MAT 365. Vector Calculus
UNCW Mathematics and Statistics Spring 2013

Thin Plate Problem

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This is problem 6 in section 5.6 of Bressoud. One has to find the area, mass, and center of mass of a thin plate of density $\rho(x, y) = x$ bounded by $x^2 - y^2 = 1$ and $x = 4$. This bounded domain is seen in Figure 1.

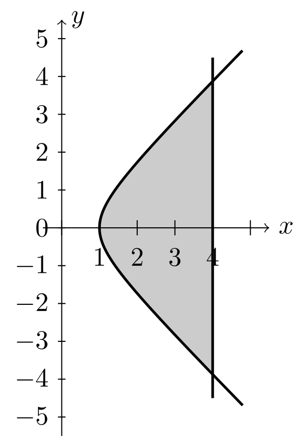

Figure 1: The domain showing the thin plate geometry.

The Domain in the Problem

We can carry out the needed integrations once we determine the limits of integration. These limits will persist throughout the problem. There are two approaches. First, one can set up the area as $A = \int_R dxdy$. For the outside integral we need to find the intersection points of the two curves. Since $x = 4$, then $y = \pm\sqrt{x^2 - 1} = \pm\sqrt{15}$. In order to find the limits of the inside integral, we consider covering the region for fixed y-values. Thus, as seen in Figure 2, we cover the region with horizontal lines and see that the region is covered from the hyperbolic curve, $x = \sqrt{1 + y^2}$, to the vertical line $x = 4$.

Therefore, the area integral becomes

$$A = \int_{-\sqrt{15}}^{\sqrt{15}} \int_{\sqrt{1+y^2}}^{4} dxdy.$$

Considering that there is symmetry in the problem about the x-axis, we could write this as

$$A = 2\int_{0}^{\sqrt{15}} \int_{\sqrt{1+y^2}}^{4} dxdy.$$

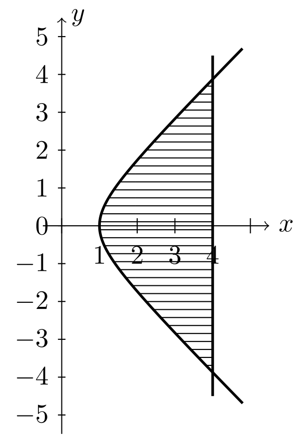

Figure 2: The domain showing lines of constant y. This is used for integrating first with respect to x.

Computing the inner integral, we have

$$A = 2\int_0^{\sqrt{15}} (4 - \sqrt{1+y^2})\, dy.$$

Another approach would be to set up the area as $A = \int_R dydx$. For the outside integral have $x = 1$ to $x = 4$.. In order to find the limits of the inside integral, we consider covering the region for fixed x-values. Thus, as seen in Figure 3, we cover the region with vertical lines and see that the region is covered from the lower part of the hyperbolic curve, $y = -\sqrt{x^2-1}$, to the upper part, $y = \sqrt{x^2-1}$.

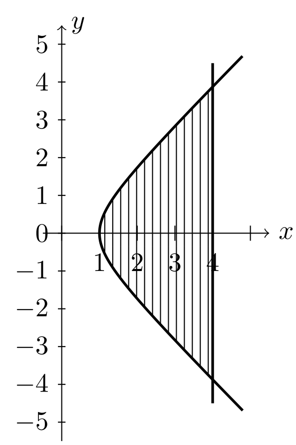

Figure 3: The domain showing lines of constant x. This is used for integrating first with respect to y.

Therefore, the area integral becomes

$$A = \int_1^4 \int_{-\sqrt{x^2-1}}^{\sqrt{x^2-1}} dydx.$$

Again, using the symmetry about the x-axis, we could write this as

$$A = 2\int_1^4 \int_0^{\sqrt{x^2-1}} dydx.$$

Carrying out the inner integration, we have

$$A = 2\int_1^4 \sqrt{x^2-1}\,dx.$$

Computing the Area Integrals

So far, we have determined that the area of the region can be computed using one of the integrals

$$\begin{aligned} A &= 2\int_0^{\sqrt{15}} (4-\sqrt{1+y^2})\,dy, && (1)\\ A &= 2\int_1^4 \sqrt{x^2-1}\,dx. && (2) \end{aligned}$$

The mass and center of mass integrals are slight variations of these. There are several methods which can be used to obtain the values of these integrals. From Calculus II you might decide that using trigonometric substitution would be the way to go. However, it is also possible to use hyperbolic function substitution.

