Chapter 4

Series Solutions

"In most sciences one generation tears down what another has built and what one has established another undoes. In mathematics alone each generation adds a new story to the old structure." - Hermann Hankel (1839-1873)

4.1 Introduction to Power Series

As noted a few times, not all differential equations have exact solutions. So, we need to resort to seeking approximate solutions, or solutions i the neighborhood of the initial value. Before describing these methods, we need to recall power series. A power series expansion about $x = a$ with coefficient sequence c_n is given by $\sum_{n=0}^{\infty} c_n (x-a)^n$. For now we will consider all constants to be real numbers with x in some subset of the set of real numbers. We review power series in the appendix.

The two types of series encountered in calculus are Taylor and Maclaurin series. A Taylor series expansion of $f(x)$ about $x = a$ is the series

Taylor series expansion.

$$f(x) \sim \sum_{n=0}^{\infty} c_n (x-a)^n, \tag{4.1}$$

where

$$c_n = \frac{f^{(n)}(a)}{n!}. \tag{4.2}$$

Note that we use $\sim$ to indicate that we have yet to determine when the series may converge to the given function.

A Maclaurin series expansion of $f(x)$ is a Taylor series expansion of $f(x)$ about $x = 0$, or

Maclaurin series expansion.

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^n, \tag{4.3}$$

where

$$c_n = \frac{f^{(n)}(0)}{n!}. \tag{4.4}$$

We note that Maclaurin series are a special case of Taylor series for which the expansion is about $x = 0$. Typical Maclaurin series, which you should know, are given in Table 4.1.

A simple example of developing a series solution for a differential equation is given in the next example.

Table 4.1: Common Mclaurin Series Expansions

Series Expansions You Should Know				
e^x	=	$1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$	=	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$
$\cos x$	=	$1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots$	=	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
$\sin x$	=	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$	=	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
$\cosh x$	=	$1 + \frac{x^2}{2} + \frac{x^4}{4!} + \ldots$	=	$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$
$\sinh x$	=	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$	=	$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$
$\frac{1}{1-x}$	=	$1 + x + x^2 + x^3 + \ldots$	=	$\sum_{n=0}^{\infty} x^n$
$\frac{1}{1+x}$	=	$1 - x + x^2 - x^3 + \ldots$	=	$\sum_{n=0}^{\infty} (-x)^n$
$\tan^{-1} x$	=	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$	=	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$
$\ln(1+x)$	=	$x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$	=	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

Example 4.1. $y'(x) = x + y(x)$, $y(0) = 1$.

We are interested in seeking solutions of this initial value problem. We note that this was already solved in Example 3.1.

Let's assume that we can write the solution as the Maclaurin series

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} \frac{y^{(n)}(x)}{n!} x^n \\ &= y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \ldots . \end{aligned} \tag{4.5}$$

We already know that $y(0) = 1$. So, we know the first term in the series expansion. We can find the value of $y'(0)$ from the differential equation:

$$y'(0) = 0 + y(0) = 1.$$

In order to obtain values of the higher order derivatives at $x = 0$, we differentiate the differential equation several times:

$$\begin{aligned} y''(x) &= 1 + y'(x). \\ y''(0) &= 1 + y'(0) = 2. \\ y'''(x) &= y''(x) = 2. \end{aligned} \tag{4.6}$$

All other values of the derivatives are the same. Therefore, we have

$$y(x) = 1 + x + 2\left(\frac{1}{2}x^2 + \frac{1}{3!}x^3 + \ldots\right).$$

This solution can be summed as

$$y(x) = 2\left(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \ldots\right) - 1 - x = 2e^x - x - 1.$$

This is the same result we had obtained before.

4.2 Power Series Method

In the last example we were able to use the initial condition to produce a series solution to the given differential equation. Even if we specified more general initial conditions, are there other ways to obtain series solutions? Can we find a general solution in the form of power series? We will address these questions in the remaining sections. However, we will first begin with an example to demonstrate how we can find the general solution to a first order differential equation.

Example 4.2. Find a general Maclaurin series solution to the ODE: $y' - 2xy = 0$.

Let's assume that the solution takes the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

The goal is to find the expansion coefficients, c_n, $n = 0, 1, \ldots$.

Differentiating, we have

$$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}.$$

Note that the index starts at $n = 1$, since there is no $n = 0$ term remaining.

Inserting the series for $y(x)$ and $y'(x)$ into the differential equation, we have

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} n c_n x^{n-1} - 2x \sum_{n=0}^{\infty} c_n x^n \\ &= (c_1 + 2c_2 x + 3c_3 x^2 + 4x^3 + \ldots) \\ &\quad -2x(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots) \\ &= c_1 + (2c_2 - c_0)x + (3c_3 - 2c_1)x^2 + (4c_4 - 2c_2)x^3 + \ldots. \end{aligned} \tag{4.7}$$

Equating like powers of x on both sides of this result, we have

$$\begin{aligned} 0 &= c_1, \\ 0 &= 2c_2 - c_0, \\ 0 &= 3c_3 - c_1, \\ 0 &= 4c_4 - 2c_2, \ldots. \end{aligned} \tag{4.8}$$

We can solve these sequentially for the coefficient of largest index:

$$c_1 = 0, c_2 = c_0, c_3 = \frac{2}{3}c_1 = 0, c_3 = \frac{1}{2}c_2 = \frac{1}{2}c_0, \ldots.$$

We note that the odd terms vanish and the even terms survive:

$$\begin{aligned} y(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots \\ &= c_0 + c_0 x^2 + \frac{1}{2}c_0 x^4 + \ldots. \end{aligned} \tag{4.9}$$

Thus, we have found a series solution, or at least the first several terms, up to a multiplicative constant.

Of course, it would be nice to obtain a few more terms and guess at the general form of the series solution. This could be done if we carried out the streps in a more general way. This is accomplished by keeping the summation notation and trying to combine all terms with like powers of x. We begin by inserting the series expansion into the differential equation and identifying the powers of x :

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} nc_n x^{n-1} - 2x \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} 2c_n x^{n+1}. \end{aligned} \tag{4.10}$$

Re-indexing a series.

We note that the powers of x in these two sums differ by 2. We can re-index the sums separately so that the powers are the same, say k. After all, when we had expanded these series earlier, the index, n, disappeared. Such an index is known as a dummy index since we could call the index anything, like $n-1$, $\ell-1$, or even $k=n-1$ in the first series. So, we can let $k=n-1$, or $n=k+1$, to write

$$\begin{aligned} \sum_{n=1}^{\infty} nc_n x^{n-1} &= \sum_{k=0}^{\infty} (k+1)c_{k+1} x^k \\ &= c_1 + 2c_2 x + 3c_3 x^2 + 4x^3 + \dots. \end{aligned} \tag{4.11}$$

Note, that re-indexing has not changed the terms in the series.

Similarly, we can let $k=n+1$, or $n=k-1$, in the second series to find

$$\begin{aligned} \sum_{n=0}^{\infty} 2c_n x^{n+1} &= \sum_{k=1}^{\infty} 2c_{k-1} x^k \\ &= 2c_0 + 2c_1 x + 2c_2 x^2 + 2c_3 x^3 + \dots. \end{aligned} \tag{4.12}$$

Combining both series, we have

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} 2c_n x^{n+1} \\ &= \sum_{k=0}^{\infty} (k+1)c_{k+1} x^k - \sum_{k=1}^{\infty} 2c_{k-1} x^k \\ &= c_1 + \sum_{k=1}^{\infty} [(k+1)c_{k+1} - 2c_{k-1}] x^k. \end{aligned} \tag{4.13}$$

Here, we have combined the two series for $k=1,2,\dots$. The $k=0$ term in the first series gives the constant term as shown.

We can now set the coefficients of powers of x equal to zero since there are no terms on the left hand side of the equation. This gives $c_1=0$ and

$$(k+1)c_{k+1} - 2c_{k-1}, \quad k=1,2,\dots.$$

This last equation is called a recurrence relation. It can be used to find successive coefficients in terms of previous values. In particular, we have

$$c_{k+1} = \frac{2}{k+1} c_{k-1}, \quad k = 1, 2, \ldots.$$

Inserting different values of k, we have

$$\begin{aligned}
k &= 1: \quad c_2 = \frac{2}{2} c_0 = c_0. \\
k &= 2: \quad c_3 = \frac{2}{3} c_1 = 0. \\
k &= 3: \quad c_4 = \frac{2}{4} c_2 = \frac{1}{2} c_0. \\
k &= 4: \quad c_5 = \frac{2}{5} c_3 = 0. \\
k &= 5: \quad c_6 = \frac{2}{6} c_4 = \frac{1}{3(2)} c_0.
\end{aligned} \tag{4.14}$$

Continuing, we can see a pattern. Namely,

$$c_k = \begin{cases} 0, & k = 2\ell + 1, \\ \frac{1}{\ell!}, & k = 2\ell. \end{cases}$$

Thus,

$$\begin{aligned}
y(x) &= \sum_{k=0}^{\infty} c_k x^k \\
&= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots \\
&= c_0 + c_0 x^2 + \frac{1}{2!} c_0 x^4 + \frac{1}{3!} c_0 x^6 + \ldots \\
&= c_0 \left(1 + x^2 + \frac{1}{2!} x^4 + \frac{1}{3!} x^6 + \ldots \right) \\
&= c_0 \sum_{\ell=0}^{\infty} \frac{1}{\ell!} x^{2\ell} \\
&= c_0 e^{x^2}.
\end{aligned} \tag{4.15}$$

This example demonstrated how we can solve a simple differential equation by first guessing that the solution was in the form of a power series. We would like to explore the use of power series for more general higher order equations. We will begin second order differential equations in the form

$$P(x) y''(x) + Q(x) y'(x) + R(x) y(x) = 0,$$

where $P(x)$, $Q(x)$, and $R(x)$ are polynomials in x. The point x_0 is called an ordinary point if $P(x_0) \neq 0$. Otherwise, x_0 is called a singular point.

Ordinary and singular points.

When x_0 is an ordinary point, then we can seek solutions of the form

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n.$$

For most of the examples, we will let $x_0 = 0$, in which case we seek solutions of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Example 4.3. Find the general Maclaurin series solution to the ODE:

$$y'' - xy' - y = 0.$$

We will look for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

The first and second derivatives of the series are given by

$$\begin{aligned} y'(x) &= \sum_{n=1}^{\infty} c_n n x^{n-1} \\ y''(x) &= \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}. \end{aligned}$$

Inserting these derivatives into the differential equation gives

$$0 = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} c_n n x^n - \sum_{n=0}^{\infty} c_n x^n.$$

We want to combine the three sums into one sum and identify the coefficients of each power of x. The last two sums have similar powers of x. So, we need only re-index the first sum. We let $k = n - 2$, or $n = k + 2$. This gives

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} = \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k.$$

Inserting this sum, and setting $n = k$ in the other two sums, we have

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} c_n n x^n - \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_k k x^k - \sum_{k=0}^{\infty} c_k x^k \\ &= \sum_{k=1}^{\infty} \left[c_{k+2}(k+2)(k+1) - c_k k - c_k\right] x^k + c_2(2)(1) - c_0 \\ &= \sum_{k=1}^{\infty} (k+1)\left[(k+2)c_{k+2} - c_k\right] x^k + 2c_2 - c_0. \end{aligned} \tag{4.16}$$

Noting that the coefficients of powers x^k have to vanish, we have $2c_2 - c_0 = 0$ and

$$(k+1)\left[(k+2)c_{k+2} - c_k\right] = 0, \quad k = 1, 2, 3, \ldots,$$

or

$$\begin{aligned} c_2 &= \frac{1}{2}c_0, \\ c_{k+2} &= \frac{1}{k+2}c_k, \quad k = 1, 2, 3, \ldots. \end{aligned} \tag{4.17}$$

Using this result, we can successively determine the coefficients to as many terms as we need.

$$\begin{aligned} k &= 1: \quad c_3 = \frac{1}{3}c_1. \\ k &= 2: \quad c_4 = \frac{1}{4}c_2 = \frac{1}{8}c_0. \\ k &= 3: \quad c_5 = \frac{1}{5}c_3 = \frac{1}{15}c_1. \\ k &= 4: \quad c_6 = \frac{1}{6}c_4 = \frac{1}{48}c_0. \\ k &= 5: \quad c_7 = \frac{1}{7}c_5 = \frac{1}{105}c_1. \end{aligned} \tag{4.18}$$

This gives the series solution as

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} c_n x^n \\ &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots \\ &= c_0 + c_1 x + \frac{1}{2}c_0 x^2 + \frac{1}{3}c_1 x^3 + \frac{1}{8}c_0 x^4 + \frac{1}{15}c_1 x^5 + \frac{1}{48}c_0 x^6 + \ldots \\ &= c_0\left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \ldots\right) + c_1\left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \ldots\right). \end{aligned} \tag{4.19}$$

We note that the general solution to this second order differential equation has two arbitrary constants. The general solution is a linear combination of two linearly independent solutions obtained by setting one of the constants equal to one and the other equal to zero.

