<div align="right">*Chapter 5*</div>

Laplace Transforms

"We could, of course, use any notation we want; do not laugh at notations; invent them, they are powerful. In fact, mathematics is, to a large extent, invention of better notations." - Richard P. Feynman (1918-1988)

5.1 The Laplace Transform

UP TO THIS POINT WE HAVE ONLY EXPLORED Fourier exponential transforms as one type of integral transform. The Fourier transform is useful on infinite domains. However, students are often introduced to another integral transform, called the Laplace transform, in their introductory differential equations class. These transforms are defined over semi-infinite domains and are useful for solving initial value problems for ordinary differential equations.

The Fourier and Laplace transforms are examples of a broader class of transforms known as integral transforms. For a function $f(x)$ defined on an interval (a, b), we define the integral transform

Integral transform on $[a, b]$ with respect to the integral kernel, $K(x, k)$.

$$F(k) = \int_a^b K(x, k) f(x) \, dx,$$

where $K(x, k)$ is a specified kernel of the transform. Looking at the Fourier transform, we see that the interval is stretched over the entire real axis and the kernel is of the form, $K(x, k) = e^{ikx}$. In Table 5.1 we show several types of integral transforms.

Laplace Transform	$F(s) = \int_0^\infty e^{-sx} f(x) \, dx$
Fourier Transform	$F(k) = \int_{-\infty}^\infty e^{ikx} f(x) \, dx$
Fourier Cosine Transform	$F(k) = \int_0^\infty \cos(kx) f(x) \, dx$
Fourier Sine Transform	$F(k) = \int_0^\infty \sin(kx) f(x) \, dx$
Mellin Transform	$F(k) = \int_0^\infty x^{k-1} f(x) \, dx$
Hankel Transform	$F(k) = \int_0^\infty x J_n(kx) f(x) \, dx$

Table 5.1: A Table of Common Integral Transforms.

It should be noted that these integral transforms inherit the linearity of integration. Namely, let $h(x) = \alpha f(x) + \beta g(x)$, where α and β are constants.

Then,

$$
\begin{aligned}
H(k) &= \int_a^b K(x,k)h(x)\,dx, \\
&= \int_a^b K(x,k)(\alpha f(x) + \beta g(x))\,dx, \\
&= \alpha \int_a^b K(x,k)f(x)\,dx + \beta \int_a^b K(x,k)g(x)\,dx, \\
&= \alpha F(x) + \beta G(x).
\end{aligned}
\tag{5.1}
$$

Therefore, we have shown linearity of the integral transforms. We have seen the linearity property used for Fourier transforms and we will use linearity in the study of Laplace transforms.

We now turn to Laplace transforms. The Laplace transform of a function $f(t)$ is defined as

$$
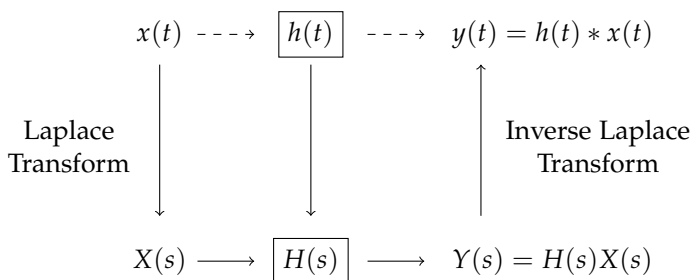F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt, \quad s > 0.
\tag{5.2}
$$

This is an improper integral and one needs

$$
\lim_{t \to \infty} f(t)e^{-st} = 0
$$

to guarantee convergence.

Laplace transforms also have proven useful in engineering for solving circuit problems and doing systems analysis. In Figure 5.1 it is shown that a signal $x(t)$ is provided as input to a linear system, indicated by $h(t)$. One is interested in the system output, $y(t)$, which is given by a convolution of the input and system functions. By considering the transforms of $x(t)$ and $h(t)$, the transform of the output is given as a product of the Laplace transforms in the s-domain. In order to obtain the output, one needs to compute a convolution product for Laplace transforms similar to the convolution operation we had seen for Fourier transforms earlier in the chapter. Of course, for us to do this in practice, we have to know how to compute Laplace transforms.

Figure 5.1: A schematic depicting the use of Laplace transforms in systems theory.

$$
\begin{array}{ccccc}
x(t) & \dashrightarrow & \boxed{h(t)} & \dashrightarrow & y(t) = h(t) * x(t) \\
& & & & \\
\text{Laplace} & & & & \text{Inverse Laplace} \\
\text{Transform} & & & & \text{Transform} \\
& & & & \\
X(s) & \longrightarrow & \boxed{H(s)} & \longrightarrow & Y(s) = H(s)X(s)
\end{array}
$$

5.2 Properties and Examples of Laplace Transforms

It is typical that one makes use of Laplace transforms by referring to a Table of transform pairs. A sample of such pairs is given in Table 5.2.

Combining some of these simple Laplace transforms with the properties of the Laplace transform, as shown in Table 5.3, we can deal with many applications of the Laplace transform. We will first prove a few of the given Laplace transforms and show how they can be used to obtain new transform pairs. In the next section we will show how these transforms can be used to sum infinite series and to solve initial value problems for ordinary differential equations.

$f(t)$	$F(s)$	$f(t)$	$F(s)$
c	$\dfrac{c}{s}$	e^{at}	$\dfrac{1}{s-a}, \quad s > a$
t^n	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$	$e^{at} \sin \omega t$	$\dfrac{\omega}{(s-a)^2 + \omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$	$e^{at} \cos \omega t$	$\dfrac{s-a}{(s-a)^2 + \omega^2}$
$t \sin \omega t$	$\dfrac{2\omega s}{(s^2 + \omega^2)^2}$	$t \cos \omega t$	$\dfrac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
$\sinh at$	$\dfrac{a}{s^2 - a^2}$	$\cosh at$	$\dfrac{s}{s^2 - a^2}$
$H(t-a)$	$\dfrac{e^{-as}}{s}, \quad s > 0$	$\delta(t-a)$	$e^{-as}, \quad a \geq 0, s > 0$

Table 5.2: Table of Selected Laplace Transform Pairs.

We begin with some simple transforms. These are found by simply using the definition of the Laplace transform.

Example 5.1. Show that $\mathcal{L}[1] = \frac{1}{s}$.

For this example, we insert $f(t) = 1$ into the definition of the Laplace transform:

$$\mathcal{L}[1] = \int_0^\infty e^{-st} \, dt.$$

This is an improper integral and the computation is understood by introducing an upper limit of a and then letting $a \to \infty$. We will not always write this limit, but it will be understood that this is how one computes such improper integrals. Proceeding with the computation, we have

$$
\begin{aligned}
\mathcal{L}[1] &= \int_0^\infty e^{-st} \, dt \\
&= \lim_{a \to \infty} \int_0^a e^{-st} \, dt \\
&= \lim_{a \to \infty} \left(-\frac{1}{s} e^{-st} \right)_0^a \\
&= \lim_{a \to \infty} \left(-\frac{1}{s} e^{-sa} + \frac{1}{s} \right) = \frac{1}{s}.
\end{aligned}
\tag{5.3}
$$

Thus, we have found that the Laplace transform of 1 is $\frac{1}{s}$. This result can be extended to any constant c, using the linearity of the transform, $\mathcal{L}[c] = c\mathcal{L}[1]$. Therefore,

$$\mathcal{L}[c] = \frac{c}{s}.$$

Example 5.2. Show that $\mathcal{L}[e^{at}] = \frac{1}{s-a}$, for $s > a$.

For this example, we can easily compute the transform. Again, we only need to compute the integral of an exponential function.

$$
\begin{aligned}
\mathcal{L}[e^{at}] &= \int_0^\infty e^{at} e^{-st} \, dt \\
&= \int_0^\infty e^{(a-s)t} \, dt \\
&= \left(\frac{1}{a-s} e^{(a-s)t} \right)_0^\infty \\
&= \lim_{t \to \infty} \frac{1}{a-s} e^{(a-s)t} - \frac{1}{a-s} = \frac{1}{s-a}.
\end{aligned}
\tag{5.4}
$$

Note that the last limit was computed as $\lim_{t \to \infty} e^{(a-s)t} = 0$. This is only true if $a - s < 0$, or $s > a$. [Actually, a could be complex. In this case we would only need s to be greater than the real part of a, $s > Re(a)$.]

Example 5.3. Show that $\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}$ and $\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$.

For these examples, we could again insert the trigonometric functions directly into the transform and integrate. For example,

$$
\mathcal{L}[\cos at] = \int_0^\infty e^{-st} \cos at \, dt.
$$

Recall how one evaluates integrals involving the product of a trigonometric function and the exponential function. One integrates by parts two times and then obtains an integral of the original unknown integral. Rearranging the resulting integral expressions, one arrives at the desired result. However, there is a much simpler way to compute these transforms.

Recall that $e^{iat} = \cos at + i \sin at$. Making use of the linearity of the Laplace transform, we have

$$
\mathcal{L}[e^{iat}] = \mathcal{L}[\cos at] + i\mathcal{L}[\sin at].
$$

Thus, transforming this complex exponential will simultaneously provide the Laplace transforms for the sine and cosine functions!

The transform is simply computed as

$$
\mathcal{L}[e^{iat}] = \int_0^\infty e^{iat} e^{-st} \, dt = \int_0^\infty e^{-(s-ia)t} \, dt = \frac{1}{s - ia}.
$$

Note that we could easily have used the result for the transform of an exponential, which was already proven. In this case, $s > Re(ia) = 0$.

