Ordinary Differential Equations | Spring 2025
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Differential Equations Project

Introduction

This document provides a list of project ideas for senior-level undergraduate students to explore systems of nonlinear first-order differential equations. These projects are designed to be completed by groups of two over a period of several weeks. The projects cover a range of applications in various fields, ensuring students can connect mathematics to real-world problems.

Instructions for Groups

The main instructions and due dates are provided at the course website. Each group should:

- Choose one project from the list below. It is expected that groups will work on different problems or variations of a given problem.
- Perform a literature review to understand the context and to determine parameters.
- Classify equilibrium points and or limit cycles.
- Simulate the system using appropriate computational tools (e.g., Maple, MATLAB, Mathematica, Simulink, Python, etc.).
- Analyze the results and discuss the implications.
- Prepare a written report summarizing your findings.

Project Topics

Biology

Predator-Prey Dynamics (Ecology)

Description: Explore and analyze the Lotka-Volterra predator-prey model and its extensions.

$$\begin{aligned}\frac{dx}{dt} &= \alpha x - \beta xy,\\ \frac{dy}{dt} &= \delta xy - \gamma y.\end{aligned}$$

For example, what if one adds logistic growth terms replacing αx and γy?

Competing Species (Ecology)

Description: Investigate competition between two species using coupled logistic equations.

$$
\begin{aligned}
\frac{dx}{dt} &= r_1 x\left(1 - \frac{x+\alpha y}{K_1}\right), \\
\frac{dy}{dt} &= r_2 y\left(1 - \frac{y+\beta x}{K_2}\right).
\end{aligned}
$$

SIR Model for Epidemics (Epidemiology)

Description: Analyze the spread of infectious diseases using the SIR model.

$$
\begin{aligned}
\frac{dS}{dt} &= -\beta S I, \\
\frac{dI}{dt} &= \beta S I - \gamma I, \\
\frac{dR}{dt} &= \gamma I.
\end{aligned}
$$

SIR Model with Vaccination (Epidemiology)

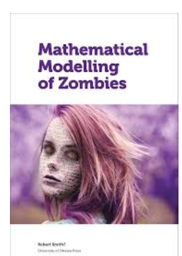
Description: Study disease spread with vaccination:

$$
\begin{aligned}
\frac{dS}{dt} &= -\beta S I - \nu S \\
\frac{dI}{dt} &= \beta S I - \gamma I \\
\frac{dR}{dt} &= \gamma I + \nu S
\end{aligned}
$$

Outbreak of a Zombie Infection (Epidemiology)

Description: In 2009 Munz, Hudea, and Smith[1] wrote a paper on modelling of an outbreak of zombie infection. For reference, see the paper, the talk, or the book. They refer to it as the SZR model, or Susceptible-Zombie-Removed(dead) model.

$$
\begin{aligned}
\frac{dS}{dt} &= \Pi - \beta S Z - \delta S \\
\frac{dI}{dt} &= \beta S Z + \zeta R - \alpha S Z \\
\frac{dR}{dt} &= \delta S + \alpha S Z - \zeta R.
\end{aligned}
$$



[1]Munz, Philip, Ioan Hudea, Joe Imad, and Robert J. Smith. "When zombies attack!: mathematical modelling of an outbreak of zombie infection." *Infectious disease modelling research progress* 4 (2009): 133-150.

There are other models. The SIZR model adds latent infection in which there is a population I of infected but not infectious bodies. You can add a quarantine for an SIZRQ or lok for a treatment therapy.

Chemistry

Coupled Chemical Reactions

Description: Study a system of coupled reactions:

$$\begin{aligned}
\frac{d[A]}{dt} &= -k_1[A] + k_2[B][C] \\
\frac{d[B]}{dt} &= k_1[A] - k_2[B][C] - k_3[B] \\
\frac{d[C]}{dt} &= k_3[B] - k_2[B][C]
\end{aligned}$$

Belousov-Zhabotinsky Reaction

From Pacific Union College

Description: This reaction involves the oscillation between the concentration of $HBrO_2$ and Br^-.

