+............ ....

37. Présentement il faut substituer dans ces formules les expressions des quantités $P_1, P_2, P_3, \ldots, Q_1, Q_2, Q_3, \ldots$, d'où, en ordonnant les termes par les quantités connues $Y_1, Y_2, Y_3, \ldots, V_1, V_2, V_3, \ldots$, on trouvera autant de suites infinies, dont chacune sera multipliée par une de ces quantités.

Soit $\frac{X}{a}$ la raison générale des indices des Y et des V au nombre m, X dénotera la partie de l'axe qui leur est correspondante dans le premier état du système; donc, si l'on emploie le signe intégral $\int$ pour exprimer la somme de toutes ces suites, on aura

$$y = \frac{2}{a}\int dx\, Y \left(\sin\frac{\varpi X}{2a}\sin\frac{\varpi x}{2a}\cos\frac{\varpi H t}{2T} + \sin\frac{2\varpi X}{2a}\sin\frac{2\varpi x}{2a}\cos\frac{2\varpi H t}{2T} + \sin\frac{3\varpi X}{2a}\sin\frac{3\varpi x}{2a}\cos\frac{3\varpi H t}{2T} + \ldots\right)$$
$$+ \frac{4T}{\varpi H a}\int dx\, V \left(\sin\frac{\varpi X}{2a}\sin\frac{\varpi x}{2a}\sin\frac{\varpi H t}{2T} + \frac{1}{2}\sin\frac{2\varpi X}{2a}\sin\frac{2\varpi x}{2a}\sin\frac{2\varpi H t}{2T} + \frac{1}{3}\sin\frac{3\varpi X}{2a}\sin\frac{3\varpi x}{2a}\sin\frac{3\varpi H t}{2T} + \ldots\right),$$

et de même pour u

$$u = -\frac{\varpi H}{aT}\int dx\, Y \left(\sin\frac{\varpi X}{2a}\sin\frac{\varpi x}{2a}\sin\frac{\varpi H t}{2T} + 2\sin\frac{2\varpi X}{2a}\sin\frac{2\varpi x}{2a}\sin\frac{2\varpi H t}{2T} + 3\sin\frac{3\varpi X}{2a}\sin\frac{3\varpi x}{2a}\sin\frac{3\varpi H t}{2T} + \ldots\right)$$
$$+ \frac{2}{a}\int dx\, V \left(\sin\frac{\varpi X}{2a}\sin\frac{\varpi x}{2a}\cos\frac{\varpi H t}{2T} + \sin\frac{2\varpi X}{2a}\sin\frac{2\varpi x}{2a}\cos\frac{2\varpi H t}{2T} + \sin\frac{3\varpi X}{2a}\sin\frac{3\varpi x}{2a}\cos\frac{3\varpi H t}{2T} + \ldots\right),$$

Figure 1: From page 100 of Lagrange's research on sound.

Lagrange's 1759 Recherches sur la nature et la propagation du son (Researches on the nature and the propagation of sound), Page 100

In this section we reinterpret Lagrange's construction of the motion of a vibrating string in a form that anticipates the modern Fourier method. His starting point is a discrete system whose motion is expressed in terms of coefficients

$$P_1, P_2, P_3, \ldots, \qquad Q_1, Q_2, Q_3, \ldots,$$

which are themselves determined by initial data. These are then reorganized in terms of quantities

$$Y_1, Y_2, Y_3, \ldots, \qquad V_1, V_2, V_3, \ldots,$$

representing, in modern language, the initial displacement and velocity projected onto normal modes.

Lagrange observes that one may pass from this discrete description to a continuous one by introducing a spatial variable x along the string and writing the sums over modes as integrals. Let X denote a point on the string corresponding to a given index in the discrete system. Then the motion may be written as a superposition of infinitely many oscillatory modes.

The displacement $r(X,t)$ takes the form

$$r(X,t) = \frac{2}{a}\int_0^a Y(x)\sum_{n=1}^{\infty} \sin\left(\frac{n\pi X}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{n\pi ct}{a}\right)\,dx$$
$$+\frac{2}{ac}\int_0^a V(x)\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi X}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi ct}{a}\right)\,dx,$$

where c is a constant depending on the physical parameters of the system.

Similarly, the velocity $u(X,t)$ is given by

$$u(X,t) = -\frac{2c}{a}\int_0^a Y(x)\sum_{n=1}^{\infty} n\sin\left(\frac{n\pi X}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi ct}{a}\right)\,dx$$
$$+\frac{2}{a}\int_0^a V(x)\sum_{n=1}^{\infty} \sin\left(\frac{n\pi X}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{n\pi ct}{a}\right)\,dx.$$

These expressions reveal several important structural features.

First, the motion is decomposed into normal modes of the form

$$\sin\left(\frac{n\pi x}{a}\right),$$

which correspond to the natural vibrations of a string fixed at its endpoints.

Second, each mode evolves independently in time, with oscillatory factors

$$\cos\left(\frac{n\pi ct}{a}\right), \qquad \sin\left(\frac{n\pi ct}{a}\right).$$

Thus the full motion is a superposition of standing waves.

Third, the initial displacement $Y(x)$ contributes to the cosine terms, while the initial velocity $V(x)$ contributes to the sine terms. In modern notation, these integrals represent the projection of the initial data onto the eigenfunctions of the system.

From a contemporary point of view, this is precisely the solution of the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < a,$$

with fixed boundary conditions

$$u(0,t) = u(a,t) = 0,$$

and initial conditions

$$u(x,0) = Y(x), \qquad u_t(x,0) = V(x).$$

Lagrange's derivation predates Fourier's work, yet already contains the essential idea: an arbitrary motion can be expressed as a superposition of harmonic modes whose amplitudes are determined by integrals against sine functions. What is missing is not the method, but the later interpretation of these integrals as coefficients of a general trigonometric series.