Sometimes one can sum the series solution obtained. In this case we note that the series multiplying c_0 can be rewritten as

$$y_1(x) = 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \ldots = 1 + \frac{x^2}{2} + \frac{1}{2}\left(\frac{x^2}{2}\right)^2 + + \frac{1}{3!}\left(\frac{x^2}{2}\right)^3 + \ldots.$$

This gives the exact solution $y_1(x) = e^{x^2/2}$.

The second linearly independent solution is not so easy. Since we know one solution, we can use the Method of Reduction of Order to obtain the second solution. One can verify that the second solution is given by

$$y_2(x) = e^{x^2/2}\int_0^{x/\sqrt{2}} e^{-t^2}\,dt = e^{x^2/2}\text{erf}\left(\frac{x}{\sqrt{2}}\right),$$

where $\text{erf}(x)$ is the error function. See Problem 3.

Example 4.4. Consider the Legendre equation

$$(1-x^2)y''-2xy'+\ell(\ell+1)y=0$$

for ℓ an integer.

Legendre's differential equation.

We first note that there are singular points for $1-x^2=0$, or $x=\pm 1$. Therefore, $x=0$ is an ordinary point and we can proceed to obtain solutions in the form of Maclaurin series expansions. Insert the series expansions

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} c_n x^n, \\ y'(x) &= \sum_{n=1}^{\infty} n c_n x^{n-1}, \\ y''(x) &= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}, \end{aligned} \tag{4.20}$$

into the differential equation to obtain

$$\begin{aligned} 0 &= (1-x^2)y''-2xy'+\ell(\ell+1)y \\ &= (1-x^2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x\sum_{n=1}^{\infty} n c_n x^{n-1} + \ell(\ell+1)\sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} \ell(\ell+1)c_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} [\ell(\ell+1)-n(n+1)]c_n x^n. \end{aligned} \tag{4.21}$$

Re-indexing the first sum with $k=n-2$, we have

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} [\ell(\ell+1)-n(n+1)]c_n x^n \\ &= \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k + \sum_{k=0}^{\infty} [\ell(\ell+1)-k(k+1)]c_k x^k \\ &= 2c_2+6c_3x+\ell(\ell+1)c_0+\ell(\ell+1)c_1x-2c_1x \\ &\quad + \sum_{k=2}^{\infty} \left((k+2)(k+1)c_{k+2}+[\ell(\ell+1)-k(k+1)]c_k\right) x^k. \end{aligned} \tag{4.22}$$

Matching terms, we have

$$\begin{aligned} k &= 0: \quad 2c_2 = -\ell(\ell+1)c_0. \\ k &= 1: \quad 6c_3 = [2-\ell(\ell+1)]c_1. \\ k &\geq 2: \quad (k+2)(k+1)c_{k+2} = [k(k+1)-\ell(\ell+1)]c_k. \end{aligned} \tag{4.23}$$

For $\ell=0$, the first equation gives $c_2=0$ and the third equation gives $c_{2m}=0$ for $m=1,2,3,\ldots$. This leads to $y_1(x)=c_0$ is a solution for $\ell=0$.

Similarly, for $\ell=1$, the second equation gives $c_3=0$ and the third equation gives $c_{2m+1}=0$ for $m=1,2,3,\ldots$. Thus, $y_1(x)=c_1x$ is a solution for $\ell=1$.

In fact, for ℓ any nonnegative integer the series truncates. For example, if $\ell = 2$, then these equations reduce to

$$\begin{aligned} k &= 0: \quad 2c_2 = -6c_0. \\ k &= 1: \quad 6c_3 = -4c_1. \\ k &\geq 2: \quad (k+2)(k+1)c_{k+2} = [k(k+1) - 2(3)]c_k. \end{aligned} \tag{4.24}$$

For $k = 2$, we have $12c_4 = 0$. So, $c_6 = c_8 = \ldots = 0$. Also, we have $c_2 = -3c_0$. This gives

$$y(x) = c_0(1 - 3x^2) + (c_1 x + c_3 x^3 + c_5 x^5 + c_7 x^7 + \ldots).$$

Therefore, there is a polynomial solution of degree 2. The remaining coefficients are proportional to c_1, yielding the second linearly independent solution, which is not a polynomial.

For other nonnegative integer values of $\ell > 2$, we have

$$c_{k+2} = \frac{k(k+1) - \ell(\ell+1)}{(k+2)(k+1)} c_k, \quad k \geq 2.$$

When $k = \ell$, the right side of the equation vanishes, making the remaining coefficients vanish. Thus, we will be left with a polynomial of degree ℓ. These are the Legendre polynomials, $P_\ell(x)$.

4.3 *Singular Points*

THE POWER SERIES METHOD does not alway give us the full general solution to a differential equation. Problems can arise when the differential equation has singular points. The simplest equations having singular points are Cauchy-Euler equations,

$$ax^2y'' + bxy' + cy = 0.$$

A few examples are sufficient to demonstrate the types of problems that can occur.

Example 4.5. Find the series solutions for the Cauchy-Euler equation,

$$ax^2y'' + bxy' + cy = 0,$$

for the cases i. $a = 1$, $b = -4$, $c = 6$, ii. $a = 1$, $b = 2$, $c = -6$, and iii. $a = 1$, $b = 1$, $c = 6$.

As before, we insert

$$y(x) = \sum_{n=0}^{\infty} d_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} n d_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) d_n x^{n-2},$$

into the differential equation to obtain

$$0 = ax^2y'' + bxy' + cy$$

$$\begin{aligned}
&= ax^2 \sum_{n=2}^{\infty} n(n-1)d_n x^{n-2} + bx \sum_{n=1}^{\infty} n d_n x^{n-1} + c \sum_{n=0}^{\infty} d_n x^n \\
&= a \sum_{n=0}^{\infty} n(n-1)d_n x^n + b \sum_{n=0}^{\infty} n d_n x^n + c \sum_{n=0}^{\infty} d_n x^n \\
&= \sum_{n=0}^{\infty} \left[an(n-1) + bn + c\right] d_n x^n. \qquad (4.25)
\end{aligned}$$

Here we changed the lower limits on the first sums as $n(n-1)$ vanishes for $n = 0, 1$ and the added terms all are zero.

Setting all coefficients to zero, we have

$$\left[an^2 + (b-a)n + c\right] d_n = 0, \quad n = 0, 1, \ldots .$$

Therefore, all of the coefficients vanish, $d_n = 0$, except at the roots of $an^2 + (b-a)n + c = 0$.

In the first case, $a = 1$, $b = -4$, and $c = 6$, we have

$$0 = n^2 + (-4-1)n + 6 = n^2 - 5n + 6 = (n-2)(n-3).$$

Thus, $d_n = 0$, $n \neq 2, 3$. This leaves two terms in the series, reducing to the polynomial $y(x) = d_2 x^2 + d_3 x^3$.

In the second case, $a = 1$, $b = 2$, and $c = -6$, we have

$$0 = n^2 + (2-1)n - 6 = n^2 + n - 6 = (n-2)(n+3).$$

Thus, $d_n = 0$, $n \neq 2, -3$. Since the n's are nonnegative, this leaves one term in the solution, $y(x) = d_2 x^2$. So, we do not have the most general solution since we are missing a second linearly independent solution. We can use the Method of Reduction of Order from Section 2.2.1, or we could use what we know about Cauchy-Euler equations, to show that the general solution is

$$y(x) = c_1 x^2 + c_2 x^{-3}.$$

Finally, the third case has $a = 1$, $b = 1$, and $c = 6$, we have

$$0 = n^2 + (1-1)n + 6 = n^2 + 6.$$

Since there are no real solutions to this equation, $d_n = 0$ for all n. Again, we could use what we know about Cauchy-Euler equations, to show that the general solution is

$$y(x) = c_1 \cos(\sqrt{6} \ln x) + c_2 \sin(\sqrt{6} \ln x).$$

In the last example, we have seen that the power series method does not always work. The key is to write the differential equation in the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0.$$

We already know that $x = 0$ is a singular point of the Cauchy-Euler equation. Putting the equation in the latter form, we have

$$y'' + \frac{a}{x} y' + \frac{b}{x^2} y = 0.$$

We see that $p(x) = a/x$ and $q(x) = b/x^2$ are not defined at $x = 0$. So, we do not expect a convergent power series solution in the neighborhood of $x = 0$.

Theorem 4.1. *The initial value problem*

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0, \quad y(x_0) = \alpha, \quad y'(x_0) = \beta$$

has a unique Taylor series solution converging in the interval $|x - x_0| < R$ if both $p(x)$ and $q(x)$ can be represented by convergent Taylor series converging for $|x - x_0| < R$. (Then, $p(x)$ and $q(x)$ are said to be analytic at $x = x_0$.) As noted earlier, x_0 is then called an ordinary point. Otherwise, if either, or both, $p(x)$ and $q(x)$ are not analytic at x_0, then x_0 is called a singular point.

Example 4.6. Determine if a power series solution exits for $xy'' + 2y' + xy = 0$ near $x = 0$.

Putting this equation in the form

$$y'' + \frac{2}{x}y' + 2y = 0,$$

we see that $a(x)$ is not defined at $x = 0$, so $x = 0$ is a singular point. Let's see how far we can get towards obtaining a series solution.

We let

$$y(x) = \sum_{n=0}^{\infty} c_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2},$$

into the differential equation to obtain

$$\begin{aligned} 0 &= xy'' + 2y' + xy \\ &= x\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 2\sum_{n=1}^{\infty} nc_n x^{n-1} + x\sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} + \sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} \\ &= 2c_1 + \sum_{n=2}^{\infty} [n(n-1) + 2n]c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1}. \end{aligned} \tag{4.26}$$

Here we combined the first two series and pulled out the first term of the second series.

We can re-index the series. In the first series we let $k = n - 1$ and in the second series we let $k = n + 1$. This gives

$$\begin{aligned} 0 &= 2c_1 + \sum_{n=2}^{\infty} n(n+1)c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} \\ &= 2c_1 + \sum_{k=1}^{\infty} (k+1)(k+2)c_{k+1}x^k + \sum_{k=1}^{\infty} c_{k-1}x^k \\ &= 2c_1 + \sum_{k=1}^{\infty} [(k+1)(k+2)c_{k+1} + c_{k-1}]x^k. \end{aligned} \tag{4.27}$$

Setting coefficients to zero, we have $c_1 = 0$ and

$$c_{k+1} = -\frac{1}{(k+12)(k+1)}c_{k-1}, \quad k = 1,2,\ldots.$$

Therefore, we have $c_n = 0$ for $n = 1,3,5,\ldots$. For the even indices, we have

$$\begin{aligned}
k &= 1: \quad c_2 = -\frac{1}{3(2)}c_0 = -\frac{c_0}{3!}. \\
k &= 3: \quad c_4 = -\frac{1}{5(4)}c_2 = \frac{c_0}{5!}. \\
k &= 5: \quad c_6 = -\frac{1}{7(6)}c_4 = -\frac{c_0}{7!}. \\
k &= 7: \quad c_8 = -\frac{1}{9(8)}c_6 = \frac{c_0}{9!}. \qquad (4.28)
\end{aligned}$$

We can see the pattern and write the solution in closed form.

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} c_n x^n \\
&= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots \\
&= c_0 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} \ldots\right) \\
&= c_0 \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \ldots\right) \\
&= c_0 \frac{\sin x}{x}. \qquad (4.29)
\end{aligned}$$

We have another case where the power series method does not yield a general solution.

In the last example we did not find the general solution. However, we did find one solution, $y_1(x) = \frac{\sin x}{x}$. So, we could use the Method of Reduction of Order to obtain the second linearly independent solution. This is carried out in the next example.

Use of the Method of Reduction of Order to obtain a second linearly independent solution. See Section 2.2.1

Example 4.7. Let $y_1(x) = \frac{\sin x}{x}$ be one solution of $xy'' + 2y' + xy = 0$. Find a second linearly independent solution.

