Introduction to the History of Mathematics

Fall 2025 - R. L. Herman

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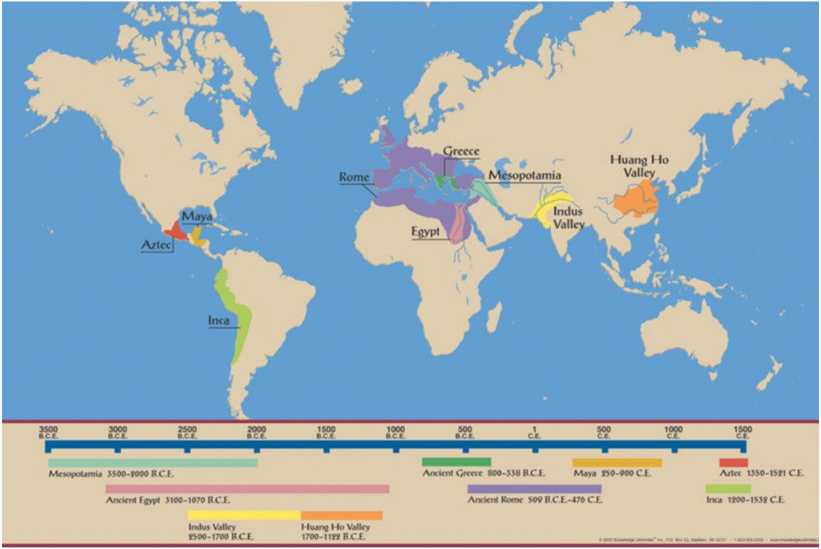
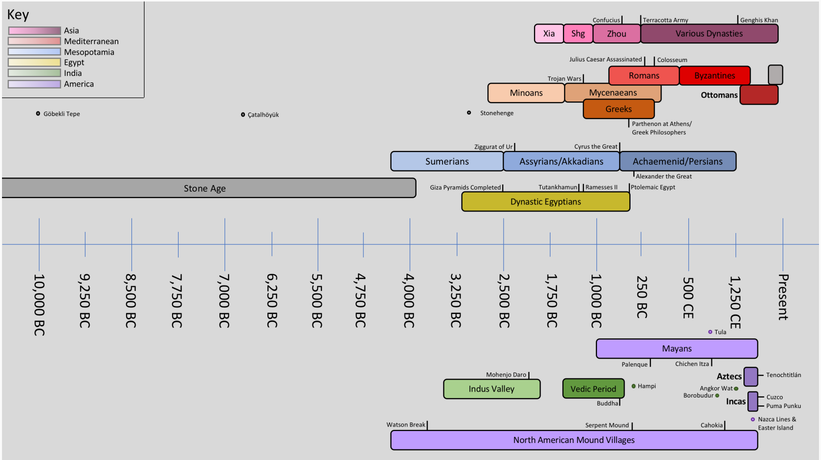
Civilizations - How Did Mathematics Develop?



A Civilization Timeline



Some Early Civilizations

- Egypt (3150-30 BCE)
- Mesopotamia (3100-539 BCE)
- Chinese (1766 BCE-220 CE)
- Indian Mathematics (500-1200)
- Mayan Mathematics (250-900)
- Aztecs and Incans (1345-1560)

Arithmetic, Geometry, No proofs.
Problems were practical or recreational.

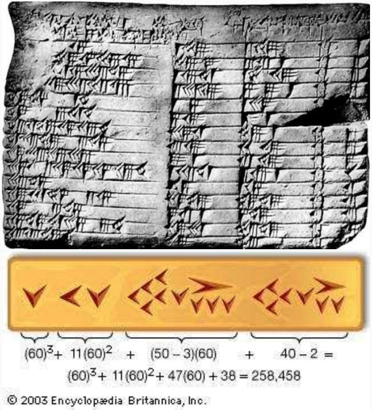


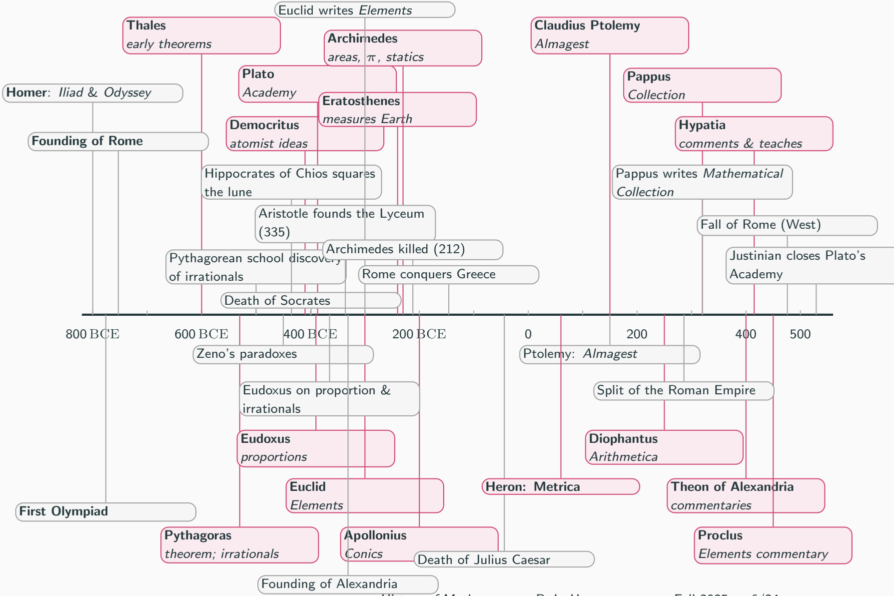
Figure 1: Babylonian Math - Base 60

Greek Civilization

- Deductive Reasoning
  - Definitions, Axioms
  - Propositions via logic
- Geometry, Trigonometry, Astronomy, Numbers, Conics
- Thales (624-546 BCE)
- Pythagoras (6th Century BCE)
- Euclid (4th Century BCE)
  Elements - geometry, numbers
- Archimedes (3rd Century BCE)
- Apollonius (2nd Century BCE)
- Heron (10-70), Diophantus (200-284), Pappas (290-350), Hypatia (370-415)

Figure 2: Euclid

Greek Mathematicians (800 BCE–530 CE)



Chinese and Indian

- Chinese Mathematics
  1300 BCE - 1800 CE
  - Pythagorean Thm
  - π estimates
  - Volumes, Applications
  - Pascal's Triangle
  - Chinese Remainder Thm
- Indian Mathematics 1200 BCE,
  mostly 500-1200 CE
  - Geometry
  - Trigonometry
  - Power series
  - Astronomy
  - π estimates
  - Number system, 0
  - Pell's Equation

Figure 3: Liu-Hong

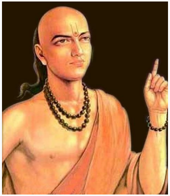


Figure 4: Aryabhatta

European Dark Ages - 400-1200 CE

- Founding of Islam - 7th Century
- Islamic mathematicians preserve/translate Greek/Asian mathematics into Arabic
- Arabic Numerals by 1000 CE
- Persian mathematicians
    - al-Khwarizmi (780-850)
      - Algebra (al-Jabr)
    - Omar Khayyam (1048-1131)
      - geometric solution of cubic

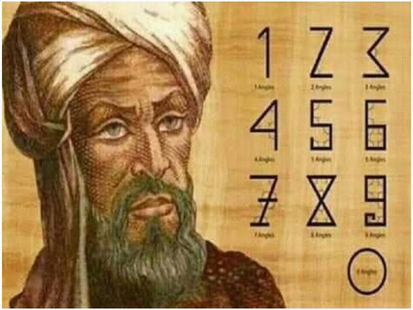
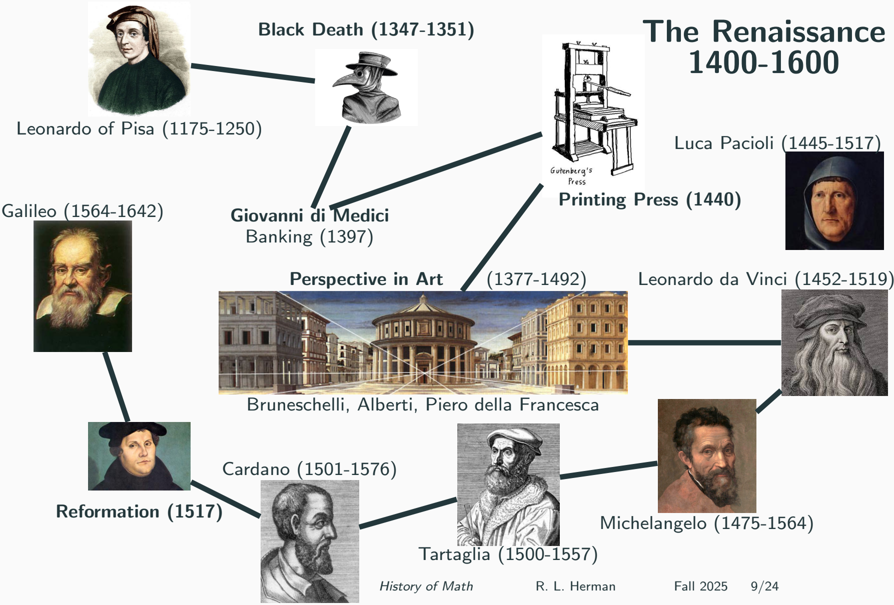


Figure 5: al-Khwarizmi

Around 10th Century - Middle Eastern Mathematics brought to Spain.

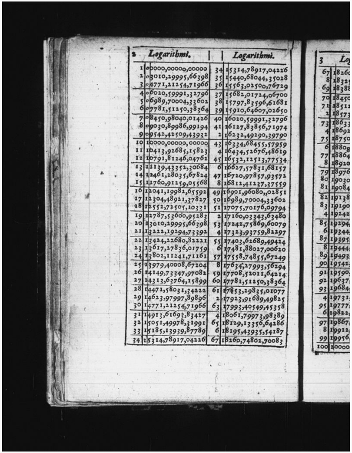
It takes 300 years to accept Hindu-Arabic numerals. - Fibonacci - 1202

The Renaissance



Beyond Numerals

- Fractions 4000 years ago
- Sexagesimal (base 60) into 17th century
- Decimal (base 10)
  - al-Uqlidisi - (920-980)
  - al-Kashi (1380-1429)
  - Simon Stevin (1548-1620)
- Logarithms
  - John Napier (1550-1617) - used a stange base
  - Henry Briggs (1561-1630)
    - Base 10 Tables
    - 54 square roots of 10 (30 decimal places)
    - Tables - 14 decimal places



2 Logarithmi. | Logarithmi.

	Logarithmi.		Logarithmi.
1	00000,00000,00000	34	15314,78917,04226
2	03010,29995,66398	35	15440,68044,35028
3	04771,21254,71966	36	15563,02500,76729
4	06020,59991,32796	37	15682,01724,06700
5	06989,70004,33602	38	15797,83596,61681
6	07781,51250,38364	39	15910,64607,02650
7	08450,98040,01426	40	16020,59991,32796
8	09030,89986,99194	41	16127,83856,71974
9	09542,42509,43932	2	16232,49290,39790
10	10000,00000,00000	43	16334,68455,57959
11	10413,92685,15823	4	16434,52676,48619
12	10791,81246,04762	45	16532,12513,77534
13	11139,43352,30684	6	16627,57831,68157
14	11461,28035,67824	47	16720,97857,93572
15	11760,91259,05568	8	16812,41237,37559
16	12041,19982,65592	49	16901,96080,,02851
17	12304,48921,37827	50	16989,70004,33602
18	12552,72505,10331	51	17075,70176,09794
19	12787,53600,95283	2	17160,03343,63480
20	13010,29995,66398	53	17242,75869,60079
21	13222,19294,73392	4	17323,93759,82297
22	13424,22680,82221	55	17403,62689,49424
23	13617,27836,01759	6	17481,88027,00620
24	13802,11241,71161	57	17558,74855,67249
25	13979,40008,67204	8	17634,27993,56294
26	14149,73347,97082	59	17708,52011,64214
27	14313,63764,15899	60	17781,51250,38364
28	14471,58031,34222	61	17853,29835,01077
29	14623,97997,89896	2	17923,91689,49825
30	14771,21254,71966	63	17993,40549,45358
31	14913,61693,83427	4	18061,79973,98389
32	15051,49978,31991	65	18129,13356,64286
33	15185,13939,87789	6	18195,43935,54187
34	15314,78917,04226	67	18260,74802,70083

Figure 6: Brigg's Tables

- Fibonacci (Leonardo of Pisa) (1170-1250) *Liber Abaci*
- Equation Solving contests
- Solutions of cubic and quartic
  - Depressed cubic
    del Ferro (1465-1526)
  - Cubic and quartic equations
    Tartaglia (1500-1557)
    Cardano (1501-1576)

    Ars Magna

    Ferrari (1522-1565)
- Bombelli (1526-1572)
  - Complex numbers
- Viète (1540-1603)
  Adriaan van Roomen Problem

Figure 7: Cardano and Tartaglia
Fight of the Century!

Unification of Geometry and Algebra

- Symbolic Algebra
  - Rhetorical until 15th century
  - Syncopated/abbrev. - 1500
  - Symbolic algebra developed 16-17th century
- Unification
  - Oresme (1320-1382) - Velocity-time graphs, $\sum \frac{1}{n}$
  - Descartes (1596-1650)
    - Rep. curves by equations
    - Coordinate systems - published *The Method*
    - Made use of variables which can vary continuously - lines.
  - Fermat (1607-1665)
    - Rep. equations by curves

Figure 8: Fermat and Descartes

The United Kingdom - Before 1700

- English
  - Thomas Harriot (c. 1560 – 1621): Algebraic equations and the use of symbols.
  - William Oughtred (1574 – 1660): Inventor of the slide rule; used math symbols.
  - William Brouncker (1620 – 1684): The first President of the Royal Society; work on series and approximations of π.
  - Isaac Barrow (1630 – 1677): Newton's teacher.
  - Christopher Wren (1632 – 1723): Architect, contributions to mathematics.
  - Edmund Halley (1656 – 1742): Celestial mechanics.
  - Abraham de Moivre (1667 – 1754): Probability theory; De Moivre's formula.
- Scottish
  - John Napier (1550-1617): Logarithms, Napier's bones, spherical trigonometry.
  - James Gregory (1638 – 1675): Series representations of trigonometric functions; Gregorian telescope.
  - James Stirling (1692-1770): Stirling numbers,approximation; cubic plane curves.
  - Colin Maclaurin (1698-1746): Known for his work in geometry and algebra.
  - Matthew Stewart (1717–1785): Geometry, Kepler's 2^{nd} law and three-body problem.
- Irish
  - Robert Boyle (1627 – 1691): Primarily a physicist and chemist, involved mathematics.
  - George Berkeley (1685 – 1753): Philosopher with an interest in mathematics.

And others - John Wallis (1616 – 1703), John Collins (1625 – 1683), Isaac Newton (1642 – 1727), Joseph Raphson (1668 – 1712), Roger Cotes (1682 – 1716) Brook Taylor (1685 – 1731), Edmund Gunter (1581 – 1626)

The Rise of Calculus

- Archimedes - 3rd century BCE
- Kepler (1571-1630)
- Cavalieri (1598-1647)
- Fermat (1607-1665)
- Wallis (1616-1673)
- Pascal (1623-1662)
- Barrow (1630-1677)
- Wren (1632-1723)
- Gregory (1638-1675)
- Newton (1642-1726)
  - *Principia* 1687
- Leibniz (1646-1716)
  - *Notation* $\frac{d}{dx}, \int$

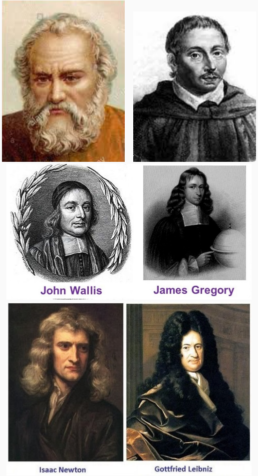
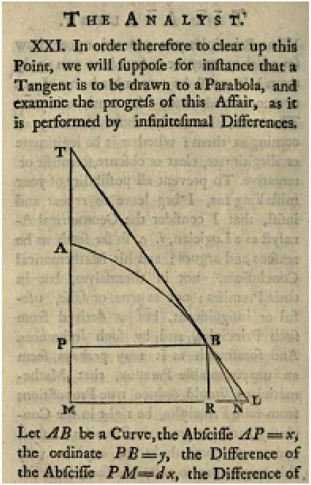


Figure 9: Archimedes, Cavalieri, Wallis, Gregory, Newton, and Leibniz

The Infinitesimal

- Hippasus 500 BCE
  - Pythagorean, $\sqrt{2}$ irrational
- Introduction of Infinitesimals
  - Cavalieri and Torricelli
  - Stevin, Wallis, Harriot
- Religious Critics
  - Era of Copernicus, Galileo
  - Jesuits in Italy, Church bans
  - George Berkeley (1685-1753) *The Analyst*, - *A Discourse Addressed to an Infidel Mathematician*, 1734
  - "Infinitesimals undermine mathematics and rationality"
- Augustin-Louis Cauchy - 1821



THE ANALYST.

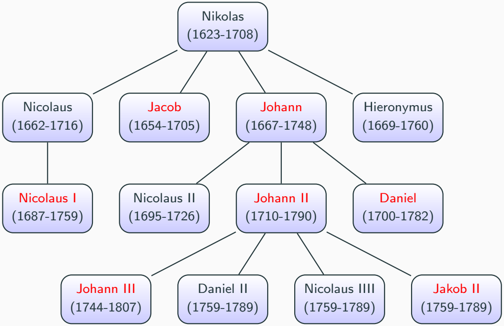
XXI. In order therefore to clear up this Point, we will suppose for instance that a Tangent is to be drawn to a Parabola, and examine the progress of this Affair, as it is performed by infinitesimal Differences.

Let AB be a Curve, the Absciſſe $AP=x$, the ordinate $PB=y$, the Difference of the Absciſſe $PM=dx$, the Difference of

Figure 10: Berkeley's *The Analyst*

Exploiting Calculus

- Bernoulli Family
- Leonhard Euler (1707-1783)
- Joseph-Louis Lagrange (1736-1813)
- Pierre-Simon Laplace (1749-1827)
- Neptune discovered using math - 1846, Le Verrier (1811-1877)



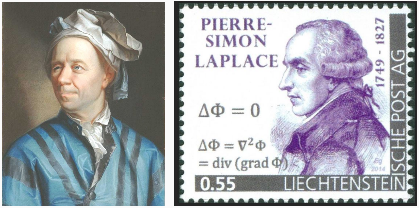
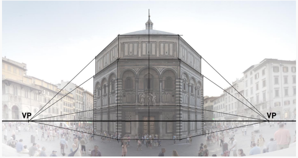


Figure 12: The Bernoulli Family

Figure 11: Euler and Laplace

Evolution of Geometry: Projective Geometry and Topology

- Perspective in Art
  - Bruneschelli (1377-1446)
  - Leon Alberti (1404-1472)
  - Girard Desarges (1591-1661)
- Birth of Topology
  - Euler - Königsberg bridge
    Geometry without distance
  - Euler Characteristic
    $\chi = V - E + F$, and
    classification of surfaces
    $\chi = 2 - 2g$, genus
- Birth of Knot Theory
  - Gauss - Intertwining curves
  - Scottish physics and Knots - Taits smoke rings.



The Birth of Rigor - 19th Century

Non-Euclidean Geometry

- Parallel Postulate
- Hyperbolic Geometry
  - Nikolai Lobachevsky (1792-1856)
  - Johann Bolyai (1802-1860)
  - Johann Carl Friedrich Gauss (1777-1855)
- Elliptic Geometry
  - Georg Friedrich Bernhard Riemann (1826-1866)
  - Prince of Mathematicians
- By 1870's Euclid in doubt!

Figure 13: Gauss, Lobachevsky, Bolyai

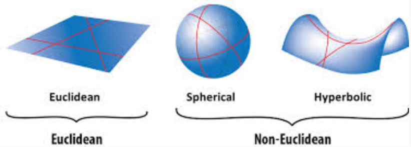


Figure 14: Different Geometries

19th Century Group Theory

The search for the general solution of the quintic.

- Joseph-Louis Lagrange (1736-1813)
- Johann Carl Friedrich Gauss (1777-1855)
- Paola Ruffini (1765-1822) - proof of unsolvability
- Augustin Cauchy (1789-1857)
- Niels Henrik Abel (1802-1829)
- Évariste Galois (1811-1832)
- Arthur Cayley (1821-1895)
- Camille Jordan (1838-1922)

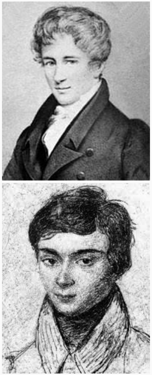
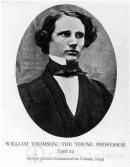


Figure 15: Abel and Galois

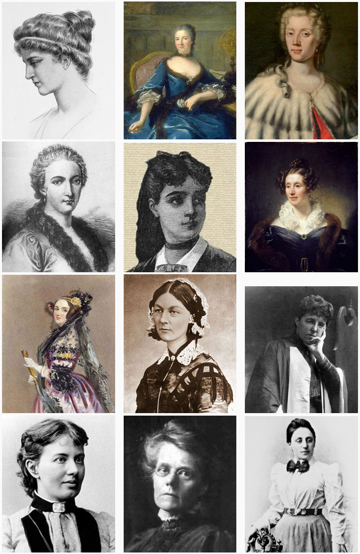
Mathematical Physics - Mathematical Physicists

- Joseph-Louis Lagrange (1736-1813)
- Pierre-Simon Laplace (1749-1827)
- Joseph Fourier (1768–1830)
- Siméon Denis Poisson (1781–1840)
- George Green (1793–1841)
- William Hamilton (1805–1865)
- George Gabriel Stokes (1819-1903)
- William Thomson (1824–1907), 1st Baron Kelvin
- James Clerk Maxwell (1831–1879)
- J. Willard Gibbs (1839–1903)
- John William Strutt (1842–1919), 3rd Baron Rayleigh
- Oliver Heaviside (1850–1925)



Famous Women Mathematicians Before 1900

- Hypatia of Alexandria (c. 350-415)
- Émilie du Châtelet (1706-1749)
- Laura Bassi (1711-1788)
- Maria Agnesi (1718-1799)
- Sophie Germain (1776-1831)
- Mary Fairfax Somerville (1780-1872)
- Ada Lovelace (1815-1852)
- Florence Nightingale (1820-1910)
- Charlotte Angas Scott (1848-1931)
- Sofia Kovalevskaya (1850-1891)
- Alicia Boole Stott (1860-1940)
- Amalie 'Emmy' Noether (1882-1935)



19th Century Analysis and Set Theory

- Jean-Baptiste Joseph Fourier (1768-1830)
- Johann Carl Friedrich Gauss (1777-1855)
- Augustin Cauchy (1789-1857)
- Karl Weierstrass (1815-1897)
- George Boole (1815-1864)
- Georg Friedrich Bernhard Riemann (1826-1866)
- Richard Dedekind (1831-1916)
- Georg Ferdinand Ludwig Philipp Cantor (1845-1918)
  - Founder of set theory
  - Defined infinite sets

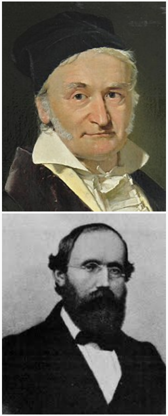


Figure 16: Gauss and Riemann

19th Century Number Theory

- Marie-Sophie Germain (1776-1831)
- Johann Carl Friedrich Gauss (1777-1855)
  Disquisitiones Arithmeticae - 1801
- Adrien-Marie Legendre (1752-1833) and
  Peter Gustav Lejeune Dirichlet (1805-1859) prove
  Fermat's Last Theorem for $n = 5$ in 1825
  - Dirichlet, $n = 14$ in 1832.
- Riemann Hypothesis, distribution of primes - 1832.
- Charles Jean de la Vallée-Poussin and Jacques Hadamard - Prime Number Theorem. 1896
- H. Minkowski: Geometry of Numbers, 1896.

Figure 17: Sophie Germain, Adrien-Marie Legendre

The Modern Era

We stop at the turn of the 20th Century: the evolution of mathematics, set theory, physics revolutions, Bourbaki, Hilbert's 23 Problems.

Explore mathematics prizes: Fields Medal, Abel Prize, Wolf Prize, Millenium Prize.

Other Sites

- Chronology of 20th Century Mathematicians
- Greatest Mathematicians born between 1860 and 1975
- Pictures of Famous 20th Century Mathematicians
- The Story of Math Website

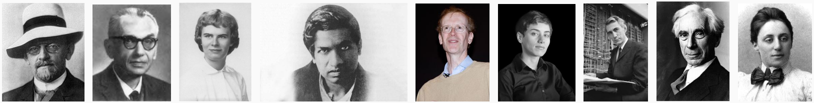


Figure 18: Hilbert, Gödel, Uhlenbeck, Ramanujan, Wiles, Mirzakhani, Shannon, Russell, Noether

Early Mathematics - Egypt and Mesopotamia

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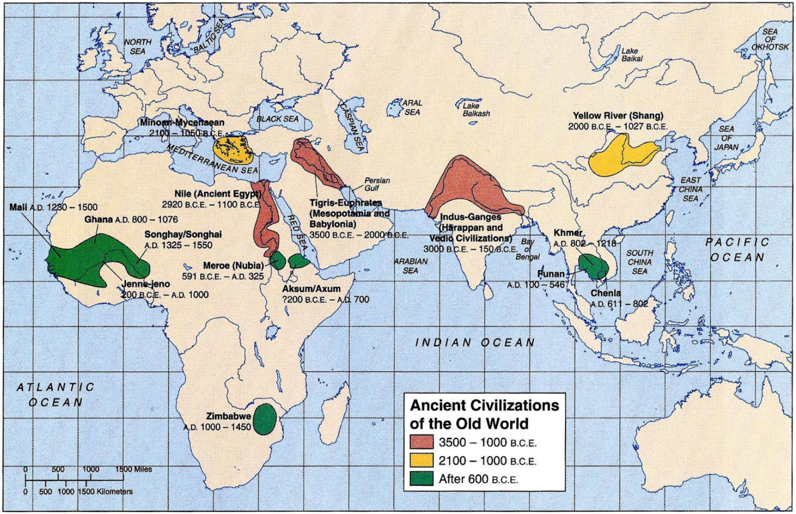
Maps of Ancient Civilizations

Ancient Civilizations of the Old World



Early Civilizations

- Ancient African ¿ 20,000 yrs
- Egypt (3150-30 BCE)
- Mesopotamia (3100-539 BCE)
- Indus (3300-1700 BCE)
- Greek (640 BCE-415 CE)
- Chinese (1766 BCE-220 CE)
- Indian Mathematics (500-1200)
- Islamic Mathematics (700-1200)
- Mayan Mathematics (250-900)
- Aztec Empire (c.1345-1521)
- Inca Civilization (1400-1560)

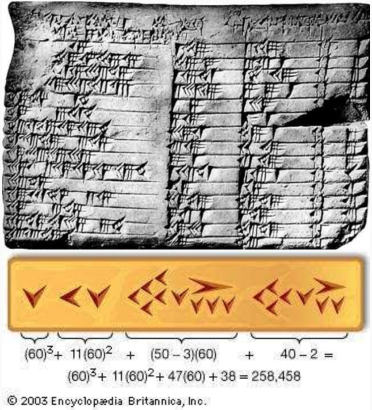
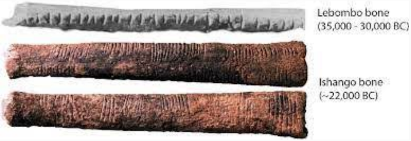


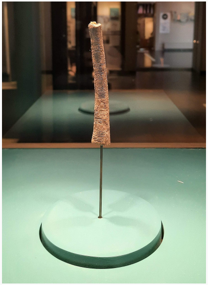
Figure 1: Babylonian tablet - Base 60

Ancient African Mathematics

- Lebombo bone, 43,000-44,200 yrs old.
  Oldest known mathematical artifact,
  29 notches on a baboon's fibula.
  Found in Border Cave, Lebombo Mountains, Swaziland.
- Ishango bone, 20,000 BCE.
  Also baboon bone.
  Ishango, Democratic Republic of Congo.
  Numerical patterns with differing interpretations.



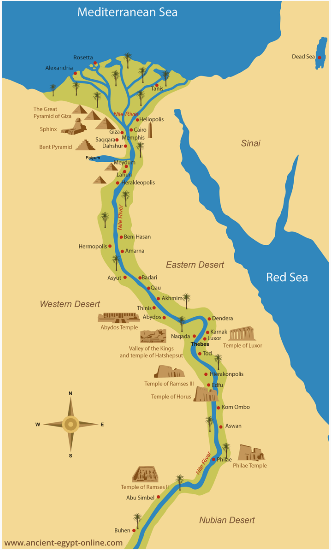
See Blog and Article



See Wikipedia.

Ancient Egypt

- Early Dynastic Period (3150–2686 BCE), writing
- Old Kingdom (2686–2181 BCE) (Great Pyramid of Giza)
- 1st Intermediate Period (2181–2055 BCE)
- Middle Kingdom (2055–1650 BCE), **Reisner Papyri** and **Moscow Papyrus**
- 2nd Intermediate Period (1650–1550 BCE), **Rhind Papyrus**
- New Kingdom (1550–1069 BCE)
- 3rd Intermediate Period (1069–664 BCE)
- Late Period (664–332 BCE)



The Papyri

- Papyri - scrolls.
  - Rhind Papyrus, 1650 BCE.
  - Moscow Papyrus, 1850 BCE.
  - Reisner Papyri, 1950 BCE.
- Reisner Papyri
  - Dr. G.A. Reisner .
  - 1901–04 - southern Egypt.
  - 4 scrolls.
  - Mostly accounts.
- Egyptian Arithmetic.
  - Base-10.
  - heiroglyphic and hieratic numerals.
  - integers, fractions.
  - surveying, building.
  - areas, volumes.

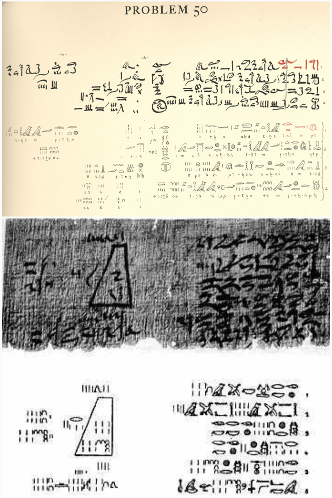
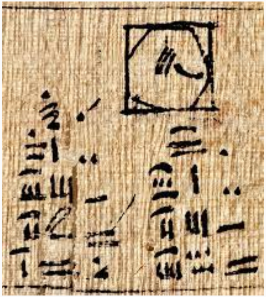


Figure 2: Papyri

The Rhind Papyrus

- Found in Thebes.
- Purchased 1858, by A. Henry Rhind.
- Size: $18' \times 13''$.
- Red and black ink.
- Geometry.
  - Areas, Volumes.
  - Ratios of sides of right triangles.
- Measures - grain.
  - 1 hekat $\approx 29,224$ in$^3 \geq \frac{1}{2}$ peck.
  - 1 ro $= \frac{1}{320}$ hekat.
- Areas of Circles - 48, 50.

$$A = \left(\frac{8}{9}D\right)^2 = \frac{256}{81}r^2 \approx 3.16049r^2.$$



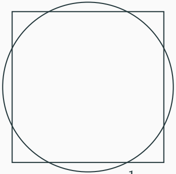


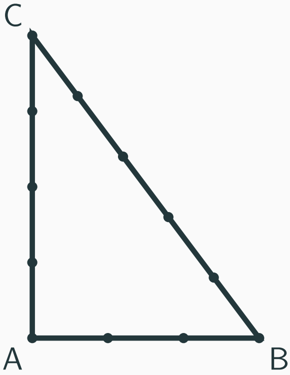
Figure 3: Problem 48

Pythagorean Triples

- Pythagorean Theorem.
- Triples (a, b, c),

$$a^2 + b^2 = c^2.$$

Examples:
  - 3-4-5.
  - 5-12-13.
- Used to Measure Perimeters.
- Knotted Ropes.
  - Loop with 12 knots.
- Other Units:
  - Finger - 1.9 cm.
  - Palm = 4 fingers - 7.5 cm.
  - Cubit = 7 palms - 52.3 cm.



Egyptian Arithmetic

Multiplication - 17×13.

1	17
2	34
4	68
8	136
13	221

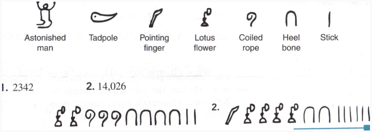
Division - $108/12$.

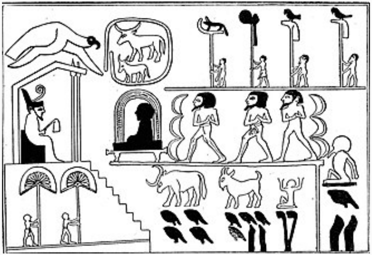
1	12
2	24
4	48
8	96
9	108

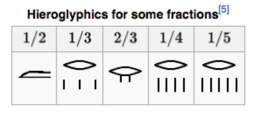
Unit fractions:

$$\frac{4}{7} = \frac{1}{2} + \frac{1}{14}.$$

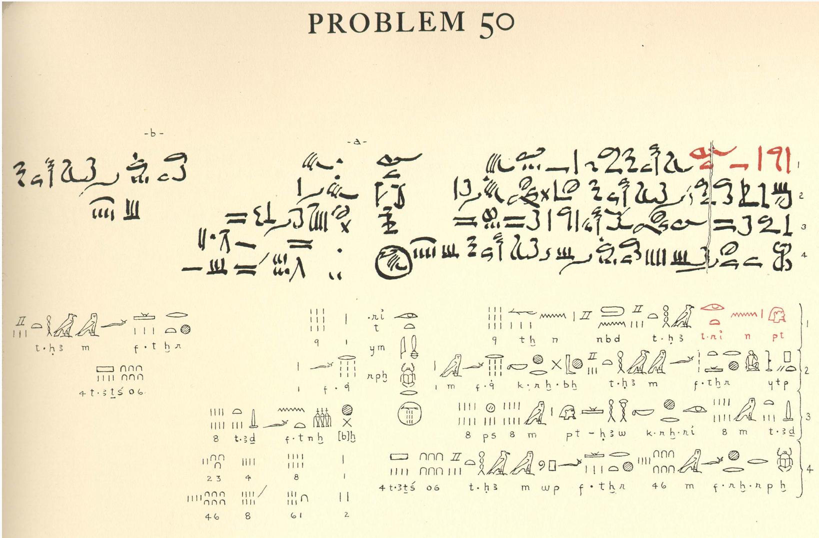
Counting and Unit Fractions







Rhind Papyrus - Problem 50



Rhind Papyrus - Problem 50

Problem 50

tp n ỉr·t ꜣḥ·t dbn n ḫt·w[1] 9 pty rḫt · f m ꜣḥ·t
Example of making a field round of khet 9. *What is the amount of it in area?*

ḫb · ḫr · k 9̇ · f m 1 ḏꜣ·t m 8 ỉr·ḫr · k wꜣḥ–tp m 8 sp 8 ḫpr·ḫr · f m 64
Take away thou ⅑ *of it, namely,* 1*; the remainder is :* 8. *Make thou the multiplication :* 8 *times* 8*; becomes it :* 64;

rḫt · f pw m ꜣḥ·t 60[2] śṯꜣ·t 4
the amount of it, this is, in area, 60 setat 4.

ỉr·t my ḫpr
The doing as it occurs:

1	9
9̇ · f *of it*	1.

ḫ[b] ḫnt · f ḏꜣ·t 8
Take away from it; the remainder is 8.

1	8
2	16
4	32
\8	64

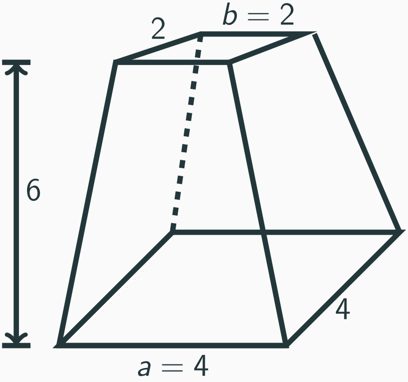
rḫt · f m ꜣḥ·t 60[2] śṯꜣ·t 4
The amount of it in area: 60 setat 4.

[1] The w suggested by the plural strokes has been omitted on the plate. The same omission occurs on the figure in Problem 51, and in Problem 52, line 2.

[2] The scribe has by mistake written here either the number 60 or the special form for 6 used in Problem 48 in writing 6 *setat*. He may have had in his mind the fact that he was actually dealing with 60 *setat* (which, however, would not properly be written in this way), and he had written the abstract number 60 a moment before at the end of the multiplication, or, remembering that 60 *setat* is written with the numeral 6, he did write 6, but used the special sign instead of the ordinary numeral.

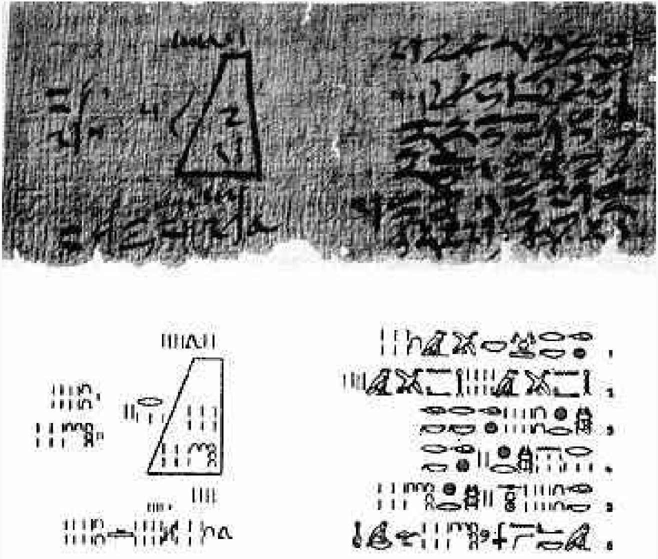
Moscow Papyrus

- From around 1850 BCE.
- Golenishchev bought in 1892 or 1893 in Thebes.
- Housed in Moscow.
- 25 Problems.
- `https://en.wikipedia.org/wiki/Moscow_Mathematical_Papyrus`
- See Problem 14:
  - Frustrum of a Pyramid



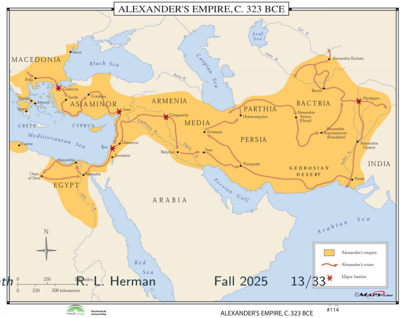
$$V = \frac{h}{3}\left(a^2 + ab + b^2\right)$$

Moscow Papyrus - Problem 14 - Frustrum of Pyramid



The Fall of the Egyptian Empire

- Argead dynasty (332–310 BCE)
    - Macedonians (700-310 BCE)
    - Alexander III of Macedon, or Alexander the Great (336–323 BCE) King of Macedonia, Pharaoh of Egypt, King of Persia and of Asia
- Ptolemaic dynasties (310–30 BCE) Cleopatra (69–30 BCE)
- Roman and Byzantine Egypt (30 BCE–641 CE)
- Sasanian (Persian) Egypt (619–629)
- Death of Mohammed (c. 570-632)
- Ruled by Caliphates (641-1517)
- Ottoman Rule (1517-1914)



Mesopotamia (2100 BCE) - Tigris and Euphrates Region

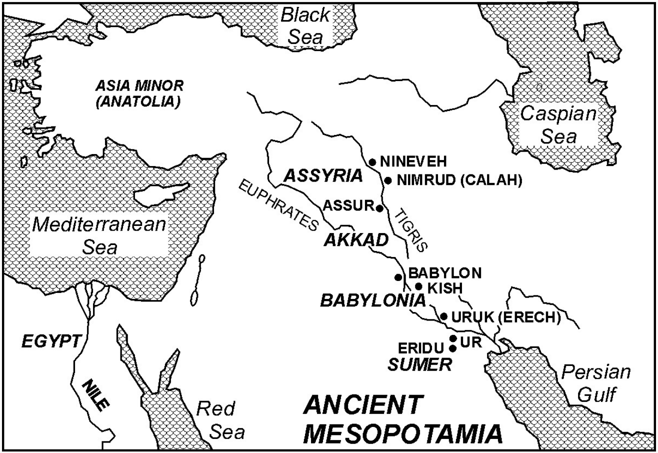


Figure 4: Tigris and Euphrates Rivers

Babylonian and Sumerian Mathematics

- More Advanced.
- Clay Tablets.
- Base 60 Arithmetic.
- Notation: $13_{60} = 1\text{\textvisiblespace}3 = 1/3$.
- Some use commas: $1,3$.
- Examples:

$$\begin{aligned}
1/3 &= 1(60) + 3 = 63 \\
1/59 &= 1(60) + 59 = 119 \\
2/49 &= 2(60) + 49 = 169 \\
3/31/49 &= 3(60^2) + 31(60) + 49
\end{aligned}$$

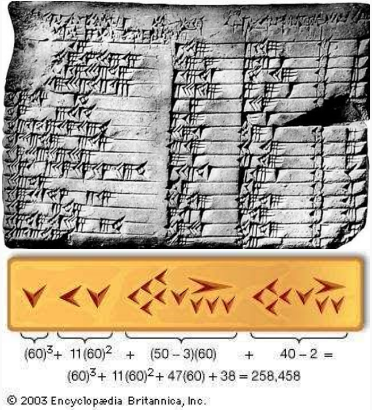


Figure 5: Babylonian tablet - Base 60

Sexagesimal Operations (Base 60)

- Ambiguities:
  - No 0's.
  - No decimal points.
- Special fractions:
  - $8.25_{10} = 8/15 = 8\frac{15}{60}$
  - $8.5_{10} = 8/30 = 8\frac{30}{60}$
  - $8.75_{10} = 8/45 = 8\frac{45}{60}$
- Addition, subtraction, multiplication.

Addition:

$$\begin{array}{r} 14/28/31 \\ +3/35/45 \\ \hline = 18/4/16. \end{array}$$

Multiplication -

$$ab = \frac{1}{4}\left[(a+b)^2 - (a-b)^2\right].$$

No division! - Use reciprocals: See Old Babylonian Multiplication and Reciprocal Tables.

Reciprocal Table

Table of reciprocals $\bar{x}$ of x, where $x\bar{x} = 60^n$, $n = 0, 1, \ldots$.

x	$\bar{x}$	x	$\bar{x}$	x	$\bar{x}$	x	$\bar{x}$
2	0/30	8	7/30	16	3/45	30	2
3	0/20	9	6/40	18	3/20	32	1/52/30
4	0/15	10	6	20	3	36	1/40
5	0/12	12	5	24	2/30	40	1/30
6	0/10	15	4	25	2/24	45	1/20

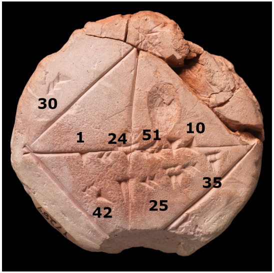
Divide 8 by 2 : $8(0/30) = 8 \times \frac{30}{60} = \frac{240}{60} = 4$, or

$0/30 + 0/30 + 0/30 + 0/30 + 0/30 + 0/30 + 0/30 + 0/30 = 1 + 1 + 1 + 1.$

Missing reciprocals: $\frac{1}{7}, \frac{1}{11}, \frac{1}{13}, \frac{1}{17}, \frac{1}{19}, \ldots$.

Sumerian Tablet - YBC 7289 - imšukkum, or "hand tablet"

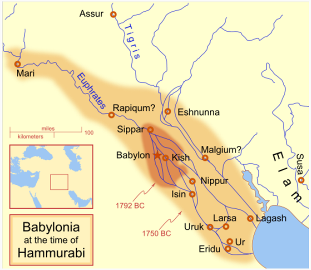
- From southern Iraq, 19th or 18th century BCE.
- Yale Peabody Museum of Natural History, 3D Print.
- Babylonians knew ratio of the side to diagonal in a square, $1 : \sqrt{2}$.



$$1/24/51/10 = 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} \approx 1.414213$$

$$42/25/35 = 42 + \frac{25}{60} + \frac{35}{60^2} \approx 42.426$$

Plimpton 322 Clay Tablet (in the news in 2017)

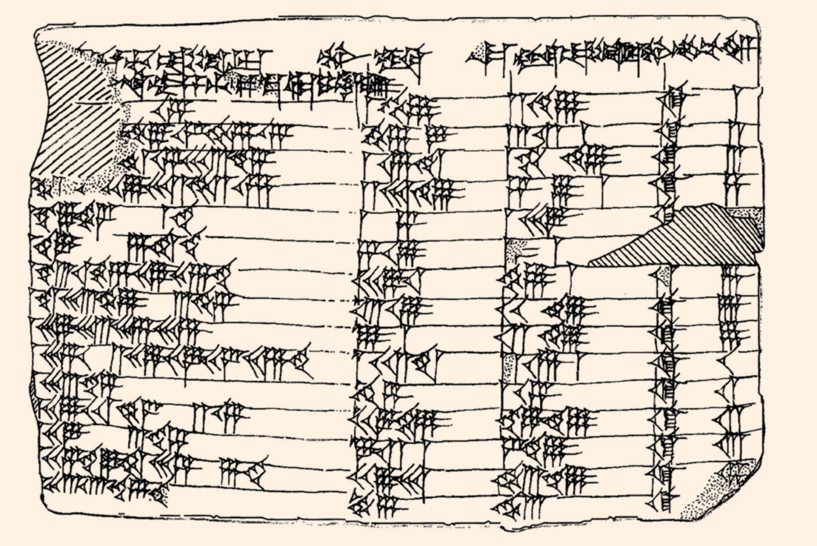


- Larsa (c. 1800 BCE) .
- Removed in 1920s.
- George Plimpton bought it, 1922.
- Left to Columbia University, 1936.
- It's about 13 by 9 by 2 cm.
  (Like a baking dish.)

- Four columns, cuneiform numbers.
- 15 rows - Pythagorean triples.
- 2nd column, side of right triangle.
- 3rd column, hypotenuse.
- 4th column, row number.
- What is the 1st Column?

Plimpton 322 Clay Tablet - Homework!

Sketch of the Plimpton 322 Tablet



Babylonian Numerals 1-100 (Base 60)

1	𒁹	26	𒎙𒐋	51	𒐐𒁹	76	𒁹 𒌋𒐋
2	𒈫	27	𒎙𒐌	52	𒐐𒈫	77	𒁹 𒌋𒐌
3	𒐈	28	𒎙𒐍	53	𒐐𒐈	78	𒁹 𒌋𒐍
4	𒐉	29	𒎙𒐎	54	𒐐𒐉	79	𒁹 𒌋𒐎
5	𒐊	30	𒌍	55	𒐐𒐊	80	𒁹 𒎙
6	𒐋	31	𒌍𒁹	56	𒐐𒐋	81	𒁹 𒎙𒁹
7	𒐌	32	𒌍𒈫	57	𒐐𒐌	82	𒁹 𒎙𒈫
8	𒐍	33	𒌍𒐈	58	𒐐𒐍	83	𒁹 𒎙𒐈
9	𒐎	34	𒌍𒐉	59	𒐐𒐎	84	𒁹 𒎙𒐉
10	𒌋	35	𒌍𒐊	60	𒁹	85	𒁹 𒎙𒐊
11	𒌋𒁹	36	𒌍𒐋	61	𒁹 𒁹	86	𒁹 𒎙𒐋
12	𒌋𒈫	37	𒌍𒐌	62	𒁹 𒈫	87	𒁹 𒎙𒐌
13	𒌋𒐈	38	𒌍𒐍	63	𒁹 𒐈	88	𒁹 𒎙𒐍
14	𒌋𒐉	39	𒌍𒐎	64	𒁹 𒐉	89	𒁹 𒎙𒐎
15	𒌋𒐊	40	𒐏	65	𒁹 𒐊	90	𒁹 𒌍
16	𒌋𒐋	41	𒐏𒁹	66	𒁹 𒐋	91	𒁹 𒌍𒁹
17	𒌋𒐌	42	𒐏𒈫	67	𒁹 𒐌	92	𒁹 𒌍𒈫
18	𒌋𒐍	43	𒐏𒐈	68	𒁹 𒐍	93	𒁹 𒌍𒐈
19	𒌋𒐎	44	𒐏𒐉	69	𒁹 𒐎	94	𒁹 𒌍𒐉
20	𒎙	45	𒐏𒐊	70	𒁹 𒌋	95	𒁹 𒌍𒐊
21	𒎙𒁹	46	𒐏𒐋	71	𒁹 𒌋𒁹	96	𒁹 𒌍𒐋
22	𒎙𒈫	47	𒐏𒐌	72	𒁹 𒌋𒈫	97	𒁹 𒌍𒐌
23	𒎙𒐈	48	𒐏𒐍	73	𒁹 𒌋𒐈	98	𒁹 𒌍𒐍
24	𒎙𒐉	49	𒐏𒐎	74	𒁹 𒌋𒐉	99	𒁹 𒌍𒐎
25	𒎙𒐊	50	𒐐	75	𒁹 𒌋𒐊	100	𒁹 𒐏

Akkadian Table of 9’s

2 Akkadian Tablet (-1700)

In the paper “Sherlock Holmes in Babylon,” *Amer. Math. Monthly* 87 (1980), 335-345, C. Buck describes Babylonian mathematics. He begins with a discussion of a clay tablet from 3700 years ago as shown in Table 2. There are four columns. You should convince yourself that this is a table of 9’s.

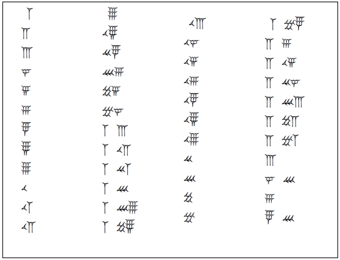


Table 2: Table of 9’s.

As an example, the last entry in the first column is 12 = . Then, $9 \times 12 = 108 =$. Note that in base 60 we have $108 = 1(60) + 48$.

In the second column is a one ( ) and 48 ( ) separated by a space. Buck introduces a slash notation to write this as 1/48.

It is easy to add in base 60. Buck gives the example $14/28/31 + 3/35/45 = 18/4/16$.

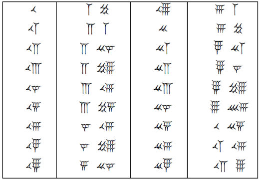
Babylonian Squares

How can a table of squares be useful? In modern notation, we see that

$$ab = \frac{1}{4}\left[(a+b)^2 - (a-b)^2\right]. \tag{1}$$

Let's find the product 11×14. Using Table 3, the formula gives

$$\begin{aligned}
11(14) &= \frac{1}{4}\left[(11+14)^2 - (11-14)^2\right] \\
&= \frac{1}{4}(25^2 - 3^2) \\
&= \frac{1}{4}(10/25 - 9) \text{ (base 60)} \\
&= \frac{1}{4}(10/16) \text{ (base 60)} \\
&= \frac{1}{4}(10(60) + 16) = \frac{616}{4} = 154.
\end{aligned} \tag{2}$$



10	1/40	19	6/1
11	2/1	20	6/40
12	2/24	21	7/21
13	2/49	22	8/4
14	3/16	23	8/49
15	3/45	24	9/36
16	4/16	25	10/25
17	4/49	26	11/16
18	5/24	27	12/9

Table 3: Table of squares with Babylonian numerals in the left table and slash notation on the right side.

4 Pythagorean Triples

Another interesting tablet from the time is the Plimpton 322 tablet shown in Figure 4. This tablet has a listing of Pythagorean triples. The last column has a list of numbers from 1 to 15. Columns two and three seem to be the hypotenuse, C, and one leg, B, of the right triangle shown in Figure 1. Recall from the Pythagorean Theorem that

$$C^2 = B^2 + D^2.$$

The triple (D, B, C) is called a Pythagorean triple.

We now know that these triples are parametrized by the pair (a, b) as follows:

$$B = a^2 - b^2, \quad C = a^2 + b^2, \quad D = 2ab,$$

since

$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2.$$

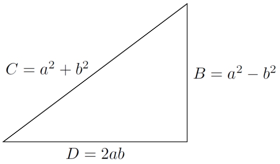


Figure 1: Right triangle with columns two and three as sides B and C, respectively. Pythagorean triples were later found to have a parametrization (a, b).

Transcription - Brackets indicate guesses

[59]/0/15	1/59	2/49	ki	1
[56/56]/58/14/50/6/15	56/7	1/20/25	ki	2
[55/7]/41/15/33/45	1/16/41	1/50/49	ki	3
53/10/29/32/52/16	3/31/49	5/9/1	ki	4
48/54/1/40	1/5	1/37	ki	[5]
47/6/41/40	5/19	8/1	[ki]	[6]
43/11/56/28/26/40	38/11	59/1	ki	7
41/33/45/14/3/45	13/19	20/49	ki	8
38/33/36/36	8/1	12/49	ki	9
35/10/2/28/27/24/26/40	1/22/41	2/16/1	ki	10
33/45	45	1/15	ki	11
29/21/54/2/15	27/59	48/49	ki	12
27/0/3/45	2/41	4/49	ki	13
25/48/51/35/6/40	29/31	53/49	ki	14
23/13/46/40	56	53	ki	[15]

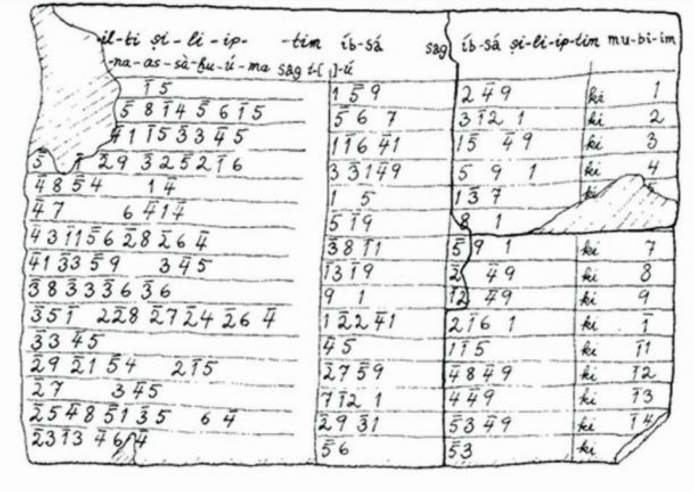


Figure 6: Arabic numerals base 60. The bars designate place holders.

Buck's Corrected Values

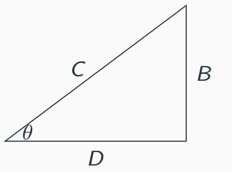
Second column - base 60 values for $(B/D)^2$ with $D^2 = C^2 - B^2$.

#	A	B	C	a	b
1	59/0/15	119	169	12	5
2	56/56/58/14/50/6/15	3367	4825	64	27
3	55/7/41/15/33/45	4601	6649	75	32
4	53/10/29/32/52/16	12709	18541	125	54
5	48/54/1/40	65	97	9	4
6	47/6/41/40	319	481	20	9
7	43/11/56/28/26/40	2291	3541	54	25
8	41/33/45/14/3/45	799	1249	32	15
9	38/33/36/36	481	769	25	12
10	35/10/2/28/27/24/26/40	4961	8161	81	40
11	33/45	45	75	1	0.5
12	29/21/54/2/15	1679	2929	48	25
13	27/0/3/45	161	289	15	8
14	25/48/51/35/6/40	1771	3229	50	27
15	23/13/46/40	56	106	9	5

First Column Computation

- Buck suggests column A is $\left(\frac{B}{D}\right)^2$.
- Others suggest $\left(\frac{C}{D}\right)^2$ and missing left part of the stone had 1's.
  Noting,

$$\left(\frac{C}{D}\right)^2 = 1 + \left(\frac{B}{D}\right)^2.$$



From row 1: 59/0/15 represents

$$\frac{59}{60} + \frac{0}{60^2} + \frac{15}{60^3} = \frac{14161}{14400}.$$

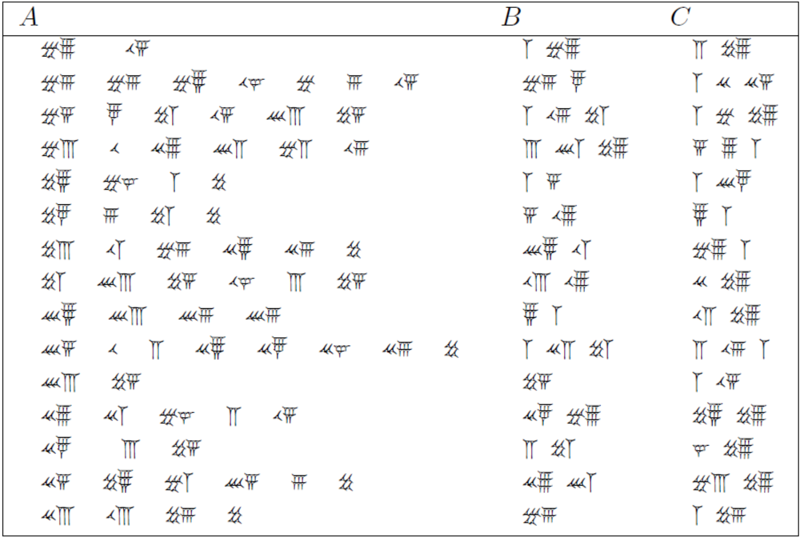
From row 1: $B = 119$, $C = 169$:

$$B^2 = 119^2 = 14161$$
$$D^2 = 169^2 - 119^2 = 14400$$
$$\left(\frac{B}{D}\right)^2 == \frac{14161}{14400}.$$

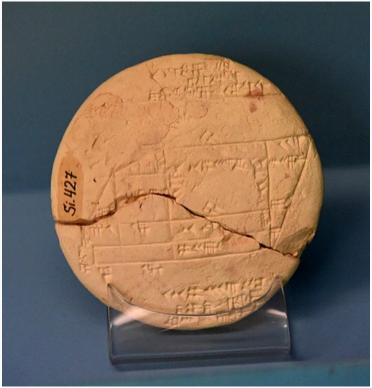
Decimal Equivalents for Column One

#	A	Decimal Value	$(B/D)^2$
1	59/0/15	0.983402777777778	0.983402777777778
2	56/56/58/14/50/6/15	0.949158552088692	0.949158552088692
3	55/7/41/15/33/45	0.918802126736111	0.918802126736111
4	53/10/29/32/52/16	0.886247906721536	0.886247906721536
5	48/54/1/40	0.815007716049383	0.815007716049383
6	47/6/41/40	0.785192901234568	0.785192901234568
7	43/11/56/28/26/40	0.719983676268862	0.719983676268862
8	41/33/45/14/3/45	0.692709418402778	0.692709418402778
9	38/33/36/36	0.642669444444444	0.642669444444444
10	35/10/2/28/27/24/26/40	0.586122566110349	0.586122566110349
11	33/45	0.562500000000000	0.562500000000000
12	29/21/54/2/15	0.489416840277778	0.489416840277778
13	27/0/3/45	0.450017361111111	0.450017361111111
14	25/48/51/35/6/40	0.430238820301783	0.430238820301783
15	23/13/46/40	0.387160493827161	0.387160493827161

Buck's Corrected Values - Babylonian Numerals



- 1900-1600 BC.
- Field plan.
- Used Pythagorean triples to make accurate right angles for measuring boundaries.
- Proposes that Plimpton 322 is the world's oldest and most accurate trigonometric table. (8/2017) Youtube
- Robson does not view it that way.

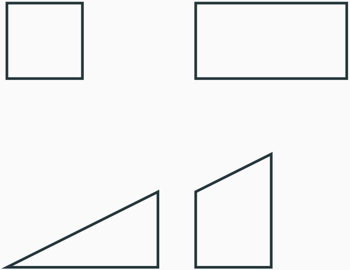


D.F. Mansfield. Plimpton 322: A Study of Rectangles. Found Sci, published online August 3, 2021; Paper.

Babylonian Geometry

- Simple shapes.
- Interested in areas.
- Fields, subdivisions.
- Inclinations, slopes.
- seked, ukuklu, run/rise.

See N. Wildberger's YouTube and reference Neugebaur and Sachs, ed., 1945, *Mathematical Cuneiform Texts*.
Neugebaur and Sachs Introduced Plimpton 322.
Wildberger and Mansfield - Babylonian trigonometry based on ratios.



Greek Mathematics I

Fall 2025 - R. L. Herman

Greek Numerals

- Decimal (Base 10).
- No zero and Positional.
- Attic Numerals (Athens),
- Ionic (Ionia): 24+3 letters

Digit	1-9	10-90	100-900
1	α	ι	ρ
2	β	κ	σ
3	γ	λ	τ
4	δ	μ	υ
5	ϵ	ν	ϕ
6	ϝ	ξ	χ
7	ζ	ο	ψ
8	η	π	ω
9	θ	ϟ, ϙ	ϡ

Arabic	Attic Greek	
1	Ι	
5	Γ Π	
10	Δ	deca
50	𐅄	
100	Η	hecto
500	𐅅	
1 000	Χ	kilo
5 000	𐅆	
10 000	Μ	

2857 = ΧΧ𐅅ΗΗΗ𐅄ΓΙΙ

761 = 𐅅ΗΗ𐅄ΔΙ

543 = $\mu\phi\gamma = \rho\rho\rho\rho\rho\kappa\kappa\alpha\beta$

$,\alpha = 1000$

$\overset{\kappa\delta}{\mathrm{M}} = 240{,}000$

Thales of Miletus (ca. 640-546 BCE)

- Ionia, Asia Minor.
- Parents were Greek or Phoenician.
- One of the Seven Sages of Greece.
- Founder of the Milesian School of natural philosophy, and the teacher of Anaximander.
- Credited with 5 theorems in geometry:
    1. A circle is bisected by any diameter.
    2. The base angles of an isosceles triangle are equal.
    3. The angles between two intersecting straight lines are equal.
    4. Two triangles are congruent if they have two angles and one side equal.
    5. An angle in a semicircle is a right angle.

According to Proclus (412-485) and others, head of Plato's Academy, commentaries on mathematicians.

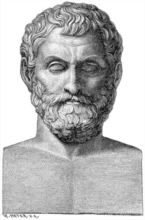


Figure 1: Thales of Miletus taught that 'all things are water.' - Aristotle

Many Other claims:
Predicted solar eclipse (585 BCE).
Measured pyramid heights.

Thales' Theorem

An angle inscribed in a semicircle is a right angle.

- 31st proposition, Book III of Euclid's Elements.
- According to Proclus and Diogenes Laërtius.
- Known earlier to Indian and Babylonian mathematicians.

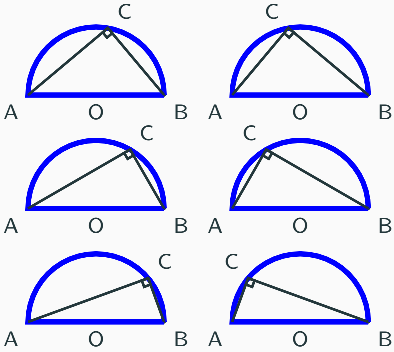


Figure 2: Thales' Theorem demonstrated.

Thales' Theorem: Inscribed Angle = 90^o

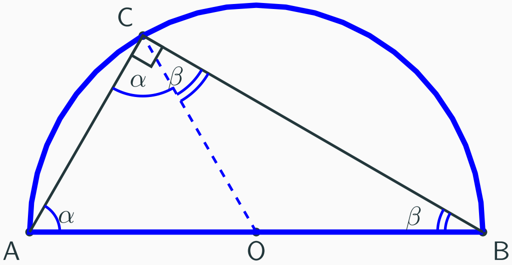


Figure 3: Proof by Picture.

Radii: $\overline{AO} = \overline{OB} = \overline{OC}$.

Isoceles triangles: AOC and OBC.

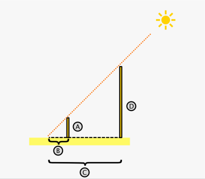
Sum of angles in ABC $= 2\alpha + 2\beta = 180^o$ implies $\alpha + \beta = 90^o$.

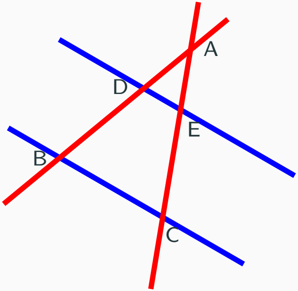
Intercept Theorem

If two (or more) parallel lines (blue) are intersected by two self-intersecting lines (red), then the ratios of the line segments of the first intersecting line is equal to the ratio of similar line segments of the second line.[1]

Prove by using similar triangles:

$$\frac{\overline{DE}}{\overline{BC}} = \frac{\overline{AE}}{\overline{AC}} = \frac{\overline{AD}}{\overline{AB}}$$





[1] "Hieronymus says that [Thales] measured the height of the pyramids by the shadow they cast, taking the observation at the hour when our shadow is of the same length as ourselves (i.e., as our own height)."

Pythagoras of Samos (570-495 BCE)

- Known from Philolaus and others.
- School in Croton, 530 BCE.
  - vegetarian, communal, secret.
  - All is number.
- Philosophy - love of wisdom.
- Mathematics - that which is learned.

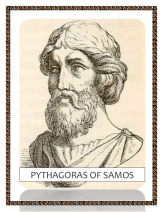


Figure 4: Pythagoras

Figure 5: Locate Samos and Croton.

Numerology - Numbers have meanings.

Even is male; Odd is female.

1. = generator
2. = opinion
3. = harmony
4. = justice
5. = marriage
6. = creation
7. = planets

10 is holiest (tetractys, tetrad, decad).

Figure 6: Tetractys

Triangular numbers:

$$1, 3, 6, 10, \ldots$$

Also the four seasons, planetary motions, music, four elements, fourth triangular number, etc.

Number Theory

- Triangular Numbers:

$$1, 3, 6, 10, \ldots$$

- Perfect Numbers [Sum factors $< n$.]:

$$\begin{aligned} 6 &= 1 + 2 + 3 \\ 10 &\neq 1 + 2 + 5 \\ 28 &= 1 + 2 + 4 + 7 + 14 \end{aligned}$$

- Amicable Numbers:

$$\begin{aligned} 220 &: \quad 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = \\ 284 &: \quad 1 + 2 + 4 + 71 + 142 = \end{aligned}$$

Pythagorean Theorem, $a^2 + b^2 = c^2$

- Known by Babylonians and Egyptians.
- Also, traces in other cultures.
- Many Proofs over the years.
- Attributed to Pythagoras.
- Pythagorean Triples (a, b, c).

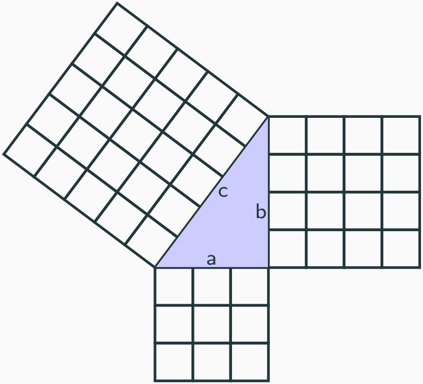


Figure 7: Euclid's Proof.

Ratios

Segments are **commensurable** if there exist a segment EF such that $\overline{AB} = p\overline{EF}$ and $\overline{CD} = q\overline{EF}$, where p and q are integers.
Therefore,

$$\frac{\overline{AB}}{\overline{CD}} = \frac{p}{q}.$$

Sometimes written as $p : q$.
Led to *Music of the Spheres*.

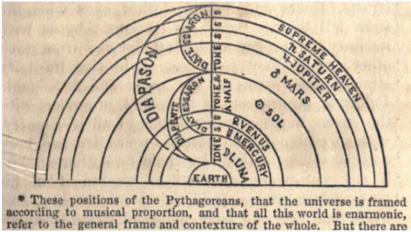


Figure 8: *A General History of the Science and Practice of Music,* Sir John Hawkins, 1853. Also provides story of Pythagoras' death.

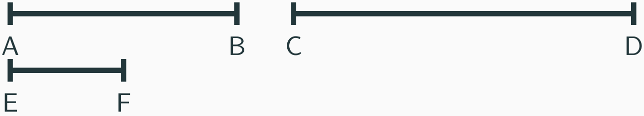


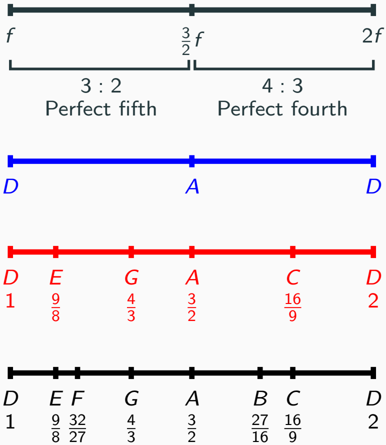
Figure 9: Commensurate Segments.

Pythagorean Scale - Series of Musical Notes

Goal - To produce a music scale.

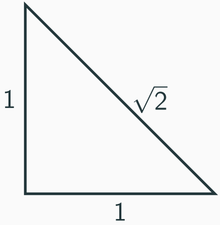
Want sounds that are pleasing when played together. Need simple ratios.

- **Octave**: From f to $2f$ (2^{nd} Harmonic).
  Ex: D goes to D, an octave higher.
- Next Notes?
  Up by **perfect fifth**. $\frac{3}{2}(1) = \frac{3}{2}$, A.
  Down by **perfect fifth**. $\frac{2}{3}(2) = \frac{4}{3}$, G.
- $\frac{3}{2}(\frac{3}{2}) = \frac{9}{4}$, wrong octave, halve.
- $\frac{2}{3}(\frac{4}{3}) = \frac{8}{9}$. wrong octave, double.
- Gives E and C.
- Pentatonic scale: D, E, G, A, C, D.
- Western Scale: D, E, F, G, A, B, C, D.
- B: $\frac{3}{2}(\frac{9}{8}) = \frac{27}{16}$, F: $\frac{2}{3}(\frac{16}{9}) = \frac{32}{27}$.



Irrational Numbers

- Hippasus of Metapontum (c. 530 - c. 450 BCE).
- Credited proving $\sqrt{2}$ is irrational.
- Drowned - possibly not an accident.
- Plato wrote Theodorus of Cyrene (c. 400 BCE) proved the irrationality of $\sqrt{3}$ to $\sqrt{17}$.
- Greeks knew sum of angles of triangle $= 2(90^o) = 180^o$.
- Construction of figures with compass and straight edge.



Classical Construction Problems

- Squaring the Circle (Quadrature) - Dinostratus (c. 390–320 BCE).
- Doubling the Cube (2×Volume) - Menaechmus (380–320 BCE).
- Trisecting a Angle (using unmarked straightedge and compass.) Hippias (460-400 BCE). Impossibility Proof: 1857, Pierre Wantzel, needs Modern Algebra.

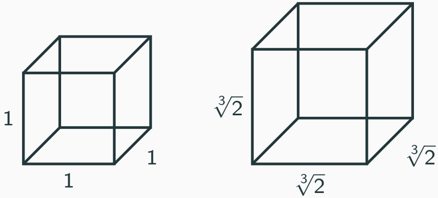


Figure 10: Doubling the Cube.

Hippocrates of Chios (c. 470 - c. 410 BCE)

- Not the Hippocrates of Kos (c. 460 - c. 370 BCE), Father of Medicine, and the Hippocratic Oath.
- Mathematician, geometer, and astronomer.
- Went to Athens.
- Used *reductio ad absurdum* arguments (proof by contradiction).
- Wrote geometry textbook, *Elements*
- Sought Quadrature of Circle.
- Quadrature of Lune.

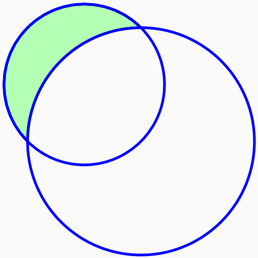


Figure 11: Lune or Crescent.

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.

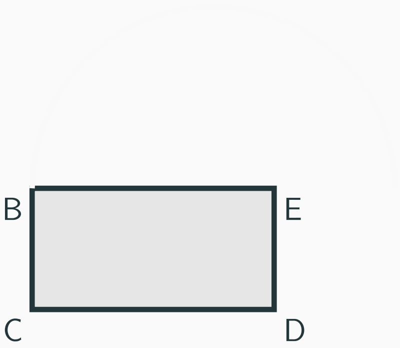


Figure 12: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.

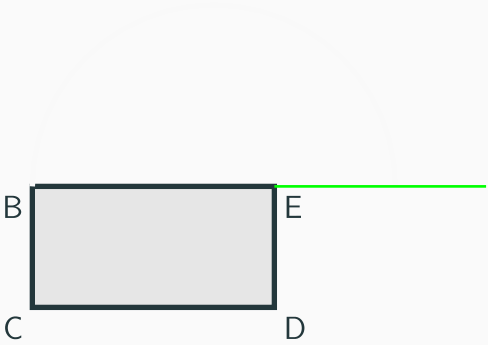


Figure 12: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.
- Get F such that $\overline{EF} = \overline{ED}$.

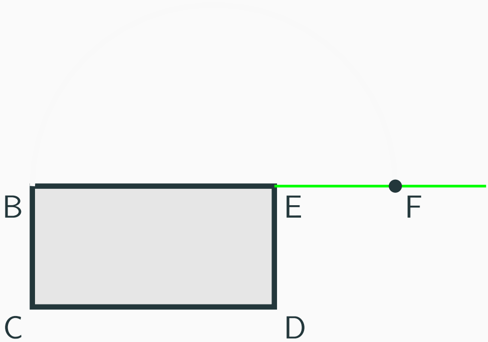


Figure 12: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.
- Get F such that $\overline{EF} = \overline{ED}$.
- How do you bisect BF?

