $P_2(x)$, $P_3(x)$, $P_4(x)$, and $P_5(x)$.

Example 7.3. Use the recursion formula to find $P_2(x)$ and $P_3(x)$, given that $P_0(x) = 1$ and $P_1(x) = x$.

We first begin by inserting $n = 1$ into Equation (7.15):

$$2P_2(x) = 3xP_1(x) - P_0(x) = 3x^2 - 1.$$

So, $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

For $n = 2$, we have

$$\begin{aligned} 3P_3(x) &= 5xP_2(x) - 2P_1(x) \\ &= \frac{5}{2}x(3x^2 - 1) - 2x \\ &= \frac{1}{2}(15x^3 - 9x). \end{aligned} \tag{7.17}$$

This gives $P_3(x) = \frac{1}{2}(5x^3 - 3x)$. These expressions agree with the earlier results.

We will prove the three term recursion relation in two ways. The first proof we mention is using the orthogonality of the Legendre polynomials and is provided in Appendix 7.7. A more algebraic proof relies on the generating function for Legendre polynomials.

7.2.2 *The Generating Function for Legendre Polynomials*

A SECOND PROOF OF THE THREE TERM RECURSION FORMULA can be obtained from the generating function of the Legendre polynomials. Many special functions have such generating functions. In this case it is given by

$$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \quad |x| \leq 1, |t| < 1. \tag{7.18}$$

This generating function occurs often in applications. In particular, it arises in potential theory, such as electromagnetic or gravitational potentials. These potential functions are $\frac{1}{r}$ type functions.

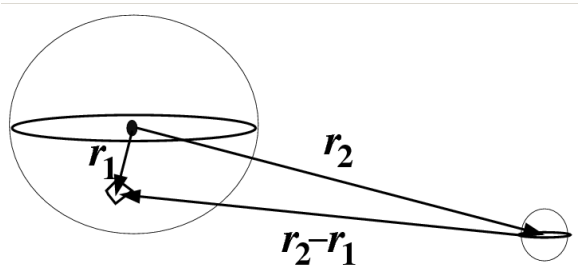


Figure 7.5: The position vectors used to describe the tidal force on the Earth due to the moon.

Legendre said that Pierre-Simon Laplace (1749-1827) introduced the potential function and Legendre provided the expansion. The polynomials were named in 1875 by Isaac Todhunter (1820-1884) as Legendre coefficients. Laplace and Legendre had written memoirs which came out in 1783 and 1785, respectively. But Legendre's work was published several years after it was written.

For example, the gravitational potential between the Earth and the moon is proportional to the reciprocal of the magnitude of the difference between their positions relative to some coordinate system. An even better example, would be to place the origin at the center of the Earth and consider the forces on the non-pointlike Earth due to the moon. Consider a piece of the Earth at position $\mathbf{r}_1$ and the moon at position $\mathbf{r}_2$ as shown in Figure 7.5. The tidal potential Φ is proportional to

$$\Phi \propto \frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|} = \frac{1}{\sqrt{(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{r}_2 - \mathbf{r}_1)}} = \frac{1}{\sqrt{r_1^2 - 2r_1 r_2 \cos\theta + r_2^2}},$$

where θ is the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$.

Typically, one of the position vectors is much larger than the other. Let's assume that $r_1 \ll r_2$. Then, one can write

$$\Phi \propto \frac{1}{\sqrt{r_1^2 - 2r_1 r_2 \cos\theta + r_2^2}} = \frac{1}{r_2}\frac{1}{\sqrt{1 - 2\frac{r_1}{r_2}\cos\theta + \left(\frac{r_1}{r_2}\right)^2}}.$$

Now, define $x = \cos\theta$ and $t = \frac{r_1}{r_2}$. We then have that the tidal potential is proportional to the generating function for the Legendre polynomials! So, we can write the tidal potential as

$$\Phi \propto \frac{1}{r_2}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r_1}{r_2}\right)^n.$$

The first term in the expansion, $\frac{1}{r_2}$, is the gravitational potential that gives the usual force between the Earth and the moon. [Recall that the gravitational potential for mass m at distance r from M is given by $\Phi = -\frac{GMm}{r}$ and that the force is the gradient of the potential, $\mathbf{F} = -\nabla\Phi \propto \nabla\left(\frac{1}{r}\right)$.] The next terms will give expressions for the tidal effects.

Now that we have some idea as to where this generating function might have originated, we can proceed to use it. First of all, the generating function can be used to obtain special values of the Legendre polynomials.

Example 7.4. Evaluate $P_n(0)$ using the generating function.

$P_n(0)$ is found by considering $g(0,t)$. Setting $x = 0$ in Equation (7.18), we have

$$\begin{aligned} g(0,t) &= \frac{1}{\sqrt{1+t^2}} \\ &= \sum_{n=0}^{\infty} P_n(0)t^n \\ &= P_0(0) + P_1(0)t + P_2(0)t^2 + P_3(0)t^3 + \ldots. \end{aligned} \tag{7.19}$$

We can use the binomial expansion to find the final answer. Namely, we have

$$\frac{1}{\sqrt{1+t^2}} = 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 + \ldots.$$

Comparing these expansions, we have the $P_n(0) = 0$ for n odd and for even integers one can show (see Problem 10) that[4]

$$P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}, \tag{7.20}$$

where $n!!$ is the *double factorial*,

$$n!! = \begin{cases} n(n-2)\ldots(3)1, & n > 0, \text{odd}, \\ n(n-2)\ldots(4)2, & n > 0, \text{even}, \\ 1 & n = 0, -1 \end{cases}.$$

[4] This example can be finished by first proving that

$$(2n)!! = 2^n n!$$

and

$$(2n-1)!! = \frac{(2n)!}{(2n)!!} = \frac{(2n)!}{2^n n!}.$$

Example 7.5. Evaluate $P_n(-1)$.

This is a simpler problem. In this case we have

$$g(-1,t) = \frac{1}{\sqrt{1+2t+t^2}} = \frac{1}{1+t} = 1 - t + t^2 - t^3 + \ldots.$$

Therefore, $P_n(-1) = (-1)^n$.

Example 7.6. Prove the three term recursion formula,

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0, \quad k = 1, 2, \ldots,$$

using the generating function.

Proof of the three term recursion formula using the generating function.

We can also use the generating function to find recurrence relations. To prove the three term recursion (7.15) that we introduced above, then we need only differentiate the generating function with respect to t in Equation (7.18) and rearrange the result. First note that

$$\frac{\partial g}{\partial t} = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \frac{x-t}{1-2xt+t^2} g(x,t).$$

Combining this with

$$\frac{\partial g}{\partial t} = \sum_{n=0}^{\infty} n P_n(x) t^{n-1},$$

we have

$$(x-t)g(x,t) = (1-2xt+t^2)\sum_{n=0}^{\infty} nP_n(x)t^{n-1}.$$

Inserting the series expression for $g(x,t)$ and distributing the sum on the right side, we obtain

$$(x-t)\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} nP_n(x)t^{n-1} - \sum_{n=0}^{\infty} 2nxP_n(x)t^n + \sum_{n=0}^{\infty} nP_n(x)t^{n+1}.$$

Multiplying out the $x-t$ factor and rearranging, leads to three separate sums:

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} - \sum_{n=0}^{\infty}(2n+1)xP_n(x)t^n + \sum_{n=0}^{\infty}(n+1)P_n(x)t^{n+1} = 0. \quad (7.21)$$

Each term contains powers of t that we would like to combine into a single sum. This is done by reindexing. For the first sum, we could use the new index $k = n-1$. Then, the first sum can be written

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} = \sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k.$$

Using different indices is just another way of writing out the terms. Note that

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} = 0 + P_1(x) + 2P_2(x)t + 3P_3(x)t^2 + \ldots$$

and

$$\sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k = 0 + P_1(x) + 2P_2(x)t + 3P_3(x)t^2 + \ldots$$

actually give the same sum. The indices are sometimes referred to as *dummy indices* because they do not show up in the expanded expression and can be replaced with another letter.

If we want to do so, we could now replace all of the k's with n's. However, we will leave the k's in the first term and now reindex the next sums in Equation (7.21). The second sum just needs the replacement $n = k$ and the last sum we reindex using $k = n+1$. Therefore, Equation (7.21) becomes

$$\sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k - \sum_{k=0}^{\infty}(2k+1)xP_k(x)t^k + \sum_{k=1}^{\infty} kP_{k-1}(x)t^k = 0. \quad (7.22)$$

We can now combine all of the terms, noting the $k=-1$ term is automatically zero and the $k=0$ terms give

$$P_1(x) - xP_0(x) = 0. \quad (7.23)$$

Of course, we know this already. So, that leaves the $k > 0$ terms:

$$\sum_{k=1}^{\infty}\left[(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x)\right]t^k = 0. \quad (7.24)$$

Since this is true for all t, the coefficients of the t^k's are zero, or

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0, \quad k = 1, 2, \ldots.$$

While this is the standard form for the three term recurrence relation, the earlier form is obtained by setting $k = n-1$.

There are other recursion relations which we list in the box below. Equation (7.25) was derived using the generating function. Differentiating it with respect to x, we find Equation (7.26). Equation (7.27) can be proven using the generating function by differentiating $g(x,t)$ with respect to x and rearranging the resulting infinite series just as in this last manipulation. This will be left as Problem 7. Combining this result with Equation (7.25), we can derive Equations (7.28)-(7.29). Adding and subtracting these equations yields Equations (7.30)-(7.31).

Recursion Formulae for Legendre Polynomials for $n = 1, 2, \ldots$.

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \tag{7.25}$$

$$(n+1)P'_{n+1}(x) = (2n+1)[P_n(x) + xP'_n(x)] - nP'_{n-1}(x) \tag{7.26}$$

$$P_n(x) = P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x) \tag{7.27}$$

$$P'_{n-1}(x) = xP'_n(x) - nP_n(x) \tag{7.28}$$

$$P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x) \tag{7.29}$$

$$P'_{n+1}(x) + P'_{n-1}(x) = 2xP'_n(x) + P_n(x). \tag{7.30}$$

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x). \tag{7.31}$$

$$(x^2-1)P'_n(x) = nxP_n(x) - nP_{n-1}(x) \tag{7.32}$$

Finally, Equation (7.32) can be obtained using Equations (7.28) and (7.29). Just multiply Equation (7.28) by x,

$$x^2P'_n(x) - nxP_n(x) = xP'_{n-1}(x).$$

Now use Equation (7.29), but first replace n with $n-1$ to eliminate the $xP'_{n-1}(x)$ term:

$$x^2P'_n(x) - nxP_n(x) = P'_n(x) - nP_{n-1}(x).$$

Rearranging gives the Equation (7.32).

Example 7.7. Use the generating function to prove

$$\|P_n\|^2 = \int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}.$$

The normalization constant.

Another use of the generating function is to obtain the normalization constant. This can be done by first squaring the generating function in order to get the products $P_n(x)P_m(x)$, and then integrating over x.

Squaring the generating function has to be done with care, as we need to make proper use of the dummy summation index. So, we first write

$$\begin{aligned}\frac{1}{1-2xt+t^2} &= \left[\sum_{n=0}^{\infty} P_n(x)t^n\right]^2 \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(x)P_m(x)t^{n+m}. \end{aligned} \tag{7.33}$$

Integrating from $x = -1$ to $x = 1$ and using the orthogonality of the Legendre polynomials, we have

$$\begin{aligned}\int_{-1}^{1}\frac{dx}{1-2xt+t^2} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t^{n+m}\int_{-1}^{1} P_n(x)P_m(x)\,dx \\ &= \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1} P_n^2(x)\,dx. \end{aligned} \tag{7.34}$$

However, one can show that[5]

$$\int_{-1}^{1}\frac{dx}{1-2xt+t^2} = \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right).$$

Expanding this expression about $t = 0$, we obtain[6]

$$\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right) = \sum_{n=0}^{\infty}\frac{2}{2n+1}t^{2n}.$$

Comparing this result with Equation (7.34), we find that

$$\|P_n\|^2 = \int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}. \tag{7.35}$$

[5] You will need the integral

$$\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx) + C.$$

[6] You will need the series expansion

$$\begin{aligned}\ln(1+x) &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots. \end{aligned}$$

7.2.3 *The Differential Equation for Legendre Polynomials*

THE LEGENDRE POLYNOMIALS SATISFY a second order linear differential equation. This differential equation occurs naturally in the solution of initial-boundary value problems in three dimensions which possess some spherical symmetry. We will see this in the next chapter. There are two approaches we could take in showing that the Legendre polynomials satisfy a particular differential equation. Either we can write down the equations and attempt to solve it,[7] or we could use the above properties to obtain the equation. For now, we will seek the differential equation satisfied by $P_n(x)$ using the above recursion relations.

We begin by differentiating Equation (7.32) and using Equation (7.28) to simplify:

$$\begin{aligned}\frac{d}{dx}\left((x^2-1)P_n'(x)\right) &= nP_n(x) + nxP_n'(x) - nP_{n-1}'(x) \\ &= nP_n(x) + n^2P_n(x) \\ &= n(n+1)P_n(x). \end{aligned} \tag{7.36}$$

[7] The standard approach to solving Legendre's Equation (7.37) is to use power series. This is typically seen in an introductory differential equations course. We insert

$$y(x) = \sum_{k=0}^{\infty} c_k x^k, \quad x \in (-1,1)$$

into Equation (7.37) to find that the coefficient satisfy a recurrence relation,

$$c_{k+1} = \frac{(n-k)(n+k+1)}{(k+1)(k+2)}c_k, \quad k \geq 0.$$

When $k = n$, the series truncates to a polynomial.

Therefore, Legendre polynomials, or Legendre functions of the first kind, are solutions of the differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0. \tag{7.37}$$

Legendre's differential equation.

As this is a linear second order differential equation, we expect two linearly independent solutions. The second solution, called the Legendre function of the second kind, is given by $Q_n(x)$ and is not well behaved at $x = \pm 1$. For example,

$$Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x}.$$

We will not need these for physically interesting examples in this book.

A generalization of the Legendre equation is given by $(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0$. Solutions to this equation, $P_n^m(x)$ and $Q_n^m(x)$, are called the associated Legendre functions of the first and second kind.

Example 7.8. Find the series solution to the Legendre equation (7.37) for n a nonnegative integer.

We first note that $x = 0$ is an ordinary point.[8] Therefore, as we learn in a first course in differential equations, we can proceed to obtain solutions in the form of Maclaurin series expansions. Insert the series expansions

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} c_k x^k, \\ y'(x) &= \sum_{k=1}^{\infty} k c_k x^{k-1}, \\ y''(x) &= \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2}, \end{aligned} \tag{7.38}$$

into the differential equation to obtain

$$\begin{aligned} 0 &= (1-x^2)y'' - 2xy' + n(n+1)y \\ &= (1-x^2)\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - 2x\sum_{k=1}^{\infty} k c_k x^{k-1} + n(n+1)\sum_{k=0}^{\infty} c_k x^k \\ &= \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - \sum_{k=2}^{\infty} k(k-1)c_k x^k - \sum_{k=1}^{\infty} 2k c_k x^k \\ &\quad + \sum_{k=0}^{\infty} n(n+1)c_k x^k \\ &= \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} + \sum_{k=0}^{\infty} [n(n+1) - k(k+1)]c_k x^k. \end{aligned} \tag{7.39}$$

Re-indexing the first sum with $\ell = k-2$, we have

$$\begin{aligned} 0 &= \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} + \sum_{k=0}^{\infty} [n(n+1) - k(k+1)]c_k x^k \\ &= \sum_{\ell=0}^{\infty} (\ell+2)(\ell+1)c_{\ell+2} x^{\ell} + \sum_{\ell=0}^{\infty} [n(n+1) - \ell(\ell+1)]c_\ell x^\ell \\ &= 2c_2 + 6c_3 x + n(n+1)c_0 + n(n+1)c_1 x - 2c_1 x \\ &\quad + \sum_{\ell=2}^{\infty} \left((\ell+2)(\ell+1)c_{\ell+2} + [n(n+1) - \ell(\ell+1)]c_\ell\right) x^\ell. \end{aligned} \tag{7.40}$$

[8] For a second order differential equations in the form

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0,$$

where $P(x)$, $Q(x)$, and $R(x)$ are polynomials in x, x_0 is called an ordinary point if $P(x_0) \neq 0$. Otherwise, x_0 is called a singular point.

Matching terms, we have

$$\begin{aligned} \ell &= 0: \quad 2c_2 = -n(n+1)c_0. \\ \ell &= 1: \quad 6c_3 = [2-n(n+1)]c_1. \\ \ell &\geq 2: \quad (\ell+2)(\ell+1)c_{\ell+2} = [\ell(\ell+1)-n(n+1)]c_\ell. \end{aligned} \tag{7.41}$$

For $n = 0$, the first equation gives $c_2 = 0$ and the third equation gives $c_{2m} = 0$ for $m = 1, 2, 3, \ldots$. This leads to $y_1(x) = c_0$ as the solution for $n = 0$.

Similarly, for $n = 1$, the second equation gives $c_3 = 0$ and the third equation gives $c_{2m+1} = 0$ for $m = 1, 2, 3, \ldots$. Thus, $y_1(x) = c_1 x$ is a solution for $n = 1$.

In fact, for n any nonnegative integer the series truncates. For example, if $n = 2$, then these equations reduce to

$$\begin{aligned} \ell &= 0: \quad 2c_2 = -6c_0. \\ \ell &= 1: \quad 6c_3 = -4c_1. \\ \ell &\geq 2: \quad (\ell+2)(\ell+1)c_{\ell+2} = [\ell(\ell+1)-2(3)]c_\ell. \end{aligned} \tag{7.42}$$

For $\ell = 2$, we have $12c_4 = 0$. So, $c_6 = c_8 = \ldots = 0$. Also, we have $c_2 = -3c_0$. This gives

$$y(x) = c_0(1-3x^2) + (c_1 x + c_3 x^3 + c_5 x^5 + c_7 x^7 + \ldots).$$

Therefore, there is a polynomial solution of degree 2. The remaining coefficients are proportional to c_1, yielding the second linearly independent solution, which is not a polynomial.

For other nonnegative integer values of $n > 2$, we have

$$c_{\ell+2} = \frac{\ell(\ell+1)-n(n+1)}{(\ell+2)(\ell+1)} c_\ell, \quad \ell \geq 2.$$

When $\ell = n$, the right side of the equation vanishes, making the remaining coefficients vanish. Thus, we will be left with a polynomial of degree n. These are the Legendre polynomials, $P_n(x)$.

7.2.4 *Fourier-Legendre Series*

With these properties of Legendre functions we are now prepared to compute the expansion coefficients for the Fourier-Legendre series representation of a given function.

Example 7.9. Expand $f(x) = x^3$ in a Fourier-Legendre series.

We simply need to compute

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} x^3 P_n(x)\, dx. \tag{7.43}$$

We first note that

$$\int_{-1}^{1} x^m P_n(x)\, dx = 0 \quad \text{for } m > n.$$

As a result, we have that $c_n = 0$ for $n > 3$. We could just compute $\int_{-1}^{1} x^3 P_m(x)\,dx$ for $m = 0, 1, 2, \ldots$ outright by looking up Legendre polynomials. But, note that x^3 is an odd function. So, $c_0 = 0$ and $c_2 = 0$.

This leaves us with only two coefficients to compute. We refer to Table 7.3 and find that

$$c_1 = \frac{3}{2}\int_{-1}^{1} x^4\,dx = \frac{3}{5}$$

$$c_3 = \frac{7}{2}\int_{-1}^{1} x^3\left[\frac{1}{2}(5x^3 - 3x)\right]\,dx = \frac{2}{5}.$$

Thus,

$$x^3 = \frac{3}{5}P_1(x) + \frac{2}{5}P_3(x).$$

Of course, this is simple to check using Table 7.3:

$$\frac{3}{5}P_1(x) + \frac{2}{5}P_3(x) = \frac{3}{5}x + \frac{2}{5}\left[\frac{1}{2}(5x^3 - 3x)\right] = x^3.$$

We could have obtained this result without doing any integration. Write x^3 as a linear combination of $P_1(x)$ and $P_3(x)$:

$$\begin{aligned} x^3 &= c_1 x + \frac{1}{2}c_2(5x^3 - 3x) \\ &= (c_1 - \frac{3}{2}c_2)x + \frac{5}{2}c_2 x^3. \end{aligned} \tag{7.44}$$

Equating coefficients of like terms, we have that $c_2 = \frac{2}{5}$ and $c_1 = \frac{3}{2}c_2 = \frac{3}{5}$.

Example 7.10. Expand the Heaviside[9] function in a Fourier-Legendre series.

The Heaviside function is defined as

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \tag{7.45}$$

In this case, we cannot find the expansion coefficients without some integration. We have to compute

$$\begin{aligned} c_n &= \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx \\ &= \frac{2n+1}{2}\int_{0}^{1} P_n(x)\,dx. \end{aligned} \tag{7.46}$$

We can make use of identity (7.31),

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x), \quad n > 0. \tag{7.47}$$

We have for $n > 0$

$$c_n = \frac{1}{2}\int_{0}^{1}[P'_{n+1}(x) - P'_{n-1}(x)]\,dx = \frac{1}{2}[P_{n-1}(0) - P_{n+1}(0)].$$

[9] Oliver Heaviside (1850-1925) was an English mathematician, physicist and engineer who used complex analysis to study circuits and was a co-founder of vector analysis. The Heaviside function is also called the step function. The modern idea of the step function is credited to Bernhard Riemann (1826-1866) and was used by Fourier and Augustin-Louis Cauchy (1789-1857). However, Heaviside's usage in his operational calculus carried over to electrical engineering and Laplace transforms.

For $n = 0$, we have

$$c_0 = \frac{1}{2}\int_0^1 dx = \frac{1}{2}.$$

This leads to the expansion

$$f(x) \sim \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}[P_{n-1}(0) - P_{n+1}(0)]P_n(x).$$

We still need to evaluate the Fourier-Legendre coefficients

$$c_n = \frac{1}{2}[P_{n-1}(0) - P_{n+1}(0)].$$

Since $P_n(0) = 0$ for n odd, the c_n's vanish for n even. Letting $n = 2k-1$, we re-index the sum, obtaining

$$f(x) \sim \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{\infty}[P_{2k-2}(0) - P_{2k}(0)]P_{2k-1}(x).$$

We can compute the nonzero Fourier coefficients, $c_{2k-1} = \frac{1}{2}[P_{2k-2}(0) - P_{2k}(0)]$, using a result from Problem 10:

$$P_{2k}(0) = (-1)^k \frac{(2k-1)!!}{(2k)!!}. \tag{7.48}$$

Namely, we have

$$\begin{aligned} c_{2k-1} &= \frac{1}{2}[P_{2k-2}(0) - P_{2k}(0)] \\ &= \frac{1}{2}\left[(-1)^{k-1}\frac{(2k-3)!!}{(2k-2)!!} - (-1)^k\frac{(2k-1)!!}{(2k)!!}\right] \\ &= -\frac{1}{2}(-1)^k\frac{(2k-3)!!}{(2k-2)!!}\left[1 + \frac{2k-1}{2k}\right] \\ &= -\frac{1}{2}(-1)^k\frac{(2k-3)!!}{(2k-2)!!}\frac{4k-1}{2k}. \end{aligned} \tag{7.49}$$

Thus, the Fourier-Legendre series expansion for the Heaviside function is given by

$$f(x) \sim \frac{1}{2} - \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2n-3)!!}{(2n-2)!!}\frac{4n-1}{2n}P_{2n-1}(x). \tag{7.50}$$

The sum of the first 21 terms of this series are shown in Figure 7.6. We note the slow convergence to the Heaviside function. Also, we see that the Gibbs phenomenon is present due to the jump discontinuity at $x = 0$. [See Section 5.5.]

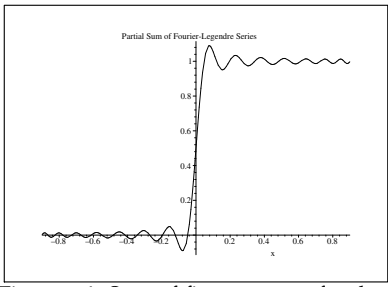


Figure 7.6: Sum of first 21 terms for the Fourier-Legendre series expansion of the Heaviside function.

7.3 Gamma Function

The name and symbol for the Gamma function were first given by Legendre in 1811. However, the search for a generalization of the factorial extends back to the 1720's when Euler provided the first representation of the factorial as an infinite product, later to be modified by others like Gauss, Weierstrass, and Legendre.

A FUNCTION THAT OFTEN OCCURS IN THE STUDY OF SPECIAL FUNCTIONS is the Gamma function. We will need the Gamma function in the next section on Fourier-Bessel series.

For $x > 0$, we define the Gamma function as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt, \quad x > 0. \tag{7.51}$$

The Gamma function is a generalization of the factorial function and a plot is shown in Figure 7.7.