We now extract the real and imaginary parts of the result using the complex conjugate of the denominator:

$$
\frac{1}{s - ia} = \frac{1}{s - ia} \frac{s + ia}{s + ia} = \frac{s + ia}{s^2 + a^2}.
$$

Reading off the real and imaginary parts, we find the sought-after transforms,

$$\mathcal{L}[\cos at] = \frac{s}{s^2 + a^2},$$
$$\mathcal{L}[\sin at] = \frac{a}{s^2 + a^2}. \tag{5.5}$$

Example 5.4. Show that $\mathcal{L}[t] = \frac{1}{s^2}$.

For this example we evaluate

$$\mathcal{L}[t] = \int_0^\infty t e^{-st}\, dt.$$

This integral can be evaluated using the method of integration by parts:

$$\int_0^\infty t e^{-st}\, dt = -t\frac{1}{s}e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$
$$= \frac{1}{s^2}. \tag{5.6}$$

Example 5.5. Show that $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$ for nonnegative integer n.

We have seen the $n = 0$ and $n = 1$ cases: $\mathcal{L}[1] = \frac{1}{s}$ and $\mathcal{L}[t] = \frac{1}{s^2}$. We now generalize these results to nonnegative integer powers, $n > 1$, of t. We consider the integral

$$\mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\, dt.$$

Following the previous example, we again integrate by parts:[1]

$$\int_0^\infty t^n e^{-st}\, dt = -t^n\frac{1}{s}e^{-st}\Big|_0^\infty + \frac{n}{s}\int_0^\infty t^{-n}e^{-st}\, dt$$
$$= \frac{n}{s}\int_0^\infty t^{-n}e^{-st}\, dt. \tag{5.7}$$

We could continue to integrate by parts until the final integral is computed. However, look at the integral that resulted after one integration by parts. It is just the Laplace transform of t^{n-1}. So, we can write the result as

$$\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}].$$

This is an example of a recursive definition of a sequence. In this case, we have a sequence of integrals. Denoting

$$I_n = \mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\, dt$$

and noting that $I_0 = \mathcal{L}[1] = \frac{1}{s}$, we have the following:

$$I_n = \frac{n}{s}I_{n-1}, \quad I_0 = \frac{1}{s}. \tag{5.8}$$

This is also what is called a difference equation. It is a first-order difference equation with an "initial condition," I_0. The next step is to solve this difference equation.

[1] This integral can just as easily be done using differentiation. We note that

$$\left(-\frac{d}{ds}\right)^n \int_0^\infty e^{-st}\, dt = \int_0^\infty t^n e^{-st}\, dt.$$

Since

$$\int_0^\infty e^{-st}\, dt = \frac{1}{s},$$
$$\int_0^\infty t^n e^{-st}\, dt = \left(-\frac{d}{ds}\right)^n \frac{1}{s} = \frac{n!}{s^{n+1}}.$$

We compute $\int_0^\infty t^n e^{-st}\, dt$ by turning it into an initial value problem for a first-order difference equation and finding the solution using an iterative method.

Finding the solution of this first-order difference equation is easy to do using simple iteration. Note that replacing n with $n - 1$, we have

$$I_{n-1} = \frac{n-1}{s} I_{n-2}.$$

Repeating the process, we find

$$
\begin{aligned}
I_n &= \frac{n}{s} I_{n-1} \\
&= \frac{n}{s} \left(\frac{n-1}{s} I_{n-2} \right) \\
&= \frac{n(n-1)}{s^2} I_{n-2} \\
&= \frac{n(n-1)(n-2)}{s^3} I_{n-3}.
\end{aligned}
\tag{5.9}
$$

We can repeat this process until we get to I_0, which we know. We have to carefully count the number of iterations. We do this by iterating k times and then figuring out how many steps will get us to the known initial value. A list of iterates is easily written out:

$$
\begin{aligned}
I_n &= \frac{n}{s} I_{n-1} \\
&= \frac{n(n-1)}{s^2} I_{n-2} \\
&= \frac{n(n-1)(n-2)}{s^3} I_{n-3} \\
&= \ldots \\
&= \frac{n(n-1)(n-2)\ldots(n-k+1)}{s^k} I_{n-k}.
\end{aligned}
\tag{5.10}
$$

Since we know $I_0 = \frac{1}{s}$, we choose to stop at $k = n$ obtaining

$$I_n = \frac{n(n-1)(n-2)\ldots(2)(1)}{s^n} I_0 = \frac{n!}{s^{n+1}}.$$

Therefore, we have shown that $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$.

Such iterative techniques are useful in obtaining a variety of integrals, such as $I_n = \int_{-\infty}^{\infty} x^{2n} e^{-x^2} \, dx$.

As a final note, one can extend this result to cases when n is not an integer. To do this, we use the Gamma function, which was discussed in Section 4.7. Recall that the Gamma function is the generalization of the factorial function and is defined as

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \, dt.
\tag{5.11}$$

Note the similarity to the Laplace transform of t^{x-1} :

$$\mathcal{L}[t^{x-1}] = \int_0^{\infty} t^{x-1} e^{-st} \, dt.$$

For $x - 1$ an integer and $s = 1$, we have that

$$\Gamma(x) = (x - 1)!.$$

Thus, the Gamma function can be viewed as a generalization of the factorial and we have shown that

$$\mathcal{L}[t^p] = \frac{\Gamma(p+1)}{s^{p+1}}$$

for $p > -1$.

Now we are ready to introduce additional properties of the Laplace transform in Table 5.3. We have already discussed the first property, which is a consequence of the linearity of integral transforms. We will prove the other properties in this and the following sections.

Laplace Transform Properties
$\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$
$\mathcal{L}[tf(t)] = -\dfrac{d}{ds}F(s)$
$\mathcal{L}\left[\dfrac{df}{dt}\right] = sF(s) - f(0)$
$\mathcal{L}\left[\dfrac{d^2f}{dt^2}\right] = s^2F(s) - sf(0) - f'(0)$
$\mathcal{L}[e^{at}f(t)] = F(s-a)$
$\mathcal{L}[H(t-a)f(t-a)] = e^{-as}F(s)$
$\mathcal{L}[(f*g)(t)] = \mathcal{L}[\int_0^t f(t-u)g(u)\,du] = F(s)G(s)$

Table 5.3:   Table of selected Laplace transform properties.

Example 5.6. Show that $\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0)$.

We have to compute

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt}e^{-st}\,dt.$$

We can move the derivative off f by integrating by parts. This is similar to what we had done when finding the Fourier transform of the derivative of a function. Letting $u = e^{-st}$ and $v = f(t)$, we have

$$
\begin{aligned}
\mathcal{L}\left[\frac{df}{dt}\right] &= \int_0^\infty \frac{df}{dt}e^{-st}\,dt \\
&= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt \\
&= -f(0) + sF(s). \tag{5.12}
\end{aligned}
$$

Here we have assumed that $f(t)e^{-st}$ vanishes for large t.

The final result is that

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0).$$

Example 6: Show that $\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0) - f'(0)$.

We can compute this Laplace transform using two integrations by parts, or we could make use of the last result. Letting $g(t) = \frac{df(t)}{dt}$, we have

$$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = \mathcal{L}\left[\frac{dg}{dt}\right] = sG(s) - g(0) = sG(s) - f'(0).$$

But,

$$G(s) = \mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0).$$

So,

$$
\begin{aligned}
\mathcal{L}\left[\frac{d^2f}{dt^2}\right] &= sG(s) - f'(0) \\
&= s\left[sF(s) - f(0)\right] - f'(0) \\
&= s^2F(s) - sf(0) - f'(0). \tag{5.13}
\end{aligned}
$$

We will return to the other properties in Table 5.3 after looking at a few applications.

5.3 Solution of ODEs Using Laplace Transforms

ONE OF THE TYPICAL APPLICATIONS OF LAPLACE TRANSFORMS is the solution of nonhomogeneous linear constant coefficient differential equations. In the following examples we will show how this works.

The general idea is that one transforms the equation for an unknown function $y(t)$ into an algebraic equation for its transform, $Y(t)$. Typically, the algebraic equation is easy to solve for $Y(s)$ as a function of s. Then, one transforms back into t-space using Laplace transform tables and the properties of Laplace transforms. The scheme is shown in Figure 5.2.

Figure 5.2: The scheme for solving an ordinary differential equation using Laplace transforms. One transforms the initial value problem for $y(t)$ and obtains an algebraic equation for $Y(s)$. Solve for $Y(s)$ and the inverse transform gives the solution to the initial value problem.
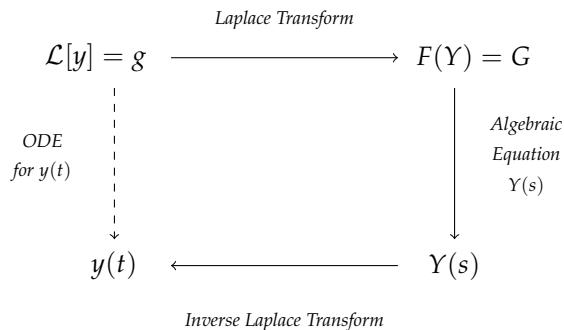
Example 5.7. Solve the initial value problem $y' + 3y = e^{2t}$, $y(0) = 1$.

