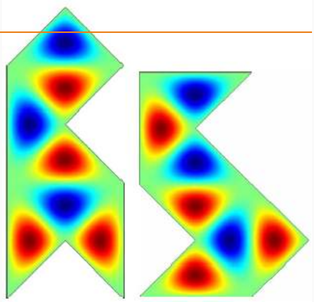
Vibrations and Fourier Analysis

Fall 2025 - R. L. Herman



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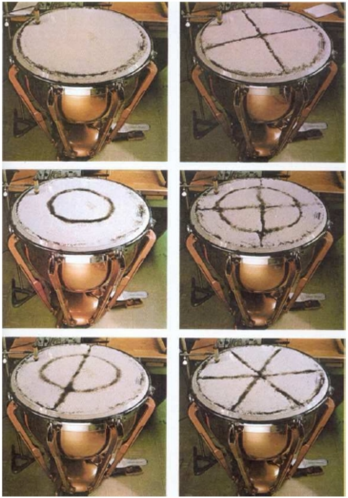
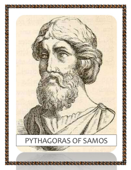
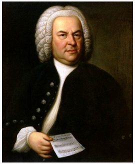


Chladni patterns on a kettledrum
from Risset, *Les instruments de l'orchestre*

Harmonics

- Pythagoras, Ptolemy.
- Galileo and Mersenne, pitch and frequency. Strings produce several tones.
- Joseph Sauveur, 1653-1716, acoustics. Introduced nodes, “harmonic.”
- Brook Taylor 1685-1731, fundamental, 1713.
- Johann Bernoulli, 1667-1748, 1727 letter to Daniel. He began studies in 1733.
- Johann Sebastian Bach, 1685-1750, 1722-44 .
- Hermann Helmholtz, 1821-1894, acoustics.
- The debates begin ...





The 1700s Debate - Mathematicians vs Physicists

- Jean le Rond d'Alembet, 1717-1783.
- Vibrating string equation and general solution, $y(x,t) = f(x+t) + g(x-t)$. BCs give $g = f$.
- Leonhard Euler's papers, 1748-9. More general equation with c, and $y(x,t) = f(x+ct) + g(x-ct)$.
- Claimed - f from ICs. $y(x,t) = \frac{1}{2}\left(Y(x+ct) + Y(x-ct) + \frac{1}{c}\int_{x+ct}^{x-ct} V(s)\,ds\right)$.
- Y, V are any curves *drawn by hand*.
- Daniel Bernoulli, 1709-1791, solutions are sums of harmonics, 1753:



$$y(x) = A_1 \sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} + A_2 \sin\frac{2\pi x}{L}\cos\frac{2\pi ct}{L} + \cdots = f(x+ct) + g(x-ct).$$

The Controversy (from Am. J. of Phys. 55, 33 (1987))

d'Alembert vs Euler

- Euler allowed corners.
- d'Alembert's first response - f must be periodic, odd, differentiable. Introduced separation of variables.
- 1761 - the attack! Use of physical arguments is prohibited.
- If slope discontinuous, then acceleration undefined.
- Euler responded 1762, 1765. For small displacement, the function at corner is infinitesimally close to differentiable.

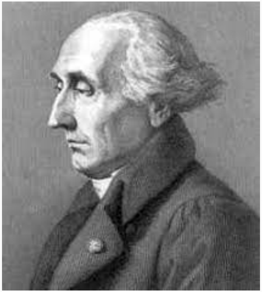
d'Alembert, Euler vs Bernoulli

- d'Alembert did not believe a sum of harmonics.
- Euler sum not general enough - snapped string.
- Bernoulli - "Listen to the string."

They all missed general periodicity.

Joseph-Louis Lagrange

- In enters another math. physicist.
- Born Luigi de la Grange Tournier (1736-1813), in Italy.
- 1759, paper on sound propagation.
- Agreed mostly with Euler, not Bernoulli.
- Avoided wave equation. Used a discrete set of masses.



$$\begin{aligned} y(x,t) &= \frac{2}{L}\int_0^L dX\, Y(X)\left[\sin\frac{\pi X}{L}\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} + \sin\frac{2\pi X}{L}\sin\frac{2\pi x}{L}\cos\frac{2\pi ct}{L} + \dots\right] \\ &+ \frac{2}{\pi c}\int_0^L dX\, V(X)\left[\sin\frac{\pi X}{L}\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} + \frac{1}{2}\sin\frac{2\pi X}{L}\sin\frac{2\pi x}{L}\cos\frac{2\pi ct}{L} + \dots\right] \end{aligned}$$

He almost discovered Fourier series in 1759. [Fourier was born, 1768.]

Jean-Baptiste Joseph Fourier (1768-1830)

- French Revolution, 1789, several arrests.
- Studied under Lagrange, Laplace, Monge.
- Succeeded Lagrange, chair of analysis and mechanics, 1797.
- Joined Napoleon's invasion of Egypt, scientific adviser with Monge, Malus.
- Organizer of French retreat from Egypt.
- Produced a multi-volume work on Egyptology.
- Studied the heat equation and series solutions.
- Almost forgotten in France, not elsewhere due to P. G. J. Dirichlet who wrote on Fourier series. Open problems led Cantor to set theory.

Siméon-Denis Poisson (1781-1840)

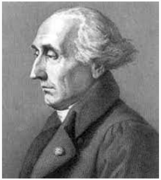
- 1798, entered École Polytechnique.
- Studied under Laplace and Lagrange.
- Degree in mathematics two years.
- Chair of mechanics, Faculty of Sciences, 1809.
- Over 300 papers: definite integrals, Fourier analysis, applied mathematics to physics (mechanics and electrostatics), probability and statistics.
- Poisson brackets, Poisson's constant, Poisson's equation, Poisson's integral, and Poisson's spot.

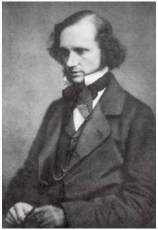
See D. H. Arnold's Work.

Corpuscular vs wave theory - 1818 Competition: Augustin-Jean Fresnel (1788-1827).