In fact, we have

$$\Gamma(1) = 1$$

and

$$\Gamma(x+1) = x\Gamma(x).$$

The reader can prove this identity by simply performing an integration by parts. (See Problem 14.) In particular, for integers $n \in Z^+$, we then have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-2) = n(n-1)\cdots 2\Gamma(1) = n!.$$

We can also define the Gamma function for negative, non-integer values of x. We first note that by iteration on $n \in Z^+$, we have

$$\Gamma(x+n) = (x+n-1)\cdots(x+1)x\Gamma(x), \quad x+n > 0.$$

Solving for $\Gamma(x)$, we then find

$$\Gamma(x) = \frac{\Gamma(x+n)}{(x+n-1)\cdots(x+1)x}, \quad -n < x < 0$$

Note that the Gamma function is undefined at zero and the negative integers.

Example 7.11. We now prove that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

This is done by direct computation of the integral:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt.$$

Letting $t = z^2$, we have

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-z^2}\, dz.$$

Due to the symmetry of the integrand, we obtain the classic integral

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^\infty e^{-z^2}\, dz,$$

which can be performed using a standard trick. Consider the integral

$$I = \int_{-\infty}^\infty e^{-x^2}\, dx.$$

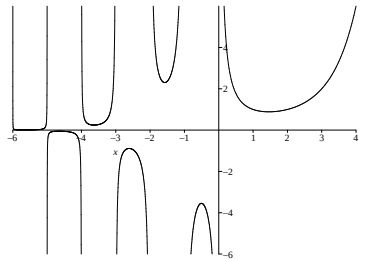


Figure 7.7: Plot of the Gamma function.

The Gamma function was a result of the problem of interpolating the factorial function. This originated with letters from Daniel Bernoulli and Leonhard Euler to Christian Goldbach (1690-1764). Bernoulli wrote October 6, 1729 with

$$x! = \lim_{n\to\infty} \left(n+1+\frac{x}{2}\right)^{x-1} \prod_{k=1}^{n} \frac{k+1}{k+x}.$$

He made a slight error computing $\frac{3}{2}!$.

October 13, 1729 Euler gave the nth approximation

$$\frac{1\cdot 2\cdot 3\cdots n(n+1)^m}{(1+m)(2+m)\cdots(n+m)}$$

and on January 8, 1730 Euler gave the integral representation

$$n! = \int_0^1 (-\ln x)^n\, dx.$$

See Problem 17.

More generally, we have

$$\int_{-\infty}^\infty e^{-\beta y^2}\, dy = \sqrt{\frac{\pi}{\beta}}.$$

For $\beta = 1$ this is referred to as the Gauss error integral, or Gauss-Poisson integral. Gauss published this result in 1809, but it was Abraham de Moivre (1667-1754) who originally discovered these types of integrals in 1733. More general Gaussian integrals are reserved for the homework such as in Problems 2 and 19.

Then,

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy.$$

Note that we changed the integration variable. This will allow us to write this product of integrals as a double integral:

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dxdy.$$

This is an integral over the entire xy-plane. We can transform this Cartesian integration to an integration over polar coordinates. The integral becomes

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,rdrd\theta.$$

This is simple to integrate and we have $I^2 = \pi$. So, the final result is found by taking the square root of both sides:

$$\Gamma\left(\frac{1}{2}\right) = I = \sqrt{\pi}.$$

In Problem 10 the reader will prove the identity

$$\Gamma(n+\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}.$$

Another useful relation, which we only state, is

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

There are many other important relations, including infinite products, which we will not need at this point. The reader is encouraged to read about these elsewhere. In the meantime, we move on to the discussion of another important special function in physics and mathematics.

7.4 *Fourier-Bessel Series*

BESSEL FUNCTIONS ARISE IN MANY PROBLEMS in physics possessing cylindrical symmetry such as the vibrations of circular drumheads and the radial modes in optical fibers. They also provide us with another orthogonal set of basis functions.

Bessel functions have a long history and were named after Friedrich Wilhelm Bessel (1784-1846).

The first occurrence of Bessel functions (zeroth order) was in the work of Daniel Bernoulli on heavy chains (1738). More general Bessel functions were studied by Leonhard Euler in 1781 and in his study of the vibrating membrane in 1764. Joseph Fourier found them in the study of heat conduction in solid cylinders and Siméon Poisson (1781-1840) in heat conduction of spheres (1823).

The history of Bessel functions does not just originate in the study of the wave and heat equations. These solutions originally came up in the study of the Kepler problem, describing planetary motion. According to G. N.

Watson in his *Treatise on Bessel Functions*, the formulation and solution of Kepler's Problem was discovered by Joseph-Louis Lagrange (1736-1813), in 1770. Namely, the problem was to express the radial coordinate and what is called the eccentric anomaly, E, as functions of time. Lagrange found expressions for the coefficients in the expansions of r and E in trigonometric functions of time. However, he only computed the first few coefficients. In 1816 Friedrich Wilhelm Bessel (1784-1846) had shown that the coefficients in the expansion for r could be given an integral representation. In 1824 he presented a thorough study of these functions, which are now called Bessel functions.

You might have seen Bessel functions in a course on differential equations as solutions of the differential equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0. \tag{7.52}$$

Solutions to this equation are obtained in the form of series expansions.[10] Namely, one seeks solutions of the form

$$y(x) = \sum_{j=0}^{\infty} a_j x^{j+n}$$

by determining the form the coefficients must take. We will leave this for a homework exercise and simply report the results.

One solution of the differential equation is the *Bessel function of the first kind of order p*, given as

$$y(x) = J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}. \tag{7.53}$$

[10] Since $x = 0$ is a regular singular point, we solve Bessel's equation using the Method of Frobenius. This differs from the series solution of Legendre's differential equation.

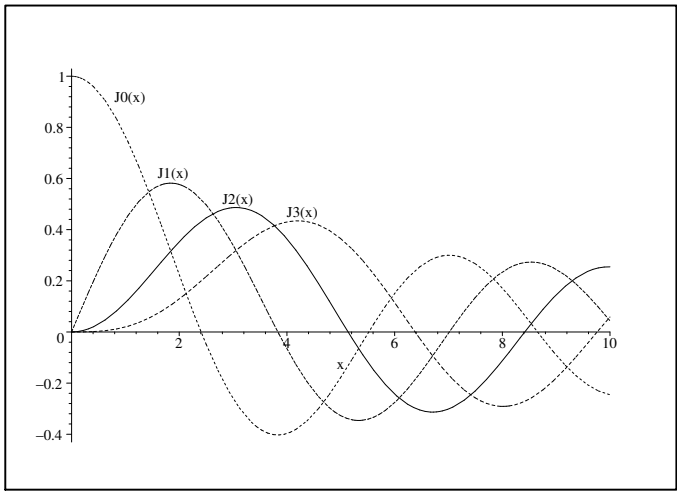


Figure 7.8: Plots of the Bessel functions $J_0(x)$, $J_1(x)$, $J_2(x)$, and $J_3(x)$.

In Figure 7.8 we display the first few Bessel functions of the first kind of integer order. Note that these functions can be described as decaying oscillatory functions.

A second linearly independent solution is obtained for p not an integer as $J_{-p}(x)$. However, for p an integer, the $\Gamma(n+p+1)$ factor leads to evaluations of the Gamma function at zero, or negative integers, when p is negative. Thus, the above series is not defined in these cases.

Another method for obtaining a second linearly independent solution is through a linear combination of $J_p(x)$ and $J_{-p}(x)$ as

$$N_p(x) = Y_p(x) = \frac{\cos \pi p J_p(x) - J_{-p}(x)}{\sin \pi p}. \tag{7.54}$$

These functions are called the Neumann functions, or Bessel functions of the second kind of order p.

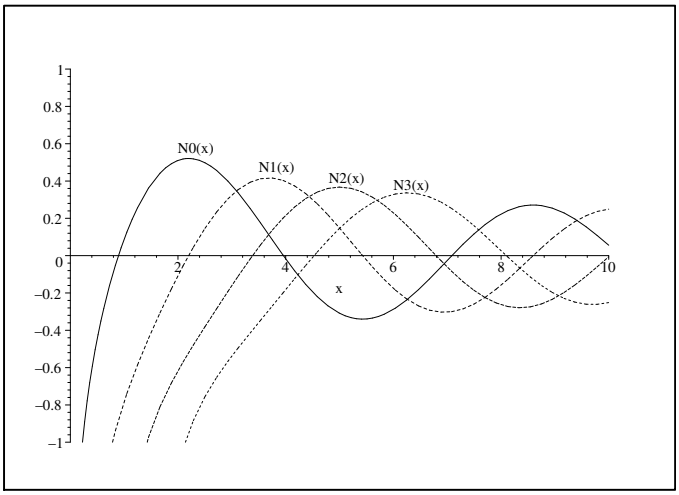


Figure 7.9: Plots of the Neumann functions $N_0(x)$, $N_1(x)$, $N_2(x)$, and $N_3(x)$.

In Figure 7.9 we display the first few Bessel functions of the second kind of integer order. Note that these functions are also decaying oscillatory functions. However, they are singular at $x = 0$.

In many applications one desires bounded solutions at $x = 0$. These functions do not satisfy this boundary condition. For example, we will later study one standard problem is to describe the oscillations of a circular drumhead. For this problem one solves the two dimensional wave equation using separation of variables in cylindrical coordinates. The radial equation leads to a Bessel equation. The Bessel function solutions describe the radial part of the solution and one does not expect a singular solution at the center of the drum. The amplitude of the oscillation must remain finite. Thus, only Bessel functions of the first kind can be used.

Bessel functions satisfy a variety of properties, which we will only list at this time for Bessel functions of the first kind. The reader will have the opportunity to prove these for homework.

Derivative Identities These identities follow directly from the manipulation of the series solution.

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x). \tag{7.55}$$

$$\frac{d}{dx}\left[x^{-p}J_p(x)\right] = -x^{-p}J_{p+1}(x). \tag{7.56}$$

Recursion Formulae The next identities follow from adding, or subtracting, the derivative identities.

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x}J_p(x). \tag{7.57}$$
$$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x). \tag{7.58}$$

Orthogonality As we will see in the next chapter, one can recast the Bessel equation into an eigenvalue problem whose solutions form an orthogonal basis of functions on $L_x^2(0,a)$. Using Sturm-Liouville theory, one can show that

$$\int_0^a xJ_p(j_{pn}\frac{x}{a})J_p(j_{pm}\frac{x}{a})\,dx = \frac{a^2}{2}\left[J_{p+1}(j_{pn})\right]^2\delta_{n,m}, \tag{7.59}$$

where j_{pn} is the nth root of $J_p(x)$, $J_p(j_{pn}) = 0$, $n = 1, 2, \ldots$. A list of some of these roots are provided in Table 7.4.

n	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
1	2.405	3.832	5.136	6.380	7.588	8.771
2	5.520	7.016	8.417	9.761	11.065	12.339
3	8.654	10.173	11.620	13.015	14.373	15.700
4	11.792	13.324	14.796	16.223	17.616	18.980
5	14.931	16.471	17.960	19.409	20.827	22.218
6	18.071	19.616	21.117	22.583	24.019	25.430
7	21.212	22.760	24.270	25.748	27.199	28.627
8	24.352	25.904	27.421	28.908	30.371	31.812
9	27.493	29.047	30.569	32.065	33.537	34.989

Table 7.4: The zeros of Bessel Functions, $J_m(j_{mn}) = 0$.

Generating Function

$$e^{x(t-\frac{1}{t})/2} = \sum_{n=-\infty}^{\infty} J_n(x)t^n, \quad x > 0, t \neq 0. \tag{7.60}$$

Integral Representation

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta - n\theta)\,d\theta, \quad x > 0, n \in \mathbb{Z}. \tag{7.61}$$

Fourier-Bessel Series

Since the Bessel functions are an orthogonal set of functions of a Sturm-Liouville problem, we can expand square integrable functions in this basis. In fact, the Sturm-Liouville problem is given in the form

$$x^2y'' + xy' + (\lambda x^2 - p^2)y = 0, \quad x \in [0, a], \tag{7.62}$$

satisfying the boundary conditions: $y(x)$ is bounded at $x = 0$ and $y(a) = 0$. The solutions are then of the form $J_p(\sqrt{\lambda}x)$, as can be shown by making the substitution $t = \sqrt{\lambda}x$ in the differential equation. Namely, we let $y(x) = u(t)$ and note that

$$\frac{dy}{dx} = \frac{dt}{dx}\frac{du}{dt} = \sqrt{\lambda}\frac{du}{dt}.$$

Then,

$$t^2u'' + tu' + (t^2 - p^2)u = 0,$$

which has a solution $u(t) = J_p(t)$.

Using Sturm-Liouville theory, one can show that $J_p(j_{pn}\frac{x}{a})$ is a basis of eigenfunctions and the resulting *Fourier-Bessel series expansion* of $f(x)$ defined on $x \in [0, a]$ is

$$f(x) = \sum_{n=1}^{\infty} c_n J_p(j_{pn}\frac{x}{a}), \tag{7.63}$$

where the Fourier-Bessel coefficients are found using the orthogonality relation as

$$c_n = \frac{2}{a^2\left[J_{p+1}(j_{pn})\right]^2}\int_0^a xf(x)J_p(j_{pn}\frac{x}{a})\,dx. \tag{7.64}$$

In the study of boundary value problems in differential equations, Sturm-Liouville problems are a bountiful source of basis functions for the space of square integrable functions.

Example 7.12. Expand $f(x) = 1$ for $0 < x < 1$ in a Fourier-Bessel series of the form

$$f(x) = \sum_{n=1}^{\infty} c_n J_0(j_{0n}x)$$

.

We need only compute the Fourier-Bessel coefficients in Equation (7.64):

$$c_n = \frac{2}{[J_1(j_{0n})]^2}\int_0^1 xJ_0(j_{0n}x)\,dx. \tag{7.65}$$

From the identity

$$\frac{d}{dx}\left[x^pJ_p(x)\right] = x^pJ_{p-1}(x). \tag{7.66}$$

we have

$$\begin{aligned}\int_0^1 xJ_0(j_{0n}x)\,dx &= \frac{1}{j_{0n}^2}\int_0^{j_{0n}} yJ_0(y)\,dy \\ &= \frac{1}{j_{0n}^2}\int_0^{j_{0n}} \frac{d}{dy}\left[yJ_1(y)\right]\,dy\end{aligned}$$

$$= \frac{1}{j_{0n}^2}\left[yJ_1(y)\right]_0^{j_{0n}}$$
$$= \frac{1}{j_{0n}}J_1(j_{0n}). \tag{7.67}$$

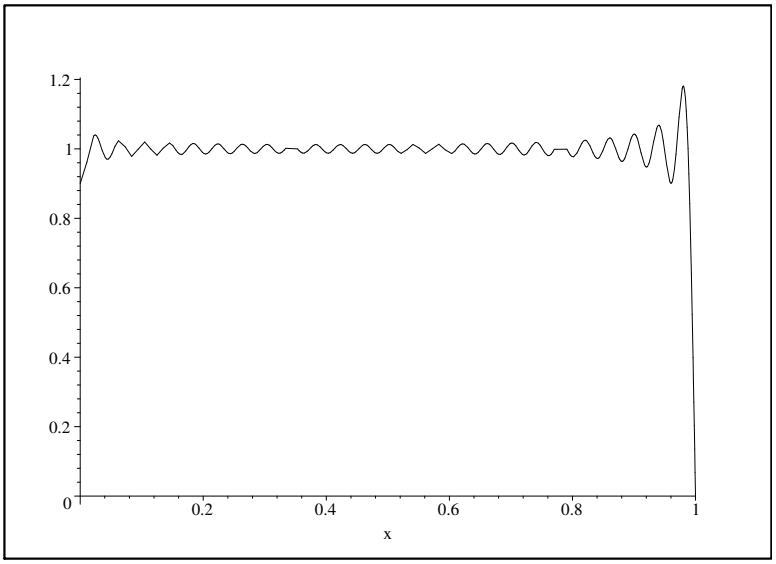


Figure 7.10: Plot of the first 50 terms of the Fourier-Bessel series in Equation (7.68) for $f(x) = 1$ on $0 < x < 1$.

As a result, the desired Fourier-Bessel expansion is given as

$$1 = 2\sum_{n=1}^{\infty}\frac{J_0(j_{0n}x)}{j_{0n}J_1(j_{0n})}, \quad 0 < x < 1. \tag{7.68}$$

In Figure 7.10 we show the partial sum for the first fifty terms of this series. Note once again the slow convergence due to the Gibbs phenomenon.

7.5 Hypergeometric Functions

HYPERGEOMETRIC FUNCTIONS ARE PROBABLY THE MOST USEFUL, but least understood, class of functions. They typically do not make it into the undergraduate curriculum and seldom in graduate curriculum. Most functions that you know can be expressed using hypergeometric functions. There are many approaches to these functions and the literature can fill books. [11]

[11] See for example *Special Functions* by G. E. Andrews, R. Askey, and R. Roy, 1999, Cambridge University Press.

In 1812 Gauss published a study of the *hypergeometric series*

$$y(x) = 1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha(1+\alpha)(1+\beta)}{2!\gamma(1+\gamma)}x^2 + \frac{\alpha(1+\alpha)(2+\alpha)\beta(1+\beta)(2+\beta)}{3!\gamma(1+\gamma)(2+\gamma)}x^3 + \ldots. \tag{7.69}$$

Here α, β, γ, and x are real numbers. If one sets $\alpha = 1$ and $\beta = \gamma$, this series reduces to the familiar geometric series

$$y(x) = 1 + x + x^2 + x^3 + \ldots.$$

The hypergeometric series is actually a solution of the differential equation

$$x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]\,y' - \alpha\beta y = 0. \tag{7.70}$$

This equation was first introduced by Euler and latter studied extensively by Gauss, Kummer and Riemann. It is sometimes called Gauss' equation. Note that there is a symmetry in that α and β may be interchanged without changing the equation. The points $x = 0$ and $x = 1$ are regular singular points. Series solutions may be sought using the Frobenius method. It can be confirmed that the above hypergeometric series results.

A more compact form for the hypergeometric series may be obtained by introducing new notation. One typically introduces the *Pochhammer symbol,* $(\alpha)_n$, satisfying (i) $(\alpha)_0 = 1$ if $\alpha \neq 0$. and (ii) $(\alpha)_k = \alpha(1+\alpha)\ldots(k-1+\alpha)$, for $k = 1, 2, \ldots$.

Consider $(1)_n$. For $n = 0$, $(1)_0 = 1$. For $n > 0$,

$$(1)_n = 1(1+1)(2+1)\ldots[(n-1)+1].$$

This reduces to $(1)_n = n!$. In fact, one can show that

$$(k)_n = \frac{(n+k-1)!}{(k-1)!}$$

for k and n positive integers. In fact, one can extend this result to noninteger values for k by introducing the gamma function:

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.$$

We can now write the hypergeometric series in standard notation as

$${}_2F_1(\alpha,\beta;\gamma;x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n} x^n.$$

Using this one can show that the general solution of Gauss' equation is

$$y(x) = A\,{}_2F_1(\alpha,\beta;\gamma;x) + B x_2^{1-\gamma} {}_2F_1(1-\gamma+\alpha, 1-\gamma+\beta; 2-\gamma; x).$$

By carefully letting β approach ∞, one obtains what is called the *confluent hypergeometric function*. This in effect changes the nature of the differential equation. Gauss' equation has three regular singular points at $x = 0, 1, \infty$. One can transform Gauss' equation by letting $x = u/\beta$. This changes the regular singular points to $u = 0, \beta, \infty$. Letting $\beta \to \infty$, two of the singular points merge.

The new confluent hypergeometric function is then given as

$${}_1F_1(\alpha;\gamma;u) = \lim_{\beta\to\infty} {}_2F_1\left(\alpha,\beta;\gamma;\frac{u}{\beta}\right).$$

This function satisfies the differential equation

$$xy'' + (\gamma - x)y' - \alpha y = 0.$$

The purpose of this section is only to introduce the hypergeometric function. Many other special functions are related to the hypergeometric function after making some variable transformations. For example, the Legendre polynomials are given by

$$P_n(x) =_2 F_1(-n, n+1; 1; \frac{1-x}{2}).$$

In fact, one can also show that

$$\sin^{-1} x = x_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right).$$

The Bessel function $J_p(x)$ can be written in terms of confluent geometric functions as

$$J_p(x) = \frac{1}{\Gamma(p+1)}\left(\frac{z}{2}\right)^p e^{-iz}\,{}_1F_1\left(\frac{1}{2}+p, 1+2p; 2iz\right).$$

These are just a few connections of the powerful hypergeometric functions to some of the elementary functions that you know.

7.6 Appendix: The Binomial Expansion

In this section we had to recall the binomial expansion. This is simply the expansion of the expression $(a+b)^p$. We will investigate this expansion first for nonnegative integer powers p and then derive the expansion for other values of p.

Lets list some of the common expansions for nonnegative integer powers.

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
&\cdots
\end{aligned} \tag{7.71}$$

We now look at the patterns of the terms in the expansions. First, we note that each term consists of a product of a power of a and a power of b. The powers of a are decreasing from n to 0 in the expansion of $(a+b)^n$. Similarly, the powers of b increase from 0 to n. The sums of the exponents in each term is n. So, we can write the $(k+1)$st term in the expansion as $a^{n-k}b^k$. For example, in the expansion of $(a+b)^{51}$ the 6th term is $a^{51-5}b^5 = a^{46}b^5$. However, we do not know the numerical coefficient in the expansion.

We now list the coefficients for the above expansions.

$$\begin{array}{lccccccccc} n=0: & & & & & 1 & & & & \\ n=1: & & & & 1 & & 1 & & & \\ n=2: & & & 1 & & 2 & & 1 & & \\ n=3: & & 1 & & 3 & & 3 & & 1 & \\ n=4: & 1 & & 4 & & 6 & & 4 & & 1 \end{array} \tag{7.72}$$

This pattern is the famous Pascal's triangle. There are many interesting features of this triangle. But we will first ask how each row can be generated.

We see that each row begins and ends with a one. Next the second term and next to last term has a coefficient of n. Next we note that consecutive pairs in each row can be added to obtain entries in the next row. For example, we have

$$\begin{array}{lccccccc} n=2: & & 1 & & & 2 & & 1 \\ & & & \searrow & \swarrow & & \searrow & \swarrow & \\ n=3: & 1 & & 3 & & & 3 & & 1 \end{array} \tag{7.73}$$

With this in mind, we can generate the next several rows of our triangle.

$$\begin{array}{lccccccccc} n=3: & & & & 1 & & 3 & & 3 & & 1 \\ n=4: & & & 1 & & 4 & & 6 & & 4 & & 1 \\ n=5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ n=6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array} \tag{7.74}$$

Of course, it would take a while to compute each row up to the desired n. We need a simple expression for computing a specific coefficient. Consider the kth term in the expansion of $(a+b)^n$. Let $r = k-1$. Then this term is of the form $C_r^n a^{n-r} b^r$. We have seen the the coefficients satisfy

$$C_r^n = C_r^{n-1} + C_{r-1}^{n-1}.$$

Actually, the coefficients have been found to take a simple form.

$$C_r^n = \frac{n!}{(n-r)!r!} = \left(\begin{array}{c} n \\ r \end{array} \right).$$

This is nothing other than the combinatoric symbol for determining how to choose n things r at a time. In our case, this makes sense. We have to count the number of ways that we can arrange the products of r b's with $n-r$ a's. There are n slots to place the b's. For example, the $r=2$ case for $n=4$ involves the six products: *aabb*, *abab*, *abba*, *baab*, *baba*, and *bbaa*. Thus, it

is natural to use this notation. The original problem that concerned Pascal was in gambling.

So, we have found that

$$(a+b)^n = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right) a^{n-r} b^r. \tag{7.75}$$

What if $a \gg b$? Can we use this to get an approximation to $(a+b)^n$? If we neglect b then $(a+b)^n \simeq a^n$. How good of an approximation is this? This is where it would be nice to know the order of the next term in the expansion, which we could state using big O notation. In order to do this we first divide out a as

$$(a+b)^n = a^n(1+\frac{b}{a})^n.$$

Now we have a small parameter, $\frac{b}{a}$. According to what we have seen above, we can use the binomial expansion to write

$$(1+\frac{b}{a})^n = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right) \left(\frac{b}{a}\right)^r. \tag{7.76}$$

Thus, we have a finite sum of terms involving powers of $\frac{b}{a}$. Since $a \gg b$, most of these terms can be neglected. So, we can write

$$(1+\frac{b}{a})^n = 1 + n\frac{b}{a} + O\left(\left(\frac{b}{a}\right)^2\right).$$

note that we have used the observation that the second coefficient in the nth row of Pascal's triangle is n.