Let $y(x) = v(x)y_1(x)$. Inserting this into the differential equation, we have

$$\begin{aligned}
0 &= xy'' + 2y' + xy \\
&= x(vy_1)'' + 2(vy_1)' + xvy_1 \\
&= x(v'y_1 + vy_1')' + 2(v'y_1 + vy_1') + xvy_1 \\
&= x(v''y_1 + 2v'y_1' + vy_1'') + 2(v'y_1 + vy_1') + xvy_1 \\
&= x(v''y_1 + 2v'y_1') + 2v'y_1 + v(xy_1'' + 2y_1' + xy_1) \\
&= x\left[\frac{\sin x}{x}v'' + 2\left(\frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}\right)v'\right] + 2\frac{\sin x}{x}v' \\
&= \sin x v'' + 2\cos x v'. \qquad (4.30)
\end{aligned}$$

This is a first order separable differential equation for $z = v'$. Thus,

$$\sin x \frac{dz}{dx} = -2z\cos x,$$

or

$$\frac{dz}{z} = -2\cot x\, dx.$$

Integrating, we have

$$\ln|z| = 2\ln|\csc x| + C.$$

Setting $C = 0$, we have $v' = z = \csc^2 x$, or $v = -\cot x$. This gives the second solution as

$$y(x) = v(x)y_1(x) = -\cot x \frac{\sin x}{x} = -\frac{\cos x}{x}.$$

4.4 The Frobenius Method

4.4.1 Introduction

It might be possible to use power series to obtain solutions to differential equations in terms of series involving noninteger powers. For example, we found in Example 4.6 that $y_1(x) = \frac{\sin x}{x}$ and $y_2(x) = \frac{\cos x}{x}$ are solutions of the differential equation $xy'' + 2y' + xy = 0$. Series expansions about $x = 0$ are given by

$$\begin{aligned} \frac{\sin x}{x} &= \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) \\ &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots. \end{aligned} \tag{4.31}$$

$$\begin{aligned} \frac{\cos x}{x} &= \frac{1}{x}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) \\ &= \frac{1}{x} - \frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \dots. \end{aligned} \tag{4.32}$$

While the first series is a Taylor series, the second one is not due to the presence of the first term, x^{-1}. We would like to be able to capture such expansions. So, we seek solutions of the form

$$y(x) = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r}$$

for some real number r. This is the basis of the Frobenius Method.

Consider the differential equation,

$$y''(x) + a(x)y'(x) + b(x)y(x) = 0.$$

If $xa(x)$ and $x^2b(x)$ are real analytic, i.e., have convergent Taylor series expansions about $x = 0$, then we can find a solution of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}, \tag{4.33}$$

for some constant r. Furthermore, r is determined from the solution of an *indicial equation*.

Example 4.8. Show that $x = 0$ is a regular singular point of the equation

$$x^2y'' + x(3+x)y' + (1+x)y = 0$$

and then find a solution in the form $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$.

If $x = 0$ is a regular singular point, then we can apply the Frobenius Method.

Rewriting the equation as

$$y' + \frac{3+x}{x}y'' + \frac{(1+x)}{x^2}y = 0,$$

we identify

$$\begin{aligned} a(x) &= \frac{3+x}{x} \\ b(x) &= \frac{(1+x)}{x^2}. \end{aligned}$$

So, $xa(x) = 3 + x$ and $x^2b(x) = 1 + x$ are polynomials in x and are therefore real analytic. Thus, $x = 0$ is a regular singular point.

Example 4.9. Solve

$$x^2y'' + x(3+x)y' + (1+x)y = 0$$

using the Frobenius Method.

In order to find a solution to the differential equation using the Frobenius Method, we assume $y(x)$ and its derivatives are of the form

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} c_n x^{n+r}, \\ y'(x) &= \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}, \\ y''(x) &= \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}. \end{aligned} \tag{4.34}$$

Inserting these series into the differential equation, we have

$$\begin{aligned} 0 &= x^2y'' + x(3+x)y' + (1+x)y = 0 \\ &= x^2 \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2} + x(3+x)\sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1} \\ &\quad + (1+x)\sum_{n=0}^{\infty} c_n x^{n+r} \\ &= \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r} + 3\sum_{n=0}^{\infty} c_n(n+r)x^{n+r} \\ &\quad + \sum_{n=0}^{\infty} c_n(n+r)x^{n+r+1} + \sum_{n=0}^{\infty} c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+1} \\ &= \sum_{n=0}^{\infty} c_n[(n+r)(n+r-1) + 3(n+r) + 1]x^{n+r} + \sum_{n=0}^{\infty} c_n[n+r+1]x^{n+r+1} \\ &= \sum_{n=0}^{\infty} c_n[(n+r)(n+r+2) + 1]x^{n+r} + \sum_{n=0}^{\infty} c_n(n+r+1)x^{n+r+1}. \end{aligned}$$

Next, we re-index the last sum using $k = n+1$ so that both sums involve the powers x^{k+r}. Therefore, we have

$$\sum_{k=0}^{\infty} c_k[(k+r)(k+r+2)+1]x^{k+r} + \sum_{k=1}^{\infty} c_{k-1}(k+r)x^{k+r} = 0. \tag{4.35}$$

We can combine both sums for $k = 1, 2, \ldots$ if we set the coefficients in the $k = 0$ term to zero. Namely,

$$c_0[r(r+2)+1] = 0.$$

If we assume that $c_0 \neq 0$, then

$$r(r+2)+1 = 0.$$

This is the indicial equation. Expanding, we have

$$0 = r^2 + 2r + 1 = (r+1)^2.$$

So, this gives $r = -1$.

We find the indicial equation from the terms with lowest powers of x. Setting $n = 0$ at the bottom of the previous page, the lowest powers are x^r and the coefficient yields $c_0[r(r+2)+1] = 0$. The indicial equation is then $r^2 + 2r + 1 = 0$.

Inserting $r = -1$ into Equation (4.35) and combining the remaining sums, we have

$$\sum_{k=1}^{\infty} \left[k^2 c_k + (k-1)c_{k-1}\right] x^{k-1} = 0.$$

Setting the coefficients equal to zero, we have found that

$$c_k = \frac{1-k}{k^2} c_{k-1}, \quad k = 1, 2, 3, \ldots.$$

So, each coefficient is a multiple of the previous one. In fact, for $k = 1$, we have that

$$c_1 = (0)c_0 = 0.$$

Therefore, all of the coefficients are zero except for c_0. This gives a solution as

$$y_0(x) = \frac{c_0}{x}.$$

We had assumed that $c_0 \neq 0$. What if $c_0 = 0$? Then, Equation (4.35) becomes

$$\sum_{k=1}^{\infty} [((k+r)(k+r+2)+1)c_k + (k+r)c_{k-1}]x^{k+r} = 0.$$

Setting the coefficients equal to zero, we have

$$((k+r)(k+r+2)+1)c_k = -(k+r)c_{k-1}, \quad k = 1, 2, 3, \ldots.$$

When $k = 1$, (the lowest power of x)

$$((1+r)(r+3)+1)c_1 = -(1+r)c_0 = 0.$$

So, $c_1 = 0$, or $0 = (1+r)(r+3)+1 = r^2 + 4r + 4 = (r+2)^2$. If $c_1 \neq 0$, this gives $r = -2$ and

$$c_k = -\frac{(k-2)}{(k-1)^2} c_{k-1}, \quad k = 2, 3, 4, \ldots.$$

Then, we have $c_2 = 0$ and all other coefficient vanish, leaving the solution as

$$y(x) = c_1 x^{1-2} = \frac{c_1}{x}.$$

We only found one solution. We need a second linearly independent solution in order to find the general solution to the differential equation. This can be found using the Method of Reduction of Order from Section 2.2.1 For completeness we will seek a solution $y_2(x) = v(x)y_1(x)$, where $y_1(x) = x^{-1}$. Then,

Method of Reduction of Order.

$$\begin{aligned} 0 &= x^2 y_2'' + x(3+x)y_2' + (1+x)y_2 \\ &= x^2(vy_1)'' + x(3+x)(vy_1)' + (1+x)vy_1 \\ &= [x^2y_1'' + x(3+x)y_1' + (1+x)y_1]v \\ &\quad +[x^2v'' + x(3+x)v']y_1 + 2x^2v'y_1' \\ &= [x^2v'' + x(3+x)v']y_1 + 2x^2v'y_1' \\ &= [x^2v'' + x(3+x)v']x^{-1} - 2x^2v'x^{-2'} \\ &= xv'' + (3+x)v' - 2v' \\ &= xv'' + (1+x)v'. \end{aligned} \tag{4.36}$$

Letting $z = v'$, the last equation can be written as

$$x\frac{dz}{dx} + (1+x)z = 0.$$

This is a separable first order equation. Separating variables and integrating, we have

$$\int \frac{dz}{z} = -\int \frac{1+x}{x}\,dx,$$

or

$$\ln|z| = -\ln|x| - x + C.$$

Exponentiating,

$$z = \frac{dv}{dx} = A\frac{e^{-x}}{x}.$$

Further integration yields

$$v(x) = A\int \frac{e^{-x}}{x}\,dx + B.$$

Thus,

$$y_2(x) = \frac{1}{x}\int \frac{e^{-x}}{x}\,dx.$$

Note that the integral does not have a simple antiderivative and defines the exponential function

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt = -\gamma - \ln x - \sum_{n=1}^\infty \frac{(-1)^2 x^n}{n!n},$$

where $\gamma = 0.5772...$ is the Euler-Mascheroni constant.

Thus, we have found the general solution

$$y(x) = \frac{c_1}{x} + \frac{c_2}{x}E_1(x).$$

Another example is that of Bessel's equation. This is a famous equation which occurs in the solution of problems involving cylindrical symmetry. We discuss the solutions more generally in Section 4.6. Here we apply the Frobenius method to obtain the series solution.

Example 4.10. Solve Bessel's equation using the Frobenius method,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$

We first note that $x = 0$ is a regular singular point. We assume $y(x)$ and its derivatives are of the form

Bessel's equation.

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} c_n x^{n+r}, \\ y'(x) &= \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}, \\ y''(x) &= \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}. \end{aligned} \tag{4.37}$$

Inserting these series into the differential equation, we have

$$\begin{aligned} 0 &= x^2y' + xy' + (x^2 - \nu^2)y \\ &= x^2 \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2} + x \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} \\ &\quad +(x^2-\nu^2) \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} c_n (n+r) x^{n+r} \\ &\quad \sum_{n=0}^{\infty} c_n x^{n+r+2} - \sum_{n=0}^{\infty} \nu^2 c_n x^{n+r} \\ &= \sum_{n=0}^{\infty} [(n+r)(n+r-1) + (n+r) - \nu^2] c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} \\ &= \sum_{n=0}^{\infty} [(n+r)^2 - \nu^2] c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2}. \end{aligned} \tag{4.38}$$

We re-index the last sum with $k = n+2$, or $n = k-2$, to obtain

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} \left([(k+r)^2 - \nu^2] c_k + c_{k-2} \right) x^{k+r} \\ &\quad +(r^2 - \nu^2) c_0 x^r + [(1+r)^2 - \nu^2] c_1 x^{r+1}. \end{aligned} \tag{4.39}$$

We again obtain the indicial equation from the $k = 0$ terms, $r^2 - \nu^2 = 0$. The solutions are $r = \pm\nu$.

We consider the case $r = \nu$. The $k = 1$ terms give

$$\begin{aligned} 0 &= [(1+r)^2 - \nu^2] c_1 \\ &= [(1+\nu)^2 - \nu^2] c_1 \\ &= [1 + 2\nu] c_1 \end{aligned}$$

For $1+2\nu \neq 0$, $c_1 = 0$. [In the next section we consider the case $\nu = -\frac{1}{2}$.]

For $k = 2, 3, \ldots,$ we have

$$[(k+\nu)^2 - \nu^2]c_k + c_{k-2} = 0,$$

or

$$c_k = -\frac{c_{k-2}}{k(k+2\nu)}.$$

Noting that $c_k = 0$, $k = 1, 3, 5, \ldots,$ we evaluate a few of the nonzero coefficients:

$$\begin{aligned}
k &= 2: \quad c_2 = -\frac{1}{2(2+2\nu)}c_0 = -\frac{1}{4(\nu+1)}c_0. \\
k &= 4: \quad c_4 = -\frac{1}{4(4+2\nu)}c_2 = -\frac{1}{8(\nu+2)}c_2 = \frac{1}{2^4(2)(\nu+2)(\nu+1)}c_0. \\
k &= 6: \quad c_6 = -\frac{1}{6(6+2\nu)}c_4 = -\frac{1}{12(\nu+3)}c_4 \\
&= -\frac{1}{2^6(6)(\nu+3))(\nu+2)(\nu+1)}c_0.
\end{aligned}$$

Continuing long enough, we see a pattern emerge,

$$c_{2n} = \frac{(-1)^n}{2^{2n}n!(\nu+1)(\nu+2)\cdots(\nu+n)}, \quad n = 1, 2, \ldots.$$

The solution is given by

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}n!(\nu+1)(\nu+2)\cdots(\nu+n)} x^{2n+\nu}.$$

As we will see later, picking the right value of c_0, this gives the Bessel function of the first kind of order ν provided ν is not a negative integer.

The case $r = -\nu$ is similar. The $k = 1$ terms give

$$\begin{aligned}
0 &= [(1+r)^2 - \nu^2]c_1 \\
&= [(1-\nu)^2 - \nu^2]c_1 \\
&= [1-2\nu]c_1
\end{aligned}$$

For $1-2\nu \neq 0$, $c_1 = 0$. [In the next section we consider the case $\nu = \frac{1}{2}$.]