The Heat Equation

- Controversy: Fourier vs Poisson
- Fourier 1805, 1807 - diffusion, series solutions ala D. Bernoulli.
- Examiners: Laplace, Lagrange skeptical.
- Poisson Review 1808.
- 1811 Prize problem. Fourier won, but still critics.
- Third version to be book, 1822. Timing affected by politics.
- 1815, Poisson writes his own paper, then book in 1823.
- Wm. Thomson defense of Fourier in 1845.



William Thomson (1824-1907)

- Father, James Thomson, taught math in Belfast and Univ. of Glasgow.
- William attended Univ. of Glasgow, 1834.
- Read Jean-Baptiste-Joseph Fourier.
- First two articles, at 16-17, defended Fourier.
- Cambridge, 1841-5, earned B.A. with high honours.
- In 1845, obtained George Green's essay and went to Paris next day.
- Chair of natural philosophy at the U. of Glasgow at 22.
- Applied Heat Eqn to Age of the Earth
- Gabriel Stokes (1819-1903) introduced him to the telegraphy problem in 1854.

WILLIAM THOMSON: THE YOUNG PROFESSOR

According to Maxwell (1873), the Stokes Thm given in Smith's Prize Examination, 1854, question 8. From 1850 letter from Thomson to Stokes: link.

Thomson's 1855 Paper

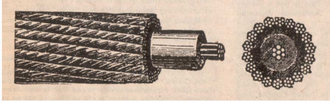
III. "On the Theory of the Electric Telegraph." By Professor WILLIAM THOMSON, F.R.S. Received May 3, 1855.

The following investigation was commenced in consequence of a letter received by the author from Prof. Stokes, dated Oct. 16, 1854. It is now communicated to the Royal Society, although only in an incomplete form, as it may serve to indicate some important practical applications of the theory, especially in estimating the dimensions of telegraph wires and cables required for long distances; and the author reserves a more complete development and illustration of the mathematical parts of the investigation for a paper on the conduction of Electricity and Heat through solids, which he intends to lay before the Royal Society on another occasion.

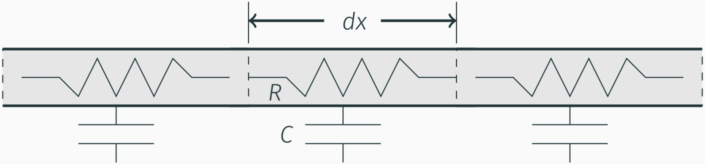
Extract from a letter to Prof. Stokes, *dated Largs, Oct.* 28, 1854.

" Let c be the electro-statical capacity per unit of length of the wire; that is, let c be such that clv is the quantity of electricity

William Thomson's Telegraph Theory - 1855



Treat the coaxial cable as a long, thin conductor, perfectly electrically insulated.



Think of the cable as a network of resistances and electrical capacity (capacitance) and use Kirchoff's laws on an infinitesimal section to derive an equation for the voltage, $v(x, t)$. [Ohm - 1827, Kirchoff - 1845.]

Thomson's Diffusion Equation

This resulted in a diffusion equation:

$$\frac{\partial v^2}{\partial x^2} = RC\frac{\partial v}{\partial t}. \tag{1}$$

It was Fourier's heat equation with solution

$$v = \frac{Q\sqrt{R}}{\sqrt{\pi C t}} e^{-RCx^2/4t}.$$

The maximum effect is at position x and time $t = \frac{1}{6}RCx^2$. This is **Thomson's law of squares**. Examples in Rayleigh's *Theory of Sound.*

Thomson solved several special cases in his correspondence with Stokes as recalled by Thomson.. Thomson's theory had many practical applications.

Further Develpoments

- Thomson had the first teaching laboratory,
- Engaged his students in testing materials and his ideas.
- Used the theory/experiment to understand underwater telegraphy.
- Explained the speed of the current in a telegraph cable,
- Dispersion caused signals of low frequency to diffuse less.

Stokes solved the more general case

$$\begin{aligned} v(x,0) &= 0, \quad 0 < x < \infty \\ v(0,t) &= f(t), \quad 0 < t < \infty, \end{aligned}$$

arriving at the solution

$$v(x,t) = \frac{x}{2\sqrt{\pi}} \int_0^t (t-t')^{-\frac{3}{2}} e^{-x^2/4(t-t')} f(t')\, dt'.$$

William Thomson - a.k.a Lord Kelvin

This was in the backdrop of the Atlantic Cable Project.

- Developed the theory, designed experiments, and obtained patents.
- Was instrumental to the success of the trans-Atlantic cable, completed 1866, after disputes with Whitehouse.
- For his work on the trans-Atlantic telegraph project:
  - Knighted by Queen Victoria, becoming Sir William Thomson, 1866.
  - Recognized for achievements in thermodynamics becoming Baron Kelvin, of Largs, 1892.

Thomson's theory of the electric telegraph remained the main theory for decades. It worked fine for long underwater cables, but to transmit human conversation, the diffusion was far too much.

Maxwell's Theory of Electricity and Magnetism

During this time scientists were beginning to move from the mechanical world of Newton and Lagrange to the world of Faraday, Oersted, Ampere, and others.

- James Clerk Maxwell (1831-1879)
- Michael Faraday (1791-1867) encouraged Maxwell.
- "A Dynamical Theory of the Electromagnetic Field," EM waves.
- "A Treatise on Electricity and Magnetism," 1873.
- Promoters of Maxwell's work: G. F. Fitzgerald (1851-1901), O. Heaviside (1850-1925), and O. Lodge (1851-1940). The Maxwellians.
- The race was on to produce electromagnetic waves, Hertz (1857-1894).
- Maxwell's theory reworked by Heaviside.

Challenge to Thomson's Theory

- The story of the attempts to connect continents with telegraph cables and Thomson's role is described by Hunt (2012, 2018, 2021).
- The subsequent contributions of Heaviside can be found in (Nahin 2002).
- In 1876 Heaviside derived the telegrapher's equation independently and updated Thomson's diffusion theory by insisting that self-inductance was important.
- This was contrary to what people working on underwater telegraphy believed.
- It led to a few disputes.

Figure 1: Who was Oliver Heaviside?

Oliver Heaviside (1850-1925)

- Heaviside left school at sixteen.
- He studied at home for two years.
- Worked as telegraph operator, Danish-Norwegian-English Telegraph Co., advice from uncle C. Wheatstone, 1868.
- He was transferred to Newcastle-on-Tyne, 1870, and later appointed Chief Operator.
- He left in 1874. Only job he would ever have.
- He spent the next couple of years working on electric theory.
- He studied and reformulated Maxwell's theory.