Summarizing, this then gives

$$\begin{aligned} (a+b)^n &= a^n(1+\frac{b}{a})^n \\ &= a^n(1+n\frac{b}{a}+O\left(\left(\frac{b}{a}\right)^2)\right) \\ &= a^n + na^n\frac{b}{a} + a^n O\left(\left(\frac{b}{a}\right)^2\right). \end{aligned} \tag{7.77}$$

Therefore, we can approximate $(a+b)^n \simeq a^n + nba^{n-1}$, with an error on the order of ba^{n-2}. Note that the order of the error does not include the constant factor from the expansion. We could also use the approximation that $(a+b)^n \simeq a^n$, but it is not as good because the error in this case is of the order ba^{n-1}.

We have seen that

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots.$$

But, $\frac{1}{1-x} = (1-x)^{-1}$. This is again a binomial to a power, but the power is not a nonnegative integer. It turns out that the coefficients of such a binomial expansion can be written similar to the form in Equation (7.75).

This example suggests that our sum may no longer be finite. So, for p a real number, we write

$$(1+x)^p = \sum_{r=0}^{\infty} \left(\begin{array}{c} p \\ r \end{array} \right) x^r. \tag{7.78}$$

However, we quickly run into problems with this form. Consider the coefficient for $r = 1$ in an expansion of $(1+x)^{-1}$. This is given by

$$\left(\begin{array}{c} -1 \\ 1 \end{array} \right) = \frac{(-1)!}{(-1-1)!1!} = \frac{(-1)!}{(-2)!1!}.$$

But what is $(-1)!$? By definition, it is

$$(-1)! = (-1)(-2)(-3)\cdots.$$

This product does not seem to exist! But with a little care, we note that

$$\frac{(-1)!}{(-2)!} = \frac{(-1)(-2)!}{(-2)!} = -1.$$

So, we need to be careful not to interpret the combinatorial coefficient literally. There are better ways to write the general binomial expansion. We can write the general coefficient as

$$\begin{aligned} \left(\begin{array}{c} p \\ r \end{array} \right) &= \frac{p!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)(p-r)!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)}{r!}. \end{aligned} \tag{7.79}$$

With this in mind we now state the theorem:

General Binomial Expansion The general binomial expansion for $(1+x)^p$ is a simple generalization of Equation (7.75). For p real, we have that

$$\begin{aligned} (1+x)^p &= \sum_{r=0}^{\infty} \frac{p(p-1)\cdots(p-r+1)}{r!} x^r \\ &= \sum_{r=0}^{\infty} \frac{\Gamma(p+1)}{r!\Gamma(p-r+1)} x^r. \end{aligned} \tag{7.80}$$

Often we need the first few terms for the case that $x \ll 1$:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2} x^2 + O(x^3). \tag{7.81}$$

7.7 *Appendix: Orthogonality Proof of Three Term Recursion Formula - Optional*

The first proof of the three term recursion formula is based upon the nature of the Legendre polynomials as an orthogonal basis, while the second proof is derived using generating functions.

WE WILL PROVE THE THREE TERM RECURSION FORMULA using the orthogonality properties of Legendre polynomials and the following lemma.

Lemma 7.1. *The leading coefficient of x^n in $P_n(x)$ is* $\frac{1}{2^n n!}\frac{(2n)!}{n!}$.

Proof. We can prove this using the Rodrigues formula. First, we focus on the leading coefficient of $(x^2-1)^n$, which is x^{2n}. The first derivative of x^{2n} is $2nx^{2n-1}$. The second derivative is $2n(2n-1)x^{2n-2}$. The jth derivative is

$$\frac{d^j x^{2n}}{dx^j} = [2n(2n-1)\dots(2n-j+1)]x^{2n-j}.$$

Thus, the nth derivative is given by

$$\frac{d^n x^{2n}}{dx^n} = [2n(2n-1)\dots(n+1)]x^n.$$

This proves that $P_n(x)$ has degree n. The leading coefficient of $P_n(x)$ can now be written as

$$\begin{aligned}\frac{[2n(2n-1)\dots(n+1)]}{2^n n!} &= \frac{[2n(2n-1)\dots(n+1)]}{2^n n!}\frac{n(n-1)\dots 1}{n(n-1)\dots 1} \\ &= \frac{1}{2^n n!}\frac{(2n)!}{n!}. \end{aligned} \tag{7.82}$$

□

Theorem 7.1. *Legendre polynomials satisfy the three term recursion formula*

$$(2n-1)xP_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x), \quad n = 1, 2, \dots. \tag{7.83}$$

Proof. In order to prove the three term recursion formula we consider the expression $(2n-1)xP_{n-1}(x) - nP_n(x)$. While each term is a polynomial of degree n, the leading order terms cancel. We need only look at the coefficient of the leading order term first expression. It is

$$\frac{2n-1}{2^{n-1}(n-1)!}\frac{(2n-2)!}{(n-1)!} = \frac{1}{2^{n-1}(n-1)!}\frac{(2n-1)!}{(n-1)!} = \frac{(2n-1)!}{2^{n-1}\left[(n-1)!\right]^2}.$$

The coefficient of the leading term for $nP_n(x)$ can be written as

$$n\frac{1}{2^n n!}\frac{(2n)!}{n!} = n\left(\frac{2n}{2n^2}\right)\left(\frac{1}{2^{n-1}(n-1)!}\right)\frac{(2n-1)!}{(n-1)!}\frac{(2n-1)!}{2^{n-1}\left[(n-1)!\right]^2}.$$

It is easy to see that the leading order terms in the expression $(2n-1)xP_{n-1}(x) - nP_n(x)$ cancel.

The next terms will be of degree $n-2$. This is because the P_n's are either even or odd functions, thus only containing even, or odd, powers of x. We conclude that

$$(2n-1)xP_{n-1}(x) - nP_n(x) = \text{ polynomial of degree } n-2.$$

Therefore, since the Legendre polynomials form a basis, we can write this polynomial as a linear combination of Legendre polynomials:

$$(2n-1)xP_{n-1}(x) - nP_n(x) = c_0P_0(x) + c_1P_1(x) + \dots + c_{n-2}P_{n-2}(x). \tag{7.84}$$

Multiplying Equation (7.84) by $P_m(x)$ for $m = 0, 1, \ldots, n-3$, integrating from -1 to 1, and using orthogonality, we obtain

$$0 = c_m \|P_m\|^2, \quad m = 0, 1, \ldots, n-3.$$

[Note: $\int_{-1}^{1} x^k P_n(x)\,dx = 0$ for $k \leq n-1$. Thus, $\int_{-1}^{1} xP_{n-1}(x)P_m(x)\,dx = 0$ for $m \leq n-3$.]

Thus, all of these c_m's are zero, leaving Equation (7.84) as

$$(2n-1)xP_{n-1}(x) - nP_n(x) = c_{n-2}P_{n-2}(x).$$

The final coefficient can be found by using the normalization condition, $P_n(1) = 1$. Thus, $c_{n-2} = (2n-1) - n = n-1$. □

7.8 Appendix: The Adomain Decomposition Method - Optional

THE ADOMIAN DECOMPOSITION METHOD (ADM) IS AN ANALYTICAL TECHNIQUE designed to solve a wide variety of linear and nonlinear differential equations, integral equations, and partial differential equations. Developed by George Adomian (1922-1996) in the 1980s, this method provides a systematic approach to decomposing complex equations into simpler subproblems, allowing for iterative refinement of the solution. ADM is particularly advantageous for tackling nonlinear problems directly without the need for linearization or perturbative methods. In this section we will provide some examples solutions of linear differential equations using ADM in order to demonstrate this method.

The core idea of ADM involves expressing the solution of a given differential equation as an infinite series of unknown functions to be determined iteratively. If u denotes the solution, it is represented as:

$$u = \sum_{n=0}^{\infty} u_n, \tag{7.85}$$

where u_n are the components to be determined. For nonlinear PDEs, the method decomposes the nonlinear terms using Adomian polynomials. If $N(u)$ represents a nonlinear operator acting on u, it is expressed as

$$N(u) = \sum_{n=0}^{\infty} A_n, \tag{7.86}$$

where the A_n are the Adomian polynomials that depend on the components u_n.

The procedure begins by rewriting the differential equation in the form:

$$Lu + Ru + N(u) = g, \tag{7.87}$$

where L is a linear operator, R is a linear but potentially inhomogeneous operator, N is a nonlinear operator, and g is a source term.

Using the decomposed forms of u and $N(u)$, the equation becomes:

$$L\left(\sum_{n=0}^{\infty} u_n\right) + R\left(\sum_{n=0}^{\infty} u_n\right) + \sum_{n=0}^{\infty} A_n = g. \tag{7.88}$$

Equating terms of the same order, we derive a set of recursive relations for the components u_n. The zeroth component u_0 is obtained by solving:

$$u_0 = L^{-1}g + \text{initial/boundary values for} u, \tag{7.89}$$

and subsequent components are determined using:

$$u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n, \quad n \geq 0. \tag{7.90}$$

This is just a formal explanation of the process. We will give several examples implementing the process to see how it works in practice.

Example 7.13. Use the ADM to solve the first order (nonlinear) initial value problem

$$\frac{dy}{dt} = 1 + y^2, \quad y(0) = 0.$$

We identify the differential equation in operator form as

$$Ly + Ry = f(t), \tag{7.91}$$

where $L = \frac{d}{dt}$, $Ry = -y^2$, and $f(t) = 1$. Now, we prepare the problem for solution.

We write $Ly = 1 + y^2$. We want to solve for y. So, we apply L^{-1} to the equation,

$$L^{-1}Ly = L^{-1}(1) + L^{-1}(y^2).$$

But what is $L^{-1}Ly$? Since $L = \dfrac{d}{dt}$, then L^{-1} is just integration. We have

$$L^{-1}Ly = \int_0^t \frac{dy(\tau)}{d\tau}\, d\tau = y(t) - y(0). \tag{7.92}$$

[Note that we needed to distinguish between the integration variable τ and the independent variable t.] Also,

$$L^{-1}(1) = \int_0^t 1\, d\tau = t.$$

So, now the solution can be written

$$y(t) = y(0) + t + L^{-1}(y^2).$$

Next, we decompose the solution,

$$y(t) = \sum_{n=0}^{\infty} y_n(t) = y_0(t) + y_1(t) + y_2(t) + \ldots. \tag{7.93}$$

Using the initial condition, $y(0) = 0$, we have

$$y_0(t) + y_1(t) + y_2(t) + \ldots = t + L^{-1}[(y_0(t) + y_1(t) + \ldots)^2].$$

Next, we formulate a recursive process to solve for the components, $y_n(t)$. First, we let $y_0(t) = t$. [If $y(0) \neq 0$, we would have included it as well.]

We then expand y^2 and group the terms so that

$$\begin{aligned} \sum_{n=0}^{\infty} A_n &= (y_0(t) + y_1(t) + y_2(t) + \ldots)^2 \\ &= y_0^2 + 2y_0y_1 + y_1^2 + 2y_0y_2 + \ldots. \end{aligned} \tag{7.94}$$

The A_n's are chosen so that they only contain y_k's with $k \leq n$. Then, we can solve for the components recursively as

$$y_{k+1}(t) = L^{-1}(A_k), \quad k \geq 0. \tag{7.95}$$

The A_k's are referred to as the Adomian polynomials. In this case we have

$$\begin{aligned} A_0 &= y_0^2 \\ A_1 &= 2y_0y_1 \\ A_2 &= y_1^2 + 2y_0y_2, \text{ etc.} \end{aligned} \tag{7.96}$$

There would be different polynomials for a nonlinear term of higher order.

Since $y_0(t) = t$, we can begin to find other components:

$$\begin{aligned} y_1(t) &= \int_0^t y_0^2 \, d\tau = \frac{t^3}{3} \\ y_2(t) &= \int_0^t 2y_0y_1 \, d\tau = \frac{2t^5}{15} \\ y_3(t) &= \int_0^t (y_1^2 + 2y_0y_2) \, d\tau = \frac{17t^7}{315} \end{aligned} \tag{7.97}$$

Summing the components, we have

$$y(t) = t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{17t^7}{315} + \ldots. \tag{7.98}$$

This series can be truncated, giving an approximate solution, or it could be summed, if you are clever.

In this case, we could easily have integrated the separable equation.

$$t = \int_0^y \frac{dz}{1+z^2} = \tan^{-1} y,$$

or $y(t) = \tan t$. Looking up the series expansion for $\tan t$, we find agreement with the series found using the ADM.

Example 7.14. Forced Oscillator Problem

As the next example, we consider the second order differential equation for forced oscillations with no damping. It is given by

$$\frac{d^2u}{dt^2} + \omega_0^2 u = A\cos(\omega t), \tag{7.99}$$

where ω_0 is the natural frequency, and $f(t) = A\cos(\omega t)$ is the forcing function with amplitude A and forcing frequency ω.

We first rewrite the differential equation in operator form as

$$Lu + Ru = f(t), \tag{7.100}$$

where $L = \dfrac{d^2}{dt^2}$ and $Ru = \omega_0^2 u$.

Before proceeding, we need to be a little careful. We wish to solve for u. So, from $Lu = f(t) - Ru$, we obtain the solution by applying L^{-1} to both sides of the equation. This gives

$$L^{-1}Lu = L^{-1}(f(t) - Ru).$$

In the same way we inverted the first order derivative operator in the previous example, we need to be careful inverting the second derivative. We use τ and z for time integration variables to obtain

$$\begin{aligned} L^{-1}Lu &= \int_0^t \int_0^z \frac{d^2u(\tau)}{d\tau^2}\, d\tau dz \\ &= \int_0^t \frac{du(\tau)}{d\tau}\Big|_{\tau=0}^{z} dz \\ &= \int_0^t \left[\frac{du(z)}{dz} - u'(0)\right] dz \\ &= u(t) - u(0) - tu'(0). \end{aligned} \tag{7.101}$$

So, we now have the solution as

$$u(t) = u(0) + tu'(0) + L^{-1}(f(t) - Ru) \tag{7.102}$$

Then, we decompose the solution,

$$u(t) = \sum_{n=0}^{\infty} u_n(t). \tag{7.103}$$

In this case the forcing term does not need to be decomposed as it does not depend on u.

Finally, we finish setting up the problem by finding the needed recursive relations. The zeroth component is determined from using the source and initial values in Equation (7.102),

$$u_0 = u(0) + tu'(0) + L^{-1}f(t). \tag{7.104}$$

Then, subsequent components are found recursively using

$$u_{n+1} = -L^{-1}Ru_n = -\omega_0^2 L^{-1}u_n. \tag{7.105}$$

The solution is found through an iterative procedure starting with the solution for u_0. For the forcing $f(t) = A\cos(\omega t)$, we have

$$u_0 = u(0) + tu'(0) + L^{-1}(A\cos(\omega t)).$$

Since, $L = \dfrac{d^2}{dt^2}$, we carefully integrate twice with respect to t as

$$L^{-1}(\cdot) = \int_0^t \int_0^z (\cdot)\, d\tau dz.$$

This gives

$$\begin{aligned} L^{-1}(A\cos(\omega t)) &= \int_0^t \int_0^z A\cos(\omega\tau)\, d\tau dz \\ &= \frac{A}{\omega}\int_0^t \sin(\omega z)\, dz \\ &= -\frac{A}{\omega^2}\cos(\omega u)\Big|_{z=0}^{t}. \\ &= -\frac{A}{\omega^2}(\cos(\omega t) - 1). \end{aligned} \tag{7.106}$$

Therefore,

$$u_0(t) = u(0) + tu'(0) - \frac{A}{\omega^2}(\cos(\omega t) - 1). \tag{7.107}$$

Next, we solve for the components $u_n(t)$, which satisfy the equations

$$\begin{aligned} u_1(t) &= -\omega_0^2 L^{-1}u_0, \\ u_2(t) &= -\omega_0^2 L^{-1}u_1, \\ u_3(t) &= -\omega_0^2 L^{-1}u_2, \text{ etc.} \end{aligned} \tag{7.108}$$

Carrying out the computations for these three terms, one finds

$$\begin{aligned} u_1(t) &= \frac{\omega_0^2\left(u'(0)\omega^4 t^3 + 3u(0)\omega^4 t^2 + 3A\omega^2 t^2 + 6A\cos(\omega t) - 6A\right)}{6\omega^4} \\ u_2(t) &= \frac{\omega_0^4\left(-u'(0)\omega^6 t^5 - 5\omega^6 u(0)\, t^4 - 5A\omega^4 t^4 + 60A\omega^2 t^2\right)}{120\omega^6} \\ &\quad + A\frac{\cos(\omega t) - 1}{\omega^6} \\ u_3(t) &= \frac{\omega_0^6\left(u'(0)\omega^8 t^7 + 7\omega^8 u(0)\, t^6 + 7A\omega^6 t^6 - 210A\omega^4 t^4 + 2520A\omega^2 t^2\right)}{5040\omega^8} \\ &\quad + A\frac{\cos(\omega t) - 1}{\omega^8}. \end{aligned} \tag{7.109}$$

Summing these, we get the approximate solution

$$\begin{aligned} u(t) &= \left(1 - \frac{1}{2}\omega_0^2 t^2 + \frac{1}{24}\omega_0^4 t^4 - \frac{1}{720}\omega_0^6 t^6\right) u(0) \\ &\quad + \left(t - \frac{1}{6}\omega_0^2 t^3 + \frac{1}{120}\omega_0^4 t^5 - \frac{1}{5040}\omega_0^6 t^7)\right) u'(0) \end{aligned}$$

$$
\begin{aligned}
&-\frac{A\cos(\omega t)}{\omega^2}-\frac{\omega_0^2 A\cos(\omega t)}{\omega^4}-\frac{\omega_0^4 A\cos(\omega t)}{\omega^6}-\frac{\omega_0^6 A\cos(\omega t)}{\omega^8}\\
&+\frac{\omega_0^4 A\,t^4}{24\omega^2}-\frac{\omega_0^4 A\,t^2}{2\omega^4}+\frac{\omega_0^6 A}{\omega^8}+\frac{\omega_0^2 A}{\omega^4}-\frac{\omega_0^2 A\,t^2}{2\omega^2}-\frac{\omega_0^6 A\,t^6}{720\omega^2}\\
&+\frac{\omega_0^6 A\,t^4}{24\omega^4}-\frac{\omega_0^6 A\,t^2}{2\omega^6}+\frac{\omega_0^4 A}{\omega^6}+\frac{A}{\omega^2}. \qquad (7.110)
\end{aligned}
$$

Of course, there is an easier way to solve this problem. The solution to the homogeneous problem is

$$u_h(t) = a\sin\omega_0 t + b\cos\omega_0 t.$$

The particular solution can be found using the Method of Undetermined Coefficients. We assume a form for the particular solution,

$$u_p(t) = B\cos\omega t.$$

[This is possible since there is no first order derivative term in the equation.] Inserting this guess, we find that $B = \dfrac{A}{\omega_0^2-\omega^2}$. So, the general solution is

$$u(t) = a\sin\omega_0 t + b\cos\omega_0 t + \frac{A}{\omega_0^2-\omega^2}\cos\omega t.$$

Finally, we need to impose the initial conditions.

$$u(0) = b + \frac{A}{\omega_0^2-\omega^2}, \qquad u'(0) = a\omega_0.$$

Solving for the coefficients, gives

$$u(t) = \frac{u'(0)}{\omega_0}\sin\omega_0 t + \left(u(0) - \frac{A}{\omega_0^2-\omega^2}\right)\cos\omega_0 t + \frac{A}{\omega_0^2-\omega^2}\cos\omega t.$$

Do these solutions seem compatible? Is the ADM truncated series an approximation to the exact solution? We should recognize from the first two lines of Equation (7.110) as $u(0)\cos\omega_0 t$ and $\dfrac{u'(0)}{\omega_0}\sin\omega_0 t$, respectively. In order to obtain the third line, we consider an expansion for B. Namely,

$$
\begin{aligned}
\frac{A}{\omega_0^2-\omega^2} &= -\frac{A}{\omega^2}\frac{1}{1-\frac{\omega_0^2}{\omega^2}}\\
&= -\frac{A}{\omega^2}\left(1+\frac{\omega_0^2}{\omega^2}+\frac{\omega_0^4}{\omega^4}+\frac{\omega_0^6}{\omega^6}+\ldots\right) \qquad (7.111)
\end{aligned}
$$

This is what multiplies a $\cos\omega t$. All that is left to account for is the term

$$-\frac{A}{\omega_0^2-\omega^2}\cos\omega_0 t.$$

However, a series expansion of this expression about $\omega_0 = 0$ captures the remaining terms.

Example 7.15. Obtain solutions to the Airy Equation,

$$u''(x) = xu(x), \quad u(0) = A, u'(0) = B. \tag{7.112}$$

This equation takes the form $Lu = xu$, where $L = \frac{d^2}{dx^2}$. Applying L^{-1} to the differential equation gives the solution

$$u(x) = L^{-1}(xu) + \text{terms involving initial conditions.}$$

As before, we find the terms involving initial conditions from

$$L^{-1}Lu = \int_0^x \int_0^z u''(\xi)\, d\xi dz = u(x) - u(0) - xu'(0).$$

Using the initial conditions, the solution takes the form

$$u(x) = \sum_{n=0}^{\infty} u_n(x) = A + Bx + L^{-1}\left(x \sum_{n=0}^{\infty} u_n(x)\right).$$

Then, the recursive scheme takes the form

$$\begin{aligned} u_0(x) &= A + Bx, \\ u_{k+1}(x) &= L^{-1}(xu_k(x)). \end{aligned} \tag{7.113}$$

We compute $u_1(x)$ as

$$\begin{aligned} u_1(x) &= \int_0^x \int_0^z \xi u_0(\xi)\, d\xi dz \\ &= \int_0^x \left(\frac{1}{3} B z^3 + \frac{1}{2} A z^2\right) dz \\ &= \frac{1}{12} B x^4 + \frac{1}{6} A x^3. \end{aligned} \tag{7.114}$$

Similarly, we find

$$\begin{aligned} u_2(x) &= \frac{1}{504} x^7 B + \frac{1}{180} x^6 A \\ u_3(x) &= \frac{1}{45360} x^{10} B + \frac{1}{12960} x^9 A. \end{aligned} \tag{7.115}$$

Adding the components, we obtain

$$\begin{aligned} u(x) &= A\left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \frac{x^9}{12960} + \dots\right) \\ &\quad + B\left(x + \frac{x^4}{12} + \frac{x^7}{504} + \frac{x^{10}}{45360} + \dots\right). \end{aligned} \tag{7.116}$$

Here we have the series solution that Airy and Stokes obtained using power series methods when trying to explain the superluminaries of a rainbow.

Example 7.16. Apply the Adomian Decomposition Method to the Mathieu Differential Equation.

The Mathieu equation is a linear second-order differential equation with periodic coefficients:

$$\frac{d^2y}{dx^2} + (a - 2q\cos(2x))y = 0,$$

where a and q are real parameters. Such equations arise in problems involving elliptical cylinder functions, parametric resonance in oscillating systems, and stability analyses in engineering.

We will apply the Adomian Decomposition Method to obtain an approximate analytical solution. In the case of the Mathieu equation, the term involving y is linear but contains a variable coefficient. So, we write the Mathieu equation in operator form as

$$Ly = -Ry,$$

where:

$$L = \frac{d^2}{dx^2}, \quad R = (a - 2q\cos(2x)).$$

We will also need the inverse operator L^{-1}, which is defined as the double integral

$$L^{-1}(f)(x) = \int_0^x \int_0^{\xi} f(\eta)\, d\eta\, d\xi,$$

assuming the initial conditions:

$$y(0) = y_0, \quad y'(0) = y_1.$$

We decompose the solution into an infinite series,

$$y(x) = \sum_{n=0}^{\infty} y_n(x),$$

and then apply L^{-1} to both sides of the Mathieu equation in operator form to obtain

$$y(x) = y_0 + y_1 x - L^{-1}\left[(a - 2q\cos(2x))y(x)\right].$$

Next we derive the recursive scheme. Assume that

$$y(x) = \sum_{n=0}^{\infty} y_n(x),$$

with

$$y_0(x) = y_0 + y_1 x, \quad y_{n+1}(x) = -L^{-1}\left[(a - 2q\cos(2x))y_n(x)\right].$$

Next, we match up terms on both sides of the equation.

The zeroth approximation is given by

$$y_0(x) = y_0 + y_1 x.$$

The first iteration is then

$$\begin{aligned} y_1(x) &= -L^{-1}\left[(a-2q\cos(2x))y_0(x)\right] = -L^{-1}\left[(a-2q\cos(2x))(y_0+y_1x)\right] \\ &= -L^{-1}\left[a(y_0+y_1x)-2q\cos(2x)(y_0+y_1x)\right]. \end{aligned} \tag{7.117}$$

Integratong term-by-term, we have

$$y_1(x) = -\int_0^x \int_0^\xi \left[a(y_0+y_1\eta)-2q\cos(2\eta)(y_0+y_1\eta)\right] d\eta\, d\xi.$$

This yields an explicit expression after computing the integrals.