$$BrO_3^- + Br^- \xrightarrow{k_1} HBrO_2 + HOBr \tag{1}$$

$$HBrO_2 + Br^- \xrightarrow{k_2} 2\, HOBr \tag{2}$$

$$BrO_3^- + HBrO_2 \xrightarrow{k_3} 2\, HBrO_2 + 2\, Ce^{4+} \tag{3}$$

$$2\, HBrO_2 \xrightarrow{k_4} BrO_3^- + HOBr \tag{4}$$

$$Ce^{4+} \xrightarrow{k_5} f Br^- \tag{5}$$

Letting $x = [HBrO_2]\,, y = [Br^-]$, and $z = \left[Ce^{4+}\right]$, this reaction leads to the system

$$\frac{dx}{dt} = k_1 ay - k_2 xy + k_3 ax - k_4 x^2 \tag{6}$$

$$\frac{dy}{dt} = -k_1 ay - k_2 xy + f k_5 z \tag{7}$$

$$\frac{dz}{dt} = 2k_3 ax - k_5 z. \tag{8}$$

Michaelis-Menten Kinetics

Description: The Michaelis-Menten kinetics reaction is a simplest approach to enzyme kinetics. A substrate S binds reversibly to an enzyme E to form an enzyme-substrate complex ES, which then reacts irreversibly to generate a product P while regenerating the free enzyme E. The reaction is given as

$$E + S \underset{k_3}{\overset{k_1}{\rightleftharpoons}} ES \underset{k_2}{\longrightarrow} E + P$$

The resulting system of equations for the chemical concentrations becomes

$$\frac{d[S]}{dt} = -k_1[E][S] + k_3[ES] \tag{9}$$

$$\frac{d[E]}{dt} = -k_1[E][S] + (k_2 + k_3)[ES] \tag{10}$$

$$\frac{d[ES]}{dt} = k_1[E][S] - (k_2 + k_3)[ES] \tag{11}$$

$$\frac{d[P]}{dt} = k_2[ES]. \tag{12}$$

Physics

Coupled Spring-Mass Systems (Physics)

Description: Analyze two masses connected by linear springs:

$$\begin{aligned}
\frac{dx_1}{dt} &= v_1 \\
\frac{dv_1}{dt} &= -\frac{k_1}{m_1}x_1 - \frac{k_2}{m_1}(x_1 - x_2) - \frac{c_1}{m_1}v_1 \\
\frac{dx_2}{dt} &= v_2 \\
\frac{dv_2}{dt} &= -\frac{k_2}{m_2}(x_2 - x_1) - \frac{c_2}{m_2}v_2
\end{aligned}$$

Double Pendulum (Physics)

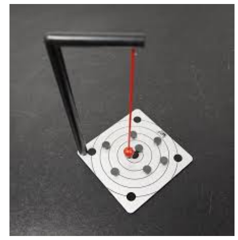
Description: Analyze the chaotic motion of a double pendulum.

$$\begin{aligned}
\frac{d\theta_1}{dt} &= \omega_1 \\
\frac{d\omega_1}{dt} &= -\frac{g}{L_1}\sin(\theta_1) - \frac{m_2 L_2}{m_1 L_1}\sin(\theta_1 - \theta_2)\omega_2^2 \\
\frac{d\theta_2}{dt} &= \omega_2 \\
\frac{d\omega_2}{dt} &= -\frac{g}{L_2}\sin(\theta_2) + \frac{L_1}{L_2}\sin(\theta_1 - \theta_2)\omega_1^2
\end{aligned}$$

Magnetic Pendulum (Physics)

Description: The magnetic pendulum has a small magnet suspended by a string above a base which contains similar magnets. The dynamics can be modelled using simple equations. See the article or IMA article. The equation for three magnets at $\mathbf{X}_n$ and the position of the pendulum magnet at $\mathbf{x}(t)$ is given below. In the equation, h is given as the average height of the swinging magnet above the plane and b is the damping constant.

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} + b\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} + \mathbf{x} = \sum_{n=1}^{3} \frac{\mathbf{X}_n - \mathbf{x}}{(|\mathbf{X}_n - \mathbf{x}|^2 + h^2)^{5/2}}.$$



Lane-Emden and Related Models (Physics)

Description: The Lane-Emden equation is an approximation for self-gravitating spheres of plasma such as stars.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n.$$

Applications:

- Gravitational potential of self-gravitating gas.
- Used by Eddington for the internal constitution of stars.
  - $0.5 \le n \le 1$, neutron stars.
  - $n = 3$, white dwarfs, Sun.
- Thomas-Fermi model of electrons in atoms, $n = \frac{3}{2}$.