Note: Heaviside and Josiah Gibbs gave us Vector Analysis and opposed quaternions introduced by Hamilton and promoted by Tait. He gave us Maxwell's Equations.

Oliver Heaviside (1850-1925)[1]

- Heaviside began publishing in 1872.
- He furthered Thomson's theory, 1876.
- Derived the telegraph equation.
- Self-induction is important in telegraphy.
- Others opposed him on this.
- He was asked to stop publishing for *The Electrician* in 1887.
- Heaviside did have some supporters including Thomson and Maxwell.

[1] Marion Camerion Gray (1902-1979) in 1923 wrote *The Equation of Telegraphy* comparing known solutions of

$$\frac{\partial^2 V}{\partial t^2} + 2\gamma \frac{\partial V}{\partial t} = a^2 \frac{\partial^2 V}{\partial x^2},$$

and used the Riemann-Green method.

Heaviside's Operational Calculus

- Used to solve partial differential equations.
- Methods were criticized - not being rigorous and hard to understand.
- First people to publish justifications of Heaviside's methods: Bromwich (1917), Wagner (1917).
- Both used complex integrals.
- Bromwich cited applications from the *Theory of Sound*, William Strutt (1894, Lord Rayleigh) and an equation similar to telegrapher's equation using a Green's function.
- Attempted to justify Heaviside's work and eventually the Laplace transform emerged.

Operational methods for differential equations and the exploration of fractional differentiation had been studied for a number of years going back to the work of Euler and Leibniz.

Some of this is summarized in Moore's 1921 text and from Carslaw and Jaeger's 1941 book on operational methods.

Example: Edmund T. Whittaker Obituary for Heaviside

Edmund T. Whittaker (1873-1956) describes how Heaviside would use operational calculus[2] to solve the differential equation.

$$\frac{d^2y(t)}{dt^2} + k^2y(t) = 0. \tag{2}$$

Let $D = \frac{d}{dt}$. We write symbolically,

$$(D^2 + k^2)y(t) = 0.$$

Now, manipulate algebraically: Multiply by D^{-2},

$$(1 + k^2D^{-2})y(t) = D^{-2}(0).$$

What is $D^{-2}(0)$? - Eventually needed fractional derivatives.

[2]According to Whittaker, Heaviside was accustomed to using symbolic differential operators. Boole (1859) devoted two chapters in *A Treatise on Differential Equations* to symbolic methods.

Fractional differentation

- In 1695 Leibniz communicated about fractional derivatives to Johann Bernoulli and l'Hôpital.
- In 1729 Euler communicated to Goldbach the general form

$$\frac{d^n x^p}{dx^n} = \frac{\Gamma(p+1)}{\Gamma(p-n+1)} x^{p-n}, \tag{3}$$

  using the Gamma function, $\Gamma(n) = n!$ for integers n.
- One definition (Riemann-Liouville)

$$f^{(q)}(x) = \frac{1}{\Gamma(k-q)} \frac{d^k}{dx^k} \int_a^x (x-t)^{k-q-1} f(t)\, dt.$$

- From Euler's formula (3) we have for $n = \frac{1}{2}$

$$D^{1/2} \cdot 1 = \frac{\Gamma(1)}{\Gamma(\frac{1}{2})} t^{-1/2} = \frac{1}{\sqrt{\pi t}}.$$

Age of the Earth

Thomson was interested in problems about the age of the Earth and Sun.

When he was sixteen he wrote that measuring the rate of heat loss from the surface of the Earth could put a bound on the age of the Earth (England, Molnar, and Richter 2007). This interest might have been sparked by reading Fourier's works.

Some of the first quantitative studies of the heat equation were by Fourier (Fourier 1808, 1820, 1822). Fourier had written on the temperature of the Earth and the diffusion of heat in a spherical solid (Godard 2017).

He later wrote a general paper about terrestrial temperatures (Fourier 1824b), which was reprinted (Fourier 1827) and translated in 1837 (Fourier 1824a). This has led to some misconceptions about his role in the origins of the greenhouse effect (Fleming 1999).

Age of the Earth (cont'd)

Naturally Thomson (1862) would use Fourier's work and in 1862 he predicted the age of the Earth based on the heat equation.

In the mid-1800's estimates of the age of the Earth went from a few thousand years to hundreds of millions based on geological estimates. Also, Darwin's theory of evolution came out in 1859.

Assuming an initial high temperature and constant diffusivity, Thomson asked how long it would take to reach the current temperature gradient at the Earth's surface of 1°F/50 ft. He came up with 98 million years (England, Molnar, and Richter 2007; Nahin 1985; Harrison 1987).

This was not long enough according to the geologists. A debate between physicists and geologists ensued based on Thomson's estimates (Jackson 2008).

Age of the Earth (cont'd)

Thomson's theory was accepted by the physics community for decades until in 1895 John Perry (1850-1920), a former assistant of Thomson, challenged Lord Kelvin (Perry 1895; England, Molnar, and Richter 2007; Shipley 2001).

Perry challenged Kelvin's assumptions: the thermal conductivity may not be constant. He found an increase in the age estimate.

This led to a debate amongst supporters of Thomson vs those of Perry.

Peter Guthrie Tait (1831 - 1901) sided with Kelvin and Heaviside took up the problem using his operational mathematics, deriving both Kelvin's and Perry's estimates.

Age of the Earth (cont'd)

Heaviside even opened the second volume of his *Electromagnetic Theory* (Heaviside 1922) with a chapter on the Age of the Earth.

It is interesting that Heaviside used a similar analysis of the diffusion equation to arrive at the age of the Earth using Thomson's data. He used took Perry's idea of a nonconstant diffusivity leading to a new equation as described in more detail in (Nahin 1985).

This allowed Perry (1895) to obtain a value for the age of the Earth of more than three times Thomson's estimate of 100 million years (Nahin 1985; Shipley 2001).