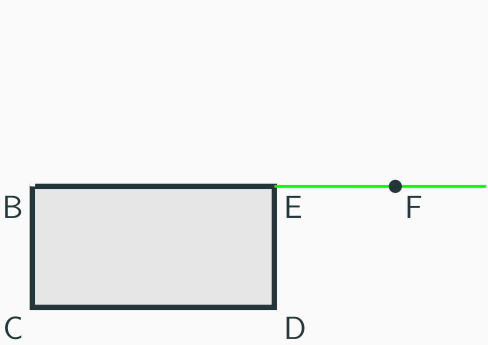


Figure 12: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.
- Get F such that $\overline{EF} = \overline{ED}$.
- How do you bisect BF?
- Bisect segment BF.

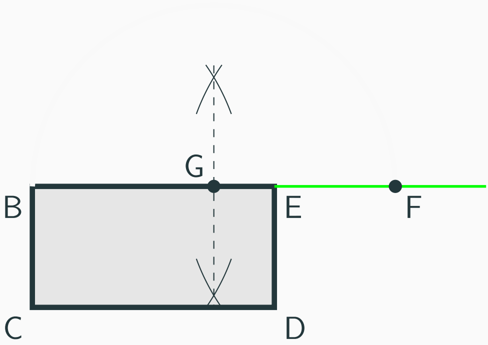


Figure 12: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.
- Get F such that $\overline{EF} = \overline{ED}$.
- How do you bisect BF?
- Bisect segment BF.
- Draw semicircle about G.

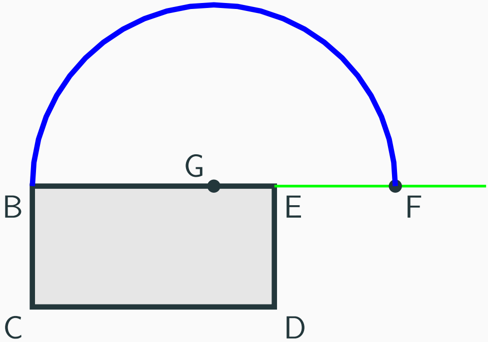


Figure 12: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.
- Get F such that $\overline{EF} = \overline{ED}$.
- How do you bisect BF?
- Bisect segment BF.
- Draw semicircle about G.
- Get point H.

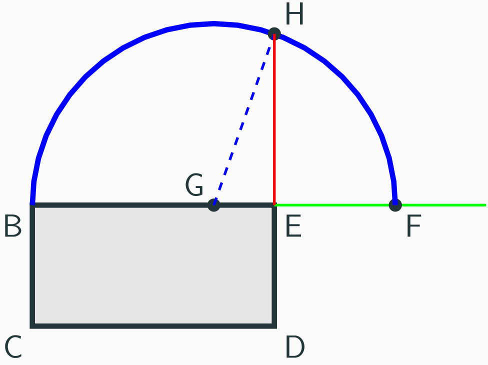


Figure 12: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.
- Get F such that $\overline{EF} = \overline{ED}$.
- How do you bisect BF?
- Bisect segment BF.
- Draw semicircle about G.
- Get point H.
- Construct square EKLH.
- Prove the areas are equal.

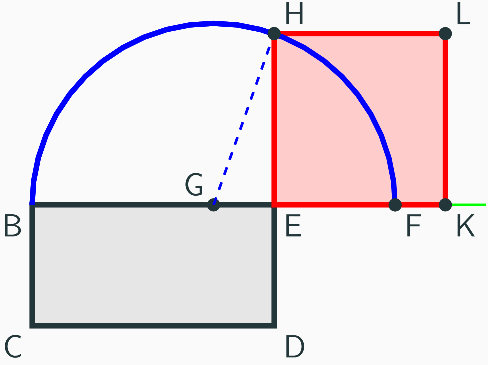


Figure 12: Quadrature of a Rectangle

Proof of Equal Areas

Label lengths a, b, c.

Area of Gray Rectangle BCDE:

$$\begin{aligned} A &= (a+b)\overline{ED} \\ &= (a+b)\overline{EF} \\ &= (a+b)(a-b) \\ &= a^2 - b^2. \end{aligned}$$

Area of Red Square EKLH:

Use Pythagorean Theorem:

$$A = c^2 = a^2 - b^2.$$

Thus, the area of the square is the same as the given rectangle; i.e., we **squared the rectangle**.

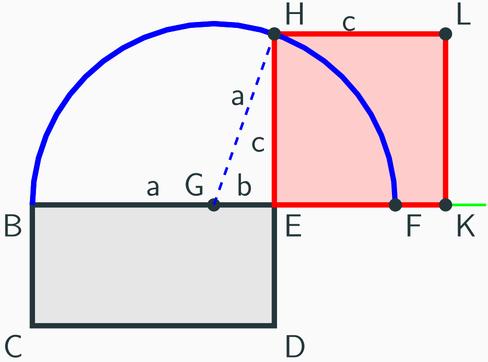
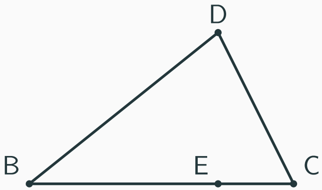


Figure 13: Quadrature of a Rectangle

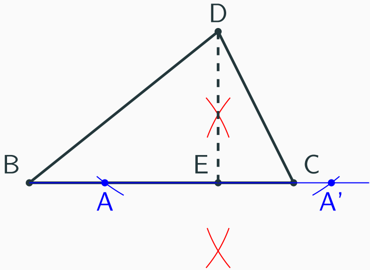
Quadrature of a Triangle

- Start with a triangle.



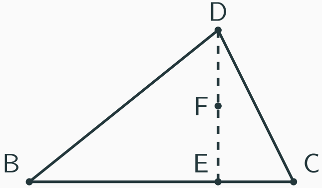
Quadrature of a Triangle

- Start with a triangle.
- Construct perpendicular DE.
    1. Draw blue arcs about D.
    2. Bisect AA' using red arcs.



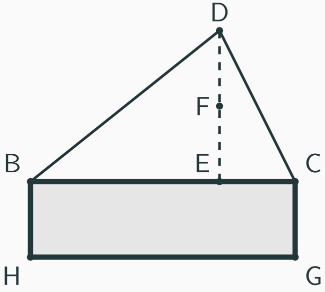
Quadrature of a Triangle

- Start with a triangle.
- Construct perpendicular DE.
  1. Draw blue arcs about D.
  2. Bisect AA' using red arcs.
- Bisect perpendicular.



Quadrature of a Triangle

- Start with a triangle.
- Construct perpendicular DE.
  1. Draw blue arcs about D.
  2. Bisect AA' using red arcs.
- Bisect perpendicular.
- Construct a rectangle with height CG = EF.

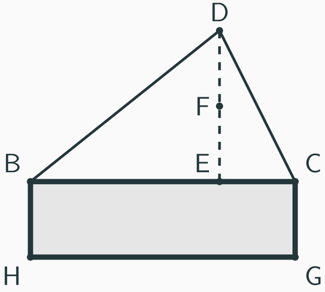


Quadrature of a Triangle

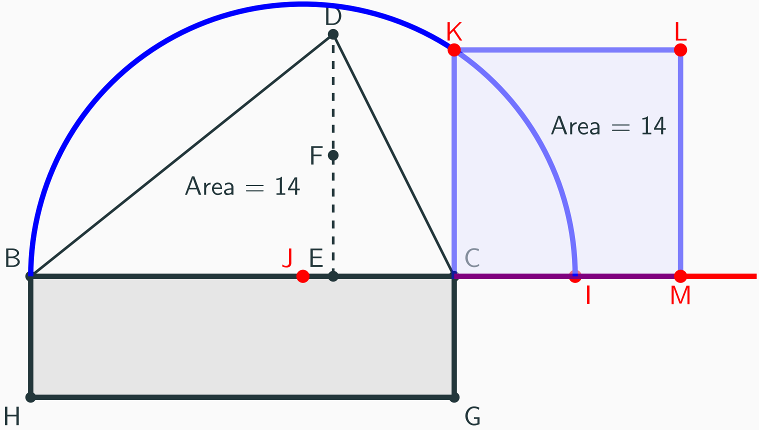
- Start with a triangle.
- Construct perpendicular DE.
  1. Draw blue arcs about D.
  2. Bisect AA' using red arcs.
- Bisect perpendicular.
- Construct a rectangle with height CG = EF.
- Area of Triangle = Area of Rectangle:

$$\begin{aligned} A(BCD) &= \frac{1}{2}\overline{BC}\,\overline{DE} \\ &= \overline{BC}\,\overline{CG} \\ &= A(BCGH). \end{aligned}$$

- Square this rectangle.



Quadrature of a Triangle - Final Construction



Quadrature of a Lune

- Lune is the figure bounded by two circular arcs.
- Hippocrates squared a special lune.
- Based on
  - Pythagorean Theorem.
  - Angle inscribed in semicircle is right.
  - Ratio of Areas of circles
  $$\frac{A_1}{A_2} = \frac{D_1^2}{D_2^2}.$$
  - Triangles are quadrable.
- Hippocrates proof not valid.

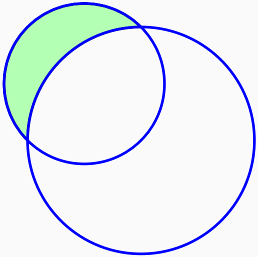


Figure 14: Lune or Crescent.

Hippocrates' Quadrature of a Lune

- $\overline{AB}^2 = \overline{AC}^2 + \overline{CB}^2 = 2\overline{AC}^2$
- Semicircle areas

$$\frac{A(AEC)}{A(ACB)} = \frac{\overline{AC}^2}{\overline{AB}^2} = \frac{1}{2}.$$

- Area of Lune = Area of ΔAOC.
- ΔAOC quadrable, so is the lune.

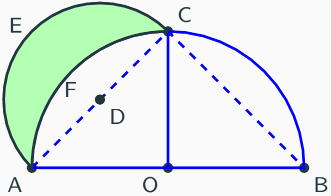


Figure 15: Lune AECF is quadrable.

Can one Square the circle?

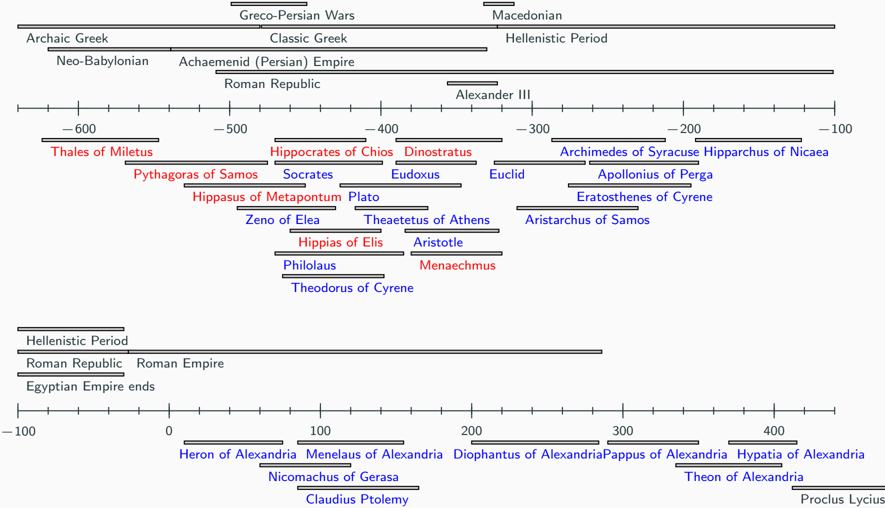
Unsolved until Ferdinand Lindemann (1852-1939).

Algebraic Numbers, solutions of polynomial equations with integer coefficients.

Ex: $x^2 - 2 = 0$ has solution $\pm\sqrt{2}$.

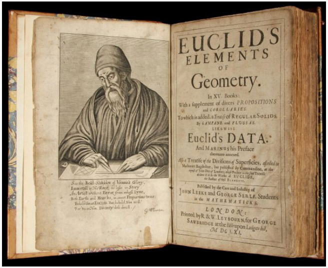
Transcendental Numbers, numbers that aren't algebraic.

Timeline of Greek Mathematicians - Where Are WE?



Euclid’s Elements

Fall 2025 - R. L. Herman



Euclid of Alexandria (c. 325 - c. 265 BCE)

- Founder of Geometry.
- Active in Alexandria, Egypt during reign of Ptolemy I (323–283 BC).
- *Elements of Geometry*
  - Most famous mathematical work of classical antiquity.
  - World's oldest continuously used mathematical textbook.
  - Geometry, proportion, and number theory.
  - 13 Books.
  - 465 Propositions.
  - 23 Definitions. (point, line, straight line, . . .)
  - 5 Postulates.
  - 5 Axioms.

Figure 1: Euclid.

The Thirteen Books

Book 1 Fundamental propositions of plane geometry.

- Congruent triangles.
- Theorems on parallel lines.
- Sum of the angles of a triangle.
- The Pythagorean theorem.

Book 2 Geometric algebra.

Book 3 Properties of circles.

- Theorems on tangents and inscribed angles.

Book 4 Inscribed and circumscribed regular polygons around circles.

Book 5 Arithmetic theory of proportion.

Book 6 Theory of proportion in plane geometry.

Book 7 Elementary number theory.

- prime numbers, greatest common denominators, etc.

Book 8 Geometric series.

Book 9 Applications and theorems on the infinitude of prime numbers, and the sum of a geometric series.

The Thirteen Books

Book 10 Incommensurable (irrational) magnitudes using the "Method of Exhaustion." [Eudoxus (390-337 BCE), Antiphon (480-411 BCE).]

Book 11 Propositions of three-dimensional geometry.

Book 12 Relative volumes of cones, pyramids, cylinders, and spheres using the Method of Exhaustion.

Book 13 The five Platonic solids.

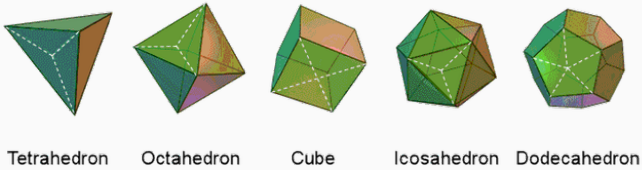


Figure 2: Platonic Solids.

Contributors to *The Elements*

The Elements - a compilation based on books by earlier Greek mathematicians. See Sir Thomas Heath

Proclus (412–485 AD), wrote in his commentary on the Elements: "Euclid, who put together the Elements, collecting many of Eudoxus' theorems, perfecting many of Theaetetus', and also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors".

Pythagoras - probably the source for most of books I and II

Hippocrates of Chios source for book III,

Eudoxus of Cnidus source for book V

Books IV, VI, XI, and XII probably from Pythagorean or Athenian mathematicians.

Definitions i

Def 1. A point is that which has no part.

Def 2. A line is breadthless length.

Def 3. The ends of a line are points.

Def 4. A straight line is a line which lies evenly with the points on itself.

Def 5. A surface is that which has length and breadth only.

Def 6. The edges of a surface are lines.

Def 7. A plane surface is a surface which lies evenly with the straight lines on itself.

Def 8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

Def 9. And when the lines containing the angle are straight, the angle is called rectilinear.

Definitions ii

Def 10. When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.

Def 11. An obtuse angle is an angle greater than a right angle.

Def 12. An acute angle is an angle less than a right angle.

Def 13. A boundary is that which is an extremity of anything.

Def 14. A figure is that which is contained by any boundary or boundaries.

Def 15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

Def 16. And the point is called the center of the circle.

Def 17. A diameter of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

Def 18. A semicircle is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.

Def 19. Rectilinear figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.

Def 20. Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.

Definitions iv

Def 21. Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.

Def 22. Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.

Def 23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Postulates

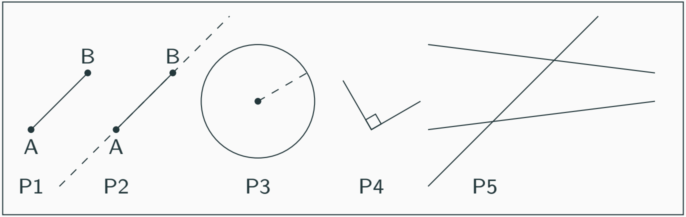
Postulate 1. To draw a straight line from any point to any point.

Postulate 2. To produce a finite straight line continuously in a straight line.

Postulate 3. To describe a circle with any center and radius.

Postulate 4. That all right angles equal one another.

Postulate 5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.



Common Notions

Notion 1. Things which equal the same thing also equal one another.

Notion 2. If equals are added to equals, then the wholes are equal.

Notion 3. If equals are subtracted from equals, then the remainders are equal.

Notion 4. Things which coincide with one another equal one another.

Notion 5. The whole is greater than the part.

Proposition 1

To construct an equilateral triangle on a given finite straight line.

- Start with segment AB.

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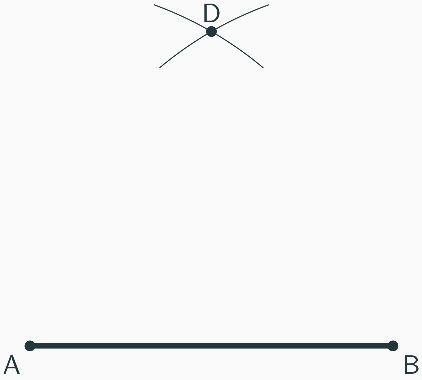
To construct an equilateral triangle on a given finite straight line.

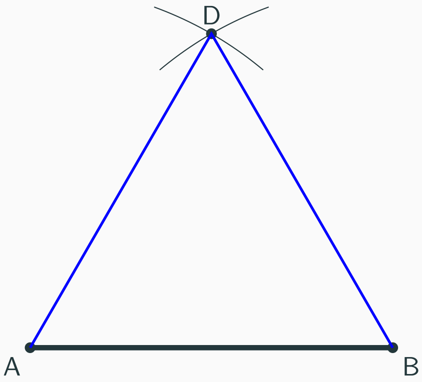
- Start with segment AB.
- Draw circular arcs about A, B of radius AB.



Proposition 1

To construct an equilateral triangle on a given finite straight line.

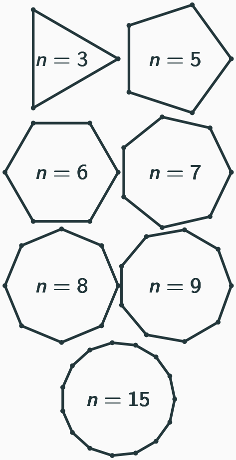
- Start with segment AB.
- Draw circular arcs about A, B of radius AB.
- Draw line segments AD, BD.



Regular Polygons

Construct using straight edge and compass.

- Triangles (Euclid I.1)
- Squares (Euclid I.46)
- Pentagons (Euclid IV.11)
- Hexagons (Euclid IV.15)
- Septagon (heptagon) (no)
- Octagon (Euclid III.30)
- Nonagon (no)
- 15-gon (Euclid IV.16) pentadecagon
- Double the number of sides of a given regular polygon, 8, 10, 12, 16, 20, 24, etc. (Euclid III.30)



Constructible regular n-gons

Is it possible to construct all regular polygons with compass and straightedge? If not, which n-gons (that is polygons with n edges) are constructible and which are not?

- Young C. F. Gauss, 1796: the regular 17-gon (Heptadecagon) is constructible.
- Theory of Gaussian periods in his *Disquisitiones Arithmeticae*. 1801.
- Gave sufficient condition for the constructibility.
- Proof of necessity - Pierre Wantzel in 1837.
- Gauss–Wantzel theorem:
  A regular n-gon can be constructed with compass and straightedge if and only if n is the product of a power of 2 and any number of distinct Fermat primes, p_ℓ:[only 3, 5, 17, 257, 65537.]

$$n = 2^m p_1 p_2 \cdots p_k, \qquad p_\ell = 2^{2^\ell} + 1.$$

Book 13 - Platonic Solids - Regular Polyhedra

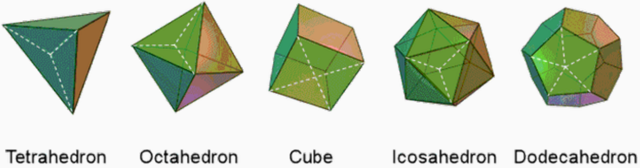


Figure 3: Platonic Solids: Fire, Air, Earth, Water, Universe.

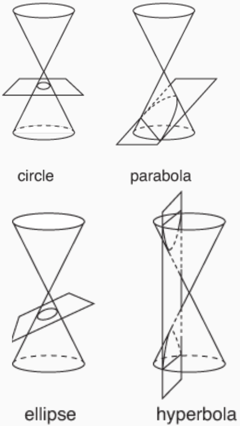
Polyhedron	Faces	Edges	Vertices
Tetrahedron	$4\ \Delta's$	6	4
Cube	6 □'s	12	8
Octahedron	$8\ \Delta's$	4	6
Dodecahedron	12 Pentagons	30	20
Icosahedron	$20\ \Delta's$	30	12

Note: Johannes Kepler (1571-1630) systematized and extended what was known about polyhedra. See *Harmonice Mundi*, 1619. Proposed relationships between six known planets and the Platonic solids.

Conic Sections

Possibly discovered by **Menaechmus**[1] (380–320 BCE) to duplicate cube: Intersect parabola $y = \frac{1}{2}x^2$ and hyperbola $xy = 1$.

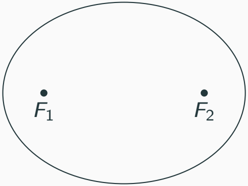
- **Euclid** - four lost books on conics.
- **Archimedes** of Syracuse (287-212 BCE) studied conics, area bounded by a parabola and a chord in *Quadrature of the Parabola*.
- **Apollonius** of Perga (262-190 BCE), eight-volume *Conics*.
  Terms: parabola, ellipse, hyperbola
- **Pappus** of Alexandria (290 – 350) - focus directrix.
- Applied by Kepler (1609), Newton (1687).



[1] To Alexander, *"O king, for travelling through the country there are private roads and royal roads, but in geometry there is one road for all."*

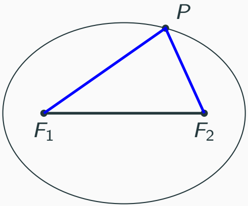
Ellipse

- Focal points: F_1, F_2.



Ellipse

- Focal points: F_1, F_2.
- $\overline{F_1P} + \overline{F_2P} = 2a$.



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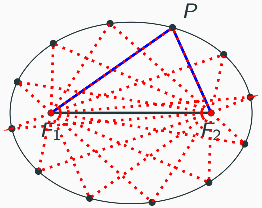
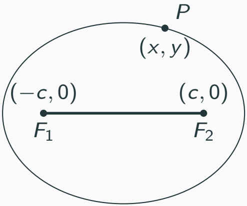


Ellipse

- Focal points: F_1, F_2.
- $\overline{F_1P} + \overline{F_2P} = 2a$.
- Algebra leads to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

a, b = semimajor/semiminor axes with $c = \sqrt{a^2 - b^2}$, $a > b$.



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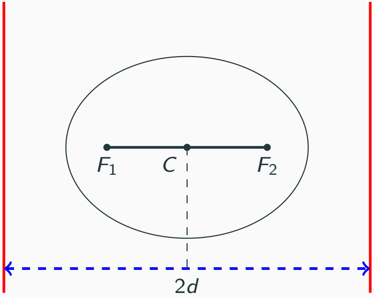
  a, $b =$ semimajor/seminor axes with $c = \sqrt{a^2 - b^2}$, $a > b$.
- Directrix $d = \frac{a^2}{c}$,



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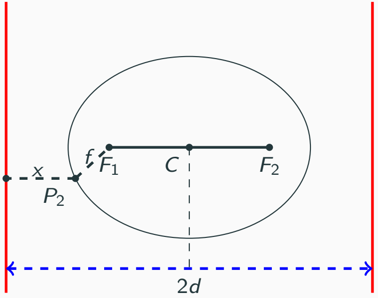
  a, $b =$ semimajor/semiminor axes with $c = \sqrt{a^2 - b^2}$, $a > b$.
- Directrix $d = \frac{a^2}{c}$,
- Eccentricity $\epsilon = \frac{f}{x} = \frac{c}{a}$.



Ellipse

- Focal points: F_1, F_2.
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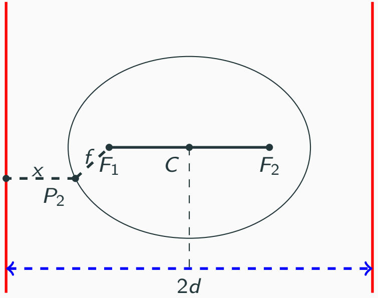
  a, $b =$ semimajor/semiminor axes with $c = \sqrt{a^2 - b^2}$, $a > b$.
- Directrix $d = \frac{a^2}{c}$,
- Eccentricity $\epsilon = \frac{f}{x} = \frac{c}{a}$.
- Eccentricty of Conics:

  $\epsilon = 0$, circle.
  $0 < \epsilon < 1$, ellipse.
  $\epsilon = 1$, parabola.
  $\epsilon > 1$, hyperbola.



Dandelin Sphere - Germinal Pierre Dandelin (1794-1847)

- Inscribed spheres tangent to cone and intersecting plane.
- Intersection is a conic.
- Tangent pts to sphere are focal points.
- Used to prove theorems of Apollonius.
    - Conic section is the set points such that the sum of the distances to two fixed points is constant.
    - The distance from the focus is proportional to the distance from a fixed line (directrix).
    - The constant of proportionality is the eccentricity

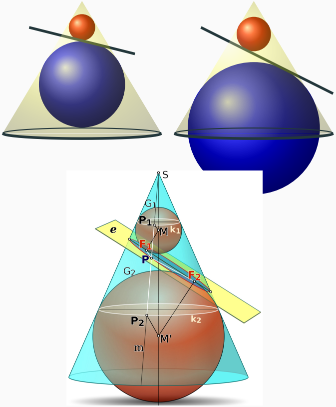
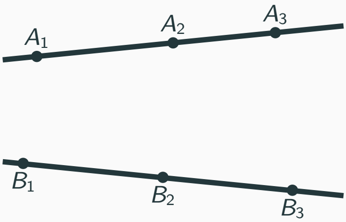


Figure 4: Dandelin Spheres (Paper in 1822)

Other Geometry Gems

- Pappas' Hexagon Theorem (290-350)
  - Connect 6 pts on two lines.
  - A_1-B_2, B_2-A_1, etc.



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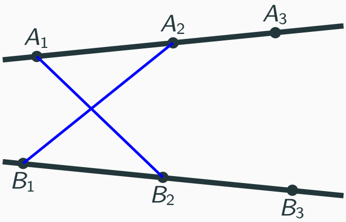

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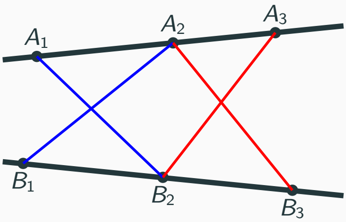

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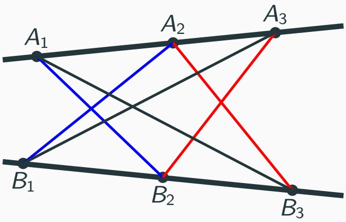

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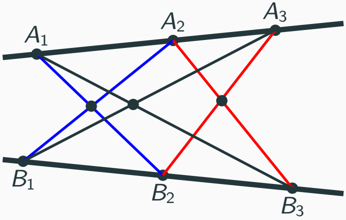

Other Geometry Gems

- Pappas' Hexagon Theorem (290-350)
  - Connect 6 pts on two lines.
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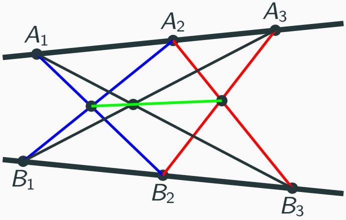
- The three points are collinear.
- The beginning of projective geometry.



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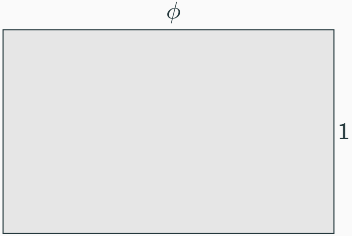
- Golden Ratio (section, mean), ϕ, τ.
  - Euclid, "extreme and mean ratio"



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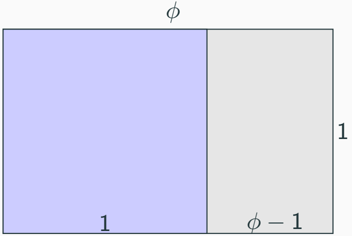
  - Golden Rectangle: $\frac{\phi}{1} = \frac{1}{\phi - 1}$. [Gray region is similar to the large rectangle.]



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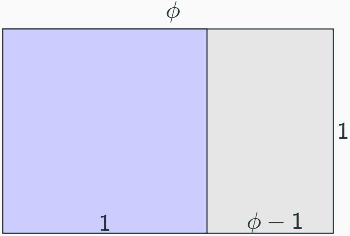
  - Euclid, "extreme and mean ratio"
  - Golden Rectangle: $\frac{\phi}{1} = \frac{1}{\phi - 1}$. [Gray region is similar to the large rectangle.]
  - Solution $\phi^2 = \phi + 1$:

$$\phi = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1+\sqrt{5}}{2} \approx 1.61803\ldots.$$



Fibonacci and Lucas Numbers

Fibonacci Numbers:

$$\begin{aligned} F_n &= 1, 1, 2, 3, 5, 8, \ldots \\ \sqrt{5}F_n &= \phi^n - (-\phi)^{-n} \qquad (1) \end{aligned}$$

Lucas Numbers:

$$\begin{aligned} L_n &= 2, 1, 3, 4, 7, \ldots \\ &= \phi^n + (-\phi)^{-n} \qquad (2) \end{aligned}$$

Ratio limits

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \phi$$

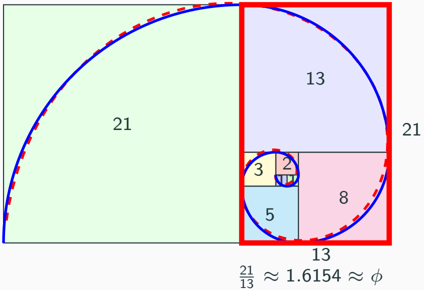


Figure 5: Golden Spiral, or Fibonacci Spiral, is approximately a logarithmic spiral, $r = a\phi^{2\theta/\pi}$.

The Parthenon

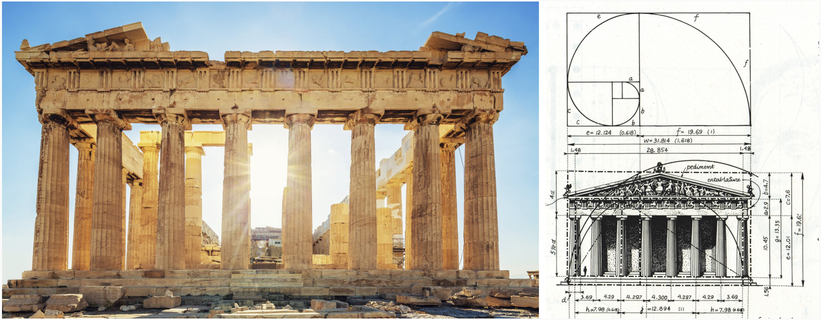


Figure 6: The Parthenon and the Golden Rectangle

Other Examples in Nature

Greek Mathematics II

Fall 2025 - R. L. Herman

Greek Number Theory

- **Pythagorean Theorem**
  $x^2 + y^2 = z^2$, x, y, z integers.
- **Diophantine Equations**
  Solve $3x + 5y = 1$, x, y integers.
- Euclid
  - Proved **# primes infinite**, Book IX, Prop 20.
  - **Perfect Numbers**, Book VII, Def 22, Book IX, Prop 36.
    Euclid proves:
    If $2^n - 1$ is prime, then $(2^n - 1)\,2^{n-1}$ is perfect.
    Mersenne prime: $2^n - 1$.
  - **Polygonal Numbers**

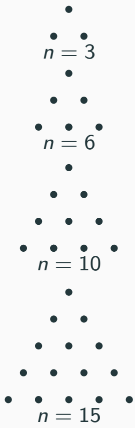


Figure 1: Polygonal Numbers.

Euclidean Algorithm - Book VII, Prop 1

- $\gcd(a,b)$: **Greatest common divisor** of a and b.
- Algorithm

$$\begin{aligned} a_1 &= \max(a,b) - \min(a,b) \\ b_1 &= \min(a,b) \\ &\quad \text{repeat} \end{aligned}$$

Terminates when $a_{i+1} = b_{i+1}$.

Example: Find $\gcd(210, 45)$.

$$\begin{aligned} a_1 &= 210 - 45 = 165 \\ b_1 &= 45 \end{aligned}$$

Continuing the computation:

a_i	b_i
210	45
165	45
120	45
75	45
30	45
15	30
15	15

We find $\gcd(210, 45) = 15$.

Euclidean Algorithm - Another Approach

Find the greatest common divisor of positive integers, a and b.

- If $a < b$, exchange a and b.
- Divide a by b and get the remainder, r. Thus,

$$a = qb + r.$$

  - If $r \neq 0$, replace a by b and b by r. Repeat the division.
  - If $r = 0$, report $gcd(a, b) = b$.

Example: Find $\gcd(210, 45)$.

$$\begin{aligned} 210 &= 4 \cdot 45 + 30 \\ 45 &= 1 \cdot 30 + 15 \\ 30 &= 2 \cdot 15 \end{aligned}$$

Thus, $\gcd(210, 45) = 15$.

Pell's Equation (1611-1685)

- $x^2 - Ny^2 = 1$,
  N is a nonsquare integer, and
  x, y are integer solutions.
- Example of a Diophantine equation.
- Related to $\sqrt{2}$: $x^2 - 2y^2 = 0$,
  $y = 1 \Rightarrow x = \sqrt{2}$.
- $x^2 - 2y^2 = 1$,
  If x, y large, then $\frac{x}{y} \approx \sqrt{2}$.
- Known to Pythagoreans,
  Diophantus, and
- Archimedes' Cattle Problem can be reduced to solving Pell's Equation.
- Brahmagupta (598-570) first to solve.

From *The New York Times*
January 18, 1931, p. 54

CATTLE PROBLEM SOLVED

Moreover, Final Conditions Set by Archimedes Can Be Worked Out

To the Editor of The New York Times:

Frank G. Nelson, whose interesting letter regarding his solution of the cattle problem of Archimedes appeared in THE TIMES, would feel flattered if he had the translation of this problem which is possessed by me, for, according to Archimedes, he is no mere "novice in numbers," since no such person could be expected to arrive at a correct solution—as has Mr. Nelson—of the first seven equations presented by the problem.

But Mr. Nelson's conclusion that the final conditions set by the problem cannot be solved is erroneous—at least according to a large number of mathematicians who have worked on it. As far back as 1860, Amthor showed that the total of the cattle would be represented by a number containing 206,545 figures, the printing of which would require about two full pages of THE NEW YORK TIMES. Since it has been calculated that it would take the work of a thousand men for a thousand years to determine the complete number, it is obvious that the world will never have a complete solution, which should relieve the mind of any linotype operator who fears that he might be called on to set it. However, the first thirty-one figures have been computed, as have the last twelve, and the solution, for those who are interested, is

7,760,271 081,800

in which the line of dots represents thirty solved and 206,502 unsolved numbers.

The above solution was worked out by the Hillsboro Mathematical Club of Hillsboro, Ill., which was formed by A. H. Bell in 1889 to labor on the problem. Nearly four years were spent by the three club members on the work, and the results were published in the American Mathematical Monthly in 1895. An interesting summary of the mathematical steps involved in the determination of these enormous numbers—there are ten altogether, each containing 206,544 or 206,545 figures—is to be found in Recreations in Mathematics, by H. E. Licks (Van Nostrand, 1917).

Archimedes was evidently fond of problems involving enormous numbers, as his book "Arenarius" discusses the solution of the problem of determining the number of grains of sand in a sphere the size of the earth. This number is, however, of insignificant size in comparison with that representing the solution of the cattle problem. In fact, it has been calculated that if the cattle represented by this number were reduced to the size of the smallest bacterium, they could not be contained in a sphere having the diameter of the Milky Way, across which astronomers calculate that it takes light, traveling at about 186,000 miles a second, 10,000 years to travel.

NORMAN MERRIMAN.
New York, Jan. 12, 1931.

Pell's Equation General Solution

- $x^2 - ny^2 = 1$,
  n is a nonsquare integer and
  x, y are integer solutions.
- Let $z = x + y\sqrt{n}$, $x, y \in \mathbb{Z}$
  and $\bar{z} = x - y\sqrt{n}$.
- $\text{Norm}(z) = z\bar{z} = x^2 - ny^2 = 1$.
- $\text{Norm}(zw) = \text{Norm}(z)\text{Norm}(w)$.

Pell's Equation General Solution

- $x^2 - ny^2 = 1$,
  n is a nonsquare integer and
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- $\text{Norm}(zw) = \text{Norm}(z)\text{Norm}(w)$.

Let

$$\begin{aligned} z &= x_1 + y_1\sqrt{n}, \\ w &= x_2 + y_2\sqrt{n}, \\ zw &= x_3 + y_3\sqrt{n}, \end{aligned}$$

Then

$$\begin{aligned} x_3 &= x_1 x_2 + n y_1 y_2, \\ y_3 &= x_1 y_2 + x_2 y_1. \end{aligned}$$

Since $\text{Norm}(zw) = 1$,
(x_3, y_3) is a solution.

Pell's Equation General Solution

- $x^2 - ny^2 = 1$,
  n is a nonsquare integer and
  x, y are integer solutions.
- Let $z = x + y\sqrt{n}$, $x, y \in \mathbb{Z}$
  and $\bar{z} = x - y\sqrt{n}$.
- $\text{Norm}(z) = z\bar{z} = x^2 - ny^2 = 1$.
- $\text{Norm}(zw) = \text{Norm}(z)\text{Norm}(w)$.
- Example: $x^2 - 3y^2 = 1$
- Guess $(2, 1)$.
  So, $z = 2 + \sqrt{3} = w$.

$$\begin{aligned} zw &= \left(2 + \sqrt{3}\right)^2 \\ &= 7 + 4\sqrt{3}. \end{aligned}$$

Let

$$\begin{aligned} z &= x_1 + y_1\sqrt{n}, \\ w &= x_2 + y_2\sqrt{n}, \\ zw &= x_3 + y_3\sqrt{n}, \end{aligned}$$

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- $\text{Norm}(zw) = \text{Norm}(z)\text{Norm}(w)$.
- Example: $x^2 - 3y^2 = 1$
- Guess $(2, 1)$.
  So, $z = 2 + \sqrt{3} = w$.

$$\begin{aligned} zw &= \left(2+\sqrt{3}\right)^2 \\ &= 7 + 4\sqrt{3}. \end{aligned}$$

Let

$$\begin{aligned} z &= x_1 + y_1\sqrt{n}, \\ w &= x_2 + y_2\sqrt{n}, \\ zw &= x_3 + y_3\sqrt{n}, \end{aligned}$$

Then

$$\begin{aligned} x_3 &= x_1x_2 + ny_1y_2, \\ y_3 &= x_1y_2 + x_2y_1. \end{aligned}$$

Since $\text{Norm}(zw) = 1$,
(x_3, y_3) is a solution.

Then, $(7, 4)$ is a solution.

Eudoxus of Cnidus (c.390 – c. 337 BCE)

- Studied under Plato.
- Taught Aristotle.
- Astronomer, Mathematician.
- Theory of Proportions:

  Circles: $A \propto r^2$,
  Spheres: $V \propto r^3$,
  Volume of a pyramid .
  Volume of a cone.
- Studied reals, continuous quantities.
- Method of Exhaustion:
  Due to Antiphon (480–411 BCE).
  Area from a sequence of inscribed polygons.

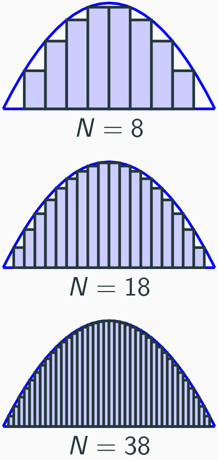
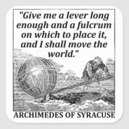
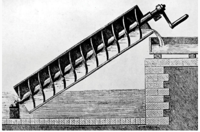


Figure 2: Method of Exhaustion.

Archimedes of Syracuse (287-212 BCE)

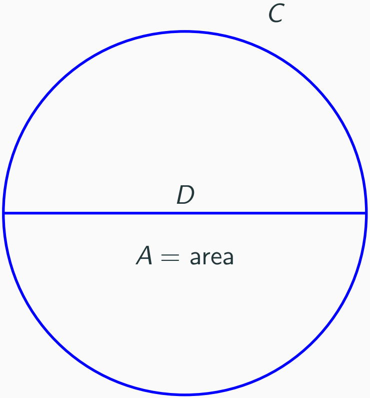
- Went to Alexandria, Egypt
  then, back to Syracuse, Sicily.
- Greatest Mathematician of Antiquity.
- Mathematician, Engineer, Inventor.
  - Archimedean screw, lever, pulley.
- King Heiro II's crown - Eureka.
  Archimedes Principle of Bouyancy.
- According to Plutarch (46-120):
  - Marcellus - Syracuse 212 BCE.
  - Claw of Archimedes.
  - Heat Ray.
  - Prone to intense concentration.
  - Death of Archimedes.





Archimedes' Mathematics

- Mastered Euclid and Eudoxus' (c. 390-337 BCE) Method of Exhaustion.
- *Measurement of a Circle*
  $\frac{C}{D} = \text{const.}, \quad \frac{A}{D^2} = \text{const.}$



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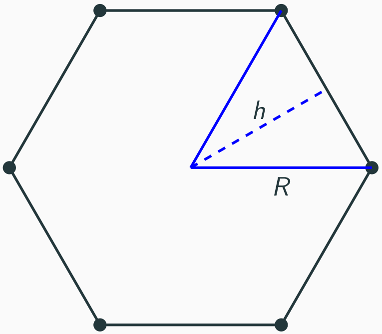
- Regular Polygons (n-gon)
  $A_p = \frac{1}{2}hQ$, $Q =$ Perimeter.



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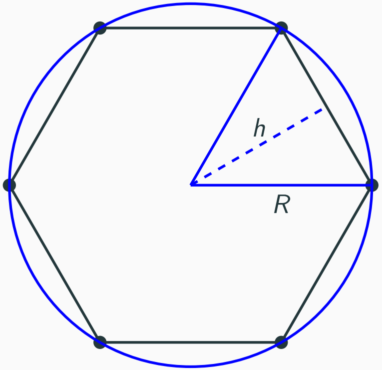
- Inscribed Polygons
  $A_p = \frac{1}{2}anh <$ area of circle.



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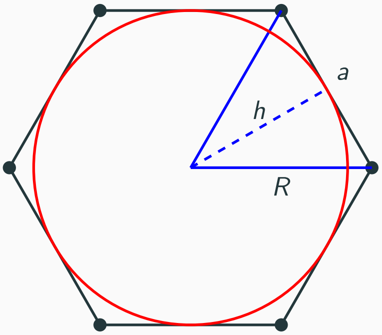
- Circumscribed Polygon
  $a = 2R\sin\frac{180}{n}, h = \sqrt{R^2 - \frac{a^2}{4}}$.



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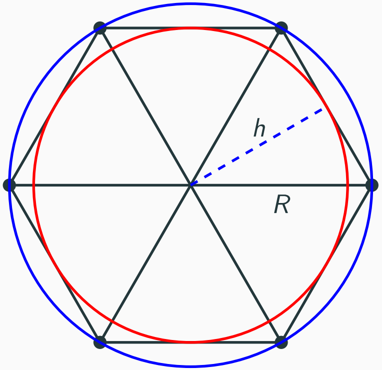
  $a = 2R\sin\frac{180}{n}, h = \sqrt{R^2 - \frac{a^2}{4}}$.
- Approximation of π,
  $\frac{A_p}{R^2} < \pi < \frac{A_p}{h^2}$.

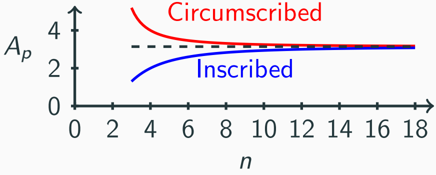


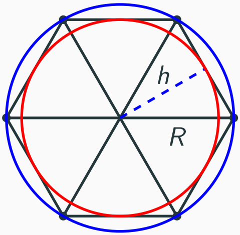
Estimating π

- Approximation of π,
  $\frac{A_p}{R^2} < \pi < \frac{A_p}{h^2}$,
- Recall
  $a = 2R \sin \frac{180}{n}$,
  $h = \sqrt{R^2 - \frac{a^2}{4}} = R \cos \frac{180}{n}$,
  $A_p = \frac{1}{2} anh = nhR \sin \frac{180}{n}$.
- Therefore,

$$\frac{n}{2} \sin \frac{360}{n} < \pi < n \tan \frac{180}{n},$$

- Hexagon ($n = 6$),
  $2.598 < \pi < 3.464$.
- Archimedes - up to 96-gon
  $3.1394 < \pi < 3.1427$.





Archimedes' Inscribed and Circumscribed n-gons

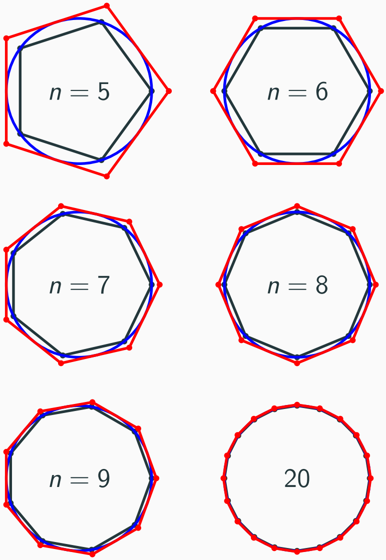
Consider a fixed circle of radius R.

- Inscribed n-gon: $h = R\cos\frac{180}{n}$, $A_i = nhR\sin\frac{180}{n}$.
- Circumscribed n-gon: $r = \frac{R}{\cos\frac{180}{n}}$, $A_c = nHr\sin\frac{180}{n}$.
- Thus,

$$\begin{aligned} A_i &= nR^2\tan\frac{180}{n}, \\ A_c &= \frac{n}{2}R^2\sin\frac{360}{n}. \end{aligned}$$

- This gives

$$\frac{n}{2}\sin\frac{360}{n} < \pi < n\tan\frac{180}{n},$$

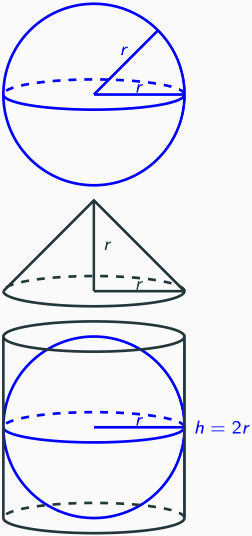


Early Approximations of π: Peripherion $\pi\epsilon\rho\iota\phi\epsilon\rho\epsilon\iota\alpha$

- Bible, $\pi \approx 3$.
- Babylonian $3 + \frac{1}{8}$.
- Egyptians, $\left(\frac{4}{3}\right)^4 = \frac{256}{81} \approx 3.1604938$.
- Sulbasutrakaras ($<$ 800 BCE), 3.08.
- Archimedes (250 BCE) $3\frac{10}{71} < \pi < 3\frac{1}{7}$.
- Aryabhata (499), $\frac{62832}{20000}$.
- Ptolemy (150), 360-gon, 3.14166.
- Chinese (430-501) $\frac{355}{113} \approx 3.14159292$.
- Hindu (1100) $\frac{3927}{1250} \approx 3.1416$.
- Viete', 393,216-gon, π to 9 places.
- van Ceulen (1540-1610) Dutch, 35 places.
- William Shanks (1873) 527 digits.

- Lambert (1728-1777) - irrationality proof.
- William Jones (1706) introduced π.
- Euler popularized notation.
- See Approximations of π .
- Leibniz-Madhava
  $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$
- Euler
  $$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$
- Ramanujan
  $$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!}{k!^4} \frac{1103 + 26390k}{396^{4k}}$$

- Spirals, Area of Parabolae.
- Volumes and Surface Areas of 3D Objects.
- A(Sphere) = 4 A(Great circle) $A = 4(\pi r^2)$.
- V(Sphere) = 4 V(Cone) $V = 4\left(\frac{1}{3}\pi r^3\right)$.
- Sphere inside Cylinder
  Cylinder Area =
  $2\pi r(2r) + 2(\pi r^2) = 6\pi r^2$
  $= \frac{3}{2}$ Sphere Area.
  Volume $= (\pi r^2)h = 2\pi r^3$
  $= \frac{3}{2}$ Sphere Volume.



Archimedes' Manuscripts

- What we know is from 3 books.
- Codex A Lost in 1564.
- Codex B Lost in 1311.
- Codex C Discovered 1906.
  - 4th century Parchment bound.
  - 10th Century Book, Constantinople housed great texts.
  - 1204 4th Crusade destroyed books.
  - 87 Sheets (43.5 goat skins).
  - 1229 Century taken apart, scraped, cut in half, written over with Christian prayer.
  - Moved to Palestine, 400 yrs.

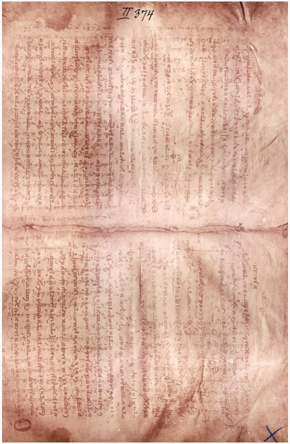


Figure 3: Codex C Page.

The Walters Museum - `http://www.archimedespalimpsest.net`

Palimpsest, http://www.archimedespalimpsest.org/about/

- 1846, It was in Istanbul, leaf removed to Cambridge.
- 1906, Johan Heiberg took pictures and translated.
- 1922, It went missing.
- 1998, Sold for $2,000,000 - Christies of NY auction.
  Moldy, Charred,
- 7 Manuscripts

  The Equilibrium of Planes, Spiral Lines, The Measurement of the Circle, Sphere and Cylinder, On Floating Bodies, The Method of Mechanical Theorems, and the Stomachion.

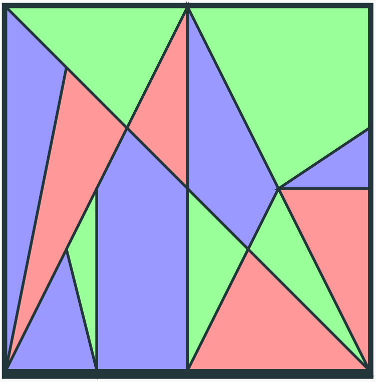


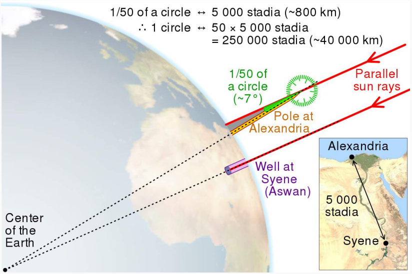
Figure 4: Number of different arrangements of the Stomachion, 17,152.

Eratosthenes of Cyrene (276 - 194 BCE)

- Chief Librarian at Library of Alexandria. (300 BCE-?). Burned 641 by Arabs?
- Sieve of Eratosthenes
- Finding primes:

$$\not{1}, 2, 3, \not{4}, 5, \not{6}, 7, \not{8}, \not{9}, \not{10}, 11, \ldots$$

- Circumference of Earth:
    - Syene - 1st day summer Sun directly over deep well.
    - Alexandria - small shadow.
    - 250,000 stadia.
      1 stade ≈ 526.37 ft
      Equals $24,466$ mi.
      Current - $24,860$ mi.



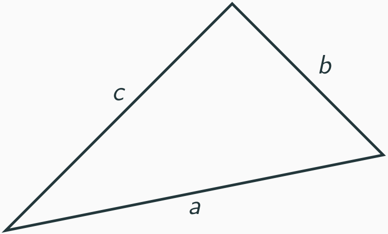
Heron of Alexandria (c. 10-70)

- Inventor.
- Aeolipile,rocket-like reaction engine.
- First-recorded steam engine.
- Hero's wind-powered organ.
- The first vending machine.
- A wind-wheel operating an organ,
- The force pump.
- A syringe-like device.
- Principle: shortest path of light:
- Standalone fountain.
- A programmable cart.

Heron's Formula

$$A = \sqrt{(s-a)(s-b)(s-c)}$$

$$S = \frac{1}{2}(a+b+c)$$



Last of the Ancient Greek Mathematicians

- Ptolemy (100-170)
  - Astronomy.
  - Geocentric model - until ...
  - Copernicus (1500). Heliocentric
- Diophantus (200's)
  - Equations with integer solutions.
  - Series of 13 books, *Arithmetica*, - algebraic equations.
- Hypatia (370-415)
  - Father - Theon.
  - Martyr.
  - Movie - *Agora*.

Romans - Little contribution to mathematics.

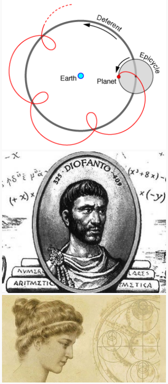
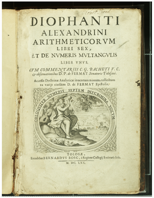


Figure 5: Epicycles, Diophantus, and Hypatia.

Diophantus' Epitaph



"Here lies Diophantus.

God gave him his boyhood one-sixth of his life;

One twelfth more as youth while whiskers grew rife;

And then yet one-seventh 'ere marriage begun.

In five years there came a bouncing new son;

Alas, the dear child of master and sage,

After attaining half the measure of his father's life, chill fate took him.

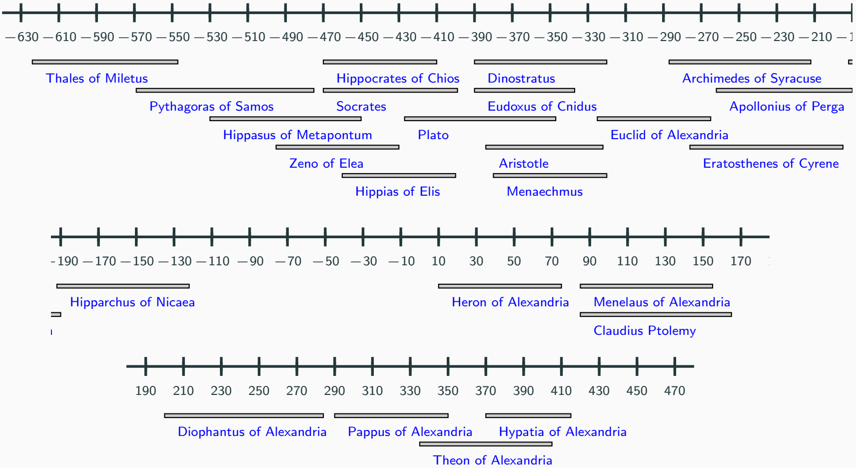
After consoling his fate by the science of numbers for four years, he ended his life."

- *Metrodorus, Greek Anthology.*

One Last Thing - The Antikythera Mechanism

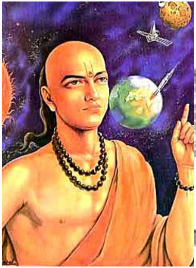
- Found in a wreck in 1900 near Antikythera.
- Recovered statues and other items.
- Small corroded fragments found.
- Fragment A has 27 gears.
- 100 years later ...
- It is an astronomical calculator.
- Predicts moon's position and phase, solar eclipses, motion of planets, and more.
- Possibly from 1st or 2nd century BCE.
- See March 2021 Paper, Freeth, et al.

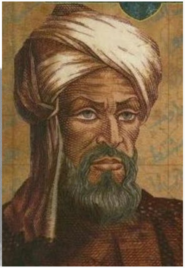
Timeline of Greek Mathematicians



Early Asian Mathematics

Fall 2025 - R. L. Herman





Overview

China

- Unique development
- *Zhoubi Suanjing* - c. 300 BCE
- *Tsinghua Bamboo Slips*, - decimal times table. 305 BCE
- Chinese abacus (<190 CE)
- After book burning (212 BCE), Han dynasty (202 BCE–220) produced mathematics works.

India

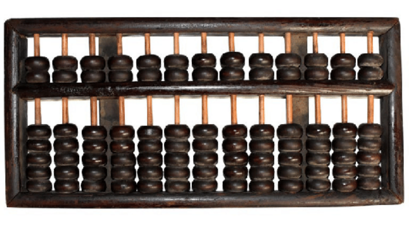
- *Pingala* (3rd–1st cent. BCE) - binary numeral system, binomial theorem, Fibonacci numbers.
- *Siddhantas*, 4th-5th cent. astronomical treatises, trigonometry.

Arabian-Islamic (330-1450)

- Preserved Greek texts
- 7th-14th cent. Development of algebra, etc.
- Hindu-Arabic numerals

Bamboo Slips and Suanpan

- *Tsinghua Bamboo Slips*, - Decimal Times Table. 305 BCE. By People from Warring States Period (476-221 BC), On right - Public Domain Image
- *Suanpan*, Chinese abacus (<190 CE), Below.



Chinese Dynasties

- Xia Dynasty (c. 2070-1600 BCE)
- Shang Dynasty (c. 1600-1046 BCE)
- Zhou Dynasty (c. 1046-256 BCE)
  Periods: Western Zhou, Spring and Autumn (770), Warring States (475)
- Qin Dynasty (221-206 BCE)
- Han Dynasty (206 BCE-220 AD)
- Six Dynasties Period (220-589)
- Sui Dynasty (581-618)
- Tang Dynasty (618-907)
- Five Dynasties and Ten Kingdoms (907-960)
- Song Dynasty (960-1279)
- Yuan Dynasty (1279-1368)
- Ming Dynasty (1368-1644)
- Qing Dynasty (1644-1912)

Figure 1: Liu Hui's circle.

Before Qin Dynasty (< 221 BCE)

- Known from myths and legends.
- Grasped numbers, figures.
- Quipu knots to record events, numbers.
- Used gnomon and compass.
- Artifacts - patterns on bone utensils.
- Plastrons (turtle shells) and oracles bones found in 1800's from Shang dynasty.
- Earliest known Chinese writing.
- Numbers up to 30,000. Special characters for numbers like 30,000, 20,000, 10,000, etc. [No zero.]
- Bone script from Zhou dynasty.

Calculations

- Ancient tools.
  - Babylonians - clay tablets.
  - Egyptians - hieroglyphs, papyri.
  - Indian, Arab - sand boards.
  - Chinese - counting rods.
- Short bamboo rods.
  - Han Dynasty: 1/10" × 6".
  - Sui Dynasty: 1/5" × 3".
  - No later than Warring States period.
  - Arranged - decimal place-value.
- Arithmetic operations. $+ - \times \div$
- Nine-nines rhyme - seen on bamboo strips. Spring and Autumn period (770-476).

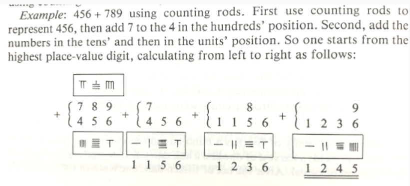
Chinese Numbers

rod	char.	digit	name
𝍩	一	1	i
𝍪	二	2	erh
𝍫	三	3	san
𝍬	四	4	ssu
𝍭	五	5	wu
𝍮	六	6	liu
𝍯	七	7	ch'i
𝍰	八	8	pa
𝍱	九	9	chiu
|	十	10	shih
—	百	100	pai
	千	1000	ch'ien

Arithmetic Operations

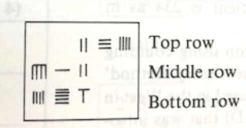
Clips from *Chinese Mathematics A Concise History* by Yan and Shiran.

Addition - $456 + 789$


Example: 456 + 789 using counting rods. First use counting rods to represent 456, then add 7 to the 4 in the hundreds' position. Second, add the numbers in the tens' and then in the units' position. So one starts from the highest place-value digit, calculating from left to right as follows:

Multiplication - 234×456

Start with 2×456



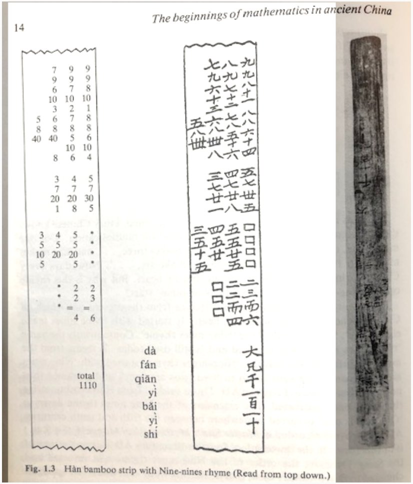
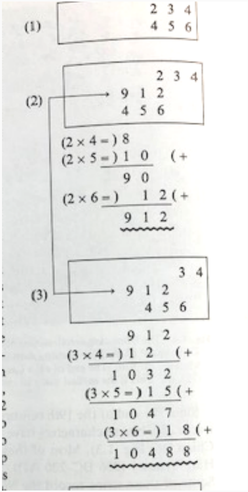
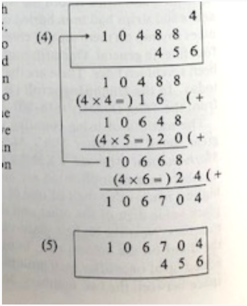


Fig. 1.3 Hàn bamboo strip with Nine-nines rhyme (Read from top down.)

Figure 2: Nine-nines Rhyme.

Multiplication 234×456





The Answer: $234 \times 456 = 106704$.

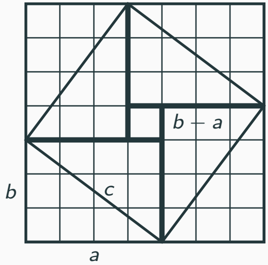
Early Mathematics

- Early works - construction techniques and statistics
- Fractions, measurements, angles, geometry, limit.
- Education - Six Gentlemanly Arts of the Zhou Dynasty (*Zhou Li - The Zhou Rites*).
    - Ritual,
    - Music,
    - Archery,
    - Horsemanship,
    - Calligraphy,
    - Mathematics.

Increased productivity in Han Dynasty (206 BCE-220 CE).

Zhoubi suanjing

- *Arithmetical Classic of the Gnomon and the Circular Paths of Heaven*,
- Written 100 BCE-100 CE.
- Uses **Gōugŭ Theorem**.
  [Around 1100 BC, Western Zhou period; Shanggao first described the Theorem.]
- Shadow gauges - vertical stakes for observing sun's shadow in Zhou.
- Fractions
  - 7 extra lunar months in 19 yr.
  - So, $12\frac{7}{19}$ mo/yr gives $29\frac{499}{940}$ da/mo.
  - Moon goes $13\frac{7}{19}^o$ per day. Then, 1 year $= 365\frac{1}{4}$ days.



$$c^2 = (b-a)^2 + 4(\tfrac{1}{2}ab) = a^2 + b^2$$

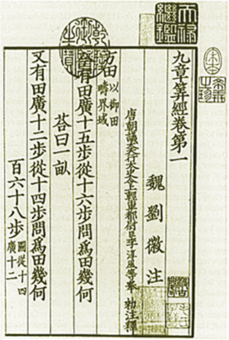
- Gŭ = vertical gauge.
- Gōu = shadow.
- Xián = Hypotenuse.

Jiuzhang Suanshu *The Nine Chapters on the Mathematical Art.*

- Complete by Eastern Han Dynasty (AD 25-220),
- Composed by generations of scholars starting 10th century BCE.
- Designated by Imperial Court as nation's standard math text - During Tang (618-907) and Song (960-1279) Dynasties.
- 246 word problems in 9 chapters on agriculture, business, geometry, engineering, surveying.
- Proof for the Pythagorean theorem.
- Formula for Gaussian elimination.
- Provides values of π. [They had approximated as 3.]
- Used negative numbers.

Chapters of the Mathematical Art

1. Rectangular Fields (Fangtian);
2. Millet and Rice (Sumi);
3. Proportional Distribution (Cuifen);
4. The Lesser Breadth (Shaoguang);
5. Consultations on Works (Shanggong);
6. Equitable Taxation (Junshu);
7. Excess and Deficit (Yingbuzu);
8. The Rectangular Array (Fangcheng); and
9. Base and Altitude (Gougu)


九章筭經卷第一 魏 劉 徽 注
唐朝議大夫行太史令上輕車都尉臣李淳風等奉勑注釋
方田 以御田疇界域
今有田廣十五步從十六步問爲田幾何
荅曰一畝
又有田廣十二步從十四步問爲田幾何
百六十八步 圖從十四廣十二

Chapters 1 and 9 summed up the accumulated knowledge of geometry and introduced a proposition - the Pythagorean theorem.

Influenced mathematical thought in China for centuries, introduced into Korea during the Sui Dynasty (581-618) and into Japan during the Tang Dynasty.

Notes and Commentaries

- Liu Hui (c. 225-295), mathematician and Li Chunfeng (602-670), astronomer and mathematician, known for commentaries on *The Nine Chapters on the Mathematical Art*.
- Chapters 2, 3 and 6: Proportion problems. 1st time.
- Chapter 7: The rule of False Double Position for linear problems. 1st time. [In Europe in the 13th century, Fibonacci.]
- World's earliest systematic explanation of fractional arithmetic.
- Chapter 8: Gaussian elimination.
  $> 1,500$ years before Carl Friedrich Gauss.
- Introduces negative numbers (1st),
  Rules for addition/ subtraction of positive and negative numbers.
  Brahmagupta (598-665) came up with the idea of negative numbers in India and Bombelli (1572) in Europe.

Summary of Chinese Mathematics i

- Shang - simple math, Oracle bones
- I Ching influenced Zhou Dynasty - use of hexagrams, binary (Leibniz).
- Decimal system since Shang Dynasty. 1st to use negative numbers.
- Suan shu shu, *A Book on Numbers and Computation*, 202-186 BCE.
  190 Bamboo strips, Found in 1984.
  Roots by False Position and Systems of equations.
- Zhoubi suanjing, *Arithmetical Classic of the Gnomon and the Circular Paths of Heaven*, 100 BCE-100 CE.
  Gougu Thm. [“Around 1100 BC, the Western Zhou period, the ancient Chinese mathematician, Shanggao, first described the Gougu Theorem. ”]
- Jiuzhang Suanshu, *The Nine Chapters on the Mathematical Art.*
- Sun Zi (400-460) *Sunzi suanjing* (Sun Zi’s Mathematical Manual) - Chinese Remainder Thm, Diophatine equations.

Summary of Chinese Mathematics ii

- Zhang Qiujian (430-490) - Manual, Sum arithmetic series, systems of 2 eqns and 3 unknowns.
- Before Han Dynasty - Addition. Subtraction, Multiplication, Division
- After Han Dynasty - Square roots and cube roots.
- Liu Hong (129-210) Calendar, Motion of moon.
- Computing π
  - Liu Xin (d. 23 AD), $\pi \approx 3.1457$.
  - Zhang Heng (78–139), $\pi \approx 3.1724, 3.162$ using $\sqrt{10}$.
  - Liu Hui (3rd century), commented on the Nine Chapters, $\pi = 3.14159$ from 96,192-gon, Exhaustion for circles, Gave method for V_{cyl} (Cavalieri's Principle).
  - Zu Chongzhi (5th century), Mathematical astronomy - Da Ming Li calendar. $\pi = 3.141592$ using 12,288-gon, Remained the most accurate value almost 1000 years. $\pi \approx \frac{355}{113}$ Gave method (Cavalieri's principle) for V_{sphere}.

Summary of Chinese Mathematics iii

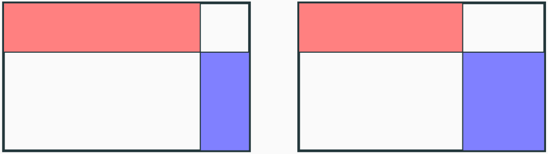
- Liu Zhuo (544-610) - quadratic interpolation.
- Yi Xing (683-727) Tangent table.
- Qin Jiushao (1202–1261) - Treatise of 81 problems
  Up to 10th degree eqns, Chinese Remainder Theorem, Euclidean Algorithm.
- Li Chih (1192-1297) 12 Chapters, 120 problems - Right triangles with circles inscribed/superscribed, geometric problems via algebra.
- Yan Hui (1238–1298), magic squares
- Zhu Shijie (1260 - 1320) - higher degree equations, binomial coefficients, uses zero digit.

One needs to view the methods from their point of view and not ours. In some cases, they provide a simpler "proof by picture."

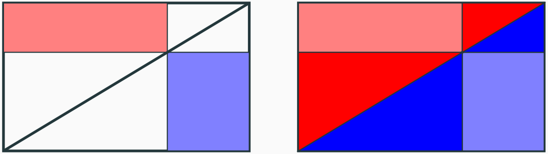
Next: The In-Out Complementary Principle.

In-Out Complementary Method

When do the red and blue rectangles have the same area?



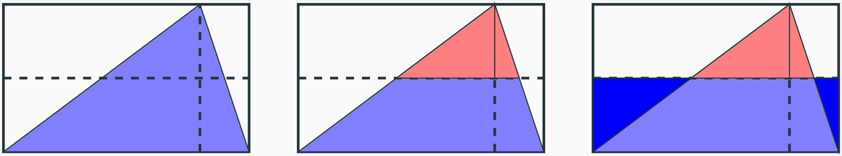
What does this suggest to you?



The case of “Equals subtracted from equals are equal.”

Area of a Triangle

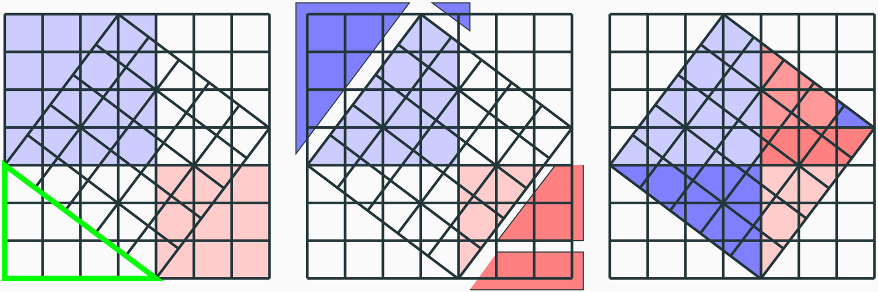
How would you prove that the area of a triangle is $A = \frac{1}{2}bh$?



Move the red "Out" triangles in the middle block to form the blue "In" triangles. The proof should be obvious at this point.

Gōugǔ Theorem

- Gǔ = vertical gauge.
- Gōu = shadow.
- Xián = hypotenuse.



According to J.W. Dauben, *Int. J. of Eng. Sci.* 36 (1998), Liu Hui explains, "The Gou-square is the red square, the Gu -square is the blue square. Putting pieces inside and outside according to their type will complement each other, then the rest (of the pieces) do not move. Composing the Xian-square, taking the square root will be Xian."

Haidao Suanjing **or** *Sea Island Mathematical Manual*

Commentary on Ch. 9 of *The Nine Chapters on the Mathematical Art*, by Liu Hui, in 263 AD.

- Surveying a sea island, set up two three zhang poles at one thousand steps apart, let the two poles and the island in a straight line. Step back from the front post 123 steps, with eye on ground level, the tip of the pole is on a straight line with the peak of island. Step back 127 steps from the rear pole, eye on ground level also aligns with the tip of pole and tip of island. What is the height of the island, and what is the distance to the pole ?
- The height of the island is four li and 55 steps, and it is 120 li and 50 steps from the pole.

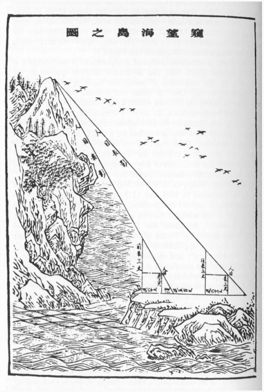


Figure 3: Liu Hui's Sea Island

Pascal's Triangle

Known by early Chinese mathematicians:

- Systems of linear equations
- Chinese Remainder Theorem
- Square roots
- Pythagorean Theorem
- Euclidean algorithm
- Pascal's Triangle
  - Typical term, $a^{n-k}b^k$, $k = 0, 1, \ldots, n$.
  - What is the coefficient?

$$
\begin{aligned}
(a+b)^0 &= 1\\
(a+b)^1 &= 1a + 1b\\
(a+b)^2 &= a^2 + 2ab + b^2\\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3\\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\end{aligned}
$$

Figure 4: Binomial Expansion,
$(a+b)^n = \sum_{k=0}^{n} C_{n,k} a^{n-k} b^k, \; n = 0, 1, \ldots.$

Yáng Huī, (ca. 1238–1298) presented Jiǎ Xiàn's (ca. 1010–1070) triangle. Used for extracting roots.
Blaise Pascal (1623–1662)

$$
\begin{aligned}
1 &= 1\\
1+1 &= 2\\
\text{Sum each row:}\quad 1+2+1 &= 4\\
\text{Sum} = 2^n.\quad 1+3+3+1 &= 8\\
1+4+6+4+1 &= 16\\
1+5+10+10+5+1 &= 32
\end{aligned}
$$

Figure 5: Pascal's Triangle, $C_{n,k} = \binom{n}{k} \equiv \frac{n!}{(n-k)!k!}$

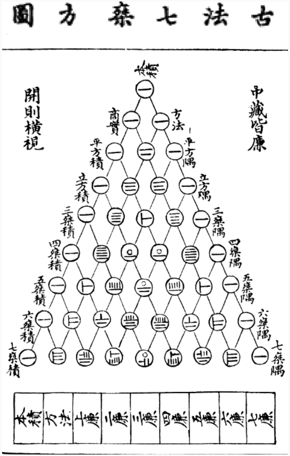


Figure 6: Jia Xian triangle published in 1303 by Zhu Shijie

Euclidean Algorithm Example

Example
One can only go 23 units or 79 units left or right. How many of each would it take to go from 0 to 1?