The first step is to perform a Laplace transform of the initial value problem. The transform of the left side of the equation is

$$\mathcal{L}[y' + 3y] = sY - y(0) + 3Y = (s + 3)Y - 1.$$

Transforming the right-hand side, we have

$$\mathcal{L}[e^{2t}] = \frac{1}{s - 2}.$$

Combining these two results, we obtain

$$(s + 3)Y - 1 = \frac{1}{s - 2}.$$

The next step is to solve for $Y(s)$:

$$Y(s) = \frac{1}{s+3} + \frac{1}{(s-2)(s+3)}.$$

Now we need to find the inverse Laplace transform. Namely, we need to figure out what function has a Laplace transform of the above form. We will use the tables of Laplace transform pairs. Later we will show that there are other methods for carrying out the Laplace transform inversion.

The inverse transform of the first term is e^{-3t}. However, we have not seen anything that looks like the second form in the table of transforms that we have compiled, but we can rewrite the second term using a partial fraction decomposition. Let's recall how to do this.

The goal is to find constants A and B such that

$$\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}. \tag{5.14}$$

We picked this form because we know that recombining the two terms into one term will have the same denominator. We just need to make sure the numerators agree afterward. So, adding the two terms, we have

$$\frac{1}{(s-2)(s+3)} = \frac{A(s+3) + B(s-2)}{(s-2)(s+3)}.$$

Equating numerators,

$$1 = A(s+3) + B(s-2).$$

There are several ways to proceed at this point.

a. Method 1.

We can rewrite the equation by gathering terms with common powers of s, we have

$$(A+B)s + 3A - 2B = 1.$$

The only way that this can be true for all s is that the coefficients of the different powers of s agree on both sides. This leads to two equations for A and B:

$$A + B = 0,$$
$$3A - 2B = 1. \tag{5.15}$$

The first equation gives $A = -B$, so the second equation becomes $-5B = 1$. The solution is then $A = -B = \frac{1}{5}$.

b. Method 2.

Since the equation $\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$ is true for all s, we can pick specific values. For $s = 2$, we find $1 = 5A$, or $A = \frac{1}{5}$. For $s = -3$, we find $1 = -5B$, or $B = -\frac{1}{5}$. Thus, we obtain the same result as Method 1, but much quicker.

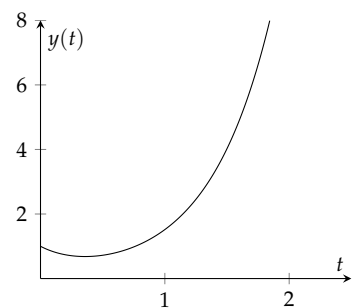This is an example of carrying out a partial fraction decomposition.

Figure 5.3: A plot of the solution to Example 5.7.

c. Method 3.

We could just inspect the original partial fraction problem. Since the numerator has no s terms, we might guess the form

$$\frac{1}{(s-2)(s+3)} = \frac{1}{s-2} - \frac{1}{s+3}.$$

But, recombining the terms on the right-hand side, we see that

$$\frac{1}{s-2} - \frac{1}{s+3} = \frac{5}{(s-2)(s+3)}.$$

Since we were off by 5, we divide the partial fractions by 5 to obtain

$$\frac{1}{(s-2)(s+3)} = \frac{1}{5}\left[\frac{1}{s-2} - \frac{1}{s+3}\right],$$

which once again gives the desired form.

Returning to the problem, we have found that

$$Y(s) = \frac{1}{s+3} + \frac{1}{5}\left(\frac{1}{s-2} - \frac{1}{s+3}\right).$$

We can now see that the function with this Laplace transform is given by

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s+3} + \frac{1}{5}\left(\frac{1}{s-2} - \frac{1}{s+3}\right)\right] = e^{-3t} + \frac{1}{5}\left(e^{2t} - e^{-3t}\right)$$

works. Simplifying, we have the solution of the initial value problem

$$y(t) = \frac{1}{5}e^{2t} + \frac{4}{5}e^{-3t}.$$

We can verify that we have solved the initial value problem.

$$y' + 3y = \frac{2}{5}e^{2t} - \frac{12}{5}e^{-3t} + 3(\frac{1}{5}e^{2t} + \frac{4}{5}e^{-3t}) = e^{2t}$$

and $y(0) = \frac{1}{5} + \frac{4}{5} = 1$.

Example 5.8. Solve the initial value problem $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 3$.

We can probably solve this without Laplace transforms, but it is a simple exercise. Transforming the equation, we have

$$\begin{aligned} 0 &= s^2Y - sy(0) - y'(0) + 4Y \\ &= (s^2 + 4)Y - s - 3. \end{aligned} \tag{5.16}$$

Solving for Y, we have

$$Y(s) = \frac{s+3}{s^2+4}.$$

We now ask if we recognize the transform pair needed. The denominator looks like the type needed for the transform of a sine or cosine.

We just need to play with the numerator. Splitting the expression into two terms, we have

$$Y(s) = \frac{s}{s^2 + 4} + \frac{3}{s^2 + 4}.$$

The first term is now recognizable as the transform of $\cos 2t$. The second term is not the transform of $\sin 2t$. It would be if the numerator were a 2. This can be corrected by multiplying and dividing by 2:

$$\frac{3}{s^2 + 4} = \frac{3}{2} \left(\frac{2}{s^2 + 4} \right).$$

The solution is then found as

$$y(t) = \mathcal{L}^{-1} \left[\frac{s}{s^2 + 4} + \frac{3}{2} \left(\frac{2}{s^2 + 4} \right) \right] = \cos 2t + \frac{3}{2} \sin 2t.$$

The reader can verify that this is the solution of the initial value problem and is shown in Figure 5.4.
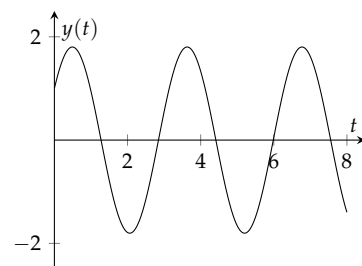

Figure 5.4: A plot of the solution to Example 5.8.

5.4 Step and Impulse Functions

5.4.1 Heaviside Step Function

OFTEN, THE INITIAL VALUE PROBLEMS THAT ONE FACES in differential equations courses can be solved using either the Method of Undetermined Coefficients or the Method of Variation of Parameters. However, using the latter can be messy and involves some skill with integration. Many circuit designs can be modeled with systems of differential equations using Kirchoff's Rules. Such systems can get fairly complicated. However, Laplace transforms can be used to solve such systems, and electrical engineers have long used such methods in circuit analysis.

In this section we add a couple more transform pairs and transform properties that are useful in accounting for things like turning on a driving force, using periodic functions like a square wave, or introducing impulse forces.

We first recall the Heaviside step function, given by

$$H(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{5.17}$$

A more general version of the step function is the horizontally shifted step function, $H(t - a)$. This function is shown in Figure 5.5. The Laplace transform of this function is found for $a > 0$ as

$$\begin{aligned} \mathcal{L}[H(t - a)] &= \int_0^\infty H(t - a)e^{-st} \, dt \\ &= \int_a^\infty e^{-st} \, dt \\ &= \frac{e^{-st}}{s} \Big|_a^\infty = \frac{e^{-as}}{s}. \end{aligned} \tag{5.18}$$
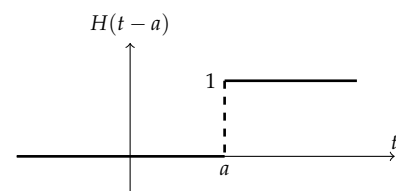

Figure 5.5: A shifted Heaviside function, $H(t - a)$.

The Laplace transform has two Shift Theorems involving the multiplication of the function, $f(t)$, or its transform, $F(s)$, by exponentials. The First and Second Shift Properties/Theorems are given by

$$
\begin{array}{rcl}
\mathcal{L}[e^{at}f(t)] & = & F(s-a), \qquad\qquad (5.19)\\
\mathcal{L}[f(t-a)H(t-a)] & = & e^{-as}F(s). \qquad\qquad (5.20)
\end{array}
$$

The Shift Theorems.

We prove the First Shift Theorem and leave the other proof as an exercise for the reader. Namely,

$$
\begin{array}{rcl}
\mathcal{L}[e^{at}f(t)] & = & \displaystyle\int_0^\infty e^{at}f(t)e^{-st}\,dt\\[2mm]
& = & \displaystyle\int_0^\infty f(t)e^{-(s-a)t}\,dt = F(s-a). \qquad (5.21)
\end{array}
$$

Example 5.9. Compute the Laplace transform of $e^{-at}\sin\omega t$.

This function arises as the solution of the underdamped harmonic oscillator. We first note that the exponential multiplies a sine function. The First Shift Theorem tells us that we first need the transform of the sine function. So, for $f(t) = \sin\omega t$, we have

$$
F(s) = \frac{\omega}{s^2+\omega^2}.
$$

Using this transform, we can obtain the solution to this problem as

$$
\mathcal{L}[e^{-at}\sin\omega t] = F(s+a) = \frac{\omega}{(s+a)^2+\omega^2}.
$$

More interesting examples can be found using piecewise defined functions. First we consider the function $H(t) - H(t-a)$. For $t < 0$, both terms are zero. In the interval $[0,a]$, the function $H(t) = 1$ and $H(t-a) = 0$. Therefore, $H(t) - H(t-a) = 1$ for $t \in [0,a]$. Finally, for $t > a$, both functions are one and therefore the difference is zero. The graph of $H(t) - H(t-a)$ is shown in Figure 5.6.