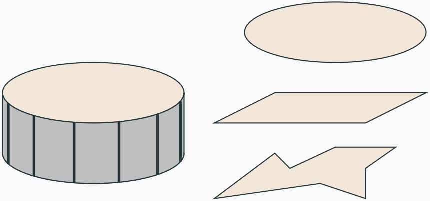
The debate continued for many years later (Jeffreys 1916, 1927).

On to applications -

- Can you hear the shape of a drum?
- How long does it take to cook a turkey?

“Can One Hear the Shape of a Drum?”

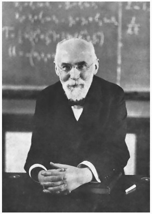
- Kac, Mark (1966). Amer. Math. Monthly. 73, Part II: 1–23.
- Title due to Lipman Bers: “If you had perfect pitch, could you hear the shape of a drum?”
- Can the frequencies (**eigenvalues**) of a resonator (**drum**) determine its shape (**geometry**)?
- Entails features of applied mathematics.
- Historical connections - from radiation theory.



Radiation Theory

- Hendrik Lorentz's (1910) 5 lectures on old/new physics. problems
- 4th - Electromagnetic Radiation Theory.
- Compared vibrations to an organ pipe.
- The number of overtones in frequency range is independent of shape, proportional to volume.
- David Hilbert's prediction (t the conjecture would not be proved in his lifetime), spectrum. His student ...
- Hermann Weyl - < 2 yrs, number $< \lambda$

$$N(\lambda) = \sum_{\lambda_n < \lambda} \sim \frac{|\Omega|}{2\pi}\lambda.$$



What Do We Hear? Frequency, $f = \omega/2\pi$,

Seek Harmonic Solutions,
[Recall $e^{i\omega t} = \cos\omega t + i\sin\omega t$.]

$$u(\mathbf{r}, t) = U(\mathbf{r})e^{i\omega t},$$

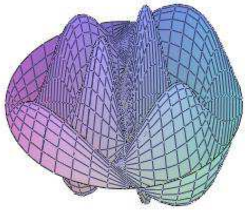
of a Wave Equation, $u(\mathbf{r}, t)$

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u.$$

Helmholtz Equation

$$\nabla^2 U = -k^2 U, \quad k^2 = \frac{\omega}{c}.$$

Eigenvalues $\sim$ frequencies



Vibrations of a String

- Ex: Violin String.
- Harmonics, $u_n(x)$.
- Wavelength, $\lambda = \frac{2L}{n}$.
- Wave Speed, $c = \sqrt{\frac{T}{\mu}}$.
- Frequency, $f = n\frac{c}{2L}$.
- A - $f = 440$ Hz, $L = 32$ cm.
  $c = 2Lf = 280$ m/s.
- Nodes, $u_n(x) = 0$

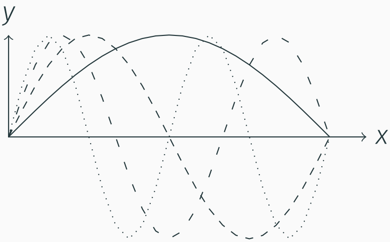


Figure 2: Plot of the eigenfunctions $u_n(x) = \sin\frac{n\pi x}{L}$ for $n = 1, 2, 3, 4$.

Solution of 1D Wave Equation

The one dimensional wave equation, given by

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial^2 x}, \quad t > 0, \quad 0 \leq x \leq L, \tag{4}$$

subject to the boundary conditions

$$u(0,t) = 0, u(L,t) = 0, \quad t > 0,$$

and the initial conditions

$$u(x,0) = f(x), u_t(x,0) = g(x), \quad 0 < x < L.$$

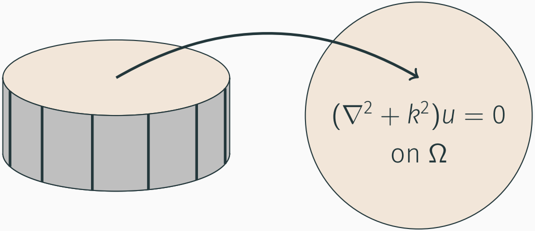
$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \omega_n t + B_n \sin \omega_n t\right] \sin \frac{n\pi x}{L}, \tag{5}$$

where $\omega_n = \frac{n\pi c}{L}$.

General 2D Membranes

- Membrane Problems.
  - Rectangular
  - Circular
  - Elliptical
  - Irregular
- Solve Helmholtz Equations
  - Normal Modes and Frequencies of Oscillation
  - Eigenvalues of Laplace Operator, $\nabla^2 u = -\lambda u$.

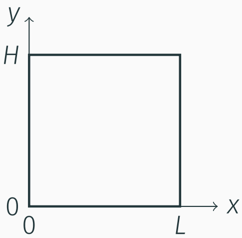
$u = 0$ on $\partial\Omega$



Vibrations of a Rectangular Membrane

- Harmonics
- Frequencies

$$\omega_{mn} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}$$



Boundary-value problem

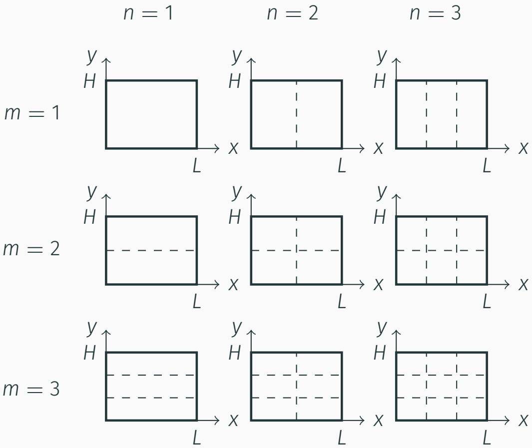
$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad t > 0, 0 < x < L, 0 < y < H, \tag{6}$$

$$u(0,y,t) = 0, \quad u(L,y,t) = 0, \quad t > 0, \quad 0 < y < H,$$
$$u(x,0,t) = 0, \quad u(x,H,t) = 0, \quad t > 0, \quad 0 < x < L,$$

$$u(x,y,t) = \sum_{n,m} (a_{nm} \cos\omega_{nm} t + b_{nm} \sin\omega_{nm} t) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}.$$

Nodes of a Rectangular Membrane

$$u_{nm}(x,y) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}, \quad f = \frac{c}{2L}\sqrt{n^2+\alpha^2 m^2}, \quad \alpha = \frac{L}{H}.$$