For $k = 2, 3, \ldots,$ we have

$$[(k-\nu)^2 - \nu^2]c_k + c_{k-2} = 0,$$

or

$$c_k = \frac{c_{k-2}}{k(2\nu-k)}.$$

Noting that $c_k = 0$, $k = 1, 3, 5, \ldots,$ we evaluate a few of the nonzero coefficients:

$$k = 2: \quad c_2 = \frac{1}{2(2\nu-2)}c_0 = \frac{1}{4(\nu-1)}c_0.$$

$$\begin{aligned}
k &= 4: \quad c_4 = \frac{1}{4(2\nu-4)}c_2 = \frac{1}{8(\nu-2)}c_2 = \frac{1}{2^4(2)(\nu-2)(\nu-1)}c_0. \\
k &= 6: \quad c_6 = \frac{1}{6(2\nu-6)}c_4 = \frac{1}{12(\nu-3)}c_4 \\
&= \frac{1}{2^6(6)(\nu-3))(\nu-2)(\nu-1)}c_0.
\end{aligned}$$

Continuing long enough, we see a pattern emerge,

$$c_{2n} = \frac{1}{2^{2n}n!(\nu-1)(\nu-2)\cdots(\nu-n)}, \quad n = 1, 2, \ldots.$$

The solution is given by

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{1}{2^{2n}n!(\nu-1)(\nu-2)\cdots(\nu-n)} x^{2n+\nu}$$

provided ν is not a positive integer. The example $\nu = 1$ is investigated in the next section.

4.4.2 *Roots of the Indicial Equation*

In this section we will consider the types of solutions one can obtain of the differential equation,

$$y''(x) + a(x)y'(x) + b(x)y(x) = 0,$$

when $x = 0$ is a regular singular point. In this case, we assume that $xa(x)$ and $x^2b(x)$ have the convergent Maclaurin series expansions

$$\begin{aligned}
xa(x) &= a_0 + a_1x + a_2x^2 + \ldots \\
x^2b(x) &= b_0 + b_1x + b_2x^2 + \ldots.
\end{aligned} \tag{4.40}$$

Using the Frobenius Method, we assume $y(x)$ and its derivatives are of the form

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} c_n x^{n+r}, \\
y'(x) &= \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1}, \\
y''(x) &= \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2}.
\end{aligned} \tag{4.41}$$

Inserting these series into the differential equation, we obtain

$$\sum_{n=0}^{\infty} c_n \left[(n+r)(n+r-1) + (n+r)xa(x) + x^2b(x)\right] x^{n+r-2} = 0.$$

Using the expansions for $xa(x)$ and $x^2b(x)$, we note that the lowest power of x is $n+r-2$ when $n = 0$. The coefficient for the $n = 0$ term must vanish:

$$c_0\left[r(r-1) + a_0r + b_0\right] = 0.$$

Assuming that $c_0 \neq 0$, we have the indicial equation

$$r(r-1) + a_0 r + b_0 = 0.$$

The roots of the indicial equation determine the type of behavior of the solution. This amounts to considering three different cases. Let the roots of the equation be r_1 and r_2. Then,

i. Distinct roots with $r_1 - r_2 \neq$ integer.

In this case we have two linearly independent solutions,

$$\begin{aligned} y_1(x) &= |x|^{r_1} \sum_{n=0}^{\infty} c_n x^n, \quad c_0 = 1, \\ y_2(x) &= |x|^{r_2} \sum_{n=0}^{\infty} d_n x^n, \quad d_0 = 1. \end{aligned}$$

ii. Equal roots: $r_1 = r_2 = r$.

The form for $y_1(x)$ is the same, but one needs to use the Method of Reduction of Order to seek the second linearly independent solution.

$$\begin{aligned} y_1(x) &= |x|^{r} \sum_{n=0}^{\infty} c_n x^n, \quad c_0 = 1, \\ y_2(x) &= |x|^{r} \sum_{n=0}^{\infty} d_n x^n + y_1(x) \ln |x|, \quad d_0 = 1. \end{aligned}$$

iii. Distinct roots with $r_1 - r_2 =$ positive integer.

Just as in the last case, one needs to find a second linearly independent solution.

$$\begin{aligned} y_1(x) &= |x|^{r_1} \sum_{n=0}^{\infty} c_n x^n, \quad c_0 = 1, \\ y_2(x) &= |x|^{r_2} \sum_{n=0}^{\infty} d_n x^n + \alpha y_1(x) \ln |x|, \quad d_0 = 1. \end{aligned}$$

The constant α can be subsequently determined and in some cases might vanish.

For solutions near regular singular points, $x = x_0$, one has a similar set of cases but for expansions of the form $y_1(x) = |x - x_0|^{r_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n$.

Example 4.11. $x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0.$

In this example $x = 0$ is a singular point. We have $a(x) = 1/x$ and $b(x) = (x^2 - 1/4)/x^2$. Thus,

$$\begin{aligned} x a(x) &= 1, \\ x^2 b(x) &= x^2 - \frac{1}{4}. \end{aligned}$$

So, $a_0 = 1$ and $b_0 = -\frac{1}{4}$. The indicial equation becomes

$$r(r-1) + r - \frac{1}{4} = 0.$$

Simplifying, we have $0 = r^2 - \frac{1}{4}$, or $r = \pm\frac{1}{2}$.

For $r = +\frac{1}{2}$, we insert the series $y(x) = \sqrt{x}\sum_{n=0}^{\infty} c_n x^n$ into the differential equation, collect like terms by re-indexing, and find

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} c_n \left[(n+\frac{1}{2})(n-\frac{1}{2}) + (n+\frac{1}{2}) + x^2 - \frac{1}{4}\right] x^{n-\frac{3}{2}} \\
&= \sum_{n=0}^{\infty} c_n \left[n^2 + n + x^2\right] x^{n-\frac{3}{2}} \\
&= \left[\sum_{n=0}^{\infty} n(n+1)c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2}\right] x^{-\frac{3}{2}} \\
&= \left[\sum_{k=0}^{\infty} k(k+1)c_k x^k + \sum_{n=2}^{\infty} c_{k-2} x^k\right] x^{-\frac{3}{2}} \\
&= 2c_1 x + \sum_{n=2}^{\infty} \left[k(k+1)c_k + c_{k-2}\right] x^{k-\frac{3}{2}}
\end{aligned} \tag{4.42}$$

This gives $c_1 = 0$ and

$$c_k = -\frac{1}{k(k+1)} c_{k-2}, \quad k \geq 2.$$

Iterating, we have $c_k = 0$ for k odd and

$$\begin{aligned}
k &= 2: \quad c_2 = -\frac{1}{3!} c_0. \\
k &= 4: \quad c_4 = -\frac{1}{20} c_2 = \frac{1}{5!} c_0. \\
k &= 6: \quad c_6 = -\frac{1}{42} c_4 = \frac{1}{7!} c_0.
\end{aligned}$$

This gives

$$\begin{aligned}
y_1(x) &= \sqrt{x} \sum_{n=0}^{\infty} c_n x^n \\
&= \sqrt{x}\left(c_0 - \frac{1}{3!} c_0 x^2 + \frac{1}{5!} c_0 x^4 - \dots\right) \\
&= \frac{c_0}{\sqrt{x}}\left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \dots\right) = \frac{c_0}{\sqrt{x}} \sin x.
\end{aligned}$$

Similarly, for for $r = -\frac{1}{2}$, one obtains the second solution, $y_2(x) = \frac{d_0}{\sqrt{x}} \cos x$. Setting $c + 0 = 1$ and $d_0 = 1$, give the two linearly independent solutions. This differential equation is the Bessel equation of order one half and the solutions are Bessel functions of order one half:

$$\begin{aligned}
J_{\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \sin x,\ x > 0, \\
J_{-\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \cos x,\ x > 0.
\end{aligned}$$

Example 4.12. $x^2 y'' + 3xy' + (1 - 2x)y = 0$, $x > 0$.

For this problem $xa(x) = 3$ and $x^2b(x) = 1 - 2x$. Thus, the indicial equation is

$$0 = r(r-1) + 3r + 1 = (r+1)^2.$$

This is a case with two equal roots, $r = -1$. A solution of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

will only result in one solution. We will use this solution form to arrive at a second linearly independent solution.

We will not insert $r = -1$ into the solution form yet. Writing the differential equation in operator form, $L[y] = 0$, we have

$$\begin{aligned}
L[y] &= x^2 y'' + 3xy' + (1-2x)y \\
&= x^2 \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2} + 3x \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} \\
&\quad + (1-2x) \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= \sum_{n=0}^{\infty} [(n+r)(n+r-1) + 2(n+r) + 1] c_n x^{n+r} - \sum_{n=0}^{\infty} 2c_n x^{n+r+1} \\
&= \sum_{n=0}^{\infty} (n+r+1)^2 c_n x^{n+r} - \sum_{n=1}^{\infty} 2c_{n-1} x^{n+r}.
\end{aligned}$$

Setting the coefficients of like terms equal to zero for $n \geq 1$, we have

$$c_n = \frac{2}{(n+r+1)^2} a_{n-1}, \quad n \geq 1.$$

Iterating, we find

$$c_n = \frac{2^n}{[(r+2)(r+3)\cdots(r+n+1)]^2} c_0, \quad n \geq 1.$$

Setting $c_0 = 1$, we have the expression

$$y(x,r) = x^r + \sum_{n=1}^{\infty} \frac{2^n}{[(r+2)(r+3)\cdots(r+n+1)]^2} x^{n+r}.$$

This is not a solution of the differential equation because we did not use the root $r = -1$. Instead, we have

$$L[y(x,c)] = (r+1)^2 x^r \tag{4.43}$$

from the $n = 0$ term. If $r = -1$, then $y(x,-1)$ is one solution of the differential equation. Namely,

$$y_1(x) = x^{-1} + \sum_{n=1}^{\infty} c_n(-1) x^{n-1}.$$

Now consider what happens if we differentiate Equation (4.43) with respect to r:

$$\begin{aligned}
\frac{\partial}{\partial r} L[y(x,r)] &= \left[\frac{\partial y(x,r)}{\partial r}\right] \\
&= 2(r+1)x^r + (r+1)^2 x^r \ln x, \quad x > 0.
\end{aligned}$$

Therefore, $\frac{\partial y(x,r)}{\partial r}$ is also a solution to the differential equation when $r=-1$.

Since

$$y(x,r) = x^r + \sum_{n=1}^{\infty} c_n(r) x^{n+r},$$

we have

$$\begin{aligned}\frac{\partial y(x,r)}{\partial r} &= x^r \ln x + \sum_{n=1}^{\infty} c_n(r) x^{n+r} \ln x + \sum_{n=1}^{\infty} c_n'(r) x^{n+r} \\ &= y(x,r) \ln x + \sum_{n=1}^{\infty} c_n'(r) x^{n+r}. \qquad (4.44)\end{aligned}$$

Therefore, the second solution is given by

$$y_2(x) = \frac{\partial y(x,r)}{\partial r}\Big|_{r=-1} = y(x,-1)\ln x + \sum_{n=1}^{\infty} c_n'(-1) x^{n-1}.$$

In order to determine the solutions, we need to evaluate $c_n(-1)$ and $c_n'(-1)$. Recall that (setting $c_0 = 1$)

$$c_n(r) = \frac{2^n}{[(r+2)(r+3)\cdots(r+n+1)]^2}, \quad n \geq 1.$$

Therefore,

$$\begin{aligned} c_n(-1) &= \frac{2^n}{[(1)(2)\cdots(n)]^2}, \quad n \geq 1, \\ &= \frac{2^n}{(n!)^2}. \qquad (4.45)\end{aligned}$$

Next we compute $c_n'(-1)$. This can be done using logarithmic differentiation. We consider

$$\begin{aligned}\ln c_n(r) &= \ln\left(\frac{2^n}{[(r+2)(r+3)\cdots(r+n+1)]^2}\right) \\ &= \ln 2^n - 2\ln(r+2) - 2\ln(r+3)\cdots - 2\ln(r+n+1).\end{aligned}$$

Differentiating with respect to r and evaluating at $r=-1$, we have

$$\begin{aligned}\frac{c_n'(r)}{c_n(r)} &= -2\left(\frac{1}{r+2} + \frac{1}{r+2} + \cdots + \frac{1}{r+3}\right) \\ c_n'(r) &= -2c_n(r)\left(\frac{1}{r+2} + \frac{1}{r+3} + \cdots + \frac{1}{r+n+1}\right). \\ c_n'(-1) &= -2c_n(-1)\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right) \\ &= -\frac{2^{n+1}}{(n!)^2} H_n, \qquad (4.46)\end{aligned}$$

where we have defined

$$H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}.$$

This gives the second solution as

$$y_2(x) = y_1(x)\ln x - \sum_{n=1}^{\infty} \frac{2^{n+1}}{(n!)^2} H_n x^{n-1}.$$

Example 4.13. $x^2y'' + xy' + (x^2-1)y = 0.$

This equation is similar to the last example, but it is the Bessel equation of order one. The indicial equation is given by

$$0 = r(r-1) + r - 1 = r^2 - 1.$$

The roots are $r_1 = 1$, $r_2 = -1$. In this case the roots differ by an integer, $r_1 - r_2 = 2$.