- $79m + 23n = 1.$
- Use Euclidean Algorithm

$$\begin{aligned} 79 &= 3 \cdot 23 + 10 \\ 23 &= 2 \cdot 10 + 3 \\ 10 &= 3 \cdot 3 + 1 \\ 3 &= 3 \cdot 1 \end{aligned}$$

In Reverse:

$$\begin{aligned} 1 &= 10 - 3 \cdot 3 \\ &= 10 - 3(23 - 2 \cdot 10) \\ &= 7 \cdot 10 - 3 \cdot 23 \\ &= 7 \cdot (79 - 3 \cdot 23) - 3 \cdot 23 \\ &= 7 \cdot 79 - 24 \cdot 23 \end{aligned}$$

Thus, $m = 7, n = -24.$

Chinese Remainder Theorem

The Chinese remainder theorem: If one knows the remainders of the Euclidean division of an integer x by several integers, then one can determine uniquely the remainder of the division of x by the product of these integers, assuming the divisors are pairwise coprime. Earliest - Sun-tzu in *Sunzi Suanjing*.

If $p_1, p_2, \ldots, p_n$ are relatively prime, then

$$\begin{aligned} x &= r_1 \pmod{p_1} \\ x &= r_2 \pmod{p_2} \\ &\vdots \\ x &= r_n \pmod{p_n} \end{aligned}$$

always has a solution.

Chinese Remainder Theorem Example

Example

$$\begin{aligned} x &= 2 \pmod 3 \\ x &= 3 \pmod 5 \\ x &= 2 \pmod 7 \end{aligned}$$

First equation means $x = 3n + 2$. Insert into second:

$$\begin{aligned} 3n + 2 &= 3 \pmod 5 \\ 3n &= 1 \pmod 5 \\ 3n &= 6 \pmod 5 \\ n &= 2 \pmod 5 \end{aligned}$$

So,

$$\begin{aligned} x &= 3n + 2 \\ &= 3(5m + 2) \\ &= 15m + 8. \end{aligned}$$

From third equation

$$\begin{aligned} 15m + 8 &= 2 \pmod 7 \\ 15m + 1 &= 2 \pmod 7 \\ 15m &= 1 \pmod 7 \\ 15m &= 15 \pmod 7 \\ m &= 1 \pmod 7 \end{aligned}$$

Therefore, $m = 7k + 1$ and $x = 105k + 23.$

Indian Mathematics (500-1200)

- Major mathematicians
  - Aryabhata (476-550?)
  - Bhaskara I (600-680)
  - Brahmagupta (598-668)
  - Bhaskara II (1114-1185)
  - Madhava (1350-1425)
- Contributions
  - Algebra
  - Geometry
  - Trigonometry
  - Spherical trigonometry
  - Diophantine Equations
  - Mathematical astronomy
  - Place-value decimal system

Brahmagupta:
$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$
$s = \frac{1}{2}(a+b+c+d)$ is semiperimeter

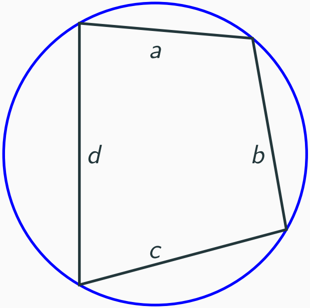


Figure 7: Cyclic Quadrilaterals

Aryabhata (476-550)

- Major work, *Aryabhatiya*, mathematics and astronomy,
  - arithmetic, algebra,
  - plane trigonometry,
  - spherical trigonometry.
  - continued fractions,
  - quadratic equations,
  - sums of power series, and
  - table of sines.
- 108 verses, 13 introductory verses
- Relativity of motion
- *Arya-siddhanta*,
  - Astronomical computations
  - Astronomical instruments

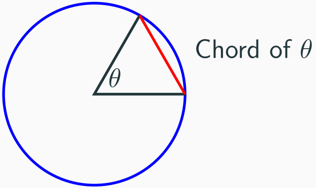
Figure 8: Aryabhata on the grounds of IUCAA, Pune.

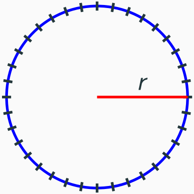
Table of Sines

- Introduction of sine
- Aryabhata's sine table
- Based on half chords
  vs Hipparchus, Menlaus, Ptolemy.
- Also, provided differences
- From Babylonians, base 60
  degrees, minutes, seconds
- Circumference $= 21600'$.
- Aryabhata, $\pi = 3.1416$
- Bhaskara I approximation

$$\sin\theta \approx \frac{4\theta(180-\theta)}{40500-\theta(180-\theta)}.$$

- Mādhava's - more accurate.





$$360^o = 21600'$$

$$r = \frac{21600}{3.1416} \approx 3438$$

Pell's Equation, $x^2 - Ny^2 = 1$, N Nonsquare

Brahmagupta (628): *samasa*, Method of Composition

$$(x_1^2 - Ny_1^2)(x_2^2 - Ny_2^2) = (x_1x_2 + Ny_1y_2)^2 - N(x_1y_2 + x_2y_1)^2$$

If $x_1^2 - Ny_1^2 = k_1$ and $x_2^2 - Ny_2^2 = k_2$, then

$$\begin{aligned} x &= x_1x_2 + Ny_1y_2 \\ y &= x_1y_2 + x_2y_1 \end{aligned}$$

solves $x^2 - Ny^2 = k_1k_2$.

This gives a composition of triples, (x_1, y_1, k_1) and (x_2, y_2, k_2) to give (x, y, k_1k_2).

Example (Brahmagupta) $x^2 - 92y^2 = 1.$

Note: $10^2 - 92(1)^2 = 8$. Thus, triple $= (10, 1, 8)$.

Pell's Equation (cont'd)

- $10^2 - 92(1)^2 = 8 \to (10, 1, 8)$.
- Compose $(10, 1, 8)$ with itself.
  $(10 \cdot 10 + 92 \cdot 1 \cdot 1, 10 \cdot 1 + 1 \cdot 10, 8 \cdot 8) = (192, 20, 64)$
- or, $192^2 - 92(20)^2 = 64$
  $24^2 - 92\left(\frac{5}{2}\right) = 1$
- Compose $(24, \frac{5}{2}, 1)$ with itself: $(1151, 120, 1)$.
- Bhaskara II (1150) - cyclic process always works - *chakravala*.
- Proved by Lagrange (1768)
  $gcd(a, b) = 1,\ a^2 - Nb^2 = k$.
- Compose (a, b, k) with $(m, 1, m^2 - N)$ gives $(am + Nb, a + bm, k(m^2 - N))$.
- Rescale

$$\left(\frac{am + Nb}{k}, \frac{a + bm}{k}, m^2 - N\right)$$

- Fermat (1657), $x^2 - 61y^2 = 1$,
  $x = 1766319049$,
  $y = 226153980$.

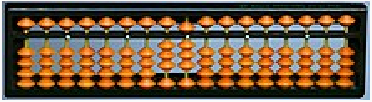
Japanese Mathematics (Wasan)

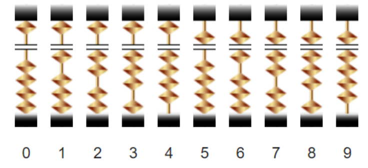
- Developed in Edo Period (1603-1867) rule of the Tokugawa shogunate.
- Economic growth, strict social order, isolationist foreign policies, a stable population, perpetual peace, and popular enjoyment of arts and culture.
- Foreign trade restrictions
- Mathematics for taxes
- Used Chinese counting rods.
- Imported Chinese texts.
- Sangi rods, adopted from Chinese rods, called saunzi.

Second period (552-1600) - Influx of Chinese learning - first through Korea.

Figure 9: Seki Takakazu.

The Soroban





- Abacus developed in Japan.
- Derived from ancient Chinese **suanpan**,
- Imported to Japan, 14th century.
- go-dama = 5, ichi-dama = 1.
- Addition, substraction, multiplication, division.

Yoshida Mitsuyoshi (1598–1672), *Jinkōki,* (1627) oldest existing Japanese math text, subject of **soroban arithmetic**, incl. square and cube roots.

Student of Kambei Mori, **first Japanese mathematician** with students Imamura Chishō, and Takahara Kisshu ("Three Arithimeticians").

Seki Takakazu (1642–1708), infinitesimal calculus and Diophantine equations, **"Japanese Newton."**

The Rise of Islam

- Mohammad was born in Mecca (570)
- Began preaching in Mecca, Escaped to Medina (622), later returned (629).
- Under Umar, Son of Al-Khattab, the empire spread. Ruled the Sasanian Empire and more than two-thirds of the Byzantine Empire
- Eventually spread from India to Spain. Main centers at Baghdad and Cordoba, Spain.
- Caliph Al-Ma'mun created Bait Al-hikma (House of Wisdom), Around 800 CE.
- Arabic became the common language.
- Translation of Greek and Hindu to Arabic (Euclid et al.) began with Al-Mansor, founder of Baghdad, grandfather of Al-Ma'mun.

Islamic Mathematics

- Caliph Al-Ma'mun appointed Al-Khwarizmi (780-850) court astronomer.
- His book, *Hisab al-jabr w'al-muqabala*, Solve linear or quadratic, 6 forms. Terms: Algebra and Algorithm.
- Al-Kindi (801-873) Arithmetic, 11 texts.
- Al-Bhattini (850-929)
  - Trigonometry, 1st Cotangent table.
- Thabit ibn Qurra (836-901) Theory of Numbers.
- Al-Kuhi (c940-1000) Archimedean and Apollonian math. Equations of degree > 2.
- Arabic numerals 1st in a book 874 and zero 2 yrs before Hindus.

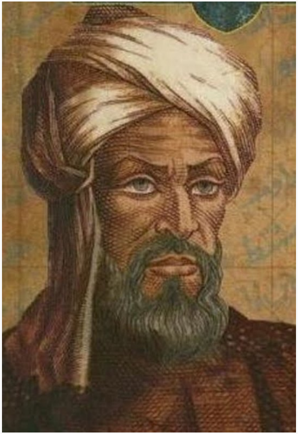


Figure 10: Al-Khwarizmi

International Year of Light 2015

Ibn al-Haytham's scientific method

Hasan Ibn al-Haytham (Latinized Alhazen)

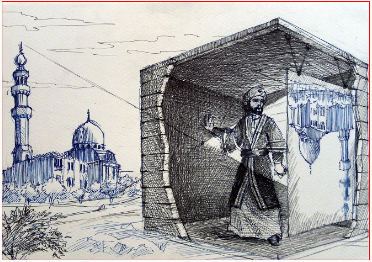
During the International Year of Light 2015, Ibn al-Haytham was celebrated at UNESCO as a pioneer of modern optics. He was a forerunner to Galileo as a physicist, almost five centuries earlier, according to Prof. S.M. Razaullah Ansari (India). Also known as Alhazen, this brilliant Arab scholar from the 10th – 11th century, made significant contributions to the principles of optics, astronomy and mathematics, and developed his own methodology: experimentation as another mode of proving the basic hypothesis or premise.

by Shaikh Mohammad Razaullah Ansari

Abū Ali al-Ḥasan Ibn al-Haytham al-Baṣrī (965-1040), known in European Middle Ages by the name of Alhazen, was called among Arab scholars as 'Second Ptolemy' (Baṭlamyūs Thānī). He was actually a scholar of many disciplines: Mathematics, physics, mechanics, astronomy, philosophy and medicine. He was one of the senior most member of the Muslim scholars' trio during 10th -11th centuries, the other two were al-Bīrūnī (973-1048) and Ibn Sīnā (980–1037).

From Basra, Ibn al- Haytham shifted to Cairo, where the Fatimid Caliph al-Ḥākim had invited him. The Caliph was a great patron of scientist-scholars, he got built an observatory for the astronomer Ibn Yūnus (d.1009) and he founded a library Dār al–'Ilm, whose fame almost equaled that of its precursor at Baghdad, Bayt al- Ḥikma(the House of Wisdom), established by the Abbasid Caliph al-Mā'mūn (reigned 813 – 833).

Ibn al-Haytham was a prolific writer. According to his own testimony, he wrote 25 works on mathematical sciences, 44 works on (Aristotelian) physics and metaphysics, also on meteorology and psychology. Moreover, his autobiographical sketch indicates clearly that he studied very thoroughly Aristotle's (natural) philosophy, logic and metaphysics of which he gave a concise account.



Ibn al-Haytham's pinhole camera and Al-Farisi's Rainbow

Later Islamic Mathematicians

- Ibn Al-Haytham (965-1039) Studied optics and visual perception.
  Used Euclid and Apollonius for Reflection, refraction, spherical mirrors, rainbows, eclipses, shadows.
  Optics works in Europe 12-13th century.
  Also, tackled 5th postulate.
- Golden Age: 9-10th centuries.
- Al-Karkhi (d. 1019-1029) Arithmetic, $\sum_n n^k,\ k = 1, 2, 3.$
- Omar Khayyam (1048-1131) studied cubic equations, and the intersection of conics, Khayyam's triangle, parallel postulate.
- Al-Kashi (1380-1429), fractions, π to 16 places, Law of Cosines.

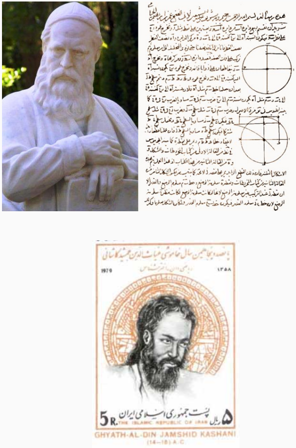


Figure 11: Omar Khayyam and Jamshīd al-Kashi stamp.

Rise of European Mathematics

- Fall of the Roman Empire
- Middle Ages, Medieval Period, 5th to the 15th century.
- Byzantine Empire (330-1453) - Church split,
- Preservation of Greek works.
- Al'Khwarizmi's work and Euclid translated into Latin.
- Crusades (1095-1291), The Plague (1347 to 1351).
- Mongols destroyed Islamic empire 1258.
- Johannes Gutenberg' printing press, 1440.
- Renaissance (1400-1600) and the Age of Discovery.
- Questioning of Aristotle
- Church of England (1534), Protestant vs Roman Catholic

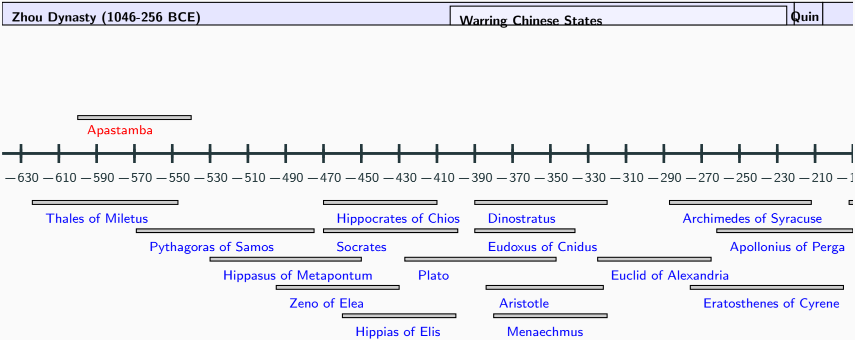
Medieval Mathematicians

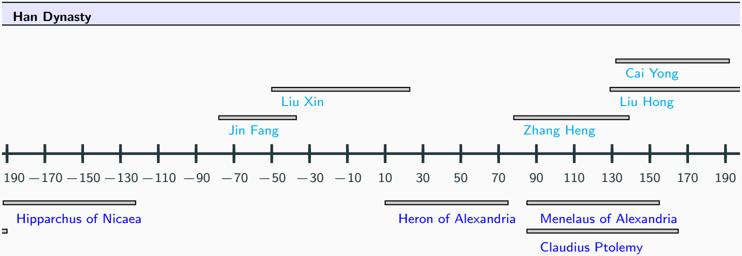
- Leonardo of Pisa (Fibonacci) (1200).
- Nicole Oresme (1323-1382), coordinate geometry, fractional exponents, infinite series.
- Johann Müller Regiomontatus (1436-1476), separated trigonometry from astronomy.
- And others:

Roger Bacon (1214-1292)	William of Ockham (1288-1348)
Filippo Brunelleschi (1377-1446)	Leone Battista Alberti (1404-1472)
Nicholas of Cusa (1401-1464)	Piero della Francesca (1420 - 1492)
Leonardo da Vinci (1452-1519)	Luca Pacioli (1445-1517)
Nicolaus Copernicus (1473-1543)	Scipione del Ferro (1465-1526)

- Rise of European Mathematics .. beginning in Italy.

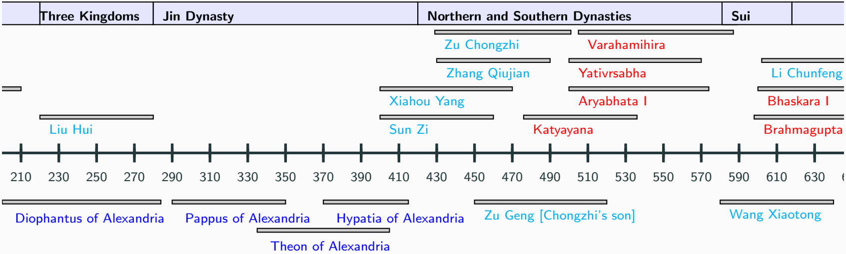
Timeline of Ancient Mathematicians i



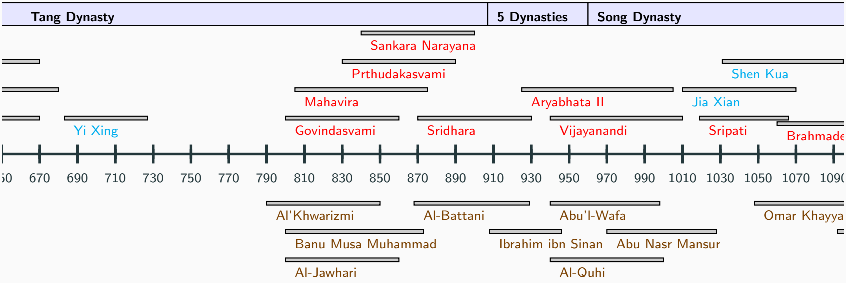


Han Dynasty
Cai Yong
Liu Xin
Liu Hong
Jin Fang
Zhang Heng
190 −170 −150 −130 −110 −90 −70 −50 −30 −10 10 30 50 70 90 110 130 150 170 190
Hipparchus of Nicaea
Heron of Alexandria
Menelaus of Alexandria
Claudius Ptolemy

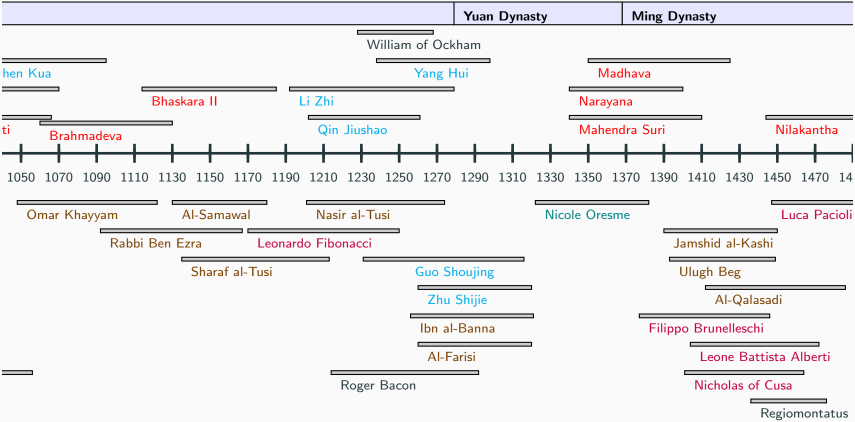
Timeline of Ancient Mathematicians iii



Timeline of Ancient Mathematicians iv



Timeline of Ancient Mathematicians v



Chinese Mathematical Classics

The Suàn shù shū (A Book on Numbers and Computations) is written on nearly 200 bamboo slips. The earliest known mathematical work in Chin, recently discovered in a tomb, dated to the early second century BCE.

The Jiu zhang suan shu (Nine Chapters on the Art of Mathematics) known from an imperfect copy printed in the Southern Song dynasty (1213 CE)

The Shi bu suan jing (Ten Books of Mathematical Classics)

- Zhoubi suanjing (Zhou Shadow Gauge Manual)
- Jiuzhang suanshu (Nine Chapters on the Mathematical Art)
- Haidao suanjing (Sea Island Mathematical Manual)
- Sunzi suanjing (Sun Zi's Mathematical Manual)
- Wucao suanjing (Mathematical Manual of the Five Administrative Departments)
- Xiahou Yang suanjing (Xiahou Yang's Mathematical Manual)
- Zhang Qiujian suanjing (Zhang Qiujian's Mathematical Manual)
- Wujing suanshu (Arithmetic methods in the Five Classics)
- Jigu suanjing (Continuation of Ancient Mathematics)
- Shushu jiyi (Notes on Traditions of Arithmetic Methods)
- Zhui shu (Method of Interpolation)
- Sandeng shu (Art of the Three Degrees; Notation of Large Numbers)

Cubic Equations

Fall 2025 - R. L. Herman



Solutions of Polynomial Equations

- Linear equations, known solutions.
- Chinese - Gaussian elimination: Systems of n linear equations and n unknowns.
- Quadratic equations: Need square roots.
- Cubic equations: Need square roots and cube roots. Solved in 16th century.
- Quintic equation: studied in 1820's. Eventually lead to group theory!

Figure 1: Leonardo da Vinci attempts Delian problem (Doubling cube).

Babylonian Method (Modern Notation)

Find x and y for a given perimeter and area.

$$\begin{aligned} x+y &= p \\ xy &= q. \end{aligned}$$

Eliminate y, $x^2+q=px$. Then,

$$x,y = \frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}.$$

Method - Compute the following

1. $\frac{x+y}{2}$
2. $\left(\frac{x+y}{2}\right)^2$
3. $\left(\frac{x+y}{2}\right)^2 - xy = \frac{(x+y)^2-4xy}{4}$
4. $\sqrt{\frac{(x+y)^2-4xy}{4}} = \frac{x-y}{2}$
5. By inspection, get x, y from p and q, since

$$\begin{aligned} \frac{x-y}{2} &= \sqrt{\frac{p^2-4q}{4}} \\ &= \sqrt{\left(\frac{p}{2}\right)^2 - q}. \end{aligned}$$

Quadratic Equations (cont'd)

- Brahmagupta (628) - Explicit
  $ax^2 + bx = c$.

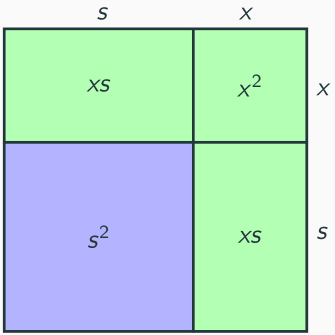
$$x = \frac{\sqrt{4ac + b^2} - b}{2a},$$

- Euclid - Prop. 28
- al'Khwarizimi

$$\begin{aligned} x^2 + 2xs &= n \\ x^2 + 2xs + s^2 &= n + s^2 \\ (x+s)^2 &= n + s^2 \end{aligned}$$

- Quadratic Irrationals

$$\frac{a + \sqrt{b}}{\sqrt{\sqrt{a} + \sqrt{b}}}$$



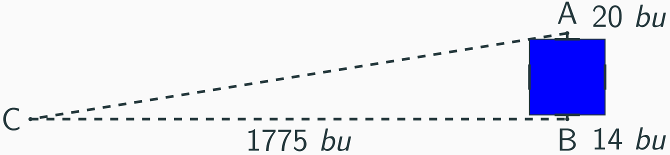
Note from the figure:
Green area $= x^2 + 2xs = n$.
No negative lengths (solutions).

Solving Quadratics Han Dynasty Style

Jiuzhang Suanshu (Gougu 20).

今有邑方不知大小，各中。出北二十步有木。出南十四步，折而西行一千七百七十五步木。：邑方何?

There is a square town of unknown dimensions. There is a gate in the middle of each side. There is a tree located 20 *bu* outside the North Gate. If one leaves the town by the South Gate, walks 14 *bu* due south, then walks due west for 1775 *bu*, the tree will just come into view. What are the dimensions of the town? [1 *bu* = 1.38 m.]



https://math-physics-problems.fandom.com/wiki/Solving_Quadratics_Han_Dynasty_Style

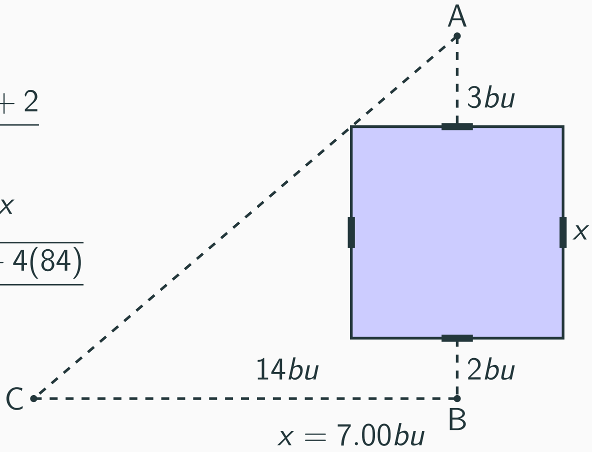
Another Example - Derivation of the Quadratic Equation

The diagram looks better if the tree located 3 *bu* outside the North Gate, one leaves the town by the South Gate, walks 2 *bu* due south, then walks due west for 14 *bu*.

$$\frac{3}{x/2} = \frac{x+3+2}{14}$$

$$84 = x^2 + 5x$$

$$x = \frac{-5 \pm \sqrt{25 + 4(84)}}{2}$$

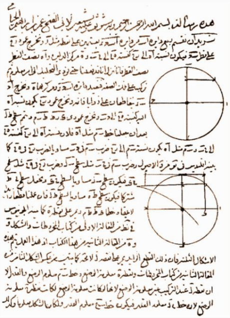


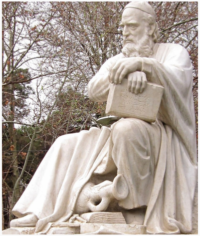
Cubic Equations

- Babylonians - Table of cubes
- Greeks - Geometric Problems
  - Duplicating cube (Delian Prob.).
  - Intersecting conics.
  - Cutting Sphere with plane.
- Omar Khayyam (1048-1131)
  - First general theory of cubics.
  - Provided 19 types of cubic.
  - Example

    $$x^3 + ax^2 = bx + c,$$

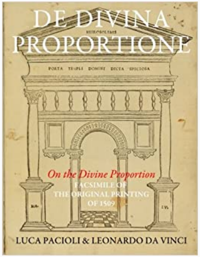
    Cube and square equals side plus number.
  - Geometric: Intersect two hyperbolae.





Pacioli, da Vinci and della Francesca

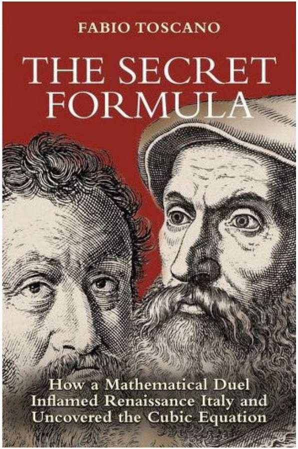
- Luca Pacioli (1445-1517)
  - Franciscan friar, tutor.
  - *Summa de arithmetica*, 1494. Father of Accounting.
  - *Divina proportione*, 1509.
  - "Solution to cubic is impossible!"
  - On table: slate, chalk, compass, dodecahedron. Hanging: Rhombicuboctahedron half-filled with water.
- Leonardo da Vinci (1452-1519)
- Piero della Francesca's (1420-1492) Painter/mathematician, met Alberti, 1451. Wrote books: algebra, perspective, Archimedean polyhedra. Pacioli used his work.



The Secret Formula

How a Mathematical Duel Inflamed Renaissance Italy and Uncovered the Cubic Equation, by Fabio Toscano, 2020.

- The Abbaco Master
  Italian Wars, Brescia, 1512, Tartaglia.
- The Rule of the Thing
  cosa (thing), *censo* (x^2), *numero*.
- The Venetian Challenge
  1535, Fior challenges Tartaglia.
- An Invitation to Milan
  Entrance of Gerolamo Cardano.
- The Old Professor's Notebook
  da Coi vs. Ferrari, del Ferro's priority:
  Solution: things and cube equal to number.
- The Final Dual
  Ars Magna, 1545. Ferrari vs. Tartaglia, 1547.



The Search for Solutions - *Enter del Ferro*

- Scipio del Ferro (1465-1526)
  - University of Bologna, notebooks
  - Printing press - Guttenberg
  - 1506/1514, solution of **depressed cubic**: $x^3 + ax = b$.
  - Public Challenges led to secrecy.
- Gave to Antonio Maria Fior (Florido).
- Tartaglia (Nicolo Fontana) (1499-1557)
  - 1512, French attack - sabre wound led to stammer.
  - Self-educated
  - 1530 da Coi wrote to him $x^3+3x^2 = 5$, $x^3+6x^2+8x = 1000$.

Figure 2: Tartaglia

Sidenotes

Pacioli, *Summa de arithmetica* 1494, Not solvable:

$$\begin{aligned} n &= ax + bx^3 \\ n &= ax^2 + bx^3 \\ n &= ax^3 + bx^4 \end{aligned}$$

Tartaglia, Abaco teacher/master but engaged in other activities.

Challenges not uncommon, but expect challenger to know solutions to their questions.

Not happy with da Coi questions since da Coi did not know answers.

The Plot Thickens

- Tartaglia boasted he could solve $x^3 + ax^2 = c$.
- Florido challenged Tartaglia
  - Each posed 30 problems
  - Florido mostly gave problems of form $x^3 + ax^2 = c$.
  - Tartaglia won by solving depressed cubic 1535, but didn't publish.
- Girolamo Cardano (1501-1576)
  - Gambler, astronomer, physician, astrologer, heretic, father of murderer.
  - Begged for solution from Tartaglia. Finally, they met in Milan.
  - Tartaglia eventually gave solution in 1539 as a Poem if it was kept secret.
  - It was not in Cardano's book.

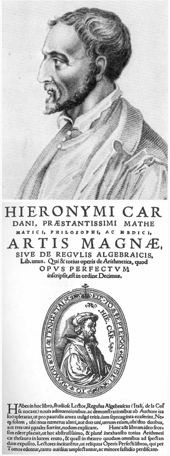
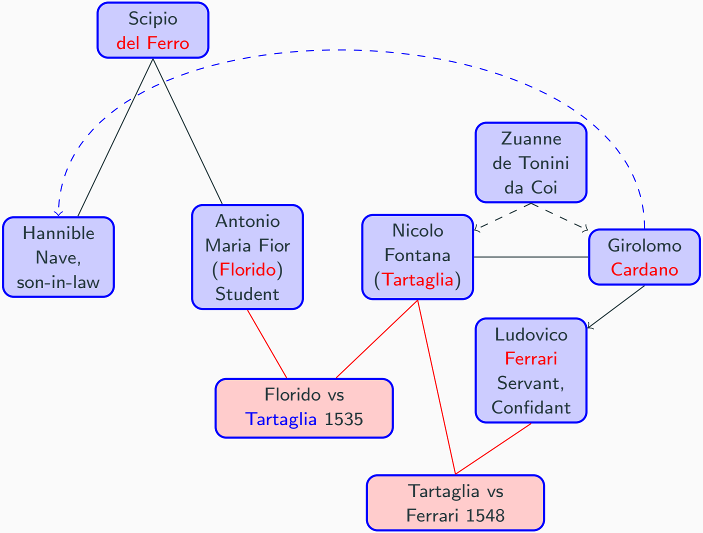


Figure 3: Cardano, *Ars Magna*.

- Ludovico Ferrari (1522-1565)
    - Servant at 14
    - Secretary, confidant
    - Worked on problems with Cardano
    - Cubic and biquadratic equations
- da Coi $\rightarrow$ Cardano $\rightarrow$ Ferrari
    - 4th degree polynomial
    - Ferrari solution involved solving cubic
    - Publishing was a problem.
- 1543 Trip to Florence, stopped in Bologna on the way.
    - Visitied Hannible Nave, del Ferro's son-in-law.
    - Saw del Ferro's notes.
    - Cardano believed he could publish in his *Ars Magna*, 1545.
- Barrage of letters from Tartaglia!

The Players in the Cubic Story



Tartaglia vs Ferrari - 1548

- Public debate in Milan, Ferrari's hometown.
- Cardano was absent.
- Tartaglia lost, blamed crowd.
- Tartaglia worked on arithmetic.
- Ferrari became professor in Bologna, 1565.
  Was poisoned 1565, white arsenic, possibly by sister.
- Cardano predicted exact date of his own death in 1576.

Figure 4: Tartaglia and Ferrari

Solution of the Quadratic $x^2 + ax + b = 0$

- Completing the square:
  $\left(x + \frac{a}{2}\right)^2 + b - \frac{a^2}{4} = 0.$
- Solution: $x + \frac{a}{2} = \pm\sqrt{\frac{a^2}{4} - b}.$
- Graph of parabola
  $y = x^2 + ax + b$
  Vertex $(-\frac{a}{2}, b - \frac{a^2}{4})$
- Number of real solutions?
- Substitute $x = u - \frac{a}{2}$:
  $0 = x^2 + ax + b$
  $0 = (u - \frac{a}{2})^2 + a(u - \frac{a}{2}) + b$
  $0 = u^2 + b - \frac{a^2}{4}.$
- Solve for u.

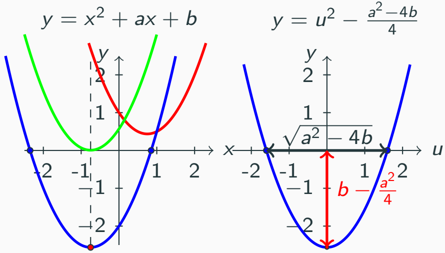


Figure 5: Plots of parabolae. Translating blue parabola on left by $\frac{a}{2}$ results in that on the right.

Plotting a Cubic Function $y = x^3 + ax^2 + bx + c$

- Set $y = 0$, then
  $x^3 + ax^2 + bx + c = 0$.
- Solutions are black points.
  Always have a real solution.
- $y' = 3x^2 + 2ax + b$ and
  $y'' = 6x + a$.
- Inflection point: $y'' = 0$ for
  $x_0 = -\frac{a}{3}$, $y_0 = c - \frac{ab}{3} + \frac{2a^3}{27}$.
- Slope of tangent at $(x_0, y_0 = q)$
  is $p = b - \frac{a^2}{3}$.
- Extrema at
  $x_\pm = -\frac{a}{3} \pm \frac{\sqrt{a^2 - 3b}}{3}$.

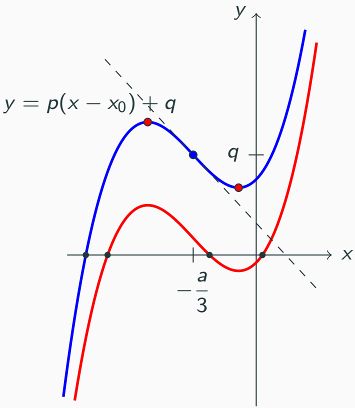


Figure 6: Plots of the cubic function exhibiting either one or three real roots.

Can you complete the cube: $(x + \alpha)^3 = x^3 + 3x^2\alpha + 3x\alpha^2 + \alpha^3$?

Solution of the Cubic $x^3 + ax^2 + bx + c = 0$.

Let $x = y - \frac{a}{3}$.

Then, $y^3 + py + q = 0$, where

$$\begin{aligned} p &= b - \frac{a^2}{3}, \\ q &= c - \frac{ab}{3} + \frac{2a^3}{27}. \end{aligned}$$

Let $y = u + v$:

$$u^3 + v^3 + (p + 3uv)(u + v) + q = 0.$$

Let $p + 3uv = 0$, then

$$\begin{aligned} u^3 v^3 &= -\frac{p^3}{27}, \\ u^3 + v^3 &= -q. \end{aligned}$$

Now, define $X = u^3$. $Y = v^3$.

We obtain

$$\begin{aligned} X + Y &= -q. \\ XY &= -\frac{p^3}{27}, \end{aligned}$$

Does this look familiar?

The solution of Cubic:

$$X, Y = u^3, v^3 = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \frac{p^3}{27}},$$

$y = u + v$,

$x = y - \frac{a}{3}$.

Example: $2x^3 - 30x^2 + 162x - 350 = 0$.

Let $x = y - \frac{b}{3a} = y + \frac{30}{6} = y + 5$.

We obtain a depressed cubic (del Ferro), $y^3 + 6y - 20 = 0$.

Letting $y = u + v$, $X, Y = u^3, v^3$, we solve

$$X + Y = -q = 20, \quad XY = -\frac{p^3}{27} = -\frac{6^3}{27} = -8.$$

Eliminating Y, $X^2 - 20X - 8 = 0$.

Solving, leads to $X = 10 \pm \sqrt{108}$, $\qquad Y = 20 - X = 10 \mp \sqrt{108}$.

So, let $u = \sqrt[3]{10 + 6\sqrt{3}} \qquad v = \sqrt[3]{10 - 6\sqrt{3}}$ and

$$y = u + v = \sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}}$$

$$x = \sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}} + 5$$

Depressed Cubic $y^3 + 6y - 20 = 0$.

Note: $y = 2$ is a solution. Therefore, $y^3 + 6y - 20 = (y-2)(y^2 + \alpha y + \beta)$.

Method 1: Expand and match coefficients.

$y^3 + 6y - 20 = y^3 + (\alpha - 2)y^2 + (\beta - 2\alpha)y - 2\beta$

$\alpha - 2 = 0, \qquad \beta - 2\alpha = 6, \qquad 2\beta = 20 \quad \Rightarrow \quad \alpha = 2, \beta = 10.$

Method 2: Long division.

$$
\begin{array}{r|l}
 & y^2 + 2y + 10 \\
\hline
y - 2 & y^3 + 6y - 20 \\
 & -y^3 + 2y^2 \\
\hline
 & 2y^2 + 6y \\
 & -2y^2 + 4y \\
\hline
 & 10y - 20 \\
 & -10y + 20 \\
\hline
 & 0
\end{array}
$$

Find other roots:

$$y^2 + 2y + 10 = 0$$

$$
\begin{aligned}
y &= \frac{-2 \pm \sqrt{4 - 40}}{2} \\
&= -1 \pm 3i.
\end{aligned}
$$

Nested Radicals

We have found that $y = \sqrt[3]{10+6\sqrt{3}} + \sqrt[3]{10-6\sqrt{3}}$. Can one simplify this?
We consider $\sqrt[3]{10 \pm 6\sqrt{3}} = \sqrt{a} \pm \sqrt{b}$. Note:

$$\begin{aligned}
(\sqrt{a} \pm \sqrt{b})^3 &= a\sqrt{a} \pm 3a\sqrt{b} + 3b\sqrt{a} \pm b\sqrt{b} \\
&= (a+3b)\sqrt{a} \pm (3a+b)\sqrt{b} \\
&= 10 \pm 6\sqrt{3}. \qquad (1)
\end{aligned}$$

From (1) $b = 3$, $a + 9 = 10$, $\quad 3a + 3 = 6$.

Then, $a = 1$ and $\sqrt[3]{10 \pm 6\sqrt{3}} = 1 \pm \sqrt{3}$.

So,

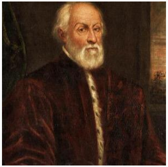
$$\begin{aligned}
y &= \sqrt[3]{10+6\sqrt{3}} + \sqrt[3]{10-6\sqrt{3}} \\
&= 1 + \sqrt{3} + 1 - \sqrt{3} \\
&= 2. \qquad (2)
\end{aligned}$$

Complex Solutions $y^2 + 2y + 10 = 0$.

We found $y^3 + 6y - 20 = (y-2)(y^2+2y+10) = 0$.

we solved the quadratic: $y = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm 3\sqrt{-1}$.

- Cardano, complex numbers
  "as subtle as they are useless."
- Raphael Bombelli (1526-1572)
  First to take seriously.
- Ex: $x^3 = 15x + 4$
  $x = \sqrt[3]{2 + 11\sqrt{-11}} + \sqrt[3]{2 - 11\sqrt{-11}}$
- But, $x = 4$ is a solution!
- Complex numbers, $a + bi$, $i = \sqrt{-1}$.
- $x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}$



L ALGEBRA
PARTE MAGGIORE
DELL'ARIMETICA
DIVISA IN TRE LIBRI
DI RAFAEL BOMBELLI
DA BOLOGNA.
Nouamente posta in luce.
IN BOLOGNA
Nella stamperia di Giouanni Rossi
MDLXXII
Con Licentia delli RR. VV. del Vesc. & Inquisit.

Figure 7: Bombelli

Cube Root of Complex Numbers

- Last Example: $x^3 = 15x + 4$
  $x = \sqrt[3]{2+11i} + \sqrt[3]{2-11i}$
- Seek: $\sqrt[3]{2+11i} = c + di$.

$$\begin{aligned}
\sqrt[3]{2+11i} &= c + di \\
2 + 11i &= (c+di)^3 \\
&= c^3 + 3c^2 di + 3c(di)^2 + (di)^3 \\
&= c^3 - 3cd^2 + i(3c^2 d - d^3).
\end{aligned}$$

Then

$$\begin{aligned}
2 &= c^3 - 3cd^2 = c(c^2 - 3d^2), \\
11 &= 3c^2 d - d^3 = d(3c^2 - d^2).
\end{aligned}$$

- Bombelli: c, d, positive integers.

Since 2 is prime, $c = 1, 2$.

If $c = 1$, $2 = 1 - 3d^2$. No!

If $c = 2$, then $d = 1$.

$$\begin{aligned}
2 &= 8 - 6d^2 \\
11 &= 12d - d^3
\end{aligned}$$

Then,

$$\begin{aligned}
x &= \sqrt[3]{2+11i} + \sqrt[3]{2-11i} \\
&= (2+i) + (2-i) = 4.
\end{aligned}$$

Françios Viète (1540-1603)

- Counselor to Henry III, IV, France
- French Wars of Religion 1562-1598
- Tutored Catherine de Parthenay (1554-1631), noblewoman, mathematician
- 1596 - Adriaan van Roomen
  "No French mathematician could solve the 45th degree polynomial."
  $x^{45} - 45x^{43} + 945x^{41} + \cdots - 3795x^3 + 45x = A.$
- Viète solved quickly:
  $2\sin(45\alpha) = A, \quad x = 2\sin\alpha.$
- Trig identity: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$
  Let $y = \cos\theta$. $4y^3 - 3y = c, |c| \leq 1$. $c = \cos 3\theta.$
  Solve for θ given c. Solution, $y = \cos\theta$.
- Use identities to rewrite $2\sin(45\alpha) = A$ in terms of $2\sin\alpha$.

Figure 8: Viète, Henry IV, and van Roomen.

Viète's Solution

Define the quantities

$$\begin{aligned} c &= 2\sin 45\theta, \qquad & y &= 2\sin 15\theta, \\ z &= 2\sin 5\theta, \qquad & x &= 2\sin\theta. \end{aligned} \tag{3}$$

Problem: Find x, given c.

Use the identities:

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha \tag{4}$$

$$\sin 5\alpha = 5\sin\alpha - 20\sin^3\alpha + 16\sin^5\alpha \tag{5}$$

Then,

$$c = 2\sin 45\theta = 6\sin 15\theta - 8\sin^3 15\theta = 3y - y^3. \tag{6}$$

$$y = 2\sin 15\theta = 6\sin 5\theta - 8\sin^3 5\theta = 3z - z^3. \tag{7}$$

$$z = 2\sin 5\theta = 10\sin\theta - 40\sin^3\theta + 32\sin^5\theta = 5x - 5x^3 + x^5. \tag{8}$$

Viète's Solution (cont'd)

Since $z = 5x - 5x^3 + x^5$., we write c in terms of x :

$$
\begin{aligned}
y &= 3z - z^3 \\
&= 3[5x - 5x^3 + x^5] - [5x - 5x^3 + x^5]^3 \\
&= -x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x. \qquad (9) \\
c &= 3y - y^3 \\
&= 3[-x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x] \\
&\quad -[-x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x]^3 \\
&= x^{45} - 45x^{43} + 945x^{41} - 12300x^{39} + 111150x^{37} - 740259x^{35} + 3764565x^{33} \\
&\quad -14945040x^{31} + 46955700x^{29} - 117679100x^{27} + 236030652x^{25} - 378658800x^{23} \\
&\quad +483841800x^{21} - 488494125x^{19} + 384942375x^{17} - 232676280x^{15} + 105306075x^{13} \\
&\quad -34512075x^{11} + 7811375x^9 - 1138500x^7 + 95634x^5 - 3795x^3 + 45x \\
&= P_{45}(x). \qquad (10)
\end{aligned}
$$

Math Symbols

- $+$ Oresme (1300)
- $-$ Widman (1400)
- $=$ Recorde (1500)
- $\times$ Outred (1500)
- Letters Viète (1500)
- Descartes (1500-1600)
  unknowns, constants a, b, c
  variables x, y, z
- $<>$ Harriot (1600)
- ∞ Wallis (1700)
- imaginary, Descartes
- $x^{3/2}, x^{-1}$, Newton 1600
- $x^2 \rightarrow xx$, Gauss 1800
- π, i, Σ. Euler 1700
- $f(x)$
- $\frac{df}{dx}, \int$ Leibniz 1600

Analytic Geometry

- Fermat (1601-1665)
- Descartes (1596-1650)
- Newton (1642-1727)

Coordinates

- Hipparchus - sky
- Apollonius - conics
- Oresme (1300s) - position, velocity plots
- Fermat-Descartes described curves in coordinate systems
- Degree 1, Linear relations

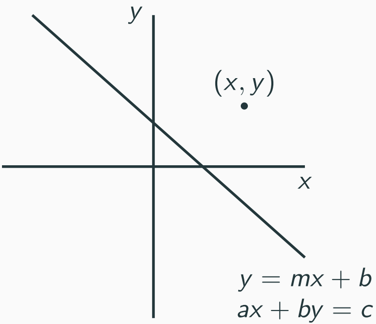


Figure 9: Cartesian system.

Curves of Degree 2 - Quadratics

$$ax^2 + \underbrace{bxy}_{rotation} + cy^2 + \underbrace{dx + ey}_{translation} + f = 0$$

- Describes Conics
- $b \neq 0$, rotation
- $d \neq 0$ or $e \neq 0$, translation
- Classification

$$D = \begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{vmatrix}$$

$D > 0$ ellipse
$D < 0$ hyperbola
$D = 0$ parabola

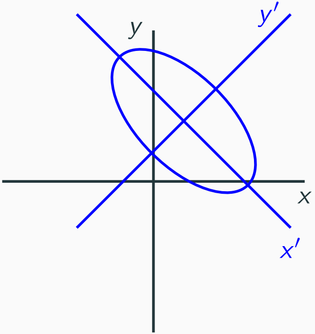


Figure 10: Rotated, translated ellipse.

Curves of Degree 3: $ax^3 + bx^2y + cxy^2 + dy^3 + \cdots = 0$, Cubics

- Newton classified cubic curves, 1710, 72 types (missed 6)
- $y = x^3$ and other types.
- Descarte's folium (leaf) $x^3 + y^3 = 3axy$
- Parametric Solutions

$$x = \frac{3at}{1+t^3}, \quad y = \frac{3at^2}{1+t^3}.$$

- Rational Points
  Ex. $x^3 + y^3 = 1$.
  Let $x = \frac{n}{p}$, $y = \frac{m}{p}$. $n^3 + m^3 = p^3$.
- Fermat - only trivial $(0,1), (1,0)$.
- Fermat's Last Theorem, 1637

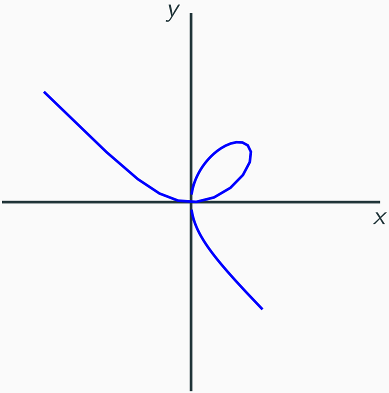


Figure 11: Descarte's Folium.

Fermat's and Bezout's Theorems

- Fermat's Last Theorem, 1637
  $x^n + y^n = z^n$
  Wiles proved in 1995.
- Bezout's Theorem. Let

$$p(x,y) = 0, \qquad \text{degree } n.$$
$$q(x,y) = 0, \qquad \text{degree } m.$$

  Then, p and q intersect in nm points.
- Elimination gives eq of degree nm.
- Need complex numbers,
  point at infinity.
  Next - Projective Geometry.

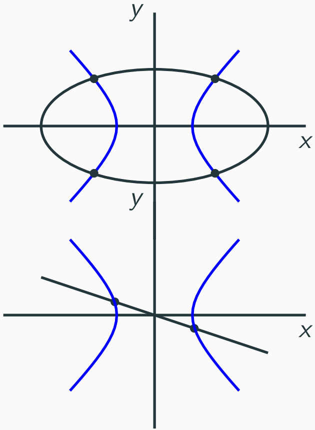


Figure 12: Intersecting Degree 2 curve (blue) with Degree 2 or 1 (black).

Projective Geometry

Fall 2025 - R. L. Herman

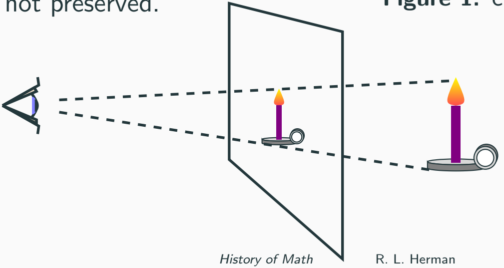
Perspective Drawing

- Art - Perspective Drawing
- Before Renaissance- no illusion of depth and space.
- 13th century Italian masters used shadowing.
- Mathematics of perspective

  Lengths not preserved.

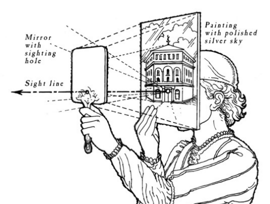
  Angles not preserved.

Figure 1: c.1308-1311



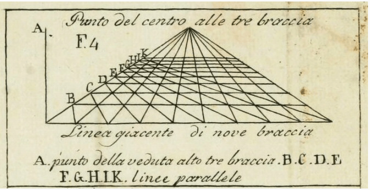
Filippo Brunelleschi (1377-1446) - Architect

- First to describe linear perspective.
- Experimented (1415-1420) using a panel with a grid of squares and a plaque with a hole at eye level.
- Drawings of the Baptistry in Florence, Place San Giovanni and other Florence landmarks.
- His method was studied by Alberti, Da Vinci & della Francesca's *The Perspective of Painting*.



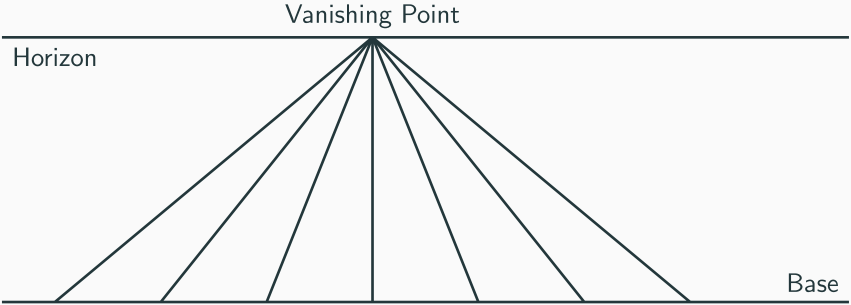
Leon Battista Alberti (1404-1472)

- Alberti's Veil
  Transparent cloth on a frame,
  Good for actual scenes not
  imaginary ones.
- Basic principles:
  1. A straight line in perspective
  remains straight.
  2. Parallel lines either remain
  parallel or converge to a point.
- Drawing a square-tiled floor, solved
  by Alberti (1436).



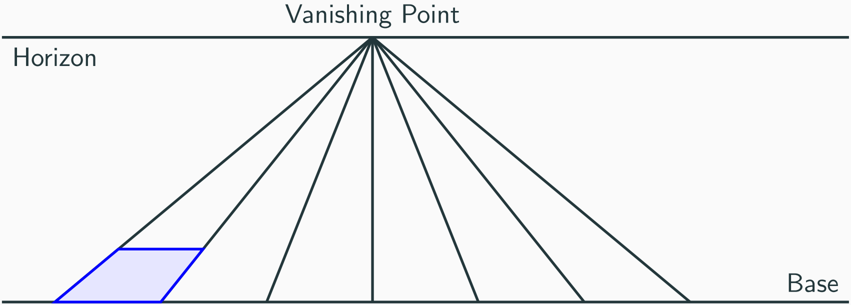
Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.



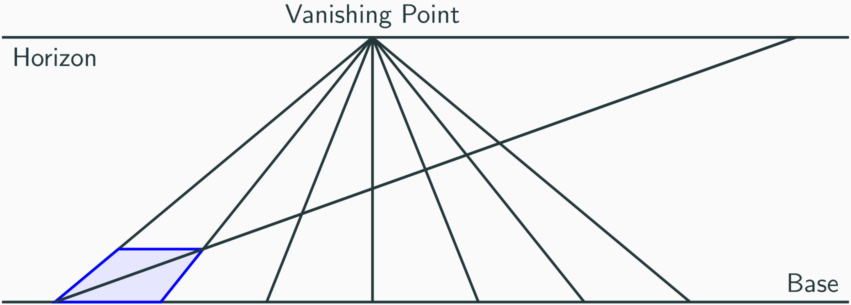
Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.
- Choose one tile.



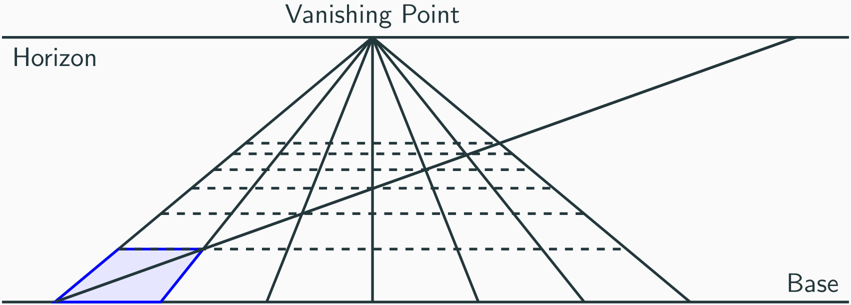
Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.
- Choose one tile.
- Extend diagonal.



Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.
- Choose one tile.
- Extend diagonal.
- Intersections determine the horizontals.



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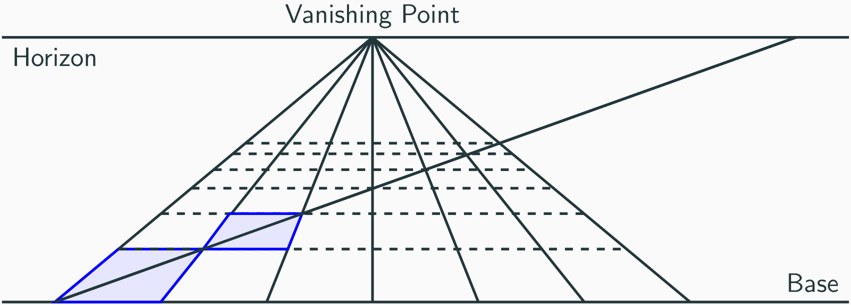

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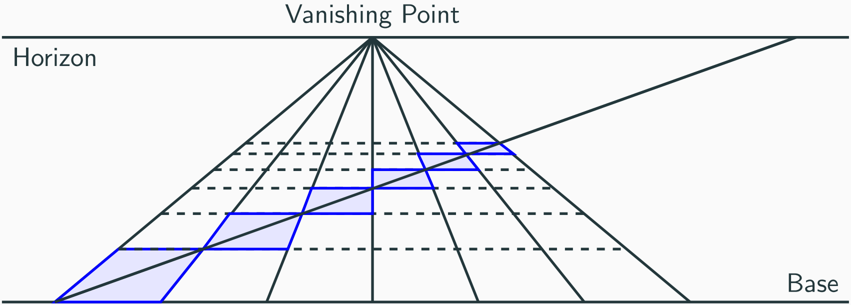

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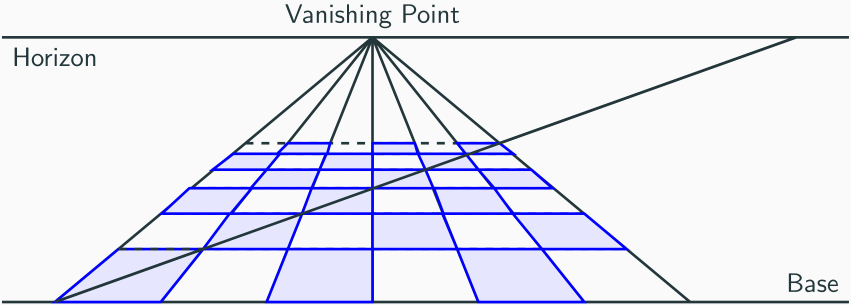


Anamorphosis

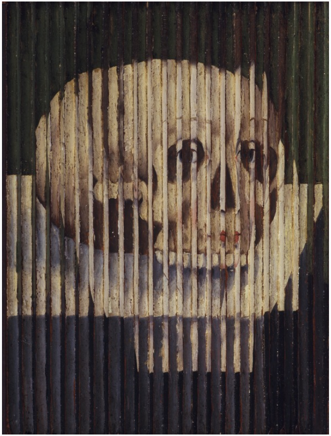
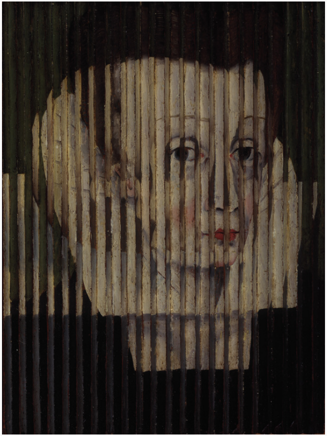


Figure 2: Mary, Queen of Scots, 1542 - 1587.

Anamorphosis



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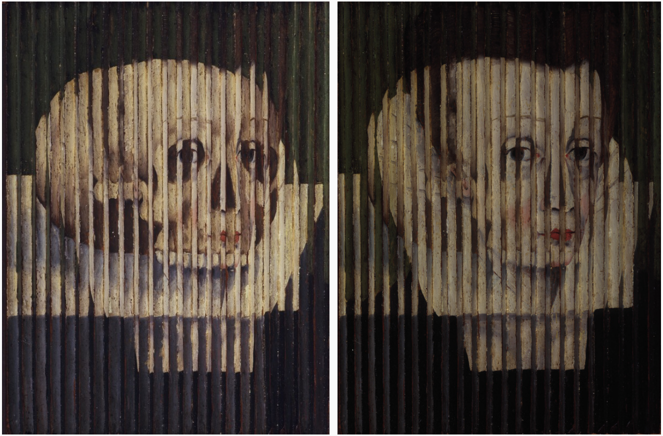


Figure 2: Mary, Queen of Scots, 1542 - 1587.

Desargues' Projective Geometry[1]

- Mathematics behind Alberti's Veil:
  Family of lines (light rays) through a point (eye) plus a plane (veil).
- Recall **Pappus' Theorem**:
  A_1, A_2, A_3, collinear;
  B_1, B_2, B_3, collinear;
  then, so are C_1, C_2, C_3.
- Blaise Pascal (1623-1662) at 16 generalized to conics.
- Desargues (1640) **Projective Geometry** only relies on a straight edge.
- Note: Piero della Francesca (c. 1415-1492) formalized rules of perspective, mid-1470s.

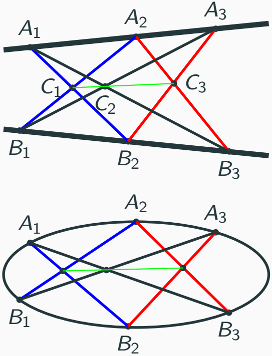
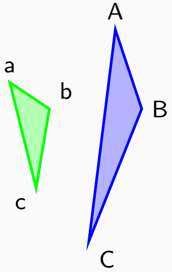


Figure 3: Pappus' and Pascal's Theorems.

[1]Two centuries ahead of his time.

Girard Desargues (1591-1661)

- Architect in Paris, Lyon and engineer.
- Desargues' Theorem in appendix of book on perspective, by friend Abraham Bosse (1602–1676).
  Two triangles are a) in ***perspective axially*** *if and only if they are b) in* ***perspective centrally****.*



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- b) Extend Aa, Bb, Cc
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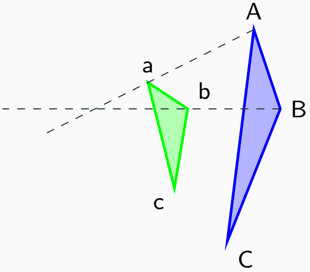

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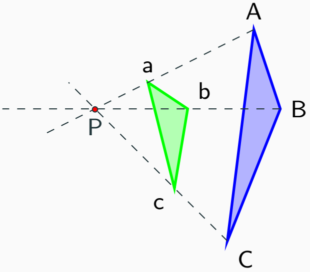

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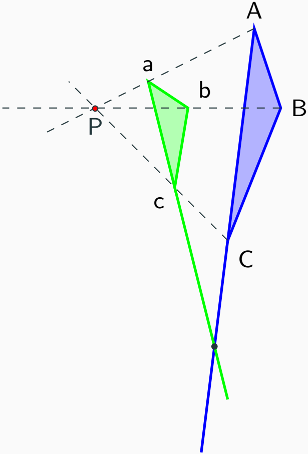

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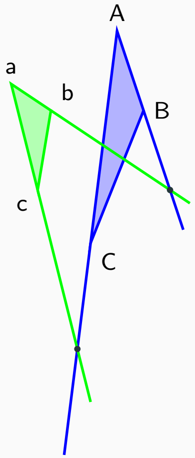

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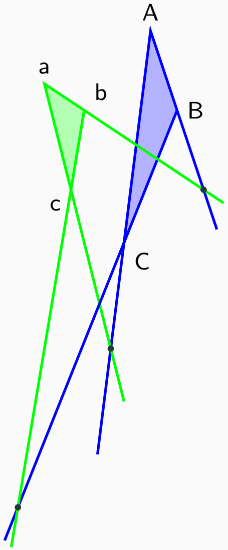

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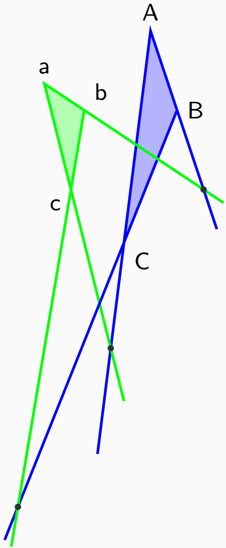

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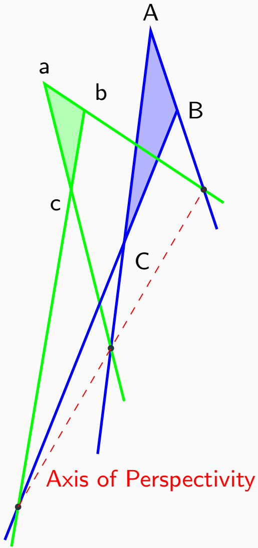
- b) Extend Aa, Bb, Cc **center of perspectivity** P.
- a) Extend pairs, AC-ac, AB-ab, etc.
- Points are collinear.



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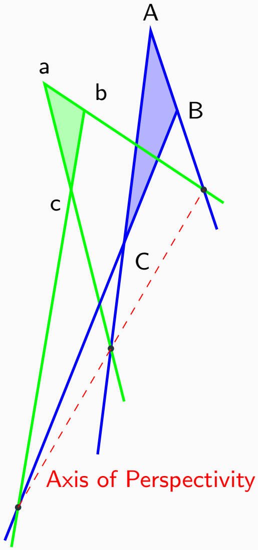
- Points are collinear.
- What if two sides are parallel?



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- b) Extend Aa, Bb, Cc
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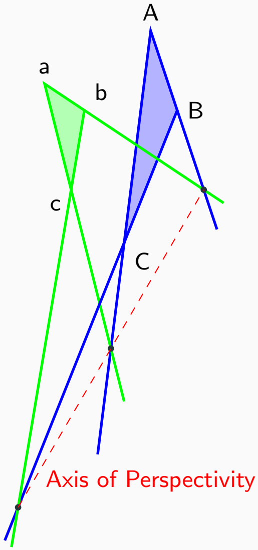
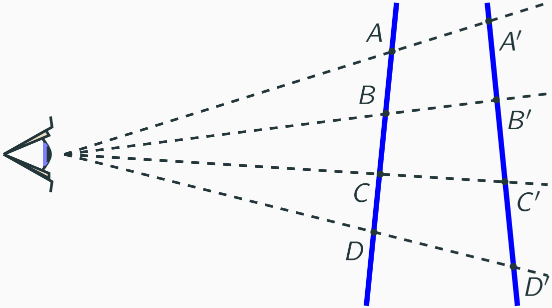
- a) Extend pairs, AC-ac, AB-ab, etc.
- Points are collinear.
- What if two sides are parallel?
- Need **Projective plane**.



Invariance of the Cross Ratio

Lengths and angles are not preserved under projection.



But, for any four points on a line, $\frac{\overline{AC}}{\overline{BC}} : \frac{\overline{AD}}{\overline{BD}}$ is invariant. That is,

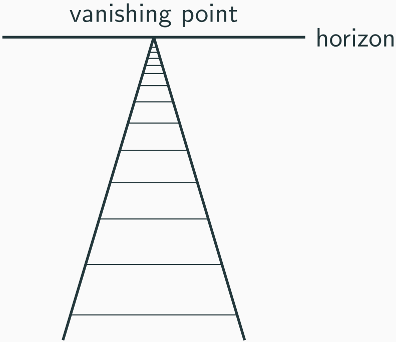
$$\frac{\overline{AC}}{\overline{BC}} : \frac{\overline{AD}}{\overline{BD}} = \frac{\overline{A'C'}}{\overline{B'C'}} : \frac{\overline{A'D'}}{\overline{B'D'}}.$$

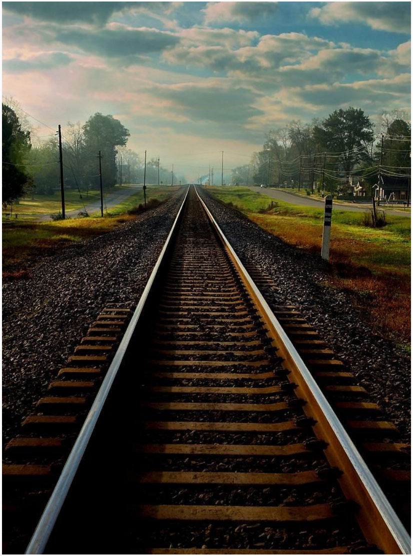
Projective Geometry Rebirth in 1800's.

Perspective

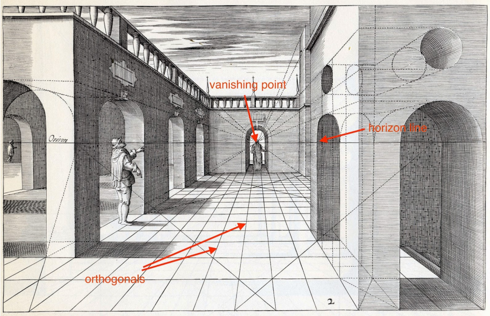
1. Parallel lines meet at a pt.
2. Lines map to lines.
3. Conics map to conics.

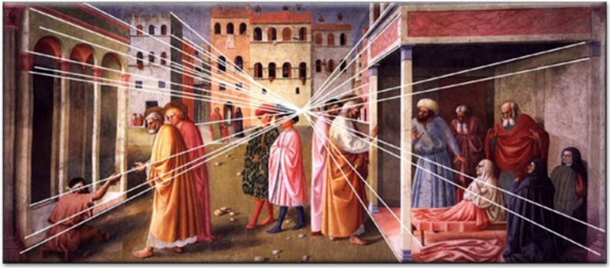
Example: Train tracks.





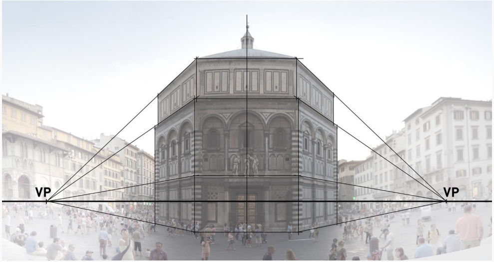
One Point Perspective - Find the Vanishing Points





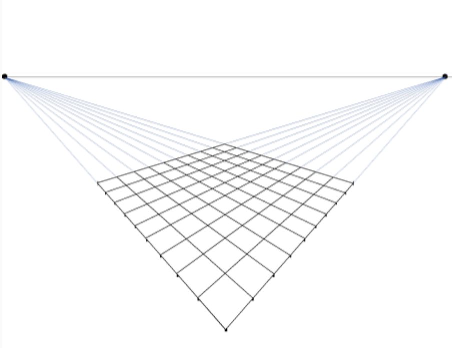
Two Point Perspective - Find the Vanishing Points

Two Point Perspective - Find the Vanishing Points



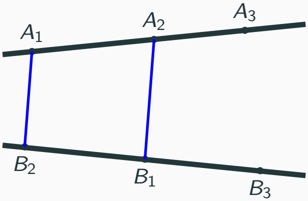
Two Point Perspective Vanishing Point(s)

Two Point Perspective Vanishing Point(s)



Points at Infinity

- Artists' use vanishing points.
- Pappus' Theorem -
  Consider parallel lines A_1B_2, A_2B_1.
  Does the theorem hold?



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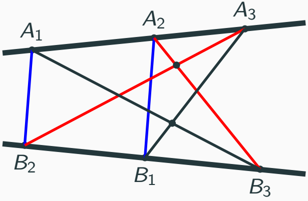

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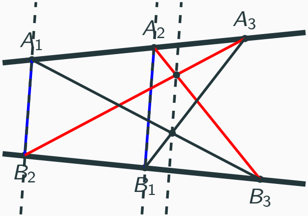

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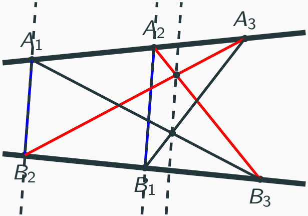
- Desargues - **line at infinity**.



Points at Infinity

- Artists' use vanishing points.
- Pappus' Theorem -
  Consider parallel lines A_1B_2, A_2B_1.
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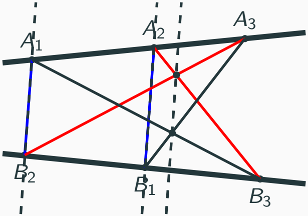
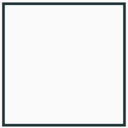
- Desargues - **line at infinity**.
- Look at a plane





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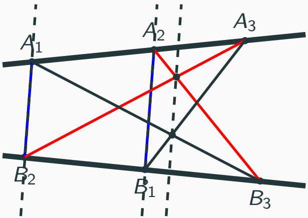

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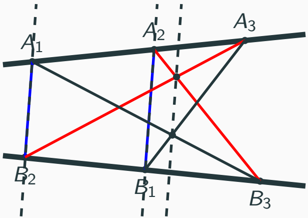
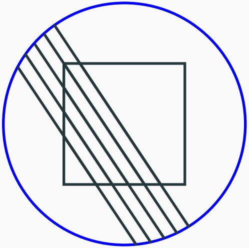
- Pappus' Theorem -
  Consider parallel lines A_1B_2, A_2B_1.
  Does the theorem hold?
- Desargues - **line at infinity**.
- Look at a plane
- Add parallel lines.
  Where do they go?
- Line at Infinity
- Plane + line at infinity =
  Projective Plane

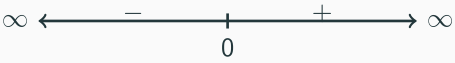




Line at infinity

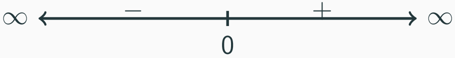
Projective Line

- Consider the real line, $\mathbb{R}$.



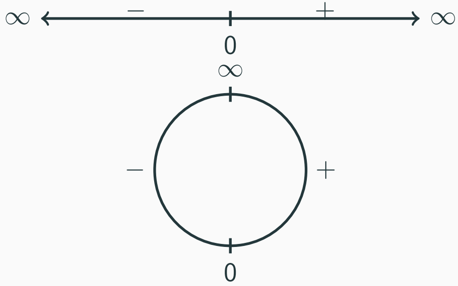
Projective Line

- Consider the real line, $\mathbb{R}$.
- Add point at infinity,
  real projective line, $\mathbb{RP}^1$.