To improve the accuracy we could carry out a second iteration.

$$y_2(x) = -L^{-1}\left[(a-2q\cos(2x))y_1(x)\right].$$

This term involves nested integrals and can be computed symbolically or numerically. Up to second order, the approximate solution is

$$y(x) \approx y_0(x) + y_1(x) + y_2(x).$$

The accuracy improves as more terms are added. ADM is particularly useful for handling periodic coefficients analytically to a reasonable approximation. Thus, the Adomian Decomposition Method allows for approximate analytical solutions of the Mathieu equation by systematically constructing a recursive series solution. While the Mathieu equation has known Floquet-type solutions, ADM offers an approach for approximating the behavior when exact solutions are intractable or when generalizations are introduced.

Problems

1. Consider the set of vectors $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$.

a. Use the Gram-Schmidt process to find an orthonormal basis for R^3 using this set in the given order.

b. What do you get if you do reverse the order of these vectors?

2. Use the Gram-Schmidt process to find the first four orthogonal polynomials satisfying the following:

a. Interval: $(-\infty,\infty)$ Weight Function: e^{-x^2}.

b. Interval: $(0,\infty)$ Weight Function: e^{-x}.

3. The Tchebysheff polynomials of the first kind are defined as $T_n(\cos\theta) = \cos(n\theta)$. Show that these functions are orthogonal on $[-1,1]$ with weight function, $(1-x^2)^{-1/2}$ as claimed in Table 7.1.

4. Let $T_n(x), n = 0,1,2,\ldots$ be solutions of the differential equation

$$(1-x^2)y'' - xy' + n^2y = 0.$$

a. Verify that $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$, satisfy the respective differential equations for $n = 0, 1, 2, 3$.

b. Show that, in general, $T_n(x) = \cos(n \cos^{-1} x), n = 1, 2, 3, \ldots$ are polynomial solutions of degree n. Hint: Use de Moivre's formula

$$(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta,$$

where $x = \cos\theta$.

5. Do the following:

a. Show that the Tchebysheff polynomials, $T_n(x) = \cos(n \cos^{-1} x)$, for $n = 1, 2, 3, \ldots$, satisfy a three term recursion formula,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

b. The generating function for the Tchebysheff polynomials is given by

$$g(x,t) = \frac{1 - tx}{1 - 2tx + t^2}.$$

Use the binomial expansion to show this is true for $n = 0, 1, 2, 3$. [See Problem 4.]

6. Find $P_4(x)$ using

a. The Rodrigues' Formula in Equation (7.13).

b. The three term recursion formula in Equation (7.15).

7. In Equations (7.25)-(7.32) we provide several identities for Legendre polynomials. Derive the results in Equations (7.26)-(7.32) as described in the text. Namely,

a. Differentiating Equation (7.25) with respect to x, derive Equation (7.26).

b. Derive Equation (7.27) by differentiating $g(x,t)$ with respect to x and rearranging the resulting infinite series.

c. Combining the last result with Equation (7.25), derive Equations (7.28)-(7.29).

d. Adding and subtracting Equations (7.28)-(7.29), obtain Equations (7.30)-(7.31).

e. Derive Equation (7.32) using some of the other identities.

8. Use the recursion relation (7.15) to evaluate $\int_{-1}^{1} xP_n(x)P_m(x)\,dx$, $n \leq m$. Namely, insert $xP_n(x) = \dfrac{n+1}{(2n+1)}P_{n+1}(x) + \dfrac{n}{(2n+1)}P_{n-1}(x)$ in the integral and use the orthogonality of the Legendre polynomials to evaluate the integral.

9. Expand the following in a Fourier-Legendre series for $x \in (-1, 1)$.

a. $f(x) = x^2$.

b. $f(x) = 5x^4 + 2x^3 - x + 3$.

c. $f(x) = \begin{cases} -1, & -1 < x < 0, \\ 1, & 0 < x < 1. \end{cases}$

d. $f(x) = \begin{cases} x, & -1 < x < 0, \\ 0, & 0 < x < 1. \end{cases}$

10. In Maple one can type **simplify(LegendreP(2*n-2,0)-LegendreP(2*n,0));** to find a value for $P_{2n-2}(0) - P_{2n}(0)$. It gives the result in terms of Gamma functions. However, in Example 7.10 for Fourier-Legendre series, the value is given in terms of double factorials! So, we have

$$P_{2n-2}(0) - P_{2n}(0) = \frac{\sqrt{\pi}(4n-1)}{2\Gamma(n+1)\Gamma\left(\frac{3}{2}-n\right)} = (-1)^n \frac{(2n-3)!!}{(2n-2)!!}\frac{4n-1}{2n}.$$

You will verify that both results are the same by doing the following:

a. Prove that $P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}$ using the generating function and a binomial expansion.

b. Prove that $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$ using $\Gamma(x) = (x-1)\Gamma(x-1)$ and iteration.

c. Verify the result from Maple that $P_{2n-2}(0) - P_{2n}(0) = \frac{\sqrt{\pi}(4n-1)}{2\Gamma(n+1)\Gamma\left(\frac{3}{2}-n\right)}$.

d. Can either expression for $P_{2n-2}(0) - P_{2n}(0)$ be simplified further?

11. In 1838 the English Astronomer Royal George Biddell Airy developed a theory for rainbows to explain the appearance of supernumeraries, faint, pastel-colored arcs that appear inside the primary rainbow due to wave interference. This led to what are now called Airy integrals or functions,

$$\text{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right) dt$$

These functions are solutions of the differential equation $\dfrac{d^2y}{dx^2} - xy = 0$.

a. Show that the Airy function is a solution of the differential equation.

b. Assume a power series solution for Airy's differential equation of the form $y(x) = \sum_{k=0}^{\infty} a_k x^k$. Find a recursion relation for the $a_k's$.

c. Show that the power series coefficients lead to the two linearly independent solutions [Compare this problem to Example 7.15.]

$$\begin{aligned} y_1(x) &= 1 + \sum_{n=1}^{\infty} \frac{1}{3^n n!\,[2\cdot 5\cdots(3n-1)]} x^{3n} \\ y_2(x) &= x + \sum_{n=1}^{\infty} \frac{1}{3^n n!\,[1\cdot 4\cdots(3n+1)]} x^{3n+1}. \end{aligned} \quad (7.118)$$

d. Plot the solutions $y_1(x)$ and $y_2(x)$ in the previous part of the problem focusing on $x \in [-10, 10]$.

12. Solve Hermite's differential equation $y'' - 2xy' + \lambda y = 0$ using a series solution about x_0. Find solutions for $\lambda = 0, 1, 2$.

13. The Hermite polynomials, $H_n(x)$, satisfy the following:

i. $< H_n, H_m >= \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \sqrt{\pi} 2^n n! \delta_{n,m}$.

ii. $H_n'(x) = 2nH_{n-1}(x)$.

iii. $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$.

iv. $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$.

Using these, show that

a. $H_n'' - 2xH_n' + 2nH_n = 0$. [Use properties ii. and iii.]

b. $\int_{-\infty}^{\infty} xe^{-x^2} H_n(x) H_m(x)\, dx = \sqrt{\pi} 2^{n-1} n! \left[\delta_{m,n-1} + 2(n+1)\delta_{m,n+1}\right]$. [Use properties i. and iii.]

c. $H_n(0) = \begin{cases} 0, & n \text{ odd}, \\ (-1)^m \frac{(2m)!}{m!}, & n = 2m. \end{cases}$ [Let $x = 0$ in iii. and iterate. Note from iv. that $H_0(x) = 1$ and $H_1(x) = 2x$.]

14. Use integration by parts to show $\Gamma(x+1) = x\Gamma(x)$.

15. Prove the double factorial identities:

$$(2n)!! = 2^n n!$$

and

$$(2n-1)!! = \frac{(2n)!}{2^n n!}.$$

16. In 1733 de Moivre added two supplements to his 1730 book after communicating with James Stirling, containing the first use of an approximation for large factorials. However, some say that the first appearance of the Stirling approximation appeared in a letter from Euler to Goldbach in 1744. We now know of the leading order approximation of the factorial for large N as

$$N! \approx \sqrt{2\pi} e^{-N} N^{N+\frac{1}{2}}.$$

Use this Stirling approximation to find an approximation for the following:

a. $\Gamma(n)$.

b. $N!!$

c. What do these approximation give for 100! and 100!!? How do they compare with the exact values? [Use a computer algebra system to answer this, do not do this by hand.]

17. Express the following as Gamma functions. Namely, noting the form $\Gamma(x+1) = \int_0^{\infty} t^x e^{-t}\, dt$ and using an appropriate substitution, each expression can be written in terms of a Gamma function.

a. $\int_0^{\infty} x^{2/3} e^{-x}\, dx$.

b. $\int_0^\infty x^5 e^{-x^2}\,dx$

c. $\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^n\,dx$

18. Show that

$$-\int_0^1 x^k \ln x\,dx = \frac{1}{(k+1)^2}, \quad k > -1$$

by first transforming the integral into a Gamma function.

19. Gaussian integrals are important in statistical mechanics and quantum mechanics. Show that

a. $\int_0^\infty x^{2n+1} e^{-ax^2}\,dx = \frac{n!}{2a^{n+1}}$.

b. $\int_0^\infty x^{2n} e^{-ax^2}\,dx = \frac{(2n-1)!!}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$.

20. Prove that

$$\int_0^\infty e^{-x^4}\,dx = \Gamma\left(\frac{5}{4}\right) = \left(\frac{1}{4}\right)!.$$

21. The coefficients C_k^p in the binomial expansion for $(1+x)^p$ are given by

$$C_k^p = \frac{p(p-1)\cdots(p-k+1)}{k!}.$$

a. Write C_k^p in terms of Gamma functions.

b. For $p = 1/2$ use the properties of Gamma functions to write $C_k^{1/2}$ in terms of factorials.

c. Confirm your answer in part b by deriving the Maclaurin series expansion of $(1+x)^{1/2}$.

22. A solution Bessel's equation, $x^2y'' + xy' + (x^2 - n^2)y = 0$, , can be found using the guess $y(x) = \sum_{j=0}^\infty a_j x^{j+n}$. One obtains the recurrence relation $a_j = \frac{-1}{j(2n+j)}a_{j-2}$. Show that for $a_0 = (n!2^n)^{-1}$ we get the Bessel function of the first kind of order n from the even values $j = 2k$:

$$J_n(x) = \sum_{k=0}^\infty \frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k}.$$

23. Use the infinite series in the last problem to derive the derivative identities (7.66) and (7.56):

a. $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$.

b. $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$.

24. Prove the following identities based on those in the last problem.

a. $J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x}J_p(x)$.

b. $J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x)$.

25. Use the derivative identities of Bessel functions,(7.66)-(7.56), and integration by parts to show that

$$\int x^3 J_0(x)\,dx = x^3 J_1(x) - 2x^2 J_2(x) + C.$$

26. Use the generating function to find $J_n(0)$ and $J_n'(0)$.

27. Bessel functions $J_p(\lambda x)$ are solutions of $x^2y'' + xy' + (\lambda^2 x^2 - p^2)y = 0$. Assume that $x \in (0,1)$ and that $J_p(\lambda) = 0$ and $J_p(0)$ is finite.

a. Show that this equation can be written in the form

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (\lambda^2 x - \frac{p^2}{x})y = 0.$$

This is the standard Sturm-Liouville form for Bessel's equation.

b. Prove that

$$\int_0^1 x J_p(\lambda x) J_p(\mu x)\, dx = 0, \quad \lambda \neq \mu$$

by considering

$$\int_0^1 \left[J_p(\mu x)\frac{d}{dx}\left(x\frac{d}{dx}J_p(\lambda x)\right) - J_p(\lambda x)\frac{d}{dx}\left(x\frac{d}{dx}J_p(\mu x)\right)\right] dx.$$

Thus, the solutions corresponding to different eigenvalues (λ, μ) are orthogonal.

c. Prove that

$$\int_0^1 x\left[J_p(\lambda x)\right]^2 dx = \frac{1}{2}J_{p+1}^2(\lambda) = \frac{1}{2}J_p'^2(\lambda).$$

[Hint: Compute the limit as $\mu \to \lambda$ of the previous result,

$$\int_0^1 x\left[J_p(\lambda x)\right]^2 dx = \lim_{\mu\to\lambda}\int_0^1 x J_p(\lambda x) J_p(\mu x)\, dx.$$

using l'Hôpital's Rule and the identities

$$\frac{dJ_n(z)}{dz} = \frac{n}{z}J_n(z) - J_{n+1}(z) = J_{n-1}(z) - \frac{n}{z}J_n(z).$$

]

28. We can rewrite Bessel functions, $J_\nu(x)$, in a form which will allow the order to be non-integer by using the gamma function. You will need the results from Problem 10b for $\Gamma\left(k + \frac{1}{2}\right)$.

a. Extend the series definition of the Bessel function of the first kind of order ν, $J_\nu(x)$, for $\nu \geq 0$ by writing the series solution for $y(x)$ in Problem 22 using the gamma function.

b. Extend the series to $J_{-\nu}(x)$, for $\nu \geq 0$. Discuss the resulting series and what happens when ν is a positive integer.

c. Use these results to obtain the closed form expressions

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x,$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x.$$

d. Use the results in part c with the recursion formula for Bessel functions to obtain a closed form for $J_{3/2}(x)$.

29. In this problem you will derive the expansion

$$x^2 = \frac{c^2}{2} + 4\sum_{j=2}^{\infty} \frac{J_0(\alpha_j x)}{\alpha_j^2 J_0(\alpha_j c)}, \quad 0 < x < c,$$

where the $\alpha_j's$ are the positive roots of $J_1(\alpha c) = 0$, by following the below steps.

a. List the first five values of α for $J_1(\alpha c) = 0$ using the Table 7.4 and Figure 7.8. [Note: Be careful determining α_1.]

b. Show that $\|J_0(\alpha_1 x)\|^2 = \frac{c^2}{2}$. Recall,

$$\|J_0(\alpha_j x)\|^2 = \int_0^c x J_0^2(\alpha_j x)\, dx.$$

c. Show that $\|J_0(\alpha_j x)\|^2 = \frac{c^2}{2}\left[J_0(\alpha_j c)\right]^2, j = 2, 3, \ldots$. (This is the most involved step.) First note from Problem 27 that $y(x) = J_0(\alpha_j x)$ is a solution of

$$x^2 y'' + xy' + \alpha_j^2 x^2 y = 0.$$

i. Verify the Sturm-Liouville form of this differential equation: $(xy')' = -\alpha_j^2 xy$.

ii. Multiply the equation in part i. by $y(x)$ and integrate from $x = 0$ to $x = c$ to obtain

$$\begin{aligned} \int_0^c (xy')' y\, dx &= -\alpha_j^2 \int_0^c xy^2\, dx \\ &= -\alpha_j^2 \int_0^c x J_0^2(\alpha_j x)\, dx. \end{aligned} \tag{7.119}$$

iii. Noting that $y(x) = J_0(\alpha_j x)$, integrate the left hand side by parts and use the following to simplify the resulting equation.

1. $J_0'(x) = -J_1(x)$ from Equation (7.56).
2. Equation (7.59).
3. $J_2(\alpha_j c) + J_0(\alpha_j c) = 0$ from Equation (7.57).

iv. Now you should have enough information to complete this part.

d. Use the results from parts b and c and Problem 16 to derive the expansion coefficients for

$$x^2 = \sum_{j=1}^{\infty} c_j J_0(\alpha_j x)$$

in order to obtain the desired expansion.

Chapter 8

Green's Functions

"I do not want to claim that every topic should be probed in detail. Following one's own talent and inclination, one should select at least one topic, and study it in depth. In the others, one should follow the example of the bee which sucks a drop of nectar from each flower ..." - Jacopo Francesco Riccati (1676-1754)

"The young theoretical physicists of a generation or two earlier subscribed to the belief that: If you haven't done something important by age 30, you never will. Obviously, they were unfamiliar with the history of George Green, the miller of Nottingham." Julian Schwinger (1918-1994)

8.1 Introduction

In this chapter we will investigate the solution of nonhomogeneous differential equations using Green's functions. Our goal is to solve the nonhomogeneous differential equation

$$L[y] = f,$$

where L is a differential operator. The solution is formally given by

$$y = L^{-1}[f].$$

The inverse of a differential operator is an integral operator, which we seek to write in the form

$$y = \int G(x, \xi) f(\xi)\, d\xi. \tag{8.1}$$

The function $G(x, \xi)$ is referred to as the kernel of the integral operator and is called the *Green's function*.

The history of the Green's function dates back to 1828, when George Green published work in which he sought solutions of Poisson's equation $\nabla^2 u = f$ for the electric potential u defined inside a bounded volume with specified boundary conditions on the surface of the volume. He introduced a function now identified as what Riemann later coined the "Green's function". A more in depth history can be found in Section 8.8.

We will restrict our discussion to Green's functions for ordinary differential equations. Extensions to partial differential equations are typically one

of the subjects of a PDE course. We will begin our investigations by examining solutions of nonhomogeneous second order linear differential equations using the Method of Variation of Parameters, which is typically seen in a first course on differential equations. We will identify the Green's function for both initial value and boundary value problems. We will then focus on boundary value Green's functions and their properties. Determination of Green's functions is also possible using Sturm-Liouville theory. This leads to series representation of Green's functions, which we will study in the last section of this chapter.

8.2 What Are Green's Functions?

Adapted from .

Russell L. Herman. Laplace Transforms and Green's Functions. In V. Martinez-Luaces, editor, *Engineering Problems and the Laplace Transform*, pages 1–112. Nova Science, 2025

A Green's function is a mathematical tool used in many branches of physics, engineering, and applied mathematics. It is a function that describes the response of a linear system to a localized input or "forcing" function. In other words, it is a solution to a differential equation with a specific forcing term that is often represented by a Dirac delta function.[1] We use the Green's function to determine the response of the system to an input by expressing the input as a sum of impulses and convolving each impulse function with the Green's function. This technique is known as the Green's function method. In Section 8.8 we review some of the history of Green's functions.

[1] Paul Adrien Maurice Dirac (1902-1984) introduced the modern concept of the delta function in 1927 [Dirac, 1927] and put it in his 1932 quantum mechanics text [Dirac, 1981]. It was a more rigorous approach than earlier attempts to describe impulse functions, leading to the modern theory of distributions. [See [Lützen, 2012]]. The Dirac delta function is defined by two properties, $\delta(x) = 0$, $x \neq 0$ and

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1.$$

We will often use the sifting property (8.2) in what follows. More details can be found in Section 8.5.1.

The Green's function itself is the response to a simple impulse. We can demonstrate this idea using a simple example. Consider what happens when we let $f(t) = \delta(t - t_0)$ in the initial value problem,

$$y'(t) + cy(t) = \delta(t - t_0), \quad y(0) = 0.$$

Here $\delta(t - t_0)$ represents a unit impulse at time $t = t_0$. Using the solution in Equation (8.1), we have

$$y(t) = \int_0^t G(t,\tau)\delta(\tau - t_0)\,d\tau = \begin{cases} G(t,t_0) & 0 < t_0 < t, \\ 0, & \text{otherwise.} \end{cases}$$

Here we made use of the sifting property of the Dirac delta function [Herman, 2016],

$$\int_a^b f(t)\delta(t - t_0)\,dt = f(t_0), \quad a < t_0 < b. \tag{8.2}$$

For a boundary value problem, we consider

$$y''(x) - a^2 y(x) = \delta(x - x'), \quad y(0) = y(L) = 0.$$

Using the general solution in Equation (8.1), we have

$$y(x) = \int_0^L G(x,\xi)\delta(\xi - x')\,d\xi = G(x,x'), \quad 0 < x' < L.$$

We further note that the boundary conditions require that $G(0,x') = 0 = G(L,x')$.

Therefore, the solutions to these problems are essentially Green's functions. In essence, the Green's function of a system is a way to "encode" the behavior of the system in response to an input in a single function.[2] It has many important applications, such as in solving partial differential equations, studying the behavior of electromagnetic fields, and modeling the behavior of mechanical systems.

8.3 The Method of Variation of Parameters

We are interested in solving nonhomogeneous second order linear differential equations of the form

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x). \tag{8.3}$$

The general solution of this nonhomogeneous second order linear differential equation is found as a sum of the general solution of the homogeneous equation,

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0, \tag{8.4}$$

and a particular solution of the nonhomogeneous equation. Recall from Chapter 1 that there are several approaches to finding particular solutions of nonhomogeneous equations. Any guess would be sufficient. An intelligent guess, based upon the Method of Undetermined Coefficients, was reviewed previously in Chapter 1. However, a more methodical method, which is first seen in a first course in differential equations, is the Method of Variation of Parameters. Also, we explored the matrix version of this method in Section 2.9. We will review this method in this section and extend it to the solution of boundary value problems.

While it is sufficient to derive the method for the general differential equation above, we will instead consider solving equations that are in Sturm-Liouville, or self-adjoint, form. Therefore, we will apply the Method of Variation of Parameters to the equation

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) = f(x). \tag{8.5}$$

Note that $f(x)$ in this equation is not the same function as in the general equation posed at the beginning of this section.

We begin by assuming that we have determined two linearly independent solutions of the homogeneous equation. The general solution is then given by

$$y(x) = c_1y_1(x) + c_2y_2(x). \tag{8.6}$$

In order to determine a particular solution of the nonhomogeneous equation, we vary the *parameters* c_1 and c_2 in the solution of the homogeneous problem by making them functions of the independent variable. Thus, we seek a particular solution of the nonhomogeneous equation in the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x). \tag{8.7}$$

[2] A number of people tried to define the notion of an impulse before Dirac. As noted by Katz and Tall [2013], Cauchy [1816] wrote in 1815 (published 1827) the formula

$$\frac{1}{2}\int_{a-\epsilon}^{a+\epsilon} F(\mu)\frac{a\,d\mu}{a^2-(\mu-a)^2} = \frac{\pi}{2}F(a).$$

However, Fourier [1808] wrote in his memoir a similar expression,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} da f(a)\int_{-\infty}^{\infty} dp\cos(px-pa).$$

More can be found in Katz and Tall [2013] and in Van der Pol and Bremmer [1950], Chapter V.

In some sense, Green [1852, 1871a] and Heaviside were also interested in specifying unit sources for their problems. Heaviside was interested in being able to find the "disturbance in any electrical system resulting from the sudden application at time $t = 0$ of a given force" [Van der Pol, 1929]. Rayleigh [1911] referenced how Kelvin [1856] derived special solutions from point sources or a point source rate.

In order for this to be a solution, we need to show that it satisfies the differential equation. We first compute the derivatives of $y_p(x)$. The first derivative is

$$y_p'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x) + c_1'(x)y_1(x) + c_2'(x)y_2(x).$$

Without loss of generality, we will set the sum of the last two terms to zero. (One can show that the same results would be obtained if we did not. See Problem 2.) Then, we have

$$c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0. \tag{8.8}$$

Now, we take the second derivative of the remaining terms to obtain

$$y_p''(x) = c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x).$$

Expanding the derivative term in Equation (8.5),

$$p(x)y_p''(x) + p'(x)y_p'(x) + q(x)y_p(x) = f(x),$$

and inserting the expressions for y_p, $y_p'(x)$, and $y_p''(x)$, we have

$$\begin{aligned} f(x) &= p(x)\left[c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right] \\ &\quad + p'(x)\left[c_1(x)y_1'(x) + c_2(x)y_2'(x)\right] + q(x)\left[c_1(x)y_1(x) + c_2(x)y_2(x)\right]. \end{aligned}$$

Rearranging terms, we find

$$\begin{aligned} f(x) &= c_1(x)\left[p(x)y_1''(x) + p'(x)y_1'(x) + q(x)y_1(x)\right] \\ &\quad + c_2(x)\left[p(x)y_2''(x) + p'(x)y_2'(x) + q(x)y_2(x)\right] \\ &\quad + p(x)\left[c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right]. \end{aligned} \tag{8.9}$$

Since $y_1(x)$ and $y_2(x)$ are both solutions of the homogeneous equation. The first two bracketed expressions vanish. Dividing by $p(x)$, we have that

$$c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = \frac{f(x)}{p(x)}. \tag{8.10}$$

Our goal is to determine $c_1(x)$ and $c_2(x)$. In this analysis, we have found that the derivatives of these functions satisfy a linear system of equations (in the c_i's):

Linear System for Variation of Parameters

$$\begin{aligned} c_1'(x)y_1(x) + c_2'(x)y_2(x) &= 0. \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) &= \frac{f(x)}{p(x)}. \end{aligned} \tag{8.11}$$

This system is easily solved to give

$$\begin{aligned} c_1'(x) &= -\frac{f(x)y_2(x)}{p(x)\left[y_1(x)y_2'(x) - y_1'(x)y_2(x)\right]} \\ c_2'(x) &= \frac{f(x)y_1(x)}{p(x)\left[y_1(x)y_2'(x) - y_1'(x)y_2(x)\right]}. \end{aligned} \tag{8.12}$$

We note that the denominator in these expressions involves the Wronskian of the solutions to the homogeneous problem. Recall that

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

Furthermore, we can show that the denominator, $p(x)W(x)$, is constant. Differentiating this expression and using the homogeneous form of the differential equation proves this assertion.

$$\begin{aligned} \frac{d}{dx}(p(x)W(x)) &= \frac{d}{dx}\left[p(x)\left(y_1(x)y_2'(x) - y_1'(x)y_2(x)\right)\right] \\ &= y_1(x)\frac{d}{dx}(p(x)y_2'(x))) + p(x)y_2'(x)y_1'(x) \\ &\quad -y_2(x)\frac{d}{dx}(p(x)y_1'(x))) - p(x)y_1'(x)y_2'(x) \\ &= -y_1(x)q(x)y_2(x) + y_2(x)q(x)y_1(x) = 0. \end{aligned} \tag{8.13}$$

Therefore,

$$p(x)W(x) = \text{ constant.}$$

So, after an integration, we find the parameters as

$$\begin{aligned} c_1(x) &= -\int_{x_0}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi \\ c_2(x) &= \int_{x_1}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi, \end{aligned} \tag{8.14}$$

where x_0 and x_1 are arbitrary constants to be determined later.