The first solution can be obtained using

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} c_n x^{n+1}, \\
y'(x) &= \sum_{n=0}^{\infty} c_n (n+1) x^n, \\
y''(x) &= \sum_{n=0}^{\infty} c_n (n+1)(n) x^{n-1}.
\end{aligned} \tag{4.47}$$

Inserting these series into the differential equation, we have

$$\begin{aligned}
0 &= x^2y'' + xy' + (x^2-1)y \\
&= x^2 \sum_{n=0}^{\infty} c_n n(n+1) x^{n-1} + x \sum_{n=0}^{\infty} c_n (n+1) x^n \\
&\quad + (x^2-1) \sum_{n=0}^{\infty} c_n x^{n+1} \\
&= \sum_{n=0}^{\infty} c_n n(n+1) x^{n+1} + \sum_{n=0}^{\infty} c_n (n+1) x^{n+1} \\
&\quad \sum_{n=0}^{\infty} c_n x^{n+3} - \sum_{n=0}^{\infty} c_n x^{n+1} \\
&= \sum_{n=0}^{\infty} [n(n+1) + (n+1) - 1] c_n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+3} \\
&= \sum_{n=0}^{\infty} [(n+1)^2 - 1] c_n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+3}.
\end{aligned} \tag{4.48}$$

We re-index the last sum with $k = n+2$, or $n = k-2$, to obtain

$$\sum_{n=2}^{\infty} \left([(k+1)^2 - 1] c_k + c_{k-2} \right) x^{k+1} + 3c_1 x^2 = 0.$$

Thus, $c_1 = 0$ and

$$c_k = -\frac{1}{(k+1)^2 - 1} c_{k-2} = -\frac{1}{k(k+2)} c_{k-2}, \quad k \geq 2.$$

Since $c_1 = 0$, all c_k's vanish for odd k. For $k = 2n$, we have

$$c_{2n} = -\frac{1}{2n(2n+2)} c_{2n-2} = -\frac{1}{4n(n+1)} c_{2(n-1)}, \quad n \geq 1.$$

$$\begin{aligned}
n &= 1: \quad c_2 = -\frac{1}{4(1)(2)}c_0. \\
n &= 2: \quad c_4 = -\frac{1}{4(2)(3)}c_2 = \frac{1}{4^2 2!3!}c_0. \\
n &= 3: \quad c_6 = -\frac{1}{4(3)(4)}c_4 = \frac{1}{4^3 3!4!}c_0.
\end{aligned}$$

Continuing, this gives

$$c_{2n} = \frac{(-1)^n}{4^n n!(n+1)!}c_0$$

and the first solution is

$$y_1(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!(n+1)!} x^{2n+1}.$$

Now we look for a second linearly independent solution of the form

$$y_2(x) = \sum_{n=0}^{\infty} d_n x^{n-1} + \alpha y_1(x) \ln x, \quad x > 0.$$

The function and its derivatives are given by

$$\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} d_n x^{n-1} + \alpha y_1(x) \ln x, \\
y_2'(x) &= \sum_{n=0}^{\infty} (n-1) d_n x^{n-2} + \alpha [y_1'(x) \ln x + y_1(x) x^{-1}], \\
y_2''(x) &= \sum_{n=0}^{\infty} (n-1) d_n (n-2) x^{n-3} \\
&\quad + \alpha [y_1''(x) \ln x + 2y_1'(x) x^{-1} - y_1(x) x^{-2}].
\end{aligned}$$

Inserting these series into the differential equation, we have

$$\begin{aligned}
0 &= x^2 y_2'' + x y_2' + (x^2 - 1) y_2 \\
&= x^2 \sum_{n=0}^{\infty} (n-1)(n-2) d_n x^{n-3} \\
&\quad + \alpha x^2 [y_1''(x) \ln x + 2y_1'(x) x^{-1} - y_1(x) x^{-2}] \\
&\quad + x \sum_{n=0}^{\infty} (n-1) d_n x^{n-2} + \alpha x [y_1'(x) \ln x + y_1(x) x^{-1}] \\
&\quad + (x^2 - 1) \left[\sum_{n=0}^{\infty} d_n x^{n-1} + \alpha y_1(x) \ln x \right] \\
&= \sum_{n=0}^{\infty} [(n-1)(n-2) + (n-1) - 1] d_n x^{n-1} + \sum_{n=0}^{\infty} d_n x^{n+1} \\
&\quad + \alpha [x^2 y_1''(x) + x y_1'(x) + (x^2 - 1) y_1(x)] \ln x \\
&\quad + \alpha [2x y_1'(x) - y_1(x)] + \alpha y_1(x) \\
&= \sum_{n=0}^{\infty} n(n-2) d_n x^{n-1} + \sum_{n=0}^{\infty} d_n x^{n+1} + 2\alpha x y_1'(x). \\
&= \sum_{n=0}^{\infty} n(n-2) d_n x^{n-1} + \sum_{n=0}^{\infty} d_n x^{n+1} + 2\alpha \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{4^n (n+1)! n!} x^{2n+1}.
\end{aligned}$$

We now try to combine like powers of x. First, we combine the terms involving d_n's,

$$-d_1 + d_0 x + \sum_{k=3}^{\infty} [k(k-2)d_k + d_{k-2}]x^{k-1} = -2\alpha \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{4^n n!(n+1)!} x^{2n+1}.$$

Since there are no even powers on the right-hand side of the equation, we find $d_1 = 0$, and $k(k-2)d_k + d_{k-2} = 0$, $k \geq 3$ and k odd. Therefore, all odd d_k's vanish.

Next, we set $k - 1 = 2n + 1$, or $k = 2n + 2$, in the remaining terms, obtaining

$$d_0 x + \sum_{n=1}^{\infty} [(2n+2)(2n)d_{2n+2} + d_{2n}]x^{2n+1} = -2\alpha x - 2\alpha \sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{4^n (n+1)! n!} x^{2n+1}.$$

Thus, $d_0 = -2\alpha$. We choose $\alpha = -\frac{1}{2}$, making $d_0 = 1$. The remaining terms satisfy the relation

$$(2n+2)(2n)d_{2n+2} + d_{2n} = \frac{(-1)^n(2n+1)}{4^n n!(n+1)!}, \quad n \geq 1$$

or

$$d_{2n+2} = \frac{d_{2n}}{4(n+1)(n)} + \frac{(-1)^n(2n+1)}{(n+1)(n)4^{n+1}(n+1)!n!}, \quad n \geq 1.$$

$$\begin{aligned}
d_4 &= -\frac{1}{4(2)(1)} d_2 - \frac{3}{(2)(1)4^2 2!1!} \\
&= -\frac{1}{4^2 2!1!}\left(4d_2 + \frac{3}{2}\right). \\
d_6 &= -\frac{1}{4(3)(2)} d_4 + \frac{5}{(3)(2)4^3 3!2!} \\
&= -\frac{1}{4(3)(2)}\left(-\frac{1}{4^2 2!1!}\left(4d_2 + \frac{3}{2}\right)\right) + \frac{5}{(3)(2)4^3 3!2!} \\
&= \frac{1}{4^3 3!2!}\left(4d_2 + \frac{3}{2} + \frac{5}{6}\right). \\
d_8 &= -\frac{1}{4(4)(3)} d_6 - \frac{7}{(4)(3)4^4 4!3!} \\
&= -\frac{1}{4^4 4!3!}\left(4d_2 + \frac{3}{2} + \frac{5}{6}\right) - \frac{7}{(4)(3)4^4 4!3!} \\
&= -\frac{1}{4^4 4!3!}\left(4d_2 + \frac{3}{2} + \frac{5}{6} + \frac{7}{12}\right).
\end{aligned} \tag{4.49}$$

Choosing $4d_2 = 1$, the coefficients take an interesting form. Namely,

$$\begin{aligned}
1 + \frac{3}{2} &= 1 + \frac{1}{2} + 1 \\
1 + \frac{3}{2} + \frac{5}{6} &= 1 + \frac{1}{2}\frac{1}{3} + 1 + \frac{1}{2} \\
1 + \frac{3}{2} + \frac{5}{6} + \frac{7}{12} &= 1 + \frac{1}{2}\frac{1}{3} + \frac{1}{4} + 1 + \frac{1}{2} + \frac{1}{3}.
\end{aligned} \tag{4.50}$$

Defining the partial sums of the harmonic series,

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad H_0 = 0,$$

these coefficients become $H_n + H_{n-1}$ and the coefficients in the expansion are

$$d_{2n} = \frac{(-1)^{n-1}(H_n + H_{n-1})}{4^n n!(n-1)!}, n = 1, 2, \ldots.$$

We can verify this by computing d_{10}:

$$\begin{aligned} d_{10} &= -\frac{1}{4(4)(3)} d_8 + \frac{9}{(5)(4)4^5 5!4!} \\ &= \frac{1}{4^5 5!4!}\left(4d_2 + \frac{3}{2} + \frac{5}{6} + \frac{7}{12}\right) + \frac{9}{(5)(4)4^5 5!4!} \\ &= \frac{1}{4^5 5!4!}\left(4d_2 + \frac{3}{2} + \frac{5}{6} + \frac{7}{12} + \frac{9}{20}\right) \\ &= \frac{1}{4^5 5!4!}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) \\ &= \frac{1}{4^5 5!4!}(H_5 + H_4). \end{aligned}$$

This gives the second solution as

$$y_2(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(H_n + H_{n-1})}{4^n n!(n-1)!} x^{2n-1} - \frac{1}{2} y_1(x) \ln x + x^{-1}.$$

4.5 *Legendre Polynomials*

THE LEGENDRE[1] POLYNOMIALS are one of a set of classical orthogonal polynomials . These polynomials satisfy a second-order linear differential equation. This differential equation occurs naturally in the solution of initial-boundary value problems in three dimensions which possess some spherical symmetry. Legendre polynomials, or Legendre functions of the first kind, are solutions of the differential equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0.$$

In Example 4.4 we found that for n an integer, there are polynomial solutions. The first of these are given by $P_0(x) = c_0$, $P_1(x) = c_1 x$, and $P_2(x) = c_2(1 - 3x^2)$. As the Legendre equation is a linear second-order differential equation, we expect two linearly independent solutions. The second solution, called the Legendre function of the second kind, is given by $Q_n(x)$ and is not well behaved at $x = \pm 1$. For example,

$$Q_0(x) = \frac{1}{2} \ln \frac{1+x}{1-x}.$$

We will mostly focus on the Legendre polynomials and some of their properties in this section.

[1] Adrien-Marie Legendre (1752-1833) was a French mathematician who made many contributions to analysis and algebra.

A generalization of the Legendre equation is given by $(1 - x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right] y = 0$. Solutions to this equation, $P_n^m(x)$ and $Q_n^m(x)$, are called the associated Legendre functions of the first and second kind.

4.5.1 Properties of Legendre Polynomials

LEGENDRE POLYNOMIALS BELONG TO THE CLASS of classical orthogonal polynomials. Members of this class satisfy similar properties. First, we have the Rodrigues Formula for Legendre polynomials:

The Rodrigues Formula.

$$\boxed{P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad n \in N_0. \qquad (4.51)}$$

From the Rodrigues formula, one can show that $P_n(x)$ is an nth degree polynomial. Also, for n odd, the polynomial is an odd function and for n even, the polynomial is an even function.

Example 4.14. Determine $P_2(x)$ from the Rodrigues Formula:

$$\begin{aligned} P_2(x) &= \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 \\ &= \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1) \\ &= \frac{1}{8}\frac{d}{dx}(4x^3-4x) \\ &= \frac{1}{8}(12x^2-4) \\ &= \frac{1}{2}(3x^2-1). \end{aligned} \qquad (4.52)$$

Note that we get the same result as we found in the last section using orthogonalization.

Table 4.2: Tabular computation of the Legendre polynomials using the Rodrigues Formula.

n	$(x^2-1)^n$	$\frac{d^n}{dx^n}(x^2-1)^n$	$\frac{1}{2^n n!}$	$P_n(x)$
0	1	1	1	1
1	x^2-1	$2x$	$\frac{1}{2}$	x
2	x^4-2x^2+1	$12x^2-4$	$\frac{1}{8}$	$\frac{1}{2}(3x^2-1)$
3	$x^6-3x^4+3x^2-1$	$120x^3-72x$	$\frac{1}{48}$	$\frac{1}{2}(5x^3-3x)$

The first several Legendre polynomials are given in Table 4.2. In Figure 4.1 we show plots of these Legendre polynomials.