We now consider the piecewise defined function:

$$
g(t) = \left\{ \begin{array}{ll} f(t), & 0 \leq t \leq a,\\ 0, & t < 0, t > a. \end{array} \right.
$$

This function can be rewritten in terms of step functions. We only need to multiply $f(t)$ by the above box function,

$$
g(t) = f(t)[H(t) - H(t-a)].
$$

We depict this in Figure 5.7.

Even more complicated functions can be written in terms of step functions. We only need to look at sums of functions of the form $f(t)[H(t-a) - H(t-b)]$ for $b > a$. This is similar to a box function. It is nonzero between a and b and has height $f(t)$.
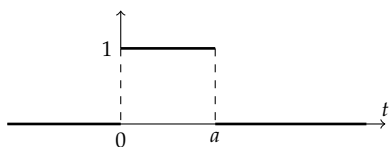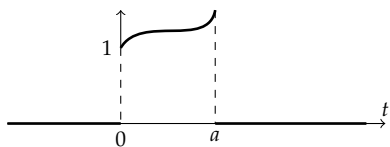

Figure 5.6: The box function, $H(t) - H(t-a)$.

Figure 5.7: Formation of a piecewise function, $f(t)[H(t) - H(t-a)]$.

We show as an example the square wave function in Figure 5.8. It can be represented as a sum of an infinite number of boxes,

$$f(t) = \sum_{n=-\infty}^{\infty} [H(t - 2na) - H(t - (2n+1)a)],$$

for $a > 0$.

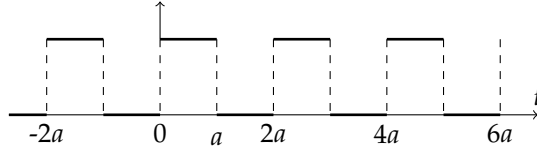**Example 5.10.** Find the Laplace Transform of a square wave "turned on" at $t = 0$.

Figure 5.8: A square wave, $f(t) = \sum_{n=-\infty}^{\infty}[H(t - 2na) - H(t - (2n+1)a)]$.

We let

$$f(t) = \sum_{n=0}^{\infty} [H(t - 2na) - H(t - (2n+1)a)], \quad a > 0.$$

Using the properties of the Heaviside function, we have

$$
\begin{aligned}
\mathcal{L}[f(t)] &= \sum_{n=0}^{\infty} [\mathcal{L}[H(t - 2na)] - \mathcal{L}[H(t - (2n+1)a)]] \\
&= \sum_{n=0}^{\infty} \left[\frac{e^{-2nas}}{s} - \frac{e^{-(2n+1)as}}{s} \right] \\
&= \frac{1 - e^{-as}}{s} \sum_{n=0}^{\infty} \left(e^{-2as} \right)^n \\
&= \frac{1 - e^{-as}}{s} \left(\frac{1}{1 - e^{-2as}} \right) \\
&= \frac{1 - e^{-as}}{s(1 - e^{-2as})} \\
&= \frac{1}{s(1 + e^{-as})}.
\end{aligned}
\tag{5.22}
$$

Note that the third line in the derivation is a geometric series. We summed this series to get the answer in a compact form since $e^{-2as} < 1$.

5.4.2   Periodic Functions*

The previous example provides us with a causal function ($f(t) = 0$ for $t < 0$.) which is periodic with period a. Such periodic functions can be teated in a simpler fashion. We will now show that

Laplace transform of periodic functions.

Theorem 5.1. *If $f(t)$ is periodic with period T and piecewise continuous on $[0, T]$, then*

$$F(s) = \frac{1}{1 - e^{-sT}} \int_0^T f(t) e^{-st} \, dt.$$

Proof.

$$
\begin{aligned}
F(s) &= \int_0^\infty f(t)e^{-st}\, dt \\
&= \int_0^T f(t)e^{-st}\, dt + \int_T^\infty f(t)e^{-st}\, dt \\
&= \int_0^T f(t)e^{-st}\, dt + \int_T^\infty f(t-T)e^{-st}\, dt \\
&= \int_0^T f(t)e^{-st}\, dt + e^{-sT}\int_0^\infty f(\tau)e^{-s\tau}\, d\tau \\
&= \int_0^T f(t)e^{-st}\, dt + e^{-sT}F(s). \qquad\qquad (5.23)
\end{aligned}
$$

Solving for $F(s)$, one obtains the desired result. $\qquad\qquad\square$

Example 5.11. Use the periodicity of

$$
f(t) = \sum_{n=0}^\infty [H(t-2na) - H(t-(2n+1)a)], \quad a > 0
$$

to obtain the Laplace transform.

We note that $f(t)$ has period $T = 2a$. By Theorem 5.1, we have

$$
\begin{aligned}
F(s) &= \int_0^\infty f(t)e^{-st}\, dt \\
&= \frac{1}{1-e^{-2as}}\int_0^{2a}[H(t) - H(t-a)]e^{-st}\, dt \\
&= \frac{1}{1-e^{-2as}}\left[\int_0^{2a} e^{-st}\, dt - \int_a^{2a} e^{-st}\, dt\right] \\
&= \frac{1}{1-e^{-2as}}\left[\left.\frac{e^{-st}}{-s}\right|_0^{2a} - \left.\frac{e^{-st}}{-s}\right|_a^{2a}\right] \\
&= \frac{1}{s(1-e^{-2as})}\left[1 - e^{-2as} + e^{-2as} - e^{-as}\right] \\
&= \frac{1-e^{-as}}{s(1-e^{-2as})} \\
&= \frac{1}{s(1+e^{-as})}. \qquad\qquad (5.24)
\end{aligned}
$$

This is the same result that was obtained in the previous example.

P. A. M. Dirac (1902-1984) introduced the δ function in his book, *The Principles of Quantum Mechanics*, 4th Ed., Oxford University Press, 1958, originally published in 1930, as part of his orthogonality statement for a basis of functions in a Hilbert space, $< \xi'|\xi'' >= c\delta(\xi' - \xi'')$ in the same way we introduced discrete orthogonality using the Kronecker delta. Historically, a number of mathematicians sought to understand the Diract delta function, culminating in Laurent Schwartz's (1915-2002) theory of distributions in 1945.

5.4.3 Dirac Delta Function

ANOTHER USEFUL CONCEPT IS THE IMPULSE FUNCTION. If we want to apply an impulse function, we can use the Dirac delta function $\delta(x)$. This is an example of what is known as a generalized function, or a distribution. Dirac had introduced this function in the 1930s in his study of quantum mechanics as a useful tool. It was later studied in a general theory of distributions and found to be more than a simple tool used by physicists. The Dirac delta function, as any distribution, only makes sense under an integral. Here will will introduce the Dirac delta function through its main properties.

The delta function satisfies two main properties:

1. $\delta(x) = 0$ for $x \neq 0$.

2. $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$.

Integration over more general intervals gives

$$\int_a^b \delta(x)\,dx = \begin{cases} 1, & 0 \in [a,b], \\ 0, & 0 \notin [a,b]. \end{cases} \qquad (5.25)$$

Another important property is the sifting property:

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)\,dx = f(a).$$

This can be seen by noting that the delta function is zero everywhere except at $x = a$. Therefore, the integrand is zero everywhere and the only contribution from $f(x)$ will be from $x = a$. So, we can replace $f(x)$ with $f(a)$ under the integral. Since $f(a)$ is a constant, we have that

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)\,dx = \int_{-\infty}^{\infty} \delta(x-a)f(a)\,dx$$

$$= f(a)\int_{-\infty}^{\infty} \delta(x-a)\,dx = f(a). \qquad (5.26)$$

Example 5.12. Evaluate: $\int_{-\infty}^{\infty} \delta(x+3)x^3\,dx$.
This is a simple use of the sifting property:

$$\int_{-\infty}^{\infty} \delta(x+3)x^3\,dx = (-3)^3 = -27.$$

Another property results from using a scaled argument, ax. In this case, we show that

$$\delta(ax) = |a|^{-1}\delta(x). \qquad (5.27)$$

As usual, this only has meaning under an integral sign. So, we place $\delta(ax)$ inside an integral and make a substitution $y = ax$:

$$\int_{-\infty}^{\infty} \delta(ax)\,dx = \lim_{L \to \infty} \int_{-L}^{L} \delta(ax)\,dx$$

$$= \lim_{L \to \infty} \frac{1}{a}\int_{-aL}^{aL} \delta(y)\,dy. \qquad (5.28)$$

If $a > 0$ then

$$\int_{-\infty}^{\infty} \delta(ax)\,dx = \frac{1}{a}\int_{-\infty}^{\infty} \delta(y)\,dy.$$

However, if $a < 0$ then

$$\int_{-\infty}^{\infty} \delta(ax)\,dx = \frac{1}{a}\int_{\infty}^{-\infty} \delta(y)\,dy = -\frac{1}{a}\int_{-\infty}^{\infty} \delta(y)\,dy.$$

The overall difference in a multiplicative minus sign can be absorbed into one expression by changing the factor $1/a$ to $1/|a|$. Thus,

Properties of the Dirac delta function:

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)\,dx = f(a).$$

$$\int_{-\infty}^{\infty} \delta(ax)\,dx = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(y)\,dy.$$

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \frac{1}{|a|} \int_{-\infty}^{\infty} \delta(y)\, dy. \tag{5.29}$$

Example 5.13. Evaluate $\int_{-\infty}^{\infty} (5x+1)\delta(4(x-2))\, dx$.