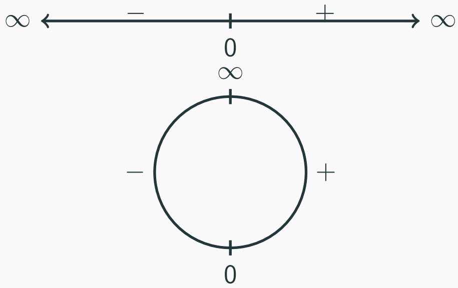
Projective Line

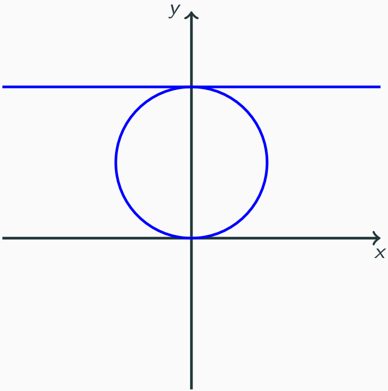
- Consider the real line, $\mathbb{R}$.
- Add point at infinity,
  real projective line, $\mathbb{RP}^1$.
- Topologically a circle!



Projective Line

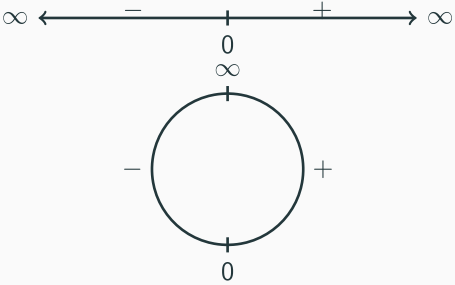
- Consider the real line, $\mathbb{R}$.
- Add point at infinity, real projective line, $\mathbb{RP}^1$.
- Topologically a circle!
- We can map the circle to $\mathbb{R}$.

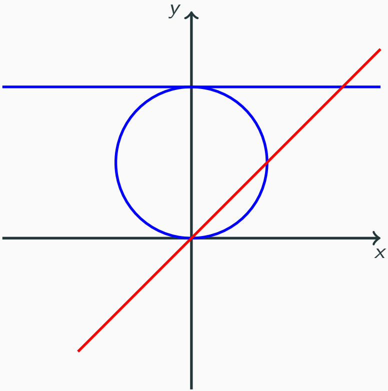




Projective Line

- Consider the real line, $\mathbb{R}$.
- Add point at infinity, real projective line, $\mathbb{RP}^1$.
- Topologically a circle!
- We can map the circle to $\mathbb{R}$.





Intersection: $y = b$, $y = mx$: $x = \frac{b}{m}$.

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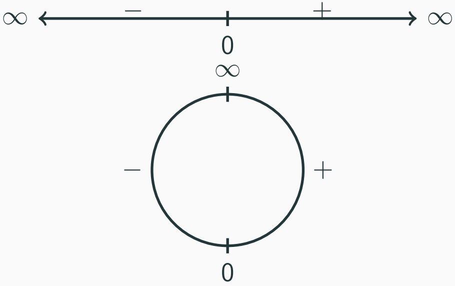
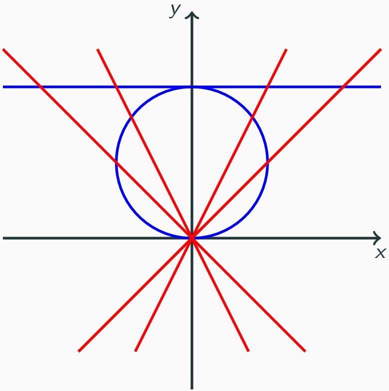


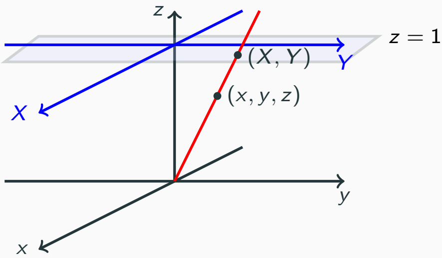


Intersection: $y = b$, $y = mx$: $x = \frac{b}{m}$.

Homogeneous Coordinates

- Point on line: (x, y, z)
- All points on line map to (X, Y) in the plane.
- (X, Y) are called homogeneous coordinates.
- Points on line are multiples, $(x', y', z') = \lambda(x, y, z)$.
- Point on plane: Let $\lambda = \frac{1}{z}$. Then, $(x', y', z') = (\frac{x}{z}, \frac{y}{z}, 1)$, or

$$X = \frac{x}{z}, \quad Y = \frac{y}{z}.$$



Curves: Given $Y = f(X)$, find $(x, y, z)-$surface.

- Curve in plane $z = 1$, $Y = X^2$.
- $X = \frac{x}{z}$, $Y = \frac{y}{z}$.
- Translates to

$$\frac{y}{z} = \left(\frac{x}{z}\right)^2.$$

- Multiply by z^2.
- This is a surface in (x, y, z)-space,

$$x^2 = yz.$$

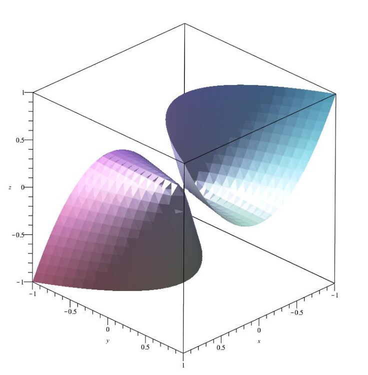


Figure 4: Surface $x^2 = yz$.

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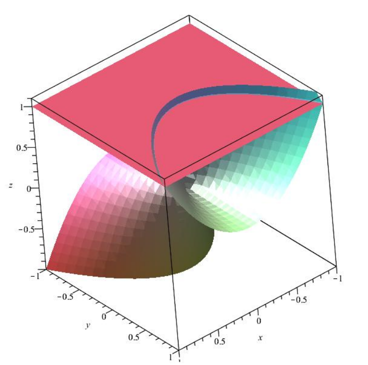

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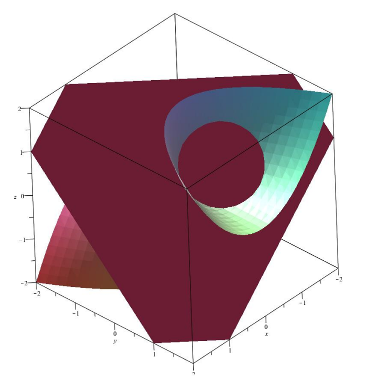


Figure 4: Surface $x^2 = yz$.

Projective Sphere: Extending $\mathbb{RP}^1$.

- Map points on a plane to points on surface of unit sphere, $\mathbb{S}^2$.
- Lines through South Pole uniquely intersect the plane and sphere.
- All points mapped except $(0,0,0)$. This point can be mapped to the line at infinity.
- Lines through origin are points of the real projective plane, $\mathbb{RP}^2$.

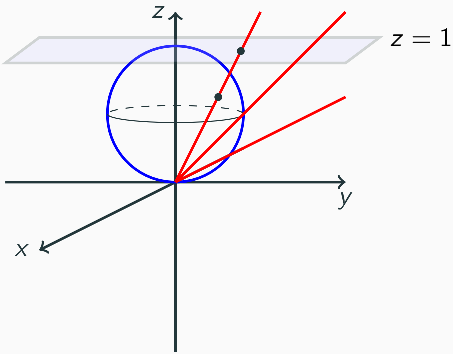
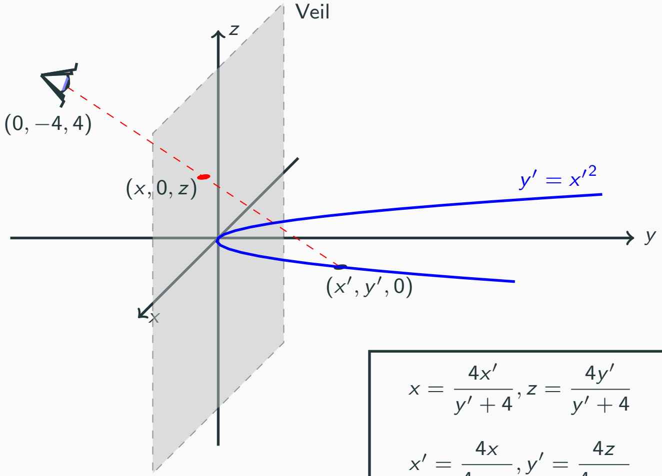


Figure 5: Stereographic Projection

Looking into the Veil - Parabola Projected



$$x = \frac{4x'}{y'+4}, z = \frac{4y'}{y'+4}$$

$$x' = \frac{4x}{4-z}, y' = \frac{4z}{4-z}$$

Figure 6: Problems 8.4.2-8.4.4

Looking into the Veil - Parabola Projected

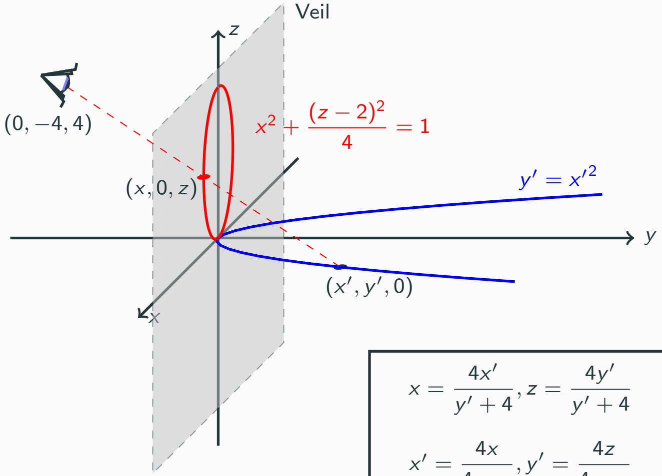


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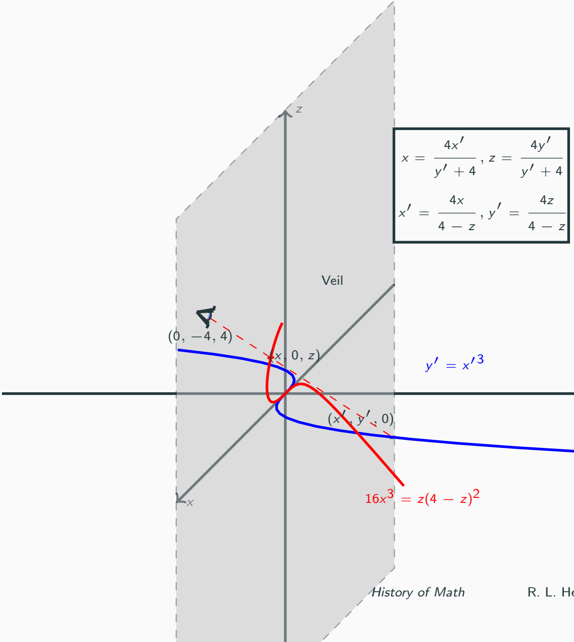
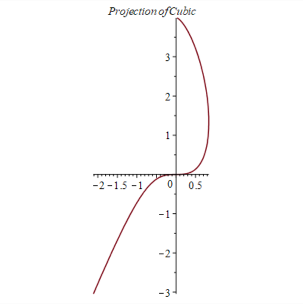
Figure 6: Problems 8.4.2-8.4.4

Viewing A Cubic in the Veil



$$x = \frac{4x'}{y' + 4}, \quad z = \frac{4y'}{y' + 4}$$

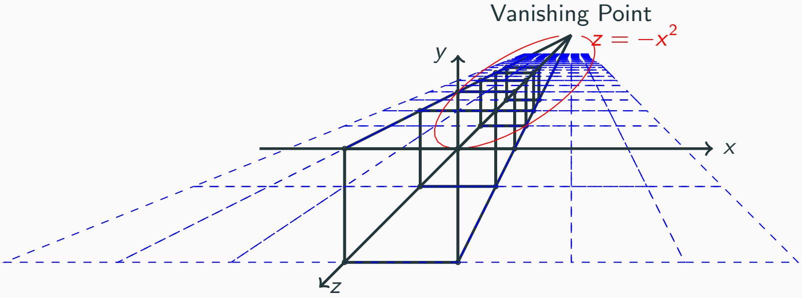
$$x' = \frac{4x}{4 - z}, \quad y' = \frac{4z}{4 - z}$$



Above: Maple plot of projected cubic to the veil.

Perspective Drawing

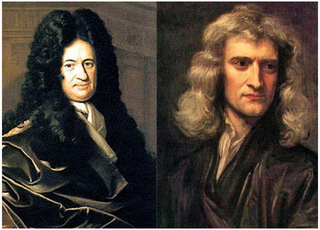
Looking at conics from a different perspective: The parabola $z = -x^2$ looks like an ellipse.



In the 1600's mathematicians had other mathematics to attend to. So, we return to geometry in the 1800's.

Emergence of Calculus

Fall 2025 - R. L. Herman



The Method of Exhaustion and the Infinite

- Zeno's Paradox of the Arrow

  "If a body moves from A to B, then before it reaches B it passes through the mid-point, say B_1 of AB. Now to move to B_1 it must first reach the mid-point B_2 of AB_1. Continue this argument to see that A must move through an infinite number of distances and so cannot move. " (450 BCE)

- Eudoxus - Method of Exhaustion.
- Archimedes - area of a segment of a parabola is $\frac{4}{3}$ the area of a triangle with the same base and vertex.
- Luca Valerio (1552-1618) published in 1608 *De quadratura parabolae.*
- Kepler (1571-1630): area as sum of lines. Inpsired Cavalieri's indivisibles.

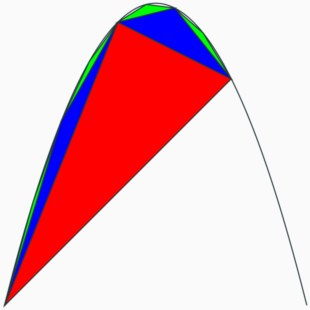


Figure 1: Archimedes: First known summation of series.

Area of blue $= \frac{1}{4}$ that of red, a. Then,

$$A = a + \frac{1}{4}a + \frac{1}{4^2}a + \ldots.$$

Developments in the 1600's

Rapid developments first 60 years of 1600's based on Greek geometry, algebra, astronomy (Kepler, Galileo). Led to unification of geometry and algebra.

- Descartes (1596-1650)
- Cavalieri (1598-1647)
- Fermat (1601-1665)
- Roberval (1602-1675)
- Wallis (1616-1703)
- Barrow (1630-1677)
- Gregory (1638-1675)
- Newton (1642-1727)
- Leibniz (1646-1716)

Two main problems

- Tangents
- Areas

Need curves

- Conics
- Archimedean spiral
- Conchoid
- Cissoid
- Cycloid

Sixteenth Century Science

- Copernicus (1473-1543)
  Commentary - 1514
  Dē revolutionibus orbium coelestium, 1542 on death bed.
- Tycho Brahe (1546-1601)
- Galileo Galilei (1564-1642)
  1609 - Telescope
  Jupiter's Moons, Moon, Saturn, Phases of Venus.
  1633 trial and judgement for heresy, under house arrest.
- Johannes Kepler (1571-1630)
  1609 - Study on Mars orbit.
  Laws of Planetary Motion

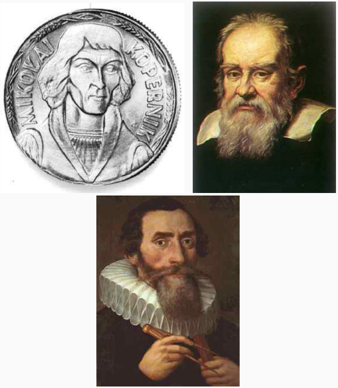


Figure 2: Copernicus, Galileo, Kepler

Seventeenth Century - French, German, English Mathematics

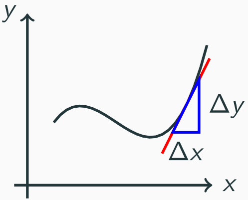
- 1590 Vièta, *The Analytic Art*
- Bonaventura Cavalieri (1598-1647)
- Evangelista Torricelli (1608-1647)
- John Napier (1550-1617) and Henry Briggs (1561-1631) - Introduced the logarithm
- French Mathematicians:
  René Descartes (1596-1650)
  Blaise Pascal (1623-1662)
  Pierre de Fermat (1601-1665)

- Descartes
  - philosopher, mathematician
  - *Discours de la méthode*, Marriage of algebra/geometry - analytic geometry
- Pascal
  - Wrote math before 16
  - Probability theory
  - Theology
- Fermat
  - Created analytic geometry
  - Contributions to Calculus
  - Number theory
  - Scribbled in Diophantus' *Arithmetica*

Tangents

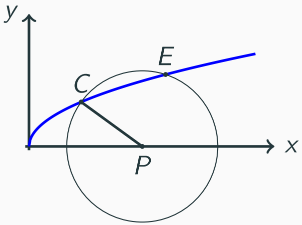
- Pierre de Fermat, René Descartes
- Both studied Apollonius' problem: construct a circle tangent to any three objects.
- Tangent line approximates curve at a point.
- Slope $\frac{\Delta y}{\Delta x}$.
- Infinitesimals - increments.
- Fermat:
  Method for maxima-minima
  1636 - Method of Tangents
- 1636 Letter: Descartes to Mersenne
  $dy = f(x+dx) - f(x) = ?dx$.

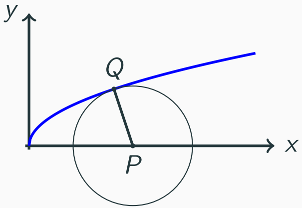
Figure 3: Fermat and Descartes



Descartes vs Fermat - Analytic Geometry, Tangents, Optics

- Decartes published *La Géométrie* - 1637
- Depicted $ax + by = c$ as a line.
- Introduced x, y.
- Fermat introduced analytic geometry earlier.
- Fermat interested in optimization.
- Fermat: lawyer in Toulouse, Math a hobby.
- Descartes denounced him and challenged him to find tangent to folium, $x^3 + y^3 = 3axy$.
- Descartes' Method of Tangents: Find circles tangent to curves.
- Fermat challenged Descartes to explain refraction. Fermat published in 1662.





Areas Under Curves

- First studied by Eudoxus, Archimedes
- Bonaventura Cavalieri (1598-1647) - *Geometria indivisibilibus continuorum nova quadam ratione promota,* 1635.

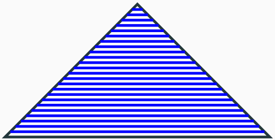
  Fill area with lines.
  But, an infinite number of lines sum to infinity.

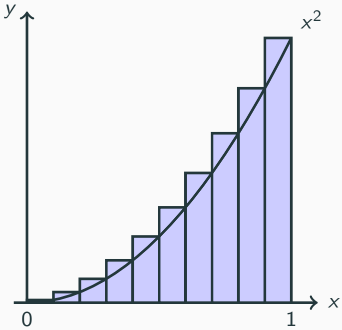
- Archimedes, John Wallis (1616-1703):

$$\int_0^1 x^2\,dx.$$

N segments of width $\frac{1}{N}$. and height $\left(\frac{k}{N}\right)^2$, $k = 1, 2, \ldots, N$.

$$A \approx \sum_{k=1}^{N} \frac{1}{N}\left(\frac{k}{N}\right)^2.$$





Cavalieri's Method of Indivisibles

John Wallis (1616-1703) - A Side Note

- 1649, Savilian professor of geometry at the University of Oxford.
- *Arithmetica Infinitorum*, "The Arithmetic of Infinitesimals", 1655
- Extended Cavalieri's law of quadrature.
- Algebraic vs Geometric approach.
- Influenced Newton.
- *Mathesis Universalis* Developed notation: introduced ∞, negative and fractional exponential notation.
- Royal Society of London in 1662.
- *Tractatus de Sectionibus Conicis,* 1659; described as curves using algebra.
- Published colleagues work on quadratures.

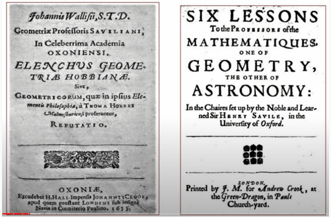

Johannis Wallisii, S.T.D.
Geometriæ Professoris SAVILIANI,
In Celeberrima Academia
OXONIENSI,
ELENCHUS GEOMETRIÆ HOBBIANÆ.
Sive,
GEOMETRICORUM, quæ in ipsius Elementis Philosophiæ, à THOMA HOBBES Malmesburiensi proferuntur,
REFUTATIO.
OXONIÆ,
SIX LESSONS
To the PROFESSORS of the
MATHEMATIQUES,
ONE OF
GEOMETRY,
THE OTHER OF
ASTRONOMY:
In the Chaires set up by the Noble and Learned Sir HENRY SAVILE, in the University of Oxford.
LONDON,
Printed by J. M. for Andrew Crook, at the Green-Dragon, in Pauls Church-yard.

Figure 4: Wallace-Hobbes Rivalry

- Thomas Hobbes (1588-1679), called book a "scab of symbols," quarter century controversy.
- Controversies with Huygens, Descartes, Fermat, Pascal.

Savilian Chairs of Geometry and Astronomy

- The Savilian Chairs of Geometry and Astronomy, University of Oxford, 1619.
- By Henry Savile (1549 - 1622)
- Click to see list.
- 1570 Lectures on Ptolemy
- " he felt that mathematics at that time was not flourishing. Students did not understand the importance of the subject, Savile wrote, there were no teachers to explain the difficult points, the texts written by the leading mathematicians of the day were not studied, and no overall approach to the teaching of mathematics had been formulated."
- Read more at MacTutor.

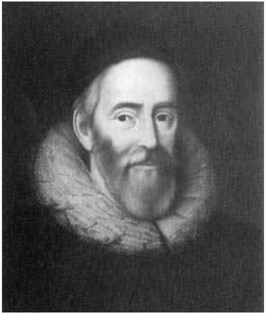


Figure 5: Henry Savile

Back to Wallis' Areas Under Curves

Find the sum

$$
\begin{aligned}
A &\approx \sum_{k=1}^{N} \frac{1}{N}\left(\frac{k}{N}\right)^2 \\
&= \frac{1}{N^3}\sum_{k=1}^{N} k^2 \\
&= \frac{1}{N^3}\frac{N(N+1)(2N+1)}{6} \\
&\sim \frac{2N^3}{6N^3} = \frac{1}{3}.
\end{aligned}
$$

Wallis showed

$$\boxed{\int_0^a x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_0^a = \frac{a^{n+1}}{n+1}}$$

for $k = 1, 2, \ldots, 9$.

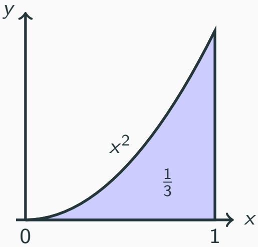
Note:

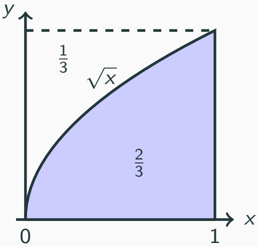
$$
\begin{aligned}
\sum_{k=1}^{N} k &= \underbrace{1 + \underbrace{2 + \cdots + (N-1)}_{2+(N-1)} + N}_{1+N} \\
&= N\frac{N+1}{2}.
\end{aligned}
$$

Figure 6: John Wallis

Integrating Powers, $\int x^k\,dx$,

- al Haytham (965-1039) $k = 1, 2, 3, 4$.
- Cavalieri (1635) knew for k up to 9.
- Proven in general by Fermat, Descartes, Roberval, 1630's.
- Fractional Powers (Fermat)
  Ex: $\int_0^1 \sqrt{x}\,dx$
  Use the symmetry in the figures.
- Areas under x^k, need sums $1^k + 2^k + \cdots n^k$.
- Volumes - use cylinders, $V = \pi r^2 h$.
  Sums needed: $1^{2k} + 2^{2k} + \cdots n^{2k}$.





Note: $1^3 + 2^3 + \cdots + k^3 = \frac{(N+1)^2 N^2}{4} = (1 + 2 + \cdots + k)^2$.

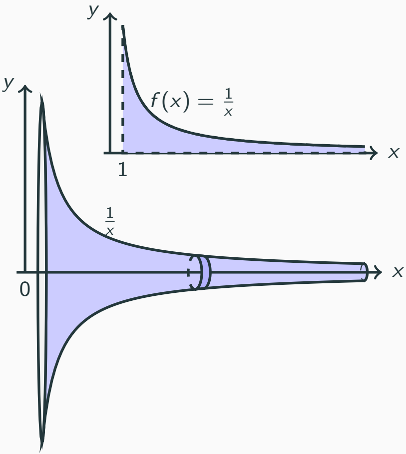
Evangelista Torricelli (1608-1647), barometer inventor

- Inverse Powers, x^{-1}
- Area under $y = \frac{1}{x}$.

$$\int_1^\infty \frac{1}{x}\,dx = \infty.$$

- 1641 Torricelli's trumpet (Gabriel's horn)

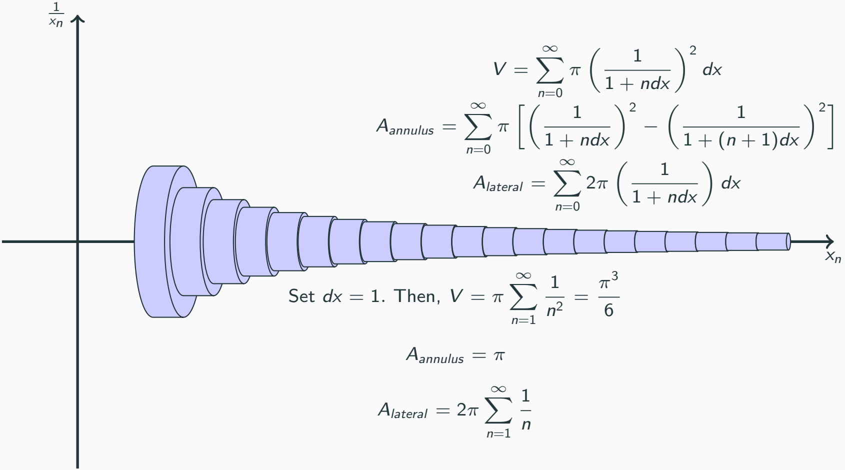
$$\begin{aligned} V &= \pi \int_1^\infty \frac{1}{x^2}\,dx = \pi. \\ A &= 2\pi \int_1^\infty \frac{dx}{x}\sqrt{1 + \left(-\frac{1}{x^2}\right)^2} \\ &> 2\pi \int_1^\infty \frac{1}{x}\,dx = \infty. \end{aligned}$$



What? You cannot paint the surface but can fill the trumpet with paint.

Hobbes - "to understand this for sense, it is not required that a man should be a geometrician or logician, but that he should be mad."

Gabriel's Wedding Cake - Discrete Case



$$V = \sum_{n=0}^{\infty} \pi \left(\frac{1}{1+ndx} \right)^2 dx$$

$$A_{annulus} = \sum_{n=0}^{\infty} \pi \left[\left(\frac{1}{1+ndx} \right)^2 - \left(\frac{1}{1+(n+1)dx} \right)^2 \right]$$

$$A_{lateral} = \sum_{n=0}^{\infty} 2\pi \left(\frac{1}{1+ndx} \right) dx$$

Set $dx = 1$. Then, $V = \pi \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^3}{6}$

$$A_{annulus} = \pi$$

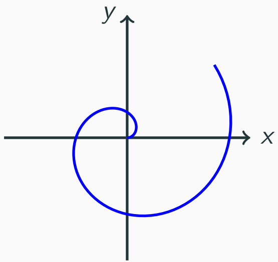
$$A_{lateral} = 2\pi \sum_{n=1}^{\infty} \frac{1}{n}$$

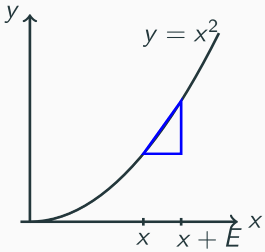
Tangents, Maxima, Minima

- Curves studied like Archimede's spiral, $r = a\theta$
- Fermat - studied polynomials
- Work simpler than Descartes
- Used infinitesimals, E
- **Example:** $y = x^2$

$$\frac{(x+E)^2 - x^2}{E} = 2x + E.$$

- Generalized to polynomials, $p(x, y) = 0$.





John Wallis' (1655) *Arithmetica Infinitorium*

- Combined Descartes' analytic geometry and Cavalieri's indivisibles.
- Some results already known.
- New approach to fractional powers.
- Ambivalent use of infinitesimals - attacked by Thomas Hobbes (1588-1679).
- 1632 Church banned infinitesimals.
- Formulae for π known by
  - Gregory, Newton, Leibniz
- Madhava (1350-1425) found π to 13 decimal places using series,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

Wallis' Formulae:

$$\begin{aligned}
\frac{\pi}{4} &= \frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots \\
\frac{\pi}{2} &= \left(\frac{2}{1}\cdot\frac{2}{3}\right)\cdot\left(\frac{4}{3}\cdot\frac{4}{5}\right)\cdot\cdots \\
\frac{4}{\pi} &= 1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cdots}}}
\end{aligned}$$

Already known formula:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Isaac Newton (1642-1727)

- Major use of infinite series
- *A Treatise of the Methods of Series and Fluxions*
- *Quadrature of the Hyperbola*
  Written in 1665,
  1st publication in 1668 by Mercator
- Akin to decimal expansions - powers of $\frac{1}{10}$ replaced by x^n.
- Example:

$$\log(1+x) = \int_0^x \frac{dt}{1+t}$$

[Note: Here $\log x = \ln x$.]

Note: Geometric series

$$\begin{aligned} 1 + t + t^2 + \cdots &= \frac{1}{1-t}, |t| < 1. \\ 1 - t + t^2 - \cdots &= \frac{1}{1+t}, |t| < 1. \end{aligned}$$

Then,

$$\begin{aligned} y &= \log(1+x) \\ &= \int_0^x (1 - t + t^2 - \cdots)\, dt \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots . \end{aligned}$$

Invert Power Series

We have for $y = \log(1+x)$,

$$y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots .$$

In order to invert the series, let $x = a_0 + a_1 y + a_2 y^2 + \cdots$. Then,

$$\begin{aligned} y &= \left(a_0 + a_1 y + a_2 y^2 + \cdots\right) - \frac{1}{2}\left(a_0 + a_1 y + a_2 y^2 + \cdots\right)^2 + \cdots \\ &= a_0 - \frac{1}{2}a_0^2 + \frac{1}{3}a_0^3 + a_1\left(a_0^2 - a_0 + 1\right) y \\ &\quad + \left[a_2\left(a_0^2 - a_0 + 1\right) + \left(a_0 - \frac{1}{2}\right)\right] y^2 \\ &\quad + \left[\frac{a_1^3}{3} + a_1 a_2 (2a_0 - 1) + a_3\left(a_0^2 - a_0 + 1\right)\right] y^3 + \ldots . \end{aligned}$$

Equate coefficients of powers of y, then ...

Series Inversion (cont'd)

We solve the resulting system of equations:

$$\begin{aligned}
0 &= a_0 - \frac{1}{2}a_0^2 + \frac{1}{3}a_0^3 \\
1 &= a_1\left(a_0^2 - a_0 + 1\right) \\
0 &= a_2\left(a_0^2 - a_0 + 1\right) + \left(a_0 - \frac{1}{2}\right) \\
0 &= \frac{a_1^3}{3} + a_1 a_2(2a_0 - 1) + a_3\left(a_0^2 - a_0 + 1\right).
\end{aligned}$$

The first equation gives $a_0 = 0$. The next two give $a_1 = 1$ and $a_2 = \frac{1}{2}$.

Continuing Newton found that

$$a_0 = 0, a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{6}, a_4 = \frac{1}{24}, \ldots, a_n = \frac{1}{n!}.$$

Newton's Series for Exponential

So far, inversion of

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots .$$

led to

$$x = y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots .$$

However,

$$y = \log(1+x) \Rightarrow x = e^y - 1.$$

So, we found the series expansion

$$e^y = 1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots .$$

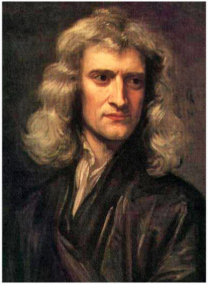


Figure 7: Newton

Newton's Series for Sine

Newton knew $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$.

Recall binomial series: $(a+b)^n = \sum_{k=0}^n C_{n,k} a^{n-k} b^k$, where the coefficients are $C_{n,k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$. Then,

$$
\begin{aligned}
(1+a)^p &= 1 + pa + \frac{p(p-1)}{2!}a^2 + \frac{p(p-1)(p-2)}{3!}a^3 + \cdots . \\
\sin^{-1} x &= \int_0^x \frac{dt}{\sqrt{1-t^2}}, \quad a = -t^2, p = -\frac{1}{2}, \\
&= \int_0^x \left(1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \cdots + \frac{-\frac{1}{2}(-\frac{3}{2})\cdots(\frac{1}{2}-k)}{k!}(-t^2)^k + \cdots \right) \\
&= x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \cdots + \frac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots 2k}\frac{x^{2k+1}}{2k+1} + \cdots
\end{aligned}
$$

Inverting, Newton found $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$.

Gottfried Wilhelm Leibniz (1646-1716)

- Librarian, philosopher, diplomat, doctorate in law.
- First papers in calculus (1684).
- Led to long dispute.
- Better notation, $\frac{dy}{dx}, \int\, dx$.
- Sum, product, quotient rules.
- Proved Fundamental Theorem of Calculus, $\frac{d}{dx}\int f(x)\,dx = f(x)$.

Figure 8: Leibniz

Infinite Series

- Geometric series,
  Known to Euclid (Zeno's paradox)
  Leonhard Euler (1707-1783)

  $$a+ar+ar^2+\cdots+ar^n+\cdots = \frac{a}{1-r}, |r| < 1.$$

- Harmonic Series - Oresme (1350)

$$\begin{aligned} & 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots \\ = & (1)+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots \\ \geq & \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots=\infty. \end{aligned}$$

- Power series - 17th Century,
  Gregory, Wallis, Taylor, Mclaurin, ...

Figure 9: Euler

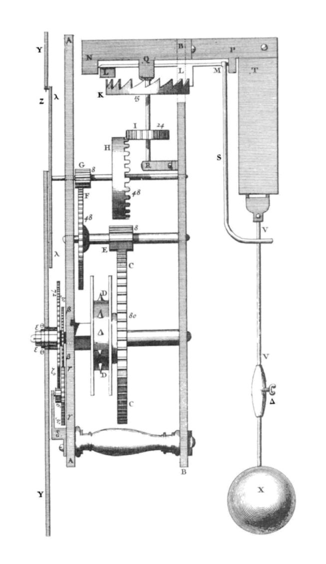
James Gregory (1638 - 1675)

- Uncle to David Gregory (1659-1708), Professor of Mathematics, University of Edinburgh at 24, filling the chair previously held by James Gregory. Savilian Professor of Astronomy, Oxford, Supported Newton in controversy and first to teach *Principia*.
- First to publish and prove Fundamental Theorem of Calculus, *Geometriae Pars Universalis* (1668) .
- Discovered 7 series before Taylor.
- *Optica Promota,* first practical reflecting (Gregorian) telescope.
- Worked with Angeli at Padua.

Figure 10: James and David Gregorie

Problems of the Day

- Pendulum clock of Galileo - Thought isochronous, time-independent period of swing. Son Vincenzio, worked on it, died 1649.
- Huygens built first pendulum driven clock, 1656.
- Tautochrone - Time taken independent of starting point, Huygens, 1659.
- Brachistochrone - Curve of fastest descent, posed by Johann Bernoulli, 1696.
- Isochrone - Connects points of equal time travel, Leibniz 1687, Jacob Bernoulli 1690.

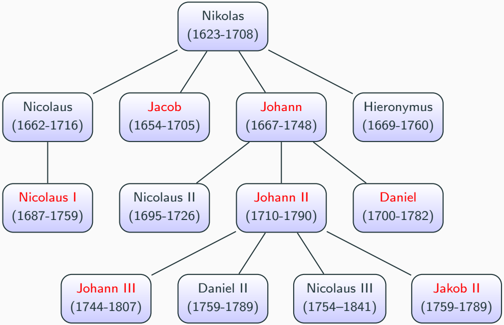


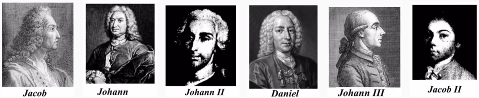
Calculus Wars

- Newton: 'the method of fluxions'.
- Paper on fluxions in 1666, but not published for decades.
- *Philosophiae naturalis principia mathematica*, published 1687.
- Little explicit calculus.
- Method of fluxions appeared in 1693.
- Leibniz, published first paper on calculus, 1684.
- Said he discovered calculus in 1670s.
- In 1695, Wallis: Leibniz learned about calculus from Newton.
- Nicolas Fatio de Duillier (1664-1753) in 1699 book, Newton's absolute priority.
- Angry responses from Johann Bernoulli and Leibniz.
- John Keill accused Leibniz of plagiarism, 1711.
- Royal Society in England gave report that Leibniz had concealed knowledge of Newton's work, 1712.
- Leibniz accused Newton and followers of stealing his calculus.
- Debate ended when Leibniz died, 1716.

The Bernoulli Family

- In three generations, there were 8 mathematicians.
- Dominated mathematics and physics, 17-18th centuries - with Newton, Leibniz, Euler, Lagrange, etc.
- Contributions: calculus, geometry, mechanics, probability, ballistics, thermodynamics, hydrodynamics, optics, elasticity, magnetism, astronomy.





Basel Problem (1644)

- Posed by Pietro Mengoli (1626-1686).

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

- Jacob and Johann Bernoulli (1704) tackled. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = ?$

Figure 11: Jacob and Johann

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n(n+1)} &= \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \left(\frac{1}{N} - \frac{1}{N+1}\right) \\
&= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots \left(\cancel{\frac{1}{N}} - \frac{1}{N+1}\right) \\
&= 1 - \frac{1}{N+1} \underset{N \to \infty}{\Longrightarrow} 1.
\end{aligned}$$

Euler's Solution of Basel Problem - 1734

- Descartes' Factor Theorem
- $p(x)$ - polynomial
- $p(r) = 0$ implies
  $p(x) = (x - r)q(x)$,
  $q(x)$ - polynomial

Proof:

$$\begin{aligned} p(x) &= a_0 + a_1 x + \cdots + a_n x^n \\ p(y) &= a_0 + a_1 y + \cdots + a_n y^n \\ p(x) - p(x) &= a_1(x - y) + \cdots + a_n(x^n - y^n) \end{aligned}$$

$$x^n - y^n = (x - y)\left(x^{n-1} + x^{n-2}y + \cdots + y^{n-1}\right)$$

Let $y = r$,

$$\begin{aligned} p(x) &= (x - r)[a_1 + a_2(x + r) + \cdots + a_n(x^{n-1} + x^{n-2}r + \cdots + r^{n-1})] \\ &= (x - r)q(x). \end{aligned}$$

Leonhard Euler's Solution of Basel Problem

$\sin x$ has roots $n\pi$, $n = 0, \pm 1, \pm 2, \dots$. - Generalize Factor Theorem:

$$\begin{aligned}
\sin x &= Ax\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\cdots \\
&= Ax\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots \\
&= A\left[x - \frac{x^3}{\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) + x^5(\cdots) - \cdots\right].
\end{aligned}$$

Compare to

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots .$$

Then $A = 1$, and

$$\frac{1}{3!} = \frac{1}{\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) \Rightarrow \zeta(2) \equiv \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

What are the next coefficients?

We need the x^5 terms in the expansion

$$\begin{aligned}\sin x &= x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)\\ &= x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\cdots\left(1-\frac{x^2}{m^2\pi^2}\right)\cdots\end{aligned}$$

We multiply $\frac{x^2}{m^2\pi^2}$ times the factors $\frac{x^2}{n^2\pi^2}$, $n\neq m$, and summing:

$$x\sum_{m=1}^{\infty}\frac{x^2}{m^2\pi^2}\sum_{n=1,n\neq m}^{\infty}\frac{x^2}{n^2\pi^2}=\frac{x^5}{\pi^4}\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{n=1,n\neq m}^{\infty}\frac{1}{n^2}.$$

$$\frac{\pi^4}{5!}=\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m^2}\left[\zeta(2)-\frac{1}{m^2}\right]=\frac{1}{2}[\zeta(2)^2-\zeta(4)].$$

So, $\zeta(4)\equiv\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{36}-\frac{\pi^4}{60}=\frac{1}{12}\left(\frac{\pi^4}{3}-\frac{\pi^4}{5}\right)=\frac{\pi^4}{90}.$

Another Approach to Obtain $\zeta(4)$

Noting that $\frac{d}{dx}(\ln\sin x) = \cot x$ and using the known series expansion for $x\cot x$ in terms of Bernoulli numbers,

$$\begin{aligned}
\sin x &= x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)\\
\ln\sin x &= \ln x + \sum_{n=1}^{\infty}\ln\left(1-\frac{x^2}{n^2\pi^2}\right)\\
x\cot x &= 1-\sum_{n=1}^{\infty}\frac{2x^2}{n^2\pi^2-x^2}\\
&= 1-\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{x^2}{n^2}\sum_{k=0}^{\infty}\left(\frac{x^2}{n^2\pi^2}\right)^k\\
1-\frac{x^2}{3}-\frac{x^4}{45}-\frac{2x^6}{945}+\dots &= 1-\frac{2}{\pi^2}\sum_{k=0}^{\infty}\left(\frac{x}{\pi}\right)^{2k+2}\zeta(2k+2)\\
x\cot x &= 1+\sum_{k=0}^{\infty}(-1)^k B_{2k}(2x)^{2k}
\end{aligned}$$

Results for the Riemann Zeta Function, $\zeta(s)$

Therefore, we have

$$\begin{aligned}\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots &= \frac{\pi^2}{6},\\ \zeta(2n) = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots &= (-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n},\end{aligned}$$

where B_{2n} are Bernoulli numbers,[1] $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, \ldots,$

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Euler (1748) - Zeta function can be defined for p prime as

$$\begin{aligned}\zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\\ &= \left(1 - \frac{1}{2^s}\right)^{-1}\left(1 - \frac{1}{3^s}\right)^{-1}\left(1 - \frac{1}{5^s}\right)^{-1}\cdots\left(1 - \frac{1}{p^s}\right)^{-1}\cdot.\end{aligned}$$

[1] Jacob Bernoulli, 1713, Seki Takakazu, 1712, published posthumously. $B_0 = 1, B_1 = -\frac{1}{2}$.

Georg Friedrich Bernhard Riemann[2] (1826-1866)

- Riemann extended Euler's zeta function, $s \in \mathbb{C}$

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots .$$

- Values

$$\begin{aligned} &\zeta(1) = \infty, \text{ harmonic series} \\ &\zeta(2) = \tfrac{\pi^2}{6} \\ &\zeta(3) \quad \text{irrational, Apery (1981)} \end{aligned}$$

- Zeros

  $\zeta(-2n) = 0$, n integer > 0.

  Riemann Hypothesis:

  $\zeta(\sigma + it) = 0$ when $\sigma = \frac{1}{2}$.

- Connection to primes?

Figure 12: Bernhard Riemann

$$\begin{aligned} \zeta(s) &= \frac{1}{\Gamma(2s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx \\ \zeta(2n) &= \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!} \\ \zeta(s) &= 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s) \\ \Gamma(s) &= \int_0^\infty x^{s-1} e^{-x}\, dx \end{aligned}$$

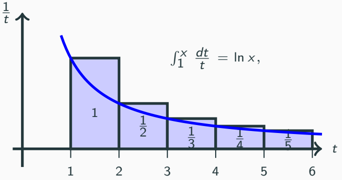
[2] *On the Number of Primes Less Than a Given Magnitude*, 1859

Connection to Primes and Other Tidbits

$$\begin{aligned}
\zeta(s) &= \left(1-\frac{1}{2^s}\right)^{-1}\left(1-\frac{1}{3^s}\right)^{-1}\cdots\left(1-\frac{1}{p^s}\right)^{-1}\cdots \\
&= \prod_{p=\text{prime}}\left(1-\frac{1}{p^s}\right)^{-1} \\
&= \prod_{p=\text{prime}}\left[1+\frac{1}{p^s}+\left(\frac{1}{p^s}\right)^2+\cdots\right]
\end{aligned}$$

- Primes less than $x \sim \int_2^x \frac{dt}{\log t}$
- Euler-Mascheroni Constant $\gamma \approx 0.577218\ldots.$
- Generalizing $n!$

$$\Gamma(n+1)=n\Gamma(n), \Gamma(0)=1.$$



$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right)=\gamma.$$

Euler Eta Function

The Euler, or Dirichlet, eta function is

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}, \quad \mathrm{Re}(s) > 0. \tag{1}$$

It is related to the zeta function by

$$\eta(s) = \left(1 - 2^{1-s}\right) \zeta(s).$$

Special values of the eta function are

$$\begin{aligned}
\eta(0) &= 1 - 1 + 1 - 1 + \cdots = \frac{1}{2}, \\
\eta(-1) &= 1 - 2 + 3 - 4 + \cdots = \frac{1}{4}, \\
\eta(1) &= 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \cdots = \ln 2. \qquad (2)
\end{aligned}$$

$1 + 2 + 3 + \cdots = -\frac{1}{12}.$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{Re}(s) > 1. \tag{3}$$

Note that

$$\zeta(-1) = 1 + 2 + 3 + \cdots .$$

But, the Riemann zeta function is not defined for $s = -1$.

So, we can use $\eta(s)$ to analytically continue $\zeta(s) = \dfrac{\eta(s)}{1 - 2^{1-s}}$.

Setting $s = -1$, we obtain

$$1 + 2 + 3 + \cdots = \frac{\eta(-1)}{1 - 2^2} = -\frac{1}{12},$$

assuming that $\eta(-1) = \dfrac{1}{4}$, using Abel summation.

Leonhard Euler (1707-1783)

Euler (at 14) studied under Johann Bernoulli, graduated in 1723.

Went to St. Petersburg in 1727, Berlin in 1741, and back to St. Petersburg in 1766.

By 1730's - lost vision in right eye and blind by 1771.

866 books and papers - 228 after death. *Opera Omnia* - over 25,000 pgs

First appearance of e - letter to Goldbach in 1731.

Euler published *Introductio in Analysin infinitorum* - 1748

Euler's Formula, $e^{ix} = \cos x + i \sin x$.

Euler's Identity, $e^{i\pi} + 1 = 0$.

Euler's constant, γ

Euler's Polyhedral Formula, $V + F = E + 2$.

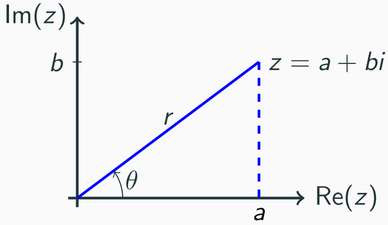
Amicable Numbers

- Recall Greeks knew 220 and 284; i.e., sum of proper factors of 220 = 284 and vice versa.
- Thabit ibn Qurra (836-901) discovered the next amicable pairs, for example 17296, 18416.
- Pierre Fermat rediscovered this pair in 1636.
- René Descartes discovered Qurra's pair 9,363,584 and 9,437,056 in 1638.
- 1747, Euler published [E100] giving 30 amicable pairs.
- By 1750 - Euler found 61 pairs!

Euler's Formula - Exponentiate $i\theta$.

- Complex numbers, polar form.
  $z = a + bi$, $a = r\cos\theta, b = r\sin\theta$

$$\begin{aligned} z &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta). \end{aligned}$$



- Exponential of imaginary number

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\ &= 1 + i\theta - \frac{(\theta)^2}{2!} - i\frac{(\theta)^3}{3!} + \cdots \\ &= \left(1 - \frac{(\theta)^2}{2!} + \cdots\right) + i\left(\theta - \frac{(\theta)^3}{3!} + \cdots\right) \\ e^{i\theta} &= \cos\theta + i\sin\theta. \end{aligned}$$

Euler's Formula Applications $e^{i\theta} = \cos\theta + i\sin\theta$.

- $\theta = \pi,\ e^{i\pi} = -1$, or $e^{i\pi} + 1 = 0$.
- $\left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n\ e^{in\theta} = \cos n\theta + i\sin n\theta$ implies **de Moivre's Theorem**

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n .$$

- **Example:** $n = 2$

$$\begin{aligned}\cos 2\theta + i\sin 2\theta &= (\cos\theta + i\sin\theta)^2 \\ &= \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta.\end{aligned}$$

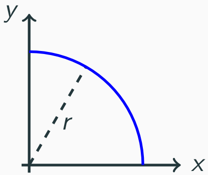
$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta.\end{aligned}$$

Rectification of a Circle - Recalling Calculus II

- Rectification = Finding arclengths
- The length of the curve $y = y(x)$:

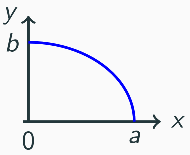
$$L = \int_a^b \sqrt{1 + y'^2}\, dx.$$

- **Example:** Circle: $x^2 + y^2 = r^2$
  $2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$



$$\begin{aligned} L &= 4\int_0^r \sqrt{1 + \frac{x^2}{y^2}}\, dx \\ &= 4\int_0^r \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx \\ &= 4r\int_0^r \frac{dx}{\sqrt{r^2 - x^2}} = 4r \sin^{-1} 1 = 4r\left(\frac{\pi}{2}\right) = 2\pi r. \end{aligned}$$

Arclength of an Ellipse



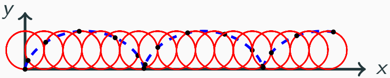
- **Example:** Ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad x \geq 0, y \geq 0.$$

$$\begin{aligned} y &= \frac{b}{a}\sqrt{a^2 - x^2} \\ y' &= \frac{bx}{a\sqrt{a^2 - x^2}} \\ 1 + y'^2 &= \frac{a^2 - k^2x^2}{a^2 - x^2}, \quad k = \frac{a^2 - b^2}{a^2} \\ L &= 4\int_0^a \sqrt{\frac{a^2 - k^2x^2}{a^2 - x^2}}\, dx. \end{aligned}$$

Historical Curves

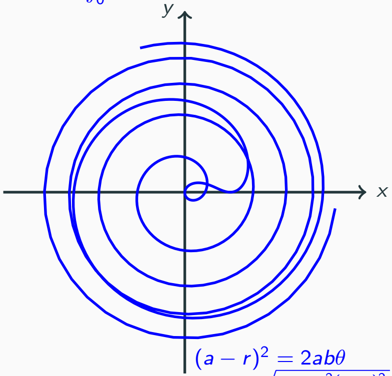
- 1609 - Kepler - Mars' orbit is an ellipse.
- 1659 - Pascal *Dimensions des lignes courbes de toutes les Roulettes.* Roulette curves.
- 1658 Proof by Wren published by Wallis in 1659 - On the rectification of the cycloid.
- 1676 - Newton - infinite series.
- 1742 - Maclaurin - expansion in eccentricities.
- 1691 - Jacob Bernoulli - parabolic spiral.

Cycloid, Parabolic Spiral, and Lemniscate



$x = r(t - \sin t),\ y = r(1 - \cos t)$

$L = \int_0^{2\pi} r\sqrt{2 - 2\cos t}\, dt = 8r$

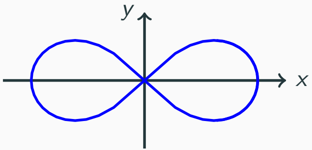


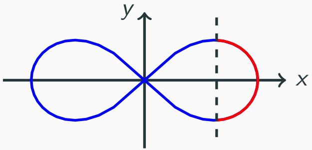
$(a - r)^2 = 2ab\theta$

$s = \int \sqrt{1 + \frac{r^2(a-r)^2}{a^2b^2}}\, dr$

- **Example:** Lemniscate, $r^2 = \cos 2\theta$

$$L = 4\int_0^1 \frac{dr}{\sqrt{1 - r^4}}$$





Elastica

Sep 1694, Jacob Bernoulli

Oct 1694, Johann Bernoulli

Elliptic Functions

- Lemniscate integral leads to new functions, $u = \int_0^x \frac{dt}{\sqrt{1-t^4}}$.
- Compare to $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$.
- Elliptic Integrals: $\int R(t, \sqrt{p(t)})\, dt$, R is rational function, $p(t)$ is polynomial of degree 3 or 4.
- Bernoulli (1694) - geometry, mechanics.
- Fagnano (1682-1766) - Doubling arc of lemniscate, 1718.
- Carl Friedrich Gauss (1777-1855) ∼1800 studied inverse $x = sl(u)$ Doubly periodic functions

$$sl(u + 2\bar{\omega}) = sl(u), \quad sl(u + 2i\bar{\omega}) = sl(u)$$

$$\bar{\omega} = 2\int_0^1 \frac{dt}{\sqrt{1-t^4}} = 2.62205\ldots.$$

- Rediscovered by Niel Henrik Abel (1802-1829) and Carl Gustav Jacobi (1804-1851) in 1820's.

Addition Theorem for Circle

- **Example** Circle

$$\begin{aligned} \sin 2u &= 2 \sin u \cos u \\ &= 2 \sin u \sqrt{1 - \sin^2 u} \end{aligned}$$

- Let $u = \sin^{-1} x$. Then,

$$\begin{aligned} 2u &= 2 \int_0^x \frac{dt}{\sqrt{1-t^2}} \\ &= \sin^{-1}\left(2 \sin u \sqrt{1 - \sin^2 u}\right) \\ &= \sin^{-1}\left(2x\sqrt{1-x^2}\right) \\ 2\int_0^x \frac{dt}{\sqrt{1-t^2}} &= \int_0^{2x\sqrt{1-x^2}} \frac{dt}{\sqrt{1-t^2}}. \end{aligned}$$

Elliptic Integral Addition Theorem for Lemniscate

In 1718 Fagnano found formula for doubling arclength of lemniscate.

He solved differential equation

$$\frac{dt}{\sqrt{1-t^4}} = \frac{2dx}{\sqrt{1-x^4}}, \Rightarrow t = \frac{2x\sqrt{1-x^2}}{1+x^4}.$$

So, if the arclength of lemniscate is

$$\int_0^x \frac{dt}{\sqrt{1-t^4}},$$

then double the arclength is

$$\int_0^{\frac{2x\sqrt{1-x^2}}{1+x^4}} \frac{dt}{\sqrt{1-t^4}}.$$

Led Euler to write extensively on elliptic integrals starting in 1752.

Elliptic Integrals

- Study of Inversions
  Gauss 1790s - $\int \frac{dt}{\sqrt{1-t^3}}$,
  Abel 1823 (pub 1827)
  Jacobi 1829 book
- Legendre - two papers 1786 (40 yrs earlier)
  Legendre classified elliptic integrals into 3 cases,
  Produced 3 volumes, 1811-1816. Examples:

$$F(\phi) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad E(\phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \theta}\, d\theta.$$

- Riemann's geometric setting, Riemann surfaces, 1851 - torus.

Gauss' *AGM* - Arithmetic-geometric mean

- Gauss's constant $G = \frac{1}{AGM(1,\sqrt{2})} = \frac{2}{\pi}\int_0^1 \frac{dx}{\sqrt{1-x^4}} = 0.8346268\ldots$
- Between 1 and $\sqrt{2}$ is $\dfrac{\pi}{\bar{\omega}} = \dfrac{1}{G}$.
- Arithmetic mean $\dfrac{a+b}{2}$.
- Geometric mean $\dfrac{a}{g} = \dfrac{g}{b} \Rightarrow g = \sqrt{ab}$.
- $AGM(a,b)$ algorithm: Start with $a_0 = a, \quad b_0 = b,$

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad n = 0, 1, \ldots.$$

- Gauss - $AGM(1,\sqrt{2}) = \dfrac{\pi}{\bar{\omega}}$ to 11 decimal places.
- Led to study of general theory, modular functions, theta functions - Ramanujan (early 1900s).

Application of $AGM(a, b)$

Example: $AGM(1, 2)$. Start with $a_0 = 1, \quad b_0 = 2,$

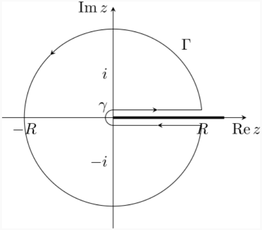
$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad n = 0, 1, \ldots.$$

a_n	b_n
1.0000	2.0000
1.5000	1.4142
1.4571	1.4565
1.4568	1.4568
$\vdots$	$\vdots$

$$AGM(a, b) = \frac{\pi}{4} \frac{a + b}{K\left(\frac{a-b}{a+b}\right)}, \quad K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

Complex Analysis

Fall 2025 - R. L. Herman



History of Complex Analysis

- Before 1600
  - Cardano 1545, quadratic
  - Bombelli 1572, cubic
  - Harriot 1600, quartic
  - Negative roots - false
  - Complex roots - impossible
- 1600s
  - Descartes, 1637, $a + b\sqrt{-1}$
  - Wallis 1685
  - Insights from geometry trigonometry, conics - justified
- 1700s
  - Bernoulli - integral transformation
  - Euler - Euler's formula, i
  - Gauss (1799, 1815) FTA, quadratic forms
  - Wessel (1797), Argand (1806) Geometric Visualization
  - Cauchy (1814, 1825) Complex Analysis
  - Riemann (1826-1866) Surfaces

Complex Numbers, $\mathbb{C}$

- $a + bi \in \mathbb{C}$, $a, b \in \mathbb{R}$, $i = \sqrt{-1}$.
- Quadratic Equation,
  $ax^2 + bx + c = 0$,
  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
  If $b^2 - 4ac < 0$,
  complex conjugate roots.
- Cubics - Role was clearer
  $y^3 = py + q$
  $y = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} +$
  $\sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}.$

Example: $x^3 = 15x + 4$

$$\begin{aligned} x &= \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} \\ &= 2 + i + 2 - i = 4. \end{aligned}$$

Bombelli (1572)

$$\begin{aligned} (2 + i)^3 &= (2 + i)(4 + 4i + i^2) \\ &= (2 + i)(3 + 4i) \\ &= 2 + 11i. \end{aligned}$$

Bernoulli's Transformations

- Johann Bernoulli (1712)

$$\begin{aligned} \frac{1}{1+z^2} &= \frac{1}{(1+iz)(1-iz)} \\ &= \frac{1}{2}\left(\frac{1}{1-iz}+\frac{1}{1+iz}\right) \\ \int \frac{dz}{1+z^2} &= \frac{1}{2}\int\left(\frac{1}{1-iz}+\frac{1}{1+iz}\right) \end{aligned}$$

- Note:

$$\int \frac{dz}{a+bz} = \frac{1}{b}\ln(a+bz).$$

So, integrating $(1+t^2)^{-1}$ gives

$$\tan^{-1} z = \frac{1}{2i}\left[\ln(1+iz) - \ln(1-iz)\right].$$

Examples: Tangent Identities

- Bernoulli studied $y = \tan n\theta$ in terms of $x = \tan\theta$.
- *Example:* $n = 2$
  $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$.
- Let $y = \tan n\theta$, Then, $n\theta = \tan^{-1} y, \theta = \tan^{-1} x$.

$$\begin{aligned} \int \frac{dy}{1+y^2} &= n\int \frac{dx}{1+x^2} \\ \ln\frac{y+i}{y-i} &= n\ln\frac{x+i}{x-i} \\ \frac{y+i}{y-i} &= A\left(\frac{x+i}{x-i}\right)^n \end{aligned}$$

$A = (-1)^{n+1}$. Solve for y.

Ex: $n = 2$:

$$\tan 2\theta = \frac{2x}{1-x^2}.$$

Ex: $n = 3$:

$$\tan 3\theta = \frac{x^3-3x}{3x^2-1}.$$

Ex: $n = 4$:

$$\tan 4\theta = \frac{4x-4x^3}{x^4-6x^2+1}.$$

Ex: $n = 5$:

$$\tan 5\theta = \frac{x^5-10x^3+5x}{5x^4-10x^2+1}.$$

The Fundamental Theorem of Algebra

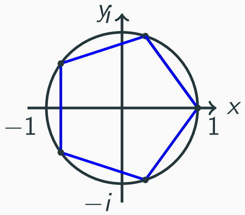
- Integration of $\frac{p(x)}{q(x)}$ for $p(x), q(x)$ polynomials
- Need Integration by parts. Assumes $q(x)$ can be factored
  - Fundamental Theorem of Algebra (FTA)
- Albert Girard (1629), *L'invention en algèbre*,
  First to claim there are always n roots of degree n polynomial.
- By 1750 - Any polynomial with real coefficients can be factored into real linear and quadratic factors.
- Nicolas II Bernoulli (1687-1759) gave a counterexample:
  $p(x) = x^4 - 4x^3 + 2x^2 + 4x + 4.$
- Euler found the factors:

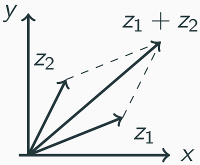
$$x^2 - \left(2 \pm \sqrt{4 + 2\sqrt{7}}\right) x + \left(1 \pm \sqrt{4 + 2\sqrt{7}} + \sqrt{7}\right)$$

He gave incorrect proof for any quartic.
His was followed by proofs from d'Alembert and Gauss.

- Cotes, de Moivre, Euler
  - $x^n - 1 = 0$. Seems $x = \sqrt[n]{1}$.
  - $x = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$, $k = 0, 1, \ldots, n-1$. **Roots of unity.**
  - Geometric Interpretation
- Caspar Wessel, surveyor.
  - Complex number = point in the complex plane, 1797.
  - Also, proposed vectors.
- Argand, 1806, visual representation, operational (translation, rotation, reflection)
- Gauss also rediscovered, 1831.

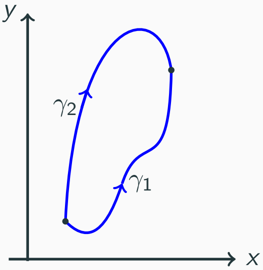
$e^{2k\pi i/5}, k = 0, 1, \ldots, 4$





Representing Complex Numbers

- Gauss (1777-1855) adopted "complex number," used i.
- Integration in $\mathbb{C}$-plane.
- $\int_\gamma \phi(z)\, dz$ is path independent for "nice" $\phi(z)$.
- Cauchy proved later, in 1814 talk, published 1827. - Now called *Cauchy's Theorem.*



Path Independence

$$\int_{\gamma_1} \phi(z)\, dz = \int_{\gamma_2} \phi(z)\, dz$$

Equivalently, for a simple, closed loop Γ, $\int_\Gamma \phi(z)\, dz = 0$.

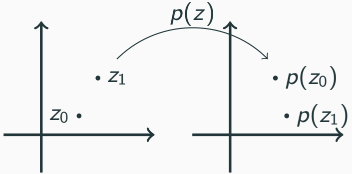
Augustin Louis Cauchy (1789-1857)

- Father of Complex Analysis
- Does $f(x) \to f(z)$ make sense?
- Integration along paths (1814) pub 1827.
- Cauchy's Theorem, Cauchy-Riemann Equations.
- Calculus of Residues (1826) - dealing with singularities.
- Convergence of infinite series.
- Use complex integration to integrate real functions.
- Path Independence (1825).
- Complex function of complex variable (1828).

Fundamental Theorem of Algebra I

Every polynomial $p(x)$ can be written as a product of linear complex factors. (Contains 1750 version)

- d'Alembert (1717-1783)
- **Lemma** $p(z_0) \neq 0$, $p(z) \neq$ constant. There exists a $z_1 = z_0 + w$ such that $|p(z_1)| < |p(z_0)|$ where $|a + bi| = \sqrt{a^2 + b^2}$.



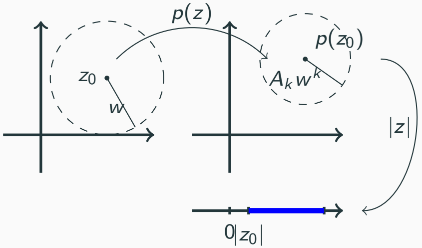
Proof

$$
\begin{aligned}
p(z) &= a_0 z^n + a_1 z^{n-1} + \cdots + a_n. \\
p(z_0 + w) &= a_0 z_0^n + a_1 z_0^{n-1} + \cdots + a_n + A_1 w + A_2 w^2 + \cdots + A_n w^n.
\end{aligned}
$$

Fundamental Theorem of Algebra II

$$p(z_0 + w) \quad = \quad a_0 z_0^n + a_1 z_0^{n-1} + \cdots + a_n + A_k w^k + \epsilon$$

Here $A_k w^k$ is the first nonzero, lowest power of w term and ϵ contains the higher powers terms in w and is small for large $|z|$.

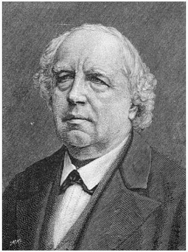


$\exists w$ such that $p(z_0) + A_k w^k$ is closer to the origin.

Let $p(z) \neq 0$. By the lemma, $\exists$ a point closer than z_0 to the origin.
$\therefore$ there exists a zero of $p(z)$.

Fundamental Theorem of Algebra III

- Gauss attempted several proofs.
- Karl Weierstrauss (1815-1897) - continuous functions on closed, bounded regions which assume maximum and minimum values.
- Gauss (1799 Thesis) considered curves $\mathrm{Re}(p(z)) = 0$, $\mathrm{Im}(p(z)) = 0$, $z = x + iy$.
- For $|z|$ large, $\mathrm{Re}(a_0 z^n) = 0$, $\mathrm{Im}(a_0 z^n) = 0$, curves are asymptotic to lines through the origin.
- Curves $\mathrm{Re}(p(z)) = 0$, $\mathrm{Im}(p(z)) = 0$, entering $|z| = r$ must come out and intersect inside disk. [See examples.]



Plotting $Re(p(z)$ and $Im(p(z))$, outside a large circle one gets alternating lines. Inside the circle they must intersect for $p(z) = 0$.

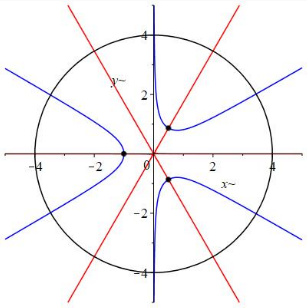


Figure 1: $p(z) = z^3 + 1$.

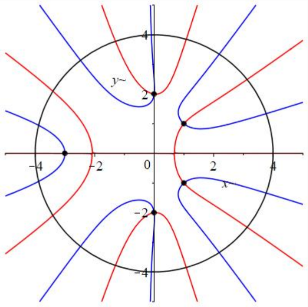


Figure 2: $p(z) = z^5 + z^4 + 10z^2 - 16z + 24 = (z - 1 - i)(z - 1 + i)(z^2 + 4)(z + 3)$.

Theory of Curves, $p(x, y) = 0$

- Descartes - linear/lines
  - quadratic/conics
- Newton - cubics
- Recall Bezout's Intersection Thm
  - ◁ Count multiplicities.
  - ◁ Intersection with ∞.
- 19th Century
  - Projective Geometry
    homogeneus coordinates
    Möbius, Plücker - 1830
  - Complex Numbers
    Gauss - FTA
  - Topological ideas
    - Riemann surfaces, 1850's

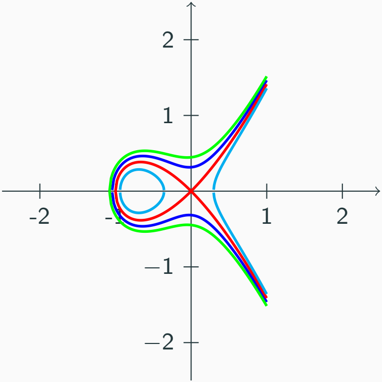
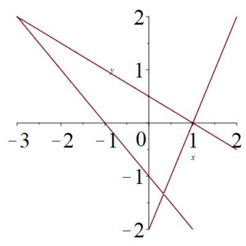


Figure 3: Cubic curves of form $y^2 = x^3 + x^2 + bx + 2b$

Cubic Curves

Consider products of linear factors or lines
$p(x, y) = (a_1 x + b_1 y + c_1)(a_2 x + b_2 y + c_2)(a_3 x + b_3 y + c_3)$

- Ex: $p(x, y) = (x + y + 1)(x + 2y - 1)(-2x + y + 2)$



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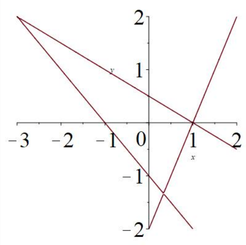
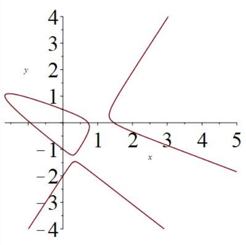
- Ex: $p(x, y) = (x + y + 1)(x + 2y - 1)(-2x + y + 2)$
- Modify: $p(x, y) = (x + y + 1)(x + 2y - 1)(-2x + y + 2) + \frac{x^2}{2}$





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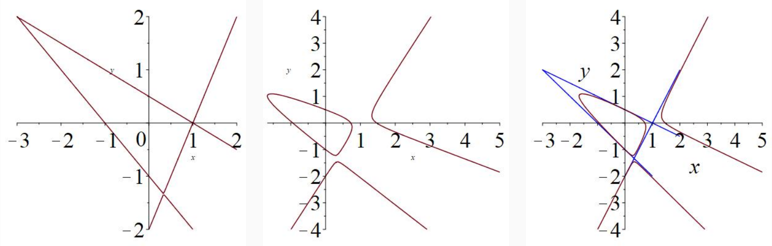
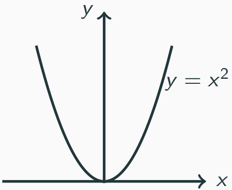
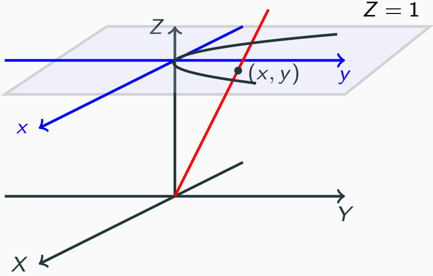
- Modify: $p(x, y) = (x + y + 1)(x + 2y - 1)(-2x + y + 2) + \frac{x^2}{2}$
- Branches go to **points at infinity**. Consider projective geometry.



Projective Geometry

- Homogeneous coordinates:
  $x = \frac{X}{Z}, y = \frac{Y}{Z}$.
- Introduced by Möbius, Pücker.
- **Example:** $y = x^2$ gives $X^2 = YZ$





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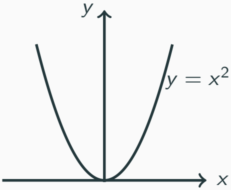
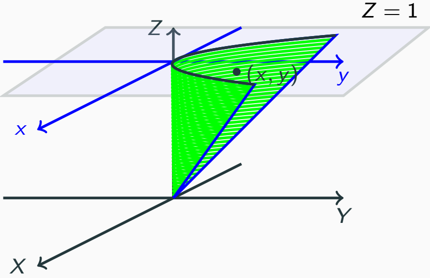
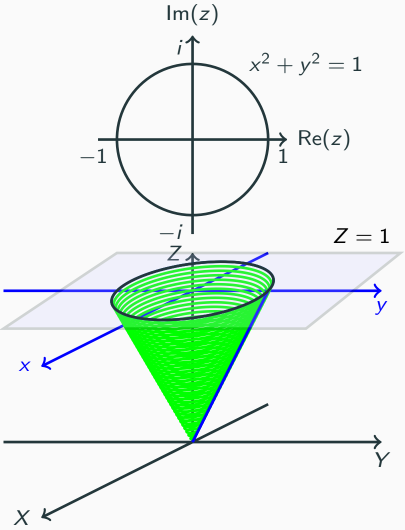
- **Example:** $y = x^2$ gives $X^2 = YZ$
- Lines thru origin (projective plane).
- $X^2 = YZ$ is a "cone"
- Points at Infinity:
  $Z = 0 \Rightarrow X = 0$,
- These points, $[0, Y, 0]$, lie on horizon.





Projective Plane and Complex Numbers

- **Example:** $x^2 + y^2 = 1$
- Projective curve: $X^2 + Y^2 = Z^2$
- Pts at infinity,
  $Z = 0 \Rightarrow X^2 + Y^2 = 0.$
- In $\mathbb{C}$, Circular pts at infinity.
  $X = 1, Y = i : I_1 = (1, i, 0)$
  $X = 1, Y = -i : I_2 = (1, -i, 0)$



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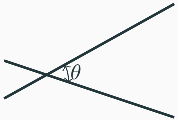
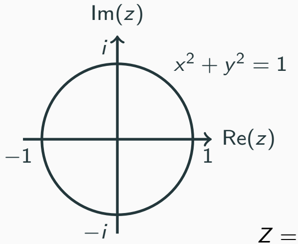
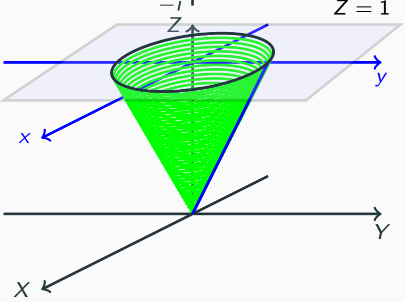
- Edmund Laguerre (1834-1886)
  - Angles, $\theta = i \log R$.
- R - Cross ratio







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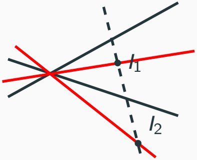
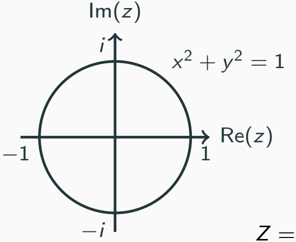
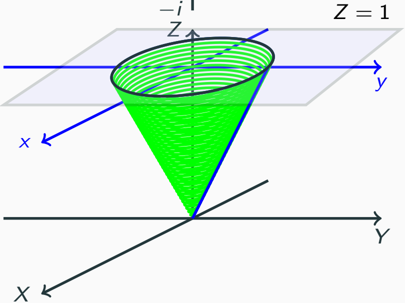


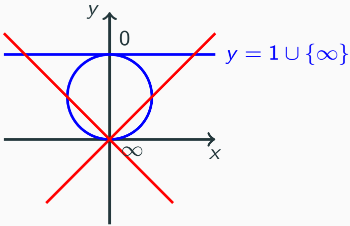


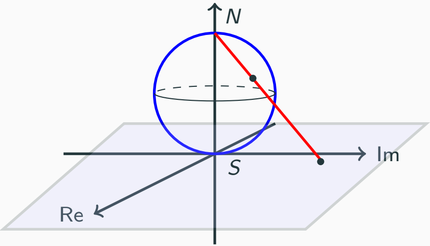


Stereographic Projection

What do complex curves look like?