The Three-Term Recursion Formula.

The classical orthogonal polynomials also satisfy a three-term recursion formula (or, recurrence relation or formula). In the case of the Legendre polynomials, we have

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad n = 1, 2, \ldots. \qquad (4.53)$$

This can also be rewritten by replacing n with $n-1$ as

$$(2n-1)xP_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x), \quad n = 1, 2, \ldots. \qquad (4.54)$$

Example 4.15. Use the recursion formula to find $P_2(x)$ and $P_3(x)$, given that $P_0(x) = 1$ and $P_1(x) = x$.

We first begin by inserting $n = 1$ into Equation (4.53):

$$2P_2(x) = 3xP_1(x) - P_0(x) = 3x^2 - 1.$$

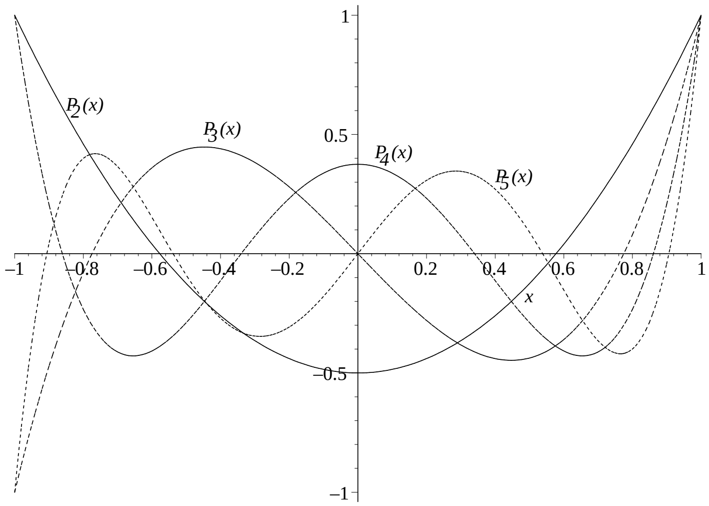


Figure 4.1: Plots of the Legendre polynomials $P_2(x)$, $P_3(x)$, $P_4(x)$, and $P_5(x)$.

So, $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

For $n = 2$, we have

$$\begin{aligned} 3P_3(x) &= 5xP_2(x) - 2P_1(x) \\ &= \frac{5}{2}x(3x^2 - 1) - 2x \\ &= \frac{1}{2}(15x^3 - 9x). \end{aligned} \tag{4.55}$$

This gives $P_3(x) = \frac{1}{2}(5x^3 - 3x)$. These expressions agree with the earlier results.

4.5.2 *The Generating Function for Legendre Polynomials*

A PROOF OF THE THREE-TERM RECURSION FORMULA can be obtained from the generating function of the Legendre polynomials. Many special functions have such generating functions. In this case, it is given by

$$g(x,t) = \frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \quad |x| \le 1, |t| < 1. \tag{4.56}$$

This generating function occurs often in applications. In particular, it arises in potential theory, such as electromagnetic or gravitational potentials. These potential functions are $\frac{1}{r}$ type functions.

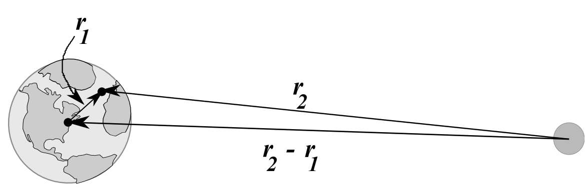


Figure 4.2: The position vectors used to describe the tidal force on the Earth due to the moon.

For example, the gravitational potential between the Earth and the moon is proportional to the reciprocal of the magnitude of the difference between their positions relative to some coordinate system. An even better example would be to place the origin at the center of the Earth and consider the forces on the non-pointlike Earth due to the moon. Consider a piece of the Earth at position $\mathbf{r}_1$ and the moon at position $\mathbf{r}_2$ as shown in Figure 4.2. The tidal potential Φ is proportional to

$$\Phi \propto \frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|} = \frac{1}{\sqrt{(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{r}_2 - \mathbf{r}_1)}} = \frac{1}{\sqrt{r_1^2 - 2r_1r_2\cos\theta + r_2^2}},$$

where θ is the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$.

Typically, one of the position vectors is much larger than the other. Let's assume that $r_1 \ll r_2$. Then, one can write

$$\Phi \propto \frac{1}{\sqrt{r_1^2 - 2r_1r_2\cos\theta + r_2^2}} = \frac{1}{r_2}\frac{1}{\sqrt{1 - 2\frac{r_1}{r_2}\cos\theta + \left(\frac{r_1}{r_2}\right)^2}}.$$

Now, define $x = \cos\theta$ and $t = \frac{r_1}{r_2}$. We then have that the tidal potential is proportional to the generating function for the Legendre polynomials! So, we can write the tidal potential as

$$\Phi \propto \frac{1}{r_2}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r_1}{r_2}\right)^n.$$

The first term in the expansion, $\frac{1}{r_2}$, is the gravitational potential that gives the usual force between the Earth and the moon. [Recall that the gravitational potential for mass m at distance r from M is given by $\Phi = -\frac{GMm}{r}$ and that the force is the gradient of the potential, $\mathbf{F} = -\nabla\Phi \propto \nabla\left(\frac{1}{r}\right)$.] The next terms will give expressions for the tidal effects.

Now that we have some idea as to where this generating function might have originated, we can proceed to use it. First of all, the generating function can be used to obtain special values of the Legendre polynomials.

Example 4.16. Evaluate $P_n(0)$ using the generating function. $P_n(0)$ is found by considering $g(0,t)$. Setting $x = 0$ in Equation (4.56), we have

$$\begin{aligned} g(0,t) &= \frac{1}{\sqrt{1+t^2}} \\ &= \sum_{n=0}^{\infty} P_n(0)t^n \\ &= P_0(0) + P_1(0)t + P_2(0)t^2 + P_3(0)t^3 + \dots. \end{aligned} \tag{4.57}$$

We can use the binomial expansion to find the final answer. Namely, we have

$$\frac{1}{\sqrt{1+t^2}} = 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 + \dots.$$

Comparing these expansions, we have the $P_n(0) = 0$ for n odd and for even integers one can show that[2]

$$P_{2n}(0) = (-1)^n\frac{(2n-1)!!}{(2n)!!}, \tag{4.58}$$

[2] This example can be finished by first proving that

$$(2n)!! = 2^n n!$$

and

$$(2n-1)!! = \frac{(2n)!}{(2n)!!} = \frac{(2n)!}{2^n n!}.$$

where $n!!$ is the *double factorial*,

$$n!! = \begin{cases} n(n-2)\ldots(3)1, & n > 0, \text{odd}, \\ n(n-2)\ldots(4)2, & n > 0, \text{even}, \\ 1, & n = 0, -1. \end{cases}.$$

Example 4.17. Evaluate $P_n(-1)$. This is a simpler problem. In this case we have

$$g(-1,t) = \frac{1}{\sqrt{1+2t+t^2}} = \frac{1}{1+t} = 1 - t + t^2 - t^3 + \ldots.$$

Therefore, $P_n(-1) = (-1)^n$.

Proof of the three-term recursion formula using the generating function.

Example 4.18. Prove the three-term recursion formula,

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0, \quad k = 1, 2, \ldots,$$

using the generating function.

We can also use the generating function to find recurrence relations. To prove the three term recursion (4.53) that we introduced above, then we need only differentiate the generating function with respect to t in Equation (4.56) and rearrange the result. First note that

$$\frac{\partial g}{\partial t} = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \frac{x-t}{1-2xt+t^2} g(x,t).$$

Combining this with

$$\frac{\partial g}{\partial t} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1},$$

we have

$$(x-t)g(x,t) = (1-2xt+t^2) \sum_{n=0}^{\infty} nP_n(x)t^{n-1}.$$

Inserting the series expression for $g(x,t)$ and distributing the sum on the right side, we obtain

$$(x-t)\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} nP_n(x)t^{n-1} - \sum_{n=0}^{\infty} 2nxP_n(x)t^n + \sum_{n=0}^{\infty} nP_n(x)t^{n+1}.$$

Multiplying out the $x-t$ factor and rearranging, leads to three separate sums:

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} - \sum_{n=0}^{\infty} (2n+1)xP_n(x)t^n + \sum_{n=0}^{\infty} (n+1)P_n(x)t^{n+1} = 0. \quad (4.59)$$

Each term contains powers of t that we would like to combine into a single sum. This is done by re-indexing. For the first sum, we could use the new index $k = n-1$. Then, the first sum can be written

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} = \sum_{k=-1}^{\infty} (k+1)P_{k+1}(x)t^k.$$

Using different indices is just another way of writing out the terms. Note that

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} = 0 + P_1(x) + 2P_2(x)t + 3P_3(x)t^2 + \ldots$$

and

$$\sum_{k=-1}^{\infty} (k+1)P_{k+1}(x)t^k = 0 + P_1(x) + 2P_2(x)t + 3P_3(x)t^2 + \ldots$$

actually give the same sum. The indices are sometimes referred to as *dummy indices* because they do not show up in the expanded expression and can be replaced with another letter.

If we want to do so, we could now replace all the k's with n's. However, we will leave the k's in the first term and now re-index the next sums in Equation (4.59). The second sum just needs the replacement $n = k$ and the last sum we re-index using $k = n + 1$. Therefore, Equation (4.59) becomes

$$\sum_{k=-1}^{\infty} (k+1)P_{k+1}(x)t^k - \sum_{k=0}^{\infty}(2k+1)xP_k(x)t^k + \sum_{k=1}^{\infty} kP_{k-1}(x)t^k = 0. \tag{4.60}$$

We can now combine all the terms, noting the $k = -1$ term is automatically zero and the $k = 0$ terms give

$$P_1(x) - xP_0(x) = 0. \tag{4.61}$$

Of course, we know this already. So, that leaves the $k > 0$ terms:

$$\sum_{k=1}^{\infty} \left[(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x)\right] t^k = 0. \tag{4.62}$$

Since this is true for all t, the coefficients of the t^k's are zero, or

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0, \quad k = 1, 2, \ldots.$$

While this is the standard form for the three-term recurrence relation, the earlier form is obtained by setting $k = n - 1$.

There are other recursion relations that we list in the box below. Equation (4.63) was derived using the generating function. Differentiating it with respect to x, we find Equation (4.64). Equation (4.65) can be proven using the generating function by differentiating $g(x,t)$ with respect to x and rearranging the resulting infinite series just as in this last manipulation. This will be left as Problem 9. Combining this result with Equation (4.63), we can derive Equations (4.66) and (4.67). Adding and subtracting these equations yields Equations (4.68) and (4.69).

Recursion Formulae for Legendre Polynomials for $n = 1, 2, \ldots$.

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \tag{4.63}$$
$$(n+1)P'_{n+1}(x) = (2n+1)[P_n(x) + xP'_n(x)] - nP'_{n-1}(x) \tag{4.64}$$
$$P_n(x) = P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x) \tag{4.65}$$
$$P'_{n-1}(x) = xP'_n(x) - nP_n(x) \tag{4.66}$$
$$P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x) \tag{4.67}$$
$$P'_{n+1}(x) + P'_{n-1}(x) = 2xP'_n(x) + P_n(x). \tag{4.68}$$
$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x). \tag{4.69}$$
$$(x^2-1)P'_n(x) = nxP_n(x) - nP_{n-1}(x) \tag{4.70}$$

Finally, Equation (4.70) can be obtained using Equations (4.66) and (4.67). Just multiply Equation (4.66) by x,

$$x^2P'_n(x) - nxP_n(x) = xP'_{n-1}(x).$$

Now use Equation (4.67), but first replace n with $n-1$ to eliminate the $xP'_{n-1}(x)$ term:

$$x^2P'_n(x) - nxP_n(x) = P'_n(x) - nP_{n-1}(x).$$

Rearranging gives the Equation (4.70).

Example 4.19. Use the generating function to prove

$$\|P_n\|^2 = \int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}.$$

Another use of the generating function is to obtain the normalization constant. This can be done by first squaring the generating function in order to get the products $P_n(x)P_m(x)$, and then integrating over x.

The normalization constant.