The more general Emden-Fowler equation,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\eta}{d\xi}\right) = \alpha\xi^\lambda\eta^n,$$

can be mapped to a two-dimensional autonomous system,

$$\begin{aligned} \dot{X} &= -X(1 + X - \alpha Y) \\ \dot{Y} &= Y(1 + \lambda + nX - \alpha Y). \end{aligned} \tag{13}$$

Here the dot represents $\frac{d}{dt}$.

Electric Circuits

Coupled RLC Circuits - Linear

Description: Analyze coupled electrical circuits. A general from might take the form

$$L_1\frac{di_1}{dt} = -R_1 i_1 - \frac{1}{C_1}\int i_1 dt + M\frac{di_2}{dt}$$
$$L_2\frac{di_2}{dt} = -R_2 i_2 - \frac{1}{C_2}\int i_2 dt + M\frac{di_1}{dt}$$

or its equivalent second order differential equation form.

Chua Circuit

Description: Study the behavior in Chua's circuit:

$$\frac{dx}{dt} = \alpha(y - x - f(x))$$
$$\frac{dy}{dt} = x - y + z$$
$$\frac{dz}{dt} = -\beta y$$

where $f(x) = bx + \frac{1}{2}(a - b)(|x + 1| - |x - 1|)$ is the piecewise-linear characteristic.

Economics

Cournot Duopoly (Economics)

Description: Model competition between two firms in a duopoly market.

In the Cournot duopoly model, two firms produce quantities q_1 and q_2 of a homogeneous good. The price $P(Q)$ is a function of total output $Q = q_1 + q_2$, typically assumed to be linear:

$$P(Q) = a - bQ, \quad a, b > 0.$$

Each firm seeks to maximize its profit, given by:

$$\pi_1 = q_1 P(Q) - C_1(q_1),$$
$$\pi_2 = q_2 P(Q) - C_2(q_2),$$

where $C_i(q_i)$ is the cost function of firm i. Assuming simple cost functions, the reaction functions satisfy:

$$\frac{dq_1}{dt} = \alpha_1 \left(Q - q_1 - \beta_1 q_2\right),$$
$$\frac{dq_2}{dt} = \alpha_2 \left(Q - q_2 - \beta_2 q_1\right).$$

Example: Suppose two firms operate in a market where the demand function is given by $P(Q) = 100 - 2Q$. Each firm has a cost function $C_i(q_i) = 10q_i$. If firms react according to the Cournot model, they will each adjust their output dynamically following:

$$\frac{dq_1}{dt} = 0.5 \left(100 - 2(q_1 + q_2) - q_1\right),$$
$$\frac{dq_2}{dt} = 0.5 \left(100 - 2(q_1 + q_2) - q_2\right).$$

You can explore the equilibrium points and stability properties of these equations through numerical simulations. What variations are possible. Think of some realistic firms and products as example.

Other

Neural Networks (Data Science/Neuroscience)

Description: Explore coupled neuron models such as the FitzHugh-Nagumo system.

$$
\begin{aligned}
\frac{dv}{dt} &= v - \frac{v^3}{3} - w + I, \\
\frac{dw}{dt} &= \epsilon(v + a - bw).
\end{aligned}
$$

This model is a simplifification of the Hodgkin–Huxley model which models the activation and deactivation dynamics of a spiking neuron. Here v is membrane voltage diminished over time by a recovery variable w after stimulation by an external current I.

Lorenz System (Meteorology)

Description: Study the Lorenz equations and their implications for weather prediction.

$$
\begin{aligned}
\frac{dx}{dt} &= \sigma(y - x), \\
\frac{dy}{dt} &= x(\rho - z) - y, \\
\frac{dz}{dt} &= xy - \beta z.
\end{aligned}
$$

Theses equations represent a two-dimensional fluid layer uniformly warmed from below and cooled from above. Here x is proportional to the rate of convection, y to the horizontal temperature variation, and z to the vertical temperature variation.

Replicator Dynamics Evolutionary Game Theory)

Description: Study evolutionary game dynamics:

$$
\frac{dx_i}{dt} = x_i \left((Ax)_i - x^T A x \right), \quad i = 1, \ldots, n
$$

where A is the payoff matrix and x_i represents the frequency of strategy i. This general theory has applications in many areas.

Possible applications

- Analyze stable evolutionary strategies.
- Study Rock-Paper-Scissors dynamics.
- Investigate cooperation emergence.
- Apply to biological competition.
- Apply to economic models.