$\alpha = 1$	1	2	3
1	1.414	2.236	3.162
2	2.236	2.828	3.606
3	3.162	3.606	4.243

$\alpha = 2$	1	2	3
1	2.236	4.123	6.083
2	2.828	4.472	6.325
3	3.606	5.000	6.708

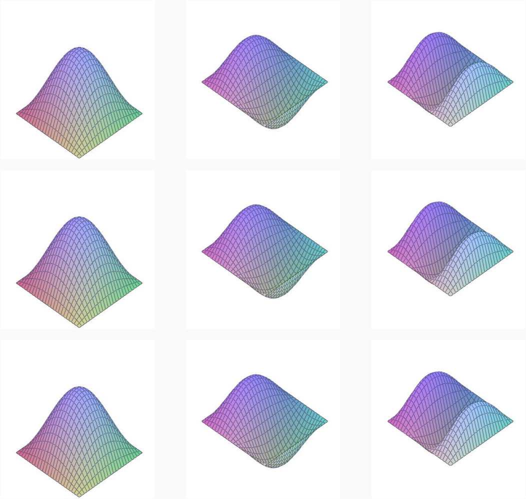
Vibrations of a Rectangular Membrane

$m = 1$ $m = 2$ $m = 3$

$n = 1$

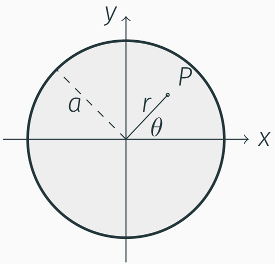
$n = 2$

$n = 3$



Vibrations of a Circular Membrane

- Circular Symmetry.
- Harmonics
- Frequencies



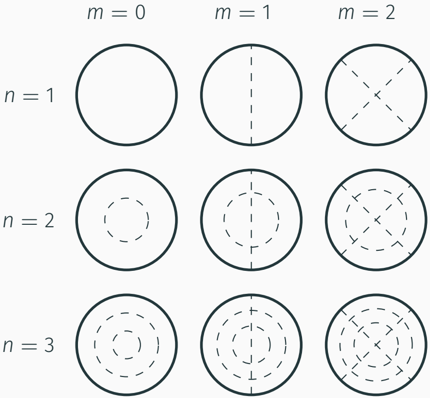
$$\omega_{mn} = \frac{j_{mn}}{a} c.$$

$$u(r,\theta,t) = \left\{ \begin{array}{c} \cos \omega_{mn} t \\ \sin \omega_{mn} t \end{array} \right\} \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} J_m(\frac{j_{mn}}{a} r). \tag{7}$$

$$J_m(j_{mn}) = 0 \quad m = 0, 1, \dots, \quad n = 1, 2, \dots .$$

Nodes of a Circular Membrane

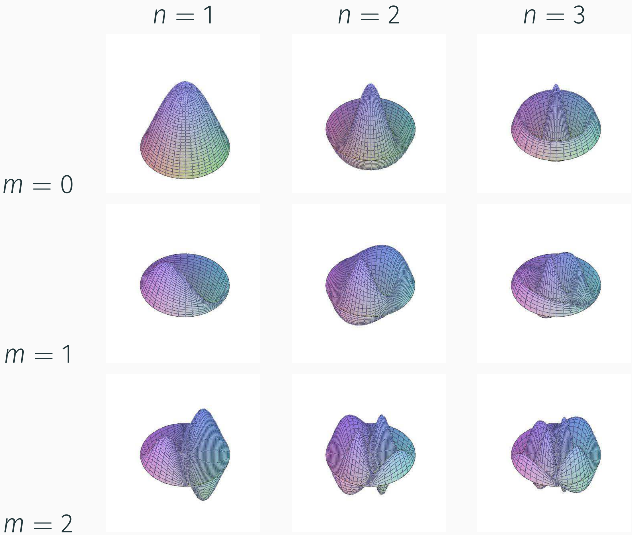
$$u_{mn}(r,\theta) = J_m\left(\frac{j_{mn}}{a}r\right)\cos m\theta, \quad f_{mn} = \frac{j_{mn}c}{2\pi a}.$$



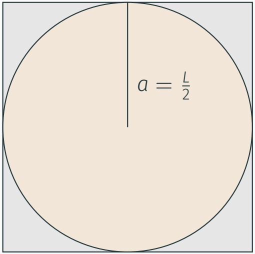
j_{mn}	0	1	2
1	2.405	3.832	5.136
2	5.520	7.016	8.417
3	8.654	10.173	11.62

f_{mn}	0	1	2
1	1.531	2.440	3.270
2	3.514	4.467	5.358
3	5.509	6.476	7.398

Vibrations of a Circular Membrane



Rectangular and Circular Membrane Frequencies



Rectangular

	1	2	3
1	1.414	2.236	3.162
2	2.236	2.828	3.606
3	3.162	3.606	4.243

Circular $a = \frac{L}{2}$

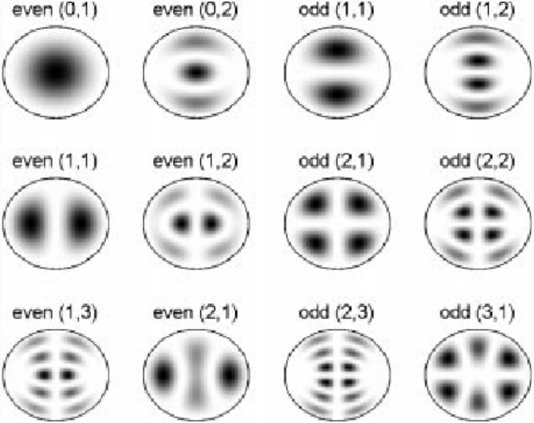
	0	1	2
1	1.531	2.440	3.270
2	3.514	4.467	5.358
3	5.509	6.476	7.398

Circular $\pi a^2 = L^2$

	0	1	2
1	1.357	2.162	2.898
2	3.114	3.958	4.749
3	4.882	5.740	6.556

Vibrations of an Elliptical Membrane

$$\left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + (kh)^2(\cosh^2 \xi - \cos^2 \eta)\right] u(\xi, \eta) = 0.$$



Vibrations of a Balloon

The wave equation takes the form

$$u_{tt} = \frac{c^2}{r^2} L u, \quad \text{where} \quad L Y_{\ell m} = -\ell(\ell+1) Y_{\ell m}$$

for the spherical harmonics $Y_{\ell m}(\theta, \phi) = P_\ell^m(\cos\theta) e^{im\phi}$, The general solution is found as

$$u(\theta, \phi, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[A_{\ell m} \cos\omega_\ell t + B_{\ell m} \sin\omega_\ell t \right] Y_{\ell m}(\theta, \phi),$$

where $\omega_\ell = \sqrt{\ell(\ell+1)} \frac{c}{R}$.