Squaring the generating function must be done with care, as we need to make proper use of the dummy summation index. So, we first write

$$\begin{aligned}\frac{1}{1-2xt+t^2} &= \left[\sum_{n=0}^{\infty} P_n(x)t^n\right]^2 \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(x)P_m(x)t^{n+m}.\end{aligned} \tag{4.71}$$

Integrating from $x=-1$ to $x=1$ and using the orthogonality of the Legendre polynomials, we have

$$\begin{aligned}\int_{-1}^{1}\frac{dx}{1-2xt+t^2} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t^{n+m}\int_{-1}^{1} P_n(x)P_m(x)\,dx \\ &= \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1} P_n^2(x)\,dx.\end{aligned} \tag{4.72}$$

The integral on the left can be evaluated by first noting

$$\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx) + C.$$

Then, we have

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right).$$

Expanding this expression about $t=0$, we obtain[3]

$$\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right) = \sum_{n=0}^{\infty} \frac{2}{2n+1} t^{2n}.$$

[3] You will need the series expansion

$$\begin{aligned} \ln(1+x) &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots. \end{aligned}$$

Comparing this result with Equation (4.72), we find that

$$\|P_n\|^2 = \int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}. \tag{4.73}$$

Finally, we can use the properties of the Legendre polynomials to obtain the Legendre differential equation. We begin by differentiating Equation (4.70) and using Equation (4.66) to simplify:

$$\begin{aligned} \frac{d}{dx}\left((x^2-1)P_n'(x)\right) &= nP_n(x) + nxP_n'(x) - nP_{n-1}'(x) \\ &= nP_n(x) + n^2P_n(x) \\ &= n(n+1)P_n(x). \end{aligned} \tag{4.74}$$

4.6 Bessel Functions

Bessel functions arise in many problems in physics possessing cylindrical symmetry, such as the vibrations of circular drumheads and the radial modes in optical fibers. They also provide us with another orthogonal set of basis functions.

Bessel functions have a long history and were named after Friedrich Wilhelm Bessel (1784-1846).

The first occurrence of Bessel functions (zeroth order) was in the work of Daniel Bernoulli on heavy chains (1738). More general Bessel functions were studied by Leonhard Euler in 1781 and in his study of the vibrating membrane in 1764. Joseph Fourier found them in the study of heat conduction in solid cylinders and Siméon Poisson (1781-1840) in heat conduction of spheres (1823).

The history of Bessel functions, did not just originate in the study of the wave and heat equations. These solutions originally came up in the study of the Kepler problem, describing planetary motion. According to G. N. Watson in his *Treatise on Bessel Functions*, the formulation and solution of Kepler's Problem was discovered by Joseph-Louis Lagrange (1736-1813), in 1770. Namely, the problem was to express the radial coordinate and what is called the eccentric anomaly, E, as functions of time. Lagrange found expressions for the coefficients in the expansions of r and E in trigonometric functions of time. However, he only computed the first few coefficients. In

1816, Friedrich Wilhelm Bessel (1784-1846) had shown that the coefficients in the expansion for r could be given an integral representation. In 1824, he presented a thorough study of these functions, which are now called Bessel functions.

You might have seen Bessel functions in a course on differential equations as solutions of the differential equation

$$x^2 y'' + xy' + (x^2 - p^2)y = 0. \tag{4.75}$$

Solutions to this equation are obtained in the form of series expansions. Namely, one seeks solutions of the form

$$y(x) = \sum_{j=0}^{\infty} a_j x^{j+n}$$

by determining the form the coefficients must take. We will leave this for a homework exercise and simply report the results.

One solution of the differential equation is the *Bessel function of the first kind of order p*, given as

$$y(x) = J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}. \tag{4.76}$$

Here $\Gamma(x)$ s the Gamma function, satisfying $\Gamma(x+1) = x\Gamma(x)$. It is a generalization of the factorial and is discussed in the next section.

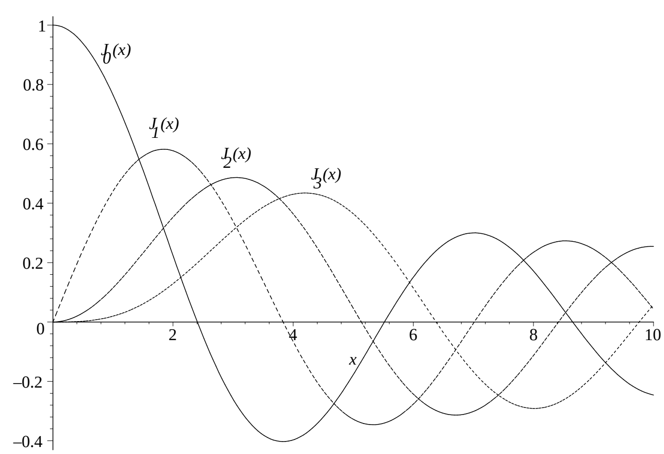


Figure 4.3: Plots of the Bessel functions $J_0(x)$, $J_1(x)$, $J_2(x)$, and $J_3(x)$.

In Figure 4.3, we display the first few Bessel functions of the first kind of integer order. Note that these functions can be described as decaying oscillatory functions.

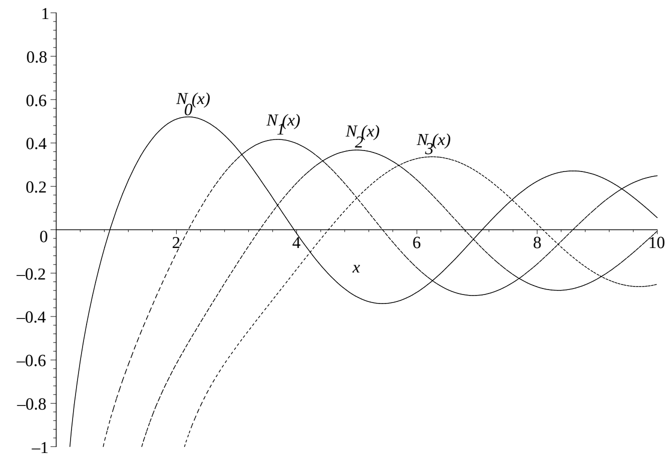
A second linearly independent solution is obtained for p not an integer as $J_{-p}(x)$. However, for p an integer, the $\Gamma(n+p+1)$ factor leads to evaluations of the Gamma function at zero, or negative integers, when p is negative. Thus, the above series is not defined in these cases.

Another method for obtaining a second linearly independent solution is through a linear combination of $J_p(x)$ and $J_{-p}(x)$ as

$$N_p(x) = Y_p(x) = \frac{\cos \pi p J_p(x) - J_{-p}(x)}{\sin \pi p}. \tag{4.77}$$

These functions are called the Neumann functions, or Bessel functions of the second kind of order p.

Figure 4.4: Plots of the Neumann functions $N_0(x)$, $N_1(x)$, $N_2(x)$, and $N_3(x)$.



In Figure 4.4, we display the first few Bessel functions of the second kind of integer order. Note that these functions are also decaying oscillatory functions. However, they are singular at $x = 0$.

In many applications, one desires bounded solutions at $x = 0$. These functions do not satisfy this boundary condition. For example, one standard problem is to describe the oscillations of a circular drumhead. For this problem one solves the two dimensional wave equation using separation of variables in cylindrical coordinates. The radial equation leads to a Bessel equation. The Bessel function solutions describe the radial part of the solution and one does not expect a singular solution at the center of the drum. The amplitude of the oscillation must remain finite. Thus, only Bessel functions of the first kind can be used.

Bessel functions satisfy a variety of properties, which we will only list at this time for Bessel functions of the first kind. The reader will have the opportunity to prove these for homework.

Derivative Identities These identities follow directly from the manipulation of the series solution.

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x). \tag{4.78}$$

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x). \tag{4.79}$$

Recursion Formulae The next identities follow from adding, or subtracting, the derivative identities.

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x). \tag{4.80}$$

$$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x). \tag{4.81}$$

Orthogonality One can recast the Bessel equation into an eigenvalue problem whose solutions form an orthogonal basis of functions on $L_x^2(0,a)$. Using Sturm-Liouville Theory, one can show that

$$\int_0^a x J_p(j_{pn}\frac{x}{a}) J_p(j_{pm}\frac{x}{a})\, dx = \frac{a^2}{2}\left[J_{p+1}(j_{pn})\right]^2 \delta_{n,m}, \tag{4.82}$$

where j_{pn} is the nth root of $J_p(x)$, $J_p(j_{pn}) = 0$, $n = 1, 2, \ldots$. A list of some of these roots is provided in Table 4.3.

n	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
1	2.405	3.832	5.136	6.380	7.588	8.771
2	5.520	7.016	8.417	9.761	11.065	12.339
3	8.654	10.173	11.620	13.015	14.373	15.700
4	11.792	13.324	14.796	16.223	17.616	18.980
5	14.931	16.471	17.960	19.409	20.827	22.218
6	18.071	19.616	21.117	22.583	24.019	25.430
7	21.212	22.760	24.270	25.748	27.199	28.627
8	24.352	25.904	27.421	28.908	30.371	31.812
9	27.493	29.047	30.569	32.065	33.537	34.989

Table 4.3: The zeros of Bessel Functions, $J_m(j_{mn}) = 0$.

Generating Function

$$e^{x(t-\frac{1}{t})/2} = \sum_{n=-\infty}^{\infty} J_n(x) t^n, \quad x > 0, t \neq 0. \tag{4.83}$$

Integral Representation

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - n\theta)\, d\theta, \quad x > 0, n \in \mathrm{Z}. \tag{4.84}$$

4.7 *Gamma Function*

A FUNCTION THAT OFTEN OCCURS IN THE STUDY OF SPECIAL FUNCTIONS is the Gamma function. We will need the Gamma function in the next section on Fourier-Bessel series.

The name and symbol for the Gamma function were first given by Legendre in 1811. However, the search for a generalization of the factorial extends back to the 1720's when Euler provided the first representation of the factorial as an infinite product, later to be modified by others like Gauß, Weierstraß, and Legendre.

For $x > 0$ we define the Gamma function as

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt, \quad x > 0. \tag{4.85}$$

The Gamma function is a generalization of the factorial function and a plot is shown in Figure 4.5. In fact, we have

$$\Gamma(1) = 1$$

and

$$\Gamma(x+1) = x\Gamma(x).$$

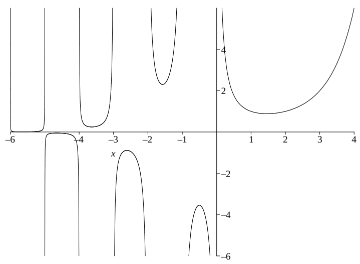


Figure 4.5: Plot of the Gamma function.

The reader can prove this identity by simply performing an integration by parts. (See Problem 13.) In particular, for integers $n \in Z^+$, we then have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-2) = n(n-1)\cdots 2\Gamma(1) = n!.$$

We can also define the Gamma function for negative, non-integer values of x. We first note that by iteration on $n \in Z^+$, we have

$$\Gamma(x+n) = (x+n-1)\cdots(x+1)x\Gamma(x), \quad x+n > 0.$$

Solving for $\Gamma(x)$, we then find

$$\Gamma(x) = \frac{\Gamma(x+n)}{(x+n-1)\cdots(x+1)x}, \quad -n < x < 0.$$

Note that the Gamma function is undefined at zero and the negative integers.

Example 4.20. We now prove that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

This is done by direct computation of the integral:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt.$$

Letting $t = z^2$, we have

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-z^2}\, dz.$$

Due to the symmetry of the integrand, we obtain the classic integral[4]

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^\infty e^{-z^2}\, dz,$$

which can be performed using a standard trick. Consider the integral

$$I = \int_{-\infty}^\infty e^{-x^2}\, dx.$$

Then,

$$I^2 = \int_{-\infty}^\infty e^{-x^2}\, dx \int_{-\infty}^\infty e^{-y^2}\, dy.$$

Note that we changed the integration variable. This will allow us to write this product of integrals as a double integral:

$$I^2 = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)}\, dxdy.$$

[4] Using a substitution $x^2 = \beta y^2$, we can show the more general result:

$$\int_{-\infty}^\infty e^{-\beta y^2}\, dy = \sqrt{\frac{\pi}{\beta}}.$$

This is an integral over the entire xy-plane. We can transform this Cartesian integration to an integration over polar coordinates. The integral becomes

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2}\, r dr d\theta.$$

This is simple to integrate and we have $I^2 = \pi$. So, the final result is found by taking the square root of both sides:

$$\Gamma\left(\frac{1}{2}\right) = I = \sqrt{\pi}.$$

In Problem 15, the reader will prove the more general identity

$$\Gamma(n + \frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}.$$

Another useful relation, which we only state, is

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

The are many other important relations, including infinite products, which we will not need at this point. The reader is encouraged to read about these elsewhere. In the meantime, we move on to the discussion of another important special function in physics and mathematics.

4.8 Hypergeometric Functions

Hypergeometric functions are probably the most useful, but least understood, class of functions. They typically do not make it into the undergraduate curriculum and seldom in graduate curriculum. Most functions that you know can be expressed using hypergeometric functions. There are many approaches to these functions and the literature can fill books. [5]

[5] See for example *Special Functions* by G. E. Andrews, R. Askey, and R. Roy, 1999, Cambridge University Press.