This is a straightforward integration:

$$\int_{-\infty}^{\infty} (5x+1)\delta(4(x-2))\, dx = \frac{1}{4} \int_{-\infty}^{\infty} (5x+1)\delta(x-2)\, dx = \frac{11}{4}.$$

The first step is to write $\delta(4(x-2)) = \frac{1}{4}\delta(x-2)$. Then, the final evaluation is given by

$$\frac{1}{4} \int_{-\infty}^{\infty} (5x+1)\delta(x-2)\, dx = \frac{1}{4}(5(2)+1) = \frac{11}{4}.$$

The Dirac delta function can be used to represent a unit impulse. Summing over a number of impulses, or point sources, we can describe a general function as shown in Figure 5.9. The sum of impulses located at points a_i, $i = 1, \ldots, n$, with strengths $f(a_i)$ would be given by

$$f(x) = \sum_{i=1}^{n} f(a_i)\delta(x - a_i).$$

A continuous sum could be written as

$$f(x) = \int_{-\infty}^{\infty} f(\xi)\delta(x - \xi)\, d\xi.$$

This is simply an application of the sifting property of the delta function.

We will investigate a case when one would use a single impulse. While a mass on a spring is undergoing simple harmonic motion, we hit it for an instant at time $t = a$. In such a case, we could represent the force as a multiple of $\delta(t - a)$.

One would then need the Laplace transform of the delta function to solve the associated initial value problem. Inserting the delta function into the Laplace transform, we find that for $a > 0$,

$$
\begin{aligned}
\mathcal{L}[\delta(t - a)] &= \int_{0}^{\infty} \delta(t - a)e^{-st}\, dt \\
&= \int_{-\infty}^{\infty} \delta(t - a)e^{-st}\, dt \\
&= e^{-as}. \tag{5.30}
\end{aligned}
$$

Example 5.14. Solve the initial value problem $y'' + 4\pi^2 y = \delta(t - 2)$, $y(0) = y'(0) = 0$.

This initial value problem models a spring oscillation with an impulse force. Without the forcing term, given by the delta function, this spring is initially at rest and not stretched. The delta function models a unit impulse at $t = 2$. Of course, we anticipate that at this time the spring will begin to oscillate. We will solve this problem using Laplace transforms.
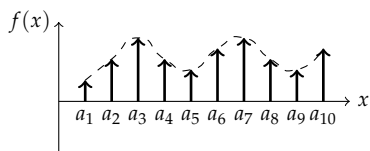

$f(x)$

$a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8\ a_9\ a_{10}$

Figure 5.9: Plot representing impulse forces of height $f(a_i)$. The sum $\sum_{i=1}^{n} f(a_i)\delta(x - a_i)$ describes a general impulse function.

$\mathcal{L}[\delta(t - a)] = e^{-as}$.

First, we transform the differential equation:

$$s^2 Y - sy(0) - y'(0) + 4\pi^2 Y = e^{-2s}.$$

Inserting the initial conditions, we have

$$(s^2 + 4\pi^2)Y = e^{-2s}.$$

Solving for $Y(s)$, we obtain

$$Y(s) = \frac{e^{-2s}}{s^2 + 4\pi^2}.$$

We now seek the function for which this is the Laplace transform. The form of this function is an exponential times some Laplace transform, $F(s)$. Thus, we need the Second Shift Theorem since the solution is of the form $Y(s) = e^{-2s}F(s)$ for

$$F(s) = \frac{1}{s^2 + 4\pi^2}.$$

We need to find the corresponding $f(t)$ of the Laplace transform pair. The denominator in $F(s)$ suggests a sine or cosine. Since the numerator is constant, we pick sine. From the tables of transforms, we have

$$\mathcal{L}[\sin 2\pi t] = \frac{2\pi}{s^2 + 4\pi^2}.$$

So, we write

$$F(s) = \frac{1}{2\pi}\frac{2\pi}{s^2 + 4\pi^2}.$$

This gives $f(t) = (2\pi)^{-1}\sin 2\pi t$.

We now apply the Second Shift Theorem, $\mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s)$, or

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left[e^{-2s}F(s)\right] \\
&= H(t-2)f(t-2) \\
&= \frac{1}{2\pi}H(t-2)\sin 2\pi(t-2). \qquad (5.31)
\end{aligned}$$

This solution tells us that the mass is at rest until $t = 2$ and then begins to oscillate at its natural frequency. A plot of this solution is shown in Figure 5.10

Example 5.15. Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 0, y'(0) = 0,$$

where

$$f(t) = \begin{cases} \cos \pi t, & 0 \le t \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

We need the Laplace transform of $f(t)$. This function can be written in terms of a Heaviside function, $f(t) = \cos \pi t H(t-2)$. In order to apply the Second Shift Theorem, we need a shifted version
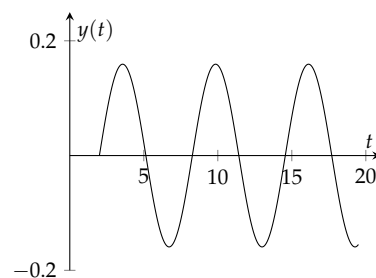

Figure 5.10: A plot of the solution to Example 5.14 in which a spring at rest experiences an impulse force at $t = 2$.

of the cosine function. We find the shifted version by noting that $\cos \pi(t-2) = \cos \pi t$. Thus, we have

$$
\begin{aligned}
f(t) &= \cos \pi t \left[H(t) - H(t-2) \right] \\
&= \cos \pi t - \cos \pi(t-2)H(t-2), \quad t \geq 0.
\end{aligned}
\tag{5.32}
$$

The Laplace transform of this driving term is

$$
F(s) = (1 - e^{-2s})\mathcal{L}[\cos \pi t] = (1 - e^{-2s})\frac{s}{s^2 + \pi^2}.
$$

Now we can proceed to solve the initial value problem. The Laplace transform of the initial value problem yields

$$
(s^2 + 1)Y(s) = (1 - e^{-2s})\frac{s}{s^2 + \pi^2}.
$$

Therefore,

$$
Y(s) = (1 - e^{-2s})\frac{s}{(s^2 + \pi^2)(s^2 + 1)}.
$$

We can retrieve the solution to the initial value problem using the Second Shift Theorem. The solution is of the form $Y(s) = (1 - e^{-2s})G(s)$ for

$$
G(s) = \frac{s}{(s^2 + \pi^2)(s^2 + 1)}.
$$

Then, the final solution takes the form

$$
y(t) = g(t) - g(t-2)H(t-2).
$$

We only need to find $g(t)$ in order to finish the problem. This is easily done using the partial fraction decomposition

$$
G(s) = \frac{s}{(s^2 + \pi^2)(s^2 + 1)} = \frac{1}{\pi^2 - 1}\left[\frac{s}{s^2 + 1} - \frac{s}{s^2 + \pi^2} \right].
$$

Then,

$$
g(t) = \mathcal{L}^{-1}\left[\frac{s}{(s^2 + \pi^2)(s^2 + 1)} \right] = \frac{1}{\pi^2 - 1}\left(\cos t - \cos \pi t \right).
$$

The final solution is then given by

$$
y(t) = \frac{1}{\pi^2 - 1}\left[\cos t - \cos \pi t - H(t-2)(\cos(t-2) - \cos \pi t) \right].
$$

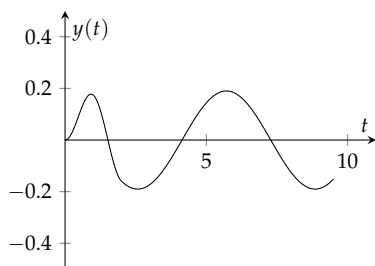A plot of this solution is shown in Figure 5.11.

Figure 5.11: A plot of the solution to Example 5.15 in which a spring at rest experiences an piecewise defined force.

5.5 The Convolution Theorem

FINALLY, WE CONSIDER THE CONVOLUTION of two functions. Often, we are faced with having the product of two Laplace transforms that we know and we seek the inverse transform of the product. For example, let's say we have

obtained $Y(s) = \frac{1}{(s-1)(s-2)}$ while trying to solve an initial value problem. In this case, we could find a partial fraction decomposition. But, there are other ways to find the inverse transform, especially if we cannot perform a partial fraction decomposition. We could use the Convolution Theorem for Laplace transforms or we could compute the inverse transform directly. We will look into these methods in the next two sections. We begin with defining the convolution.

We define the convolution of two functions defined on $[0, \infty)$ much the same way as we had done for the Fourier transform. The convolution $f * g$ is defined as

$$(f * g)(t) = \int_0^t f(u)g(t-u)\,du. \tag{5.33}$$

Note that the convolution integral has finite limits as opposed to the Fourier transform case.