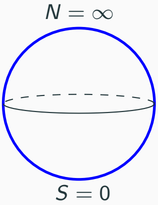
- Projective lines:
  Lines thru origin
  Topologically looks like a circle,
  S^1, after adding point at infinity
- Extend to $\mathbb{C}$ - topologically, S^2
- Stereographic Projection
  Connect pts in $\mathbb{C}$ to North Pole.
- N mapped to pt at ∞.
- Möbius (1790-1868) Image of
  circle = circle.

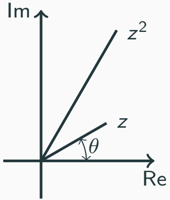




Mapping Functions onto Surfaces

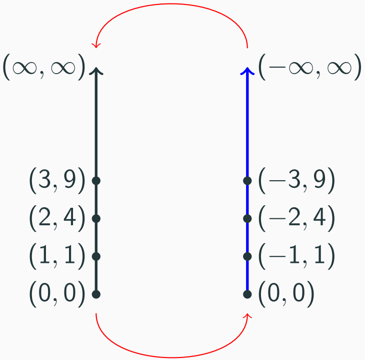
- Riemann (1826-1866)
  - Riemannian manifolds
  - Curved spaces - Gauss
  - Complex Analysis - Riemann surfaces [Cauchy (1788-1857)]
  - Number theory - $\zeta(s)$
- Start with a Sphere
- Extend $f : \mathbb{C} \to \mathbb{C}$ to $g : S^2 \to S^2$.
- Complex function, $f(z) = z^2$
  Let $z = re^{i\theta}$.
  [$\theta =$ argument, $r =$ modulus, $|z|$.]
  Then, $f(z) = r^2 e^{2i\theta}$.

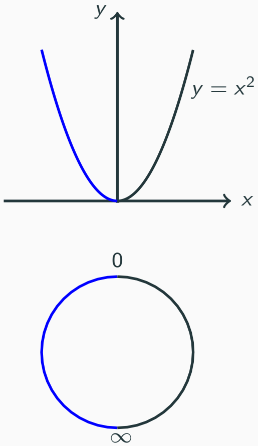




Real Function, $y = f(x)$, Mapped to S^1

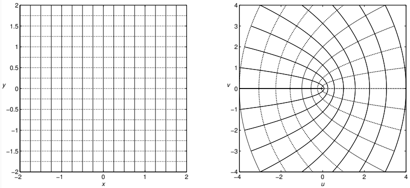
- **Example** $f(x) = x^2$.

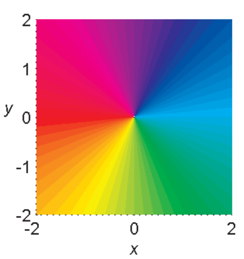


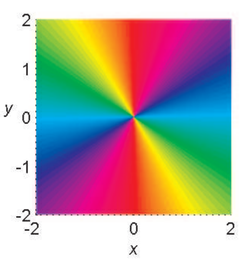


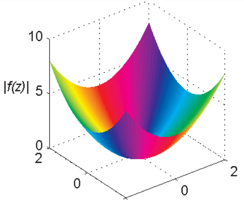
Visualizing Complex Functions: $w = f(z) = u(x, y) + iv(x, y)$

- What is $f(z) = z^2$?
- Map xy-plane to uv-plane.
- $(x + iy)^2 = x^2 - y^2 + 2ixy$.
- $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$
- Domain Coloring



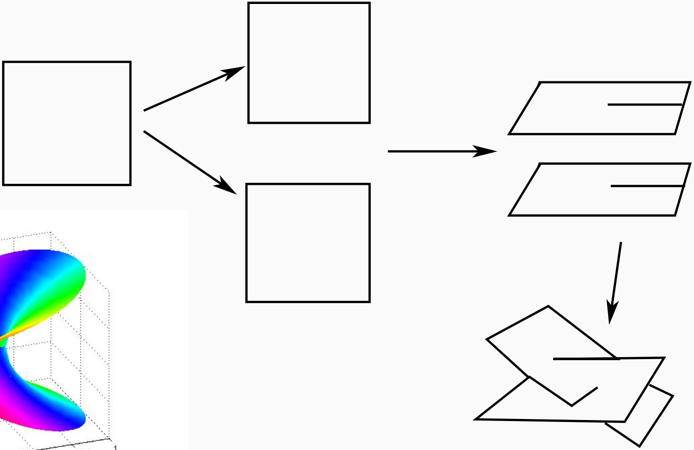


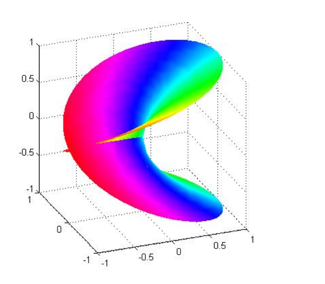




Riemann Surfaces and the Square Root Function

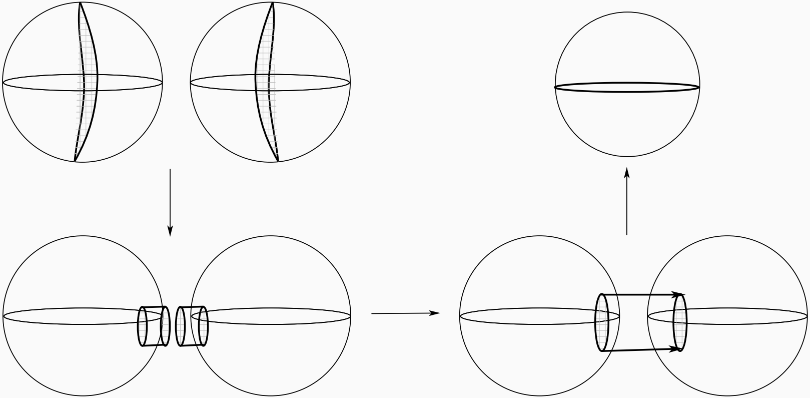
- Riemann Sheets - Two copies of $\mathbb{C}$. Riemann's Dissertation, 1851.
- **Example:** $w = \sqrt{z}$.





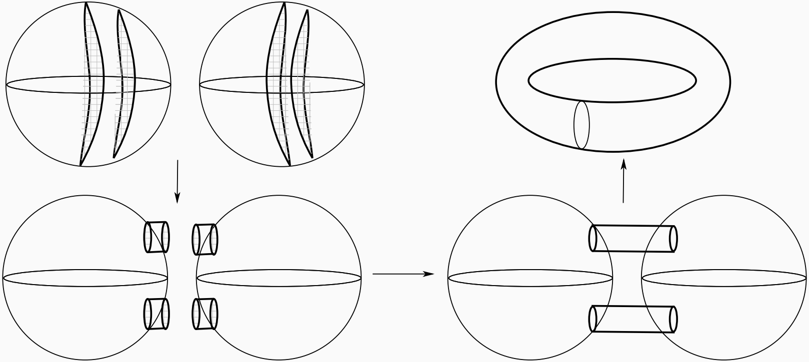
Mapping $f(z) = z^2$ to S^2.

- **Example:** $f(z) = z^2$.



Riemann Surfaces and Elliptic Integrals

- **Example:** $\int_0^x \frac{dz}{\sqrt{z(z-a)(z-b)(z-c)}}$.
- $w^2 = z(z-a)(z-b)(z-c)$
- Beginning of topology.



Merging Two Cut Riemann Spheres

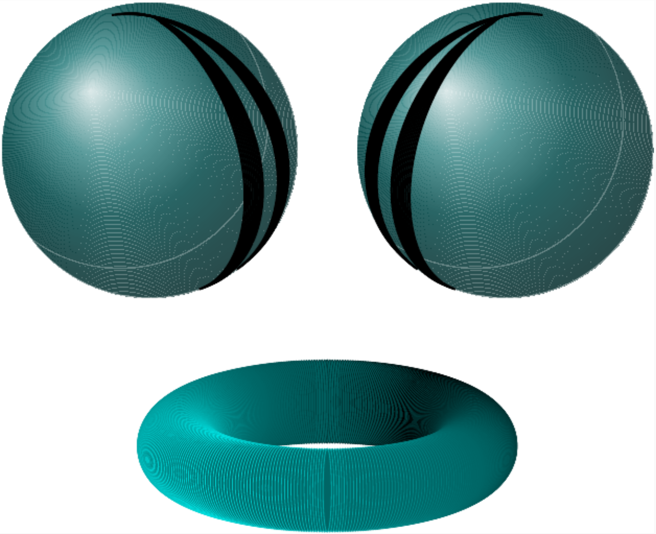
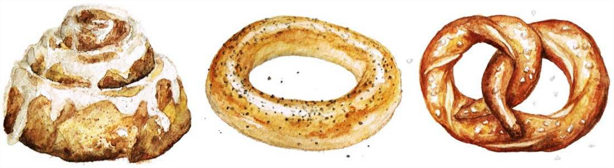
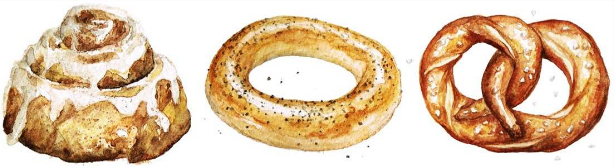


Figure 4: Genus $g = 1, 2, 3$.



Beginnings of Topology ...

Figure 4: Genus $g = 1, 2, 3$.



Women in Mathematics in the 1800s

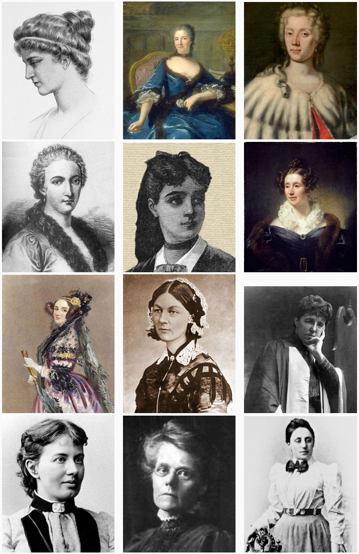
Fall 2025 - R. L. Herman





Famous Women Mathematicians Before 1900

- Hypatia of Alexandria (c. 350-415)
- Émilie du Châtelet (1706-1749)
- Laura Bassi (1711-1788)
- Maria Agnesi (1718-1799)
- Sophie Germain (1776-1831)
- Mary Fairfax Somerville (1780-1872)
- Ada Lovelace (1815-1852)
  (Augusta Byron, Countess of Lovelace)
- Florence Nightingale (1820-1910)
- Charlotte Angas Scott (1848-1931)
- Sofia Kovalevskaya (1850-1891)
- Alicia Boole Stott (1860-1940)
- Amalie 'Emmy' Noether (1882-1935)



Pandrosion (4th c. CE) and Hypatia (c. 350–370 – 415 CE)

Pandrosion, Alexandria

- Earliest *documented* woman mathematician on record (from Pappus's *Collection*);
- Active as a teacher in Alexandria.
- Pappus criticized her students' method for *doubling the cube*.
- Associated with a construction of a *mean proportional*.
- Misidentified for centuries: due to editorial errors;
- Recent philological work confirms feminine address and name.

Hypatia, Alexandria

- Neoplatonist *philosopher, astronomer, and mathematician*;
- Father was Theon.
- Led a respected teaching circle.
- Produced *commentaries* (e.g., *Diophantus's Arithmetica*, *Apollonius's Conics*);
- Public intellectual and civic advisor;
- *Murdered in 415 CE*, later symbol of intellectual independence and vulnerability to sectarian violence.

Context: Pandrosion predates Hypatia as the earliest recorded woman mathematician; Hypatia remains the first whose intellectual life is well attested.

Émilie du Châtelet (1706-1749)

- Gabrielle-Émilie Le Tonnelier de Breteuil
- Father - official at the Court of Louis XIV at Versailles.
- Husband - Marquis Florent-Claude Chastellet, military man, governor of Semur-en-Auxois in Burgundy.
- Lovers: Pierre Louis Moreau de Maupertuis (1698-1759), Alexis Clairaut (1713-1765) and François-Marie Arouet (Voltaire) (1694-1778).
- Wrote on Newton, Leibniz, and the propagation of fire.
- Translation of the *Principia* into French.
- Debated Euler and others over *vis viva*, "living force," or kinetic energy Σmv^2.

Figure 1: Gabrielle Émilie Le Tonnelier de Breteuil Marquise du Châtelet [Émilie du Châtelet]

Laura Bassi and Marie Agnessi

Both promoted by Pope Benedict XIV (Prospero Lambertini).

- Laura Bassi (1711-1788)
  - 1st female physics professor. Studied Newton, electricity. Student Galvani, Volta's Ph.D.
  - Second in the world: Ph.D., 1732.

    1st - philosopher Elena Cornaro Piscopia, 1678.
  - 1st woman: doctorate in science.
- Maria Agnessi (1718-99).
  - 1st woman: math handbook.
  - 1st woman math professor.
  - First book on both differential and integral calculus.
  - Witch of Agnesi curve.

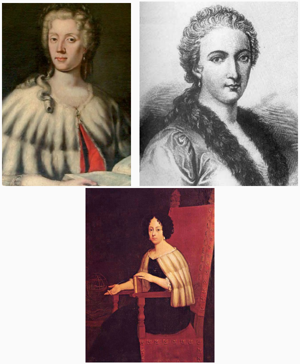


Figure 2: Bassi, Agnessi, Piscopia.

Marie Sophie Germain (1776-1831)

- Self-taught, French revolution.
- 1794 - École Polytechnique opened - for men, obtained .notes
  Signed HW - Monsieur Le Blanc.
- Joseph-Louis Lagrange (1736-1813).
- Adrien-Marie Legendre (1752-1833).
- Gauss (1777-1855)
  - letters 1804-12; saved his life.
- Germain Primes -
  If p is prime, then so is $2p+1$
  Ex: $5 = 2(2)+1$, $7 = 2(3)+1$,
  ~~$9 = 2(4)+1$~~, $11 = 2(4)+1$.
- Elasticity work did not get her name on **Eiffel Tower**.

Figure 3: Sophie Germain

- Fermat's Last Theorem.
- Chladni Plates, elasticity.
- Competitions 1811, 1813, 1815.

Mary Fairfax Somerville (1780-1872)

- Mathematics and astronomy.
- Wrote books
- Jointly - the first female member of the Royal Astronomical Society with Caroline Herschel.
- First to sign petition to Parliament to give women the right to vote.
- Experiments to explore the relationship between light and magnetism.
- Translated/expanded Laplace's work, 1831, *The Mechanism of the Heavens*.
- First Geography text, 1848.

Figure 4: Mary Sommerville.

Ada Lovelace (1815-1852)

- Daughter of Lord Byron, (poet, died 1824) and
- Mathematician Anne Isabelle Milbanke, self-named as "princess of parallelograms."
- She wrote papers and the first computer programs.

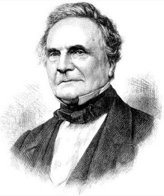
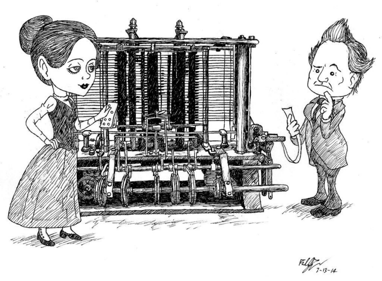


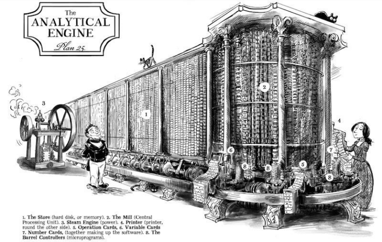
Figure 5: Charles Babbage

Figure 6: Augusta Ada King, Countess of Lovelace [Augusta Ada Byron]

Ada Lovelace and Charles Babbage

- Charles Babbage (1791-1881)
  - English mathematician, philosopher, engineer.
  - 1833 Difference Engine.
  - 1844 Analytical Engine.
  - Designed, never Built.
- Lovelace first algorithm for a machine.
- Translated Luigi Menabrea's article on the engine (1842-1843). Added notes containing first computer program.
- Loops, recursion - Bernoulli numbers, systems of linear equations.
- 1980's - Ada, programming language.





Florence Nightingale (1820-1910)

- Born in Florence, Tuscany, Italy.
- Studied under famous mathematicians.
- Crimean War (1853-1856), Britain was at war with Russia.
- Supervised 38 nurses.
- Used statistics - mortality rates
- Pioneer in data visualization, polar area diagrams.
- National heroine, 1883 recipient of the Royal Red Cross, and later others.

Figure 7: Florence Nightingale

Sofia Kovalevskaya (1850-1891)

- Born Sofya Vasilyevna Korvin-Krukovskaya in Moscow.
- Education in Europe.
- Teachers - Hermann von Helmholtz, Gustav Kirchhoff and Robert Bunsen.
- Advisor - Weierstrass (1874) - 3 papers PDEs, elliptic integrals, Saturn's rings.
- 1st woman to get doctorate in math outside Italy. - not enrolled! 1874.
- 1883 Teaching position, U. Stockholm.
- 1889 1st to hold chair in European university since Laura Bassi and Maria Agnessi.

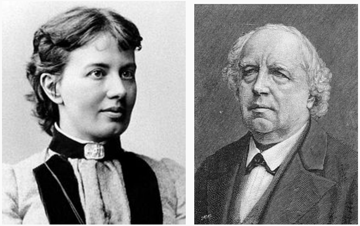


Figure 8: Sofia Kovalevskaya and Karl Weierstrass (1815-1897).

Sofia Kovalevskaya (1850-1891)

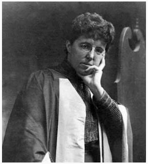
- Light waves, tops, wrote books.
- 1886 - French Competition - spinning tops.
- 1889 - Swedish Academy of Science Prize Chebyshev got her membership in Imperial Academy of Sciences.
- 1891 - On vacation, Influenza - pneumonia.
- Cauchy–Kovalevskya Theorem: local existence and uniqueness theorem for Cauchy problem in PDEs.
- Kowalevski top - a symmetric top with a particular ratio of the moments of inertia: $I_1 = I_2 = 2I_3$.

Figure 9: Sofia Kovalevskaya and her top.

Turn of Century - Charlotte Scott and Alicia Stott

Charlotte Angas Scott (1848-1931)

- One of 1st woman to obtain a doctorate in England.
- Studied under Arthur Cayley.
- Algebraic curves of degree higher than two.
- 1885 - 1st mathematician at Bryn Mawr College, dept head.
- A founder of AMS.



Alicia Boole Stott (1860-1940)

- Parents: George Boole (1815-1864) and Mary Everest Boole (1832-1916).
- Four-dimensional polytopes.
- Exactly six regular polytopes in four dimensions
- Worked with Harold Coxeter,(1907–2003).

Amalie ‘Emmy’ Noether (1882-1935)

- German mathematician
- Abstract algebra - theories of rings, fields, and algebras.
- Noether’s theorem - connects symmetry and conservation laws.
- Mathematical Institute of Erlangen, 1908–1915 - without pay.
- University of Göttingen, 1915-1933, First four years lecturing under Hilbert’s name.
- Bryn Mawr - 1933-5.
- Lectured at Institute for Advanced Study in Princeton.

Figure 10: Emmy Noether

That's *Not* All Folks! (Click on the Names)

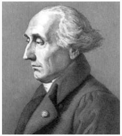
- Elizaveta Fedorovna Litvinova (1845-1919)
- Christine Ladd-Franklin (1847-1930)
- Ellen Amanda Hayes (1851-1930)
- Helen Abbot Merrill (1864-1949)
- Grace Chisholm Young (1868-1944)
- Ada Isabell Maddison (1869-1950)
- Mary Frances Winston Newson (1869-1959)
- Mary Emily Sinclair (1878-1955)
- Anna Johnson Pell Wheeler (1883-1966)
    - Marion Cameron Gray (1902-1979)
      Gray codes, telegrapher eqn, p 34, Herman, 2025.
- Pauline Sperry (1885-1967)
- Women in the AMS before 1900

Non-Euclidean Geometry and Group Theory

Fall 2025 - R. L. Herman








Euclidean Geometry

- 300 BCE - Euclid's *Elements*
- Five Postulates.
- 5th Postulate - not needed in first 28 propositions.

If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.

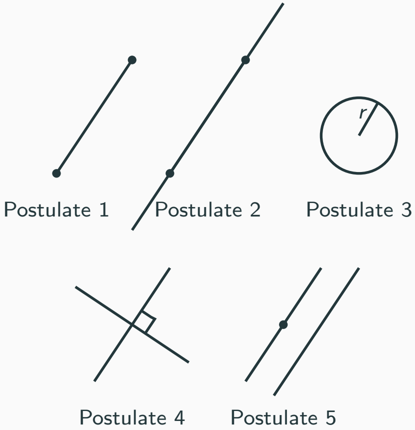


Figure 1: Euclid's 5 Postulates.

Statements of Parallel Axiom in Text

$\mathbf{P}_1$ For each straight line L and point P outside L there is exactly one line through P that does not meet L.

Equivalent statements

- The angle sum of a triangle $= \pi$. - Euclid.
- The locus of points equidistant from a straight line is a straight line. - al-Haytham.
- Similar triangles of different sizes exist. - Wallis

Saccheri (1733) - provided two alternatives to arrive at proof by contradiction.

$\mathbf{P}_0$ There is not line through P that does not meet L.

$\mathbf{P}_2$ There are at least two lines through P that do not meet L.

Parallel Postulate

- Proclus (410-485) Equivalent postulate. Revived as Playfair axiom.
- William Ludlam (1785): Two straight lines, meeting at a point, are not both parallel to a third line.
- John Playfair, *Elements of Geometry* (1795): **Playfair's axiom:** Two straight lines which intersect one another cannot be both parallel to the same straight line.
- Many false attempts to prove based on other four postulates.
- 1663 John Wallis "To each triangle, there exists a similar triangle of arbitrary magnitude."
- Giralomo Saccheri (1667-1733) Assume 5th postulate false and get contradiction.
- Used assumption - lines are infinite. Led to contradiction of $\mathbf{P}_1$, almost $\mathbf{P}_2$.
- d'Alembert, 1767 - "The scandal of elementary geometry."

Spherical Geometry

- Lines = geodesics,
  Lie on great circles.
- Euclidean triangles, $a + b + c = \pi$.
- Spherical triangles, $a + b + c > \pi$.
- Thomas Harriot (1560-1621), astronomy, mathematics, and navigation
- Johann Heinrich Lambert (1726-1777)
  - General properties of map projections.
  - hyperbolic functions
  - π is irrational
  - optics

$a + b + c = \pi + \frac{A}{R^2}$.

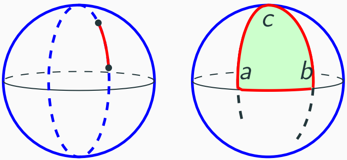


Figure 2: Harriot and Lambert.

Other Geometries

- Ferdinand Karl Schweikart (1780) Astral geometry, sum of three angles of a triangle is less than two right angles.
- Wrote to Gauss, 1818, via student Christian Ludwig Gerling (1788-1864).
- Franz Taurinus (1784-1854), Schweikart's nephew. Proposed geometry on a sphere of imaginary radius, logarithmic-spherical geometry.
- 1826, hyperbolic law of cosines in *Geometriae prima elementa.*
- Wrote to Gauss. after being encouraged, he sent copies of his works with no reply.
- Later he burned copies of his book.

Figure 3: Gerling and Schweikart

Parallel Postulate Revisited

- Carl Friedrich Gauss (1777-1885) started on it in 1799; was convinced it was independent of first 4.
- Discussed with Farkas Bolyai (1775 - 1856) - told his son no to waste his time.
- János Bolyai (1802-1860) - Believed a non-Euclidean geometry existed.
- Nikolai Lobachevsky (1792-1856) - independently 1840 new 5th postulate: There exists two lines parallel to a given line through a given point not on the line.

  Developed trig identities, hyperbolic geometry.



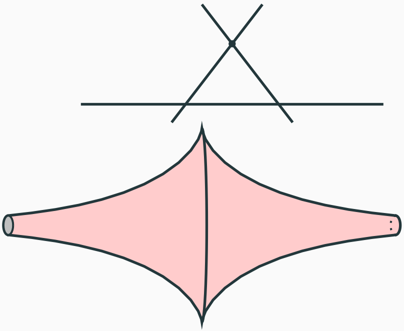
Figure 4: Gauss, Bolyai, Lobachevsky

Riemannian Geometry

- Georg Friedrich Bernhard Riemann (1826-1866)

    Published in 1868 Lecture
    Spherical geometry
    Riemannian geometry $\rightarrow$ differential geometry
    Every line through a point not on a given line meets the line.

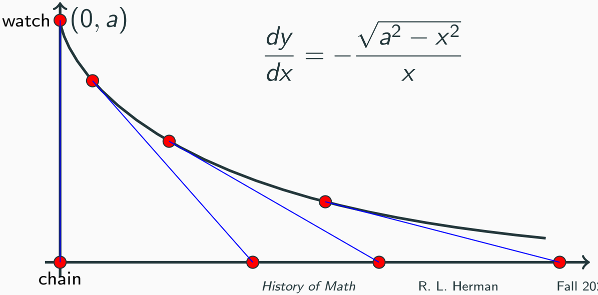
- Eugenio Beltrami (1835-1900) Published interpretations of non-Euclidean geometry - introduced pseudosphere in 1868 using a **tractrix**.



tractrix $(a(t - \tanh t), a \operatorname{sech} t)$

Aside: The Tractrix

- Claude Perrault [brother Charles author of *Cinderella*, *Puss-in-Boots*] in 1693, Paris, placed a watch in the middle of a table and pulled its chain along the edge of the table. What was the curve traced out ?
- Studied by Newton (1676), Huygens (1692) and Leibniz (1693). Euler gave complete theory in 1788. [Am. Math. Monthly, 72(10) (1965), 1065-1071.]
- Huygens coined name from Latin, *tractus*.



$$\frac{dy}{dx} = -\frac{\sqrt{a^2 - x^2}}{x}$$

Curvature

- $k = 0,\ k > 0,\ k < 0.$
- sums of angles of triangles $a + b + c - \pi = kA$.

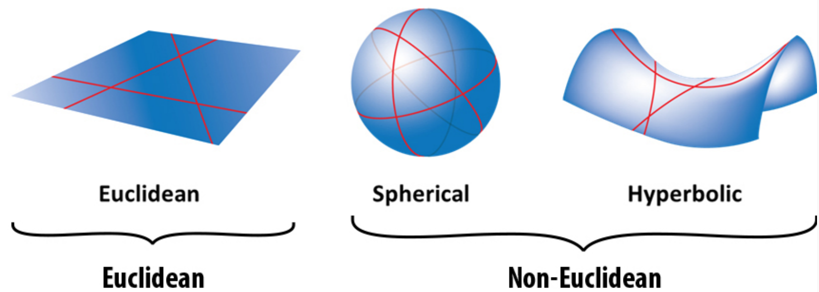


Figure 5: Surfaces of Constant Curvature.

Hyperbolic Geometry

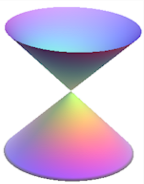
- Sphere

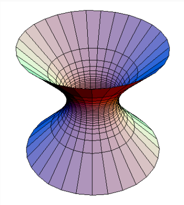
$$x^2 + y^2 + z^2 = \text{const}$$

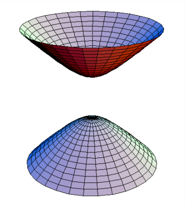
- Modify

$$x^2 + y^2 - z^2 = K$$

- $K = 0$, $z^2 = x^2 + y^2$. Cones.
- $K = 1$, $x^2 + y^2 - z^2 = 1$.
  Hyperboloid of one sheet
- $K = 1$, $z^2 - x^2 - y^2 = 1$.
  Hyperboloid of two sheets.





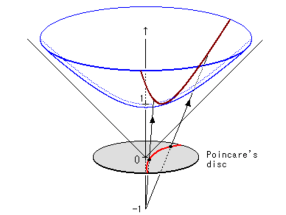


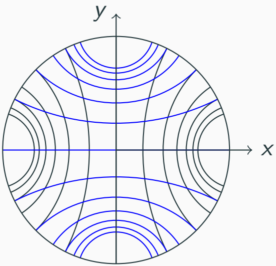
Beltrami-Poincaré Model

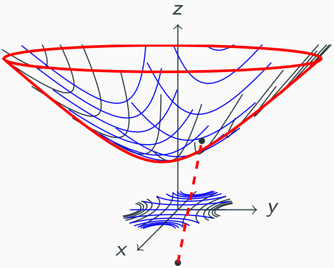
- Poincaré's Disks

  $$(x, y, z) = (c \cosh t, \sinh t, \sqrt{1 + c^2} \cosh t)$$

- Stereographic Projection thru $(0, 0, -1)$ to $z = 0 : (x, y, z) \to \frac{(x,y)}{1+z}$.
- Hyperbolic geometry.



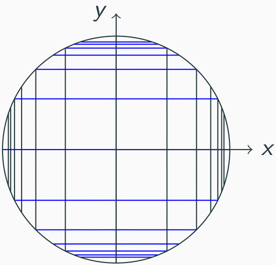


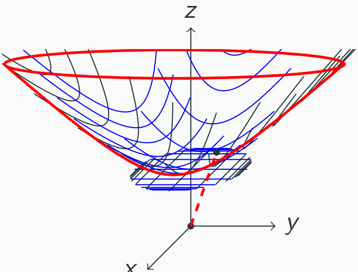


Beltrami-Poincaré Model

Beltrami-Klein Model

- Stereographic Projection thru $(0,0,0)$ to $z=1:(x,y,z)\to\frac{(x,y)}{z}$.
- Klein's Disks
  Projection to $(0,0,1)$

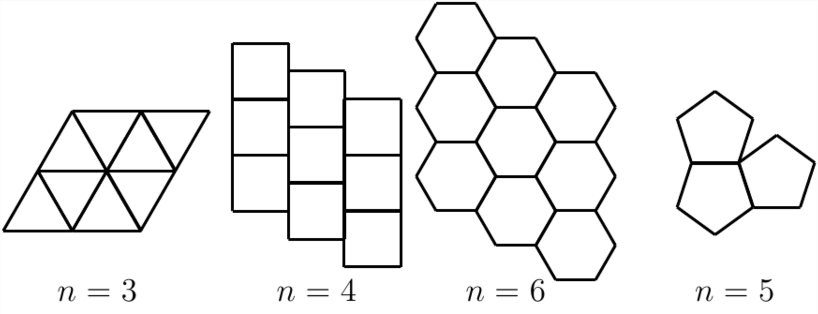




Beltrami-Klein Model

Tiling the Plane

One can tile the plane with a single polygon with sides 3, 4, and 6. However, one cannot fit pentagons together. As seen below, the angles do not allow for a fit. For large *n*, the interior angles are too small.



Other Tilings

- Johannes Kepler (1571-1630)
  - Studied Tilings
  - *Harmonicae Mundi* (Harmony of the World).
  - Planned in 1599.
  - Published 1619 - delay by Tycho Brahe to look as orbit of Mars.
- Roger Penrose (1931-)
  - 2020 Nobel Prize
  - 70's Inspired by Tilings - Penrose tilings. In 80's found in nature.
  - and M. C. Escher (1889-1972)
  - Circle Limit - Tiling Hyperbolic Plane.
- Others - Polyominoes and Pentominoes.

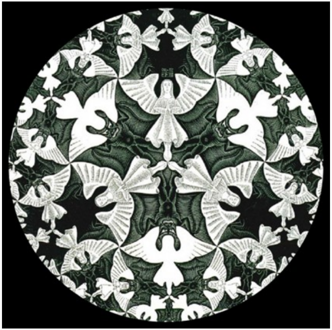


Figure 6: Circle Limit IV

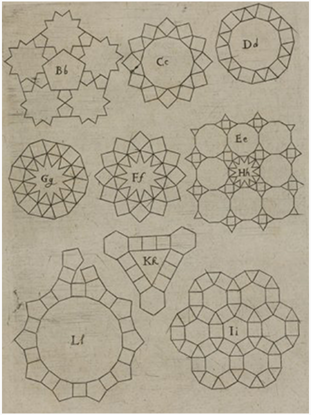
Hyperbolic Tessellations

- Tessellation = cover plane by tiles, no tiles overlap and no space empty.
- Schläfli symbol: $\{n, m\}$,
  $n =$ number of sides on the tile,
  $m =$ number of tiles that meet at a vertex.
  Euclidean: $\dfrac{1}{n} + \dfrac{1}{m} = \dfrac{1}{2}$,
  Hyperbolic: $\dfrac{1}{n} + \dfrac{1}{m} < \dfrac{1}{2}$.

Figure 7: Circle Limits I-IV.

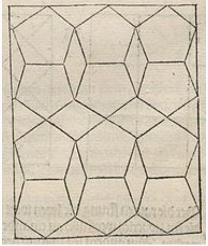
Kepler's Tiling

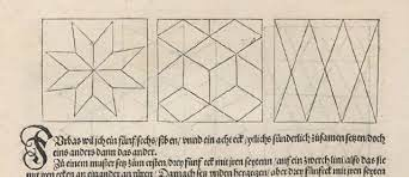
- 1619 Johannes Kepler published the first classification regular polygon tilings, Book II of *Harmonices Mundi*.
- First catalog of the 11 uniform tilings of the plane.
- See online discussion.



Albrecht Dürer's Tilings

- 1525, *Underweysung der Messung mit dem Zirckel und Richtscheyt* (A Course in the Art of Measurement with Compass and Ruler), the Painter's Manual.
- Constructed various curves and regular polygons with a ruler and compass.
- Illustrates three regular tilings (squares, triangles and hexagons), octagon tiling, uniform tiling with a six pointed star pattern, and rhomb tiling.





Aperiodic Tiling

- Non-periodic tiling that does not contain arbitrarily large periodic regions.
- 1964 Robert Berger, 20,426 Wang tiles.Later reduced his set to 104.
- 1966 Hans Läuchli, 40 Wang tiles.
- 1967 Raphael M. Robinson, 104.
- 1968, Donald Knuth, 96.
- 1971, Robinson, 6 tiles.
- 1974 Penrose, 6 tiles. P1.

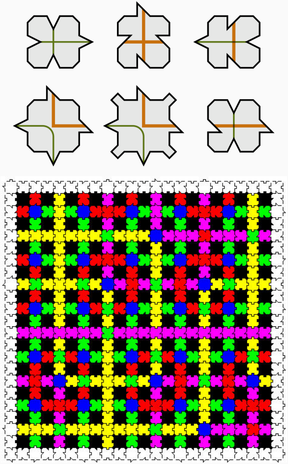
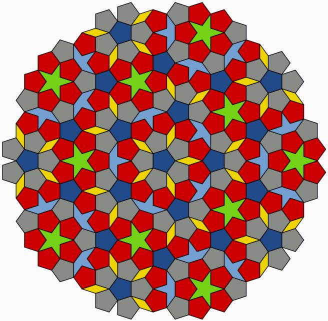


Figure 8: Robinson's tiles.

Penrose Tiling P1

- Penrose's first tiling, 1978.
- Uses pentagons and three other shapes:
  a five-pointed star,
  a "boat"
  and a "diamond".
- Need matching rules specifying how tiles meet each other to give non-periodic tilings.
- There are three different types of matching rules for the pentagonal tiles.
- Treating these as different prototiles gives a set of six.
- Indicate the three different pentagonal tiles using different colors.



Penrose Tiling P2

- Penrose introduced aperiodic tiling with two tiles.
- P2: Used quadrilaterals, "kite" and "dart." Can be combined to make a rhombus.
- Need matching rules.
- A. Color the vertices and require that adjacent tiles have matching vertices.
- B. Use circular arcs to constrain the placement of tiles. The patterns must match at these edges.

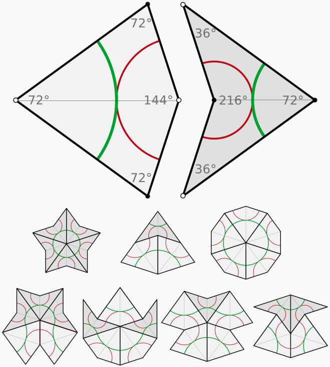


Figure 9: Penrose Kites and Darts. Use to create shown shapes: star, ace, sun, king, jack, queen, deuce.

Penrose Tiling P3

- Rhomus tiles.
- Thin rhombus with angles of 36, 144, 36, and 144 degrees.
- Thick rhombus with angles of 72, 108, 72, and 108 degrees.
- Tiles must be assembled such that the curves on the faces match in color and position across an edge.

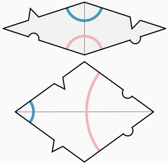


Figure 10: Thin and thick rhombs.

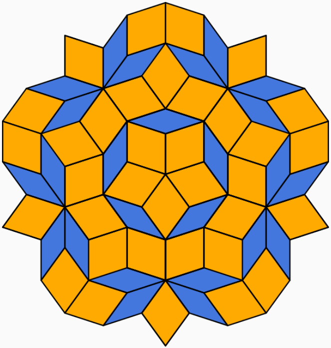
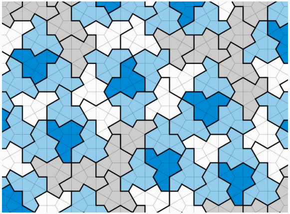
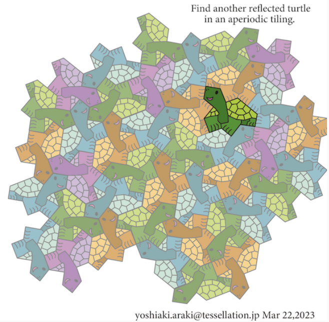


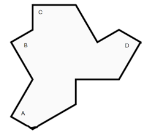
Figure 11: Penrose Rhombus Tiling.

Einstein Tiles

- The search for tiling with one tile.
- David Smith, Joseph Samuel Myers, Craig S. Kaplan, Chaim Goodman-Strauss, `https://arxiv.org/abs/2303.10798`, March 20, 2023.
- *An Aperiodic Monotile*
- Proved that "the hat" is an aperiodic monotile, called an einstein (one stone).
- Involves the hat and its mirror image, noted by Yoshiaki Araki.

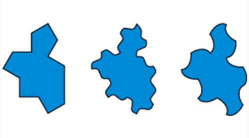


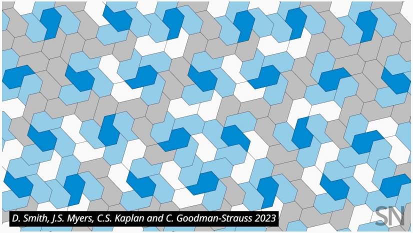




Spectre Tiles

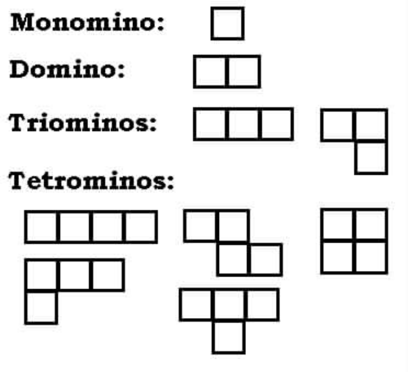
- David Smith, Joseph Samuel Myers, Craig S. Kaplan, Chaim Goodman-Strauss found a new tile `https://arxiv.org/abs/2305.17743`, May 28, 2023.
- *A Chiral Aperiodic Monotile*
- Is not accompanied by its reflection, a "vampire einstein."

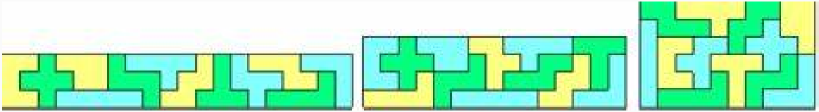




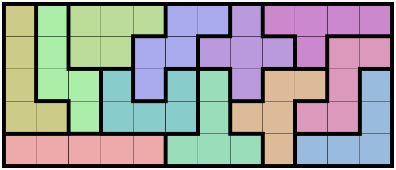
Polyominoes

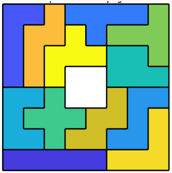
- A plane geometric figure formed by joining one or more equal squares edge to edge.
- Used in puzzles since at least 1907.
- Name *polyomino* invented by Solomon W. Golomb in 1953.
- Some types: domino, triomino, tetromino, pentomino, etc,
- Wikipedia page

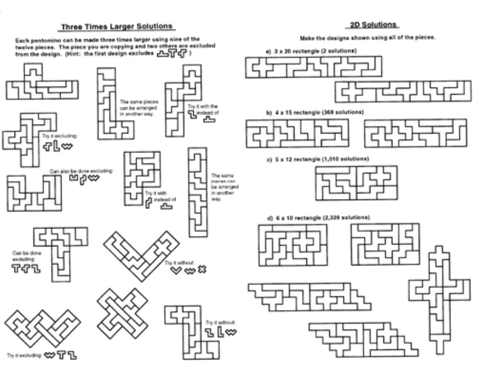




Pentomino Puzzles







Return to the Quintic

- In the meantime, the search for solving the quintic continues.
- The general quintic cannot be solved algebraically in terms of a finite number of additions, subtractions, multiplications, divisions, and root extractions.
- Malfatti (1731-1807) was the first to solve a solvable quintic using a resolvent of sixth degree, 1771.
- The general quintic was solved in terms of Jacobi theta functions by Hermite in 1858. See story.
- Our story begins with Gauss and Lagrange.

Carl Friedrich Gauss (1777-1855)

- *Disquisitions Arithmeticae* - 1801
- Summary and Extension on Number Theory.
- Initiated finite Abelian groups.
  - closed, identity, inverse, associative,
  plus commutative.
- Proved Fermat's Little Theorem: If p is prime, then for any integer a, $a^{p-1} \equiv 1 \mod(p)$.
- Represented integers as quadratic forms, like Fermat Primes ($4n + 1 = x^2 + y^2$.) for x and y integers.
- Binary quadratic forms - $ax^2 + bxy + cy^2$ - for a, b, c integers.
  - composition has properties of an abelian group.
- Did not have a general theory of groups.

Figure 12: List of things named after Gauss

Joseph Louis Lagrange (1736-1813)

- Born in Turin, Italy.
- Professor at 19 (artillery school).
- 1766 Frederick the Great (Prussia) sought a great mathematician to replace Euler.
- Lagrange went to Berlin for 20 yrs.
- Invited by Louis XVI to Paris, 1786.
- 1793, Reign of Terror, saved by Lavoisier.
- 1795 - established dept. at École Normal.
- 1797 - established dept. at École Polytechnique.
- Napoleon made him senator, count, and he received many other honors.
- Sought solution of quintic by studying cubic and quartic.

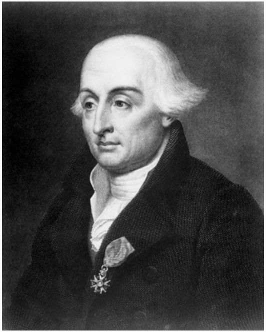
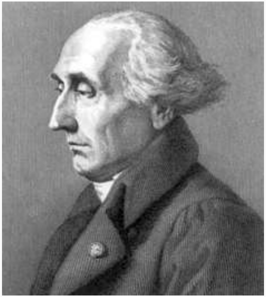


Figure 13: List of things named after Lagrange

Resolvents

- Consider $x^3 + nx + p = 0$. Let $x = y - \frac{n}{3y}$.
- Yields 6^{th} degree polynomial, $y^6 + py^3 - \frac{n^3}{27} = 0$, the resolvent.
- Let $r = y^3$, $r^2 + pr - \frac{n^3}{27} = 0$.
- Has roots r_1, r_2, where $r_2 = -\left(\frac{n}{3}\right)^3 \frac{1}{r_1}$.
- Then, $x = \sqrt[3]{r_1} + \sqrt[3]{r_2}$
- Cardano got this real root but did not seek complex solutions.
- Lagrange knew there should be 3 roots. $\sqrt[3]{r}, \omega\sqrt[3]{r}, \omega^2\sqrt[3]{r}$, where ω is a cube root of unity, $\omega^3 = 1$. Then,

$$\begin{aligned} x_1 &= \sqrt[3]{r_1} + \sqrt[3]{r_2} \\ x_2 &= \omega\sqrt[3]{r_1} + \omega\sqrt[3]{r_2} \\ x_3 &= \omega^2\sqrt[3]{r_1} + \omega^2\sqrt[3]{r_2} \end{aligned}$$



Permutation of Roots

- Lagrange then wrote roots of the resolvent
  $y = x_i + \omega x_j + \omega^2 x_k, \quad i, j, k = 1, 2, 3, \quad i \neq j \neq k.$
- 3!=6 permutations of cubic roots.
- In $y^6 + py^3 - \frac{n^3}{27} = 0$, the coefficients of y^5, y^4, y^2, y are
  $x_1 + x_2 + x_3$, $p = x_1x_2x_3$, and $\dfrac{n^3}{27} = \dfrac{(x_1x_2 + x_1x_3 + x_2x_3)^3}{27}$.
- Resolvent coefficients are rational functions of the cubic roots.
- Lagrange obtained similar results for the quartic.
- Lagrange sought solutions of higher order equations using symmetric functions of the roots and permutations.
- Paola Ruffini (1765 – 1822) - 1802, 1805, 1813 - gave proofs that quintic can't be solved. Proofs not understood.

Niels Henrick Abel (1802-1829)

- Born in Norway into poverty and had a pulmonary condition.
- Mathematical ability discovered by his teacher.
- Toured Europe after college and published 5 papers in *Journal für die reine und angewandte Mathematik*.
- Studied convergence of infinite series, the theory of doubly periodic functions, elliptic functions, elliptic functions and the theory of equations.
- Could not get employment, so tutored.
- At university, thought he had solution of quintic. Then, proved no solution existed.
- Died of tuberculosis before completing work.

Évariste Galois (1811-1832)

- Born Oct 25, 1811
- Interest in math at 14.
- Read Adrien-Marie Legendre (1752-1833).
- 1828 Failed to get into École Polytechnique.
- 1829 Paper on continued fractions.
- Studied polynomial equations.
- Wrote two papers.
  Reviewed by Arthur Cayley (1821-1895)
  Entered competition.
- 1830 Submitted to Joseph Fourier (1768–1830) - got lost.
  Winners - Niels Henrik Abel (1802-1829) and Carl Gustav Jacobi (1804-1851).
- Published 3 papers.

Figure 14: Évariste Galois

Évariste Galois (cont'd)

- Political turmoil in France.
- Student uprising - Galois left school.
- He was arrested and acquitted.
- Arrested Jul 1831 - April 29, 1832.
- Siméon Denis Poisson (1781-1840) asked him to submit work 1831.
- July 4 - declared work incomprehensible.
- Galois found out in October.
- Stayed up all night; wrote letters and note to Auguste Chevalier.
- On May 30, fought in duel and lost.
- Chevalier forwarded papers for publication by Joseph Liouville.

Figure 15: Legendre, Cayley, Fourier, Jacobi, Poisson, Liouville

Group Theory

1843 - Joseph Louiville (1809-1882) reviewed Galois' delayed manuscript, published 1846. - introduction of groups and fields.

- Multiplicative group modulo n.
- Euler - Fermat's Little Theorem
  p prime, $(a, p) = 1$,
  $$a^{p-1} \equiv 1 (\text{mod } p).$$
- Euler's ϕ function:
  $$\phi(n) = \#\{k \in \{1, 2, \ldots, n-1\} | (k, n) = 1\}.$$
  $\phi(5) = 4$, $\{1, 2, 3, 4\}$,
  $\phi(8) = 4$, $\{1, 3, 5, 7\}$.
- Group Properties:

  closed, identity, inverse, associative

Examples: Mod 5 and 8.

x	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

x	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

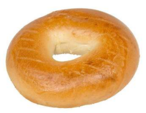
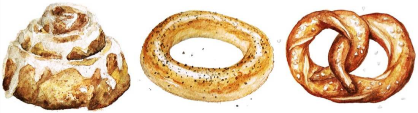
Symmetry Groups

- Levi ben Gorshun (1321) Number of permutations of n objects $= n!$
- Leads to Symmetric Group.
- Felix Klein (1872) extended groups to geometry - studied invariants of groups of transformations.
- Sophus Lie (1842-1899) continuous groups of transformations, applied to differential equations.
- Emmy Noether (1882-1935) related symmetries to constants of motion in physics.

Figure 16: Sophus Lie and Emmy Noether.

Topology and Knot Theory

Fall 2025 - R. L. Herman

What is Topology?

From Wikipedia

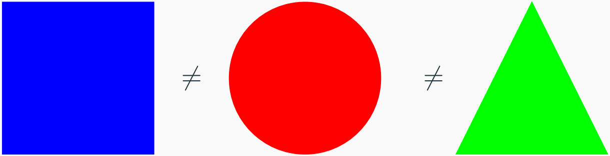
"In mathematics, topology (from the Greek *t*opos, 'place', and *l*ogos, 'study') is concerned with the properties of a geometric object that are preserved under continuous deformations, such as stretching, twisting, crumpling and bending, but not tearing or gluing."

Nobel Prize in Physics 2016

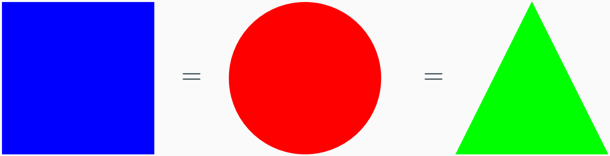
Figure 1: Nobel Prize in Physics 2016

Geometry vs Topology

Geometry



Topology



Types of Topology

General topology (Point Set Topology) Study of basic topological properties derived from properties such as connectivity, compactness, and continuity.

Metric topology Study of distance in different spaces.

Algebraic topology (Combinatorial Topology) Study of topologies using abstract algebra like constructing complex spaces from simpler ones and the search for algebraic invariants to classify topological spaces.

Geometric topology Study of manifolds and their embeddings.

Network topology Study of topology discrete math. Network topologies are graphs consisting of nodes and edges.

Differential Topology Study of manifolds with smoothness at each point to allow calculus.

Origins of Topology

The search for a type of geometry where distance is not relevant.

- Euler - Graphs, Polyhedra
- Gauss, Maxwell - Physics
- Thomson, Tait - Knot Theory
- Riemann - 2D Surfaces in 3D
- Betti - Higher Dimensions
- Klein - Geometry and Groups
- Poincaré - Algebraic Topology
- Noether - Homology Groups

Leonhard Euler (1707-1783) - Königsberg Bridges

- 1736 Correspondences with Carl Gottlieb Ehler (1685-1753)
- Ehler's Letter

  "You would render to me and our friend Kuhn a most valuable service, putting us greatly in your debt, most learned sir, if you would send us the solution, which you know well, to the problem of the **seven Königsberg bridges** together with a proof. It would prove to an outstanding example of the calculus of position [calculi situs] worthy of your great genius. I have added a sketch of the said bridges."

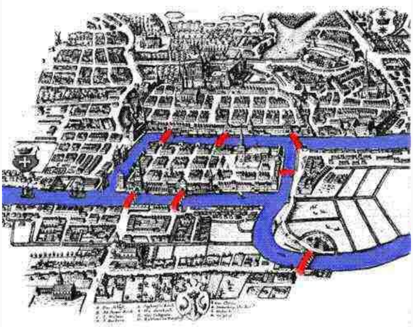
Leonhard Euler (1707-1783) - Königsberg Bridges

- 1736 Correspondences with Carl Gottlieb Ehler (1685-1753)
- Euler's reply

  "Thus you see, most noble sir, how this type of solution bears little relationship to mathematics and I do not understand why you expect a mathematician to produce it rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle. Because of this, I do not know why even questions which bear so little relationship to mathematics are solved more quickly by mathematicians than by others. In the meantime most noble sir, you have a assigned this question to the geometry of position but I am ignorant as to what this new discipline involves, and as to which types of problem Leibniz and Wolff expected to see expressed this way."
- Based on Liebniz's *geometria situs* and *analysis situs*.
- Geometry of position: concerned only with the determination of position and does not involve using distances.

Königsberg Bridges Problem

Can the seven bridges over the river Preger in the city of Königsberg (formerly in Prussia but now known as Kaliningrad, Russia) all be traversed in a single trip without doubling back and ending where you started?



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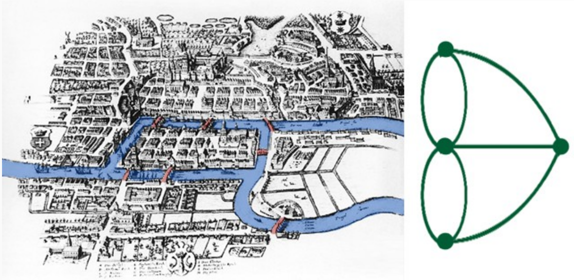

- Euler's result did not depend on the lengths of the bridges or on their distance from one another, but only on connectivity.

Königsberg Bridges Problem

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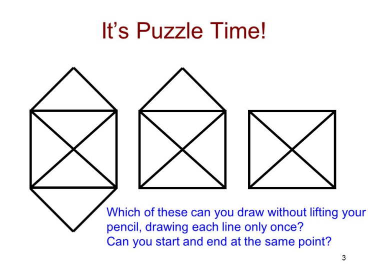

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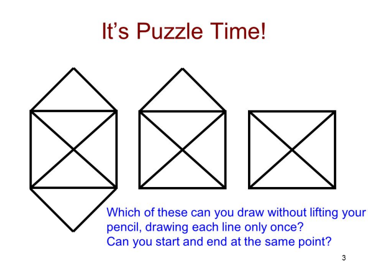


- A connected graph has an Euler cycle
  $\Leftrightarrow$ every vertex has even degree.

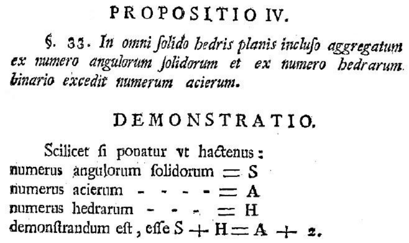
Euler's Polyhedron Formula

- 1750, Euler wrote Christian Goldbach (1690-1764)
- For polyhedron, like Platonic solids,

$$V - E + F = 2.$$

- Published papers in 1752.
- Not known before.
  - Descartes was close (1676).
- Euler characteristic: $\chi = V - E + F$.

Shape	Vertices	Edges	Faces
Tetrahedron	4	6	4
Cube/Hexahedron	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12

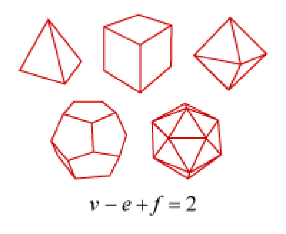


PROPOSITIO IV.

§. 33. *In omni ſolido hedris planis incluſo aggregatum ex numero angulorum ſolidorum et ex numero hedrarum binario excedit numerum acierum.*

DEMONSTRATIO.

Scilicet ſi ponatur vt hactenus:
numerus angulorum ſolidorum = S
numerus acierum - - - - = A
numerus hedrarum - - - = H
demonſtrandum eſt, eſſe S + H = A + 2.



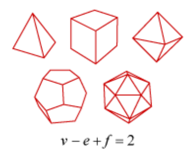
There Are Exactly Five Regular Polyhedra

Proof:

- Let $n = \#$ of sides of each face.
- Let $m = \#$ of faces meeting each vertex.
- $E = \frac{1}{2}Fn$ and $V = \frac{1}{m}Fn$.
- Since $V - E + F = 2$,

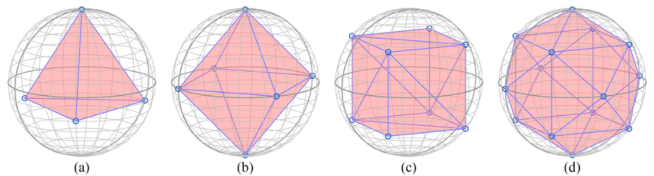
$$F = \frac{4m}{2n - mn + 2m.}$$

- $2n - mn + 2m > 0$ and $n \geq 3$.
  $n = 3$: $2n - mn + 2m = 6 - m$.
  So, $m = 3, 4, 5$.
  $n = 4$: $2n - mn + 2m = 8 - 2m$.
  So, $m = 3$.
  $n = 5$: $2n - mn + 2m = 10 - 3m,\ m = 3$.



Solutions (n, m)
$(3, 3)$ tetrahedron,
$(3, 4)$ octahedron,
$(3, 5)$ icosahedron,
$(4, 3)$ cube,
$(5, 3)$ docecahedron.

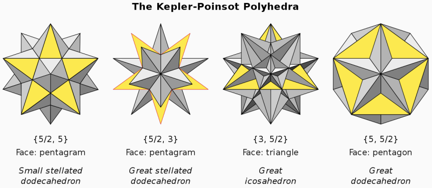
Euler Characteristic of a Sphere

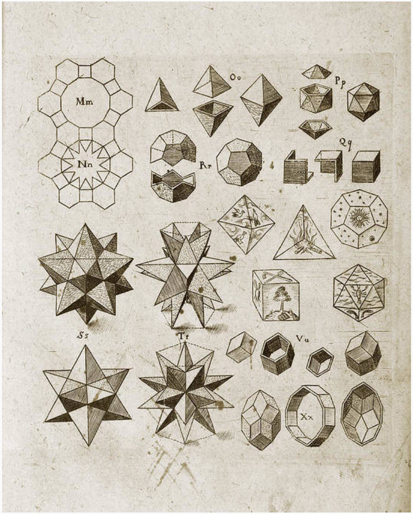


- Inscribe Platonic solids in a sphere.
- a to b: Add 2 vertices, 4 faces, 6 edges. $\Delta\chi = 2 - 6 + 4 = 0$
- Push faces to sphere surface.
- Euler characteristic, $\chi = V - E + F = 2$.

Kepler's Polyhedra - 1619 *Harmonice Mundi*

- Johannes Kepler (1571-1630) systematized and extended polyhedra.
- He defined classes of polyhedra and proved that his set was complete.
- Kepler–Poinsot polyhedron - regular star polyhedra.
- Stellated polyhedra - extend edges or faces until they meet to form a new polyhedron.





Kepler's Mysterium Cosmographicum

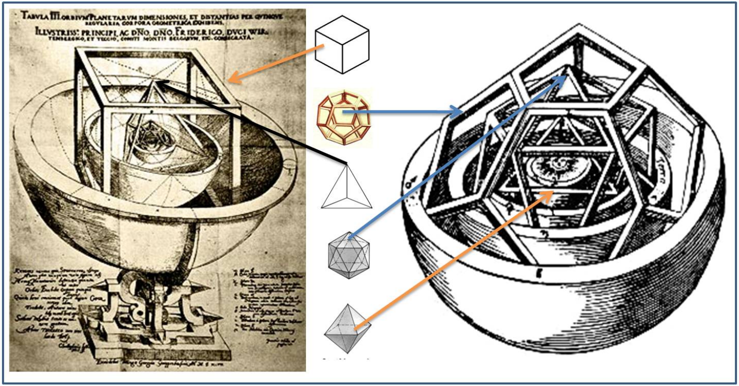


Figure 2: Published at Tübingen in 1596. Second edition, 1621. The orbits of the planets (Mercury, Venus, Earth, Mars, Jupiter and Saturn) were arranged in spheres nested around the five Platonic solids: octahedron, icosahedron, dodecahedron, tetrahedron and cube.

The Thirteen Archimedean Solids: $V - E + F = ?$

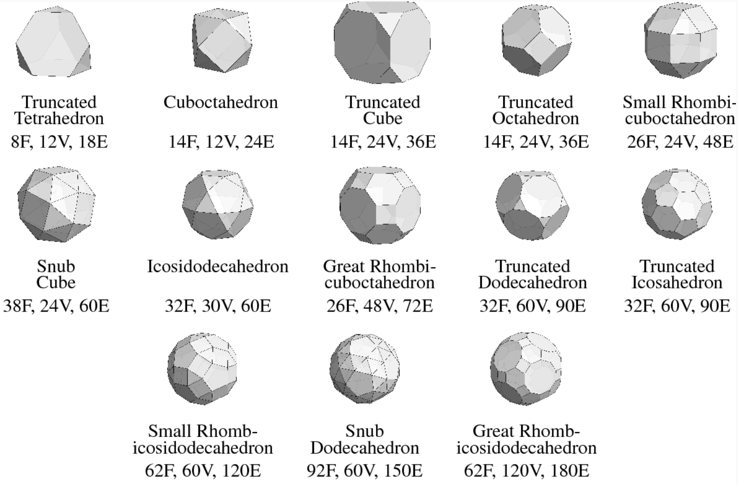
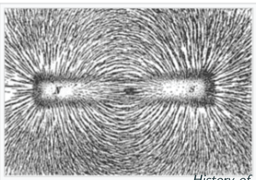


Figure 3: The 13 Archimedean solids. The duals are called Catalan Solids.

The Birth of Electromagnetism

- 1785, Coulomb's Law.
- 1820, New discoveries:
- Ørsted: Electric current deflects compass.
- Biot-Savart: Currents produce magnetic fields.
- Amperè: Parallel wires carrying currents attract or repel. 1827 *électrodynamiques.*
- 1821, Faraday: Electromagnetic rotation.
- 1831, Electromagnetic induction, Faraday's Law.
- Field lines.



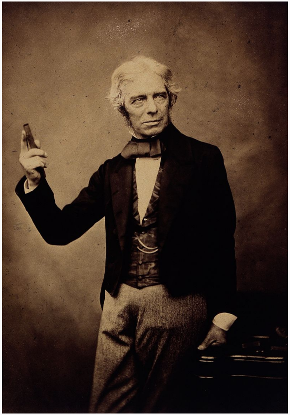
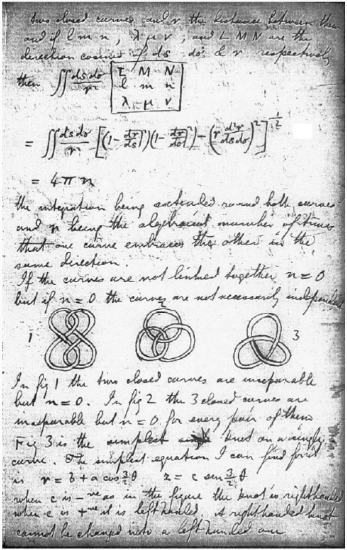


Figure 4: Michael Faraday (1791 – 1867)

Carl Friedrich Gauss - 1831 - EM Induction.

- Gauss and Weber.
    - First telegraph 1833 to communicate 3 km.
    - Mapped Earth's Magnetic Field.
    - Weber - 1856, $c =$ speed of light.
- Gauss introduced formula for two intertwining curves.



two closed curves and r the distance between them and if l m n, λ μ ν, and L M N are the direction cosines of ds dσ & r respectively

then $\iint \frac{ds\,d\sigma}{r^2} \begin{vmatrix} L & M & N \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix}$

$= \iint \frac{ds\,d\sigma}{r^2} \left[\left(1-\frac{dr}{ds}\right)^2 \left(1-\frac{dr}{d\sigma}\right)^2 - \left(r\frac{d^2r}{ds\,d\sigma}\right)^2 \right]^{\frac{1}{2}}$

$= 4\pi n$

the integration being extended round both curves and n being the algebraical number of times that one curve embraces the other in the same direction.

If the curves are not linked together n = 0 but if n = 0 the curves are not necessarily independent

1 3

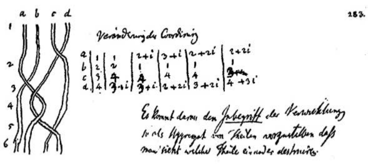
In fig 1 the two closed curves are inseparable but n = 0. In fig 2 the 3 closed curves are inseparable but n = 0 for every pair of them. Fig 3 is the simplest knot on a single curve. The simplest equation I can find for it is $r = b + a\cos\frac{3}{2}\theta$ $z = c\sin\frac{3}{2}\theta$ when c is −ve as in the figure the knot is righthanded when c is +ve it is lefthanded. A righthanded knot cannot be changed into a left handed one

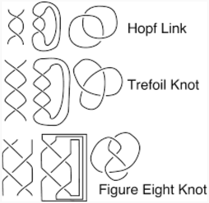
http://mathematiciancfgauss.blogspot.com/2016/11/contribution-of-math-gauss-published-in.html

Gauss' Linking Number and Braids

$$\int\int \frac{(x'-x)\,dydz'-dzdy')+(y'-y)\,(dzdx'-dzdz')+(z'-z)\,(dxdy'-dydx'))}{((x-x')^2+(y-y')^2+(z-z')^2)^{-3/2}} = 4m\pi$$

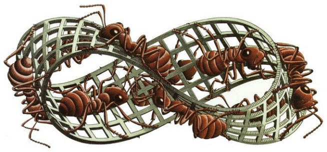
- 1833 Entered in notebook.
- Published in 1867.
- No proof or reason given.
- Possibly from E&M or astronomy. Orbits of Ceres and Pallas 1801
- Gauss studied linked orbits.
- Maxwell sent postcard to Tait.
- Braids (Artin, 1926) drawn in unpublished notebooks.





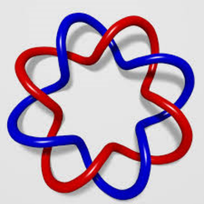
Two of Gauss' Students - Möbius and Listing

Möbius Band - 1858 - independently discovered by both. Möbius went on to study under Pfaff, Gauss' advisor.

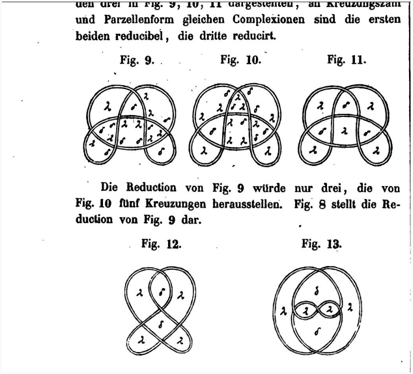


First Use of 'Topologie'

- Johann Listing gave Gauss' *geometria situs* a new name:
- 1847 -*Vorstudien zur Topologie*.
- Studied Connectivity and
- Link Invariants.



Inversion (rotation) and perversion (reflection)



den drei in Fig. 9, 10, 11 dargestellten, an Kreuzungszahl und Parzellenform gleichen Complexionen sind die ersten beiden reducibel, die dritte reducirt.

Fig. 9. Fig. 10. Fig. 11.

Die Reduction von Fig. 9 würde nur drei, die von Fig. 10 fünf Kreuzungen herausstellen. Fig. 8 stellt die Reduction von Fig. 9 dar.

Fig. 12. Fig. 13.

How do you distinguish knots and their symmetries?

Hermann Ludwig Ferdinand Helmhotz (1821-1894)

- German Physicist, Physician.
- Mathematics of the eye, theories of vision, perception of sound, electrodynamics, thermodynamics.
- In 1858 Helmholtz wrote on **vortex dynamics**, translated by Tait into English. *On Integrals of the Hydrodynamical Equations, which Express Vortex-motion*

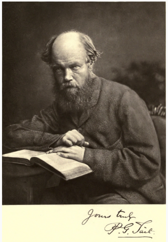
The evolution of a magnetic field $\boldsymbol{B}$ is similar to the evolution of vorticity ω, the curl of the flow velocity, **u.**

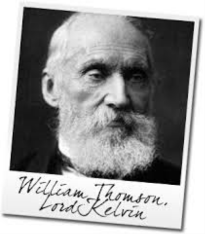
$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}(\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}.$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \mathrm{curl}(\mathbf{u} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega}$$

Scottish Physics and Knots

James Clerk Maxwell (1831-1879), Peter Guthrie Tait (1831-1901), and William Thomson (1824-1907)





https://www.gutenberg.org/files/39373/39373-h/39373-h.htm

Maxwell and Tait met at Edinburgh Academy, went to University 1847.

Thomson (22) elected to Glasgow College Chair of Natural Philosophy.

James Clerk Maxwell and Helmholtz's Water Vortices

Maxwell read Gauss' work and referred to the work of Liebniz, Euler, and Vandermonde on *geometria situs.*

Maxwell wrote to Tait about Helmhotz's paper.

Tait's interest in Helmholtz was from recalling reading Hamilton's *Lectures* in 1853 and remembering formulae.

Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen.

(Von Herrn *H. Helmholtz.*)

Es sind bisher Integrale der hydrodynamischen Gleichungen fast nur unter der Voraussetzung gesucht worden, dafs die rechtwinkligen Componenten der Geschwindigkeit jedes Wassertheilchens gleich gesetzt werden können den nach den entsprechenden Richtungen genommenen Differentialquotienten einer bestimmten Function, welche wir das *Geschwindigkeitspotential* nennen wollen. Allerdings hat schon *Lagrange* *) nachgewiesen, dafs diese Voraussetzung zulässig ist, so oft die Bewegung der Wassermasse unter dem Einflusse von Kräften entstanden ist und fortgesetzt wird, welche selbst als Differentialquotienten eines *Kräftepotentials* dargestellt werden können, und dafs auch der Einflufs bewegter fester Körper, welche mit der Flüssigkeit in

GLENLAIR
DALBEATTIE,
Nov. 13, 1867.

Dear Tait

If you have any spare copies of your translation of Helmholtz on "Water Twists" I should be obliged if you could send me one.

I set [sic] the Helmholtz dogma to the Senate House in '66, and got it very nearly done by some men, completely as to the calculation, nearly as to the interpretation.

Thomson has set himself to spin the chains of destiny out of a fluid plenum as M. Scott set an eminent person to spin ropes from the sea sand, and I saw you had put your calculus in it too. May you both prosper and disentangle your formulae in proportion as you entangle your worbles. But I fear the simplest indivisible whirl is either two embracing worbles or a worble embracing itself.

For a simple closed worble may be easily split and the parts separated

but two embracing worbles preserve each others solidarity thus

though each may split into many, every one of the one set must embrace every one of the other. So does a knotted one.

yours truly
J. CLERK MAXWELL

Sir William Rowan Hamilton (1805–1865)

This led Tait to work on quaternions.

Hamilton discovered **quaternions** in 1843, an extension of complex numbers: $w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, where w, x, y, z are real and $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ satisfy the bridge equations.

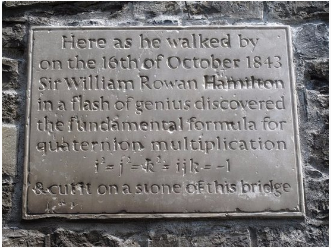


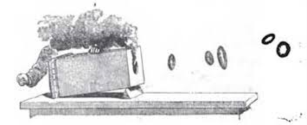
Figure 5: Hamilton carved his equations into the stone of the Brougham Bridge while on a walk.

Back to vortex rings ...

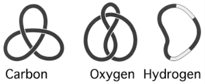
From Vortex Rings to Vortex Atoms

https://i.imgur.com/Y64h8ol.mp4 Movie

Tait experimented with smoke rings in 1867.

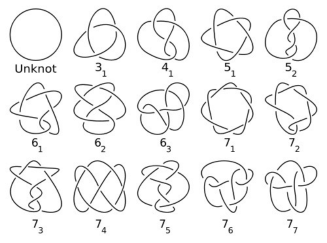


Thomson wrote to Helmholtz ...
later wrote *On vortex atoms*
as vortex rings in the aether.



Tait - Classification of Knots

- Knots up to 7 crossings reduced to 8 different knots.
- Periodic table of knot elements [Mendeleev - 1869].
- Tables up to 10 alternating crossings.
- Aether disproved in 1887 by Michelson Morley Experiment.



Put ideas in envelope for the Royal Society of Edinburgh [Open by 15/10/1987].

Tait Conjectures[1]

1. Reduced alternating diagrams have minimal link crossing number.
2. Any two reduced alternating diagrams of a given knot have equal writhe.
3. The flyping conjecture, which states that the number of crossings is the same for any reduced diagram of an alternating knot.

In 1987 one of Tait's conjectures was found in the envelope.

1,2 proved by Kauffman, Murasugi, and Thistlethwaite 1987.

3 proved by Menasco and Thistlethwaite, 1991 using Jones polynomials, 1984.

Perko Pair 1974

Links to Papers. and Gresham College Lecture about Tait.

[1] From Mathworld

More Knots - Reidemeister Moves

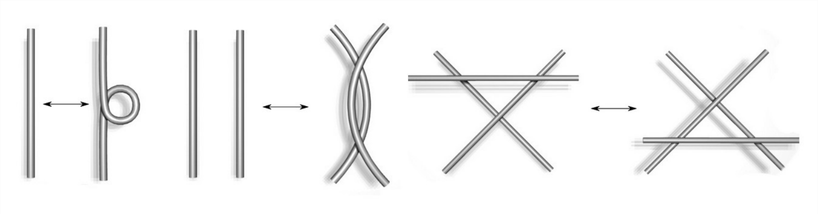


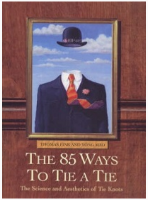
Figure 6: Reidemeister Moves: Untwist, Poke, Slide.