Therefore, the particular solution of (8.5) can be written as

$$y_p(x) = y_2(x)\int_{x_1}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(x)\int_{x_0}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \tag{8.15}$$

As a further note, we usually do not rewrite the initial value problems in self-adjoint form. Recall that for an equation of the form

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = g(x). \tag{8.16}$$

we obtained the self-adjoint form by multiplying the equation by

$$\frac{1}{a_2(x)}e^{\int \frac{a_1(x)}{a_2(x)}\,dx} = \frac{1}{a_2(x)}p(x).$$

This gives the standard form

$$(p(x)y'(x))' + q(x)y(x) = f(x),$$

where

$$f(x) = \frac{1}{a_2(x)}p(x)g(x).$$

With this in mind, Equation (8.15) becomes

$$y_p(x) = y_2(x)\int_{x_1}^{x} \frac{g(\xi)y_1(\xi)}{a_2(\xi)W(\xi)}\,d\xi - y_1(x)\int_{x_0}^{x} \frac{g(\xi)y_2(\xi)}{a_(\xi)W(\xi)}\,d\xi. \tag{8.17}$$

Example 8.1. Consider the nonhomogeneous differential equation

$$y'' - y' - 6y = 20e^{-2x}.$$

We seek a particular solution to this equation. First, we note two linearly independent solutions of this equation are

$$y_1(x) = e^{3x}, \quad y_2(x) = e^{-2x}.$$

So, the particular solution takes the form

$$y_p(x) = c_1(x)e^{3x} + c_2(x)e^{-2x}.$$

We just need to determine the c_i's.

Since this problem is not in self-adjoint form, we will use

$$\frac{f(x)}{p(x)} = \frac{g(x)}{a_2(x)} = 20e^{-2x}$$

as seen above. Then the linear system we have to solve is

$$\begin{aligned} c_1'(x)e^{3x} + c_2'(x)e^{-2x} &= 0. \\ 3c_1'(x)e^{3x} - 2c_2'(x)e^{-2x} &= 20e^{-2x}. \end{aligned} \tag{8.18}$$

Multiplying the first equation by 2 and adding the equations yields

$$5c_1'(x)e^{3x} = 20e^{-2x},$$

or

$$c_1'(x) = 4e^{-5x}.$$

Inserting this back into the first equation in the system, we have

$$4e^{-2x} + c_2'(x)e^{-2x} = 0,$$

leading to

$$c_2'(x) = -4.$$

These equations are easily integrated to give

$$c_1(x) = -\frac{4}{5}e^{-5x}, \quad c_2(x) = -4x.$$

Therefore, the particular solution has been found as

$$\begin{aligned} y_p(x) &= c_1(x)e^{3x} + c_2(x)e^{-2x} \\ &= -\frac{4}{5}e^{-5x}e^{3x} - 4xe^{-2x} \\ &= -\frac{4}{5}e^{-2x} - 4xe^{-2x}. \end{aligned} \tag{8.19}$$

Noting that the first term can be absorbed into the solution of the homogeneous problem. So, the particular solution can simply be written as

$$y_p(x) = -4xe^{-2x}.$$

This is the answer you would have found had you used the Modified Method of Undetermined Coefficients from Section 1.1.4.

Example 8.2. Revisiting the last example, $y'' - y' - 6y = 20e^{-2x}$.

The formal solution in Equation (8.15) was not used in the last example. Instead, we proceeded from the Linear System for Variation of Parameters earlier in this section. This is the more natural approach towards finding the particular solution of the nonhomogeneous equation. Since we will be using Equation (8.15) to obtain solutions to initial value and boundary value problems, it might be useful to use it to solve this problem.

From the last example we have

$$y_1(x) = e^{3x}, \quad y_2(x) = e^{-2x}.$$

We need to compute the Wronskian:

$$W(x) = W(y_1, y_2)(x) = \begin{vmatrix} e^{3x} & e^{-2x} \\ 3e^{3x} & -2e^{-2x} \end{vmatrix} = -5e^{x}.$$

Also, we need $p(x)$, which is given by

$$p(x) = \exp\left(-\int dx\right) = e^{-x}.$$

So, we see that $p(x)W(x) = -5$. It is indeed constant, just as we had proven earlier.

Finally, we need $f(x)$. Here is where one needs to be careful as the original problem was not in self-adjoint form. We have from the original equation that $g(x) = 20e^{-2x}$ and $a_2(x) = 1$. So,

$$f(x) = \frac{p(x)}{a_2(x)} g(x) = 20e^{-3x}.$$

Now we are ready to construct the solution.

$$\begin{aligned} y_p(x) &= y_2(x) \int_{x_1}^{x} \frac{f(\xi) y_1(\xi)}{p(\xi) W(\xi)}\, d\xi - y_1(x) \int_{x_0}^{x} \frac{f(\xi) y_2(\xi)}{p(\xi) W(\xi)}\, d\xi \\ &= e^{-2x} \int_{x_1}^{x} \frac{20e^{-3\xi} e^{3\xi}}{-5}\, d\xi - e^{3x} \int_{x_0}^{x} \frac{20e^{-3\xi} e^{-2\xi}}{-5}\, d\xi \\ &= -4e^{-2x} \int_{x_1}^{x} d\xi + 4e^{3x} \int_{x_0}^{x} e^{-5x}\, d\xi \\ &= -4\xi e^{-2x}\Big|_{x_1}^{x} - \frac{4}{5} e^{3x} e^{-5\xi}\Big|_{x_0}^{x} \\ &= -4xe^{-2x} - \frac{4}{5} e^{-2x} + 4x_1 e^{-2x} + \frac{4}{5} e^{-5x_0} e^{3x}. \end{aligned} \tag{8.20}$$

Note that the first two terms we had found in the last example. The remaining two terms are simply linear combinations of y_1 and y_2. Thus, we really have the solution to the homogeneous problem contained within the solution when we use the arbitrary constant limits in the integrals. In the next section we will make use of these constants when solving initial value and boundary value problems.

In the next section we will determine the unknown constants subject to either initial conditions or boundary conditions. This will allow us to combine the two integrals and then determine the appropriate Green's functions.

8.4 Initial and Boundary Value Green's Functions

We begin with the particular solution (8.15) of the nonhomogeneous differential equation (8.5). This can be combined with the general solution of the homogeneous problem to give the general solution of the nonhomogeneous differential equation:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_2(x) \int_{x_1}^{x} \frac{f(\xi) y_1(\xi)}{p(\xi) W(\xi)}\, d\xi - y_1(x) \int_{x_0}^{x} \frac{f(\xi) y_2(\xi)}{p(\xi) W(\xi)}\, d\xi. \tag{8.21}$$

As seen in the last section, an appropriate choice of x_0 and x_1 could be found so that we need not explicitly write out the solution to the homogeneous problem, $c_1 y_1(x) + c_2 y_2(x)$. However, setting up the solution in this form will allow us to use x_0 and x_1 to determine particular solutions which satisfies certain homogeneous conditions.

We will now consider initial value and boundary value problems. Each type of problem will lead to a solution of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \int_a^b G(x, \xi) f(\xi)\, d\xi, \tag{8.22}$$

where the function $G(x, \xi)$ will be identified as the Green's function and the integration limits will be found on the integral. Having identified the Green's function, we will look at other methods in the last section for determining the Green's function.

8.4.1 Initial Value Green's Function

We begin by considering the solution of the initial value problem

$$\begin{aligned} \frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) &= f(x). \\ y(0) = y_0, \quad y'(0) &= v_0. \end{aligned} \tag{8.23}$$

Of course, we could have studied the original form of the differential equation without writing it in self-adjoint form. However, this form is useful when studying boundary value problems. We will return to this point later.

We first note that we can solve this initial value problem by solving two separate initial value problems. We assume that the solution of the homogeneous problem satisfies the original initial conditions:

$$\begin{aligned} \frac{d}{dx}\left(p(x)\frac{dy_h(x)}{dx}\right) + q(x)y_h(x) &= 0. \\ y_h(0) = y_0, \quad y_h'(0) &= v_0. \end{aligned} \tag{8.24}$$

We then assume that the particular solution satisfies the problem

$$\begin{aligned} \frac{d}{dx}\left(p(x)\frac{dy_p(x)}{dx}\right) + q(x)y_p(x) &= f(x). \\ y_p(0) = 0, \quad y_p'(0) &= 0. \end{aligned} \tag{8.25}$$

Since the differential equation is linear, then we know that $y(x) = y_h(x) + y_p(x)$ is a solution of the nonhomogeneous equation. However, this solution satisfies the initial conditions:

$$y(0) = y_h(0) + y_p(0) = y_0 + 0 = y_0,$$

$$y'(0) = y_h'(0) + y_p'(0) = v_0 + 0 = v_0.$$

Therefore, we need only focus on solving for the particular solution that satisfies homogeneous initial conditions.

Recall Equation (8.15) from the last section,

$$y_p(x) = y_2(x) \int_{x_1}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(x) \int_{x_0}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \tag{8.26}$$

We now seek values for x_0 and x_1 which satisfies the homogeneous initial conditions, $y_p(0) = 0$ and $y_p'(0) = 0$.

First, we consider $y_p(0) = 0$. We have

$$y_p(0) = y_2(0) \int_{x_1}^{0} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(0) \int_{x_0}^{0} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \tag{8.27}$$

Here, $y_1(x)$ and $y_2(x)$ are taken to be any solutions of the homogeneous differential equation. Let's assume that $y_1(0) = 0$ and $y_2 \neq (0) = 0$. Then we have

$$y_p(0) = y_2(0) \int_{x_1}^{0} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi. \tag{8.28}$$

We can force $y_p(0) = 0$ if we set $x_1 = 0$.

Now, we consider $y_p'(0) = 0$. First we differentiate the solution and find that

$$y_p'(x) = y_2'(x) \int_{0}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1'(x) \int_{x_0}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi, \tag{8.29}$$

since the contributions from differentiating the integrals will cancel. Evaluating this result at $x = 0$, we have

$$y_p'(0) = -y_1'(0) \int_{x_0}^{0} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \tag{8.30}$$

Assuming that $y_1'(0) \neq 0$, we can set $x_0 = 0$.

Thus, we have found that

$$\begin{aligned} y_p(x) &= y_2(x) \int_{0}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(x) \int_{0}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \\ &= \int_0^x \left[\frac{y_1(\xi)y_2(x) - y_1(x)y_2(\xi)}{p(\xi)W\xi)}\right] f(\xi)\,d\xi. \end{aligned} \tag{8.31}$$

This result is in the correct form and we can identify the temporal, or initial value, Green's function. So, the particular solution is given as

$$y_p(x) = \int_0^x G(x,\xi)f(\xi)\,d\xi, \tag{8.32}$$

where the *initial value Green's function* is defined as

$$G(x,\xi) = \frac{y_1(\xi)y_2(x) - y_1(x)y_2(\xi)}{p(\xi)W\xi)}.$$

We summarize

Solution of Initial Value Problem (8.23)

The solution of the initial value problem (8.23) takes the form

$$y(x) = y_h(x) + \int_0^x G(x,\xi)f(\xi)\,d\xi, \tag{8.33}$$

where

$$G(x,\xi) = \frac{y_1(\xi)y_2(x) - y_1(x)y_2(\xi)}{p(\xi)W\xi)}$$

and the solution of the homogeneous problem satisfies the initial conditions,

$$y_h(0) = y_0, \quad y_h'(0) = v_0.$$

Here $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation satisfying

$$y_1(0) = 0, y_2(0) \neq 0, y_1'(0) \neq 0, y_2'(0) = 0.$$

Example 8.3. Solve the forced oscillator problem

$$x'' + x = 2\cos t, \quad x(0) = 4, \quad x'(0) = 0.$$

This is a simple problem which could be solved using the Modified Method of Undetermined Coefficients in Section 1.1.4. Here we will determine the initial value Green's function. We first solve the homogeneous problem with nonhomogeneous initial conditions:

$$x_h'' + x_h = 0, \quad x_h(0) = 4, \quad x_h'(0) = 0.$$

The solution is easily seen to be $x_h(t) = 4\cos t$.

Next, we construct the Green's function. We need two linearly independent solutions, $y_1(x)$, $y_2(x)$, to the homogeneous differential equation satisfying $y_1(0) = 0$ and $y_2'(0) = 0$. So, we pick $y_1(t) = \sin t$ and $y_2(t) = \cos t$. The Wronskian is found as

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = -\sin^2 t - \cos^2 t = -1.$$

Since $p(t) = 1$ in this problem, we have

$$\begin{aligned} G(t,\tau) &= \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{p(\tau)W\tau)} \\ &= \sin t\cos\tau - \sin\tau\cos t \\ &= \sin(t-\tau). \end{aligned} \tag{8.34}$$

Note that the Green's function depends on $t - \tau$. While this is useful in some contexts, we will use the expanded form.

We can now determine the particular solution of the nonhomogeneous differential equation. We have

$$\begin{aligned} x_p(t) &= \int_0^t G(t,\tau) f(\tau)\, d\tau \\ &= \int_0^t (\sin t \cos\tau - \sin\tau \cos t)\,(2\cos\tau)\; d\tau \\ &= 2\sin t \int_0^t \cos^2\tau d\tau - 2\cos t \int_0^t \sin\tau\cos\tau d\tau \\ &= 2\sin t \left[\frac{\tau}{2} + \frac{1}{2}\sin 2\tau\right]_0^t - 2\cos t \left[\frac{1}{2}\sin^2\tau\right]_0^t \\ &= t\sin t. \end{aligned} \tag{8.35}$$

Therefore, the particular solution is $x(t) = 4\cos t + t\sin t$. This is what you would obtain using the Modified Method of Undetermined Coefficients.

Remember, you should verify your solutions. First, $x(0) = 4$ and $x'(0) = -4\sin(0) + \sin(0) = 0$. Furthermore,

$$x'' = -2\cos t - t\sin t = 2\cos t - x(t).$$

So, $x'' + x = 2\cos t$ and $x(0) = 4, x'(0) = 0$.

As noted in an earlier section, we usually are not given the differential equation in self-adjoint form. Generally, it takes the form

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = g(x). \tag{8.36}$$

Using Equation (6.9), the driving term becomes

$$f(x) = p(x)\frac{g(x)}{a_2(x)}.$$

Inserting this into the Green's function form of the particular solution, we obtain the following:

Solution Using the Initial Value Green's Function

The solution of the initial value problem,

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = g(x), \quad y_h(0) = y_0, y_h'(0) = v_0$$

takes the form

$$y(x) = y_h(x) + \int_0^x G(x,\xi)g(\xi)\, d\xi, \tag{8.37}$$

where the Green's function is the piecewise defined function

$$G(x,\xi) = \frac{y_1(\xi)y_2(x) - y_1(x)y_2(\xi)}{a_2(\xi)W(\xi)} \tag{8.38}$$

and $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation satisfying

$$y_1(0) = 0, y_2(0) \neq 0, y_1'(0) \neq 0, y_2'(0) = 0.$$

8.4.2 Boundary Value Green's Function

We now turn to boundary value problems. We will focus on the problem

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) = f(x), \quad a < x < b,$$
$$y(a) = 0, \quad y(b) = 0. \tag{8.39}$$

However, the general theory works for other forms of homogeneous boundary conditions.

Once again, we seek x_0 and x_1 in the form

$$y(x) = y_2(x)\int_{x_1}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(x)\int_{x_0}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi$$

so that the solution to the boundary value problem can be written as a single integral involving a Green's function. Here we absorb $y_h(x)$ into the integrals with an appropriate choice of lower limits on the integrals.

We first pick solutions of the homogeneous differential equation such that $y_1(a) = 0$, $y_2(b) = 0$ and $y_1(b) \neq 0$, $y_2(a) \neq 0$. So, we have

$$\begin{aligned} y(a) &= y_2(a)\int_{x_1}^{a} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(a)\int_{x_0}^{a} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi \\ &= y_2(a)\int_{x_1}^{a} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi. \end{aligned} \tag{8.40}$$

This expression is zero if $x_1 = a$.

At $x = b$ we find that

$$\begin{aligned} y(b) &= y_2(b)\int_{x_1}^{b} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(b)\int_{x_0}^{b} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi \\ &= -y_1(b)\int_{x_0}^{b} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \end{aligned} \tag{8.41}$$

This vanishes for $x_0 = b$.

So, we have found that

$$y(x) = y_2(x)\int_{a}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(x)\int_{b}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \tag{8.42}$$

We are seeking a Green's function so that the solution can be written as one integral. We can move the functions of x under the integral. Also, since $a < x < b$, we can flip the limits in the second integral. This gives

$$y(x) = \int_{a}^{x} \frac{f(\xi)y_1(\xi)y_2(x)}{p(\xi)W(\xi)}\,d\xi + \int_{x}^{b} \frac{f(\xi)y_1(x)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \tag{8.43}$$

This result can be written in a compact form:

Boundary Value Green's Function

The solution of the boundary value problem takes the form

$$y(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi, \tag{8.44}$$

where the Green's function is the piecewise defined function

$$G(x,\xi) = \begin{cases} \dfrac{y_1(\xi)y_2(x)}{pW}, & a \le \xi \le x \\ \dfrac{y_1(x)y_2(\xi)}{pW} & x \le \xi \le b \end{cases}. \tag{8.45}$$

The Green's function satisfies several properties, which we will explore further in the next section. For example, the Green's function satisfies the boundary conditions at $x = a$ and $x = b$. Thus,

$$G(a,\xi) = \frac{y_1(a)y_2(\xi)}{pW} = 0,$$

$$G(b,\xi) = \frac{y_1(\xi)y_2(b)}{pW} = 0.$$

Also, the Green's function is symmetric in its arguments. Interchanging the arguments gives

$$G(\xi,x) = \begin{cases} \dfrac{y_1(x)y_2(\xi)}{pW}, & a \le x \le \xi \\ \dfrac{y_1(\xi)y_2(x)}{pW} & \xi \le x \le b \end{cases}. \tag{8.46}$$

But a careful look at the original form shows that

$$G(x,\xi) = G(\xi,x).$$

This property is often referred to as reciprocity. Another was to look at this is to say that the effect at x from a source at ξ is the same as the effect at ξ from a source at x. This is a consequence of Green's Reciprocation Theorem, as found in Problem 1.11 in .

John David Jackson. *Classical Electrodynamics*. John Wiley & Sons, 1962

We will make use of these properties in the next section to quickly determine the Green's functions for other boundary value problems.

Example 8.4. Solve the boundary value problem $y'' = x^2$, $y(0) = 0 = y(1)$ using the boundary value Green's function.

We first solve the homogeneous equation, $y'' = 0$. After two integrations, we have $y(x) = Ax + B$, for A and B constants to be determined.

We need one solution satisfying $y_1(0) = 0$ Thus, $0 = y_1(0) = B$. So, we can pick $y_1(x) = x$, since A is arbitrary.

The other solution has to satisfy $y_2(1) = 0$. So, $0 = y_2(1) = A + B$. This can be solved for $B = -A$. Again, A is arbitrary and we will choose $A = -1$. Thus, $y_2(x) = 1 - x$.

For this problem $p(x) = 1$. Thus, for $y_1(x) = x$ and $y_2(x) = 1 - x$,

$$p(x)W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = x(-1) - 1(1-x) = -1.$$

Note that $p(x)W(x)$ is a constant, as it should be.

Now we construct the Green's function. We have

$$G(x,\xi) = \begin{cases} -\xi(1-x), & 0 \le \xi \le x \\ -x(1-\xi), & x \le \xi \le 1 \end{cases}. \tag{8.47}$$

Notice the symmetry between the two branches of the Green's function. Also, the Green's function satisfies homogeneous boundary conditions: $G(0,\xi) = 0$, from the lower branch, and $G(1,\xi) = 0$, from the upper branch.

Finally, we insert the Green's function into the integral form of the solution:

$$\begin{aligned} y(x) &= \int_0^1 G(x,\xi) f(\xi)\, d\xi \\ &= \int_0^1 G(x,\xi)\xi^2\, d\xi \\ &= -\int_0^x \xi(1-x)\xi^2\, d\xi - \int_x^1 x(1-\xi)\xi^2\, d\xi \\ &= -(1-x)\int_0^x \xi^3\, d\xi - x\int_x^1 (\xi^2 - \xi^3)\, d\xi \\ &= -(1-x)\left[\frac{\xi^4}{4}\right]_0^x - x\left[\frac{\xi^3}{3} - \frac{\xi^4}{4}\right]_x^1 \\ &= -\frac{1}{4}(1-x)x^4 - \frac{1}{12}x(4-3) + \frac{1}{12}x(4x^3 - 3x^4) \\ &= \frac{1}{12}(x^4 - x). \end{aligned} \tag{8.48}$$

8.5 Properties of Green's Functions

We have noted some properties of Green's functions in the last section. In this section we will elaborate on some of these properties as a tool for quickly constructing Green's functions for boundary value problems. Here is a list of the properties based upon our previous solution. We now show how a knowledge of these properties allows one to quickly construct a Green's function.

Properties of the Green's Function

1. **Differential Equation:**[3]
   $$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = 0, x \neq \xi$$
   For $x < \xi$ we are on the second branch and $G(x,\xi)$ is proportional to $y_1(x)$. Thus, since $y_1(x)$ is a solution of the homogeneous equation, then so is $G(x,\xi)$. For $x > \xi$ we are on the first branch and $G(x,\xi)$ is proportional to $y_2(x)$. So, once again $G(x,\xi)$ is a solution of the homogeneous problem.

2. **Boundary Conditions:**

[3] Note that this is a partial differential equation since $G(x,\xi)$ is a function of two variables. We can treat ξ as a parameter and solve the eqwuation as if it were an ordinary differential equation.

For $x = a$ we are on the second branch and $G(x, \xi)$ is proportional to $y_1(x)$. Thus, whatever condition $y_1(x)$ satisfies, $G(x, \xi)$ will satisfy. A similar statement can be made for $x = b$.

3. **Symmetry or Reciprocity:** $G(x, \xi) = G(\xi, x)$
   We had shown this in the last section.

4. **Continuity of G at** $x = \xi$**:** $G(\xi^+, \xi) = G(\xi^-, \xi)$
   Here we have defined

$$G(\xi^+, x) = \lim_{x \downarrow \xi} G(x, \xi), \quad x > \xi,$$

$$G(\xi^-, x) = \lim_{x \uparrow \xi} G(x, \xi), \quad x < \xi.$$

   Setting $x = \xi$ in both branches, we have

$$\frac{y_1(\xi)y_2(\xi)}{pW} = \frac{y_1(\xi)y_2(\xi)}{pW}.$$

5. **Jump Discontinuity of** $\dfrac{\partial G}{\partial x}$ **at** $x = \xi$**:**

$$\frac{\partial G(\xi^+, \xi)}{\partial x} - \frac{\partial G(\xi^-, \xi)}{\partial x} = \frac{1}{p(\xi)}$$

   This case is not as obvious. We first compute the derivatives by noting which branch is involved and then evaluate the derivatives and subtract them. Thus, we have

$$\begin{aligned} \frac{\partial G(\xi^+, \xi)}{\partial x} - \frac{\partial G(\xi^-, \xi)}{\partial x} &= -\frac{1}{pW} y_1(\xi) y_2'(\xi) + \frac{1}{pW} y_1'(\xi) y_2(\xi) \\ &= -\frac{y_1'(\xi)y_2(\xi) - y_1(\xi)y_2'(\xi)}{p(\xi)(y_1(\xi)y_2'(\xi) - y_1'(\xi)y_2(\xi))} \\ &= \frac{1}{p(\xi)}. \end{aligned} \tag{8.49}$$

Example 8.5. Construct the Green's function for the problem

$$y'' + \omega^2 y = f(x), \quad 0 < x < 1,$$

$$y(0) = 0 = y(1),$$

with $\omega \neq 0$.