The convolution operation has two important properties:

1. The convolution is commutative: $f * g = g * f$

    Proof. The key is to make a substitution $y = t - u$ in the integral. This makes f a simple function of the integration variable.

$$\begin{aligned}(g * f)(t) &= \int_0^t g(u)f(t-u)\,du \\ &= -\int_t^0 g(t-y)f(y)\,dy \\ &= \int_0^t f(y)g(t-y)\,dy \\ &= (f * g)(t). \tag{5.34}\end{aligned}$$

□

2. The Convolution Theorem: The Laplace transform of a convolution is the product of the Laplace transforms of the individual functions:

$$\mathcal{L}[f * g] = F(s)G(s).$$

    Proof. Proving this theorem takes a bit more work. We will make some assumptions that will work in many cases. First, we assume that the functions are causal, $f(t) = 0$ and $g(t) = 0$ for $t < 0$. Second, we will assume that we can interchange integrals, which needs more rigorous attention than will be provided here. The first assumption will allow us to write the finite integral as an infinite integral. Then a change of variables will allow us to split the integral into the product of two integrals that are recognized as a product of two Laplace transforms.

    Carrying out the computation, we have

$$\mathcal{L}[f * g] = \int_0^\infty \left(\int_0^t f(u)g(t-u)\,du \right) e^{-st}\,dt$$

The convolution is commutative.

The Convolution Theorem for Laplace transforms.

$$= \int_0^\infty \left(\int_0^\infty f(u)g(t-u)\, du \right) e^{-st}\, dt$$

$$= \int_0^\infty f(u) \left(\int_0^\infty g(t-u)e^{-st}\, dt \right) du \qquad (5.35)$$

Now, make the substitution $\tau = t - u$. We note that

$$int_0^\infty f(u) \left(\int_0^\infty g(t-u)e^{-st}\, dt \right) du = \int_0^\infty f(u) \left(\int_{-u}^\infty g(\tau)e^{-s(\tau+u)}\, d\tau \right) du$$

However, since $g(\tau)$ is a causal function, we have that it vanishes for $\tau < 0$ and we can change the integration interval to $[0, \infty)$. So, after a little rearranging, we can proceed to the result.

$$\begin{aligned}
\mathcal{L}[f * g] &= \int_0^\infty f(u) \left(\int_0^\infty g(\tau)e^{-s(\tau+u)}\, d\tau \right) du \\
&= \int_0^\infty f(u)e^{-su} \left(\int_0^\infty g(\tau)e^{-s\tau}\, d\tau \right) du \\
&= \left(\int_0^\infty f(u)e^{-su}\, du \right) \left(\int_0^\infty g(\tau)e^{-s\tau}\, d\tau \right) \\
&= F(s)G(s). \qquad (5.36)
\end{aligned}$$

$\square$

We make use of the Convolution Theorem to do the following examples.

Example 5.16. Find $y(t) = \mathcal{L}^{-1}\left[\frac{1}{(s-1)(s-2)} \right]$.

We note that this is a product of two functions:

$$Y(s) = \frac{1}{(s-1)(s-2)} = \frac{1}{s-1}\frac{1}{s-2} = F(s)G(s).$$

We know the inverse transforms of the factors:

$$f(t) = e^t \text{ and } g(t) = e^{2t}.$$

Using the Convolution Theorem, we find $y(t) = (f * g)(t)$. We compute the convolution:

$$\begin{aligned}
y(t) &= \int_0^t f(u)g(t-u)\, du \\
&= \int_0^t e^u e^{2(t-u)}\, du \\
&= e^{2t} \int_0^t e^{-u}\, du \\
&= e^{2t}[-e^t + 1] = e^{2t} - e^t. \qquad (5.37)
\end{aligned}$$

One can also confirm this by carrying out a partial fraction decomposition.

Example 5.17. Consider the initial value problem, $y'' + 9y = 2\sin 3t$, $y(0) = 1$, $y'(0) = 0$.

The Laplace transform of this problem is given by

$$(s^2 + 9)Y - s = \frac{6}{s^2 + 9}.$$

Solving for $Y(s)$, we obtain

$$Y(s) = \frac{6}{(s^2 + 9)^2} + \frac{s}{s^2 + 9}.$$

The inverse Laplace transform of the second term is easily found as $cos(3t)$; however, the first term is more complicated.

We can use the Convolution Theorem to find the Laplace transform of the first term. We note that

$$\frac{6}{(s^2 + 9)^2} = \frac{2}{3} \frac{3}{(s^2 + 9)} \frac{3}{(s^2 + 9)}$$

is a product of two Laplace transforms (up to the constant factor). Thus,

$$\mathcal{L}^{-1} \left[\frac{6}{(s^2 + 9)^2} \right] = \frac{2}{3}(f * g)(t),$$

where $f(t) = g(t) = sin3t$. Evaluating this convolution product, we have

$$
\begin{aligned}
\mathcal{L}^{-1} \left[\frac{6}{(s^2 + 9)^2} \right] &= \frac{2}{3}(f * g)(t) \\
&= \frac{2}{3} \int_0^t \sin 3u \sin 3(t - u) \, du \\
&= \frac{1}{3} \int_0^t [\cos 3(2u - t) - \cos 3t] \, du \\
&= \frac{1}{3} \left[\frac{1}{6} \sin(6u - 3t) - u \cos 3t \right]_0^t \\
&= \frac{1}{9} \sin 3t - \frac{1}{3} t \cos 3t. \tag{5.38}
\end{aligned}
$$

Combining this with the inverse transform of the second term of $Y(s)$, the solution to the initial value problem is

$$y(t) = -\frac{1}{3} t \cos 3t + \frac{1}{9} \sin 3t + \cos 3t.$$

Note that the amplitude of the solution will grow in time from the first term. You can see this in Figure 5.12. This is known as a resonance.

Example 5.18. Find $\mathcal{L}^{-1}[\frac{6}{(s^2+9)^2}]$ using partial fraction decomposition.

If we look at Table 5.2, we see that the Laplace transform pairs with the denominator $(s^2 + \omega^2)^2$ are

$$\mathcal{L}[t \sin \omega t] = \frac{2\omega s}{(s^2 + \omega^2)^2},$$

and

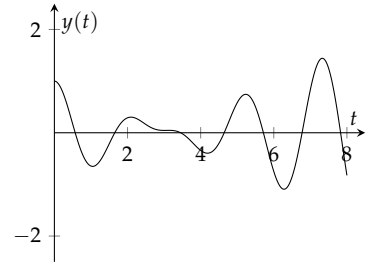$$\mathcal{L}[t \cos \omega t] = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}.$$

Figure 5.12: Plot of the solution to Example 5.17 showing a resonance.

So, we might consider rewriting a partial fraction decomposition as

$$\frac{6}{(s^2+9)^2} = \frac{A6s}{(s^2+9)^2} + \frac{B(s^2-9)}{(s^2+9)^2} + \frac{Cs+D}{s^2+9}.$$

Combining the terms on the right over a common denominator, we find

$$6 = 6As + B(s^2-9) + (Cs+D)(s^2+9).$$

Collecting like powers of s, we have

$$Cs^3 + (D+B)s^2 + 6As + (D-B) = 6.$$

Therefore, $C = 0$, $A = 0$, $D + B = 0$, and $D - B = \frac{2}{3}$. Solving the last two equations, we find $D = -B = \frac{1}{3}$.

Using these results, we find

$$\frac{6}{(s^2+9)^2} = -\frac{1}{3}\frac{(s^2-9)}{(s^2+9)^2} + \frac{1}{3}\frac{1}{s^2+9}.$$

This is the result we had obtained in the last example using the Convolution Theorem.

5.6 Systems of ODEs*

LAPACE TRANSFORMS ARE ALSO USEFUL for solving systems of differential equations. We will study linear systems of differential equation in Chapter 6. For now, we will just look at simple examples of the application of Laplace transforms.

An example of a system of two differential equations for two unknown functions, $x(t)$ and $y(t)$, is given by the pair of coupled differential equations

$$\begin{aligned} x' &= 3x + 4y, \\ y' &= 2x + y. \end{aligned} \tag{5.39}$$

Neither equation can be solved on its own without knowledge of the other unknown function. This is why they are called couple. We will also need initial values for the system. We will choose $x(0) = 1$ and $y(0) = 0$.

Now, what would happen if we were to take the Laplace transform of each equation? We can apply the rules as before. Letting the Laplace transforms of $x(t)$ and $y(t)$ be $X(t)$ and $Y(t)$, respectively, we have

$$\begin{aligned} sX - 1 &= 3X + 4Y, \\ sY &= 2X + Y. \end{aligned} \tag{5.40}$$

We have obtained a system of algebraic equations for X and Y. Using standard methods, like Cramer's Method, we can solve this system of two equations and two unknowns. First, we rewrite the equations as

$$\begin{aligned} (s-3)X - 4Y &= 1, \\ -2X + (s-1)Y &= 0. \end{aligned} \tag{5.41}$$

Using Cramer's (determinant) Rule for solving such systems, we have

$$X = \frac{\begin{vmatrix} 1 & -4 \\ 0 & s-1 \end{vmatrix}}{\begin{vmatrix} s-3 & -4 \\ -2 & s-1 \end{vmatrix}}, \qquad Y = \frac{\begin{vmatrix} s-3 & 1 \\ -2 & 0 \end{vmatrix}}{\begin{vmatrix} s-3 & -4 \\ -2 & s-1 \end{vmatrix}}. \qquad (5.42)$$

Note that the denominator in each solution is a 2×2 determinant consisting of the coefficients of X and Y in the appropriate order. The numerators are the same determinant but with the right-hand side of the equation replacing the respective columns.