1. Trigonometric Substitution in Equation (1).

   For this integral we would use a tangent substitution, $y = \tan\theta$, $dy = \sec^2\theta\,d\theta$, in the second integral. This would reduce the square root to $\sqrt{1+y^2} = \sec\theta$. Therefore, we have

$$\begin{aligned} A &= 2\int_0^{\sqrt{15}} (4-\sqrt{1+y^2})\,dy,\\ &= 8\sqrt{15} - 2\int_0^{\sqrt{15}} (\sqrt{1+y^2})\,dy,\\ &= 8\sqrt{15} - 2\int_0^{\tan^{-1}\sqrt{15}} \sec^3\theta\,d\theta. \end{aligned}$$

   Now we need the result

$$\int \sec^3 x\,dx = \frac{1}{2}\left[\sec x\tan x + \ln|\sec x + \tan x|\right] + C.$$

   This integral was determined in Calculus II. Let $u = \sec x$ and $v = \tan x$ in the integration by parts formula. Then,[1]

$$
\begin{aligned}
\int \sec^3 x\,dx &= \sec x \tan x - \int \sec x \tan^2 x\,dx \\
&= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx \\
&= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx \\
2\int \sec^3 x\,dx &= \sec x \tan x + \int \sec x\,dx \\
&= \sec x \tan x + \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx \\
&= \sec x \tan x + \int \frac{dy}{y}, \quad y = \sec x + \tan x, \\
\int \sec^3 x\,dx &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A &= 8\sqrt{15} - 2\int_0^{\tan^{-1}\sqrt{15}} \sec^3\theta\,d\theta. \\
&= 8\sqrt{15} - [\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|]_0^{\tan^{-1}\sqrt{15}}.
\end{aligned}
$$

We can evaluate these terms using the triangle in Figure 4. For the lower limit, $\theta = 0$, the terms vanish. Also, we have

$$
\tan\theta = y \quad \text{and} \quad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \sqrt{1+y^2}.
$$

Therefore,

$$
\tan(\tan^{-1}\sqrt{15}) = \sqrt{15} \quad \text{and} \quad \sec(\tan^{-1}\sqrt{15}) = \sqrt{1+15^2} = 4.
$$

$$
\begin{aligned}
A &= 8\sqrt{15} - [\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|]_0^{\tan^{-1}\sqrt{15}}, \\
&= 8\sqrt{15} - [4\sqrt{15} + \ln(4+\sqrt{15})] \\
&= 4\sqrt{15} - \ln(4+\sqrt{15}).
\end{aligned}
$$

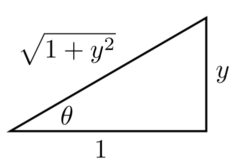

Figure 4: Triangle for the tangent substitution.

[1] Within the proof is a proof that $\int \sec x\,dx = \ln|\sec x + \tan x| + C$.

2. Trigonometric Substitution in Equation (2).

   For Equation (2) we use the trigonometric substitution

$$x = \sec\theta \quad \text{and} \quad dx = \sec\theta\tan\theta\,d\theta.$$

   Then, $\sqrt{x^2-1} = \tan\theta$ and the integral is given by

$$\begin{aligned} A &= 2\int_1^4 \sqrt{x^2-1}\,dx, \\ &= 2\int_0^{\sec^{-1}4} \sec\theta\tan^2\theta\,d\theta \\ &= 2\int_0^{\sec^{-1}4} \sec\theta(\sec^2\theta-1)\,d\theta \\ &= 2\int_0^{\sec^{-1}4} (\sec^3\theta-\sec\theta)\,d\theta \\ &= 2\left(\frac{1}{2}\left[\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\right]-\ln|\sec\theta+\tan\theta|\right)_0^{\sec^{-1}4} \\ &= \left[\sec\theta\tan\theta-\ln|\sec\theta+\tan\theta|\right]_0^{\sec^{-1}4} \end{aligned}$$

   Here we used

$$\boxed{\int \sec x\,dx = \ln|\sec x+\tan x| + C}$$

   and the previously used integral of $\sec^3 x$.

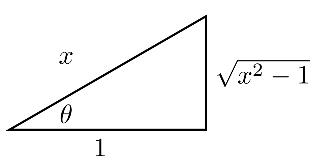

Figure 5: Triangle for the secant substitution.

