Summing the Unsumable: Laplace Transforms and Divergent Series from Euler to Feynman

Russell L. Herman

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Abstract

Infinite series have long fascinated mathematicians, from Zeno's paradoxes and Euler's manipulations to the divergent expansions of Stokes and the integral tricks of Feynman. In this article we explore how the Laplace transform provides a unifying framework: it can sum convergent series, give meaning to divergent ones through Borel summation, and reveal surprising connections to physical problems in optics and quantum theory. This account builds on themes developed more fully in [12, 13].

1 Introduction

Certain infinite series have long captured the imagination of mathematicians and scientists. From antiquity onward, they served as both puzzles and tools. Zeno of Elea (490–430 BCE) posed his paradox of Achilles and the tortoise, which today is easily resolved by summing the convergent geometric series [20].

Centuries later, series again drew attention when Pietro Mengoli (1626–1686) proposed the problem of finding

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots,$$

now famous as the Basel problem [17, 18]. Leonhard Euler, in 1734, astonished the mathematical world by showing that this infinite series converges to $\pi^2/6$ [8, 4]. The elegance of Euler's solution, using ideas that anticipated

later developments in analysis, remains one of the celebrated triumphs of eighteenth-century mathematics.

Yet not all series behave so well. The harmonic series,

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots ,$$

was shown by Nicole Oresme in the fourteenth century to diverge. Even more unsettling were series like

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots ,$$

for which mathematicians debated whether any sense could be made at all. In modern times, the provocative identity

$$1 + 2 + 3 + \cdots = -\tfrac{1}{12}$$

has circulated widely, leading to spirited discussions about what it could possibly mean [15].

If these examples highlight the richness and strangeness of infinite series, they also set the stage for the central theme of this article: the role of the Laplace transform as a unifying device for both summing convergent series and assigning meaning to divergent ones. What begins with simple manipulations of integrals leads, unexpectedly, to applications in physics—rainbow supernumeraries, Airy functions, and quantum field theory. The journey from Zeno to Feynman illustrates how ideas in mathematics can evolve from paradox to essential tool.

2 Early Encounters with Divergent Series

The eighteenth century witnessed Euler's bold use of divergent series. In 1746 he considered

$$\sum_{n=0}^{\infty} (-1)^n n! = 1 - 1 + 2 - 6 + 24 - 120 + \cdots ,$$

whose partial sums grow erratically as $1, 0, 2, -4, 20, -100, \ldots$ [7]. By ordinary standards the series diverges, yet Euler succeeded in attaching a finite

value to it by methods that anticipated analytic continuation. His willingness to treat divergent expressions as meaningful foreshadowed the modern theory of summability [4].

Other mathematicians, such as James Stirling, encountered divergent expansions in their attempts to approximate factorials. The famous Stirling series for $\ln n!$ is divergent but when truncated it gives superb approximations. The apparent contradiction—that a divergent expansion can be practically useful—was not lost on physicists of the nineteenth century, especially when similar series appeared in optics and wave theory. George Stokes, studying the fine fringes of light below a rainbow's arc, confronted divergent asymptotic series and laid the groundwork for methods that would later reappear in quantum theory [21, 1, 2].

3 Laplace Transforms as a Summation Tool

Wheelon systematically used Laplace transforms to sum series [23, 24]. Later expositions and extensions include [5, 6, 14, 11]. Suppose we wish to sum a series of the form

$$S = \sum_{n=1}^{\infty} F(n),$$

where each term $F(n)$ can be expressed as a Laplace transform

$$F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt.$$

By substituting this representation into the sum and interchanging summation and integration, we obtain

$$\sum_{n=1}^{\infty} F(n) = \int_0^{\infty} f(t) \left(\sum_{n=1}^{\infty} e^{-nt} \right) dt.$$

The geometric series provides the inner sum, leaving an integral that often can be evaluated more directly than the original series.

Example 1: Alternating Harmonic Series

Consider the alternating harmonic series,

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

We recall that $1/s$ is the Laplace transform of the constant function 1. Thus

$$\frac{1}{n} = \int_0^\infty e^{-nt}\, dt,$$

and the series becomes

$$S = \sum_{n=1}^{\infty} (-1)^{n+1} \int_0^\infty e^{-nt}\, dt.$$

Interchanging sum and integral and evaluating the resulting geometric series gives

$$S = \int_0^\infty \frac{e^{-t}}{1+e^{-t}}\, dt.$$

With the substitution $u = e^{-t}$, this reduces to

$$S = \int_0^1 \frac{1}{1+u}\, du = \ln 2.$$

Thus the Laplace transform converts a slowly convergent alternating series into a straightforward integral [18].