Computing the determinants, using

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

we have

$$X = \frac{1}{(s-3)(s-1)-8}, \qquad Y = \frac{2}{(s-3)(s-1)-8},$$

or

$$X = \frac{s-1}{s^2 - 4s - 5}, \qquad Y = \frac{2}{s^2 - 4s - 5}.$$

We now know the Laplace transforms of the solutions, so a simple inverse Laplace transform is in order. The denominators are the same,

$$s^2 - 4s - 5 = (s-5)(s+1).$$

We can apply a partial fraction decomposition to each function to obtain

$$
\begin{aligned}
X &= \frac{s-1}{(s-5)(s+1)} \\
&= \frac{s-5+4}{(s-5)(s+1)} \\
&= \frac{1}{s+1} + \frac{4}{(s-5)(s+1)} \\
&= \frac{1}{s+1} + \frac{2}{3}\left[\frac{1}{s-5} - \frac{1}{s+1}\right] \\
&= \frac{2}{3}\frac{1}{s-5} + \frac{1}{3}\frac{1}{s+1}. \\
Y &= \frac{2}{(s-5)(s+1)} \\
&= \frac{1}{3}\left[\frac{1}{s-5} - \frac{1}{s+1}\right].
\end{aligned}
$$

So, the solutions to the system of differential equations is given by

$$
\begin{aligned}
x(t) &= \frac{2}{3}e^5 t + \frac{1}{3}e^{-t}. \\
y(t) &= \frac{1}{3}(e^{5t} - e^{-t}).
\end{aligned}
$$

We can verify that $x(0) = 1$ and $y(0) = 0$.

$$
\begin{aligned}
x' &= \frac{10}{3}e^{5t} - \frac{1}{3}e^{-t} \\
3x + 4y &= (2e^{5t} + e^{-t}) + \frac{4}{3}(e^{5t} - e^{-t}) \\
&= \frac{10}{3}e^{5t} - \frac{1}{3}e^{-t}. \\
y' &= \frac{5}{3}e^{5t} + \frac{1}{3}e^{-t} \\
2x + y &= (\frac{4}{3}e^{5t} + \frac{2}{3}e^{-t}) + \frac{1}{3}(e^{5t} - e^{-t}) \\
&= \frac{5}{3}e^{5t} + \frac{1}{3}e^{-t}.
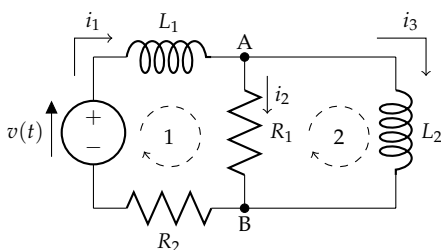\end{aligned}
$$

$$(5.43)$$

Figure 5.13: A circuit with two loops containing two resistors and two inductors in parallel.

Example 5.19. Determine the current in Figure 5.13 for the following values: $i_1(0) = i_2(0) = i_3(0) = 0$ and

$$
v(t) = \begin{cases} v_0, & 0 \le t \le 3.0 \\ 0, & \text{otherwise.} \end{cases}
$$

The problem can be modeled by a system of differential equations. In Figure 5.13 there are three currents indicated. Kirchoff's Point (Junction) Rule indicates that $i_1 = i_2 + i_3$.

In order to apply Kirchoff's Loop Rule , we need to tally the potential drops and rises. For resistors, these come from Ohm's Law, $v = iR$, and for inductors, this comes from Faraday's Law, $v = L\frac{di}{dt}$. For the left loop (2), we have

$$L_2 i_3' = R_1 i_2,$$

where the prime denotes the time derivative. For the right loop (1), we have

$$L_1 i_1' + R_1 i_2 + R_2 i_1 = v(t).$$

We can use the Point Rule to eliminate one of the currents, $i_2 = i_1 - i_3$, leaving the model as two first order differential equations,

$$
\begin{aligned}
L_2 i_3' - R_1(i_1 - i_3) &= 0 \\
L_1 i_1' + R_1(i_1 - i_3) + R_2 i_1 &= v(t),
\end{aligned}
$$

or

$$
\begin{aligned}
L_2 i_3' - R_1 i_1 + R_1 i_3 &= 0 \\
L_1 i_1' + (R_1 + R_2)i_1 - R_1 i_3 &= v_0(1 - H(t - 3)),
\end{aligned}
$$

where $H(t)$ is the Heaviside function.

Taking the Laplace transform, assuming that $i_1(0) = i_2(0) = 0$, we obtain the algebraic system of equations

$$
\begin{aligned}
-R_1 I_1 + (sL_2 + R_1)I_3 &= 0 \\
(sL_1 + R_1 + R_2)I_1 - R_1 I_3 &= \frac{v_0}{s}\left(1 - e^{-3s}\right).
\end{aligned}
$$

Here $I_1(s)$ and $I_3(s)$ are the Laplace transforms of $i_1(t)$ and $i_3(t)$, respectively.

As before, we use Cramer's Rule to find the solutions.

$$I_1 = \frac{\begin{vmatrix} 0 & sL_2 + R_1 \\ \frac{v_0}{s}(1 - e^{-3s}) & -R_1 \end{vmatrix}}{\begin{vmatrix} -R_1 & sL_2 + R_1 \\ sL_1 + R_1 + R_2 & -R_1 \end{vmatrix}},$$

$$= \frac{-v_0(sL_2 + R_1)(1 - e^{-3s})}{s[R_1^2 - (sL_2 + R_1)(sL_1 + R_1 + R_2)]}$$

$$= \frac{v_0(sL_2 + R_1)(1 - e^{-3s})}{s[(L_1L_2)s^2 + (R_1L_1 + L_2(R_1 + R_2))s + R_1R_2]}.$$

$$I_3 = \frac{\begin{vmatrix} -R_1 & 0 \\ sL_1 + R_1 + R_2 & \frac{v_0}{s}(1 - e^{-3s}) \end{vmatrix}}{\begin{vmatrix} -R_1 & sL_2 + R_1 \\ sL_1 + R_1 + R_2 & -R_1 \end{vmatrix}}$$

$$= \frac{v_0R_1(1 - e^{-3s})}{s[R_1^2 - (sL_2 + R_1)(sL_1 + R_1 + R_2)]}$$

$$= \frac{-v_0R_1(1 - e^{-3s})}{s[(L_1L_2)s^2 + (R_1L_1 + L_2(R_1 + R_2))s + R_1R_2]}. \qquad (5.44)$$

The denominator in these expressions cannot be factored. So, to make any further progress, one needs specific values for the constants. Let $R_1 = 2.00\Omega$, $R_2 = 18.0\Omega$, $L_1 = 48.0$ H, $L_2 = 6.00$ H. and $v_0 = 18$ V. Then,

$$I_1 = \frac{3s + 1}{s(2s + 1)(4s + 1)}\left(1 - e^{-3s}\right)$$

$$I_3 = -\frac{1}{s(2s + 1)(4s + 1)}\left(1 - e^{-3s}\right)$$

Using partial fractions on the coefficient of $\left(1 - e^{-3s}\right)$, we find that

$$\frac{3s + 1}{s(2s + 1)(4s + 1)} = \frac{1}{s} - \frac{2}{4s + 1} - \frac{1}{2s + 1},$$

$$\frac{1}{s(2s + 1)(4s + 1)} = \frac{1}{s} - \frac{8}{4s + 1} + \frac{2}{2s + 1}.$$

This gives

$$I_1 = \left(\frac{1}{s} - \frac{1}{2}\frac{1}{s + \frac{1}{4}} - \frac{1}{2}\frac{1}{s + \frac{1}{2}}\right)(1 - e^{-3s})$$

$$I_3 = \left(-\frac{1}{s} + \frac{2}{s + \frac{1}{4}} - \frac{1}{s + \frac{1}{2}}\right)(1 - e^{-3s})$$

Taking the inverse Laplace transform, we find the solutions

$$i_1 = 1 - \frac{1}{2}e^{-\frac{t}{4}} - \frac{1}{2}e^{-\frac{t}{2}} + \left(-1 + \frac{1}{2}e^{-\frac{t-3}{4}} + \frac{1}{2}e^{-\frac{t-3}{2}}\right)H(t - 3)$$
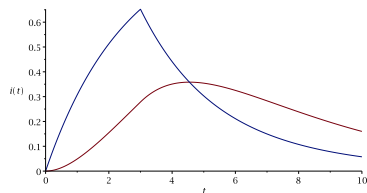
$$= \begin{cases} 1 - \frac{1}{2}e^{-\frac{t}{4}} - \frac{1}{2}e^{-\frac{t}{2}}, & t \leq 3, \\ -\frac{1}{2}(1 - e^{\frac{3}{4}})e^{-\frac{t}{4}} - \frac{1}{2}(1 - e^{\frac{3}{2}})e^{-\frac{t}{2}}, & t \geq 3. \end{cases}$$

$$i_3 = -1 + 2e^{-\frac{t}{4}} - e^{-\frac{t}{2}} + \left(1 - 2e^{-\frac{t-3}{4}} + e^{-\frac{t-3}{2}}\right)H(t-3).$$

$$= \begin{cases} -1 + 2e^{-\frac{t}{4}} - e^{-\frac{t}{2}}, & t \leq 3, \\ 2(1 - e^{\frac{3}{4}})e^{-\frac{t}{4}} - (1 - e^{\frac{3}{2}})e^{-\frac{t}{2}}, & t \geq 3. \end{cases}$$

Figure 5.14: A plot of the currents vs time for Example 5.19 with the voltage $v(t) = v_0(1 - H(t-3))$. The taller curve represents i_1 and the other curve is $-i_3$.