   We can evaluate the result using the triangle in Figure 5. For the lower limit, $\theta = 0$, the terms vanish. Also, we have

$$\tan\theta = \sqrt{x^2-1} \quad \text{and} \quad \sec\theta = x.$$

   Therefore,

$$\tan(\sec^{-1}4) = \sqrt{15} \quad \text{and} \quad \sec(\sec^{-1}4) = 4.$$

   Using these values in the area computation, we have

$$\begin{aligned} A &= \left[\sec\theta\tan\theta-\ln|\sec\theta+\tan\theta|\right]_0^{\sec^{-1}4} \\ &= 4\sqrt{15}-\ln(4+\sqrt{15}). \end{aligned}$$

Voilà! This is the same answer obtained doing the integration in reverse order. However, it is not the answer in the back of the book. The book gives the answer $A = 4\sqrt{15} - \cosh^{-1} 4$. Are these really different answers?

Recall, $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$.[2] Then, $\cosh^{-1} 4 = \ln\left(4 + \sqrt{4^2 - 1}\right) = \ln(4 + \sqrt{15})$. Therefore, our answers agree with the book.

3. Hyperbolic Function Substitution in Equation (1).

So, why did the book not report the answer using logarithms? It is possible that the answers were not found using trigonometric substitutions. Sometimes one can use hyperbolic function substitutions. As usual, one needs to use identities to simplify the radical. These are similar to the trigonometric identities:

$$
\begin{aligned}
\cosh^2 u - \sinh^2 u &= 1, && (3)\\
\tanh^2 u + \operatorname{sech}^2 u &= 1, && (4)\\
\cosh(u \pm v) &= \cosh u \cosh v \pm \sinh u \sinh v, && (5)\\
\sinh(u \pm v) &= \sinh u \cosh v \pm \sinh v \cosh u, && (6)\\
\cosh 2u &= \cosh^2 u + \sinh^2 u, && (7)\\
\sinh 2u &= 2 \sinh u \cosh u, && (8)\\
\cosh^2 u &= \frac{1}{2}\left(\cosh 2u + 1\right), && (9)\\
\sinh^2 u &= \frac{1}{2}\left(\cosh 2u - 1\right). && (10)
\end{aligned}
$$

For Equation (1) we look at the radical, $\sqrt{1 + y^2}$. Identity (3) suggests $y = \sinh u$. So, $dy = \cosh u\, du$ and $\sqrt{1 + y^2} = \cosh u$.

The integral becomes

$$
\begin{aligned}
A &= 2 \int_0^{\sqrt{15}} (4 - \sqrt{1 + y^2})\, dy,\\
&= 8\sqrt{15} - 2 \int_0^{\sinh^{-1} \sqrt{15}} \cosh^2 u\, du.
\end{aligned}
$$

[2] This is readily shown as follows: Let $y = \cosh^{-1} x$. Then,

$$x = \cosh y = \frac{e^y + e^{-y}}{2}.$$

Rewriting,

$$(e^y)^2 - 2xe^y + 1 = 0.$$

Solving for e^y,

$$e^y = \frac{1}{2}(2x + \sqrt{4x^2 - 4}).$$

Then, $y = \ln(x + \sqrt{x^2 - 1})$.

Similarly, one can show that

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

and

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{x + 1}{x - 1}.$$

We can use identity (9) to do the integral.

$$\begin{aligned} A &= 8\sqrt{15} - 2\int_0^{\sinh^{-1}\sqrt{15}} \cosh^2 u\, du. \\ &= 8\sqrt{15} - \int_0^{\sinh^{-1}\sqrt{15}} (\cosh 2u + 1)\, du. \\ &= 8\sqrt{15} - \left[\frac{1}{2}\sinh 2u + u\right]_0^{\sinh^{-1}\sqrt{15}} \end{aligned}$$

Using identity (8), the result can be written in terms of $\cosh u$ and $\sinh u$.

$$\begin{aligned} A &= 8\sqrt{15} - \left[\frac{1}{2}\sinh 2u + u\right]_0^{\sinh^{-1}\sqrt{15}} \\ &= 8\sqrt{15} - [\sinh u \cosh u + u]_0^{\sinh^{-1}\sqrt{15}} \\ &= 8\sqrt{15} - \left[4\sqrt{15} + \sinh^{-1}\sqrt{15}\right] \\ &= 4\sqrt{15} - \sinh^{-1}\sqrt{15}. \end{aligned}$$