Modes for a Vibrating Spherical Membrane (Balloon?)

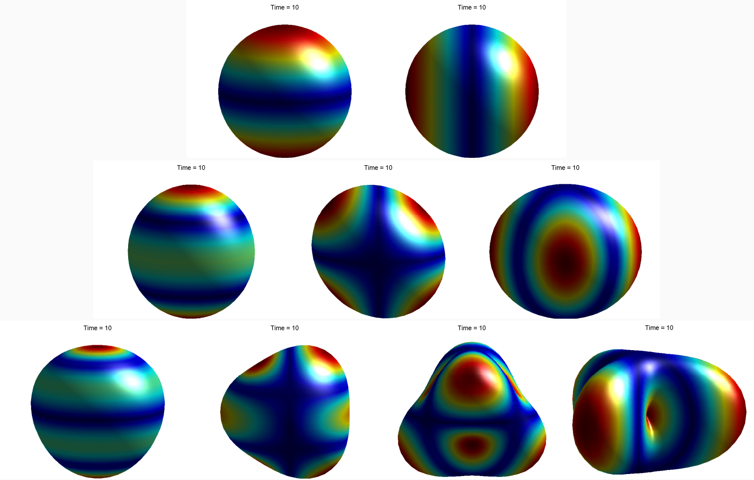
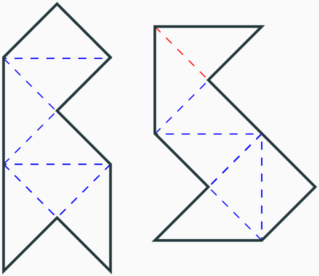
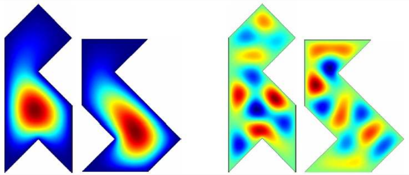


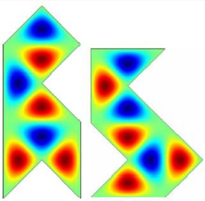
Figure 3: *http://people.uncw.edu/hermanr/pde1/sphmem/*
Row 1: $(1,0),(1,1)$; Row 2: $(2,0),(2,1),(2,2)$;
Row 3 $(3,0),(3,1),(3,2),(3,3)$.

Vibrations of a Irregular Membranes

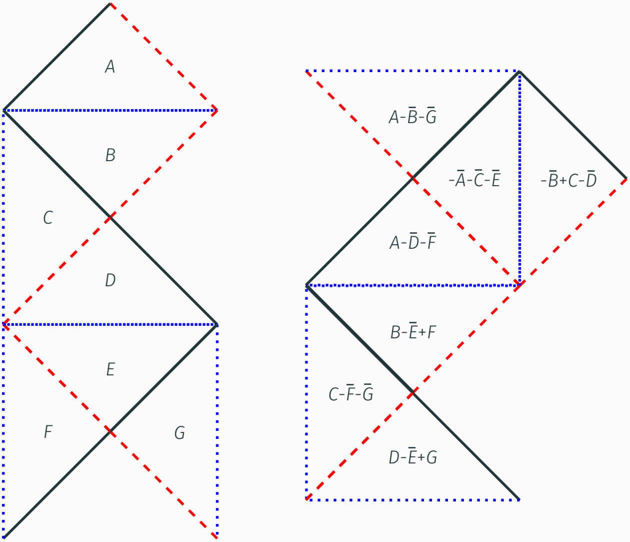
- Gordon, C., Webb, D., and Wolpert, S.(1992) - *You Cannot Hear the Shape of a Drum*
- Shapes on right have same set of frequencies - **isospectral drums.**





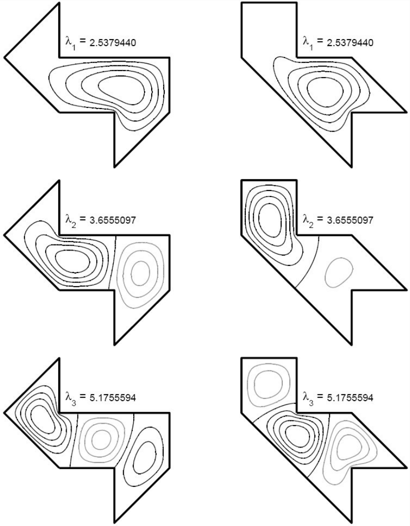


Isospectral Drums



Spectra of Isospectral Drums

$\lambda = 2.5379440, 3.6555097, 5.1755594.$



Other Isospectral Drums

2250 Olivier Giraud and Koen Thas: Hearing shapes of drums: Mathematical and ...

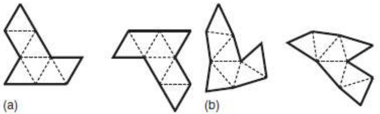


FIG. 25. Pair 7_2. Sunada triple $G=\mathbf{PSL}(3,2)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 1)(2\ 5)$, $b_1=(1\ 5)(3\ 4)$, $c_1=(0\ 4)(1\ 6)$, $a_2=(0\ 4)(2\ 3)$, $b_2=(0\ 6)(1\ 4)$, and $c_2=(0\ 2)(1\ 5)$.

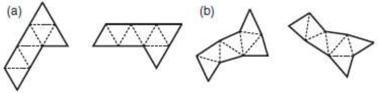


FIG. 26. Pair 7_3. Sunada triple $G=\mathbf{PSL}(3,2)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(2\ 5)(4\ 6)$, $b_1=(1\ 5)(3\ 4)$, $c_1=(0\ 4)(1\ 6)$, $a_2=(0\ 3)(2\ 4)$, $b_2=(0\ 6)(1\ 4)$, and $c_2=(0\ 2)(1\ 5)$.