In Figure 5.14 we plot the currents vs time. The taller curve represents i_1 and the other curve is $-i_3$. We note that the derived current, i_3, is negative, indicating a flow in reverse of the direction shown in Figure 5.13. Not the sudden change in $i[1]$ at $t = 3$, the time that the voltage is turned on.

One can easily change the time that the voltage is applied. Namely, if

$$v(t) = \begin{cases} v_0, & 0 \leq t \leq t_0 \\ 0, & \text{otherwise,} \end{cases}$$

then the solutions are given by

$$i_1 = 1 - \frac{1}{2}e^{-\frac{t}{4}} - \frac{1}{2}e^{-\frac{t}{2}} + \left(-1 + \frac{1}{2}e^{-\frac{t-t_0}{4}} + \frac{1}{2}e^{-\frac{t-t_0}{2}}\right)H(t-t_0)$$

$$i_3 = -1 + 2e^{-\frac{t}{4}} - e^{-\frac{t}{2}} + \left(1 - 2e^{-\frac{t-t_0}{4}} + e^{-\frac{t-t_0}{2}}\right)H(t-t_0).$$

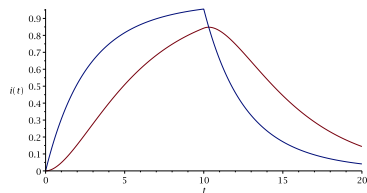A plot of the currents for $t_0 = 10$ are shown in Figure 5.15.

Figure 5.15: A plot of the currents vs time for Example 5.19 for the voltage given by $v(t) = v_0(1 - H(t-10))$. The taller curve represents i_1 and the other curve is $-i_3$.

Problems

1. Find the Laplace transform of the following functions:

   a. $f(t) = 9t^2 - 7$.

   b. $f(t) = e^{5t-3}$.

   c. $f(t) = \cos 7t$.

   d. $f(t) = e^{4t} \sin 2t$.

   e. $f(t) = e^{2t}(t + \cosh t)$.

   f. $f(t) = t^2 H(t-1)$.

   g. $f(t) = \begin{cases} \sin t, & t < 4\pi, \\ \sin t + \cos t, & t > 4\pi. \end{cases}$

   h. $f(t) = \int_0^t (t-u)^2 \sin u \, du$.

   i. $f(t) = \int_0^t \cosh u \, du$.

   j. $f(t) = (t+5)^2 + te^{2t} \cos 3t$ and write the answer in the simplest form.

2. Find the inverse Laplace transform of the following functions using the properties of Laplace transforms and the table of Laplace transform pairs.

   a. $F(s) = \dfrac{18}{s^3} + \dfrac{7}{s}$.

b. $F(s) = \dfrac{1}{s-5} - \dfrac{2}{s^2+4}$.

c. $F(s) = \dfrac{s+1}{s^2+1}$.

d. $F(s) = \dfrac{3}{s^2+2s+2}$.

e. $F(s) = \dfrac{1}{(s-1)^2}$.

f. $F(s) = \dfrac{e^{-3s}}{s^2-1}$.

g. $F(s) = \dfrac{1}{s^2+4s-5}$.

h. $F(s) = \dfrac{s+3}{s^2+8s+17}$.

3. Use Laplace transforms to solve the following initial value problems. Where possible, describe the solution behavior in terms of oscillation and decay.

    a. $y'' - 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 0$.

    c. $y'' + 2y' + 5y = 0$, $y(0) = 1$, $y'(0) = 0$.

    b. $y'' - y = te^{2t}$, $y(0) = 0$, $y'(0) = 1$.

    c. $y'' - 3y' - 4y = t^2$, $y(0) = 2$, $y'(0) = 1$.

    d. $y''' - 3y' - 2y = e^t$, $y(0) = 1$, $y'(0) = 0$.

4. Use Laplace transforms to solve the following initial value problems. Where possible, describe the solution behavior in terms of oscillation and decay.

    a. $y'' + 4y = \delta(t-1)$, $y(0) = 3$, $y'(0) = 0$.

    b. $y'' - 4y' + 13y = \delta(t-1)$, $y(0) = 0$, $y'(0) = 2$.

    c. $y'' + 6y' + 18y = 2H(\pi - t)$, $y(0) = 0$, $y'(0) = 0$.

    d. $y'' + 4y = f(t)$, $y(0) = 1$, $y'(0) = 0$, where $f(t) = \begin{cases} 1, & 0 < t < 1, \\ 0, & t > 1. \end{cases}$

5. For the following problems, draw the given function and find the Laplace transform in closed form.

    a. $f(t) = 1 + \displaystyle\sum_{n=1}^{\infty} (-1)^n H(t-n)$.

    b. $f(t) = \displaystyle\sum_{n=0}^{\infty} [H(t-2n+1) - H(t-2n)]$.

    c.

$$
\begin{aligned}
f(t) &= \sum_{n=0}^{\infty} (t-2n)[H(t-2n) - H(t-2n-1)] \\
&\quad + \sum_{n=0}^{\infty} (2n+2-t)[H(t-2n-1) - H(t-2n-2)].
\end{aligned}
$$

6. The period, T, and the function defined on its first period are given. Sketch several periods of these periodic functions. Make use of the periodicity to find the Laplace transform of each function.

    a. $f(t) = \sin t$, $T = 2\pi$.

    b. $f(t) = t$, $T = 1$.

    c. $f(t) = \begin{cases} t, & 0 \le t \le 1, \\ 2 - t, & 1 \le t \le 2, \end{cases}$ $T = 2$.

    d. $f(t) = t[H(t) - H(t-1)]$, $T = 2$.

    e. $f(t) = \sin t[H(t) - H(t-\pi)]$, $T = \pi$.

7. Compute the convolution $(f * g)(t)$ (in the Laplace transform sense) and its corresponding Laplace transform $\mathcal{L}[f * g]$ for the following functions:

    a. $f(t) = t^2$, $g(t) = t^3$.

    b. $f(t) = t^2$, $g(t) = \cos 2t$.

    c. $f(t) = 3t^2 - 2t + 1$, $g(t) = e^{-3t}$.

    d. $f(t) = \delta\left(t - \frac{\pi}{4}\right)$, $g(t) = \sin 5t$.

8. Use the Convolution Theorem to compute the inverse transform of the following:

    a. $F(s) = \dfrac{2}{s^2(s^2 + 1)}$.

    b. $F(s) = \dfrac{e^{-3s}}{s^2}$.

    c. $F(s) = \dfrac{1}{s(s^2 + 2s + 5)}$.

9. Find the inverse Laplace transform in two different ways: (i) Use tables. (ii) Use the Convolution Theorem.

    a. $F(s) = \dfrac{1}{s^3(s + 4)^2}$.

    b. $F(s) = \dfrac{1}{s^2 - 4s - 5}$.

    c. $F(s) = \dfrac{s + 3}{s^2 + 8s + 17}$.

    d. $F(s) = \dfrac{s + 1}{(s - 2)^2(s + 4)}$.

    e. $F(s) = \dfrac{s^2 + 8s - 3}{(s^2 + 2s + 1)(s^2 + 1)}$.

10. A linear Volterra integral equation, introduced by Vito Volterra (1860-1940), is of the form

$$y(t) = f(t) + \int_0^t K(t - \tau)y(\tau)\, d\tau,$$

where $y(t)$ is an unknown function and $f(t)$ and the "kernel," $K(t)$, are given functions. The integral is in the form of a convolution integral and such equations can be solved using Laplace transforms. Solve the following Volterra integral equations.

a. $y(t) = e^{-t} + \int_0^t \cos(t - \tau)y(\tau)\,d\tau$.

b. $y(t) = t - \int_0^t (t - \tau)y(\tau)\,d\tau$.

c. $y(t) = t + 2\int_0^t e^{t-\tau}y(\tau)\,d\tau$.

d. $\sin t = \int_0^t e^{t-\tau}y(\tau)\,d\tau$. Note: This is a Volterra integral equation of the first kind.

11. Use Laplace transforms to convert the following system of differential equations into an algebraic system and find the solution of the differential equations.

$$\begin{aligned}
x'' &= 3x - 6y, & x(0) = 1, & \quad x'(0) = 0, \\
y'' &= x + y, & y(0) = 0, & \quad y'(0) = 0.
\end{aligned}$$

12. Use Laplace transforms to convert the following nonhomogeneous systems of differential equations into an algebraic system and find the solutions of the differential equations.

a.

$$\begin{aligned}
x' &= 2x + 3y + 2\sin 2t, & x(0) = 1, \\
y' &= -3x + 2y, & y(0) = 0.
\end{aligned}$$

b.

$$\begin{aligned}
x' &= -4x - y + e^{-t}, & x(0) = 2, \\
y' &= x - 2y + 2e^{-3t}, & y(0) = -1.
\end{aligned}$$

c.

$$\begin{aligned}
x' &= x - y + 2\cos t, & x(0) = 3, \\
y' &= x + y - 3\sin t, & y(0) = 2.
\end{aligned}$$

13. Redo Example 5.19 using the values $R_1 = 1.00\Omega$, $R_2 = 1.40\Omega$, $L_1 = 0.80$ H, $L_2 = 1.00$ H. and $v_0 = 100$ V in $v(t) = v_0(1 - H(t - t_0))$. Plot the currents as a function of time for several values of t_0.