This is not quite the answer in the back of the book either! Let's look closer at $z = \sinh^{-1}\sqrt{15}$. This gives $\sinh z = \sqrt{15}$. From identity (3) we have

$$\cosh^2 z = 1 + \sinh^2 z = 16.$$

So, $z = \sinh^{-1}\sqrt{15} = \cosh^{-1} 4$ and the answer is

$$A = 4\sqrt{15} - \cosh^{-1} 4.$$

4. Hyperbolic Function Substitution in Equation (2).

   The last integral (2) can also be evaluated using hyperbolic function substitution. Let $x = \cosh u$. Then, $dx = \sinh u\, du$ and $\sqrt{x^2 - 1} = \sinh u$. Then,

$$\begin{aligned} A &= 2\int_1^4 \sqrt{x^2 - 1}\, dx, \\ &= 2\int_0^{\cosh^{-1} 4} \sinh^2 u\, du \\ &= \int_0^{\cosh^{-1} 4} (\cosh 2u - 1)\, du \\ &= \left[\frac{1}{2}\sinh 2u - u\right]_0^{\cosh^{-1} 4} \\ &= [\sinh u \cosh u - u]_0^{\cosh^{-1} 4} \\ &= 4\sqrt{15} - \cosh^{-1} 4. \end{aligned} \tag{11}$$

   In this case we actually got the answer directly.

The Other Integrals

As with the area integral, we can evaluate the other integrals in four different ways. We will obtain the results using the limits and order of integration in Equation (2) and apply hyperbolic function integration. The needed identities can be found in the list above.

1. The Mass of the Plate

   The mass is found using a simple substitution in the integral

$$\begin{aligned} M &= \int_R \rho(x,y)\,dA \\ &= 2\int_1^4 \int_0^{\sqrt{x^2-1}} x\,dydx \\ &= 2\int_1^4 x\sqrt{x^2-1}\,dx \\ &= \frac{2}{3}(x^2-1)^{3/2}\Big|_1^4 = \frac{2}{3}15^{3/2} = 10\sqrt{15}. \end{aligned}$$

2. The Center of Mass of the Plate

   Due to the symmetry of the plate, it is easy to see that $\bar{y} = 0$. So, we need only compute $\bar{x}$. Again, we will need the substitutions $x = \cosh u$, $dx = \sinh u\,du$, and $\sqrt{x^2-1} = \sinh u$. Then, we have

$$\begin{aligned} \bar{x} &= \frac{1}{M}\int_R x\rho(x,y)\,dA \\ &= \frac{2}{M}\int_1^4 \int_0^{\sqrt{x^2-1}} x^2\,dydx \\ &= \frac{2}{M}\int_1^4 x^2\sqrt{x^2-1}\,dx \\ &= \frac{2}{M}\int_0^{\cosh^{-1}4} \cosh^2 u \sinh^2 u\,du \end{aligned}$$

   The integration can be handled using the hyperbolic function identities. Identities (8) and (10) allow for the integration,

$$\begin{aligned} \bar{x} &= \frac{2}{M}\int_0^{\cosh^{-1}4} \cosh^2 u \sinh^2 u\,du \\ &= \frac{2}{M}\int_0^{\cosh^{-1}4} (\frac{1}{2}\sinh 2u)^2\,du \\ &= \frac{1}{2M}\int_0^{\cosh^{-1}4} \sinh^2 2u\,du \\ &= \frac{1}{4M}\int_0^{\cosh^{-1}4} (\cosh 4u - 1)\,du \\ &= \frac{1}{4M}\left[\frac{1}{4}\sinh 4u - u\right]_0^{\cosh^{-1}4} \end{aligned}$$

Identities (7) and (8) allow this result to be written in terms of $\cosh u$ and $\sinh u$.

$$\begin{aligned}
\bar{x} &= \frac{1}{4M}\left[\frac{1}{4}\sinh 4u - u\right]_0^{\cosh^{-1}4} \\
&= \frac{1}{4M}\left[\frac{1}{2}\sinh 2u\cosh 2u - u\right]_0^{\cosh^{-1}4} \\
&= \frac{1}{4M}\left[\sinh u\cosh u(\cosh^2 u + \sinh^2 u) - u\right]_0^{\cosh^{-1}4} \\
&= \frac{1}{40\sqrt{15}}[4\sqrt{15}(4^2+15) - \cosh^{-1}4] \\
&= \frac{124\sqrt{15} - \cosh^{-1}4}{40\sqrt{15}}.
\end{aligned}$$

The results are

$$\begin{aligned}
A &= 4\sqrt{15} - \cosh^{-1}4 \\
&= 4\sqrt{15} - \sinh^{-1}\sqrt{15} \\
&= 4\sqrt{15} - \ln(4+\sqrt{15}) \\
M &= 10\sqrt{15}, \\
\bar{x} &= \frac{124\sqrt{15} - \cosh^{-1}4}{40\sqrt{15}}, \\
\bar{y} &= 0.
\end{aligned}$$