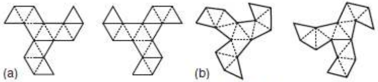


FIG. 27. Pair 13_1. Sunada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 12)(1\ 10)(3\ 5)(6\ 7)$, $b_1=(0\ 10)(2\ 9)(3\ 4)(5\ 8)$, $c_1=(0\ 4)(1\ 6)(2\ 11)(9\ 12)$, $a_2=(0\ 4)(2\ 3)(6\ 8)(9\ 10)$, $b_2=(0\ 1\ 2)(1\ 4)(5\ 11)(6\ 9)$, and $c_2=(0\ 10)(1\ 5)(2\ 7)(3\ 12)$.

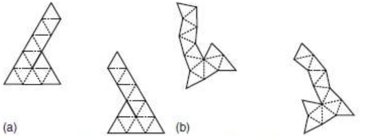


FIG. 31. Pair 13_5. Sunada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(1\ 7)(3\ 5)(4\ 9)(6\ 10)$, $b_1=(0\ 5)(1\ 2)(6\ 12)(9\ 11)$, $c_1=(0\ 4)(1\ 6)(2\ 11)(9\ 12)$, $a_2=(0\ 9)(4\ 10)(6\ 8)(7\ 12)$, $b_2=(0\ 11)(1\ 8)(2\ 7)(3\ 4)$, and $c_2=(0\ 10)(1\ 5)(2\ 7)(3\ 12)$.

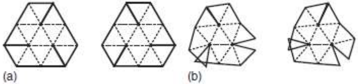


FIG. 32. Pair 13_6. Sunada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 2)(1\ 7)(3\ 6)(5\ 10)$, $b_1=(0\ 6)(2\ 4)(3\ 8)(5\ 9)$, $c_1=(0\ 5)(1\ 2)(6\ 12)(9\ 11)$, $a_2=(0\ 7)(3\ 11)(6\ 8)(9\ 12)$, $b_2=(0\ 8)(1\ 10)(5\ 11)(7\ 9)$, and $c_2=(0\ 11)(1\ 8)(2\ 7)(3\ 4)$.

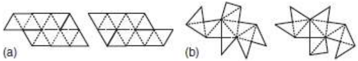


FIG. 33. Pair 13_7. Sun+ada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 2)(1\ 7)(3\ 6)(5\ 10)$, $b_1=(0\ 4)(2\ 3)(6\ 8)(9\ 10)$, $c_1=(0\ 5)(1\ 2)(6\ 12)(9\ 11)$, $a_2=(0\ 7)(3\ 11)(6\ 8)(9\ 12)$, $b_2=(0\ 12)(1\ 1\ 0)(3\ 5)(6\ 7)$, and $c_2=(0\ 11)(1\ 8)(2\ 7)(3\ 4)$.

No!

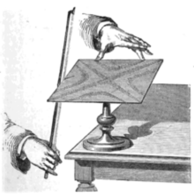
Membranes - Rectangular, circular, elliptical, irregular

Never look at MATLAB logo the same way again - Why?

MATLAB

Chladni Plates

- Recall Sophie Germain, 1776-1831.
- Ernst Chladni, 1756-1827, physicist and musician.
- In 1808, Chladni demonstrated vibrating plates at the Academy of Science in Paris.
- Napoleon, who attended, proposed a prize.
- Lagrange, Laplace and others – felt that it was beyond reach.
- Germain only one to try.
- 1816, two more tries, first woman awarded Grand Prize in Mathematics of the Paris Academy of Sciences.



Heat Equation vs Wave Equation

1D Wave Equation

$$u_{tt} = c^2 u_{xx}$$

1D Heat Equation

$$u_t = k u_{xx}$$

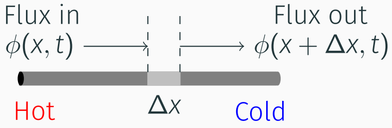
History of Heat Equation

Developed by Joseph Fourier (1768-1830)

- Discovered in early 1807 and published later in 1822

  Afterwards, diffusion processes studied outside of France.
  Lead to research in partial differential equations.
- Describes conduction and storage of heat (energy) in a body.
- Involves heat exchange with surroundings, conservation of energy.
- Leads to temperature changes inside body (diffusion).
- Uses the relation of heat energy to temperature (gradient), Fourier Law of Heat Conduction.

Heat Equation - Mathematics



Rate of change of heat energy = Flux in - Flux out

$$\frac{dQ}{dt} = \phi(x,t) - \phi(x+\Delta x,t).$$

Flux density = conductivity $\times$ temperature gradient

$$\phi = k\frac{dT}{dx}.$$

Heat energy is proportional to temperature

$$Q = mcT.$$

$q =$ Heat energy per vol, $u =$ temperature per vol

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \quad D = \frac{k}{mc}.$$

Thanksgiving Turkey!

- Native to North America.
- Introduced in Spain in 1500's.
- Benjamin Franklin - national bird.
- Holiday bird in Europe in 1800's
  - replacing goose.
- Turkeys mostly walk.
- Harold McGee: Breast 155-160 F, Legs 180 F.
- Cooking times

  Constant oven temp, diffusivity constant, Turkey plump

  Small - 20 min/lb + 20.

  Large - 15 min/lb + 15.

  $t \sim M^{2/3}$.

How long does it take to cook a turkey?

Example 1
If it takes 4 hours to cook a 10 pound turkey in a 350^o F oven, then how long would it take to cook a 20 pound turkey at the same conditions?

Figure 4: A Thanksgiving turkey - From 2015.

Panofsky Equation

- Pief Panofsky [SLAC Director Emeritus] *SLAC Today*, Nov 26, 2008 *http://today.slac.stanford.edu/a/2008/11-26.htm* For a stuffed turkey at 325° F

$$t = \frac{W^{2/3}}{1.5}$$

  vs. 30 minutes/lb.
- Also, check out WolframAlpha *http://www.wolframalpha.com/input/?i=how+long+should+you+cook+a+turkey*
- Musings of an Energy Nerd *http://www.greenbuildingadvisor.com/blogs/dept/musings/heat-transfer-when-roasting-turkey*

Consider a Spherical Turkey

Figure 5: The depiction of a spherical turkey.