I. **Find solutions to the homogeneous equation.**

A general solution to the homogeneous equation is given as

$$y_h(x) = c_1 \sin \omega x + c_2 \cos \omega x.$$

Thus, for $x \neq \xi$,

$$G(x, \xi) = c_1(\xi) \sin \omega x + c_2(\xi) \cos \omega x.$$

II. **Boundary Conditions.**

First, we have $G(0,\xi) = 0$ for $0 \le x \le \xi$. So,

$$G(0,\xi) = c_2(\xi)\cos\omega x = 0.$$

So,

$$G(x,\xi) = c_1(\xi)\sin\omega x, \quad 0 \le x \le \xi.$$

Second, we have $G(1,\xi) = 0$ for $\xi \le x \le 1$. So,

$$G(1,\xi) = c_1(\xi)\sin\omega + c_2(\xi)\cos\omega. = 0$$

A solution can be chosen with

$$c_2(\xi) = -c_1(\xi)\tan\omega.$$

This gives

$$G(x,\xi) = c_1(\xi)\sin\omega x - c_1(\xi)\tan\omega\cos\omega x.$$

This can be simplified by factoring out the $c_1(\xi)$ and placing the remaining terms over a common denominator. The result is

$$\begin{aligned} G(x,\xi) &= \frac{c_1(\xi)}{\cos\omega}\left[\sin\omega x\cos\omega - \sin\omega\cos\omega x\right] \\ &= -\frac{c_1(\xi)}{\cos\omega}\sin\omega(1-x). \end{aligned} \tag{8.50}$$

Since the coefficient is arbitrary at this point, as can write the result as

$$G(x,\xi) = d_1(\xi)\sin\omega(1-x), \quad \xi \le x \le 1.$$

We note that we could have started with $y_2(x) = \sin\omega(1-x)$ as one of the linearly independent solutions of the homogeneous problem in anticipation that $y_2(x)$ satisfies the second boundary condition.

III. **Symmetry or Reciprocity**

We now impose that $G(x,\xi) = G(\xi,x)$. To this point we have that

$$G(x,\xi) = \begin{cases} c_1(\xi)\sin\omega x, & 0 \le x \le \xi \\ d_1(\xi)\sin\omega(1-x), & \xi \le x \le 1 \end{cases}.$$

We can make the branches symmetric by picking the right forms for $c_1(\xi)$ and $d_1(\xi)$. We choose $c_1(\xi) = C\sin\omega(1-\xi)$ and $d_1(\xi) = C\sin\omega\xi$. Then,

$$G(x,\xi) = \begin{cases} C\sin\omega(1-\xi)\sin\omega x, & 0 \le x \le \xi \\ C\sin\omega(1-x)\sin\omega\xi, & \xi \le x \le 1 \end{cases}.$$

Now the Green's function is symmetric and we still have to determine the constant C. We note that we could have gotten to this point using the Method of Variation of Parameters result where $C = \dfrac{1}{pW}$.

IV. **Continuity of $G(x,\xi)$**

We note that we already have continuity by virtue of the symmetry imposed in the last step.

V. **Jump Discontinuity in $\frac{\partial}{\partial x}G(x,\xi)$.**

We still need to determine C. We can do this using the jump discontinuity of the derivative:

$$\frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} = \frac{1}{p(\xi)}.$$

For our problem $p(x) = 1$. So, inserting the Green's function, we have

$$\begin{aligned} 1 &= \frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} \\ &= \frac{\partial}{\partial x}\left[C\sin\omega(1-x)\sin\omega\xi\right]_{x=\xi} - \frac{\partial}{\partial x}\left[C\sin\omega(1-\xi)\sin\omega x\right]_{x=\xi} \\ &= -\omega C\cos\omega(1-\xi)\sin\omega\xi - \omega C\sin\omega(1-\xi)\cos\omega\xi \\ &= -\omega C\sin\omega(\xi+1-\xi) \\ &= -\omega C\sin\omega. \end{aligned} \tag{8.51}$$

Therefore,

$$C = -\frac{1}{\omega\sin\omega}.$$

Finally, we have the Green's function:

$$G(x,\xi) = \begin{cases} -\dfrac{\sin\omega(1-\xi)\sin\omega x}{\omega\sin\omega}, & 0 \le x \le \xi \\ -\dfrac{\sin\omega(1-x)\sin\omega\xi}{\omega\sin\omega}, & \xi \le x \le 1 \end{cases}. \tag{8.52}$$

It is instructive to compare this result to the Variation of Parameters result. We have the functions $y_1(x) = \sin\omega x$ and $y_2(x) = \sin\omega(1-x)$ as the solutions of the homogeneous equation satisfying $y_1(0) = 0$ and $y_2(1) = 0$. We need to compute pW:

$$\begin{aligned} p(x)W(x) &= y_1(x)y_2'(x) - y_1'(x)y_2(x) \\ &= -\omega\sin\omega x\cos\omega(1-x) - \omega\cos\omega x\sin\omega(1-x) \\ &= -\omega\sin\omega \end{aligned} \tag{8.53}$$

Inserting this result into the Variation of Parameters result for the Green's function leads to the same Green's function as above.

8.5.1 *The Dirac Delta Function*

We will develop a more general theory of Green's functions for ordinary differential equations which encompasses some of the listed properties. The Green's function satisfies a homogeneous differential equation for $x \neq \xi$,

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = 0, \quad x \neq \xi. \tag{8.54}$$

When $x = \xi$, we saw that the derivative has a jump in its value. This is similar to the step, or Heaviside, function,

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}.$$

In the case of the step function, the derivative is zero everywhere except at the jump. At the jump, there is an infinite slope, though technically, we have learned that there is no derivative at this point. We will try to remedy this by introducing the Dirac delta function,

$$\delta(x) = \frac{d}{dx}H(x).$$

We will then show that the Green's function satisfies the differential equation

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = \delta(x-\xi). \tag{8.55}$$

The Dirac delta function, $\delta(x)$, is one example of what is known as a *generalized function*, or a *distribution*. Dirac [1927] had introduced this function in the 1920's in his study of quantum mechanics as a useful tool. It was later studied in a general theory of distributions and found to be more than a simple tool used by physicists. The Dirac delta function, as any distribution, only makes sense under an integral.

Before defining the Dirac delta function and introducing some of its properties, we will look at some representations that lead to the definition. We will consider the limits of two sequences of functions.

First we define the sequence of functions

$$f_n(x) = \begin{cases} 0, & |x| > \frac{1}{n} \\ \frac{n}{2}, & |x| < \frac{1}{n} \end{cases}.$$

This is a sequence of functions as shown in Figure 8.1. As $n \to \infty$, we find the limit is zero for $x \neq 0$ and is infinite for $x = 0$. However, the area under each member of the sequences is one since each box has height $\frac{n}{2}$ and width $\frac{2}{n}$. Thus, the limiting function is zero at most points but has area one. (At this point the reader who is new to this should be doing some head scratching!)

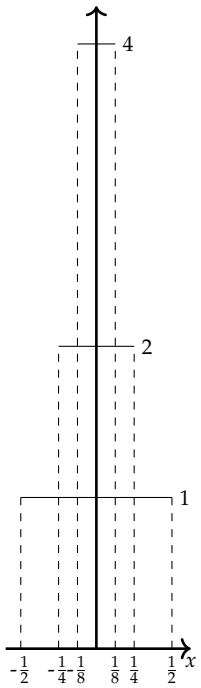


Figure 8.1: A plot of the functions $f_n(x)$ for $n = 2, 4, 8$.

The limit is not really a function. It is a *generalized function*. It is called the *Dirac delta function*, which is defined by

1. $\delta(x) = 0$ for $x \neq 0$.
2. $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$.

Another example is the sequence defined by

$$D_n(x) = \frac{2\sin nx}{x}. \tag{8.56}$$

We can graph this function. We first rewrite this function as

$$D_n(x) = 2n\frac{\sin nx}{nx}.$$

Now it is easy to see that as $x \to 0$, $D_n(x) \to 2n$. For large x, The function tends to zero. A plot of this function is in Figure 8.2. For large n the peak grows and the values of $D_n(x)$ for $x \neq 0$ tend to zero as show in Figure 8.3.

We note that in the limit $n \to \infty$, $D_n(x) = 0$ for $x \neq 0$ and it is infinite at $x = 0$. However, using complex analysis one can show that the area is

$$\int_{-\infty}^{\infty} D_n(x)\,dx = 2\pi.$$

Thus, the area is constant for each n.

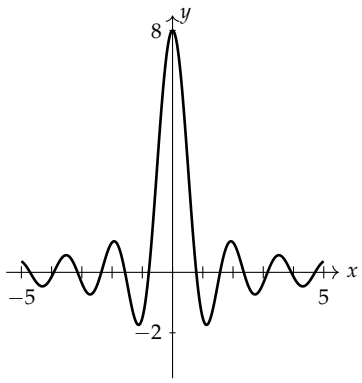


Figure 8.2: A plot of the function $D_n(x)$ for $n = 4$.

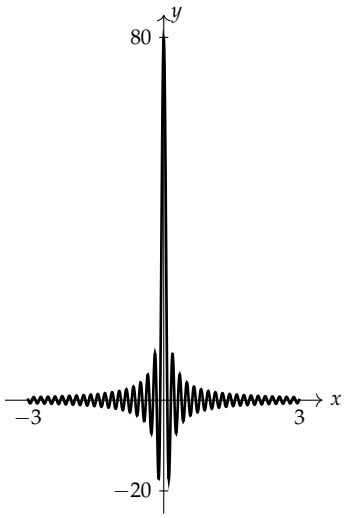


Figure 8.3: A plot of the function $D_\Omega(x)$ for $\Omega = 40$.

There are two main properties that define a Dirac delta function. First one has that the area under the delta function is one,

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1.$$

Integration over more general intervals gives

$$\int_a^b \delta(x)\,dx = 1, \quad 0 \in [a,b]$$

and

$$\int_a^b \delta(x)\,dx = 0, \quad 0 \notin [a,b].$$

Another common property is what is sometimes called the *sifting property*. Namely, integrating the product of a function and the delta function "sifts" out a specific value of the function. It is given by

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)\,dx = f(a).$$

This can be seen by noting that the delta function is zero everywhere except at $x = a$. Therefore, the integrand is zero everywhere and the only contribution from $f(x)$ will be from $x = a$. So, we can replace $f(x)$ with $f(a)$ under the integral. Since $f(a)$ is a constant, we have that

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)\,dx = \int_{-\infty}^{\infty} \delta(x-a)f(a)\,dx = f(a)\int_{-\infty}^{\infty} \delta(x-a)\,dx = f(a).$$

Another property results from using a scaled argument, ax. In this case we show that

$$\delta(ax) = |a|^{-1}\delta(x). \tag{8.57}$$

As usual, this only has meaning under an integral sign. So, we place $\delta(ax)$ inside an integral and make a substitution $y = ax$:

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(ax)\,dx &= \lim_{L\to\infty}\int_{-L}^{L} \delta(ax)\,dx \\ &= \lim_{L\to\infty}\frac{1}{a}\int_{-aL}^{aL} \delta(y)\,dy.\end{aligned} \tag{8.58}$$

If $a > 0$ then

$$\int_{-\infty}^{\infty} \delta(ax)\,dx = \frac{1}{a}\int_{-\infty}^{\infty} \delta(y)\,dy.$$

However, if $a < 0$ then

$$\int_{-\infty}^{\infty} \delta(ax)\,dx = \frac{1}{a}\int_{\infty}^{-\infty} \delta(y)\,dy = -\frac{1}{a}\int_{-\infty}^{\infty} \delta(y)\,dy.$$

The overall difference in a multiplicative minus sign can be absorbed into one expression by changing the factor $1/a$ to $1/|a|$. Thus,

$$\int_{-\infty}^{\infty} \delta(ax)\,dx = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(y)\,dy. \tag{8.59}$$

Example 8.6. Evaluate $\int_{-\infty}^{\infty}(5x+1)\delta(4(x-2))\,dx$. This is a straight forward integration:

$$\int_{-\infty}^{\infty}(5x+1)\delta(4(x-2))\,dx = \frac{1}{4}\int_{-\infty}^{\infty}(5x+1)\delta(x-2)\,dx = \frac{11}{4}.$$

A more general scaling of the argument takes the form $\delta(f(x))$. The integral of $\delta(f(x))$ can be evaluated depending upon the number of zeros of $f(x)$. If there is only one zero, $f(x_1) = 0$, then one has that

$$\int_{-\infty}^{\infty} \delta(f(x))\,dx = \int_{-\infty}^{\infty} \frac{1}{|f'(x_1)|}\delta(x-x_1)\,dx.$$

This can be proven using the substitution $y = f(x)$ and is left as an exercise for the reader. This result is often written as

$$\delta(f(x)) = \frac{1}{|f'(x_1)|}\delta(x-x_1).$$

Example 8.7. Evaluate $\int_{-\infty}^{\infty}\delta(3x-2)x^2\,dx$.

This is not a simple $\delta(x-a)$. So, we need to find the zeros of $f(x) = 3x-2$. There is only one, $x = \frac{2}{3}$. Also, $|f'(x)| = 3$. Therefore, we have

$$\int_{-\infty}^{\infty}\delta(3x-2)x^2\,dx = \int_{-\infty}^{\infty}\frac{1}{3}\delta(x-\frac{2}{3})x^2\,dx = \frac{1}{3}\left(\frac{2}{3}\right)^2 = \frac{4}{27}.$$

Note that this integral can be evaluated the long way by using the substitution $y = 3x-2$. Then, $dy = 3dx$ and $x = (y+2)/3$. This gives

$$\int_{-\infty}^{\infty}\delta(3x-2)x^2\,dx = \frac{1}{3}\int_{-\infty}^{\infty}\delta(y)\left(\frac{y+2}{3}\right)^2 dy = \frac{1}{3}\left(\frac{4}{9}\right) = \frac{4}{27}.$$

More generally, one can show that when $f(x_j) = 0$ and $f'(x_j) \neq 0$ for x_j, $j = 1, 2, \ldots, n$, (i.e.; when one has n simple zeros), then

$$\delta(f(x)) = \sum_{j=1}^{n}\frac{1}{|f'(x_j)|}\delta(x-x_j).$$

Example 8.8. Evaluate $\int_0^{2\pi} \cos x\, \delta(x^2 - \pi^2)\, dx$.

In this case the argument of the delta function has two simple roots. Namely, $f(x) = x^2 - \pi^2 = 0$ when $x = \pm\pi$. Furthermore, $f'(x) = 2x$. Therefore, $|f'(\pm\pi)| = 2\pi$. This gives

$$\delta(x^2 - \pi^2) = \frac{1}{2\pi}[\delta(x - \pi) + \delta(x + \pi)].$$

Inserting this expression into the integral and noting that $x = -\pi$ is not in the integration interval, we have

$$\begin{aligned} \int_0^{2\pi} \cos x\, \delta(x^2 - \pi^2)\, dx &= \frac{1}{2\pi} \int_0^{2\pi} \cos x\, [\delta(x - \pi) + \delta(x + \pi)]\, dx \\ &= \frac{1}{2\pi} \cos \pi = -\frac{1}{2\pi}. \end{aligned} \tag{8.60}$$

Finally, we previously noted there is a relationship between the Heaviside, or step, function and the Dirac delta function.

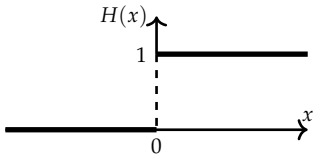


Figure 8.4: The Heaviside step function, $H(x)$.

Example 8.9. Show $H'(x) = \delta(x)$, where the Heaviside function (or, step function) is defined as

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

and is shown in Figure 8.4.

Looking at the plot, it is easy to see that $H'(x) = 0$ for $x \neq 0$ and is undefined for $x = 0$. In order to check that this gives the delta function, we need to compute the area integral. Therefore, we have

$$\int_{-\infty}^{\infty} H'(x)\, dx = H(x)\Big|_{-\infty}^{\infty} = 1 - 0 = 1.$$

Thus, $H'(x)$ satisfies the two properties of the Dirac delta function.

This argument might not be satisfying to some. Here we seem to have

$$H'(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

This is not a normal function. It is what is called a distribution, or generalized function. It is zero everywhere except it has an infinite spike at $x = 0$.

Distributions only make sense when integrated against a test function. So, we need a class of test functions, $\phi(x)$. These are infinitely differentiable functions, C^∞, which vanish at infinity, $\lim x \to \pm\infty = 0$.

Now, we consider $H'(x)$ under the an integral with a test function,.

$$\begin{aligned} \int_{-\infty}^{\infty} H'(x)\phi(x)\, dx &= H(x)\phi(x)\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H(x)\phi'(x)\, dx \\ &= -\int_0^{\infty} \phi'(x)\, dx = \phi(0). \end{aligned} \tag{8.61}$$

Thus, we see that $H'(x)$ sifts out the value of the test function at $x = 0$. This is precisely the behavior we have seen with the Dirac delta function.

8.5.2 Green's Function Differential Equation

[4] At the end of this section we show that the Green's function is the response (solution) to the impulse (source term) as governed by the differential equation (8.62.

As noted, the Green's function satisfies the differential equation[4]

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right)+q(x)G(x,\xi)=\delta(x-\xi) \tag{8.62}$$

and satisfies homogeneous conditions. We have used the Green's function to solve the nonhomogeneous equation

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right)+q(x)y(x)=f(x). \tag{8.63}$$

These equations can be written in the more compact forms

$$\begin{aligned}\mathcal{L}[y]&=f(x)\\ \mathcal{L}[G]&=\delta(x-\xi).\end{aligned} \tag{8.64}$$

Multiplying the first equation by $G(x,\xi)$, the second equation by $y(x)$, and then subtracting, we have

$$G\mathcal{L}[y]-y\mathcal{L}[G]=f(x)G(x,\xi)-\delta(x-\xi)y(x).$$

Now, integrate both sides from $x=a$ to $x=b$. The left side becomes

$$\int_a^b\left[f(x)G(x,\xi)-\delta(x-\xi)y(x)\right]\,dx=\int_a^b f(x)G(x,\xi)\,dx-y(\xi)$$

and, using Green's Identity, the right side is

$$\int_a^b\left(G\mathcal{L}[y]-y\mathcal{L}[G]\right)\,dx=\left[p(x)\left(G(x,\xi)y'(x)-y(x)\frac{\partial G}{\partial x}(x,\xi)\right)\right]_{x=a}^{x=b}.$$

Combining these results and rearranging, we obtain

$$y(\xi)=\int_a^b f(x)G(x,\xi)\,dx-\left[p(x)\left(y(x)\frac{\partial G}{\partial x}(x,\xi)-G(x,\xi)y'(x)\right)\right]_{x=a}^{x=b}. \tag{8.65}$$

Next, one uses the boundary conditions in the problem in order to determine which conditions the Green's function needs to satisfy. For example, if we have the boundary condition $y(a)=0$ and $y(b)=0$, then the boundary terms yield

$$\begin{aligned}y(\xi)&=\int_a^b f(x)G(x,\xi)\,dx-\left[p(b)\left(y(b)\frac{\partial G}{\partial x}(b,\xi)-G(b,\xi)y'(b)\right)\right]\\&\quad+\left[p(a)\left(y(a)\frac{\partial G}{\partial x}(a,\xi)-G(a,\xi)y'(a)\right)\right]\\&=\int_a^b f(x)G(x,\xi)\,dx+p(b)G(b,\xi)y'(b)-p(a)G(a,\xi)y'(a).\end{aligned} \tag{8.66}$$

The right hand side will only vanish if $G(x,\xi)$ also satisfies these homogeneous boundary conditions. This then leaves us with the solution

$$y(\xi)=\int_a^b f(x)G(x,\xi)\,dx.$$

We should rewrite this as a function of x. So, we replace ξ with x and x with ξ. This gives

$$y(x) = \int_a^b f(\xi)G(\xi, x)\, d\xi. \tag{8.67}$$

However, this is not yet in the desirable form. The arguments of the Green's function are reversed. But, $G(x, \xi)$ is symmetric in its arguments. So, we can simply switch the arguments getting the desired result.[5]

We can now see that the theory works for other boundary conditions. If we had $y'(a) = 0$, then the $y(a)\frac{\partial G}{\partial x}(a, \xi)$ term in the boundary terms could be made to vanish if we set $\frac{\partial G}{\partial x}(a, \xi) = 0$. So, this confirms that other boundary value problems can be posed besides the one elaborated upon in the chapter so far.

We can even adapt this theory to nonhomogeneous boundary conditions. We first rewrite Equation (8.65) as

$$y(x) = \int_a^b G(x, \xi)f(\xi)\, d\xi - \left[p(\xi)\left(y(\xi)\frac{\partial G}{\partial \xi}(x, \xi) - G(x, \xi)y'(\xi)\right)\right]_{\xi=a}^{\xi=b}. \tag{8.68}$$

Let's consider the boundary conditions $y(a) = \alpha$ and $y'(b) = \beta$. We also assume that $G(x, \xi)$ satisfies homogeneous boundary conditions,

$$G(a, \xi) = 0, \quad \frac{\partial G}{\partial \xi}(b, \xi) = 0.$$

in both x and ξ since the Green's function is symmetric in its variables. Then, we need only focus on the boundary terms to examine the effect on the solution. We have

$$\begin{aligned} & \left[p(\xi)\left(y(\xi)\frac{\partial G}{\partial \xi}(x, \xi) - G(x, \xi)y'(\xi)\right)\right]_{\xi=a}^{\xi=b} \\ = & \left[p(b)\left(y(b)\frac{\partial G}{\partial \xi}(x, b) - G(x, b)y'(b)\right)\right] \\ - & \left[p(a)\left(y(a)\frac{\partial G}{\partial \xi}(x, a) - G(x, a)y'(a)\right)\right] \\ = & -\beta p(b)G(x, b) - \alpha p(a)\frac{\partial G}{\partial \xi}(x, a). \end{aligned} \tag{8.69}$$

Therefore, we have the solution

$$y(x) = \int_a^b G(x, \xi)f(\xi)\, d\xi + \beta p(b)G(x, b) + \alpha p(a)\frac{\partial G}{\partial \xi}(x, a). \tag{8.70}$$

This solution satisfies the nonhomogeneous boundary conditions. Let's see how it works.

Example 8.10. Modify Example 8.4 to solve the boundary value problem $y'' = x^2$, $y(0) = 1$, $y(1) = 2$ using the boundary value Green's function that we found:

$$G(x, \xi) = \begin{cases} -\xi(1 - x), & 0 \le \xi \le x \\ -x(1 - \xi), & x \le \xi \le 1 \end{cases}. \tag{8.71}$$

[5] If $f(x) = \delta(x - x_0)$ in Equation (8.63), then the solution is $f(x) = G(x, x_0)$. However, from Equation (8.67) we have

$$\begin{aligned} G(x, x_0) &= \int_a^b \delta(\xi - x_0)G(\xi, x)\, d\xi \\ &= G(x_0, x). \end{aligned}$$

Therefore, we have established reciprocity. Note that this works this way because $\mathcal{L}$ is self-adjoint. If it were not, then we would need the adjoint Green's function.

We insert the Green's function into the solution and use the given conditions to obtain

$$\begin{aligned} y(x) &= \int_0^1 G(x,\xi)\xi^2\,d\xi - \left[y(\xi)\frac{\partial G}{\partial \xi}(x,\xi) - G(x,\xi)y'(\xi)\right]_{\xi=0}^{\xi=1} \\ &= \int_0^x (x-1)\xi^3\,d\xi + \int_x^1 x(\xi-1)\xi^2\,d\xi \\ &\quad + y(0)\frac{\partial G}{\partial \xi}(x,0) - y(1)\frac{\partial G}{\partial \xi}(x,1) \\ &= \frac{1}{12}(x^4-x) + (1-x) + 2x \\ &= \frac{x^4}{12} + \frac{11}{12}x + 1. \end{aligned} \tag{8.72}$$

Of course, this problem can be solved more directly by direct integration. The general solution is

$$y(x) = \frac{x^4}{12} + c_1 x + c_2.$$

Inserting this solution into each boundary condition yields the same result.