In 1812 Gauss published a study of the *hypergeometric series*

$$\begin{aligned} y(x) = 1 \quad &+ \quad \frac{\alpha\beta}{\gamma}x + \frac{\alpha(1+\alpha)\beta(1+\beta)}{2!\gamma(1+\gamma)}x^2 \\ &+ \quad \frac{\alpha(1+\alpha)(2+\alpha)\beta(1+\beta)(2+\beta)}{3!\gamma(1+\gamma)(2+\gamma)}x^3 + \ldots . \end{aligned} \tag{4.86}$$

Here α, β, γ, and x are real numbers. If one sets $\alpha = 1$ and $\beta = \gamma$, this series reduces to the familiar geometric series

$$y(x) = 1 + x + x^2 + x^3 + \ldots .$$

The hypergeometric series is actually a solution of the differential equation

$$x(1-x)y'' + [\gamma - (\alpha + \beta + 1)x]\, y' - \alpha\beta y = 0. \tag{4.87}$$

This equation was first introduced by Euler and latter studied extensively by Gauss, Kummer and Riemann. It is sometimes called Gauss' equation.

Note that there is a symmetry in that α and β may be interchanged without changing the equation. The points $x = 0$ and $x = 1$ are regular singular points. Series solutions may be sought using the Frobenius method. It can be confirmed that the above hypergeometric series results.

A more compact form for the hypergeometric series may be obtained by introducing new notation. One typically introduces the *Pochhammer symbol*, $(\alpha)_n$, satisfying

i. $(\alpha)_0 = 1$ if $\alpha \neq 0$.

ii $(\alpha)_k = \alpha(1+\alpha)\ldots(k-1+\alpha)$, for $k = 1, 2, \ldots$.

This symbol was introduced by Leo August Pochhammer (1841-1920).

Consider $(1)_n$. For $n = 0$, $(1)_0 = 1$. For $n > 0$,

$$(1)_n = 1(1+1)(2+1)\ldots[(n-1)+1].$$

This reduces to $(1)_n = n!$. In fact, one can show that

$$(k)_n = \frac{(n+k-1)!}{(k-1)!}$$

for k and n positive integers. In fact, one can extend this result to noninteger values for k by introducing the gamma function:

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.$$

We can now write the hypergeometric series in standard notation as

$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n} x^n.$$

For $\gamma > \beta > 0$, one can express the hypergeometric function as an integral:

$${}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tx)^{-\alpha}\, dt.$$

Using this notation, one can show that the general solution of Gauss' equation is

$$y(x) = A\, {}_2F_1(\alpha, \beta; \gamma; x) + Bx^{1-\gamma}\, {}_2F_1(1-\gamma+\alpha, 1-\gamma+\beta; 2-\gamma; x).$$

By carefully letting β approach ∞, one obtains what is called the *confluent hypergeometric function*. This in effect changes the nature of the differential equation. Gauss' equation has three regular singular points at $x = 0, 1, \infty$. One can transform Gauss' equation by letting $x = u/\beta$. This changes the regular singular points to $u = 0, \beta, \infty$. Letting $\beta \to \infty$, two of the singular points merge.

The new confluent hypergeometric function is then given as

$${}_1F_1(\alpha; \gamma; u) = \lim_{\beta\to\infty} {}_2F_1\left(\alpha, \beta; \gamma; \frac{u}{\beta}\right).$$

This function satisfies the differential equation

$$xy'' + (\gamma - x)y' - \alpha y = 0.$$

The purpose of this section is only to introduce the hypergeometric function. Many other special functions are related to the hypergeometric function after making some variable transformations. For example, the Legendre polynomials are given by

$$P_n(x) =_2 F_1(-n, n+1; 1; \frac{1-x}{2}).$$

In fact, one can also show that

$$\sin^{-1} x = x_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right).$$

The Bessel function $J_p(x)$ can be written in terms of confluent geometric functions as

$$J_p(x) = \frac{1}{\Gamma(p+1)}\left(\frac{z}{2}\right)^p e^{-iz}\,{}_1F_1\left(\frac{1}{2}+p, 1+2p; 2iz\right).$$

These are just a few connections of the powerful hypergeometric functions to some of the elementary functions that you know.

Problems

1. Find the first four terms in the Taylor series expansion of the solution to

a. $y'(x) = y(x) - x$, $y(0) = 2$.

b. $y'(x) = 2xy(x) - x^3$, $y(0) = 1$.

c. $(1 + x)y'(x) = py(x)$, $y(0) = 1$.

d. $y'(x) = \sqrt{x^2 + y^2(x)}$, $y(0) = 1$.

e. $y''(x) - 2xy'(x) + 2y(x) = 0$, $y(0) = 1$, $y'(0) = 0$.

2. Use the power series method to obtain power series solutions about the given point.

a. $y' = y - x$, $y(0) = 2$, $x_0 = 0$.

b. $(1 + x)y'(x) = py(x)$, $x_0 = 0$.

c. $y'' + 9y = 0$, $y(0) = 1$, $y'(0) = 0$, $x_0 = 0$.

d. $y'' + 2x^2y' + xy = 0$, $x_0 = 0$.

e. $y'' - xy' + 3y = 0$, $y(0) = 2$, $x_0 = 0$.

f. $xy'' - xy' + y = e^x$, $y(0) = 1$, $y'(0) = 2$, $x_0 = 0$.

g. $x^2y'' - xy' + y = 0$, $x_0 = 1$.

3. In Example 4.3 we found the general Maclaurin series solution to

$$y'' - xy' - y = 0.$$

a. Show that one solution of this problem is $y_1(x) = e^{x^2/2}$.

b. Find the first five nonzero terms of the Maclaurin series expansion for $y_1(x)$ and

c. According to Maple, a second solution is $\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) e^{x^2/2}$. Use the Method of Reduction of Order to find this second linearly independent solution. Note: The error function is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

d. Verify that this second solution is consistent with the solution found in Example 4.3.

4. Find at least one solution about the singular point $x = 0$ using the power series method. Determine the second solution using the method of reduction of order.

a. $x^2y'' + 2xy' - 2y = 0$.

b. $xy'' + (1 - x)y' - y = 0$.

c. $x^2y'' - x(1 - x)y' + y = 0$.

5. List the singular points in the finite plane of the following:

a. $(1-x^2)y'' + \frac{3}{x+2}y' + \frac{(1-x)^2}{x+3}y = 0.$

b. $\frac{1}{x}y'' + \frac{3(x-4)}{x+6}y' + \frac{x^2(x-2)}{x-1}y = 0.$

c. $y'' + xy = 0.$

d. $x^2(x-2)y'' + 4(x-2)y' + 3y = 0.$

6. Sometimes one is interested in solutions for large x. This leads to the concept of the *point at infinity*.

a. Let $z = \frac{1}{x}$ and $y(x) = v(z)$. Using the Chain Rule, show that

$$\begin{aligned} \frac{dy}{dx} &= -z^2\frac{dv}{dz}, \\ \frac{d^2y}{dx^2} &= z^4\frac{d^2v}{dz^2} + 2z^2\frac{dv}{dz}. \end{aligned}$$

b. Use the transformation in part (a) to transform the differential equation $x^2y'' + y = 0$ into an equation for $w(z)$ and classify the point at infinity by determining if $w = 0$ is an ordinary point, a regular singular point, or an irregular singular point.

c. Classify the point at infinity for the following equations:

i. $y'' + xy = 0.$

ii. $x^2(x-2)y'' + 4(x-2)y' + 3y = 0.$

7. Find the general solution of the following equations using the Method of Frobenius at $x = 0$.

a. $4xy'' + 2y' + y = 0.$

b. $y'' + \frac{1}{4x^2}y = 0.$

c. $xy'' + 2y' + xy = 0.$

d. $y'' + \frac{1}{2x}y' - \frac{x+1}{2x^2}y = 0.$

d. $4x^2y'' + 4xy' + (4x^2-1)y = 0.$

e. $2x(x+1)y'' + 3(x+1)y' - y = 0.$

f. $x^2y'' - x(1+x)y' + y = 0.$

g. $xy'' - (4+x)y' + 2y = 0.$

8. Find $P_4(x)$ using

a. The Rodrigues Formula in Equation (4.51).

b. The three-term recursion formula in Equation (4.53).

9. In Equations (4.63) through (4.70) we provide several identities for Legendre polynomials. Derive the results in Equations (4.64) through (4.70) as described in the text. Namely,

a. Differentiating Equation (4.63) with respect to x, derive Equation (4.64).

b. Derive Equation (4.65) by differentiating $g(x,t)$ with respect to x and rearranging the resulting infinite series.

c. Combining the previous result with Equation (4.63), derive Equations (4.66) and (4.67).

d. Adding and subtracting Equations (4.66) and (4.67), obtain Equations (4.68) and (4.69).

e. Derive Equation (4.70) using some of the other identities.

10. Use the recursion relation (4.53) to evaluate $\int_{-1}^{1} xP_n(x)P_m(x)\,dx$, $n \leq m$.

11. Consider the Hermite equation

$$y'' - 2xy' + 2ny = 0.$$

Determine the recursion formula for the coefficients in a series solution, $y(x) = \sum_{k=0}^{\infty} c_k x^k$. Show that if n is an integer, then one of the linearly independent solutions is a polynomial.

12. Using the power series method to find the general solution of Airy's equation,

$$y'' - xy = 0.$$

13. Use integration by parts to show $\Gamma(x+1) = x\Gamma(x)$.

14. Prove the double factorial identities:

$$(2n)!! = 2^n n!$$

and

$$(2n-1)!! = \frac{(2n)!}{2^n n!}.$$

15. Using the property $\Gamma(x+1) = x\Gamma(x)$, $x > 0$, and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, prove that

$$\Gamma(n+\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}.$$

16. Express the following as Gamma functions. Namely, noting the form $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt$ and using an appropriate substitution, each expression can be written in terms of a Gamma function.

a. $\displaystyle\int_0^\infty x^{2/3} e^{-x}\,dx.$

b. $\displaystyle\int_0^\infty x^5 e^{-x^2}\,dx.$

c. $\displaystyle\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^n\,dx.$

17. A solution of Bessel's equation, $x^2y'' + xy' + (x^2 - n^2)y = 0,$, can be found using the guess $y(x) = \sum_{j=0}^{\infty} a_j x^{j+n}$. One obtains the recurrence relation $a_j = \frac{-1}{j(2n+j)} a_{j-2}$. Show that for $a_0 = (n!2^n)^{-1}$, we get the Bessel function of the first kind of order n from the even values $j = 2k$:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k}.$$

18. Use the infinite series in Problem 17 to derive the derivative identities (4.78) and (4.79):

a. $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$.

b. $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$.

19. Prove the following identities based on those in Problem 18.

a. $J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$.

b. $J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x)$.

20. Use the derivative identities of Bessel functions, (4.78) and (4.79), and integration by parts to show that

$$\int x^3 J_0(x)\, dx = x^3 J_1(x) - 2x^2 J_2(x) + C.$$

21. We can rewrite the series solution for Bessel functions,

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k},$$

in a form which will allow the order to be non-integer, $n = \nu$, by using the Gamma function. You will need

$$\Gamma\left(k + \frac{1}{2}\right) = \frac{(2k-1)!!}{2^k}\sqrt{\pi}$$

.

a. Extend the series definition of the Bessel function of the first kind of order ν, $J_\nu(x)$, for $\nu \geq 0$ by writing the series solution for $y(x)$ in Problem 17 using the Gamma function.

b. Extend the series to $J_{-\nu}(x)$, for $\nu \geq 0$. Discuss the resulting series and what happens when ν is a positive integer.

c. Use these results to obtain the closed form expressions

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x,$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

d. Use the results in part c with the recursion formula for Bessel functions,

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x),$$

to obtain a closed form for $J_{3/2}(x)$.

22. Show that setting $\alpha = 1$ and $\beta = \gamma$ in ${}_2F_1(\alpha, \beta; \gamma; x)$ leads to the geometric series.

23. Prove the following:

a. $(a)_n = (a)_{n-1}(a + n - 1)$, $n = 1, 2, \ldots$, $a \neq 0$.

b. $(a)_n = a(a+1)_{n-1}$, $n = 1, 2, \ldots$, $a \neq 0$.

24. Verify the following relations by transforming the hypergeometric equation into the equation satisfied by each function.

a. $P_n(x) =_2 F_1(-n, n+1; 1; \frac{1-x}{2})$.

b. $\sin^{-1} x = x_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right)$.

c. $J_p(x) = \dfrac{1}{\Gamma(p+1)} \left(\dfrac{z}{2}\right)^p e^{-iz}\, {}_1F_1\left(\dfrac{1}{2} + p, 1 + 2p; 2iz\right)$.