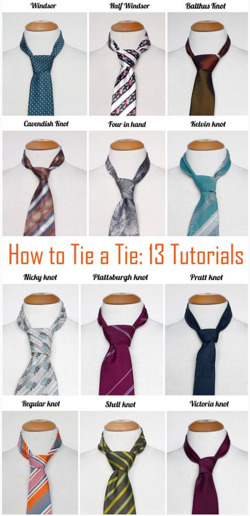
John Conway (1937-2020) - the theory of finite groups, knot theory, number theory, combinatorial game theory, coding theory, recreational mathematics - The Game of Life.

Graduate Student Solves Decades-Old Conway Knot Problem

How to Tie a Tie - Recent Application

- Two physicists, “The 85 Ways to Tie a Tie,” 1999.
- The Man Who Invented Fifteen Hundred Necktie Knots, The New Yorker. Nov. 2023.
- Boris Mocka, doorman, $>$ 1500 knots.
- More ties than we thought, 266,682 distinct tie-knots, 2015.





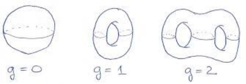
Connection to Surfaces

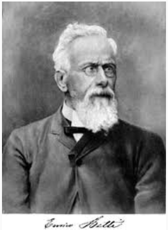


Torus: $V - E + F = 0$
Two-hole Torus: $V - E + F = -2$
Three-hole Torus: $V - E + F = -4$

Bernhard Riemann and Enrico Betti

- Riemann and Betti - connectedness.
- Surfaces with curvature - Manifolds.
- Can we classify surfaces up to a continuous transformation?
- Genus g and Euler Characteristic $\chi = 2 - 2g$.
- Enrico Betti (1823-1892)
- Betti number - maximum number of cuts that can be made without dividing a surface into two separate pieces.





Betti Numbers

- β_0 - number of connected components.
- β_1 - number of handles.
- β_2 - number of voids or. cavities

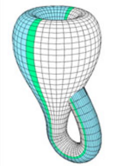
β_0	1	1	1	1
β_1	0	1	0	2
β_2	0	0	1	1

Poincaré Polynomial - Generator of Betti numbers.

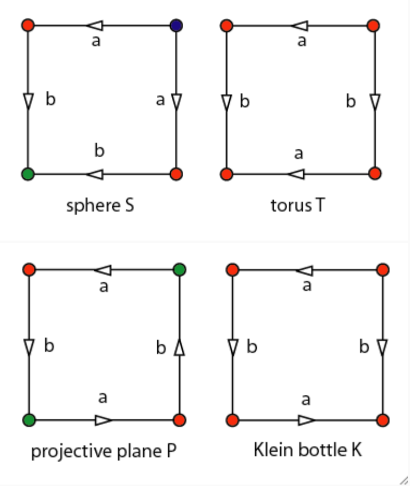
Ex: Torus: $T^2 = S^1 \times S^1$
$p(x) = \beta_0 + \beta_1 x + \beta_2 x^2 = 1 + 2x + x^2 = (1+x)^2.$

Felix Klein (1849-1925)

- 1872 Felix Klein Public Lecture
  - Erlangen Program, Geometry.
  - Symmetry groups $\rightarrow$ invariants.
  - Euclidean Geometry - invariant under translations, rotations.
  - Topology - invariants under continuous transformations.
  - Klein Bottle.



Equivalence Relations



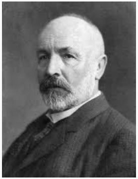
Henri Poincaré (1854-1912)

- 1895 Start of Algebraic Topology.
  - *Analysis Situs*
  - Brings rigor, better Betti Numbers.
- History of Poincaré's Mistakes.
  - 1888 King Oscar II, Sweden, Offered Prize.
  - Judges: Mittag-Leffler, Weierstrass, and Hermite.
  - *Acta Mathematica* - 3 body problem stable.
  - Oops! .... Chaos!
- Topological methods for differential equations.

See Stillwell on Early History of 3-Manifolds

Felix Klein and Georg Cantor

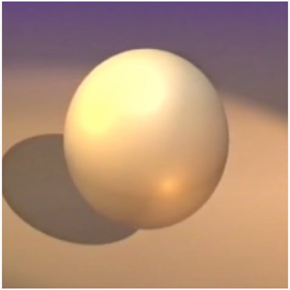
- 1872 Cantor - Open and closed sets.
  - Introduced Set Theory.
  - Infinite and transfinite numbers
  - Cardinality.
- 1902 Hilbert - Neighborhoods.
- 1906 Frechet - Compactness, metric spaces have open and closed sets.
- Riesz 1909 and Hausdorff 1914 - abstract topological spaces.
- 1926 Emmy Noether - Homological groups, corrected Poincaré.



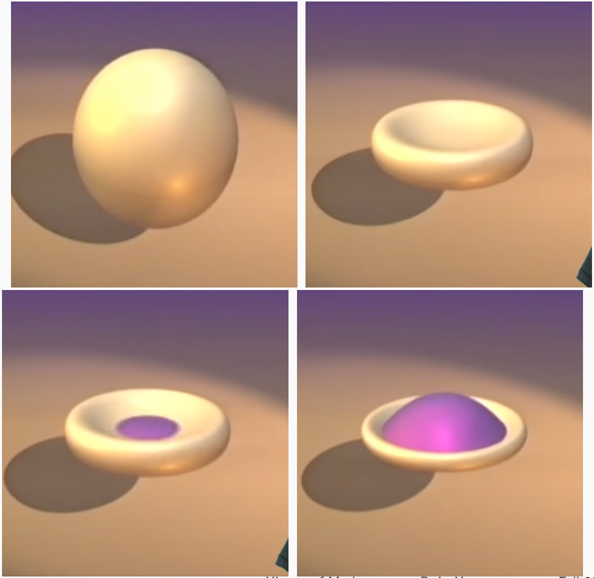

How Do You Turn a Sphere Inside-out?

Sphere Eversion - a continuous deformation, allowing the surface to pass through itself, without puncturing, ripping, creasing, or pinching.

An existence proof for crease-free sphere eversion was first created by Stephen Smale (1957). See Video - *Outside In*



How Do You Turn a Sphere Inside-out?



How Do You Turn a Sphere Inside-out?

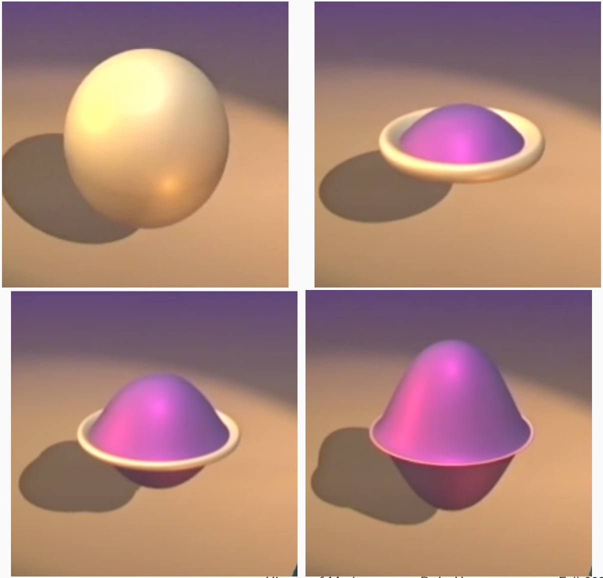
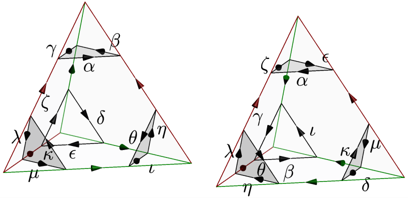


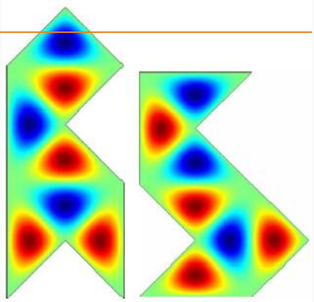
Figure Eight Knot Complement - Honors Thesis

- Start with a Figure Eight Knot.
- Thurston noted knot is the second most commonly occurring knot in garden hoses and vacuum cords.
- What is the space outside the knot?
- 1973 Robert Riley, a graduate student, showed that the figure-eight knot complement had a hyperbolic structure.
- 1978, William Thurston (1936-2012) provides construction.
- Tiled by 2 hyperbolic tetrahedra.



Vibrations and Fourier Analysis

Fall 2025 - R. L. Herman



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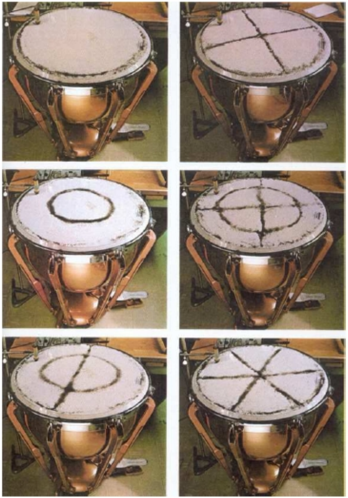
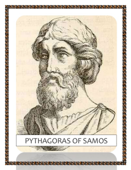
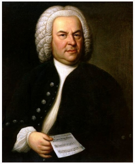


Chladni patterns on a kettledrum
from Risset, *Les instruments de l'orchestre*

Harmonics

- Pythagoras, Ptolemy.
- Galileo and Mersenne, pitch and frequency. Strings produce several tones.
- Joseph Sauveur, 1653-1716, acoustics. Introduced nodes, "harmonic."
- Brook Taylor 1685-1731, fundamental, 1713.
- Johann Bernoulli, 1667-1748, 1727 letter to Daniel. He began studies in 1733.
- Johann Sebastian Bach, 1685-1750, 1722-44 .
- Hermann Helmholtz, 1821-1894, acoustics.
- The debates begin ...





The 1700s Debate - Mathematicians vs Physicists

- Jean le Rond d'Alembet, 1717-1783.
- Vibrating string equation and general solution, $y(x,t) = f(x+t) + g(x-t)$. BCs give $g = f$.
- Leonhard Euler's papers, 1748-9. More general equation with c, and $y(x,t) = f(x+ct) + g(x-ct)$.
- Claimed - f from ICs. $y(x,t) = \frac{1}{2}\left(Y(x+ct) + Y(x-ct) + \frac{1}{c}\int_{x+ct}^{x-ct} V(s)\,ds\right)$.
- Y, V are any curves *drawn by hand*.
- Daniel Bernoulli, 1709-1791, solutions are sums of harmonics, 1753:



$$y(x) = A_1 \sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} + A_2 \sin\frac{2\pi x}{L}\cos\frac{2\pi ct}{L} + \cdots = f(x+ct) + g(x-ct).$$

The Controversy (from Am. J. of Phys. 55, 33 (1987))

d'Alembert vs Euler

- Euler allowed corners.
- d'Alembert's first response - f must be periodic, odd, differentiable. Introduced separation of variables.
- 1761 - the attack! Use of physical arguments is prohibited.
- If slope discontinuous, then acceleration undefined.
- Euler responded 1762, 1765. For small displacement, the function at corner is infinitesimally close to differentiable.

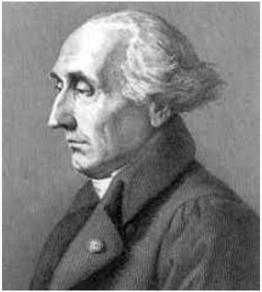
d'Alembert, Euler vs Bernoulli

- d'Alembert did not believe a sum of harmonics.
- Euler sum not general enough - snapped string.
- Bernoulli - "Listen to the string."

They all missed general periodicity.

Joseph-Louis Lagrange

- In enters another math. physicist.
- Born Luigi de la Grange Tournier (1736-1813), in Italy.
- 1759, paper on sound propagation.
- Agreed mostly with Euler, not Bernoulli.
- Avoided wave equation. Used a discrete set of masses.



$$
\begin{aligned}
y(x,t) &= \frac{2}{L}\int_0^L dX\,Y(X)\left[\sin\frac{\pi X}{L}\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L}+\sin\frac{2\pi X}{L}\sin\frac{2\pi x}{L}\cos\frac{2\pi ct}{L}+\dots\right]\\
&+ \frac{2}{\pi c}\int_0^L dX\,V(X)\left[\sin\frac{\pi X}{L}\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L}+\frac{1}{2}\sin\frac{2\pi X}{L}\sin\frac{2\pi x}{L}\cos\frac{2\pi ct}{L}+\dots\right]
\end{aligned}
$$

He almost discovered Fourier series in 1759. [Fourier was born, 1768.]

Jean-Baptiste Joseph Fourier (1768-1830)

- French Revolution, 1789, several arrests.
- Studied under Lagrange, Laplace, Monge.
- Succeeded Lagrange, chair of analysis and mechanics, 1797.
- Joined Napoleon's invasion of Egypt, scientific adviser with Monge, Malus.
- Organizer of French retreat from Egypt.
- Produced a multi-volume work on Egyptology.
- Studied the heat equation and series solutions.
- Almost forgotten in France, not elsewhere due to P. G. J. Dirichlet who wrote on Fourier series. Open problems led Cantor to set theory.

Siméon-Denis Poisson (1781-1840)

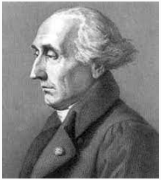
- 1798, entered École Polytechnique.
- Studied under Laplace and Lagrange.
- Degree in mathematics two years.
- Chair of mechanics, Faculty of Sciences, 1809.
- Over 300 papers: definite integrals, Fourier analysis, applied mathematics to physics (mechanics and electrostatics), probability and statistics.
- Poisson brackets, Poisson's constant, Poisson's equation, Poisson's integral, and Poisson's spot.

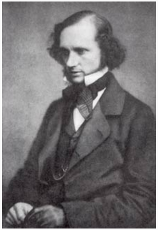
See D. H. Arnold's Work.

Corpuscular vs wave theory - 1818 Competition: Augustin-Jean Fresnel (1788-1827).

The Heat Equation

- Controversy: Fourier vs Poisson
- Fourier 1805, 1807 - diffusion, series solutions ala D. Bernoulli.
- Examiners: Laplace, Lagrange skeptical.
- Poisson Review 1808.
- 1811 Prize problem. Fourier won, but still critics.
- Third version to be book, 1822. Timing affected by politics.
- 1815, Poisson writes his own paper, then book in 1823.
- Wm. Thomson defense of Fourier in 1845.



William Thomson (1824-1907)

- Father, James Thomson, taught math in Belfast and Univ. of Glasgow.
- William attended Univ. of Glasgow, 1834.
- Read Jean-Baptiste-Joseph Fourier.
- First two articles, at 16-17, defended Fourier.
- Cambridge, 1841-5, earned B.A. with high honours.
- In 1845, obtained George Green's essay and went to Paris next day.
- Chair of natural philosophy at the U. of Glasgow at 22.
- Applied Heat Eqn to Age of the Earth
- Gabriel Stokes (1819-1903) introduced him to the telegraphy problem in 1854.

WILLIAM THOMSON: THE YOUNG PROFESSOR

According to Maxwell (1873), the Stokes Thm given in Smith's Prize Examination, 1854, question 8. From 1850 letter from Thomson to Stokes: link.

Thomson's 1855 Paper

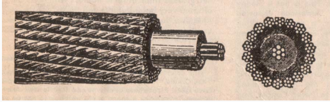
III. "On the Theory of the Electric Telegraph." By Professor WILLIAM THOMSON, F.R.S. Received May 3, 1855.

The following investigation was commenced in consequence of a letter received by the author from Prof. Stokes, dated Oct. 16, 1854. It is now communicated to the Royal Society, although only in an incomplete form, as it may serve to indicate some important practical applications of the theory, especially in estimating the dimensions of telegraph wires and cables required for long distances; and the author reserves a more complete development and illustration of the mathematical parts of the investigation for a paper on the conduction of Electricity and Heat through solids, which he intends to lay before the Royal Society on another occasion.

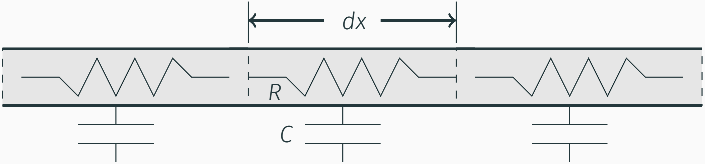
Extract from a letter to Prof. Stokes, *dated Largs, Oct.* 28, 1854.

"Let c be the electro-statical capacity per unit of length of the wire; that is, let c be such that clv is the quantity of electricity

William Thomson's Telegraph Theory - 1855



Treat the coaxial cable as a long, thin conductor, perfectly electrically insulated.



Think of the cable as a network of resistances and electrical capacity (capacitance) and use Kirchoff's laws on an infinitesimal section to derive an equation for the voltage, $v(x, t)$. [Ohm - 1827, Kirchoff - 1845.]

Thomson's Diffusion Equation

This resulted in a diffusion equation:

$$\frac{\partial v^2}{\partial x^2} = RC\frac{\partial v}{\partial t}. \tag{1}$$

It was Fourier's heat equation with solution

$$v = \frac{Q\sqrt{R}}{\sqrt{\pi C t}} e^{-RCx^2/4t}.$$

The maximum effect is at position x and time $t = \frac{1}{6}RCx^2$. This is **Thomson's law of squares**. Examples in Rayleigh's *Theory of Sound.*

Thomson solved several special cases in his correspondence with Stokes as recalled by Thomson.. Thomson's theory had many practical applications.

Further Develpoments

- Thomson had the first teaching laboratory,
- Engaged his students in testing materials and his ideas.
- Used the theory/experiment to understand underwater telegraphy.
- Explained the speed of the current in a telegraph cable,
- Dispersion caused signals of low frequency to diffuse less.

Stokes solved the more general case

$$\begin{aligned} v(x,0) &= 0, \quad 0 < x < \infty \\ v(0,t) &= f(t), \quad 0 < t < \infty, \end{aligned}$$

arriving at the solution

$$v(x,t) = \frac{x}{2\sqrt{\pi}} \int_0^t (t-t')^{-\frac{3}{2}} e^{-x^2/4(t-t')} f(t')\, dt'.$$

William Thomson - a.k.a Lord Kelvin

This was in the backdrop of the Atlantic Cable Project.

- Developed the theory, designed experiments, and obtained patents.
- Was instrumental to the success of the trans-Atlantic cable, completed 1866, after disputes with Whitehouse.
- For his work on the trans-Atlantic telegraph project:
  - Knighted by Queen Victoria, becoming Sir William Thomson, 1866.
  - Recognized for achievements in thermodynamics becoming Baron Kelvin, of Largs, 1892.

Thomson's theory of the electric telegraph remained the main theory for decades. It worked fine for long underwater cables, but to transmit human conversation, the diffusion was far too much.

Maxwell's Theory of Electricity and Magnetism

> During this time scientists were beginning to move from the mechanical world of Newton and Lagrange to the world of Faraday, Oersted, Ampere, and others.

- James Clerk Maxwell (1831-1879)
- Michael Faraday (1791-1867) encouraged Maxwell.
- "A Dynamical Theory of the Electromagnetic Field," EM waves.
- "A Treatise on Electricity and Magnetism," 1873.
- Promoters of Maxwell's work: G. F. Fitzgerald (1851-1901), O. Heaviside (1850-1925), and O. Lodge (1851-1940). The Maxwellians.
- The race was on to produce electromagnetic waves, Hertz (1857-1894).
- Maxwell's theory reworked by Heaviside.

Challenge to Thomson's Theory

- The story of the attempts to connect continents with telegraph cables and Thomson's role is described by Hunt (2012, 2018, 2021).
- The subsequent contributions of Heaviside can be found in (Nahin 2002).
- In 1876 Heaviside derived the telegrapher's equation independently and updated Thomson's diffusion theory by insisting that self-inductance was important.
- This was contrary to what people working on underwater telegraphy believed.
- It led to a few disputes.

Figure 1: Who was Oliver Heaviside?

Oliver Heaviside (1850-1925)

- Heaviside left school at sixteen.
- He studied at home for two years.
- Worked as telegraph operator, Danish-Norwegian-English Telegraph Co., advice from uncle C. Wheatstone, 1868.
- He was transferred to Newcastle-on-Tyne, 1870, and later appointed Chief Operator.
- He left in 1874. Only job he would ever have.
- He spent the next couple of years working on electric theory.
- He studied and reformulated Maxwell's theory.

Note: Heaviside and Josiah Gibbs gave us Vector Analysis and opposed quaternions introduced by Hamilton and promoted by Tait. He gave us Maxwell's Equations.

Oliver Heaviside (1850-1925)[1]

- Heaviside began publishing in 1872.
- He furthered Thomson's theory, 1876.
- Derived the telegraph equation.
- Self-induction is important in telegraphy.
- Others opposed him on this.
- He was asked to stop publishing for *The Electrician* in 1887.
- Heaviside did have some supporters including Thomson and Maxwell.

[1] Marion Camerion Gray (1902-1979) in 1923 wrote *The Equation of Telegraphy* comparing known solutions of

$$\frac{\partial^2 V}{\partial t^2} + 2\gamma \frac{\partial V}{\partial t} = a^2 \frac{\partial^2 V}{\partial x^2},$$

and used the Riemann-Green method.

Heaviside's Operational Calculus

- Used to solve partial differential equations.
- Methods were criticized - not being rigorous and hard to understand.
- First people to publish justifications of Heaviside's methods: Bromwich (1917), Wagner (1917).
- Both used complex integrals.
- Bromwich cited applications from the *Theory of Sound*, William Strutt (1894, Lord Rayleigh) and an equation similar to telegrapher's equation using a Green's function.
- Attempted to justify Heaviside's work and eventually the Laplace transform emerged.

Operational methods for differential equations and the exploration of fractional differentiation had been studied for a number of years going back to the work of Euler and Leibniz.

Some of this is summarized in Moore's 1921 text and from Carslaw and Jaeger's 1941 book on operational methods.

Example: Edmund T. Whittaker Obituary for Heaviside

Edmund T. Whittaker (1873-1956) describes how Heaviside would use operational calculus[2] to solve the differential equation.

$$\frac{d^2y(t)}{dt^2} + k^2y(t) = 0. \tag{2}$$

Let $D = \frac{d}{dt}$. We write symbolically,

$$(D^2 + k^2)y(t) = 0.$$

Now, manipulate algebraically: Multiply by D^{-2},

$$(1 + k^2D^{-2})y(t) = D^{-2}(0).$$

What is $D^{-2}(0)$? - Eventually needed fractional derivatives.

[2] According to Whittaker, Heaviside was accustomed to using symbolic differential operators. Boole (1859) devoted two chapters in *A Treatise on Differential Equations* to symbolic methods.

Fractional differentation

- In 1695 Leibniz communicated about fractional derivatives to Johann Bernoulli and l'Hôpital.
- In 1729 Euler communicated to Goldbach the general form

$$\frac{d^n x^p}{dx^n} = \frac{\Gamma(p+1)}{\Gamma(p-n+1)} x^{p-n}, \tag{3}$$

  using the Gamma function, $\Gamma(n) = n!$ for integers n.
- One definition (Riemann-Liouville)

$$f^{(q)}(x) = \frac{1}{\Gamma(k-q)} \frac{d^k}{dx^k} \int_a^x (x-t)^{k-q-1} f(t)\, dt.$$

- From Euler's formula (3) we have for $n = \frac{1}{2}$

$$D^{1/2} \cdot 1 = \frac{\Gamma(1)}{\Gamma(\frac{1}{2})} t^{-1/2} = \frac{1}{\sqrt{\pi t}}.$$

Age of the Earth

Thomson was interested in problems about the age of the Earth and Sun.

When he was sixteen he wrote that measuring the rate of heat loss from the surface of the Earth could put a bound on the age of the Earth (England, Molnar, and Richter 2007). This interest might have been sparked by reading Fourier's works.

Some of the first quantitative studies of the heat equation were by Fourier (Fourier 1808, 1820, 1822). Fourier had written on the temperature of the Earth and the diffusion of heat in a spherical solid (Godard 2017).

He later wrote a general paper about terrestrial temperatures (Fourier 1824b), which was reprinted (Fourier 1827) and translated in 1837 (Fourier 1824a). This has led to some misconceptions about his role in the origins of the greenhouse effect (Fleming 1999).

Age of the Earth (cont’d)

Naturally Thomson (1862) would use Fourier’s work and in 1862 he predicted the age of the Earth based on the heat equation.

In the mid-1800’s estimates of the age of the Earth went from a few thousand years to hundreds of millions based on geological estimates. Also, Darwin’s theory of evolution came out in 1859.

Assuming an initial high temperature and constant diffusivity, Thomson asked how long it would take to reach the current temperature gradient at the Earth’s surface of 1°F/50 ft. He came up with 98 million years (England, Molnar, and Richter 2007; Nahin 1985; Harrison 1987).

This was not long enough according to the geologists. A debate between physicists and geologists ensued based on Thomson’s estimates (Jackson 2008).

Age of the Earth (cont'd)

Thomson's theory was accepted by the physics community for decades until in 1895 John Perry (1850-1920), a former assistant of Thomson, challenged Lord Kelvin (Perry 1895; England, Molnar, and Richter 2007; Shipley 2001).

Perry challenged Kelvin's assumptions: the thermal conductivity may not be constant. He found an increase in the age estimate.

This led to a debate amongst supporters of Thomson vs those of Perry.

Peter Guthrie Tait (1831 - 1901) sided with Kelvin and Heaviside took up the problem using his operational mathematics, deriving both Kelvin's and Perry's estimates.

Age of the Earth (cont'd)

Heaviside even opened the second volume of his *Electromagnetic Theory* (Heaviside 1922) with a chapter on the Age of the Earth.

It is interesting that Heaviside used a similar analysis of the diffusion equation to arrive at the age of the Earth using Thomson's data. He used took Perry's idea of a nonconstant diffusivity leading to a new equation as described in more detail in (Nahin 1985).

This allowed Perry (1895) to obtain a value for the age of the Earth of more than three times Thomson's estimate of 100 million years (Nahin 1985; Shipley 2001).

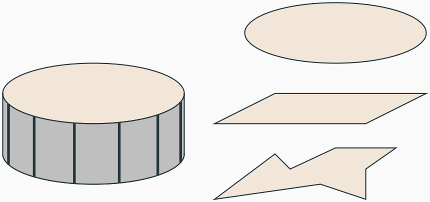
The debate continued for many years later (Jeffreys 1916, 1927).

On to applications -

- Can you hear the shape of a drum?
- How long does it take to cook a turkey?

"Can One Hear the Shape of a Drum?"

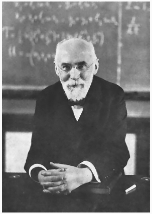
- Kac, Mark (1966). Amer. Math. Monthly. 73, Part II: 1–23.
- Title due to Lipman Bers: "If you had perfect pitch, could you hear the shape of a drum?"
- Can the frequencies (**eigenvalues**) of a resonator (**drum**) determine its shape (**geometry**)?
- Entails features of applied mathematics.
- Historical connections - from radiation theory.



Radiation Theory

- Hendrik Lorentz's (1910) 5 lectures on old/new physics. problems
- 4th - Electromagnetic Radiation Theory.
- Compared vibrations to an organ pipe.
- The number of overtones in frequency range is independent of shape, proportional to volume.
- David Hilbert's prediction (t the conjecture would not be proved in his lifetime), spectrum. His student ...
- Hermann Weyl - < 2 yrs, number $< \lambda$

$$N(\lambda) = \sum_{\lambda_n < \lambda} \sim \frac{|\Omega|}{2\pi}\lambda.$$



What Do We Hear? Frequency, $f = \omega/2\pi$,

Seek Harmonic Solutions,
[Recall $e^{i\omega t} = \cos\omega t + i\sin\omega t$.]

$$u(\mathbf{r},t) = U(\mathbf{r})e^{i\omega t},$$

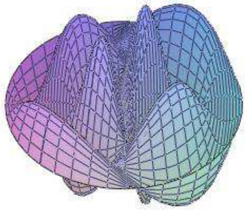
of a Wave Equation, $u(\mathbf{r},t)$

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u.$$

Helmholtz Equation

$$\nabla^2 U = -k^2 U, \quad k^2 = \frac{\omega}{c}.$$

Eigenvalues $\sim$ frequencies



Vibrations of a String

- Ex: Violin String.
- Harmonics, $u_n(x)$.
- Wavelength, $\lambda = \frac{2L}{n}$.
- Wave Speed, $c = \sqrt{\frac{T}{\mu}}$.
- Frequency, $f = n\frac{c}{2L}$.
- A - $f = 440$ Hz, $L = 32$ cm.
  $c = 2Lf = 280$ m/s.
- Nodes, $u_n(x) = 0$

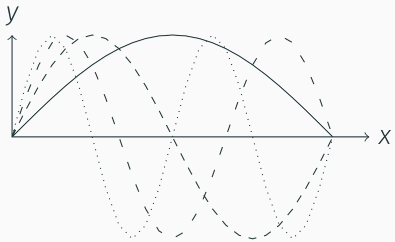


Figure 2: Plot of the eigenfunctions $u_n(x) = \sin\frac{n\pi x}{L}$ for $n = 1, 2, 3, 4$.

Solution of 1D Wave Equation

The one dimensional wave equation, given by

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial^2 x}, \quad t > 0, \quad 0 \leq x \leq L, \tag{4}$$

subject to the boundary conditions

$$u(0,t) = 0, u(L,t) = 0, \quad t > 0,$$

and the initial conditions

$$u(x,0) = f(x), u_t(x,0) = g(x), \quad 0 < x < L.$$

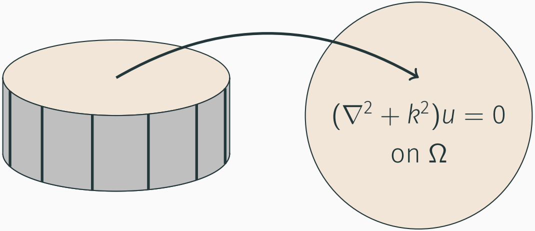
$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \omega_n t + B_n \sin \omega_n t\right] \sin \frac{n\pi x}{L}, \tag{5}$$

where $\omega_n = \frac{n\pi c}{L}$.

General 2D Membranes

- Membrane Problems.
  - Rectangular
  - Circular
  - Elliptical
  - Irregular
- Solve Helmholtz Equations
  - Normal Modes and Frequencies of Oscillation
  - Eigenvalues of Laplace Operator, $\nabla^2 u = -\lambda u$.

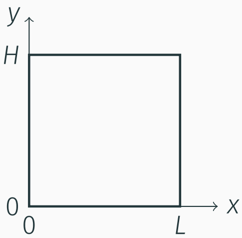
$u = 0$ on $\partial\Omega$



Vibrations of a Rectangular Membrane

- Harmonics
- Frequencies

$$\omega_{mn} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}$$



Boundary-value problem

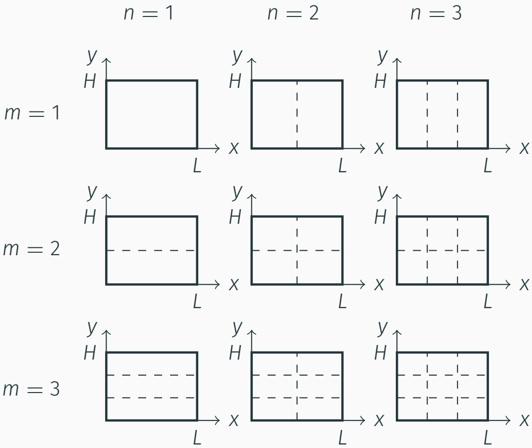
$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad t > 0, 0 < x < L, 0 < y < H, \tag{6}$$

$$u(0, y, t) = 0, \quad u(L, y, t) = 0, \quad t > 0, \quad 0 < y < H,$$
$$u(x, 0, t) = 0, \quad u(x, H, t) = 0, \quad t > 0, \quad 0 < x < L,$$

$$u(x, y, t) = \sum_{n,m} (a_{nm} \cos \omega_{nm} t + b_{nm} \sin \omega_{nm} t) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}.$$

Nodes of a Rectangular Membrane

$$u_{nm}(x,y) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}, \quad f = \frac{c}{2L}\sqrt{n^2+\alpha^2 m^2}, \quad \alpha = \frac{L}{H}.$$



$\alpha = 1$	1	2	3
1	1.414	2.236	3.162
2	2.236	2.828	3.606
3	3.162	3.606	4.243

$\alpha = 2$	1	2	3
1	2.236	4.123	6.083
2	2.828	4.472	6.325
3	3.606	5.000	6.708

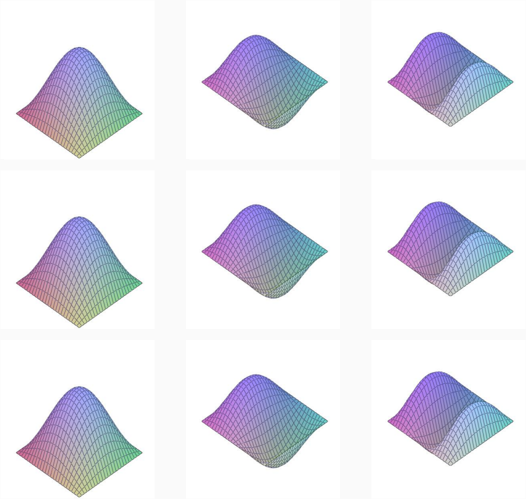
Vibrations of a Rectangular Membrane

$m = 1$ $m = 2$ $m = 3$

$n = 1$

$n = 2$

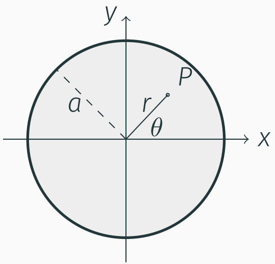
$n = 3$



Vibrations of a Circular Membrane

- Circular Symmetry.
- Harmonics
- Frequencies

$$\omega_{mn} = \frac{j_{mn}}{a} c.$$

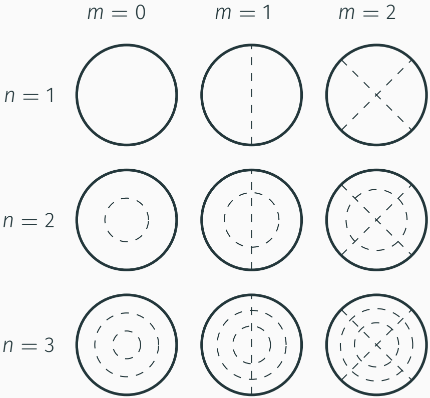


$$u(r, \theta, t) = \left\{ \begin{array}{c} \cos \omega_{mn} t \\ \sin \omega_{mn} t \end{array} \right\} \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} J_m\left(\frac{j_{mn}}{a} r\right). \tag{7}$$

$$J_m(j_{mn}) = 0 \quad m = 0, 1, \ldots, \quad n = 1, 2, \ldots.$$

Nodes of a Circular Membrane

$$u_{mn}(r,\theta) = J_m\left(\frac{j_{mn}}{a}r\right)\cos m\theta, \quad f_{mn} = \frac{j_{mn}c}{2\pi a}.$$

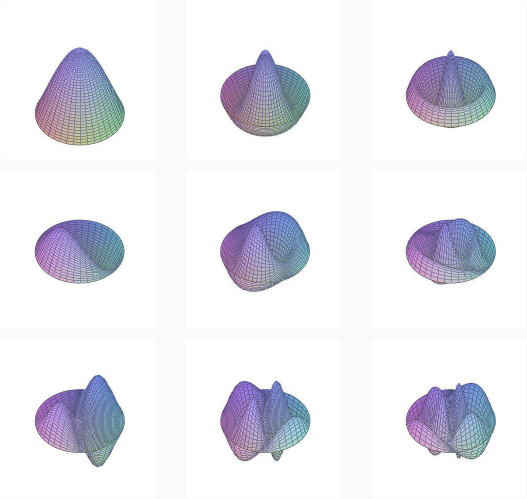


j_{mn}	0	1	2
1	2.405	3.832	5.136
2	5.520	7.016	8.417
3	8.654	10.173	11.62

f_{mn}	0	1	2
1	1.531	2.440	3.270
2	3.514	4.467	5.358
3	5.509	6.476	7.398

Vibrations of a Circular Membrane

$n = 1$ $n = 2$ $n = 3$

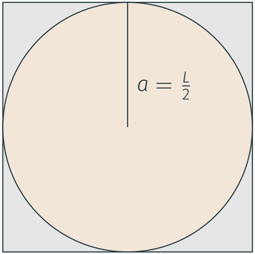


$m = 0$

$m = 1$

$m = 2$

Rectangular and Circular Membrane Frequencies



Rectangular

	1	2	3
1	1.414	2.236	3.162
2	2.236	2.828	3.606
3	3.162	3.606	4.243

Circular $a = \frac{L}{2}$

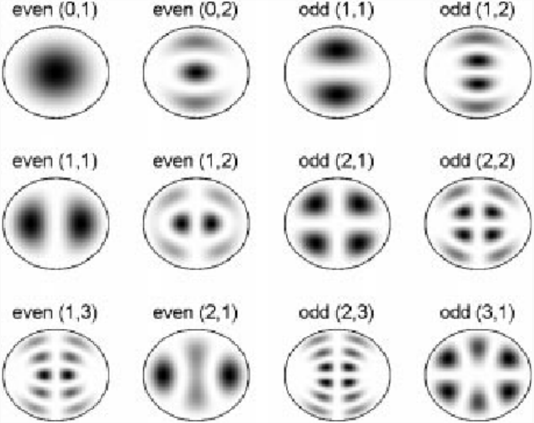
	0	1	2
1	1.531	2.440	3.270
2	3.514	4.467	5.358
3	5.509	6.476	7.398

Circular $\pi a^2 = L^2$

	0	1	2
1	1.357	2.162	2.898
2	3.114	3.958	4.749
3	4.882	5.740	6.556

Vibrations of an Elliptical Membrane

$$\left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + (kh)^2(\cosh^2 \xi - \cos^2 \eta)\right] u(\xi, \eta) = 0.$$



Vibrations of a Balloon

The wave equation takes the form

$$u_{tt} = \frac{c^2}{r^2} L u, \quad \text{where} \quad LY_{\ell m} = -\ell(\ell+1) Y_{\ell m}$$

for the spherical harmonics $Y_{\ell m}(\theta, \phi) = P_\ell^m(\cos\theta) e^{im\phi}$, The general solution is found as

$$u(\theta, \phi, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[A_{\ell m} \cos \omega_\ell t + B_{\ell m} \sin \omega_\ell t \right] Y_{\ell m}(\theta, \phi),$$

where $\omega_\ell = \sqrt{\ell(\ell+1)} \frac{c}{R}$.

Modes for a Vibrating Spherical Membrane (Balloon?)

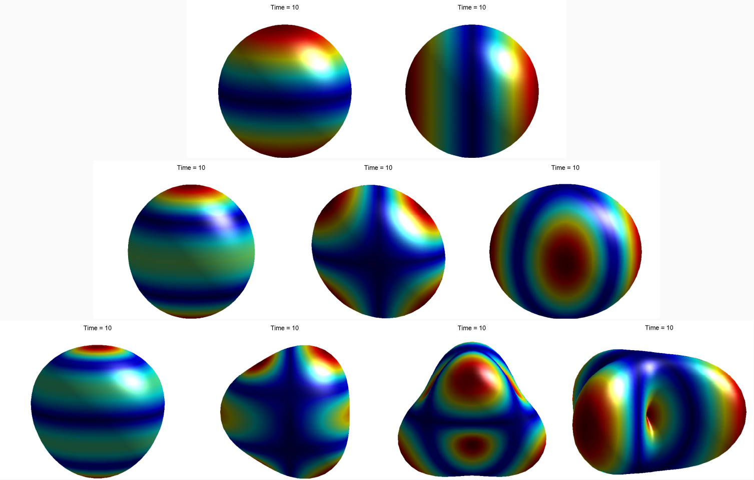
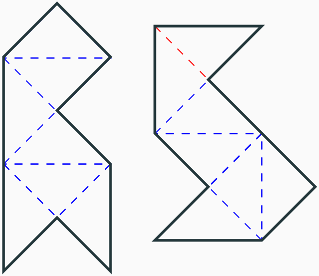
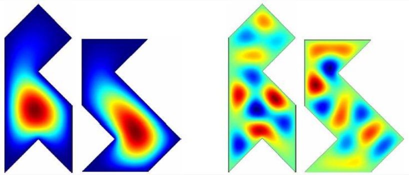


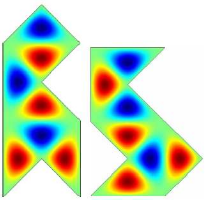
Figure 3: *http://people.uncw.edu/hermanr/pde1/sphmem/*
Row 1: $(1, 0), (1, 1)$; Row 2: $(2, 0), (2, 1), (2, 2)$;
Row 3 $(3, 0), (3, 1), (3, 2), (3, 3)$.

Vibrations of a Irregular Membranes

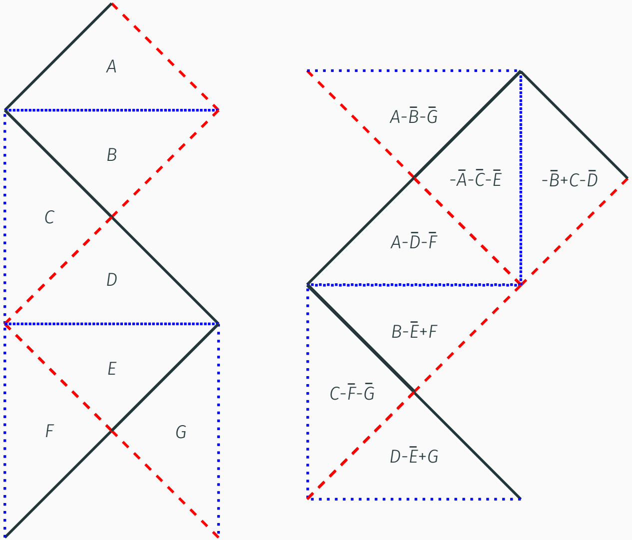
- Gordon, C., Webb, D., and Wolpert, S.(1992) - *You Cannot Hear the Shape of a Drum*
- Shapes on right have same set of frequencies - **isospectral drums.**





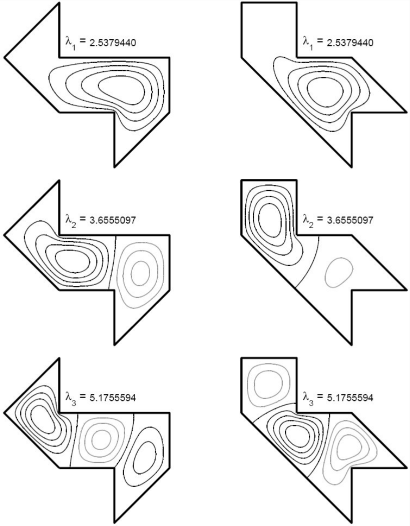


Isospectral Drums



Spectra of Isospectral Drums

$\lambda = 2.5379440, 3.6555097, 5.1755594.$



Other Isospectral Drums

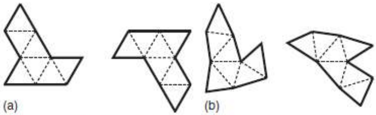


FIG. 25. Pair 7_2. Sunada triple $G=\mathbf{PSL}(3,2)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 1)(2\ 5)$, $b_1=(1\ 5)(3\ 4)$, $c_1=(0\ 4)(1\ 6)$, $a_2=(0\ 4)(2\ 3)$, $b_2=(0\ 6)(1\ 4)$, and $c_2=(0\ 2)(1\ 5)$.

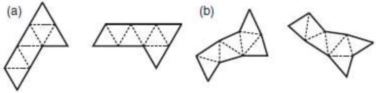


FIG. 26. Pair 7_3. Sunada triple $G=\mathbf{PSL}(3,2)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(2\ 5)(4\ 6)$, $b_1=(1\ 5)(3\ 4)$, $c_1=(0\ 4)(1\ 6)$, $a_2=(0\ 3)(2\ 4)$, $b_2=(0\ 6)(1\ 4)$, and $c_2=(0\ 2)(1\ 5)$.

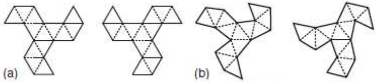


FIG. 27. Pair 13_1. Sunada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 12)(1\ 10)(3\ 5)(6\ 7)$, $b_1=(0\ 10)(2\ 9)(3\ 4)(5\ 8)$, $c_1=(0\ 4)(1\ 6)(2\ 11)(9\ 12)$, $a_2=(0\ 4)(2\ 3)(6\ 8)(9\ 10)$, $b_2=(0\ 1\ 2)(1\ 4)(5\ 11)(6\ 9)$, and $c_2=(0\ 10)(1\ 5)(2\ 7)(3\ 12)$.

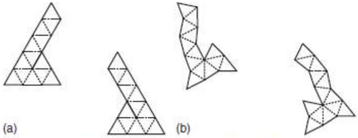


FIG. 31. Pair 13_5. Sunada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(1\ 7)(3\ 5)(4\ 9)(6\ 10)$, $b_1=(0\ 5)(1\ 2)(6\ 12)(9\ 11)$, $c_1=(0\ 4)(1\ 6)(2\ 11)(9\ 12)$, $a_2=(0\ 9)(4\ 10)(6\ 8)(7\ 12)$, $b_2=(0\ 11)(1\ 8)(2\ 7)(3\ 4)$, and $c_2=(0\ 10)(1\ 5)(2\ 7)(3\ 12)$.

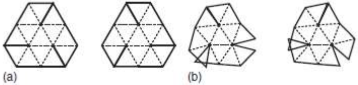


FIG. 32. Pair 13_6. Sunada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 2)(1\ 7)(3\ 6)(5\ 10)$, $b_1=(0\ 6)(2\ 4)(3\ 8)(5\ 9)$, $c_1=(0\ 5)(1\ 2)(6\ 12)(9\ 11)$, $a_2=(0\ 7)(3\ 11)(6\ 8)(9\ 12)$, $b_2=(0\ 8)(1\ 10)(5\ 11)(7\ 9)$, and $c_2=(0\ 11)(1\ 8)(2\ 7)(3\ 4)$.

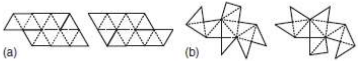


FIG. 33. Pair 13_7. Sun+ada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 2)(1\ 7)(3\ 6)(5\ 10)$, $b_1=(0\ 4)(2\ 3)(6\ 8)(9\ 10)$, $c_1=(0\ 5)(1\ 2)(6\ 12)(9\ 11)$, $a_2=(0\ 7)(3\ 11)(6\ 8)(9\ 12)$, $b_2=(0\ 12)(1\ 1\ 0)(3\ 5)(6\ 7)$, and $c_2=(0\ 11)(1\ 8)(2\ 7)(3\ 4)$.

Can one hear the shape of a drum? -

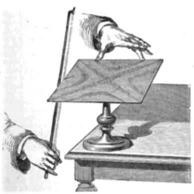
No!

Membranes - Rectangular, circular, elliptical, irregular

Never look at MATLAB logo the same way again - Why?

Chladni Plates

- Recall Sophie Germain, 1776-1831.
- Ernst Chladni, 1756-1827, physicist and musician.
- In 1808, Chladni demonstrated vibrating plates at the Academy of Science in Paris.
- Napoleon, who attended, proposed a prize.
- Lagrange, Laplace and others – felt that it was beyond reach.
- Germain only one to try.
- 1816, two more tries, first woman awarded Grand Prize in Mathematics of the Paris Academy of Sciences.



Heat Equation vs Wave Equation

1D Wave Equation

$$u_{tt} = c^2 u_{xx}$$

1D Heat Equation

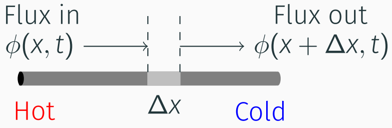
$$u_t = k u_{xx}$$

History of Heat Equation

Developed by Joseph Fourier (1768-1830)

- Discovered in early 1807 and published later in 1822

  Afterwards, diffusion processes studied outside of France.
  Lead to research in partial differential equations.
- Describes conduction and storage of heat (energy) in a body.
- Involves heat exchange with surroundings, conservation of energy.
- Leads to temperature changes inside body (diffusion).
- Uses the relation of heat energy to temperature (gradient), Fourier Law of Heat Conduction.



Rate of change of heat energy = Flux in - Flux out

$$\frac{dQ}{dt} = \phi(x,t) - \phi(x+\Delta x,t).$$

Flux density = conductivity $\times$ temperature gradient

$$\phi = k\frac{dT}{dx}.$$

Heat energy is proportional to temperature

$$Q = mcT.$$

$q =$ Heat energy per vol, $u =$ temperature per vol

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \quad D = \frac{k}{mc}.$$

Thanksgiving Turkey!

- Native to North America.
- Introduced in Spain in 1500's.
- Benjamin Franklin - national bird.
- Holiday bird in Europe in 1800's
  - replacing goose.
- Turkeys mostly walk.
- Harold McGee: Breast 155-160 F, Legs 180 F.
- Cooking times

  Constant oven temp, diffusivity constant, Turkey plump

  Small - 20 min/lb + 20.

  Large - 15 min/lb + 15.

  $t \sim M^{2/3}$.

How long does it take to cook a turkey?

Example 1

If it takes 4 hours to cook a 10 pound turkey in a 350^o F oven, then how long would it take to cook a 20 pound turkey at the same conditions?

Figure 4: A Thanksgiving turkey - From 2015.

Panofsky Equation

- Pief Panofsky [SLAC Director Emeritus] *SLAC Today*, Nov 26, 2008 *http://today.slac.stanford.edu/a/2008/11-26.htm*
  For a stuffed turkey at 325° F

  $$t = \frac{W^{2/3}}{1.5}$$

  vs. 30 minutes/lb.
- Also, check out WolframAlpha *http://www.wolframalpha.com/input/?i=how+long+should+you+cook+a+turkey*
- Musings of an Energy Nerd *http://www.greenbuildingadvisor.com/blogs/dept/musings/heat-transfer-when-roasting-turkey*

Consider a Spherical Turkey

Figure 5: The depiction of a spherical turkey.

Scaling a Spherically Symmetric Turkey

The baking follows the heat equation.

Rescale the coordinates (r, t) to (ρ, τ) :

$$r = \beta\rho \text{ and } t = \alpha\tau.$$

Then, the heat equation rescales as

$$u_\tau = \frac{\alpha}{\beta^2}\frac{k}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right).$$

- Invariance of heat equation implies $\alpha = \beta^2$.
- So, if the radius increases by a factor of β, then the time to cook the turkey increases by β^2.

Problem Solution

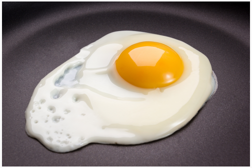
Example 1
If it takes 4 hours to cook a 10 pound turkey in a 350^o F oven, then how long would it take to cook a 20 pound turkey at the same conditions?

- The weight doubles $\Rightarrow$ the volume doubles.
  (if density $=$ constant).
- $V \propto r^3 \Rightarrow r$ increases by factor: $2^{1/3}$.
- Therefore, the time increases by a factor of $2^{2/3} \approx 1.587$.
- If 4 lb turkey takes 4 hrs, then a 20 lb turkey takes

$$t = 4(2^{2/3}) = 2^{8/3} \approx 6.35 \text{ hours.}$$

- In general, if the weight increases by a factor of x, then the time increases by $x^{2/3}$.

Eggs

Omelettes

Egg Protein

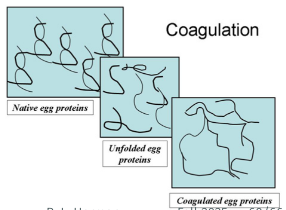
Proteins in eggs can be used

- to help food set (e.g. egg custards),
- as a foam to add air and volume (e.g. sponge cakes),
- as an emulsifier (e.g. mayonnaise).

Two different major proteins, egg white (albumin) and egg yolk,

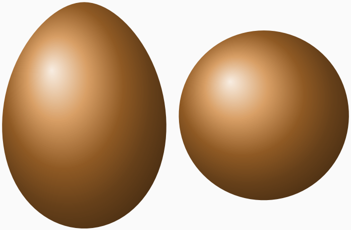
- Albumin starts coagulating at 63^oC
- Yolks start at 70^oC

Coagulation - protein unfolds, denaturation.
As heat increases the proteins rearrange and coagulate.
Egg albumin turns from clear to cloudy white.



Egg Cooking Time

Peter Barnham, *The Science of Cooking* & Dr. Charles Williams of Exeter:

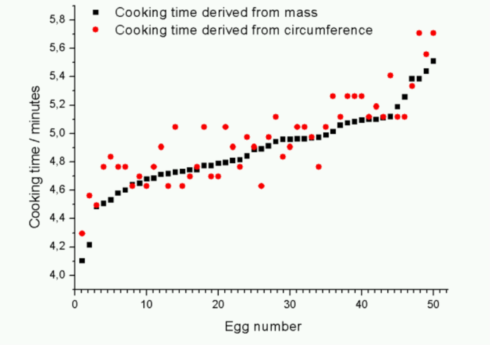


$$t = 0.0152 d^2 \log\left[2 \times \frac{(T_{\text{water}} - T_0)}{T_{\text{water}} - T_{\text{yolk}}}\right],$$

$$t = 0.451 M^{2/3} \log\left[0.76 \times \frac{(T_{\text{water}} - T_0)}{T_{\text{water}} - T_{\text{yolk}}}\right],$$

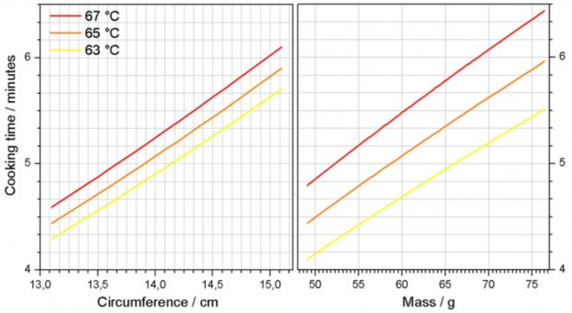
for t min, diameter d cm, M g, and temperatures in ^{o}C.

From Khymos *Towards the perfect soft boiled egg* by Martin Lersch, April 9th, 2009. See also University of Oslo Applet



50 eggs with $T_{yolk} = 63^oC$, $T_{water} = 100^oC$ and $T_{egg} = 4^oC$.

Egg Cooking Time - Formula



Given circumference or mass to reach to reach 63, 65 and 67^o C, respectively, at the yolk-white boundary with $T_{water} = 100^o$ C and $T_{egg} = 4^o$ C.

Egg Consistency

Temp	White	Yolk
62	Begins to set, runny	Liquid
64	Partly set, runny	Begins to set
66	Largely set, still runny	Soft solid
70	Tender solid	Soft solid, waxy
80	Firm	Firm
90	Rubbery solid	Crumbly texture

At sea level, boiling water is 100^o C. At higher altitudes, the boiling temperature of water is lowered 0.3^o C for each additional 100 m above sea level.

Fast Fourier Transform - FFT

- One of top algorithms of 20th Century.
- Developed by Cooley and Tukey, 1965, to compute DFT (Discrete fourier transform)
- Some traced the ideas back to Gauss.
- Limit of Fourier series = Fourier Transform.
- Related to Laplace transform.

$$\begin{aligned} F(k) &= \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx, \\ f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ikx}\, dk. \\ F(s) &= \int_{0}^{\infty} f(t) e^{-st}\, dt. \end{aligned} \tag{8}$$

Left for another course!

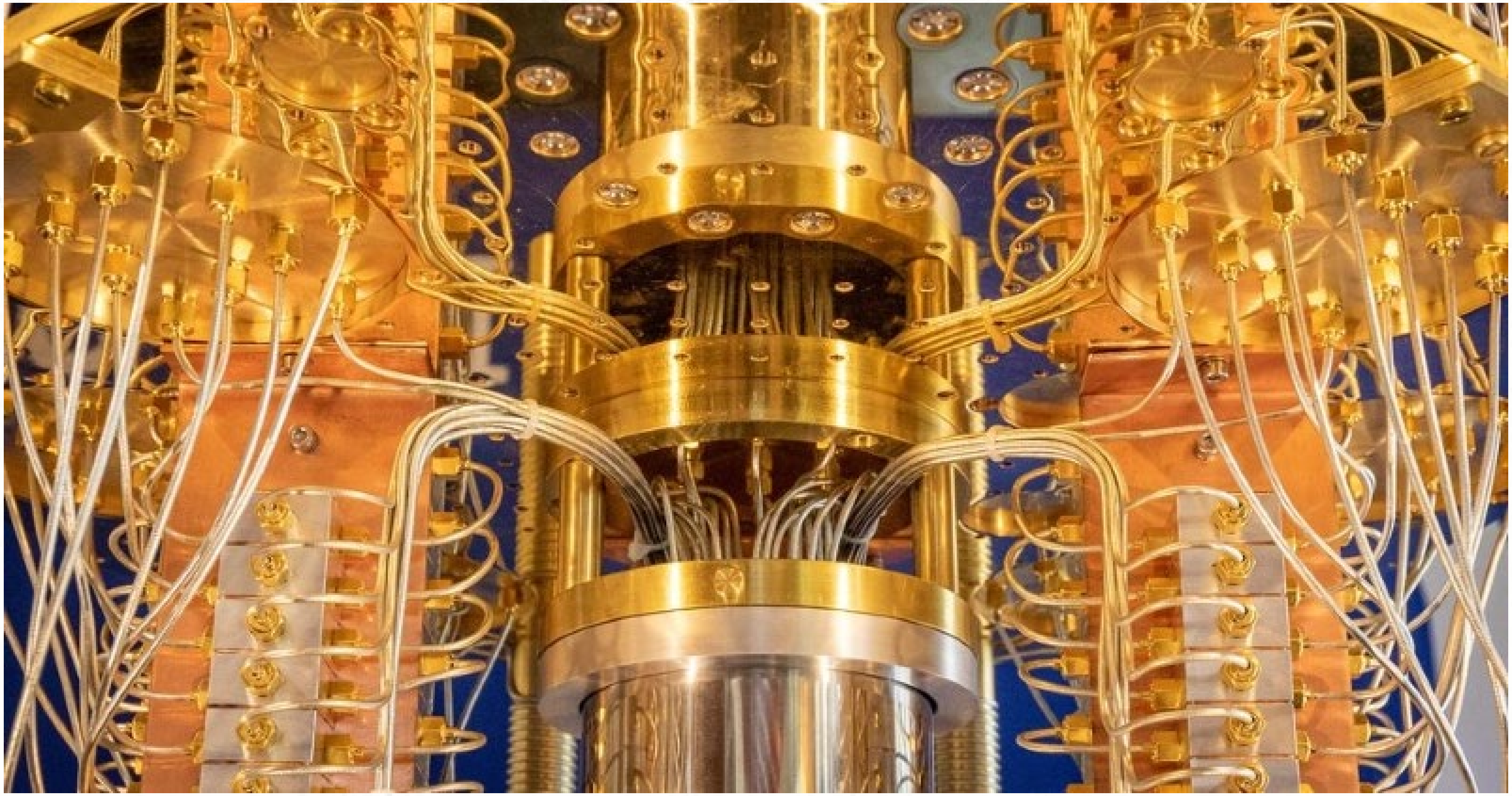
References for Drums

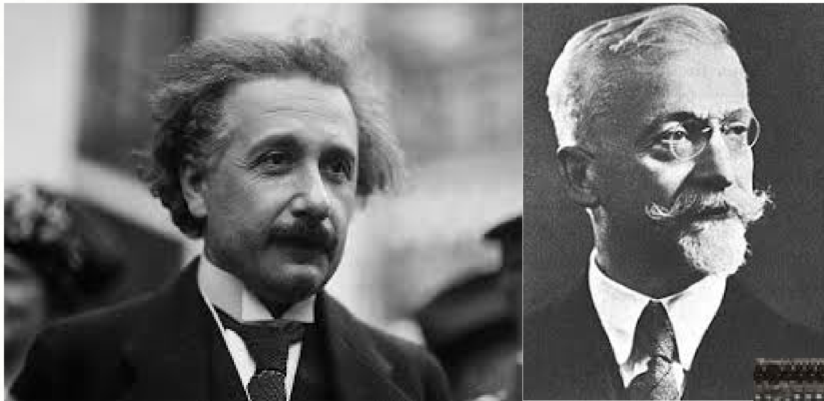
- S. J. Chapman, Drums that sound the same, *Amer. Math. Monthly* 102 (1995), 124-138.
- Tobin Driscoll, Eigenmodes of isospectral drums, SIAM Review 39 (1997), 1-17.
- Carolyn Gordon, David Webb, Scott Wolpert, One cannot hear the shape of a drum, *Bull. Amer. Math. Soc.* 27 (1992), 134-138.
- Marc Kac, Can one hear the shape of a drum?, *Amer. Math. Monthly* 73 (1966), 1-23.
- Cleve Moler, The MathWorks logo is an eigenfunction of the wave equation (2003).
- Lloyd N. Trefethen and Timo Betcke, Computed eigenmodes of planar regions (2005).

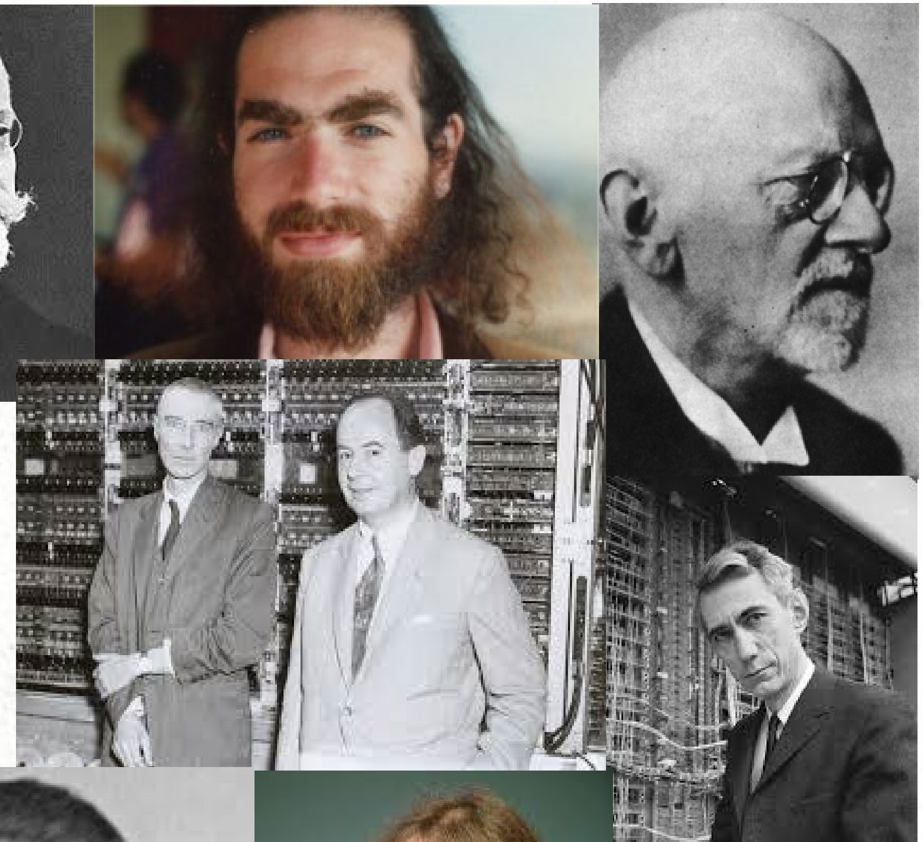
History of Mathematics - 1900s

Dr. R. L. Herman, UNCW

MAT 346, Fall 2023





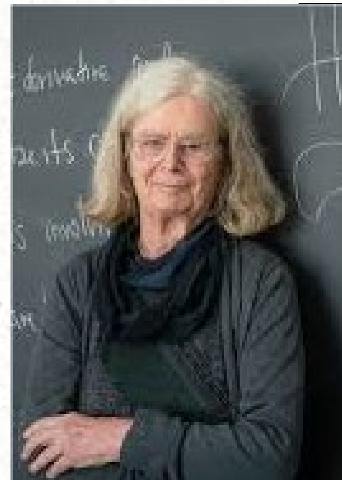
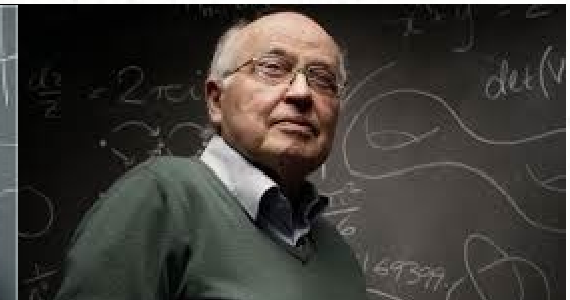


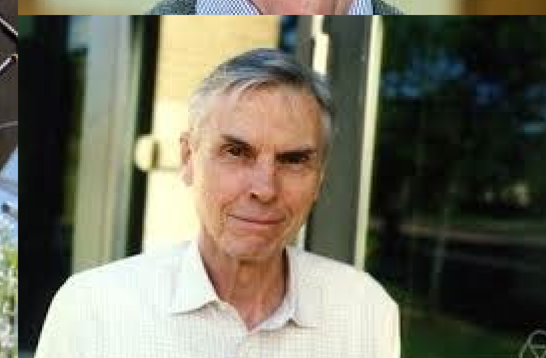
Outline

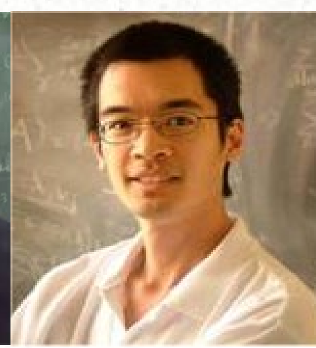
- The Rise of Rigor
- Set Theory
- N. Bourbaki
- Physics Revolutions
- Hilbert’s Problems
- Mathematics Prizes

Mathematics Prizes

- Fields Medal
- Abel Prize
- Wolf Prize
- Millenium Prize

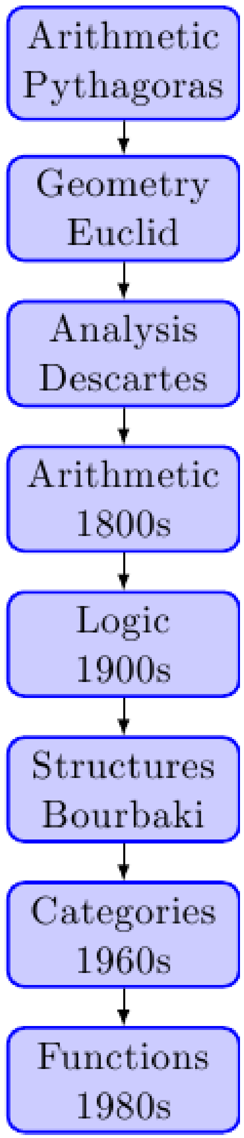



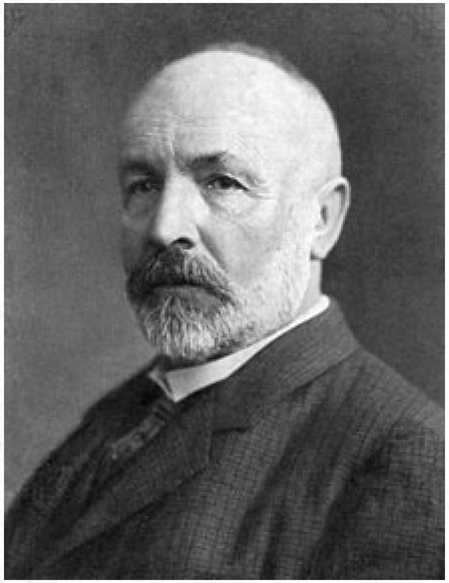



Evolution of Mathematics

- Pythagoreans - Arithmetic
- Euclid - Geometry
- Descartes – Analytic Geometry
- Newton – Calculus, Mechanics
- Euler – Numbers, Applications
- Gauss – Noneuclidean geometry
- Cantor – Set Theory, Infinity
- Frege – Predicate Logic
- Russell – *Principia Mathematica*
- Gödel – Incompleteness Theorems



Georg Cantor (1845-1918)



- Created set theory.

  [How did Cantor Discover Set Theory and Topology?](#)

  – Connection to Fourier series.

- Cardinality of sets.

- Not all infinities have same cardinality.

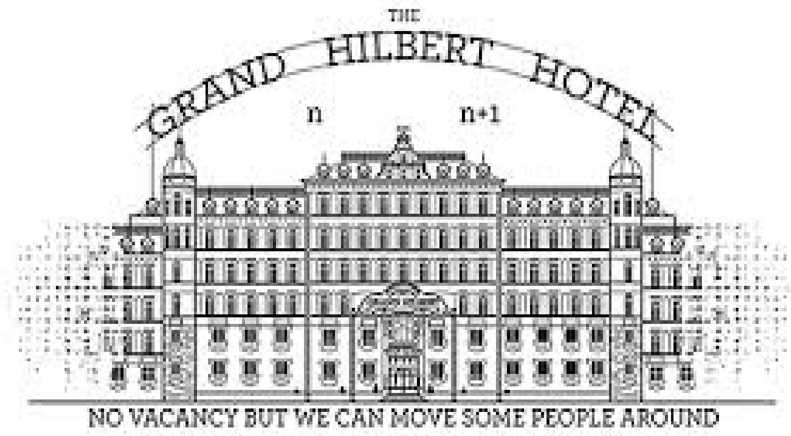
– Natural Numbers, Rationals, Reals

- Transfinite numbers and the Continuum hypothesis $\aleph_0$

  There are no intermediate cardi $\aleph_1$ numbers between (aleph-null) and the cardinality of the continuum (set of real numbers).

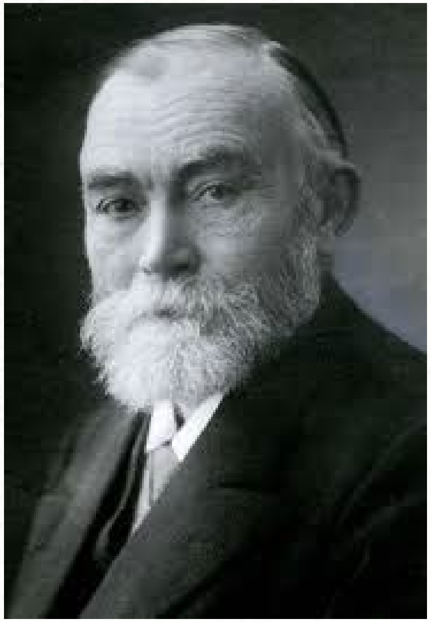
Hilbert's Grand Hotel - 1924

- David Hilbert (1862-1923)
- Infinite # rooms fully occupied with infinite # people.
- First add one person.
- Then, a bus with an infinite number of guests.
- What about an infinite number of buses filled with an infinite number of people?



Gottlob Frege (1848-1925)

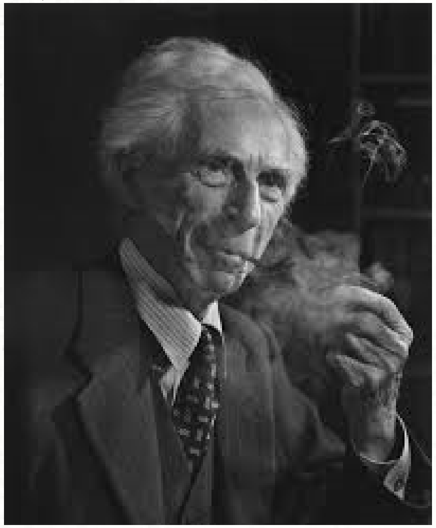
Invented axiomatic predicate logic,
Essential to



- Principia Mathematica (1910–13) Bertrand Russell, (1872–1970), and Alfred North Whitehead, (1861–1947).
- Kurt Gödel's (1906–78) incompleteness theorems.
- Alfred Tarski's (1901–83) theory of truth.
- Development of set theory.

Bertrand Russell, (1872–1970)

- Philosopher, logician, mathematician, historian, writer, essayist, social critic, political activist, and Nobel laureate



- Liar's Paradox

  “This sentence is a lie.”

- Russell's Paradox

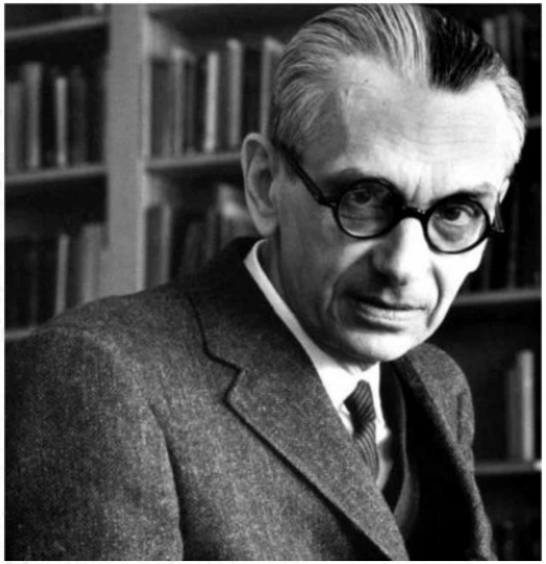
  Consider the set of all sets (power set) that are not members of themselves.

$$\text{Let } R = \{x \mid x \notin x\}, \text{ then } R \in R \iff R \notin R$$

- Avoid paradox - Zermelo–Fraenkel set theory

Kurt Gödel (1906–78)

- 1930 Incompleteness Theorems
- 1. If a logical, or axiomatic formal, system is consistent, it cannot be complete.
- 2. The consistency of axioms cannot be proved within their own system.
- Met Einstein 1933.
- Moved to Princeton 1940.
- 1949 Rotating universes and time travel.