We have seen how the introduction of the Dirac delta function in the differential equation satisfied by the Green's function, Equation (8.62), can lead to the solution of boundary value problems. The Dirac delta function also aids in our interpretation of the Green's function. We note that the Green's function is a solution of an equation in which the nonhomogeneous function is $\delta(x-\xi)$. Note that if we multiply the delta function by $f(\xi)$ and integrate we obtain

$$\int_{-\infty}^{\infty} \delta(x-\xi)f(\xi)\,d\xi = f(x).$$

We can view the delta function as a unit impulse at $x = \xi$ which can be used to build $f(x)$ as a sum of impulses of different strengths, $f(\xi)$. Thus, the Green's function is the response to the impulse as governed by the differential equation and given boundary conditions.

In particular, the delta function forced equation can be used to derive the jump condition. We begin with the equation in the form

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = \delta(x-\xi). \tag{8.73}$$

Now, integrate both sides from $\xi-\epsilon$ to $\xi+\epsilon$ and take the limit as $\epsilon \to 0$. Then,

$$\begin{aligned} \lim_{\epsilon\to 0}\int_{\xi-\epsilon}^{\xi+\epsilon}\left[\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi)\right]dx &= \lim_{\epsilon\to 0}\int_{\xi-\epsilon}^{\xi+\epsilon}\delta(x-\xi)\,dx \\ &= 1. \end{aligned} \tag{8.74}$$

Since the $q(x)$ term is continuous, the limit of that term vanishes. Using the Fundamental Theorem of Calculus, we then have

$$\lim_{\epsilon\to 0}\left[p(x)\frac{\partial G(x,\xi)}{\partial x}\right]_{\xi-\epsilon}^{\xi+\epsilon} = 1. \tag{8.75}$$

This is the jump condition that we have been using!

8.6 The Adjoint Green's Function

IN THE PREVIOUS DISCUSSION, we have relied on the differential operator to be self-adjoint and constructed Green's functions based on properties of Sturm-Liouville operators. However, we can start with a general differential operator and not put it into Sturm-Liouville form. So, if $G(x,\xi)$ is the Green's function satisfying $LG(x,\xi) = \delta(x-\xi)$, then there is an adjoint Green's function, $G^A(x,\xi)$, satisfying $L^\dagger G^A(x,\xi) = \delta(x-\xi)$. We will see that the adjoint Green's function will come in handy to prove that we can use $G(x,\xi)$ in ways similar to the earlier examples.

We will show how this works by solving a specific problem of the form

$$y'' + 2y' + 2y = f(x), \quad y(0) = 0, y\left(\frac{\pi}{2}\right) = 0. \tag{8.76}$$

This problem takes the form $Ly = f$, where $L = D^2 + 2D + 2$ is not a self-adjoint operator. The adjoint operator is given by $L^\dagger = D^2 - 2D + 2$. We can also put this problem in Sturm-Liouville form,

$$\mathcal{L}y(x) = \frac{d}{dx}\left(e^{2x}\frac{dy(x)}{dx}\right) + 2e^{2x}y(x) = e^{2x}f(x). \tag{8.77}$$

We seek the Green's functions associated with each operator satisfying homogeneous boundary conditions and the equations

$$\begin{aligned} LG(x,\xi) &= \delta(x-\xi) \\ L^\dagger G^A(x,\xi) &= \delta(x-\xi) \\ \mathcal{L}\mathcal{G}(x,\xi) &= \delta(x-\xi). \end{aligned} \tag{8.78}$$

We then use $G(x,\xi)$ and $\mathcal{G}(x,\xi)$ to construct solutions to the boundary value problem.

We first seek the Green's function, $G(x,\xi)$, satisfying

$$\frac{\partial^2 G(x,\xi)}{\partial x^2} + 2\frac{\partial G(x,\xi)}{\partial x} + 2G(x,\xi) = \delta(x-\xi), \tag{8.79}$$

and the boundary conditions $G(0,\xi) = 0$, $G\left(\frac{\pi}{2},\xi\right) = 0$. We will see that this Green's function is not symmetric.

We will then find the adjoint Green's function, $G^A(x,\xi)$, satisfying

$$\frac{\partial^2 G^A(x,\xi)}{\partial x^2} - 2\frac{\partial G^A(x,\xi)}{\partial x} + 2G^A(x,\xi) = \delta(x-\xi), \tag{8.80}$$

and the boundary conditions $G^A(0,\xi) = 0, G^A\left(\frac{\pi}{2},\xi\right) = 0$. We will show that $G(\xi,x) = G^A(x,\xi)$ and use both functions to find the solution to the boundary value problem. We will then show that this solution is the same as using the Sturm-Liouville operator.

Example 8.11. Find the Green's function satisfying Equation (8.79).

Defining $g(x) = G(x, \xi)$, then for $x \neq \xi$,

$$g'' + 2g' + 2g = 0, \quad g(0) = 0, g\left(\frac{\pi}{2}\right) = 0.$$

The characteristic equation is $r^2 + 2r + 2 = 0$. So, $r = -1 \pm i$. This gives the general solution as

$$g(x) = e^{-x}(a\cos x + b\sin x).$$

For $0 \leq x \leq \xi$, we find the solution $g_1(x)$ satisfying the boundary condition $g_1(0) = 0$.

$$g_1(0) = e^0(a\cos 0 + b\sin 0) = a = 0.$$

So, $g_1(x) = be^{-x}\sin x$.

Similarly, we find the solution $g_2(x)$, $\xi \leq x \leq \frac{\pi}{2}$, satisfying the boundary condition $g_2\left(\frac{\pi}{2}\right) = 0$. In this case we find $g_2(x) = ae^{-x}\cos x$.

Now we construct the Green's function. So far, we have the piecewise defined function

$$G(x, \xi) = \begin{cases} be^{-x}\sin x, & 0 \leq x \leq \xi, \\ ae^{-x}\cos x, & \xi \leq x \leq \frac{\pi}{2}. \end{cases} \tag{8.81}$$

The first condition is that $G(x, \xi)$ be continuous at $x = \xi$. This gives

$$be^{-\xi}\sin\xi = ae^{-\xi}\cos\xi.$$

This can be satisfied by defining

$$a = c\sin\xi, \quad b = c\cos\xi.$$

So, we have

$$G(x, \xi) = \begin{cases} ce^{-x}\sin x\cos\xi, & 0 \leq x \leq \xi, \\ ce^{-x}\sin\xi\cos x, & \xi \leq x \leq \frac{\pi}{2}. \end{cases} \tag{8.82}$$

The next condition is that $\frac{\partial G(x,\xi)}{\partial x}$ is discontinuous at $x = \xi$. We show this by integrating Equation (8.79) over the interval $x \in [\xi - \epsilon, \xi + \epsilon]$. Using the definition of the Dirac delta function and continuity of $G(x, \xi)$, we let ϵ approach zero to obtain

$$\begin{aligned} \int_{\xi-\epsilon}^{\xi+\epsilon}\left[\frac{\partial^2 G(x,\xi)}{\partial x^2} + 2\frac{\partial G(x,\xi)}{\partial x} + 2G(x,\xi)\right]dx &= \int_{\xi-\epsilon}^{\xi+\epsilon}\delta(x-\xi)\,dx \\ \lim_{\epsilon\to 0}\left[\frac{\partial G(x,\xi)}{\partial x} + 2G(x,\xi)\right]_{\xi-\epsilon}^{\xi+\epsilon} &= \int_{-\infty}^{\infty}\delta(x-\xi)\,dx \\ \left[\frac{\partial G(x,\xi)}{\partial x}\right]_{\xi^-}^{\xi^+} &= 1 \end{aligned} \tag{8.83}$$

This gives a jump condition for the discontinuity of the derivative at $x = \xi$ where ξ^+ is the value above $x = \xi$ and ξ^- is the value below $x = \xi$.

We can apply this to $G(x, \xi)$. Namely, we have

$$\begin{aligned} 1 &= \left[\frac{\partial G(x,\xi)}{\partial x}\right]_{\xi^-}^{\xi^+} \\ &= ce^{-\xi}\left[-\sin\xi\cos\xi - \sin^2\xi - (-\sin\xi\cos\xi + \cos^2\xi)\right] \\ &= -ce^{-\xi}. \end{aligned}$$

So, $c = -e^{\xi}$ and the Green's function is

$$G(x,\xi) = \begin{cases} -e^{\xi - x}\sin x\cos\xi, & 0 \le x \le \xi, \\ -e^{\xi - x}\sin\xi\cos x, & \xi \le x \le \frac{\pi}{2}. \end{cases} \tag{8.84}$$

We see that $G(x, \xi)$ is not symmetric, $G(x, \xi) \neq G(\xi, x)$. Thus, it seems that $G(x, \xi)$ does not satisfy a reciprocity condition.

Example 8.12. Find the adjoint Green's function satisfying Equation (8.80).

The derivation parallels that for $G(x, \xi)$. However, in this case we start with the general solution $g^A(x) = G^A(x, \xi)$, for

$$g^{A\prime\prime} - 2g^{A\prime} + 2g^A = 0, \quad x \neq \xi, \quad g^A(0) = 0, g^A\left(\frac{\pi}{2}\right) = 0$$

as

$$g^A(x) = e^x(a\cos x + b\sin x).$$

Following the same steps as before, we find the adjoint Green's function,

$$G^A(x,\xi) = \begin{cases} -e^{x - \xi}\sin x\cos\xi, & 0 \le x \le \xi, \\ -e^{x - \xi}\sin\xi\cos x, & \xi \le x \le \frac{\pi}{2}. \end{cases} \tag{8.85}$$

While $G^A(x, \xi) \neq G^A(\xi, x)$, we do have that $G(x, \xi) = G^A(\xi, x)$ gives the form of a reciprocity condition.

Example 8.13. Derive a general relation between $G(x, \xi)$ and $G^A(x, \xi)$.

From the last example we have found $G(x, \xi) = G^A(\xi, x)$ for a specific problem. In general these Green's functions satisfy the equations for $x \in [a, b]$:

$$\begin{aligned} LG(x,\xi) &= \delta(x - \xi) \\ L^A G^A(x,\xi') &= \delta(x - \xi'). \end{aligned} \tag{8.86}$$

Multiply the first equation by $G^A(x, \xi')$ and the second equation by $G(x, \xi)$. Subtract and integrate

$$\begin{aligned} &\int_a^b \left[G^A(x,\xi')LG(x,\xi) - G(x,\xi)L^A G^A(x,\xi')\right]\,dx \\ &= \int_a^b \left[G^A(x,\xi')\delta(x - \xi) - G(x,\xi)\delta(x - \xi')\right]\,dx. \end{aligned} \tag{8.87}$$

Assuming appropriate boundary conditions, we have

$$\int_a^b G^A(x,\xi')LG(x,\xi)\,dx = \int_a^b G(x,\xi)L^AG^A(x,\xi')\,dx.$$

So, after applying the Dirac delta function integrations, we have

$$G^A(\xi,\xi') = G(\xi',\xi).$$

Now we can return to the original problem but adding nonhomogeneous boundary conditions.

Example 8.14. Use the adjoint Green's function to solve

$$y'' + 2y' + 2y = f(x), \quad y(0) = A, y\left(\frac{\pi}{2}\right) = B. \tag{8.88}$$

Defining $L = D^2 + 2D + 2$, we have

$$\begin{aligned} Ly(x) &= f(x) \\ L^AG^A(x,\xi) &= \delta(x-\xi). \end{aligned} \tag{8.89}$$

As with the previous example, we multiply the first equation by $G^A(x,\xi)$ and the second equation by $y(x)$. Subtracting and integrating we have

$$\begin{aligned} &\int_0^{\pi/2} \Big[G^A(x,\xi)\left(y''(x) + 2y'(x) + 2y(x)\right) \\ &\quad -y\left(\frac{\partial^2 G^A(x,\xi)}{\partial x^2} - 2\frac{\partial G^A(x,\xi)}{\partial x} + 2G^A(x,\xi)\right)\Big]\,dx \\ = &\int_0^{\pi/2} \Big[G^A(x,\xi)f(x) - y(x)\delta(x-\xi)\Big]\,dx \end{aligned} \tag{8.90}$$

Cleaning this up, we find that

$$\begin{aligned} &\int_0^{\pi/2} \frac{\partial}{\partial x}\left[G^A(x,\xi)y'(x) - \frac{\partial G^A(x,\xi)}{\partial x}y(x) + 2y(x)G^A(x,\xi)\right]\,dx \\ = &\int_0^{\pi/2} G^A(x,\xi)f(x)\,dx - y(\xi), \end{aligned} \tag{8.91}$$

or, after applying boundary conditions,

$$\begin{aligned} y(\xi) &= \int_0^{\pi/2} G^A(x,\xi)f(x)\,dx \\ &\quad - \left[G^A(x,\xi)y'(x) - \frac{\partial G^A(x,\xi)}{\partial x}y(x) + 2y(x)G^A(x,\xi)\right]_0^{\pi/2} \\ &= \int_0^{\pi/2} G^A(x,\xi)f(x)\,dx + \frac{\partial G^A(\frac{\pi}{2},\xi)}{\partial x}B - \frac{\partial G^A(0,\xi)}{\partial x}A. \end{aligned} \tag{8.92}$$

Since $G^A(x,\xi) = G(\xi,x)$, we can exchange variables to obtain the solution in terms of the Green's function, $G(x,\xi)$:

$$\begin{aligned} y(x) &= \int_0^{\pi/2} G^A(\xi,x)f(\xi)\,d\xi + \frac{\partial G^A(\frac{\pi}{2},x)}{\partial \xi}B - \frac{\partial G^A(0,x)}{\partial \xi}A \\ &= \int_0^{\pi/2} G(x,\xi)f(\xi)\,d\xi + \frac{\partial G(x,\frac{\pi}{2})}{\partial \xi}B - \frac{\partial G(x,0)}{\partial \xi}A. \end{aligned} \tag{8.93}$$

We now apply this general solution to a specific problem.

Example 8.15. Use Equation (8.93) with the Green's function in Equation (8.84) to solve

$$y'' + 2y' + 2y = 5\sin x, \quad y(0) = 2e, y\left(\frac{\pi}{2}\right) = 0. \tag{8.94}$$

$$\begin{aligned}
y(x) &= \int_0^{\pi/2} G(x,\xi) f(\xi)\, d\xi + \frac{\partial G(x,\frac{\pi}{2})}{\partial \xi} B - \frac{\partial G(x,0)}{\partial \xi} A \\
&= \int_0^x G(x,\xi) f(\xi)\, d\xi + \int_x^{\pi/2} G(x,\xi) f(\xi)\, d\xi - 2e \frac{\partial G(x,0)}{\partial \xi} \\
&= \int_0^x \left[-e^{\xi - x}\sin\xi\cos x\right] 5\sin(\xi)\, d\xi + \int_x^{\pi/2} \left[-e^{\xi-x}\sin x\cos\xi\right] 5\sin(\xi)\, d\xi \\
&\quad -2e\frac{\partial}{\partial \xi}\left[-e^{\xi-x}\sin\xi\cos x\right]_{\xi=0} \\
&= -5e^{-x}\cos x \int_0^x e^{\xi}\sin^2\xi\, d\xi - 5e^{-x}\sin x \int_x^{\pi/2} e^{\xi}\sin\xi\cos\xi\, d\xi + 2e^{1-x}\cos x \\
&= -5e^{-x}\left[\left(-\frac{2}{5} + \frac{1}{5}e^x\sin^2 x - \frac{2}{5}e^x\sin x\cos x + \frac{2}{5}e^x\right)\cos x \right. \\
&\quad \left. + \left(-\frac{1}{10}e^x\sin 2x + (\frac{1}{5}e^x\cos 2x + \frac{1}{5}e^{\frac{\pi}{2}}\right)\sin x\right] + 2e^{1-x}\cos x \\
&= 2e^{-x}\cos x - 2\cos x + \sin x - e^{\frac{\pi}{2}-x}\sin x + 2e^{1-x}\cos x \\
&= 2\left[e^{-x}(1+e) - 1\right]\cos x + \left(1 - e^{\frac{\pi}{2}-x}\right)\sin x.
\end{aligned} \tag{8.95}$$

We can check this solution by using the Method of Undetermined Coefficients to obtain the solution. We know the solution to the homogeneous equation is

$$y_h(x) = e^{-x}(c_1\sin x + c_2\cos x).$$

We seek a particular solution,

$$y_p(x) = c_3\sin x + c_4\cos x.$$

Inserting into the differential equation, we have

$$[c_3 - 2c_4]\sin x + [c_4 + 2c_3]\cos x = 5\sin x.$$

This is true when $c_3 = 1$ and $c_4 = -2$. So, the general solution to the nonhomogeneous equation is

$$y(x) = e^{-x}(c_1\sin x + c_2\cos x) + \sin x - 2\cos x.$$

For the solution of the boundary value problem, we need to satisfy the boundary conditions.

$$\begin{aligned}
y(0) &= c_2 - 2 = 2e, \\
y\left(\frac{\pi}{2}\right) &= e^{-\frac{\pi}{2}}c_1 + 1 = 0.
\end{aligned} \tag{8.96}$$

So, $c_2 = 2(1+e)$ and $c_1 = -e^{\frac{\pi}{2}}$ and the solution is

$$\begin{aligned} y(x) &= e^{-x}(-e^{\frac{\pi}{2}}\sin x + 2(1+e)\cos x) + \sin x - 2\cos x \\ &= 2\left[e^{-x}(1+e) - 1\right]\cos x + \left(1 - e^{\frac{\pi}{2}-x}\right)\sin x. \end{aligned} \tag{8.97}$$

So, the solutions agree.

We have seen how we can solve for and use the Green's function and adjoint Green's function in an example inolving a non-Hermitian operator. However, we also know that we can cast the problem in Sturm-Liouville form. So, how do these methods differ if we used the Sturm-Liouville operator and its Green's function?

Example 8.16. Consider the boundary value problem

$$y'' + 2y' + 2y = f(x), \quad y(0) = A, y\left(\frac{\pi}{2}\right) = B. \tag{8.98}$$

Put this in Sturm-Liouville form, find its Green's function, and write the solution in terms of the Green's function.

The Sturm-Liouville form of the differential equation is

$$\frac{d}{dx}\left(e^{2x}\frac{dy(x)}{dx}\right) + 2e^{2x}y(x) = e^{2x}f(x). \tag{8.99}$$

The associated Green's function would then satisfy

$$\frac{\partial}{\partial x}\left(e^{2x}\frac{\partial \mathcal{G}(x,\xi)}{\partial x}\right) + 2e^{2x}\mathcal{G}(x,\xi) = \delta(x-\xi). \tag{8.100}$$

Following Example 8.11, we find the Green's function satisfying the homogeneous boundary conditions, $G(0,\xi) = 0$, $G\left(\frac{\pi}{2},\xi\right) = 0$, takes the form

$$\mathcal{G}(x,\xi) = \begin{cases} ce^{-x}\sin x\cos\xi, & 0 \le x \le \xi, \\ ce^{-x}\sin\xi\cos x, & \xi \le x \le \frac{\pi}{2}. \end{cases} \tag{8.101}$$

So far, this function is continuous at $x = \xi$.

We need to derive the jump condition for the discontinuity of the derivative of the Green's function. As before, we integrate the Green's function equation over the interval $x \in [\xi - \epsilon, \xi + \epsilon]$ to obtain

$$\begin{aligned} \int_{\xi-\epsilon}^{\xi+\epsilon}\left[\frac{\partial}{\partial x}\left(e^{2x}\frac{\partial \mathcal{G}(x,\xi)}{\partial x}\right) + 2e^{2x}\mathcal{G}(x,\xi)\right]dx &= \int_{\xi-\epsilon}^{\xi+\epsilon}\delta(x-\xi)\,dx \\ \lim_{\epsilon\to 0}\left[e^{2x}\frac{\partial \mathcal{G}(x,\xi)}{\partial x}\right]_{\xi-\epsilon}^{\xi+\epsilon} &= \int_{-\infty}^{\infty}\delta(x-\xi)\,dx \\ \left[e^{2x}\frac{\partial \mathcal{G}(x,\xi)}{\partial x}\right]_{\xi^-}^{\xi^+} &= 1 \end{aligned} \tag{8.102}$$

We can apply this to $\mathcal{G}(x,\xi)$. Namely, we have

$$\begin{aligned} 1 &= \left[e^{2x}\frac{\partial \mathcal{G}(x,\xi)}{\partial x}\right]_{\xi^-}^{\xi^+} \\ &= ce^{2\xi}e^{-\xi}\left[-\sin\xi\cos\xi - \sin^2\xi - (-\sin\xi\cos\xi + \cos^2\xi)\right] \\ &= -ce^{\xi}. \end{aligned}$$

So, $c = -e^{-\xi}$ and the Green's function is

$$\mathcal{G}(x,\xi) = \begin{cases} -e^{-x-\xi}\sin x\cos\xi, & 0 \le x \le \xi, \\ -e^{-x-\xi}\sin\xi\cos x, & \xi \le x \le \frac{\pi}{2}. \end{cases} \tag{8.103}$$

We see that $\mathcal{G}(x,\xi)$ is symmetric, $\mathcal{G}(x,\xi) = \mathcal{G}(\xi,x)$.

We can now find the solution to Equation (8.98) using this Green's function. Defining

$$\mathcal{L} = \frac{d}{dx}\left(e^{2x}\frac{d}{dx}\right) + 2e^{2x},$$

$y(x)$ and $\mathcal{G}$ satisfy the differential equations

$$\begin{aligned} \mathcal{L}y(x) &= e^{2x}f(x) \\ \mathcal{L}\mathcal{G}(x,\xi) &= \delta(x-\xi). \end{aligned} \tag{8.104}$$

As with the previous example, we multiply the first equation by $\mathcal{G}(x,\xi)$ and the second equation by $y(x)$. Subtracting and integrating we have

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \left[\mathcal{G}(x,\xi)\mathcal{L}y(x) - y(x)\mathcal{L}\mathcal{G}(x,\xi)\right] dx &= \int_0^{\frac{\pi}{2}} \left[e^{2x}f(x)\mathcal{G}(x,\xi) - y(x)\delta(x-\xi)\right] dx \\ \left[e^{2x}\left(\mathcal{G}(x,\xi)\frac{dy(x)}{dx} - y(x)\frac{\partial\mathcal{G}(x,\xi)}{\partial x}\right)\right]_0^{\frac{\pi}{2}} &= \int_0^{\frac{\pi}{2}} e^{2x}f(x)\mathcal{G}(x,\xi)\,dx - y(\xi) \\ y(\xi) &= \int_0^{\frac{\pi}{2}} e^{2x}f(x)\mathcal{G}(x,\xi)\,dx \\ &\quad + e^{\pi}B\frac{\partial\mathcal{G}(\frac{\pi}{2},\xi)}{\partial x} - A\frac{\partial\mathcal{G}(0,\xi)}{\partial x}. \end{aligned} \tag{8.105}$$

Computing the derivative of the Green's function,

$$\frac{\partial\mathcal{G}(x,\xi)}{\partial x} = \begin{cases} e^{-x-\xi}(\sin x\cos\xi - \cos x\cos\xi), & 0 \le x \le \xi, \\ e^{-x-\xi}(\sin\xi\cos x + \sin\xi\sin x), & \xi \le x \le \frac{\pi}{2}. \end{cases} \tag{8.106}$$

and evaluating it at the boundary points, we have

$$\frac{\partial\mathcal{G}(0,\xi)}{\partial x} = -e^{-\xi}\cos\xi, \quad \frac{\partial\mathcal{G}(\frac{\pi}{2},\xi)}{\partial x} = e^{-\frac{\pi}{2}-\xi}\sin\xi.$$

The solution can be written as

$$y(\xi) = \int_0^{\frac{\pi}{2}} e^{2x}f(x)\mathcal{G}(x,\xi)\,dx + Be^{\frac{\pi}{2}-\xi}\sin\xi + Ae^{-\xi}\cos\xi$$

or

$$y(x) = \int_0^{\frac{\pi}{2}} e^{2\xi}f(\xi)\mathcal{G}(x,\xi)\,dx + Be^{\frac{\pi}{2}-x}\sin x + Ae^{-x}\cos x.$$

Let's compare this with the solution in Equation (8.93). We need

$$\frac{\partial G(x,\xi)}{\partial\xi} = \begin{cases} -e^{\xi-x}(\sin x\cos\xi - \sin x\sin\xi), & 0 \le x \le \xi, \\ -e^{\xi-x}(\sin\xi\cos x + \cos\xi\cos x), & \xi \le x \le \frac{\pi}{2}. \end{cases} \tag{8.107}$$

Then,

$$\frac{\partial G(x,0)}{\partial \xi} = -e^{-x}\cos x, \quad \frac{\partial G(x,\frac{\pi}{2})}{\partial x} = e^{\frac{\pi}{2}-x}\sin x.$$

Inserting these values and noting that $G(x,\xi) = e^{2\xi}\mathcal{G}(x,\xi)$

$$\begin{aligned} y(x) &= \int_0^{\pi/2} G(x,\xi)f(\xi)\,d\xi + \frac{\partial G(x,\frac{\pi}{2})}{\partial \xi}B - \frac{\partial G(x,0)}{\partial \xi}A \\ &= \int_0^{\pi/2} e^{2\xi}\mathcal{G}(x,\xi)f(\xi)\,d\xi + Be^{\frac{\pi}{2}-x}\sin x + Ae^{-x}\cos x. \end{aligned} \tag{8.108}$$

Thus, the solutions using the Sturm-Liouville form of the problem and the original form are the same.