Example 2: The Basel Problem

The celebrated Basel problem asks for

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Since the Laplace transform of t is $1/s^2$, we can write

$$\frac{1}{n^2} = \int_0^\infty t e^{-nt}\, dt.$$

Inserting this into the series yields

$$S = \int_0^\infty t \left(\sum_{n=1}^{\infty} e^{-nt} \right) dt.$$

The geometric series gives

$$S = \int_0^\infty \frac{t}{e^t - 1}\, dt,$$

an integral representation of the Riemann zeta function at $s = 2$. Evaluating the integral leads to Euler's famous result [8, 4],

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

These two examples illustrate a general principle: once a term $1/n^s$ is expressed via the Laplace transform, summing over n produces integral formulas of the form

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{\,s-1}}{e^t - 1}\, dt,$$

which connect the Riemann zeta function with integrals appearing in statistical mechanics (the Bose–Einstein and Fermi–Dirac distributions) [16]. Thus a tool developed for solving differential equations unexpectedly illuminates both number theory and physics.

4 From Euler to Feynman: Tricks with Parameters

Feynman's trick of differentiating under the integral sign, made famous in his work on quantum electrodynamics [9, 10], is a natural extension of Laplace transform methods. Related techniques were developed by Schwinger [19] and have become standard in quantum field theory [22].

For example, one identity reads

$$\frac{1}{(a+bn)^2} = -\frac{\partial}{\partial a} \int_0^\infty e^{-s(a+bn)}\, ds.$$

More generally,

$$\frac{1}{(a+bn)^{k+1}} = \frac{(-1)^k}{k!} \frac{\partial^k}{\partial a^k} \int_0^\infty e^{-s(a+bn)}\, ds.$$

Such expressions reduce summations involving rational terms to manageable integrals.

As an illustration,

$$S = \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \int_0^{\infty} \frac{e^{-t}}{1+e^{-t}}\, dt = \ln 2.$$

This continuity from Euler's expansions to Feynman's parameters underscores the shared structure underlying diverse applications.

5 Borel Summation and Divergent Series

Divergent series are not merely curiosities; they often arise naturally in analysis and physics. One of Euler's most striking examples was the factorial series [7],

$$1 - 1 + 2 - 6 + 24 - 120 + \cdots,$$

whose terms grow as $n!$. Ordinary convergence fails, yet Euler managed to assign a value through bold analytic reasoning. His manipulations anticipated what is now formalized as *Borel summation*.

The Borel transform of a series $\sum a_n$ is

$$\mathcal{B}(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n,$$

and the Borel sum is defined as

$$S = \int_0^{\infty} e^{-t}\, \mathcal{B}(t)\, dt.$$

In Euler's factorial example, $\mathcal{B}(t) = 1/(1+t)$, so

$$S = \int_0^{\infty} \frac{e^{-t}}{1+t}\, dt = \ln 2.$$

Stokes's studies of rainbow fringes [21, 1] revealed similar divergent asymptotics. His identification of the "Stokes phenomenon" showed how values of divergent expansions shift across lines in the complex plane, anticipating modern asymptotic theory [2].

In recent decades, resurgence theory, initiated by Écalle [3], has shown how singularities in the Borel plane encode hidden analytic structure. Perturbative series in quantum mechanics and field theory, such as the anharmonic oscillator, become meaningful once analyzed through Borel–Laplace techniques.

6 Conclusion

From the paradoxes of Zeno to Euler's bold manipulations, from Stokes's analysis of the rainbow to Feynman's parameter tricks in quantum electrodynamics, divergent series have moved steadily from the fringes of mathematics into its very core. What once seemed nonsensical has become indispensable. At the center of this story lies the Laplace transform, a tool originally devised to solve differential equations, but which reappears whenever infinite processes threaten to outrun their definitions.

Through the Laplace transform we can sum convergent series by turning them into integrals, recover finite values for divergent factorial series through Borel summation, and reveal hidden analytic structure in perturbation expansions. Its quiet universality underscores an important theme: mathematics evolves not by discarding the divergent, but by finding ways to interpret it.

The history traced here shows remarkable continuity. Euler's ingenuity foreshadowed Borel's systematic methods; Stokes's insights into asymptotics anticipated modern resurgence; Feynman's playful tricks became standard tools in physics. Each generation inherited and transformed the techniques of the last. Thus the Laplace transform is not merely a computational device, but a thread weaving together analysis, history, and physics. It is, in a very real sense, the bridge that makes the unsummable sum.

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