Henri Cartan


André Weil


Jean Dieudonné


Claude Chevalley


René de Possel


Charles Ehresmann


Pierre Samuel

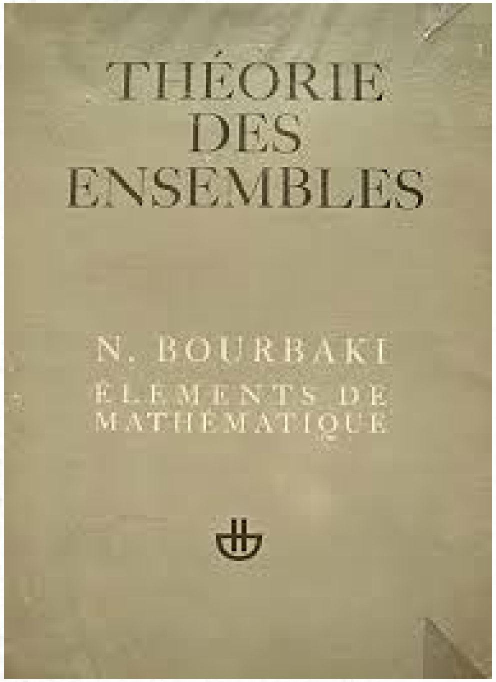

Jean-Pierre Serre


Laurent Schwartz


Adrien Douady

Nicolas Bourbaki (1935 -)

École Normale Supérieure, Paris



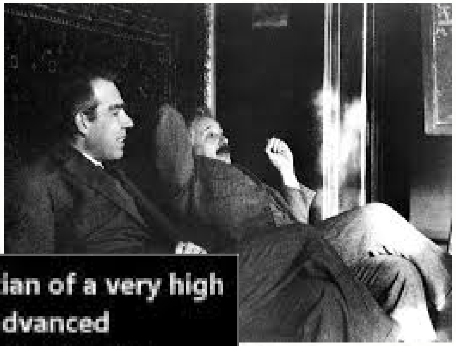
- Founders
  - Henri Cartan,
  - Claude Chevalley,
  - Jean Coulomb,
  - Jean Delsarte,
  - Jean Dieudonné,
  - Charles Ehresmann,
  - René de Possel, Szolem Mandelbrojt,
  - André Weil.
- Notable participants in later days:
  - Schwartz, Serre, Grothendieck, Eilenberg, and Lang.

Bourbaki (1935 -)

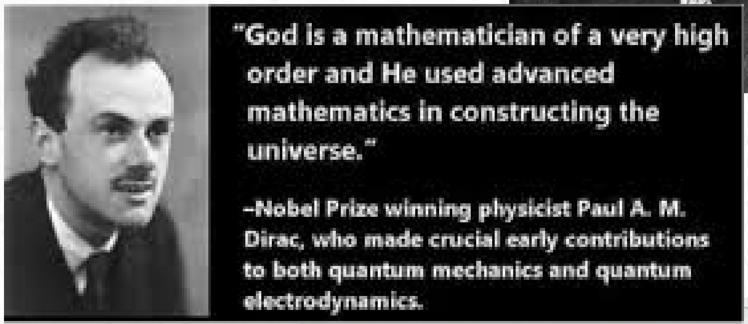
École Normale Supérieure, Paris

- Sought to write better analysis texts (Henri Cartan complained to Weil).
- Fall of French mathematics –due to WWI loss of a generation of French mathematicians.
- The rise of German Mathematics (and physicists).
- Bourbaki – a secret society, name based on past French general.
- Original plan -one volume of 1000 pages, collectively written, – a treatise on analysis. plan and became *Éléments de Mathématique*
- *By 1967:* 10 books in several volumes, over 60 chapters, publications starting in 1939. set Theory, Algebra, General Topology, Real Analysis, Topological Vector Spaces, Integration, Commutative Algebra, Varieties, Lie Groups and Algebras, Spectral Theory.
- In 2016 – Algebraic Topology; 2019 Revised Spectral Theory.

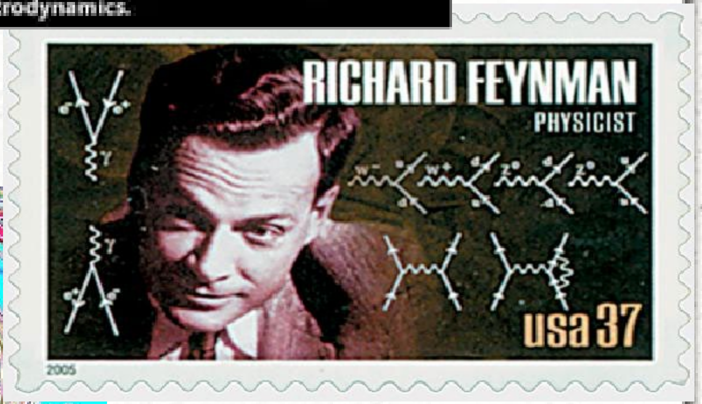
Revolutions – Paradigm Shifts



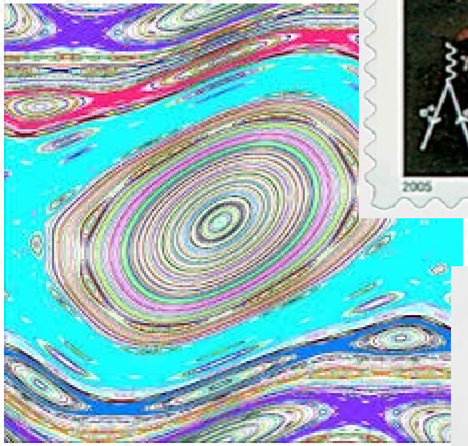
- Quantum Theory



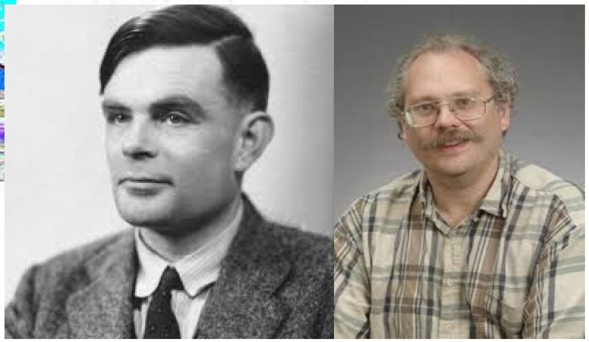
- Special Relativity
- General Relativity



- String Theory Revolution



- Nonlinear Dynamics and Chaos
- Information Age

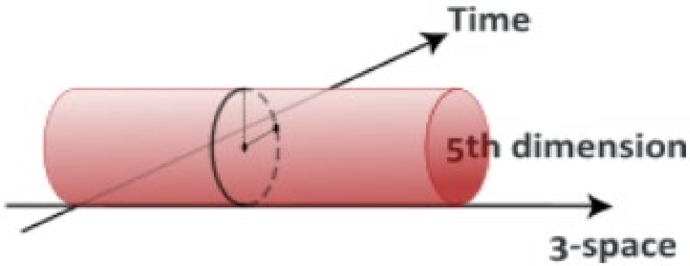


- *Focus on higher dimensions …*

Kaluza-Klein to Calabi-Yau

The Search for Higher Dimensions

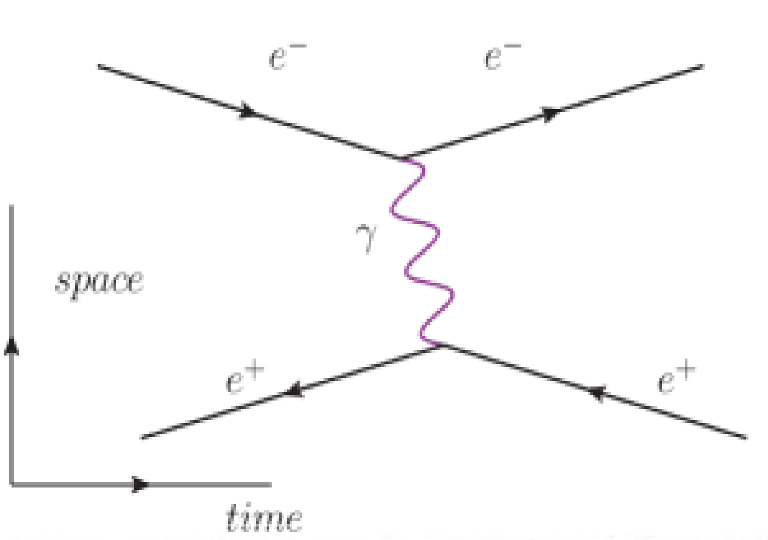
- The 4th Dimension
  - Special Relativity 1905;
  - Minkowski – 4D spacetime.
  - General Relativity 1915.



- Theodor Kaluza (1885-1954) In 1921
  - Solved Einstein Equations in 5D.
  - Unified General Relativity with E&M.
- Oskar Klein (1894-1977) – QM Interpretation
  - Fifth dimension - curled up, microscopic
- Led to gauge theories on fiber bundles.

Feynman Diagrams and QED

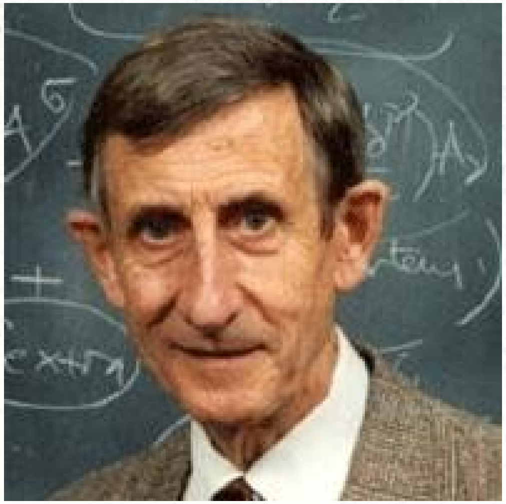
- Richard Feynman (1918-1988)
- Quantum Electrodynamics (QED) – 1948-9
  – Interaction of light and matter
  – Nobel Prize with Schwinger, Tomonaga 1965



Not to F. Dyson

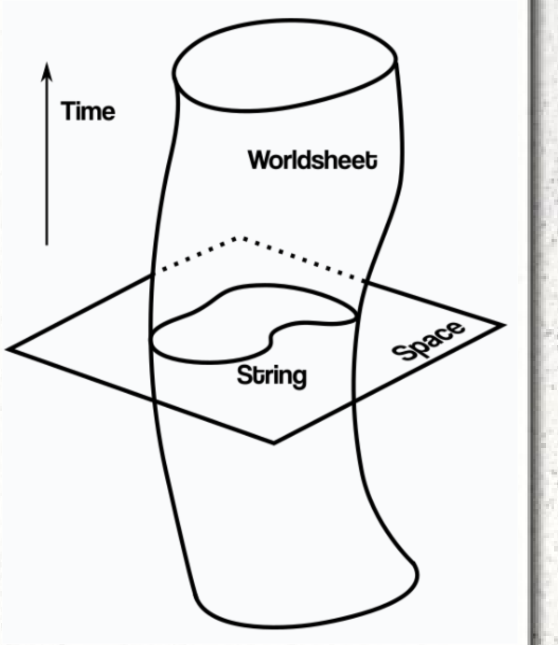
Freeman Dyson (1923-2020)
1972 Gibbs Lecture

I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has ended in divorce.



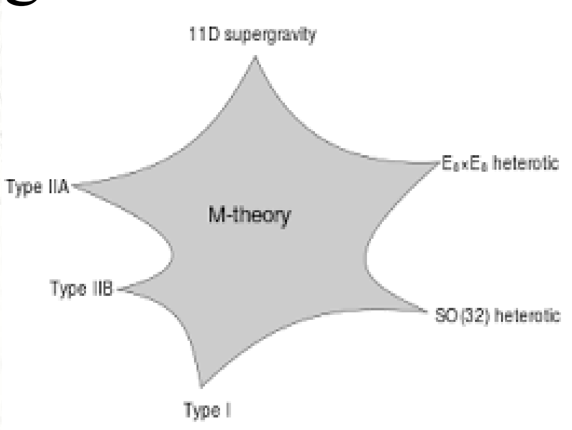
String Theory

- Gabriele Veneziano (1942-)
- In 1968 "Thumbing through old math books, they [Mahiko Suzuki] stumbled by chance on the **[Euler] Beta function** …," Michio Kaku



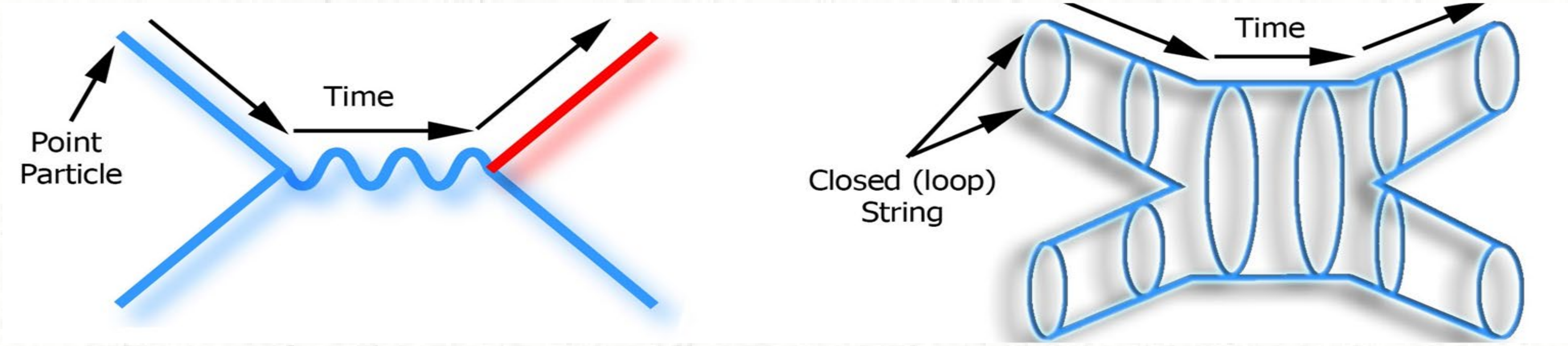
$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

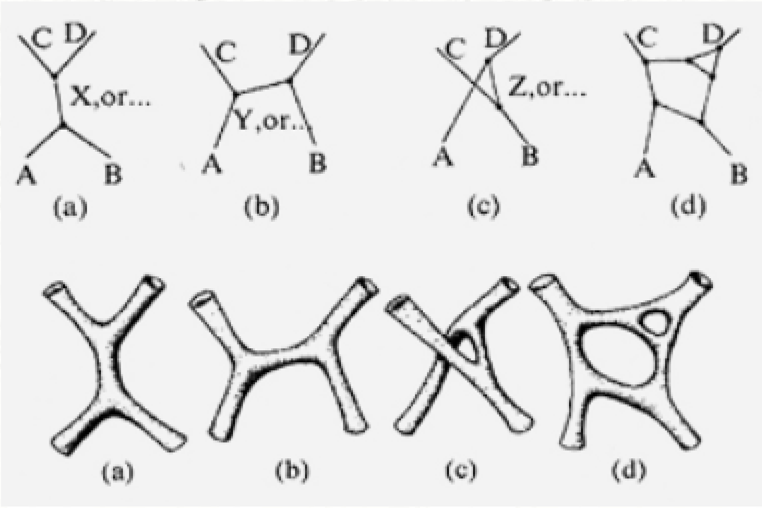
- Point-like particles modeled as 1D strings.
- Extra Dimensions 70s-90s
  - Bosonic string theory, 26-dimensional.
  - Superstring theory, 10-dimensional.
  - M-theory, 11-dimensional.



Feynman Diagrams to Strings

- Extend Feynman diagrams



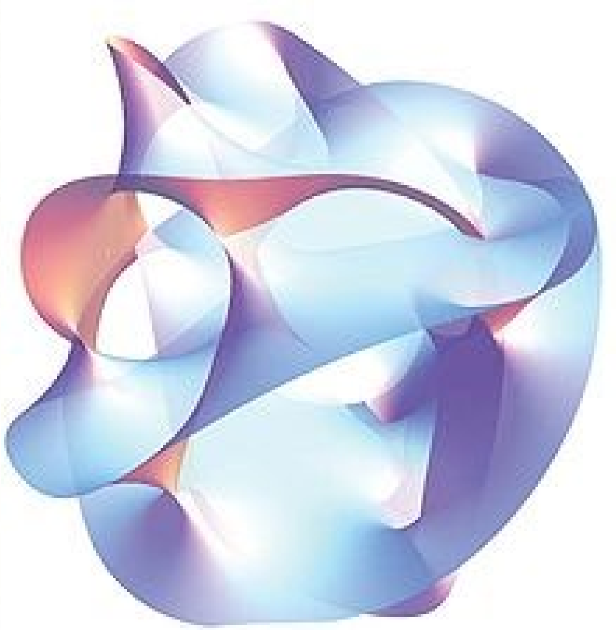


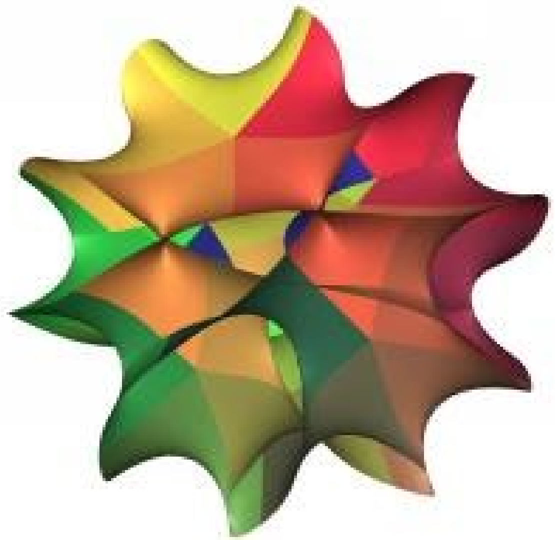
- Differential Geometry
- Topology
- Knot Theory

Calabi-Yau Manifolds

- Compactification (curl up extra 6 dimensions).
- Eugenio Calabi and Shing-Tung Yau, mathematicians

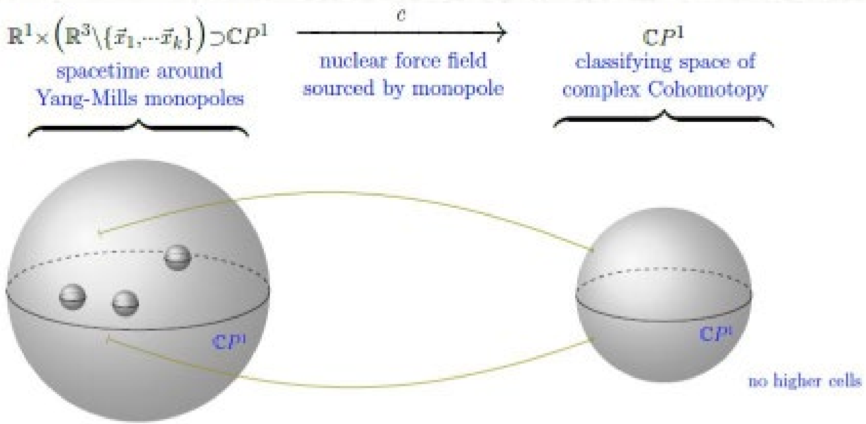
$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + \psi x_1 x_2 x_3 x_4 = 1$$





Mathematics and Physics

- Not Divorced

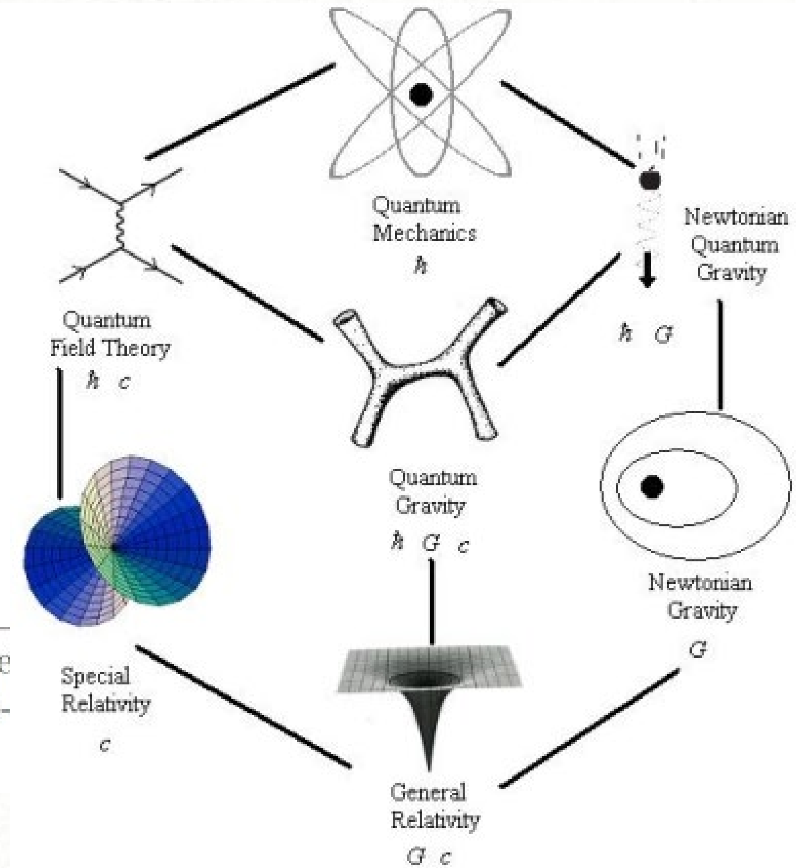


$$[c] \in \left\{\mathbb{C}P^1 \longrightarrow \mathbb{C}P^1\right\}/_{\sim \text{homotopy}} \simeq \mathbb{Z}$$

charge = homotopy class

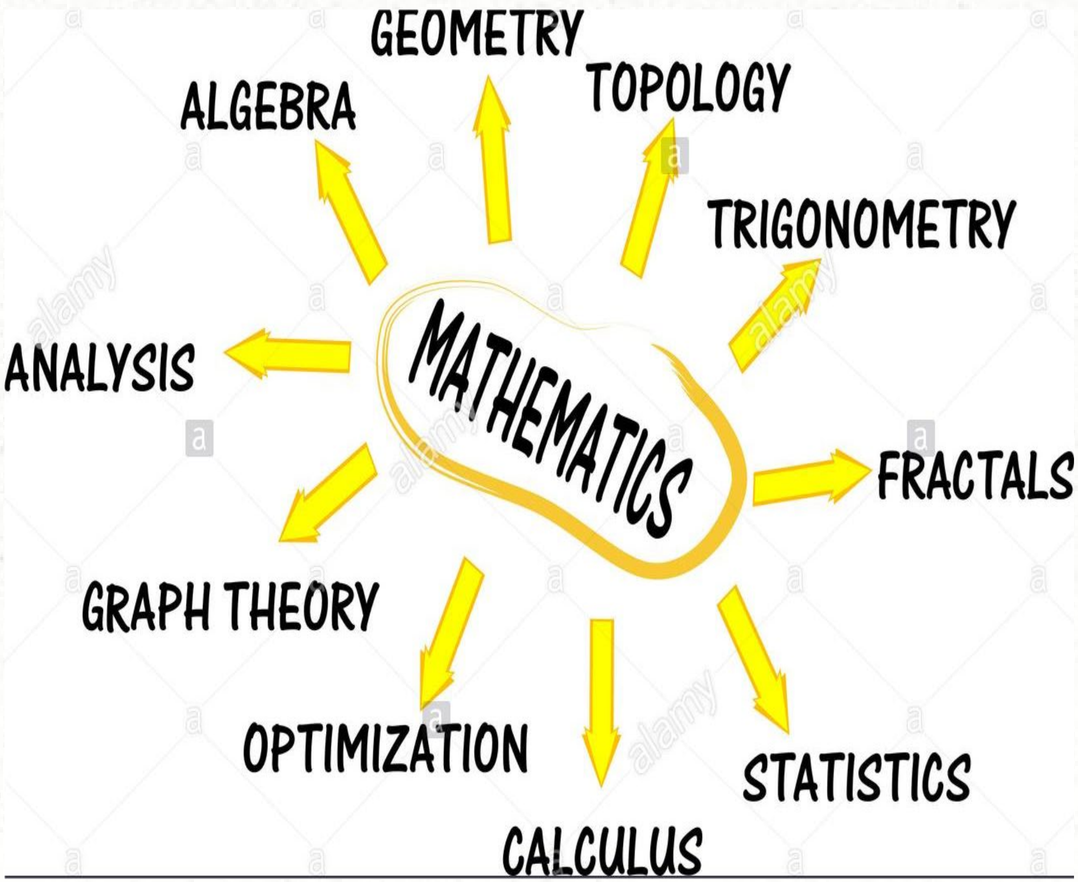
charge lattice

Atiyah-Hitchin charge quantization – The moduli space of SU(2) Yang-Mills monopoles is the cocycle space of complex-rational Cohomotopy of any sphere enclosing them.



Michael Atiyah (1929-2019)

The Greatest Problems and Prizes



David Hilbert (1862-1943)

- Opened the International Congress of Mathematicians in Paris in the year 1900.
- Outlined 23 major mathematical problems to provide solutions for in the coming new century.

Hilbert's Problems 1-6

1. Cantor's problem of the cardinal number of the continuum. (partially resolved)
2. The compatibility of the arithmetic axioms.
3. The equality of two volumes of two tetrahedra of equal bases and equal altitudes. - (resolved)
4. Problem of the straight line as the shortest distance between two points. (vague)
5. Lie's concept of a continuous group of transformations without the assumption of the differentiability of the functions defining the group. (i.e., are continuous groups automatically differential groups?)
6. Mathematical treatment of the axioms of physics.

Hilbert's Problems 7-14

7. Irrationality and transcendence of certain numbers.
8. Problems (with the distribution) of prime numbers.
9. Proof of the most general law of reciprocity in any number field.

Red - Unresolved

10. Determination of the solvability of a diophantine equation.
11. Quadratic forms with any algebraic numerical coefficients.
12. Extension of Kronecker's theorem on abelian fields.
13. Impossibility of the solution of the general equation of the 7th degree.
14. Proof of the finiteness of certain complete systems of functions.

Hilbert's Problems 15-23

15. Rigorous foundation of Schubert's calculus.
16. Problem of the topology of algebraic curves and surfaces.
17. Expression of definite forms by squares.
18. Building space from congruent polyhedra.
19. Are the solutions of regular problems in the calculus of variations always necessarily analytic?
20. The general problem of boundary curves.
21. Proof of the existence of linear differential equations having a prescribed monodromic group.
22. Uniformization of analytic relations by means of automorphic functions.
23. Further development of the methods of the calculus of variations.

Fields Medal

Prize (John Charles Fields)

- 2-4 mathematicians under 40 yrs.
- The International Congress of the International Mathematical Union
- Every four years.

Transire suum pectus mundoque potiri.
Rise above oneself and grasp the world.

See [Fields Medalists]

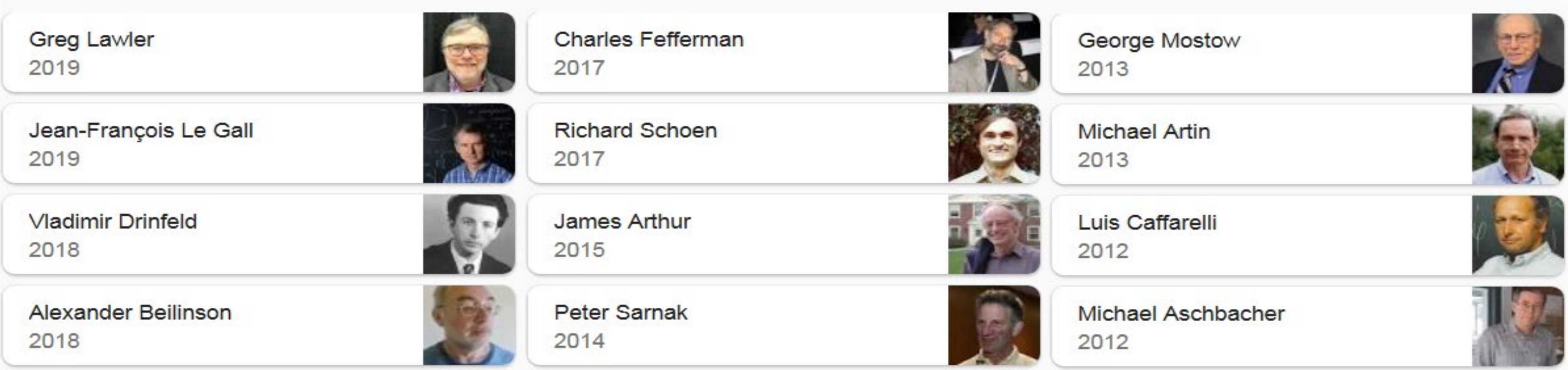
Abel Prize

- 1899 Proposed by the Norwegian mathematician Sophus Lie (1842-1899).
- He learned of Alfred Nobel's plans.
- First Awarded 2003
- See Laureates

Niels Henrik Abel (1802-1829)

Wolf Prize in Mathematics

- Awarded almost annually by the Wolf Foundation, in Israel.
- One of the 6 Wolf Prizes since 1978;
  - Agriculture, Chemistry, Medicine, Physics and Arts.
- See [Winners]

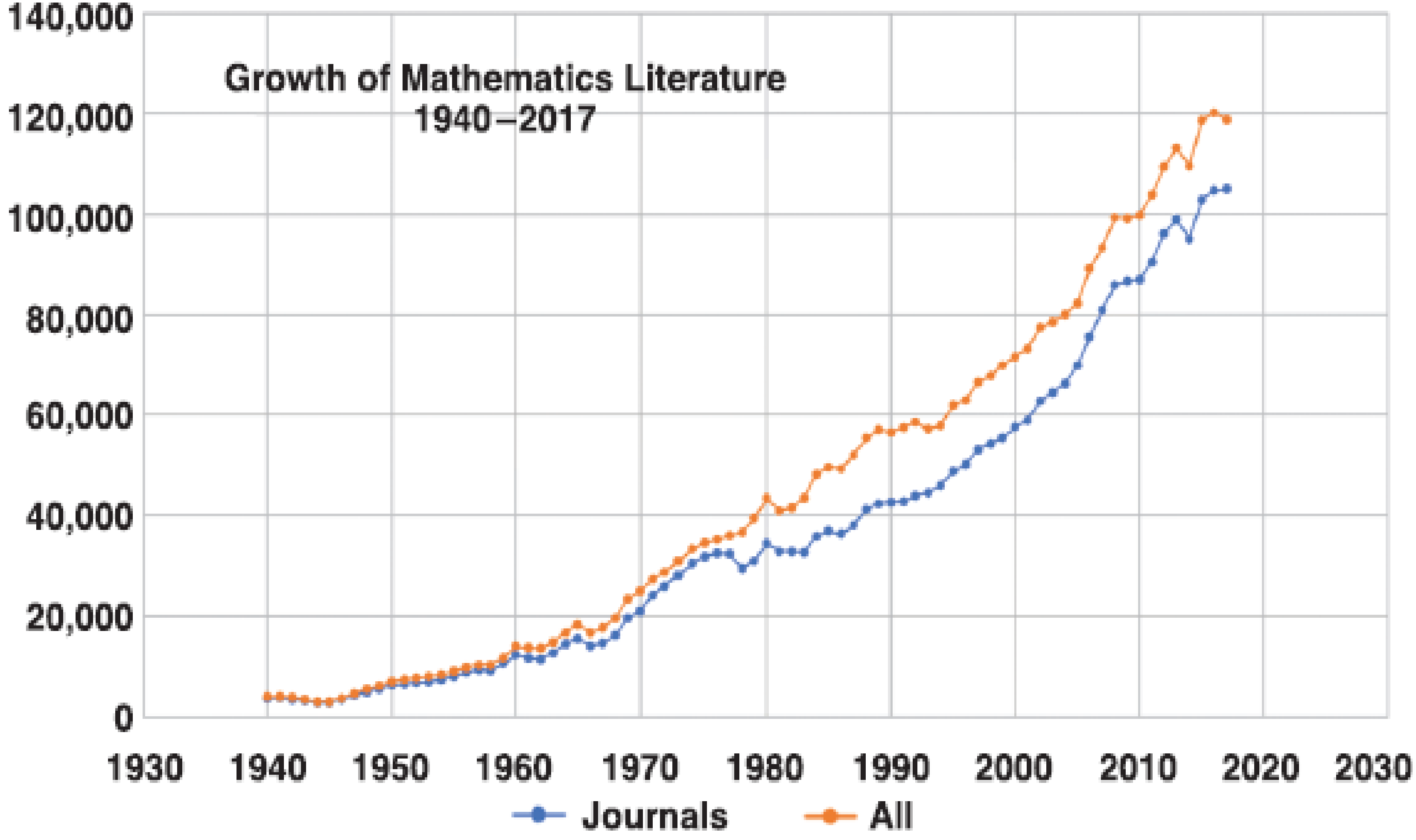


Millennium Prize Problems

Stated by the Clay Mathematics Institute on May 24, 2000 – for One Million Dollars

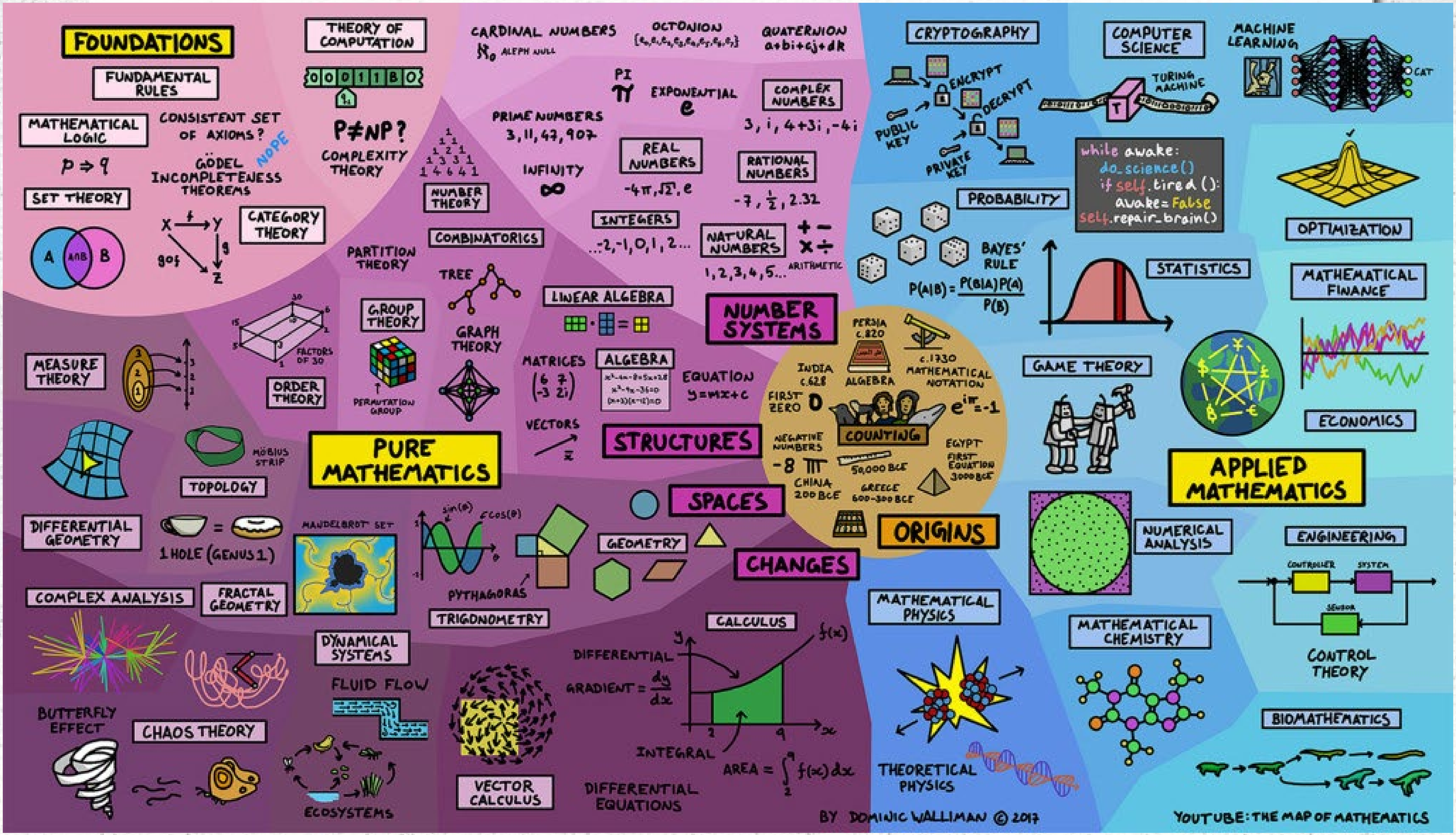
- Birch and Swinnerton-Dyer conjecture,
- Hodge conjecture,
- Navier–Stokes existence and smoothness,
- P versus NP problem,
- Poincaré conjecture (Solved! - Perelman),
- Riemann hypothesis, and
- Yang–Mills existence and mass gap

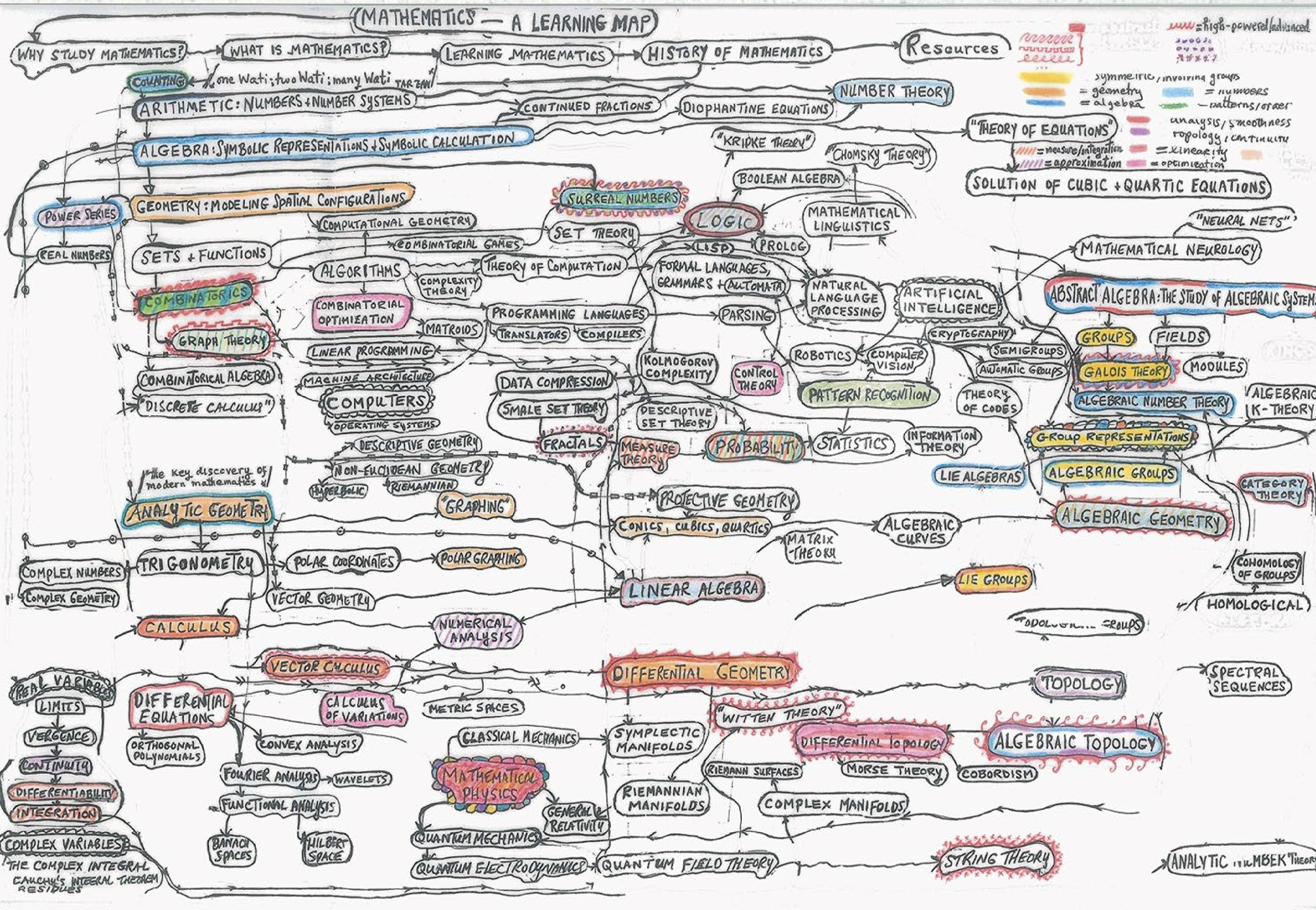
Mathematics Publications



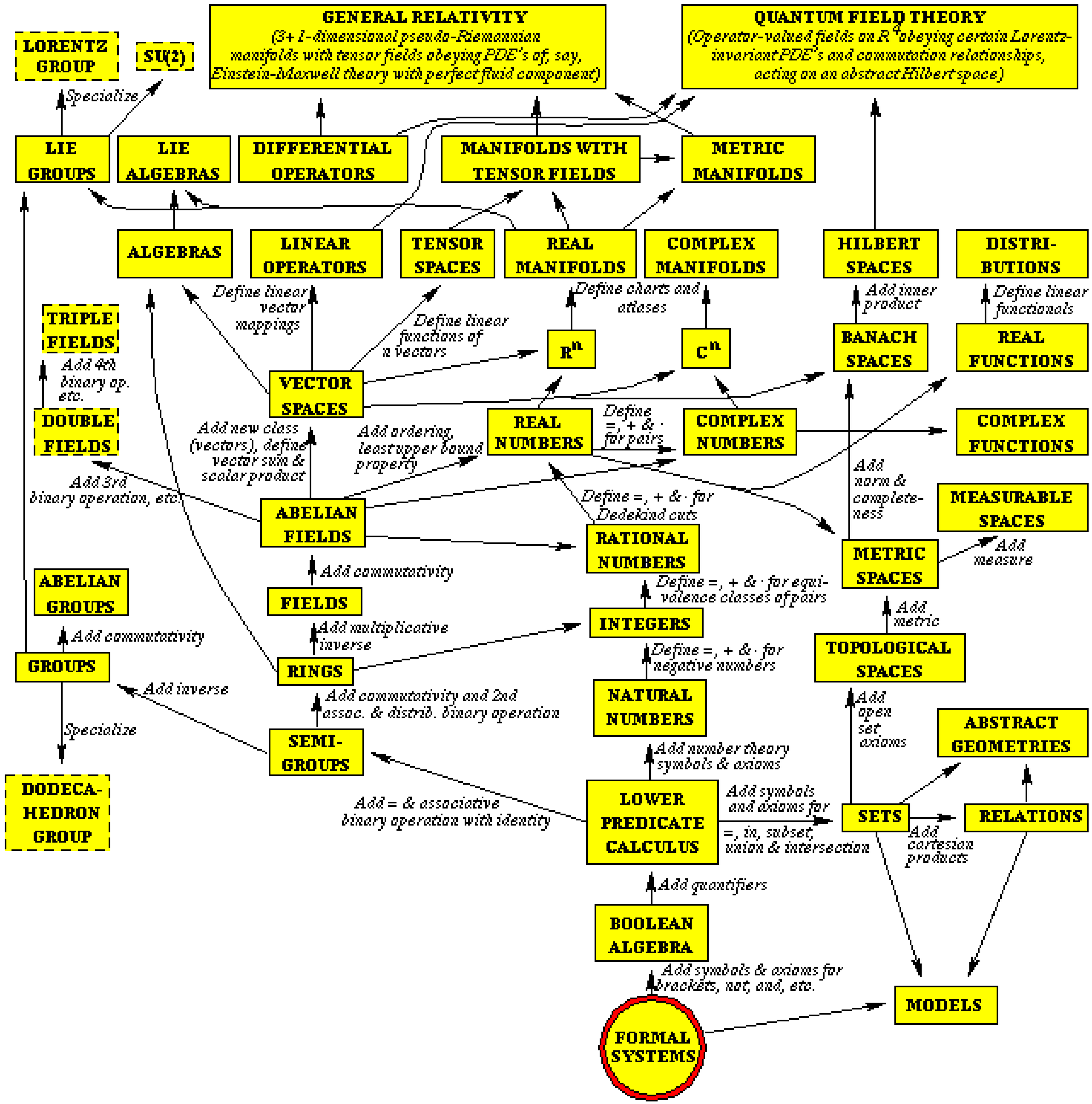
Mind Maps of Mathematics

https://www.sciencealert.com/this-mind-boggling-map-explains-how-everything-in-mathematics-is-connected-3





https://math.stackexchange.com/questions/166862/detailed-diagram-with-mathematical-fields-of-study


LORENTZ GROUP
SU(2)
GENERAL RELATIVITY
(3+1-dimensional pseudo-Riemannian manifolds with tensor fields obeying PDE's of, say, Einstein-Maxwell theory with perfect fluid component)
QUANTUM FIELD THEORY
(Operator-valued fields on R4 obeying certain Lorentz-invariant PDE's and commutation relationships, acting on an abstract Hilbert space)
Specialize
LIE GROUPS
LIE ALGEBRAS
DIFFERENTIAL OPERATORS
MANIFOLDS WITH TENSOR FIELDS
METRIC MANIFOLDS
ALGEBRAS
LINEAR OPERATORS
TENSOR SPACES
REAL MANIFOLDS
COMPLEX MANIFOLDS
HILBERT SPACES
DISTRI-BUTIONS
Define linear vector mappings
Define linear functions of n vectors
Define charts and atlases
Add inner product
Define linear functionals
TRIPLE FIELDS
Add 4th binary op. etc.
DOUBLE FIELDS
VECTOR SPACES
Rn
Cn
BANACH SPACES
REAL FUNCTIONS
Add new class (vectors), define vector sum & scalar product
Add ordering, least upper bound property
REAL NUMBERS
Define =, + & · for pairs
COMPLEX NUMBERS
COMPLEX FUNCTIONS
Add 3rd binary operation, etc.
Add norm & complete-ness
MEASURABLE SPACES
ABELIAN FIELDS
Define =, + & · for Dedekind cuts
RATIONAL NUMBERS
METRIC SPACES
Add measure
Add commutativity
FIELDS
Define =, + & · for equi-valence classes of pairs
Add metric
ABELIAN GROUPS
Add commutativity
Add multiplicative inverse
INTEGERS
TOPOLOGICAL SPACES
GROUPS
RINGS
Define =, + & · for negative numbers
Add inverse
Add commutativity and 2nd assoc. & distrib. binary operation
NATURAL NUMBERS
Add open set axioms
Specialize
SEMI-GROUPS
ABSTRACT GEOMETRIES
Add number theory symbols & axioms
DODECA-HEDRON GROUP
Add = & associative binary operation with identity
LOWER PREDICATE CALCULUS
Add symbols and axioms for =, in, subset, union & intersection
SETS
RELATIONS
Add cartesian products
Add quantifiers
BOOLEAN ALGEBRA
Add symbols & axioms for brackets, not, and, etc.
MODELS
FORMAL SYSTEMS

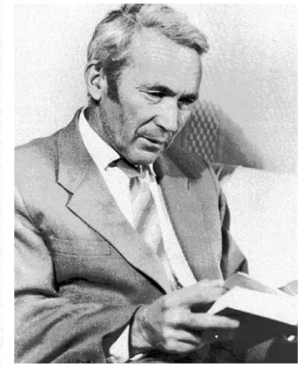
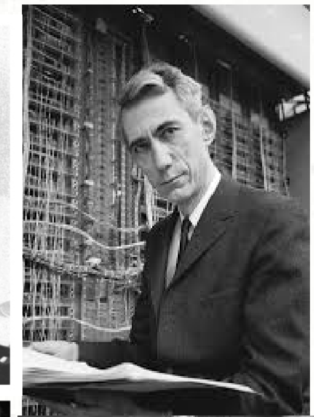
Top Numerical Algorithms (2000)

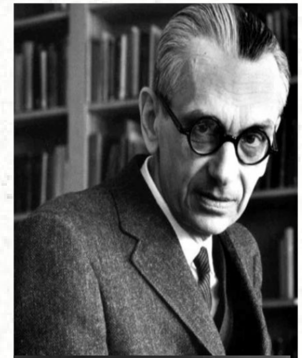
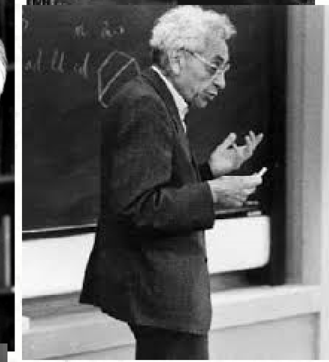
Another list (2015)

1. Newton and quasi-Newton methods
2. Matrix factorizations (LU, Cholesky, QR)
3. Singular value decomposition, QR and QZ algorithms
4. Monte-Carlo methods
5. Fast Fourier transform
6. Krylov subspace methods (conjugate gradients, Lanczos, GMRES, minres)
7. JPEG
8. PageRank
9. Simplex algorithm
10. Kalman filter

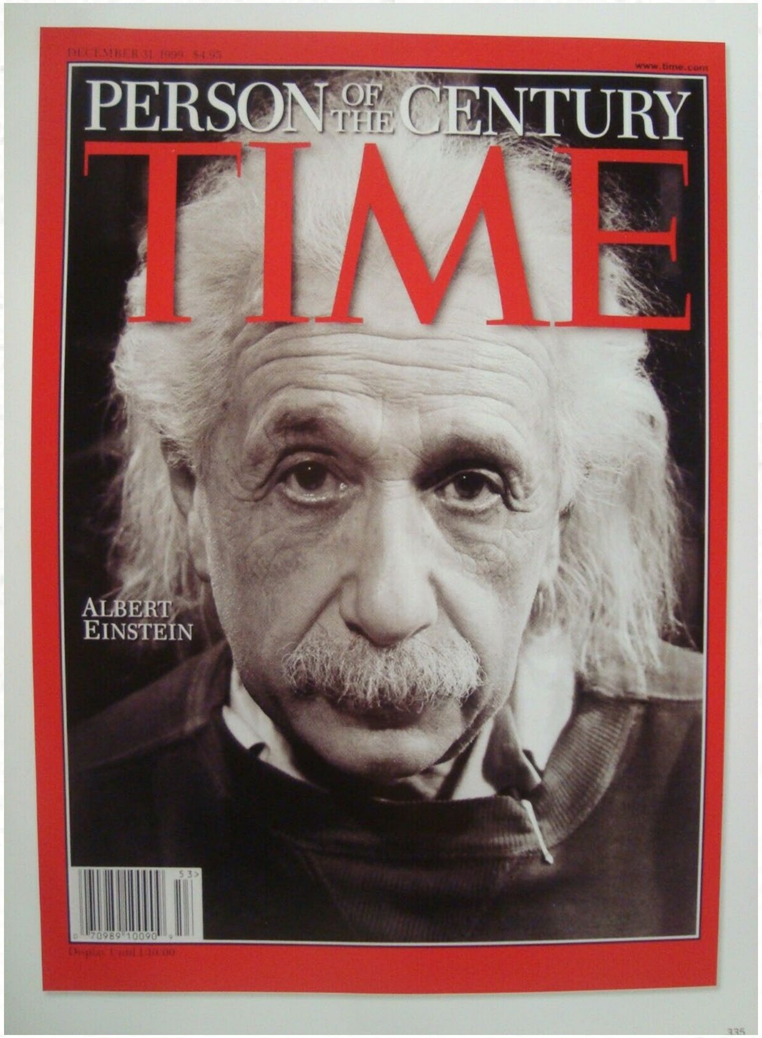
Greatest Mathematicians of 20th Century?

- John von Neumann
- Andrey Kolmogorov
- Claude Shannon
- Alexander Grothendieck
- Kurt Godel
- Paul Erdos
- Alan Turing
- Hermann Weyl
- Srinivasa Ramanujan

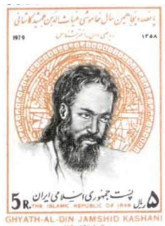




Person of the Century



Good Luck on Your Finals!

al-Haitham
965-1039



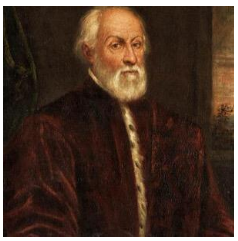
al-Kashi
1380-1429

al-Khwarizmi
780-850

Apollonius
262-190 BCE

Archimedes
287-212 BCE

Aryabhata
476-550



Bombelli
1526-1572

Cardano
1501-1576

Cavalieri
1598-1647

Diophantus
200-284

Eratosthenes
276-194 BCE

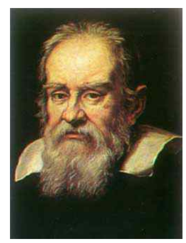
Euclid
325-265 BCE

Euler
1707-1783

Fermat
1601-1665

Ferrari
1522-1565

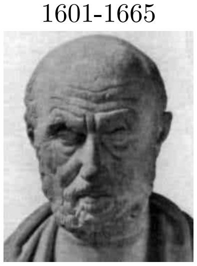
Fibonacci
1170-1250



Galileo
1564-1642

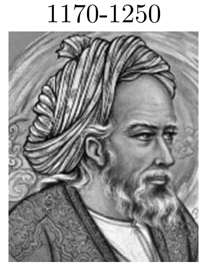
Gauss
1777-1855

Harriot
1560-1621

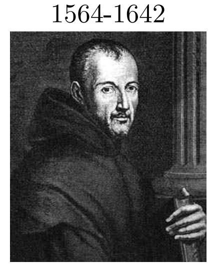


Hippocrates
470-410 BCE

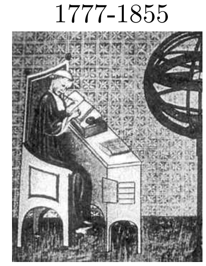
Hypatia
370-415



Khayyam
1048-1131

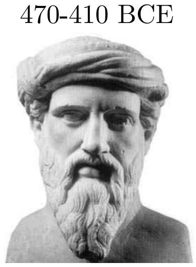


Mersenne
1588-1648



Oresme
1323-1382

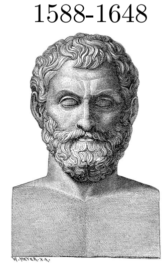
Ptolemy
85-165



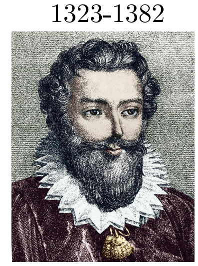
Pythagorus
570-490 BCE

Seki
1642-1708

Tartaglia
1500-1557



Thales
624-547 BCE



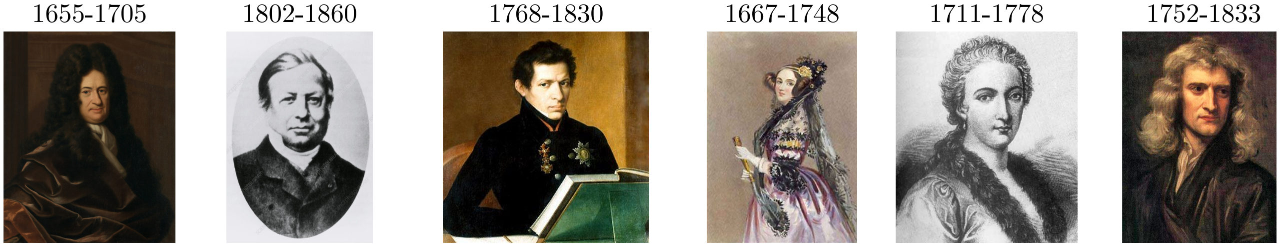
Viete
1540-1603

Note: There are more from these times but the images were not available.

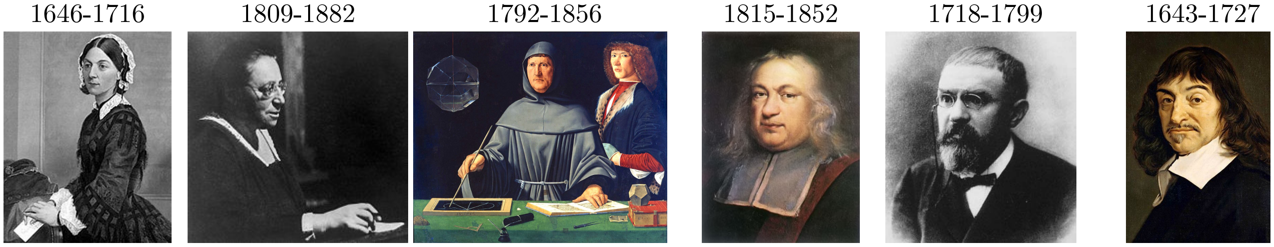
Cayley	Beltrami	Riemann	Jacobi	Babbage	Chatelet
1821-1895	1835-1900	1826-1866	1804-1851	1791-1871	1706-1749

Daniel Bernoulli	Litvinova	Euler	Galois	Germain	Huygens
1700-1782	1845–1919	1707-1783	1811-1832	1776-1831	1629-1695

Jacob Bernoulli	Bolyai	Fourier	Johann Bernoulli	Bassi	Legendre
1655-1705	1802-1860	1768-1830	1667-1748	1711-1778	1752-1833



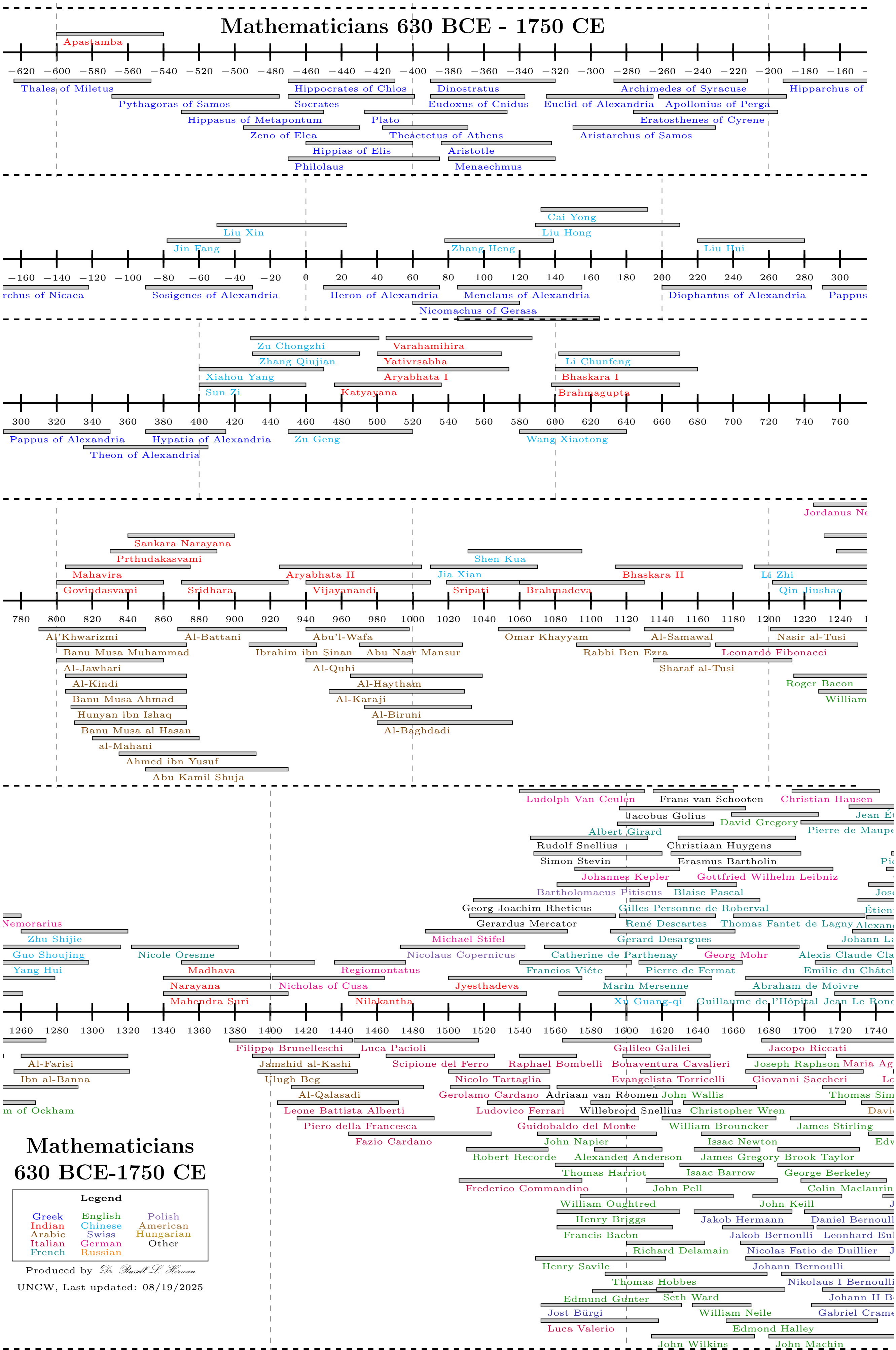
Leibniz	Liouville	Lobachevski	Lovelace	Agnesi	Newton
1646-1716	1809-1882	1792-1856	1815-1852	1718-1799	1643-1727



Nightingale	Noether	Pacioli	Fermat	Poincaré	Descartes
1820-1910	1882-1935	1445-1517	1601-1665	1854 - 1912	1596-1650

Poisson	Somerville	Lie	Tschirnhaus	Abel	Gauss
1781-1840	1780-1872	1842-1899	1651-1708	1802-1829	1777-1855

Lagrange	Weierstrass	Wallis	Lambert	Laplace	d'Alembert
1736-1813	1815-1897	1616-1703	1728-1777	1749-1827	1717-1783



Mathematicians 630 BCE - 1750 CE
Apastamba
Thales of Miletus
Pythagoras of Samos
Hippasus of Metapontum
Zeno of Elea
Hippocrates of Chios
Socrates
Plato
Theaetetus of Athens
Hippias of Elis
Philolaus
Dinostratus
Eudoxus of Cnidus
Aristotle
Menaechmus
Euclid of Alexandria
Aristarchus of Samos
Archimedes of Syracuse
Apollonius of Perga
Eratosthenes of Cyrene
Hipparchus of
Liu Xin
Jin Fang
Zhang Heng
Cai Yong
Liu Hong
Liu Hui
rchus of Nicaea
Sosigenes of Alexandria
Heron of Alexandria
Nicomachus of Gerasa
Menelaus of Alexandria
Diophantus of Alexandria
Pappus
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Zhang Qiujian
Xiahou Yang
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Yativrsabha
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Bhaskara II
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Jordanus Ne
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Banu Musa Muhammad
Al-Jawhari
Al-Kindi
Banu Musa Ahmad
Hunyan ibn Ishaq
Banu Musa al Hasan
al-Mahani
Ahmed ibn Yusuf
Abu Kamil Shuja
Al-Battani
Ibrahim ibn Sinan
Abu'l-Wafa
Abu Nasr Mansur
Al-Quhi
Al-Haytham
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Rabbi Ben Ezra
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William
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Zhu Shijie
Guo Shoujing
Yang Hui
Nicole Oresme
Madhava
Narayana
Mahendra Suri
Regiomontatus
Nicholas of Cusa
Nilakantha
Michael Stifel
Nicolaus Copernicus
Jyesthadeva
Georg Joachim Rheticus
Gerardus Mercator
Ludolph Van Ceulen
Frans van Schooten
Christian Hausen
Jacobus Golius
David Gregory
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Albert Girard
Pierre de Maupe
Rudolf Snellius
Christiaan Huygens
Simon Stevin
Erasmus Bartholin
Johannes Kepler
Gottfried Wilhelm Leibniz
Bartholomaeus Pitiscus
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Gilles Personne de Roberval
René Descartes
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Gerard Desargues
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Al-Qalasadi
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Henry Savile
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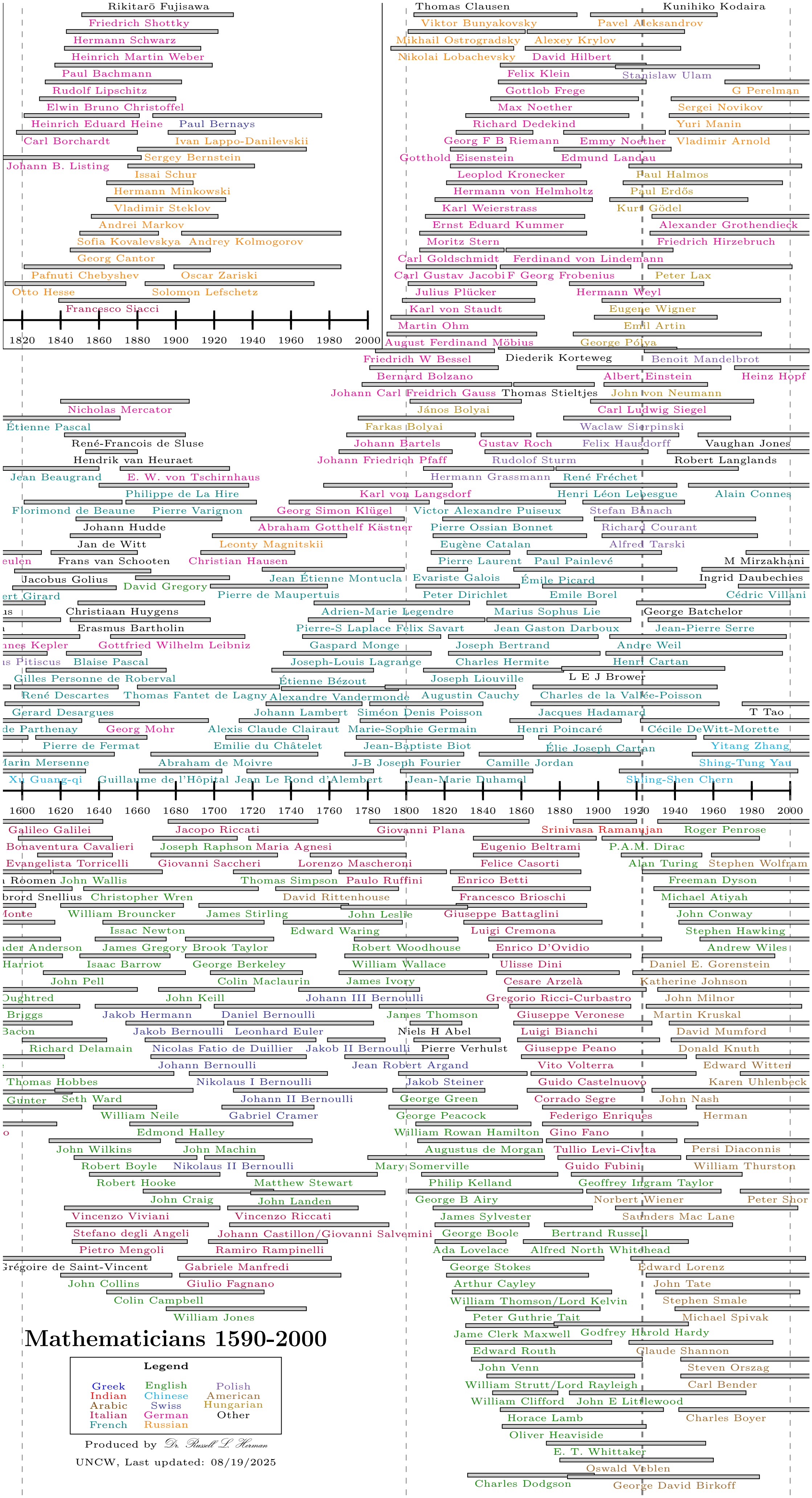

Rikitarō Fujisawa
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Hermann Schwarz
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Paul Bachmann
Rudolf Lipschitz
Elwin Bruno Christoffel
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Johann B. Listing
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Vladimir Steklov
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Nikolai Lobachevsky
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G Perelman
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Sergei Novikov
Richard Dedekind
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Karl Weierstrass
Kurt Gödel
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Alexander Grothendieck
Moritz Stern
Friedrich Hirzebruch
Carl Goldschmidt
Ferdinand von Lindemann
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Peter Lax
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Karl von Staudt
Eugene Wigner
Martin Ohm
Emil Artin
August Ferdinand Möbius
George Pólya
Friedrich W Bessel
Diederik Korteweg
Benoit Mandelbrot
Bernard Bolzano
Albert Einstein
Heinz Hopf
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Alain Connes
Florimond de Beaune
Pierre Varignon
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Victor Alexandre Puiseux
Stefan Banach
Johann Hudde
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Jan de Witt
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Nikolaus II Bernoulli
Mary Somerville
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Philip Kelland
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John Landen
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Norbert Wiener
Peter Shor
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Saunders Mac Lane
Stefano degli Angeli
Johann Castillon/Giovanni Salvemini
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French
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Steven Orszag
William Strutt/Lord Rayleigh
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E. T. Whittaker
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Charles Dodgson
George David Birkoff

The following are past talks with some embedded history.

What is π?

Mathematics and Statistics Club

3.14159265358979323846264338327950288419716939937510

Dr. R. L. Herman

March 21, 2023
Mathematics & Statistics
UNC Wilmington

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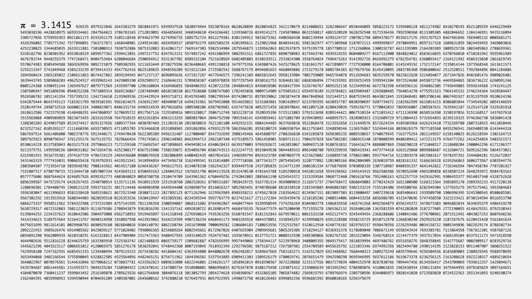


Figure 1: 10000 digits of π.

Introduction

What is π?

How do you compute π?

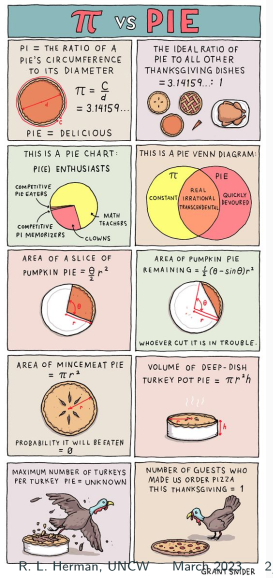
How many digits do we know?

Memorization record?

Who is Akira Haraguchi?

What is your favorite π joke?

In the beginning there was ...
Archimedes.



Eudoxus of Cnidus (c.390 – c. 337 BCE)

- Studied under Plato.
- Taught Aristotle.
- Astronomer, Mathematician.
- Theory of Proportions:

  Circles: $A \propto r^2$,
  Spheres: $V \propto r^3$,
  Volume of a pyramid .
  Volume of a cone.
- Studied reals, continuous quantities.
- Method of Exhaustion:
  Due to Antiphon (480–411 BCE).
  Area from a sequence of inscribed polygons.

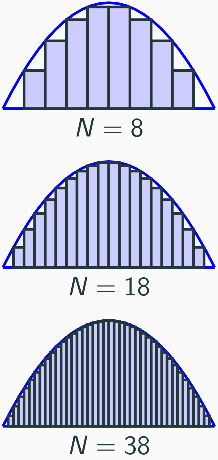
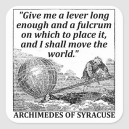
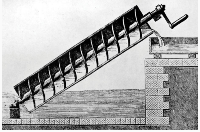


Figure 2: Method of Exhaustion.

Archimedes of Syracuse (287-212 BCE)

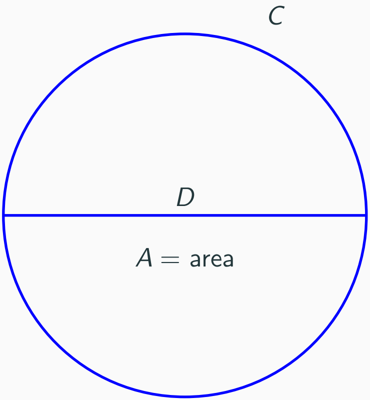
- Went to Alexandria, Egypt
  then, back to Syracuse, Sicily.
- Greatest Mathematician of Antiquity.
- Mathematician, Engineer, Inventor.
  - Archimedean screw, lever, pulley.
- King Heiro II's crown - Eureka.
  Archimedes Principle of Bouyancy.
- According to Plutarch (46-120):
  - Marcellus - Syracuse 212 BCE.
  - Claw of Archimedes.
  - Heat Ray.
  - Prone to intense concentration.
  - Death of Archimedes.





Archimedes' Mathematics

- Mastered Euclid and Eudoxus' (c. 390-337 BCE) Method of Exhaustion.
- *Measurement of a Circle*
  $\frac{C}{D} = \text{const.}, \quad \frac{A}{D^2} = \text{const.}$



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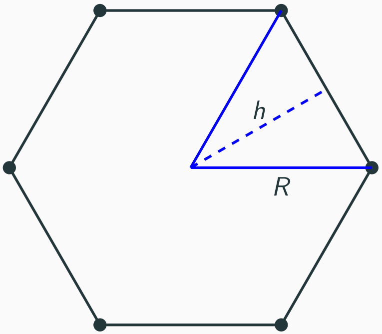
  $A_p = \frac{1}{2}hQ$, $Q =$ Perimeter.



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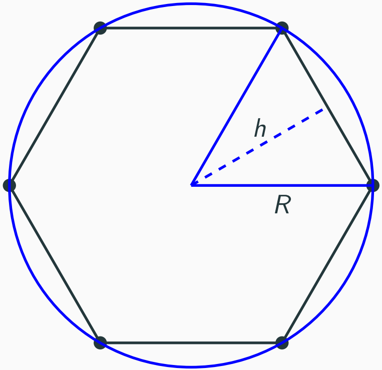

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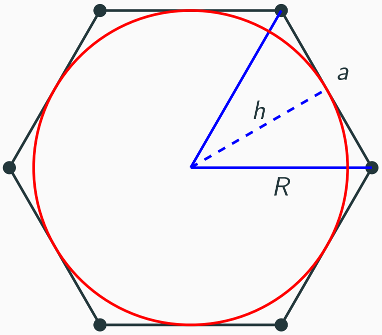
  $a = 2R\sin\frac{180}{n}, h = \sqrt{R^2 - \frac{a^2}{4}}$.



