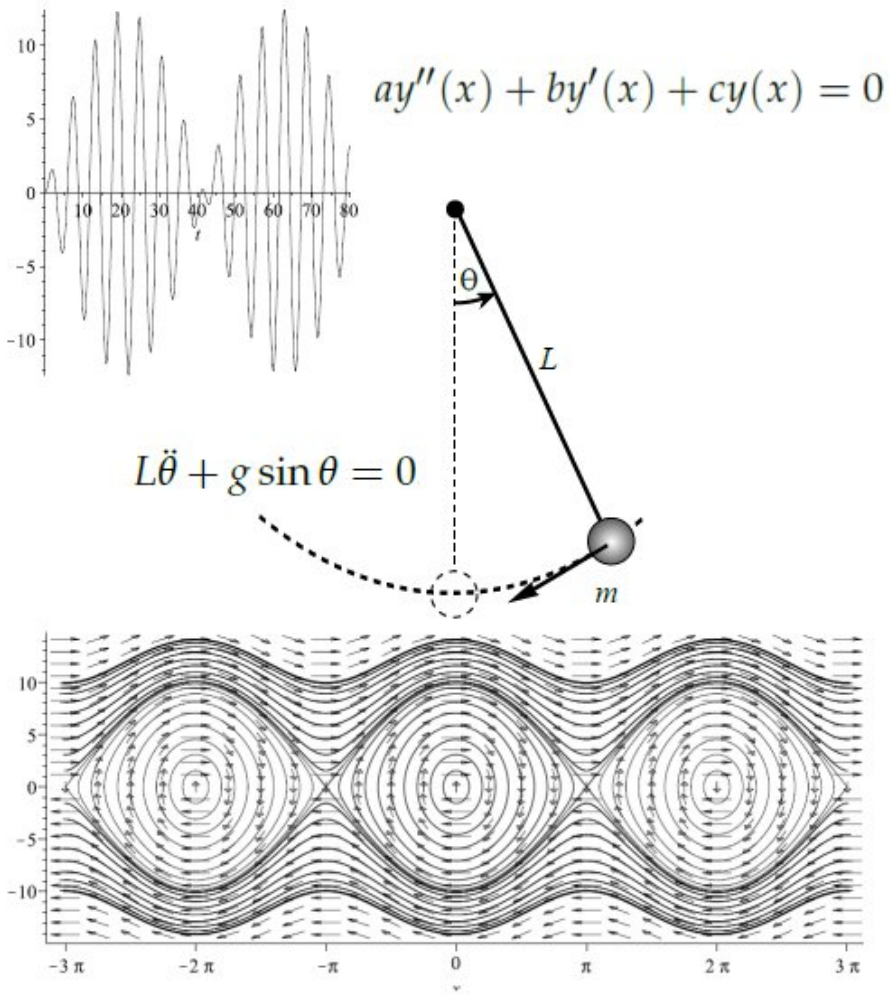
RUSSELL L. HERMAN

A SECOND COURSE IN ORDINARY DIFFERENTIAL EQUATIONS: DYNAMICAL SYSTEMS AND BOUNDARY VALUE PROBLEMS



R. L. HERMAN - VERSION DATE: APRIL 25, 2025

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A SECOND COURSE IN ORDINARY DIFFERENTIAL EQUATIONS: DYNAMICAL SYSTEMS AND BOUNDARY VALUE PROBLEMS by Russell Herman. Versions of these notes have been posted online since Spring 2005.
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April 2025

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Dedicated to those students who have endured previous versions of my notes and to current students who take the time to read this revised edition.