8.7 Series Representations of Green's Functions

THERE ARE TIMES THAT IT MIGHT NOT BE SO SIMPLE to find the Green's function in the simple closed form that we have seen so far. However, there is a method for determining the Green's functions of Sturm-Liouville boundary value problems in the form of an eigenfunction expansion. We will finish our discussion of Green's functions for ordinary differential equations by showing how one obtains such series representations. (Note that we are really just repeating the steps towards developing eigenfunction expansion which we had seen in Chapter 6.3.)

We will make use of the complete set of eigenfunctions of the differential operator, $\mathcal{L}$, satisfying the homogeneous boundary conditions:

$$\mathcal{L}[\phi_n] = -\lambda_n\sigma\phi_n, \quad n = 1,2,\ldots$$

We want to find the particular solution y satisfying $\mathcal{L}[y] = f$ and homogeneous boundary conditions. We assume that

$$y(x) = \sum_{n=1}^{\infty} a_n\phi_n(x).$$

Inserting this into the differential equation, we obtain

$$\mathcal{L}[y] = \sum_{n=1}^{\infty} a_n\mathcal{L}[\phi_n] = -\sum_{n=1}^{\infty}\lambda_n a_n\sigma\phi_n = f.$$

This has resulted in the generalized Fourier expansion

$$f(x) = \sum_{n=1}^{\infty} c_n\sigma\phi_n(x)$$

with coefficients

$$c_n = -\lambda_n a_n.$$

We have seen how to compute these coefficients earlier in the text. We multiply both sides by $\phi_k(x)$ and integrate. Using the orthogonality of the eigenfunctions,

$$\int_a^b \phi_n(x)\phi_k(x)\sigma(x)\,dx = N_k\delta_{nk},$$

one obtains the expansion coefficients (if $\lambda_k \neq 0$)

$$a_k = -\frac{(f, \phi_k)}{N_k \lambda_k},$$

where $(f, \phi_k) \equiv \int_a^b f(x)\phi_k(x)\,dx$.

As before, we can rearrange the solution to obtain the Green's function. Namely, we have

$$y(x) = \sum_{n=1}^{\infty} \frac{(f, \phi_n)}{-N_n \lambda_n} \phi_n(x) = \int_a^b \underbrace{\sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{-N_n \lambda_n}}_{G(x,\xi)} f(\xi)\,d\xi$$

Therefore, we have found the Green's function as an expansion in the eigenfunctions:

$$G(x, \xi) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{-\lambda_n N_n}. \tag{8.109}$$

Example 8.17. Eigenfunction Expansion Example

We will conclude this discussion with an example. Consider the boundary value problem

$$y'' + 4y = x^2, \quad x \in (0, 1), \quad y(0) = y(1) = 0.$$

The Green's function for this problem can be constructed fairly quickly for this problem once the eigenvalue problem is solved. We will solve this problem three different ways in order to summarize the methods we have used in the text.

The eigenvalue problem is

$$\phi''(x) + 4\phi(x) = -\lambda\phi(x),$$

where $\phi(0) = 0$ and $\phi(1) = 0$. The general solution is obtained by rewriting the equation as

$$\phi''(x) + k^2\phi(x) = 0,$$

where

$$k^2 = 4 + \lambda.$$

Solutions satisfying the boundary condition at $x = 0$ are of the form

$$\phi(x) = A \sin kx.$$

Forcing $\phi(1) = 0$ gives

$$0 = A \sin k \Rightarrow k = n\pi, \quad k = 1, 2, 3 \ldots.$$

So, the eigenvalues are

$$\lambda_n = n^2\pi^2 - 4, \quad n = 1, 2, \ldots$$

and the eigenfunctions are

$$\phi_n = \sin n\pi x, \quad n = 1, 2, \ldots.$$

We need the normalization constant, N_n. We have that

$$N_n = \|\phi_n\|^2 = \int_0^1 \sin^2 n\pi x = \frac{1}{2}.$$

We can now construct the Green's function for this problem using Equation (8.109).

$$G(x,\xi) = 2\sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi\xi}{(4-n^2\pi^2)}. \tag{8.110}$$

We can use this Green's function to determine the solution of the boundary value problem. Thus, we have

$$\begin{aligned} y(x) &= \int_0^1 G(x,\xi) f(\xi)\, d\xi \\ &= \int_0^1 \left(2\sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi\xi}{(4-n^2\pi^2)} \right) \xi^2\, d\xi \\ &= 2\sum_{n=1}^{\infty} \frac{\sin n\pi x}{(4-n^2\pi^2)} \int_0^1 \xi^2 \sin n\pi\xi\, d\xi \\ &= 2\sum_{n=1}^{\infty} \frac{\sin n\pi x}{(4-n^2\pi^2)} \left[\frac{(2-n^2\pi^2)(-1)^n - 2}{n^3\pi^3} \right]. \end{aligned} \tag{8.111}$$

We can compare this solution to the one one would obtain if we did not employ Green's functions directly. The eigenfunction expansion method for solving boundary value problems, which we saw in Section 6.3 proceeds as follows. We assume that the solution is in the form

$$y(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

Inserting this into the differential equation $\mathcal{L}[y] = x^2$ gives

$$\begin{aligned} x^2 &= \mathcal{L}\left[\sum_{n=1}^{\infty} c_n \sin n\pi x \right] \\ &= \sum_{n=1}^{\infty} c_n \left[\frac{d^2}{dx^2}(\sin n\pi x) + 4\sin n\pi x \right] \\ &= \sum_{n=1}^{\infty} c_n [4 - n^2\pi^2] \sin n\pi x \end{aligned} \tag{8.112}$$

We need the Fourier sine series expansion of x^2 on $[0,1]$ in order to determine the c_n's. The Fourier sine series coefficients are given by

$$\begin{aligned} b_n &= \frac{2}{1}\int_0^1 x^2 \sin n\pi x \\ &= 2\left[\frac{(2-n^2\pi^2)(-1)^n - 2}{n^3\pi^3} \right], \quad n = 1, 2, \ldots. \end{aligned} \tag{8.113}$$

Therefore,

$$x^2 = 2\sum_{n=1}^{\infty} \left[\frac{(2-n^2\pi^2)(-1)^n - 2}{n^3\pi^3} \right] \sin n\pi x.$$

Inserting this series in Equation (8.112), we find

$$2\sum_{n=1}^{\infty}\left[\frac{(2-n^2\pi^2)(-1)^n-2}{n^3\pi^3}\right]\sin n\pi x=\sum_{n=1}^{\infty}c_n[4-n^2\pi^2]\sin n\pi x.$$

Due to the linear independence of the eigenfunctions, we can solve for the unknown coefficients to obtain

$$c_n=2\frac{(2-n^2\pi^2)(-1)^n-2}{(4-n^2\pi^2)n^3\pi^3}.$$

Therefore, the solution using the eigenfunction expansion method is

$$\begin{aligned}y(x) &= \sum_{n=1}^{\infty}c_n\phi_n(x)\\ &= 2\sum_{n=1}^{\infty}\frac{\sin n\pi x}{(4-n^2\pi^2)}\left[\frac{(2-n^2\pi^2)(-1)^n-2}{n^3\pi^3}\right].\end{aligned}\tag{8.114}$$

We note that this is the same solution as in Equation 8.111 which we had obtained using the Green's function as an eigenfunction expansion.

One remaining question is the following: Is there a closed form for the Green's function and the solution to this problem? The answer is yes! We note that the differential operator is a special case of the Example 8.5 done is section 8.4.2. Namely, we pick $\omega=2$. The Green's function was already found in that section. For this special case, we have

$$G(x,\xi)=\begin{cases}-\dfrac{\sin 2(1-\xi)\sin 2x}{2\sin 2}, & 0\le x\le\xi\\ -\dfrac{\sin 2(1-x)\sin 2\xi}{2\sin 2}, & \xi\le x\le 1\end{cases}.\tag{8.115}$$

What about the solution to the boundary value problem? This solution is given by

$$\begin{aligned}&y(x)\\ =&\int_0^1 G(x,\xi)f(\xi)\,d\xi\\ =&-\int_0^x\frac{\sin 2(1-x)\sin 2\xi}{2\sin 2}\xi^2\,d\xi+\int_x^1\frac{\sin 2(\xi-1)\sin 2x}{2\sin 2}\xi^2\,d\xi\\ =&-\frac{1}{4\sin 2}\left[-x^2\sin 2-\sin 2\cos^2 x+\sin 2+\cos 2\sin x\cos x+\sin x\cos x\right].\\ =&-\frac{1}{4\sin 2}\left[-x^2\sin 2+(1-\cos^2 x)\sin 2+\sin x\cos x(1+\cos 2)\right].\\ =&-\frac{1}{4\sin 2}\left[-x^2\sin 2+2\sin^2 x\sin 1\cos 1+2\sin x\cos x\cos^2 1)\right].\\ =&-\frac{1}{8\sin 1\cos 1}\left[-x^2\sin 2+2\sin x\cos 1(\sin x\sin 1+\cos x\cos 1)\right].\\ =&\frac{x^2}{4}-\frac{\sin x\cos(1-x)}{4\sin 1}.\end{aligned}\tag{8.116}$$

In Figure 8.5 we show a plot of this solution along with the first five terms of the series solution. The series solution converges quickly.

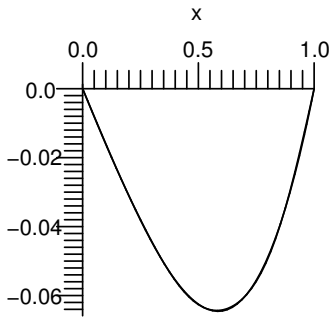


Figure 8.5: Plots of the exact solution to Example 8.17 with the first five terms of the series solution.

As one last check, we solve the boundary value problem directly using the Method of Undetermined Coefficients. Again, the problem is

$$y'' + 4y = x^2, \quad x \in (0,1), \quad y(0) = y(1) = 0.$$

The problem has the general solution

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + y_p(x),$$

where y_p is a particular solution of the nonhomogeneous differential equation. Using the Method of Undetermined Coefficients, we assume a solution of the form

$$y_p(x) = Ax^2 + Bx + C.$$

Inserting this in the nonhomogeneous equation, we have

$$2A + 4(Ax^2 + Bx + C) = x^2,$$

Thus, $B = 0$, $4A = 1$ and $2A + 4C = 0$. The solution of this system is

$$A = \frac{1}{4}, \quad B = 0, \quad C = -\frac{1}{8}.$$

So, the general solution of the nonhomogeneous differential equation is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{x^2}{4} - \frac{1}{8}.$$

We now determine the arbitrary constants using the boundary conditions. We have

$$\begin{aligned} 0 &= y(0) \\ &= c_1 - \frac{1}{8} \\ 0 &= y(1) \\ &= c_1 \cos 2 + c_2 \sin 2 + \frac{1}{8} \end{aligned} \tag{8.117}$$

Thus, $c_1 = \frac{1}{8}$ and

$$c_2 = -\frac{\frac{1}{8} + \frac{1}{8}\cos 2}{\sin 2}.$$

Inserting these constants in the solution we find the same solution as before.

$$\begin{aligned} y(x) &= \frac{1}{8}\cos 2x - \left[\frac{\frac{1}{8}+\frac{1}{8}\cos 2}{\sin 2}\right]\sin 2x + \frac{x^2}{4} - \frac{1}{8} \\ &= \frac{\cos 2x \sin 2 - \sin 2x \cos 2 - \sin 2x}{8\sin 2} + \frac{x^2}{4} - \frac{1}{8} \\ &= -\frac{\sin^2 x \sin 1 + \sin x \cos x \cos 1}{4\sin 1} + \frac{x^2}{4} \\ &= \frac{x^2}{4} - \frac{\sin x \cos(1-x)}{4\sin 1}. \end{aligned} \tag{8.118}$$

8.8 Appendix: The Rise of Green's Functions

GREEN'S FUNCTIONS WERE FIRST INTRODUCED by the British mathematician George Green (1793-1841) in 1828 [Green, 1889]. George Green sought solutions of Poisson's equation

$$\nabla^2 u(\mathbf{r}) = f(\mathbf{r})$$

for the electric potential u defined inside a bounded volume with specified boundary conditions on the surface of the volume. He introduced a function now identified as what Riemann later coined the "Green's function"[6] [Cannell, 2001, Cannell and Lord, 1993, Duffy, 2015]. However, the concept of Green's functions was not widely understood or used until much later, and it wasn't until the 20th century that they became a central tool in many areas of physics and applied mathematics.[7]

George Green was self-taught as he had about one year of formal schooling between 8 and 9. [See [Cannell, 2001, Cannell and Lord, 1993].] He lived most of his life in Sneinton, Nottinghamshire. His father, George, was a baker who built and owned a brick windmill to grind grain. In 1828 the younger George Green published his famous essay, *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*. He published it privately at his expense and sold 51 copies to mostly friends. [For more on the history, see [Cannell, 2001, Cannell and Lord, 1993, Challis and Sheard, 2003, Tazzioli, 2001].]

Green's father died in 1829 and left an inheritance to his son and daughter. Green later met a wealthy landowner and mathematician, Edward Bromhead (1789 - 1855), who had bought a copy of the essay and encouraged Green to further his studies. But, Green did not contact Bromhead for two years when in 1833 Green enrolled in Gonville and Caius College, Cambridge. He wrote several papers but in his final years at Cambridge, Green became ill and in 1840 he returned to Sneinton, only to die a year later [Cannell, 1999, 2001, Cannell and Lord, 1993, Grattan-Guinness, 1995].

Green's work was not well known during his lifetime. He had done little to promote his essay. But he had shared copies with Carl Jacobi (1804-1851) and William Hopkins (1793-1866) [Cannell and Lord, 1993, Craik, 2008,

Adapted from .

Russell L. Herman. Laplace Transforms and Green's Functions. In V. Martinez-Luaces, editor, *Engineering Problems and the Laplace Transform*, pages 1–112. Nova Science, 2025

[6] Grattan-Guinness [1995] suggests that both Riemann and Carl Neumann (1832-1925) contributed to the name.

[7] Green also introduced potential theory and Green's Theorem in his studies of electricity and magnetism.

Grattan-Guinness, 1995]. In 1845 a young William Thomson (1824-1907), later to be known as Lord Kelvin and who was interested in electricity and magnetism, saw a footnote in Robert Murphy's (1806-1843) 1832 paper, *On the inverse method of definite integrals, with physical applications,* referring to Green's essay. He searched for the essay and on the evening before going to Paris on a fellowship he had obtained copies of the essay from William Hopkins [8] . While in Paris he met with a number of mathematicians sharing what he had found, including Joseph Liouville (1809-1882), Michel Chasles (1793-1880). Jacques Charles François Sturm (1803-1855), Augustin-Louis Cauchy (1789-1857) and Jean-Baptiste Biot (1775-1862).

[8] William Hopkins (1793-1866) was famous for coaching students at Cambridge University to become senior wranglers by scoring the best on the Mathematical Tripos. Thomson and Stokes were two of many to later be notable mathematicians.

Thomson saw the importance of Green's work and saw to it that it was published and popularised. The essay was published in three parts in Crelle's *Journal* 1850, 1852, and 1854 [Cannell, 1999]. In 1871 Norman M. Ferrers (1829 - 1903) assembled *The Mathematical Papers of the Late George Green*[9] [Green, 1871a].

[9] Green's essay can now be found at arXiv:0807.0088.

There were several contributions that Green made in these papers. In his 1928 essay Green introduced potential functions, although there were others working in the area [Grattan-Guinness, 1995].It is believed that Green coined the word "potential." He also introduced what we now call Green's Theorem and Green's identities. He made use of these in the introduction of what we now call Green's functions.

Green later wrote a paper on electrostatics and then turned to attractions of ellipsoids, elasticity, and water waves. In one of his papers, *On the Motion of Waves in a Variable Canal ...,* Green gave an early version of the Wentzel–Kramers–Brillouin (WKB) approximation[10] [Merzbacher, 1998, Berera and Del Debbio, 2021]. He wrote on the motion of small water waves in a thin, shallow rectangular canal. According to Cannell and Lord [1993], he was first to note that "particles in a deep sea wave execute circular motion" and was an early commentator on John Scott Russell (1808 - 1882) work on solitary waves [Green, 1871b, Herman, 1992]. Also, in 1839 Green contributed to the study of vibrations of an elastic solid and the Cauchy-Green tensor is named after him.

[10] The WKB Approximation, named after Wentzel–Kramers–Brillouin, is a method to approximate solutions to linear differential equations with spatially varying coefficients. According to Schlissel [1977], this method was discussed by several researchers such as George Green (1793-1841), Joseph Liouville (1809-1882), Lord Rayleigh (1842-1909), Richard Gans (1880-1954), Harold Jeffreys (1891-1989), only to have the names of Gregor Wentzel (1898-1978), Henrich A. Kramers (1884-1952), and Leon Brillouin (1889-1969) stick as the "WKB" approximation (or, method).

William Thomson and George Stokes (1819-1903), both prominent mathematicians and physicists of the 19th century, did not explicitly use the term "Green's function" in their work, as the concept was not yet widely understood or named as such. However, they both made significant contributions to the development of the underlying mathematical concepts that would later become known as Green's functions.

Thomson, for example, developed what is now known as the Kelvin method, which is a technique for solving Laplace's equation by decomposing the solution into a sum of simpler functions. This method is closely related to the use of Green's functions to solve partial differential equations. Thomson also introduced the method of images and the Kelvin transformation, or inversion in a sphere.

Stokes, on the other hand, made important contributions to the study of fluid dynamics and the behavior of viscous fluids. He introduced what is

now known as the Stokes equation, which describes the motion of a viscous fluid in response to an external force. The solution to this equation involves the use of integral operators that are closely related to Green's functions. He is best know to calculus students for an integral theorem, Stokes' Theorem. [11]

So while neither Thomson nor Stokes explicitly used the term "Green's function" in their work, they both made important contributions to the underlying mathematical concepts and techniques that would later become central to the use of Green's functions in physics and applied mathematics. Also, Kirchhoff [1857, 1883] mentions Green's functions, has a delta function and references work by J. Fröhlich [1878] and Woldemar Voigt (1850–1918)[12] [Voigt, 1878]. Gustav Kirchhoff (1824-1887) introduced what is now known as the Kirchhoff integral theorem, which is essentially a form of Green's function. Similarly, Bernhard Riemann (1826-1866) introduced the concept of a "potentialschrift," which is essentially a Green's function for the Laplace equation. In [Riemann, 1857] he used Abelian integrals to determine a Green's function. Bottazzini and Tazzioli [1995] describe Riemann's work in natural philosophy (physics) and how his "complex function theory is deeply connected with potential theory in two dimensions."

These researchers used Green's functions to study a range of physical phenomena, such as the propagation of waves [Riemann, 1860], the behavior of electric and magnetic fields, and the diffusion of heat. One of the earliest uses of Green's functions for the heat equation was by Bernhard Minnigerode (1837–1896)) [Minnigerode, 1862, 1886].

So while the term "Green's function" may not have been in use in the 1800s, the concept itself was certainly being explored and applied by a number of researchers in the field of mathematics and physics. More recently, Green's functions found there way into quantum field theory. In 1928, P. A. M. Dirac (1902-1984) used Green's functions to develop his theory of quantum mechanics [Dirac, 1928]. Richard Feynman (1918-1988) used Green's functions to develop his path integral formulation of quantum mechanics [Feynman, 1950, Feynman et al., 2010]. Julian Schwinger (1918-1994) used Green's functions to develop his theory of quantum electrodynamics [Schwinger, 1948, 2001] and Freeman Dyson (1923-2020) [Dyson, 1949, 2001] added to the understanding of these works. The Schwinger–Dyson equation is used to find the Green functions of quantum field theories. Green's functions are also commonly taught in electromagnetism [Franklin, 2017].

[11] Histories of this theorem can be found in [Crowe, 1984, Katz, 1979]. Maxwell sat for the Smith Prize in 1854 and took first place with Edward Routh and Maxwell [1873], stated "This theorem was given by Professor Stokes. Smith's Prize Examination, 1854, question 8. It is proved in Thomson and Tait's Natural Philosophy, § 190 (f)" [Kelvin and Tait, 1867]. Copies of Smith Prize questions are in [Stokes, 1905]. [Tait, 1870] later relates Stokes' Theorem to Green's Theorem using quaternions and the ∇ operator.

[12] Kirchhoff, Voight and Minnigerode were students of Franz Neumann (1798-1895), father of Carl Neumann.

Problems

1. Use the Method of Variation of Parameters to determine the general solution for the following problems.

a. $y'' + y = \tan x$.

b. $y'' - 4y' + 4y = 6xe^{2x}$.

2. Instead of assuming that $c_1'y_1 + c_2'y_2 = 0$ in the derivation of the solution using Variation of Parameters, assume that $c_1'y_1 + c_2'y_2 = h(x)$ for an arbitrary function $h(x)$ and show that one gets the same particular solution.

3. Find the solution of each initial value problem using the appropriate initial value Green's function.

a. $y'' - 3y' + 2y = 20e^{-2x}, \quad y(0) = 0, \quad y'(0) = 6.$

b. $y'' + y = 2\sin 3x, \quad y(0) = 5, \quad y'(0) = 0.$

c. $y'' + y = 1 + 2\cos x, \quad y(0) = 2, \quad y'(0) = 0.$

d. $x^2y'' - 2xy' + 2y = 3x^2 - x, \quad y(1) = \pi, \quad y'(1) = 0.$

4. Consider the problem $y'' = \sin x$, $y'(0) = 0$, $y(\pi) = 0$.

a. Solve by direct integration.

b. Determine the Green's function.

c. Solve the boundary value problem using the Green's function.

d. Change the boundary conditions to $y'(0) = 5$, $y(\pi) = -3$.

i. Solve by direct integration.

ii. Solve using the Green's function.

5. Consider the problem:

$$\frac{\partial^2 G}{\partial x^2} = \delta(x - x_0), \quad \frac{\partial G}{\partial x}(0, x_0) = 0, \quad G(\pi, x_0) = 0.$$

a. Solve by direct integration.

b. Compare this result to the Green's function in part b of the last problem.

c. Verify that G is symmetric in its arguments.

6. In this problem you will show that the sequence of functions

$$f_n(x) = \frac{n}{\pi}\left(\frac{1}{1 + n^2x^2}\right)$$

approaches $\delta(x)$ as $n \to \infty$. Use the following to support your argument:

a. Show that $\lim_{n\to\infty} f_n(x) = 0$ for $x \neq 0$.

b. Show that the area under each function is one.

7. Verify that the sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$, defined by $f_n(x) = \frac{n}{2}e^{-n|x|}$, approaches a delta function.

8. Evaluate the following integrals:

a. $\int_0^{\pi} \sin x\,\delta\left(x - \frac{\pi}{2}\right)\,dx.$

b. $\int_{-\infty}^{\infty} \delta\left(\frac{x-5}{3}e^{2x}\right)\left(3x^2 - 7x + 2\right)\,dx.$

c. $\int_0^{\pi} x^2\delta\left(x + \frac{\pi}{2}\right)\,dx.$

d. $\int_0^\infty e^{-2x}\delta(x^2-5x+6)\,dx$. [See Problem 10.]

e. $\int_{-\infty}^{\infty}(x^2-2x+3)\delta(x^2-9)\,dx$. [See Problem 10.]

9. Find a Fourier series representation of the Dirac delta function, $\delta(x)$, on $[-L, L]$.

10. For the case that a function has multiple simple roots, $f(x_i) = 0$, $f'(x_i) \neq 0$, $i = 1, 2, \ldots$, it can be shown that

$$\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|f'(x_i)|}.$$

Use this result to evaluate $\int_{-\infty}^{\infty} \delta(x^2-5x+6)(3x^2-7x+2)\,dx$.

11. Consider the boundary value problem: $y'' - y = x$, $x \in (0,1)$, with boundary conditions $y(0) = y(1) = 0$.

a. Find a closed form solution without using Green's functions.

b. Determine the closed form Green's function using the properties of Green's functions. Use this Green's function to obtain a solution of the boundary value problem.

c. Determine a series representation of the Green's function. Use this Green's function to obtain a solution of the boundary value problem.

d. Confirm that all of the solutions obtained give the same results.

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