Scaling a Spherically Symmetric Turkey

The baking follows the heat equation.

Rescale the coordinates (r, t) to (ρ, τ) :

$$r = \beta\rho \text{ and } t = \alpha\tau.$$

Then, the heat equation rescales as

$$u_\tau = \frac{\alpha}{\beta^2}\frac{k}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right).$$

- Invariance of heat equation implies $\alpha = \beta^2$.
- So, if the radius increases by a factor of β, then the time to cook the turkey increases by β^2.

Problem Solution

Example 1

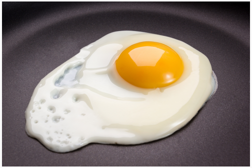
If it takes 4 hours to cook a 10 pound turkey in a 350^o F oven, then how long would it take to cook a 20 pound turkey at the same conditions?

- The weight doubles $\Rightarrow$ the volume doubles.
  (if density = constant).
- $V \propto r^3 \Rightarrow r$ increases by factor: $2^{1/3}$.
- Therefore, the time increases by a factor of $2^{2/3} \approx 1.587$.
- If 4 lb turkey takes 4 hrs, then a 20 lb turkey takes

$$t = 4(2^{2/3}) = 2^{8/3} \approx 6.35 \text{ hours.}$$

- In general, if the weight increases by a factor of x, then the time increases by $x^{2/3}$.

Eggs


Omelettes

Egg Protein

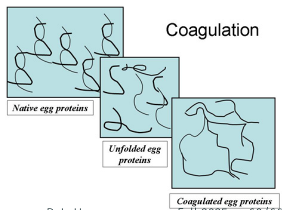
Proteins in eggs can be used

- to help food set (e.g. egg custards),
- as a foam to add air and volume (e.g. sponge cakes),
- as an emulsifier (e.g. mayonnaise).

Two different major proteins, egg white (albumin) and egg yolk,

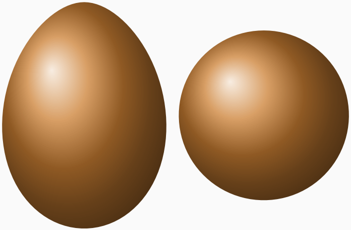
- Albumin starts coagulating at 63^oC
- Yolks start at 70^oC

Coagulation - protein unfolds, denaturation.
As heat increases the proteins rearrange and coagulate.
Egg albumin turns from clear to cloudy white.



Egg Cooking Time

Peter Barnham, *The Science of Cooking* & Dr. Charles Williams of Exeter:

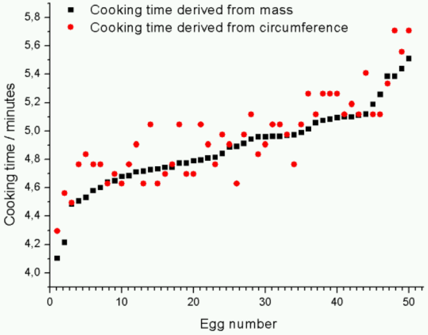


$$t = 0.0152 d^2 \log \left[2 \times \frac{(T_{\text{water}} - T_0)}{T_{\text{water}} - T_{\text{yolk}}} \right],$$
$$t = 0.451 M^{2/3} \log \left[0.76 \times \frac{(T_{\text{water}} - T_0)}{T_{\text{water}} - T_{\text{yolk}}} \right],$$

for t min, diameter d cm, M g, and temperatures in ^{o}C.

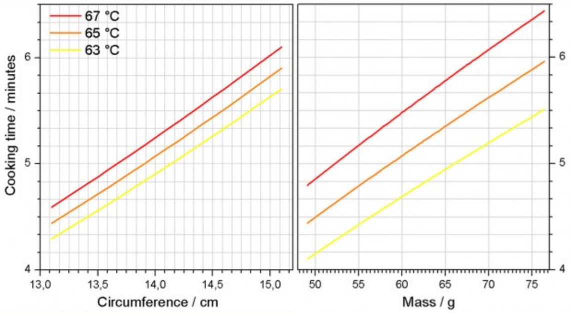
Egg Cooking Time - Data

From Khymos *Towards the perfect soft boiled egg* by Martin Lersch, April 9th, 2009. See also University of Oslo Applet



50 eggs with $T_{yolk} = 63^oC$, $T_{water} = 100^oC$ and $T_{egg} = 4^oC$.

Egg Cooking Time - Formula



Given circumference or mass to reach to reach 63, 65 and 67^o C, respectively, at the yolk-white boundary with $T_{water} = 100^o$ C and $T_{egg} = 4^o$ C.

Egg Consistency

Temp	White	Yolk
62	Begins to set, runny	Liquid
64	Partly set, runny	Begins to set
66	Largely set, still runny	Soft solid
70	Tender solid	Soft solid, waxy
80	Firm	Firm
90	Rubbery solid	Crumbly texture

At sea level, boiling water is 100^o C. At higher altitudes, the boiling temperature of water is lowered 0.3^o C for each additional 100 m above sea level.

Fast Fourier Transform - FFT

- One of top algorithms of 20th Century.
- Developed by Cooley and Tukey, 1965, to compute DFT (Discrete fourier transform)
- Some traced the ideas back to Gauss.
- Limit of Fourier series = Fourier Transform.
- Related to Laplace transform.

$$\begin{aligned} F(k) &= \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx, \\ f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ikx}\, dk. \\ F(s) &= \int_{0}^{\infty} f(t) e^{-st}\, dt. \end{aligned} \tag{8}$$

Left for another course!

References for Drums

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- Tobin Driscoll, Eigenmodes of isospectral drums, SIAM Review 39 (1997), 1-17.
- Carolyn Gordon, David Webb, Scott Wolpert, One cannot hear the shape of a drum, *Bull. Amer. Math. Soc.* 27 (1992), 134-138.
- Marc Kac, Can one hear the shape of a drum?, *Amer. Math. Monthly* 73 (1966), 1-23.
- Cleve Moler, The MathWorks logo is an eigenfunction of the wave equation (2003).
- Lloyd N. Trefethen and Timo Betcke, Computed eigenmodes of planar regions (2005).