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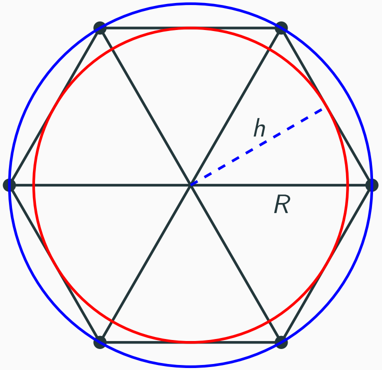
- Circumscribed Polygon
  $a = 2R\sin\frac{180}{n}, h = \sqrt{R^2 - \frac{a^2}{4}}$.
- Approximation of π,
  $\frac{A_p}{R^2} < \pi < \frac{A_p}{h^2}$.

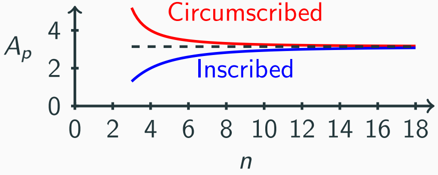


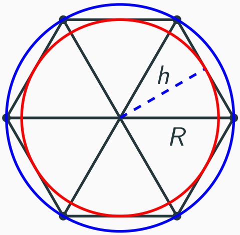
Estimating π

- Approximation of π,
  $\frac{A_p}{R^2} < \pi < \frac{A_p}{h^2}$,
- Recall
  $a = 2R \sin \frac{180}{n}$,
  $h = \sqrt{R^2 - \frac{a^2}{4}} = R \cos \frac{180}{n}$,
  $A_p = \frac{1}{2} anh = nhR \sin \frac{180}{n}$.
- Therefore,

$$\frac{n}{2} \sin \frac{360}{n} < \pi < n \tan \frac{180}{n},$$

- Hexagon $(n = 6)$,
  $2.598 < \pi < 3.464$.
- Archimedes - up to 96-gon
  $3.1394 < \pi < 3.1427$.





Archimedes' Inscribed and Circumscribed n-gons

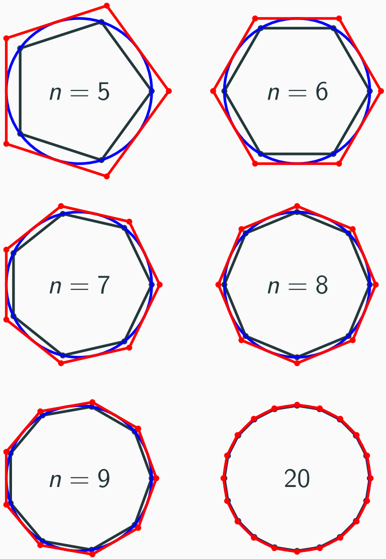
Consider a fixed circle of radius R.

- Inscribed n-gon: $h = R\cos\frac{180}{n}$, $A_i = nhR\sin\frac{180}{n}$.
- Circumscribed n-gon: $r = \frac{R}{\cos\frac{180}{n}}$, $A_c = nHr\sin\frac{180}{n}$.
- Thus,

$$\begin{aligned} A_i &= nR^2\tan\frac{180}{n}, \\ A_c &= \frac{n}{2}R^2\sin\frac{360}{n}. \end{aligned}$$

- This gives

$$\frac{n}{2}\sin\frac{360}{n} < \pi < n\tan\frac{180}{n},$$



Early Approximations of π: Peripherion $\pi\epsilon\rho\iota\phi\epsilon\rho\epsilon\iota\alpha$

- Bible, $\pi \approx 3$.
- Babylonian $3 + \frac{1}{8}$.
- Egyptians, $\left(\frac{4}{3}\right)^4 = \frac{256}{81} \approx 3.1604938$.
- Sulbasutrakaras ($<$ 800 BCE), 3.08.
- Archimedes (250 BCE) $3\frac{10}{71} < \pi < 3\frac{1}{7}$.
- Aryabhata (499), $\frac{62832}{20000}$.
- Ptolemy (150), 360-gon, 3.14166.
- Chinese (430-501) $\frac{355}{113} \approx 3.14159292$.
- Hindu (1100) $\frac{3927}{1250} \approx 3.1416$.
- Viéte, 393,216-gon, π to 9 places.
- van Ceulen (1540-1610) Dutch, 35 places.
- William Shanks (1873) 527 digits.
- Lambert (1728-1777) - irrationality proof.
- William Jones (1706) introduced π.
- Euler popularized notation.
- See Approximations of π .
- Leibniz-Madhava
  $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$
- Euler
  $$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$
- Ramanujan
  $$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!}{k!^4} \frac{1103 + 26390k}{396^{4k}}$$
- A Complete Chronology

Buffon's Needle

1777 essay, Georges-Louis LeClerc, the Comte de Buffon (1707–1788). [See MAA Convergence]

If a needle of length ℓ is thrown randomly onto a floor marked with parallel lines, set at distance d apart, what is the probability that the needle will cross one of the lines? $p = \dfrac{2\ell}{\pi d}$.

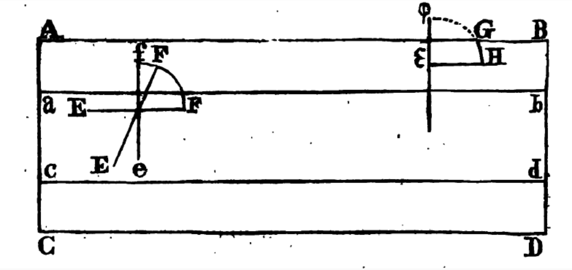


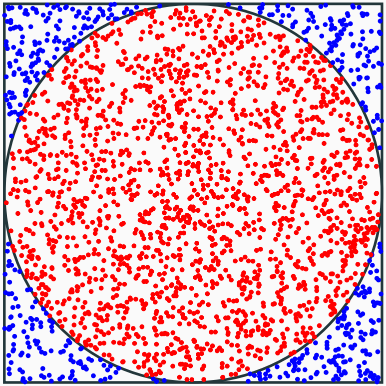
Figure 3: Buffon's sketch of the Needle Problem.

Monte Carlo Approximation

- Simulate random points in the plane with domain as a square of side $2r$ units centered on $(0, 0)$.
- Inscribe a circle with radius r
- Find the number points that lie inside the circle.
- Then,

$$\frac{\text{Area of circle}}{\text{Area of square}} = \frac{\pi r^2}{4r^2} = \frac{\pi}{4}.$$

$\pi \approx 3.15039$



Tikz Code for Monte Carlo

```
\begin{tikzpicture}[scale=.5]
\def\r{5}
\def\n{2500}
\pgfmathsetmacro\s{\r*\r}
\edef\k{0}
\draw[line width=1] (0,0) circle (\r);
\draw[line width=1] (-\r,-\r) rectangle (\r,\r);

\foreach \n in {1,2,...,\n}{
\pgfmathsetmacro\myx{\r*rand};
\pgfmathsetmacro\myy{\r*rand};
\pgfmathsetmacro\d{\myx*\myx+\myy*\myy};
\pgfmathsetmacro{\col}{\d<\s ? "red" : "blue" }
\pgfmathsetmacro{\c}{\d<\s ? 1 : 0 }
\pgfmathparse{\k+\c}
\xdef\k{\pgfmathresult}

\draw[fill,color=\col] (\myx,\myy) circle (.05);
}
\pgfmathsetmacro{\p}{\k/\n*4}
\node at (0,6) {$\pi\approx$ \p};
\end{tikzpicture}
```

More Ways to Compute π

- Let a_n = length of circumscribed, regular $6 \cdot 2^n$-gon.
- Let b_n =length of inscribed, regular $6 \cdot 2^n$-gon about circle of radius 1/2.
- $a_0 = 2\sqrt{3},\ b_0 = 3,$

$$\begin{aligned} a_{n+1} &= \frac{2a_nb_n}{a_n+b_n} \\ b_{n+1} &= \sqrt{a_{n+1}b_n} \end{aligned}$$

- $a_4 = 3.1427,\ b_n = 3.1410$ Like 96-gon
- François Viéte (1540-1603)

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}$$

- John Wallis (1616-1703)

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdots}$$

- Lord Brouckner (1620-1684)

$$\pi = \frac{4}{1+\frac{1}{2+\frac{9}{2+\frac{25}{2+\cdots}}}}$$

- James Gregory (1638-1675)

$$\tan^{-1}x = \int_0^x \frac{dt}{1+t^2} = x - \frac{x^3}{3} + \frac{x^3}{3} - \cdots.$$

- John Machin (1680-1752) - 100 digits

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

Ramanujan-Type Series

- Ramanujan (1887-1920) used elliptic integral approximations to find

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!}{k!^4} \frac{1103 + 26390k}{396^{4k}}$$

Gives eight correct digits per term. Gosper (1985) used this to find 17 million digits of π

- David and Gregory Chudnovsky (1989) found

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(13591409 + 545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}}$$

Gives 14 correct digits per term. See The New Yorker, March 1992. The Mountains of Pi
Led to record (1994) 4 billion digits.

Breaking Records - Digits of π

- 2016 World record of 22,459,157,718,361 digits using 1,583,677,621,196 terms, Peter Trueb.
- In 2019, 31.4 trillion digits, Emma Haruka Iwao .
- In 2021, scientists at the University of Applied Sciences of the Grisons calculated another 31.4 trillion digits of the constant, bringing the total up to 62.8 trillion decimal places.
- 2022 another record: 100 trillion digits of π. See here.

BBP Formula

Compute the dth digit of π and other transcendental numbers in various bases without computing all of the preceding digits.

Based on identities introduced by D. Bailey, P. Borwein and S. Plouffe [2].

The identity used to obtain the binary and hexidecimal digits of π is given by

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right). \tag{1}$$

F. Bellard introduced an identity which produced the 40 trillionth bit of π [3]. 43% faster than that using the BBP formula.

$$\begin{aligned} \pi &= \frac{1}{2^6} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{10k}} \left(-\frac{2^5}{4k+1} - \frac{1}{4k+3} + \frac{2^8}{10k+1} - \frac{2^6}{10k+3} \right. \\ &- \left. \frac{2^2}{10k+5} - \frac{2^2}{10k+7} + \frac{1}{10k+9} \right). \end{aligned} \tag{2}$$

Bellard Note

Bellard shows these are easy to find using the series expansion

$$-\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}, \text{ for } |x| < 1.$$

Further noting that $\tan^{-1}\left(\frac{1}{x-1}\right) = \Im\left(\ln(1 - \frac{1+i}{x})\right)$ and setting $x = 2$, the BBP Equation (1) results.

Combining this series expansion for $\tan^{-1}\left(\frac{1}{x-1}\right)$ with the relation

$$\frac{\pi}{4} = 2\tan^{-1}(\frac{1}{2}) - \tan^{-1}(\frac{1}{7}),$$

one can obtain Bellard's Equation (2).

This is just a variation of Machin's 1709 formula

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}.$$

Integration Formula

These formulae appear to be found through inspired guess work. M. Hirschhorn has shown how formula (1) can be obtained using simple integration formulae [6] using

$$\sum_{k=0}^{\infty} \frac{x^{8k+r}}{8k+r} = \int_0^x \frac{u^{r-1}}{1-u^8}\, du.$$

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{x^{8k+r}}{8k+r} &= \sum_{k=0}^{\infty} x^{8k+r} \int_0^{\infty} e^{-(8k+r)s}\, ds \\
&= \int_0^{\infty} (xe^{-s})^r \sum_{k=0}^{\infty} ((xe^{-s})^8)^k\, ds \\
&= \int_0^x \frac{u^{r-1}}{1-u^8}\, du,
\end{aligned}$$

where $u = xe^{-s}$, $du = -u\, ds$.

Other Curiosities - Borwein Integrals - 3Blue1Brown

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\,dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\,dx = \frac{\pi}{2}$$

This pattern continues ...

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\cdots\frac{\sin(x/13)}{x/13}\,dx = \frac{\pi}{2}.$$

At the next step the pattern fails,

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\cdots\frac{\sin(x/15)}{x/15}\,dx = ?.$$

Other Sites

Thanks for listening!!!

- BBP Formula at Wikipedia
- Plouffe's new paper at Simon Plouffe's pages
- John Baez blog

Some References i

- V. Adamchick, S. Wagon, Pi: A 2000-Year Search Changes Direction, American Math. Monthly 104 (1997) 852-854.
- D. H. Bailey, P. B. Borwein and S. Plouffe, On the Rapid Computation of Various Polylogarithmic Constants, Mathematics Computation, 1997.
- F. Bellard, A New Formula to Compute the nth Binary Digit of Pi, `http://www-stud.enst.fr/~bellard/pi/pi-bin.html`, 1997
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- S. Finch, The Miraculous Bailey-Borwein-Plouffe Pi Algorithm, http://www.mathsoft.com/asolve/plouffe/plouffe.html
- M. Hirschhorn, Australian Math. Society Gazette 25 (1998) 82-83.

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- I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals Series and Products, 4th Ed., Academic Press, 1965, 64-65
- M. Preston, *The Mountains of Pi*, The New Yorker, 1992, https://www.newyorker.com/magazine/1992/03/02/the-mountains-of-pi.

The Mathematics of Rainbows and Caustics

Fall 2021 - R. L. Herman





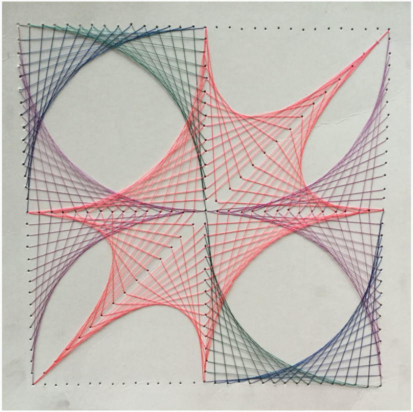
What Do You Know About Rainbows?

Questions asked by MIT Professor Walter Lewin:

- What is the radius of the rainbow?
- What is the color sequence in the rainbow?
- What is the sky darkness and brightness?
- Is there a second bow? What is the color sequence?
- Are rainbows polarized?

The Rainbow - What Do You See?

Table of Contents



Greeks

- Aristotle (384-322 BCE)
  First to rationally explain
  Reflection of light from clouds
- Alexander of Aphrodisias
  (3rd-2nd century BCE)
  `https://plato.stanford.edu/entries/alexander-aphrodisias/`
- Alexander's Band

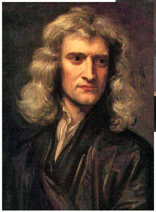
Early Arabic and European Studies

- Kamāl al-Dīn al-Fārisī (1267-1319)

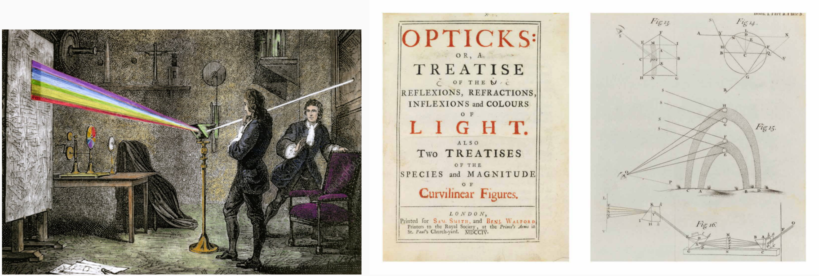
  Pupil of Qutb Al-Din al-Shirāzī (1236–1311), Student of Nasir al-Din al-Tusi (1201-1274) - trigonometry creator.

  Reformed *Book of Optics*, Alhazen [Hasan Ibn al-Haytham] (965-1039) and work of Avicenna (980-1037).

  Scattering due to raindrops.
- Roger Bacon (1214-1294) - 42^o
- Theodoric of Freiberg (1250-1310)
  Due to raindrops not clouds.
- Willebrord Snellius (1580-1626) Refraction
- René Descartes (1596-1650)
  Rediscovered - internal reflections.
- Pierre de Fermat (1607-1665)
  Theory of refraction - Snell's Law.
- Isaac Newton (1642-1727) - Theory of color.



Newton's Opticks



OPTICKS:
OR, A
TREATISE
OF THE
REFLEXIONS, REFRACTIONS,
INFLEXIONS and COLOURS
OF
LIGHT.
ALSO
Two TREATISES
OF THE
SPECIES and MAGNITUDE
OF
Curvilinear Figures.
LONDON,
Printed for SAM. SMITH, and BENJ. WALFORD, Printers to the Royal Society, at the Prince's Arms in St. Paul's Church-yard. MDCCIV.

Figure 1: Newton explained dispersion and rainbow optics.

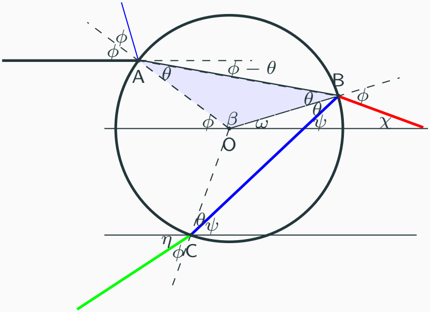
- Index of refraction - $n = \frac{c}{v}$,
- For water, $n = 1.35$ for violet to $n = 1.33$ for red.
- Snell's Law $n_{air} \sin\theta_{air} = n_{prism} \sin\theta_{prism}$.
- Wave Theory: Huygens (1678), Young (1803), Fresnel (1818), Maxwell (1862)

Geometric Optics

- Horizontally incident ray at A.
  $\phi =$ Angle of incidence
- Refracted into the droplet.
  $\theta =$ angle of refraction is
- At B: Either refracted or reflected.
- Laws of Optics

  Incident $\angle =$ Reflected $\angle$

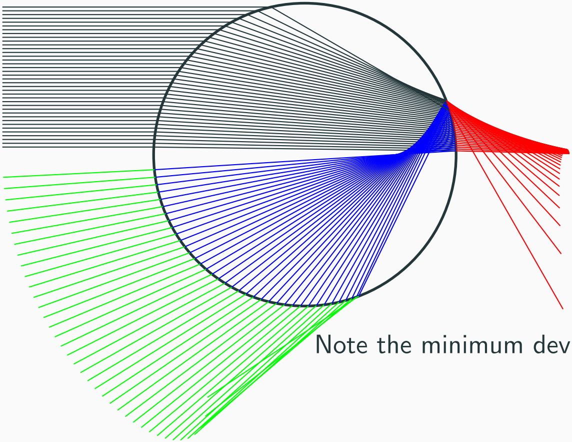
  Snell's Law: $\sin\phi = n\sin\theta$.
- At A, the ray is bent by $\phi - \theta$.
- At B the ray reflects and turns by $\beta = \pi - 2\theta$.
- At C the ray is bent by $\phi - \theta$.
- k reflections: $\Phi = |2(\phi - \theta) + k\beta|$



The scattering angle is

$$\Phi = 2(\phi - \theta) + \beta = \pi + 2\phi - 4\theta.$$

Multiple Incident Rays - Varying Impact Parameters

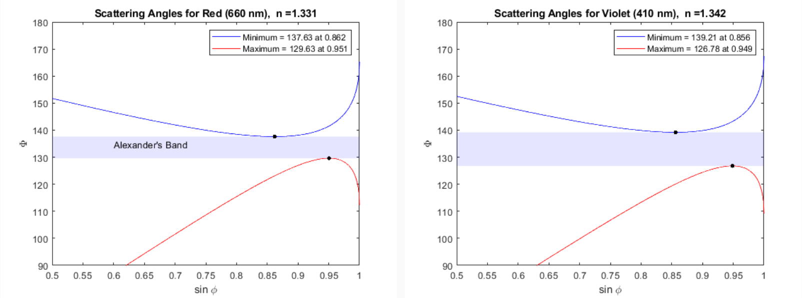


Note the minimum deviation.

Note that **caustics** are formed by the reflected and refracted rays.

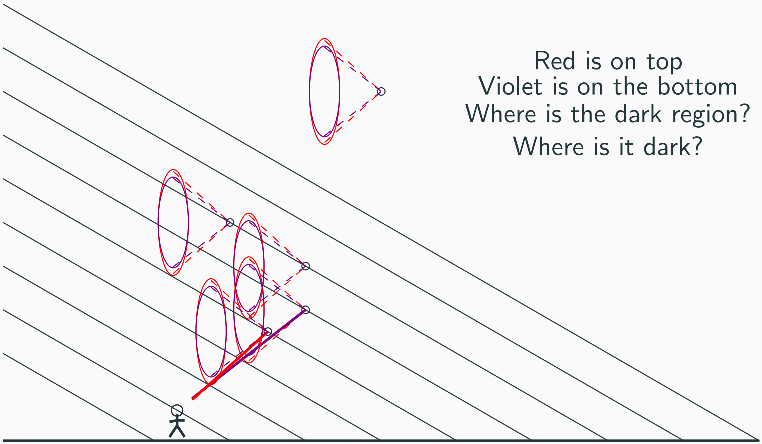
Minimum Deviation

Scattering angles for red (660 nm) and violet (410 nm) showing the extrema of the scattering angles. Extrema: $\sin\theta_c = \sqrt{\frac{(k+1)^2 - n^2}{(k+1)^2 - 1}}$, $k = 1, 2$.



The upper curves are for the primary rainbow and the lower curves are for the secondary rainbow.

Rays emerge between 40^o (violet) and 42^o (red).



Red is on top
Violet is on the bottom
Where is the dark region?
Where is it dark?

Figure 2: Following the scattering cones from multiple raindrops.

Secondary Rainbow

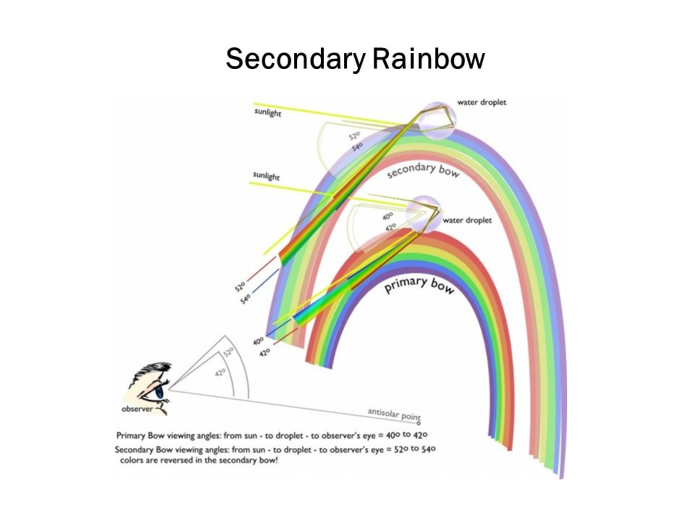
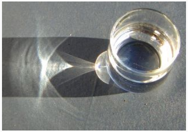


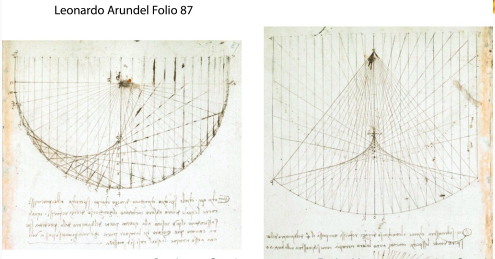
Figure 3: From `https://slideplayer.com/slide/8496591/`.

Caustics

Caustic from the Greek $\kappa\alpha\nu\sigma o\zeta$ for burnt.

- Reflection - Catacaustics
- Refraction - Diacaustics
- Leonardo da Vinci sketches.
- MathWorld Catacaustics

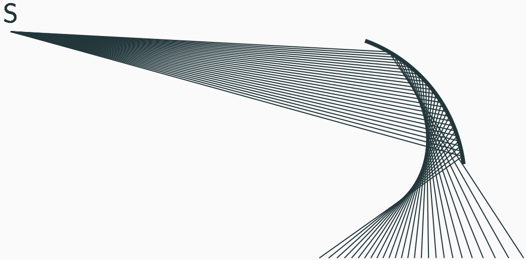




Light rays from a point source

Consider the development of a caustic from a point source S.

Equation of reflected ray: $y - y_P = (x - x_P)\tan\delta$, $y_P = y_P(x_P)$.



Then, $F(x_P, y(x_P)) = 2y_P'(y - y_P) + (x - x_P)(1 - y_P'^2) = 0$.

For envelopes, set $\dfrac{dF}{dx_P} = 0$, to find $x = x_P - \dfrac{y_P'}{y_P''}$, $y = y_P + \dfrac{1 - y_P'^2}{2y_P''}$.

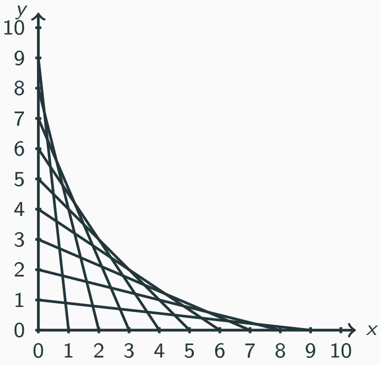
Envelope Example: String Art

- Connect $(0, k)$ with $(10 - k, 0)$.
- Equation of each line

$$y = -\frac{k}{10-k}x + k.$$

- Rewrite the family of curves, $k = t$.

$$\begin{aligned} y &= -\frac{t}{10-t}x + t \\ 1 &= \frac{x}{10-t} + \frac{y}{t} \\ t(10-t) &= xt + (10-t)y \end{aligned}$$



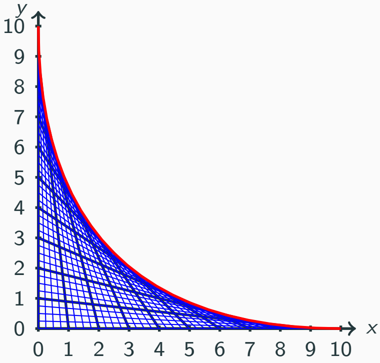
Let $F(x, y; t) \equiv t^2 + (x - y - 10)t + 10y = 0.$

String Art: Envelope Equation

- Solve $F(x,y;t)=0$, $\dfrac{\partial F(x,y;t)}{\partial t}=0$.
  $F(x,y;t)\equiv t^2+(x-y-10)t+10y=0$.
  $\dfrac{\partial F(x,y;t)}{\partial t}=2t+(x-y-10)=0$.
- Eliminating t, we have the envelope

$$(x-y-10)^2-40y=0.$$

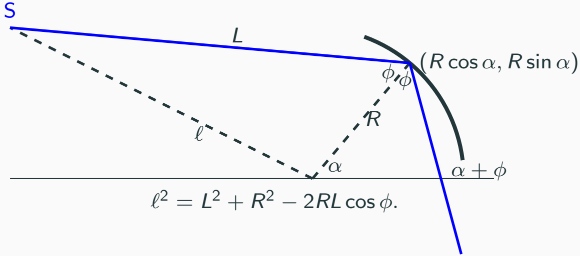
- The red curve is a parabola.
  The general conic is
  $x^2-2xy+y^2-20x-20y+100=0$.
  Letting $x=\dfrac{u-v}{\sqrt{2}}$, $y=\dfrac{u+v}{\sqrt{2}}$,
  we obtain $20\sqrt{2}u=2v^2+100$.



Equation of Reflected Ray

Equation:

$$\begin{aligned} y &= (x - R\cos\alpha)\tan(\phi+\alpha) + R\sin\alpha \\ &= \frac{x\sin(\phi+\alpha) - R\cos\alpha\sin(\phi+\alpha) + R\sin\alpha\cos(\phi+\alpha)}{\cos(\phi+\alpha)} \\ &= \frac{x\sin(\phi+\alpha) - R\sin\phi}{\cos(\phi+\alpha)} \qquad (1) \end{aligned}$$



$$\ell^2 = L^2 + R^2 - 2RL\cos\phi.$$

Seeking Caustic Equation

From $y = \dfrac{x\sin(\phi+\alpha) - R\sin\phi}{\cos(\phi+\alpha)}$, we define
$F(x, y; \alpha) = x\sin(\phi+\alpha) - y\cos(\phi+\alpha) - R\sin\phi = 0.$

Then,
$F_\alpha(x, y; \alpha) = [x\cos(\phi+\alpha) + y\sin(\phi+\alpha)](\phi_\alpha + 1) - R\phi_\alpha\cos\phi = 0.$
Solve the system

$$\begin{aligned} x\sin(\phi+\alpha) - y\cos(\phi+\alpha) &= R\sin\phi, \\ x\cos(\phi+\alpha) + y\sin(\phi+\alpha) &= \frac{R\phi_\alpha\cos\phi}{\phi_\alpha+1}. \end{aligned} \tag{2}$$

$$\begin{aligned} x &= R\sin\phi\sin(\phi+\alpha) + R_2\cos\phi\cos(\phi+\alpha), \\ y &= R_2\cos\phi\sin(\phi+\alpha) - R\sin\phi\cos(\phi+\alpha), \end{aligned} \tag{3}$$

where $R_2 = \frac{\phi_\alpha}{\phi_\alpha+1}R$.

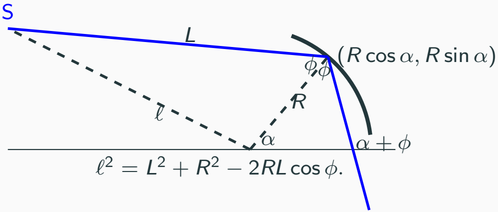
Finding ϕ_α

From $\ell^2 = L^2 + R^2 - 2RL\cos\phi$, we have $L\,dL - R\cos\phi\,dL + RL\sin\phi\,d\phi = 0$, or $d\phi = \dfrac{R\cos\phi - L}{RL\sin\phi}dL$.

Let S be the point (x_0, y_0). Since $L^2 = (R\cos\alpha - x_0)^2 + (R\sin\alpha - y_0)^2$, $L\,dL = (x_0\sin\alpha - y_0\cos\alpha)R\,d\alpha$.

So,

$$\begin{aligned}\frac{d\phi}{d\alpha} &= \frac{(R\cos\phi - L)}{RL\sin\phi}\frac{(x_0\sin\alpha - y_0\cos\alpha)R}{L} \\ &= \frac{(R\cos\phi - L)(x_0\sin\alpha - y_0\cos\alpha)}{L^2\sin\phi}. \end{aligned} \tag{4}$$



Catacaustic - Mathematics

- Appollonius' *Conica* (c. 200 BCE).
- Catacaustics introduced by Ehrenfried Walther von Tschirnhaus (1682).
- Important for early calculus.
- Jacob Bernoulli (1692) - logarithmic and parabolic spirals.
- Two chapters of Guillaume de l'Hôpital's (1696) calculus text.
- Christiaan Huygens (1673) introduced evolutes in *Horologium Oscillatorum* - cycloiods - ideal pendulum clock.
- Leibniz, Euler, Poisson, Puiseux, etc.

Spherical Mirror

The equations of the reflected rays are given by

$$y = (x - R\cos\phi)\tan 2\phi + R\sin\phi, \tag{5}$$

where ϕ is the angle of incidence. Rearranging, the reflected rays satisfy

$$F(x, y; \phi) = x\sin 2\phi - y\cos 2\phi - R\sin\phi = 0.$$

$$F_\phi(x, y; \phi) = x\cos 2\phi + y\sin 2\phi - R\cos\phi = 0.$$

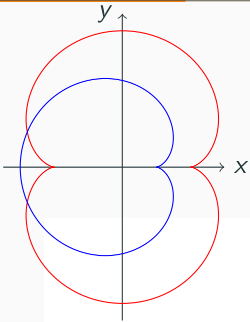
We can solve this system of equations to obtain the caustic envelope:

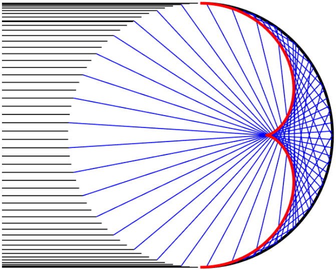
$$\begin{aligned} x &= R\sin\phi\sin 2\phi + \frac{1}{2}\cos\phi\cos 2\phi = \frac{R}{4}\left(3\cos\phi - \cos 3\phi\right). \\ y &= \frac{1}{2}R\cos\phi\sin 2\phi + \sin\phi\cos 2\phi = \frac{R}{4}\left(3\sin\phi - \sin 3\phi\right). \end{aligned} \tag{6}$$

The curve that results is a member of the family of epicycloids.

$$\begin{aligned} x(\phi) &= r(k+1)\cos\phi - r\cos\left((k+1)\phi\right) \\ y(\phi) &= r(k+1)\sin\phi - r\sin\left((k+1)\phi\right). \end{aligned} \tag{7}$$

Nephroid Catacaustics for Spherical Surface





Examples: Cardiod ($k = 1$) and a Nephroid ($k = 2$).

Prism i

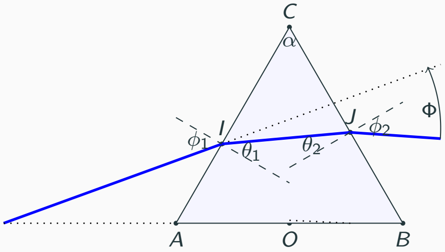
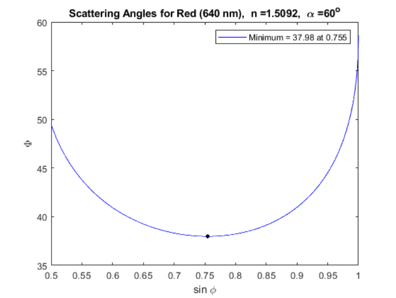


Figure 4: The bending of light through a prism.

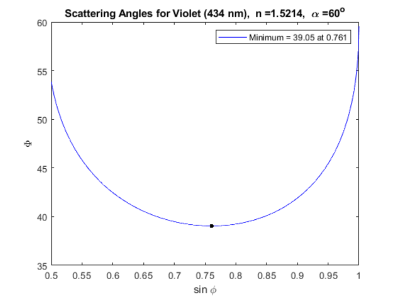
The angle of deviation, or scattering angle, is

$$\Phi = \phi_1 + \sin^{-1}\left(n \sin\left[\alpha - \sin^{-1}\left(\frac{\sin\phi_1}{n}\right)\right]\right) - \alpha.$$

Prism ii

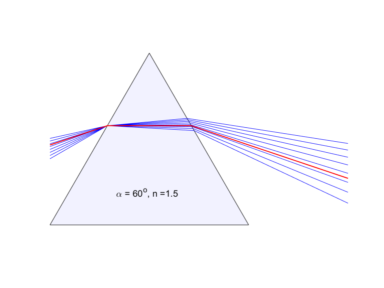


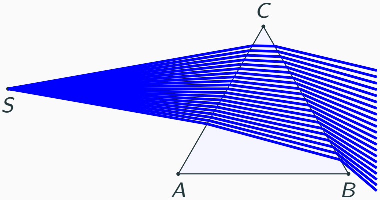
Scattering Angles for Red (640 nm), n =1.5092, α =60^o
Minimum = 37.98 at 0.755
Φ
sin ϕ



Scattering Angles for Violet (434 nm), n =1.5214, α =60^o
Minimum = 39.05 at 0.761
Φ
sin ϕ

Prism iii



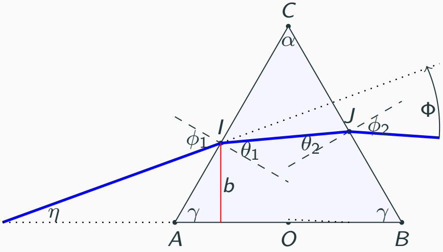


The scattered rays through a prism do not form a real caustic.

We begin with the equation for the emerging rays. The incident light enters at height b at point $I : (x_0, y_0) = (b \tan \frac{\alpha}{2} - a \sin \frac{\alpha}{2}, b)$ along the incident ray $y = (x - x_0) \tan \eta + b$.

We have that $\eta = \phi_1 - \frac{\alpha}{2}$, where ϕ_1 is the incident angle since external angle $\gamma = \eta + \frac{\pi}{2} - \phi_1$. But from the prism, $\gamma = \frac{\pi - \alpha}{2}$.

Prism iv



The ray is internally refracted by $\phi_1 - \theta_1$ from the incident ray.

The resulting angle is $\eta - (\phi_1 - \theta_1) = \theta_1 - \frac{\alpha}{2}$, giving the equation of the line:

$$y = (x - x_0)\tan(\theta_1 - \frac{\alpha}{2}) + b,$$

where $n \sin\theta_1 = \sin\phi_1$.

Prism v

At the second face at $J : (x_1, y_1)$,

$$\begin{aligned} x_1 &= \frac{x_0 \tan(\theta_1 - \frac{\alpha}{2}) + a\cos\frac{\alpha}{2} - b}{\cot\frac{\alpha}{2} + \tan(\theta_1 - \frac{\alpha}{2})} \\ &= \frac{(a\cos\frac{\alpha}{2} - b)\tan\frac{\alpha}{2}\cos\theta_1}{cos(\theta_1 - \alpha)}, \\ y_1 &= (x_1 - x_0)\tan(\theta_1 - \frac{\alpha}{2}) + b \\ &= \frac{a\sin\alpha\sin(\theta_1 - \frac{\alpha}{2}) + b\cos\theta_1}{cos(\theta_1 - \alpha)}. \end{aligned} \tag{8}$$

The second refracted ray starts at J and is bent by $\phi_2 - \theta_2$. Overall,

$$\theta_1 - \frac{\alpha}{2} - (\phi_2 - \theta_2) = \theta_1 + \theta_2 - \frac{\alpha}{2} - \phi_2.$$

Since $\theta_1 + \theta_2 = \alpha$, the emerging rays are

$$y = (x - x_1)\tan(\frac{\alpha}{2} - \phi_2) + y_1.$$

Prism vi

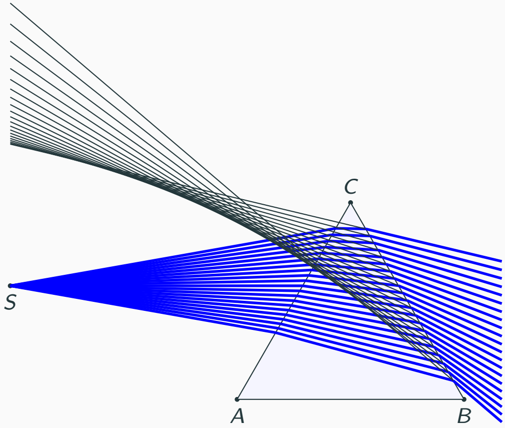


Figure 5: Extending the emerging rays backwards reveals a virtual caustic.

We can seek the equations of the caustic. As before, we define

$$F(x, y; \phi_1) = (x - x_1)\sin(\frac{\alpha}{2} - \phi_2) - y\cos(\frac{\alpha}{2} - \phi_2) + y_1\cos(\frac{\alpha}{2} - \phi_2),$$

where x_1, y_1 are functions of θ_1 and θ_1, ϕ_2 are functions of ϕ_1 [See Equation (8)].Differentiating F with respect to phi_1,

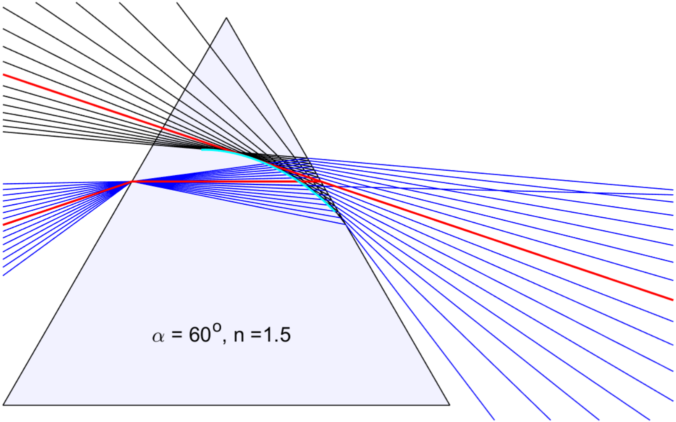
$$\frac{dF}{d\phi_1} = \frac{\partial F}{\partial \phi_1} + \frac{dF}{d\theta_1}\frac{d\theta_1}{d\phi_1} + + \frac{dF}{d\phi_2}\frac{d\phi_2}{d\phi_1} = 0,$$

where

$$\begin{aligned} \frac{d\theta_1}{d\phi_1} &= \frac{\cos\phi_1}{n\cos\theta_1} \\ \frac{d\phi_2}{d\phi_1} &= -\frac{n\cos(\alpha - \theta_1)}{\cos\phi_2}\frac{d\theta_1}{d\phi_1}. \end{aligned} \quad (9)$$

This with $F(x, y; \phi_1) = 0$ leads to parametric equations for the virtual caustic. These are determined using MATLAB and then used to superimpose the solution on Figure figprism4 as shown in Figure 6.

Prism viii



MATLAB Code

```
% Search for Caustic Solution in MATLAB
a = 3.0;    b = 0.5*a;   alpha = 60*pi/180; n = 1.5;

% Determine caustic symbolically
syms p t z x y dtdp dzdp

% f - equation for refracted rays, g = f_phi.
% u=x1, v=y1
u = @(t) cos(t)/cos(t-alpha)*(a*cos(alpha/2)-b)*tan(alpha/2);
v = @(t) (a*sin(alpha)*sin(t-alpha/2)+b*cos(t))/cos(t-alpha);
f = (x-u(t))*sin(alpha/2-z)+v(t)*cos(alpha/2-z) ...
     -y*cos(alpha/2-z);
g = diff(f,p)+diff(f,t)*dtdp+diff(f,z)*dzdp;

% (X,Y) - parametric equations for caustic
[X,Y] = solve([f==0,g==0],[x,y]);
```

Rainbow Caustics i

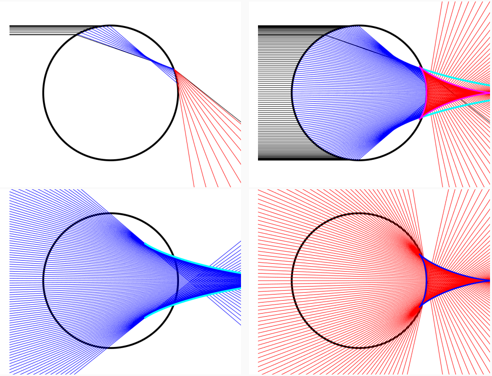


Figure 7: Light rays and caustics after passage with no reflection.

Rainbow Caustics ii

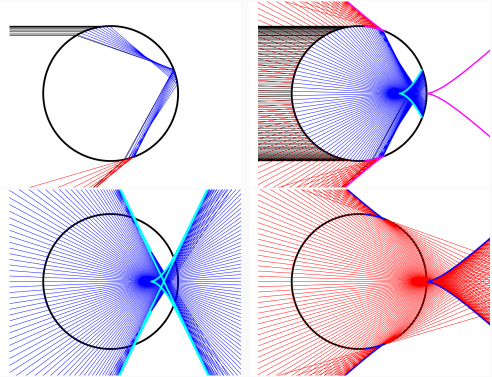


Figure 8: Light rays and caustics after passage with one reflection.

Rainbow Caustics iii

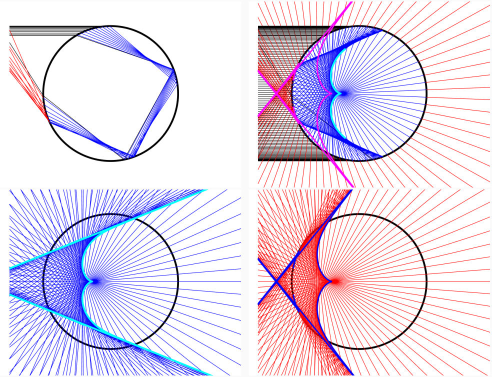


Figure 9: Light rays and caustics after passage with two reflections.

Rainbow Caustics iv

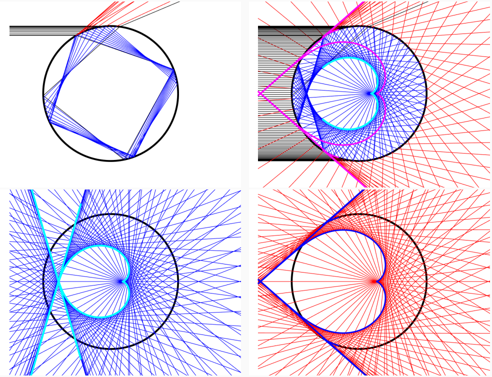


Figure 10: Light rays and caustics after passage with three reflections.

Rainbow Caustics v

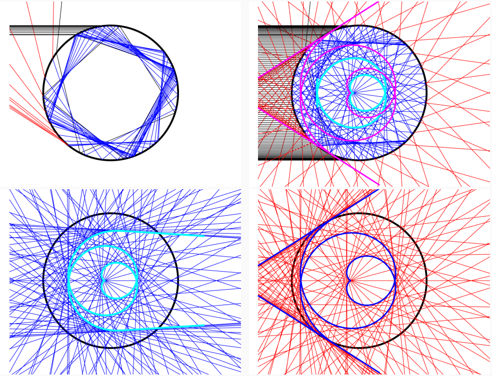
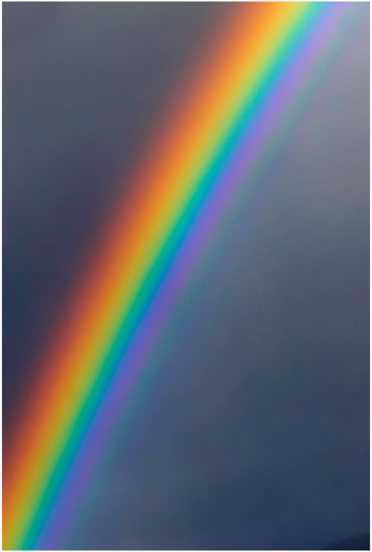
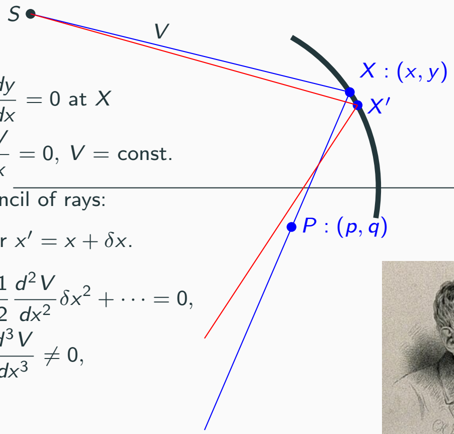


Figure 11: Light rays and caustics after passage through the droplet with six reflections.

Supernumaries

- Green, pink and purple fringes.
- Thomas Young (1803) - suggested due to wave interference.
- R Potter (1835) *Mathematical Considerations on the Problem of the Rainbow.*
- George Biddle Airy, *On the intensity of light in the neighbourhood of a caustic,* 1838.
- Later, improved theory - Mie Scattering and Debye series.
- Rayleigh scattering - why the sky is blue.





$$\frac{dV}{dx} = \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}\frac{dy}{dx} = 0 \text{ at } X$$

Convergence to focus: $\dfrac{dV}{dx} = 0, \; V = \text{const.}$

Convergence of pencil of rays:

$$\frac{dV}{dx} = 0, \; \frac{dV}{dx'} = 0, \text{ for } x' = x + \delta x.$$

So, $\dfrac{dV}{dx'} = \dfrac{dV}{dx} + \dfrac{dV}{dx}\delta x + \dfrac{1}{2}\dfrac{d^2V}{dx^2}\delta x^2 + \cdots = 0,$

$$\frac{d^2V}{dx^2} = 0, \text{ but } \frac{d^3V}{dx^3} \neq 0,$$

Figure 12: A summary of what lead Airy first to an equation which holds for the convergence of rays based on his Figure I.

Airy Functions

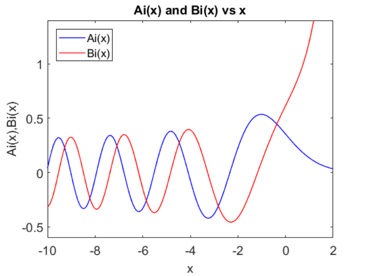
- Airy - Caustic singularity corrected with diffraction.
- Used Thomas Young's 1801 superposition.
- Airy's Integral

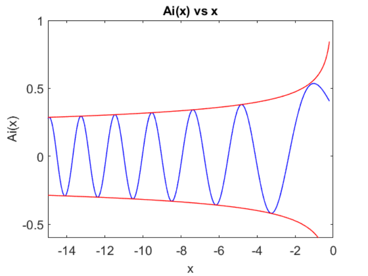
$$Ai(x) = \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right)\, dt$$

- Convergent series

$$\begin{aligned} Ai(x) &= A\left\{1 + \frac{9x+3}{2\cdot 3} + \frac{9^2 x^6}{2\cdot 3\cdot 5\cdot 6} + \frac{9^3 x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \cdots\right\} \\ &\quad + B\left\{x + \frac{9x^4}{3\cdot 4} + \frac{9^2 x^7}{3\cdot 4\cdot 6\cdot 7} + \frac{9^3 x^{10}}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10} + \cdots\right\} \end{aligned}$$

Airy Functions





$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt$$

$$Bi(x) = \frac{1}{\pi} \int_0^\infty \left[e^{-\frac{t^3}{3}+xt} + \sin\left(\frac{t^3}{3} + xt\right)\right] dt$$

Airy Differential Equation

$$\frac{d^2y}{dx^2} - xy = 0,$$

Solve using power series method: $y = \sum_{m=0}^{\infty} a_m y^m$:

$$2a_2 + \sum_{m=1}^{\infty} \left[(m+1)(m+2)a_{m+2} + a_{m-1}\right] x^m = 0.$$

Equate coefficients to zero and Solve for a_m's.

$$\begin{aligned} y_1 &= 1 + \sum_{m=1}^{\infty} \frac{(-1)^n}{3^n n! [2 \cdot 5 \cdot \cdots \cdot (3n-1)]} x^{3m}, \\ y_2 &= \sum_{m=1}^{\infty} \frac{(-1)^n}{3^n n! [1 \cdot 4 \cdot \cdots \cdot (3n+1)]} x^{3m+1} \end{aligned}$$

Sir George Gabriel Stokes (1819-1903)

- When x large, computation tedious and slow convergence.
- Airy - only two bands.
- But 30 bands measured!
- Stoke's Expansion (1850)

$$
\begin{aligned}
Ai(x) &= Cx^{-1/4}e^{-2x^{3/2}}\left\{1-\frac{1\cdot5\cdot7\cdot11}{1\cdot144x^{\frac{3}{2}}}+\frac{1\cdot5}{1\cdot2\cdot144^2x^3}+\cdots\right\}\\
&\quad +Dx^{-1/4}e^{2x^{3/2}}\left\{1+\frac{1\cdot5\cdot7\cdot11}{1\cdot144x^{\frac{3}{2}}}+\frac{1\cdot5}{1\cdot2\cdot144^2x^3}+\cdots\right\}
\end{aligned}
$$

$$
\mathrm{Ai}(z)\sim\frac{e^{-\frac{2}{3}z^{\frac{3}{2}}}}{2\sqrt{\pi}\,z^{\frac{1}{4}}}\left[\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})\left(\frac{3}{4}\right)^n}{2\pi n!z^{3n/2}}\right],|arg(z)|<\frac{\pi}{3}.
$$

A letter Stokes wrote, London, March 19, 1857.[1]

When the cat's away the mice may play. You are the cat and I am the mouse. I have been doing what I guess you won't let me do when we are married, sitting up till 3 o'clock in the morning fighting hard against a mathematical difficulty. Some years ago I attacked an integral of Airy's, and after a severe trial reduced it to a readily calculable form. But there was one difficulty about it which, though I tried till I almost made myself ill, I could not get over, and at last I had to give it up and profess myself unable to master it. I took it up again a few days ago, and after a two or three days' fight, the last of which I sat up till 3, I at last mastered it. I don't say you won't let me work at such things, but you will keep me to more regular hours. A little out of the way now and then does not signify, but there should not be too much of it. It is not the mere sitting up but the hard thinking combined with it.

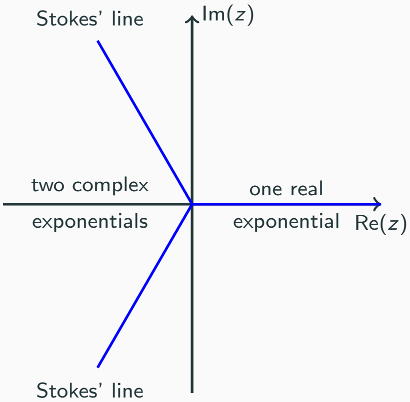
[1] Sir George Gabriel Stokes, *Memoirs and Scientific Correspondence*, Cambridge, 1907, Vol. 1, p. 62.

Stokes' Lines

- Stokes (1858) considered

$$Ai(z) = \int_{-\infty}^{\infty} ds\, e^{i(s^3/3+zs)}, \quad z \in \mathbb{C}.$$

- $z \in \mathbb{R}$ for optics.
- Now called WKB method, or stationary phase method.
- Stationary points $s = \pm(-z)^{1/2}$.
- z real and $\mathrm{Re}(z) \gg 0$, one contributes, exponentially small.
- z real and $\mathrm{Re}(z) \ll 0$, both contribute, oscillatory behavior.
- So, a second exponential has to be born between $\pm x$-axes.



Divergent Series[2]

$$f(x) = \sum_{n=0}^{N-1} a_n (x - x_0)^n + R_N(x)$$

Convergent: $|R_N(x)| \to 0$ as $N \to \infty$, x fixed.

Asymptotic: $|R_N(x)| \ll (x - x_0)^N$ as $x \to x_0$, N fixed.

Airy functions - factorial divergence.

$$\left.\begin{array}{c} 2Ai(x) \\ Bi(x) \end{array}\right\} = \frac{\mp e^{\frac{2}{3}x^{3/2}}}{2\pi^{3/2}x^{1/4}} \sum_{n=0}^{\infty} (\mp 1)^n \frac{\Gamma\left(n+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{n!(\frac{4}{3}x^{3/2})^n}$$

Connection formula.

$$Ai(e^{\mp\frac{2\pi i}{3}}x) = \pm\frac{i}{2}e^{\mp\frac{\pi i}{3}}Bi(x) + \frac{1}{2}e^{\mp\frac{\pi i}{3}}Ai(x)$$

[2] *Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever.* - Niels Abel (1802–1829)

Analytic Continuation Using Borel Transform

$$f(g) = \sum_{n=0}^{\infty} a_n g^{n+1}$$

Borel Transform

$$\mathcal{B}[f](s) = \sum_{n=0}^{\infty} \frac{a_n}{n!} s^n$$

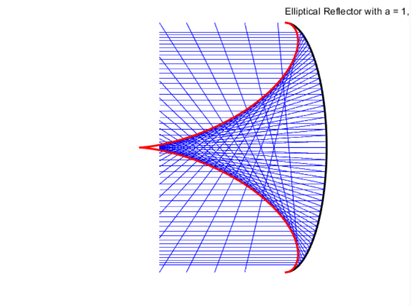
Laplace Transform

$$Sf(g) = \frac{1}{g} \int_0^{\infty} e^{-\zeta/g} \mathcal{B}[f](\zeta)\, d\zeta$$

Asymptotic Expansion

Review

- Brief History of Rainbows
- Geometric Optics
- Caustics - Other Surfaces
- Airy Functions
- Divergent Series
- Maxwell's Equations (1860's)
- Fresnel Equations, Polarization.
- Scattering Theory - Rayleigh, Mie, Debye
- Take away for students
    - Review sources and history.
    - LaTeX- Figures from Tikz, MATLAB,
    - Presentation in Beamer



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Genesis of Differential Equations

Spring 2025 - R. L. Herman





In the beginning there was Calculus

- Eudoxus: Method of exhaustion.
- Bonaventura Cavalieri (1598–1647), follower of Galileo Galilei (1564–1642), 1635, *Geometria indivisibilibus continuorum* - method of indivisibles for integration.
- Fermat had described in 1629, *Methodus ad disquirendam maximam et minimam,* method to find maxima and minima.

  Also found method to obtain the area under the curve by dividing it into an infinite number of rectangles
- John Wallis (1616–1703), published in 1656 *Arithmetica Infinitorum,* area under x^k is $\dfrac{x^{k+1}}{k+1}$.

  So, one can find the quadrature of any function represented as a power series expansion.
- Proofs of the fundamental theorem of calculus: James Gregory (1638–1675), Isaac Barrow (1630–1677),
- Integral as the antiderivative had to wait for Newton and Leibniz's calculus.

Early contributors: Newton, Leibniz and the Bernoulli brothers

- Isaac Newton (1643–1727)

  Developed the fundamentals of differential calculus in the second half around 1666,

  Wrote *Methodus fluxionum et serierum infinitarum* of 1671 (published 1736).

  He would also write the *De analysi per aequationes numero terminorum infinitas* in 1669 (published in 1711), and

  A geometrical form of his differential calculus in section I of book I of the Principia, 1687.

- In *Methodus fluxionum ...* - fluxion equations using infinte series.

- Gottfried Leibniz (1646–1716)

  His works were based on those of Descartes, Blaise Pascal and Fermat.

  Derived 1674-1676, but published 1684 in *Acta Eruditorum* titled *Nova Methodus pro Maximis et Minimis.*

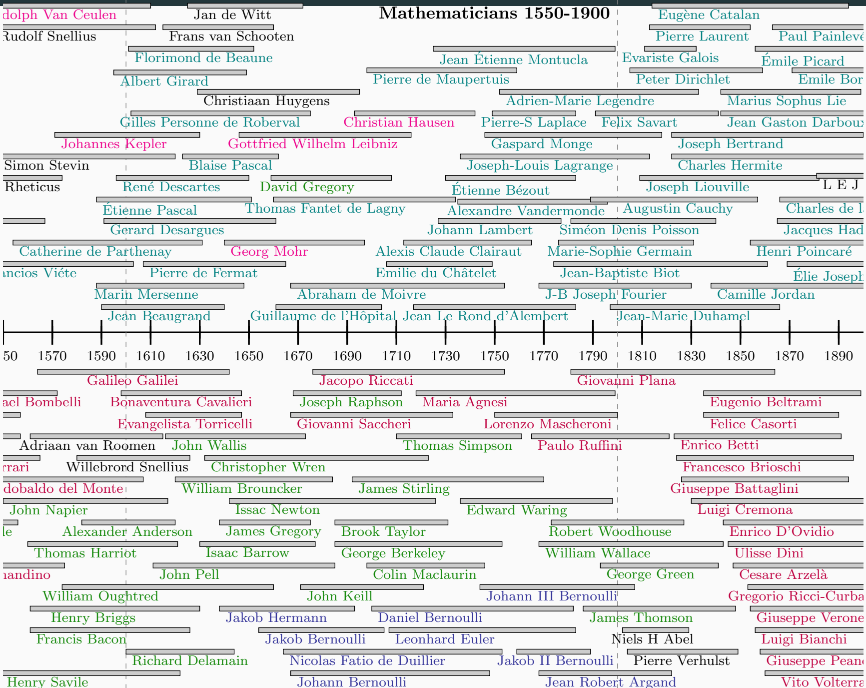
  October and November 1675: introduced d and $\int$.

- First text, *De constructione Aequationum Differentialium Primi Gradus*, 1707 by Gabriele Manfredi (1681–1761).

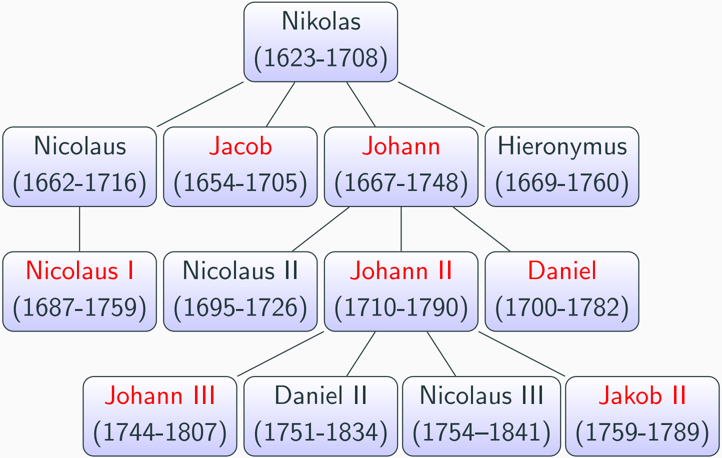
- Later developers: Euler, D. Bernoulli, Lagrange and Laplace, ...

- Euler's book *Institutionum Calculi Integralis*, 1768–70.

Mathematicians - 1550-1900



The Bernoulli Family



Jakob Bernoulli (aka James or Jacques or Jacob)

Johann Bernoulli (aka Jean or John)

Newton's Classification

Newton gives three types of problems in 1736 translation of 1671 work on fluxions, *The method of fluxions and infinite series*:

> *13. But in respect of this Problem Equations may be distinguish'd into three Orders.*
>
> *14. First: In which two Fluxions of Quantities, and only one of their flowing Quantities are involved.*
>
> *15. Second: In which the two flowing Quantities are involved, together with their Fluxions.*
>
> *16. Third: In which the Fluxions of more than two Quantities are involved.*

These are often cited as Newton giving three classifications of differential equations in terms of fluxions and fluents. Here Newton referred to a flowing quantity as a fluent and to its instantaneous rate of change as a fluxion. Thus, he wrote $\dot{x}$ and $\dot{y}$.

Newton's Problems

When Newton needed the slope of the curve $y = y(x)$, he sought $\frac{\dot{y}}{\dot{x}}$. The classification is often summarized as [see Krishnachandran.]

$dy/dx = f(x)$:

$dy/dx = f(y)$:

$dy/dx = f(x, y)$:

However, in his work on fluxions, he listed a number of problem types and examples. He relied on infinite series to obtain solutions to the various differential equations he considered.

Note: Newton gave a 'geometrical form' of his differential calculus in section I of book I of the *Principia* of 1687. Online 1846 translation.

The method of fluxions and infinite series, pg 29.

> 19. So propoſing the Equation $\dot{y}\dot{y} = \dot{x}\dot{y} + \dot{x}\dot{x}xx$; I ſuppoſe x to be the Correlate Quantity, and the Equation being accordingly reduced, we ſhall have $\frac{\dot{y}}{\dot{x}} = 1 + x^2 - x^4 + 2x^6$, &c. Now I multiply the Value of $\frac{\dot{y}}{\dot{x}}$ into x, and there ariſes $x + x^3 - x^5 + 2x^7$, &c. which Terms I divide ſeverally by their number of Dimenſions, and the Reſult $x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{2}{7}x^7$, &c. I put $= y$. And by

Newton solves $\dot{y}\dot{y} = \dot{x}\dot{y} + \dot{x}\dot{x}xx$, or $\left(\frac{\dot{y}}{\dot{x}}\right)^2 = \frac{\dot{y}}{\dot{x}} + x^2$, for $\frac{\dot{y}}{\dot{x}}$ as a series:

$$\frac{\dot{y}}{\dot{x}} = \frac{1}{2} + \frac{\sqrt{4x^2+1}}{2} = 1 + x^2 - x^4 + 2x^6 + \mathcal{O}(x^8).$$

"Integrate," $y(0) = 0$:

$$y = x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{2}{7}x^7 + \mathcal{O}(x^9).$$

Maple:

$$\frac{x}{2} \pm \frac{x\sqrt{4x^2+1}}{4} \pm \frac{\operatorname{arcsinh}(2x)}{8}$$

Footnotes

Newton discovered binomial series in 1665.

Hyperbolic functions introduced 1757 by Vincenzo Riccati (1707-1775). 1768 - Johann Heinrich Lambert (1728-1777)

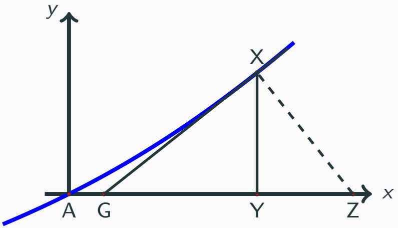
Now, we go to continental Europe starting with Descartes.

La Géométrie - 1637, online - On algebraic operations and geometric constructions

- René Descartes (1596-1650)
- Book I: *Problems Which Can Be Constructed by Means of Circles and Straight Lines Only.*
  He introduced algebraic notation: x, y, z, etc. denote unknown variables, a, b, c, etc. denote constants.
- Book II: *On the Nature of Curved Lines*,
  Descartes described two kinds of curves, called by him geometrical and mechanical.
  Algebraic method for finding the normal at any point of a curve whose equation is known. The construction of the tangents to the curve follows.
- Book III. *On the Construction of Solid and Supersolid Problems*
  Introduction to nature of equations and their solution.

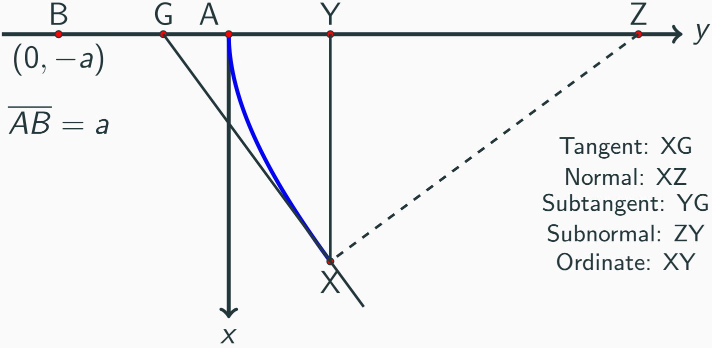
de Beaune's Second Problem

- Erasmus Bartholin (1625-1698) edited *Introduction to the geometry of Descartes* by Frans van Schooten (1615 - 1660).
- During the years 1664- 1674 he produced several volumes of a book *Dissertatio de problematibus geometricis* that consisted of theses he had proposed for his students.
- In 1672 he gave a proof of the **second problem of de Beaune**.
- Originally posed in a letter from Florimond de Beaune (1601 - 1652) to Marin Mersenne (1588 - 1648) in 1638. **1st. inverse tangent problem.**



Inverse Tangent Problem

The curve AX with vertex A and axis AY is determined as follows: From the arbitrary point X (with the axis intersections G and Z) and specify a fixed distance AB. Then there is always the ratio ZY: XY = AB: (XY - AY). [Note: ZY:XY = XY: YG.]



Tangent: XG
Normal: XZ
Subtangent: YG
Subnormal: ZY
Ordinate: XY

The First Differential Equation?

Inverse tangent problem: Find a curve given properties of its tangents.

In modern notation, we seek the solution to a differential equation.

Debeaune's second problem can be written as

$$\frac{dy}{dx} = \frac{x - y}{a}, \tag{1}$$

for some constant a.

We would now solve this as

$$y = x + a\left(e^{-x/a} - 1\right). \tag{2}$$

K. M. Pedersen [*Centaurus* 22 (2) (1978), 99-107] suggests Bartholin's geometric proof may be Debeaune's original proof which he sent to Descartes. Debeaune sent Bartholin papers for safe keeping shortly before his death in 1652.

Leibniz's Solution

René Descartes, fond of Debeaune, supposedly solved Debeaune's problem in 1639.

Gottfried Wilhelm Leibniz discusses Debeaune's problem in his first calculus publication (1684), "A New Method for Finding Minima and Maxima," in *Acta Eruditorum*, in the form [See Leibniz's Figure on next page. See paper online and a translation and next slides.]

> *to find a line WW of such a nature that, drawn to the axis tangent to WC, XC is always equal to the same straight line constant a.*

quisquam pari facilitate tractabit. Appendicis loco placet adjicere solutionem Problematis, quod Cartesius a Beaunio sibi propositum, Tom.3. Epist. tentavit, sed non solvit. Lineam invenire WW talis naturæ, ut ducta ad axem tangente WC, sit XC semper æqualis eidem rectæ constanti, a. Jam. XW seu w ad XC seu a, ut d w ad d x: Ergo si dx (quæ assumi potest pro arbitrio) assumatur constans sive semper eadem nempe b, seu si ipsæ x sive AX crescant uniformiter, fiet W æqu. $\frac{a}{b}$dw, quæ erunt ipsæ W ordinatæ, ipsis dw, suis incrementis sive differentiis, proportionales, hoc est si x sint progressionis arithmeticæ, erunt w progressionis Geometricæ, seu si w sint numeri, x erunt logarithmi: linea ergo WW logarithmica est.

Leibniz (1684) *Nova methodus pro maximis et minimis, ...*

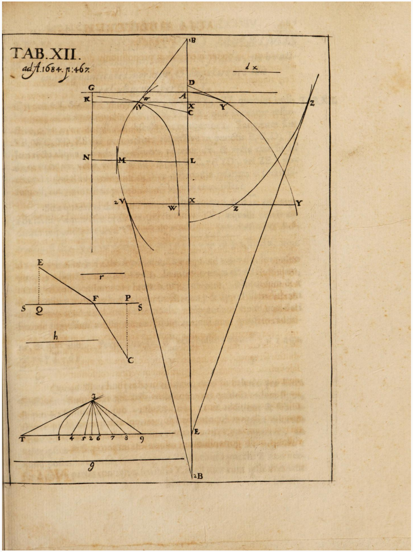
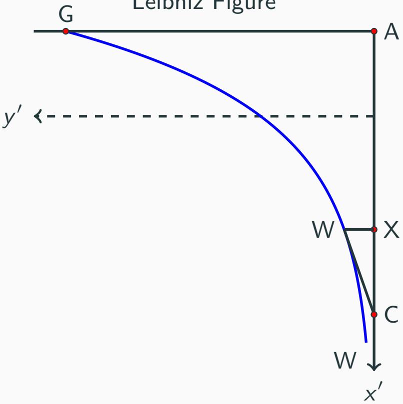


Figure 1: The curve WW is used to discuss the de Beaune problem.

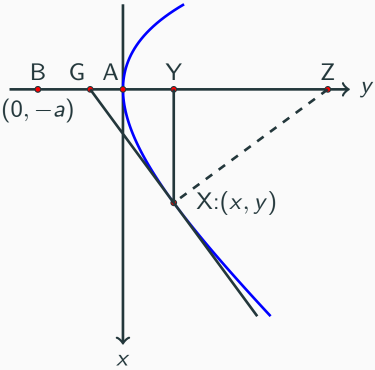
Problem Comparison

Leibniz solves De Beaune's inverse-tangent problem: What kind of curve will have a constant subtangent? He said that it is "logarithmica."

Leibniz Figure



de Beaune Figure



Leibniz says (in Latin)

Appendicis loco placet adjicere solutionem Problematis, quod Cartesius a Beaunio sibi propositum Tom. 3. Epist. tentavit, sed non solvit : Lineam invenire WW talis naturae, ut ducta ad axem tangente WC, sit XC semper aequalis eidem rectae constanti a. Jam XW seu w ad XC seu a, ut dw ad dx; ergo si dx (quae assumi potest pro arbitrio) assumatur constans sive semper eadem, nempe b, seu si ipsae x sive AX crescant uniformiter, fiet w aequ. a/b dw , quae erunt ipsae w ordinatae ipsis dw, suis incrementis sive differentiis proportionales, hoc est si x sint progressionis arithmeticae, erunt w progressionis Geometricae, seu si w sint numeri, x erunt logarithmi: linea ergo WW logarithmica est.

Ian Bruce (2014) translates

It pleases to add the solution of a problem as an appendix, which De Baune proposed to Decartes to attempt himself, in Vol. 3 of his letters, but which he did not solve: To find the line of such a kind WW, [adapted from the first figure] so that with the tangent WC drawn to the axis, XC shall always be equal to the same constant right line a. Now XW or w shall be to XC or a, as dw to dx; therefore if dx (which can be taken by choice) may be assumed constant or always the same, truly b, or if x itself or if AX may increase uniformly, w will be made equal to a/b dw, and b the ordinates w themselves which will proportional to their increments, or differentials, from dw, that is if the x shall be in an arithmetic progression, the w shall be in a geometric progression, or if w shall be numbers, x will be their logarithms: therefore the line WW is logarithmic.

The Differential Equation

For Leibniz's formulation of the problem,

$$\frac{dw}{dx} = -\frac{w}{a}, \tag{3}$$

whose solution is readily found as

$$w = Ae^{-x/a}. \tag{4}$$

However, back in the late 1600s it was not known how to "integrate" $\frac{1}{x}$. This was noticeable in Bernoulli's solution of the problem.

Often referred to *logarithmica.* [We had to wait for Euler to give e in 1736.]

This solution in Equation (2) can be written as $y = x + a\left(\frac{w}{A} - 1\right)$. We can map the diagrams in onto each other.

Leibniz - 1686

pendent a lectione generali rationis, ſeu a Lo-
one generali anguli, ſeu ab arcubus circuli, aliæ
uæſtionibus magis compoſitis) ideo præter li-
dhuc tertiam ut v, quæ tranſcendentem quan-
x his tribus formo æquationem generalem ad
qua lineæ tangentem quæro, ſecundum meam
m in Actis Octobr. 84 publicatam, quæ nec
tur. Deinde id quod invenio comparans cum
entium curvæ, reperio non tantum literas aſ-
d & ſpecialem tranſcendentis naturam. Quan-
o fieri poſſit, ut plures adhibendæ ſint tranſ-
andoque inter ſe diverſæ, & dentur tranſcen-
um, & omnino talia procedant in infinitum, ta-
tilioribus contenti eſſe poſſumus; & plerumq;
uti licet ad calculum contrahendum, proble-
terminos ſimplices revocandum, quæ non ſunt
em methodo ad Tetragoniſmos applicata, ſeu
rum quadratricium (in quibus utique ſemper
as data eſt) patet non tantum, quomodo inve-
indefinita ſit Algebraice impoſſibilis, ſed &
tate hac deprehenſa reperiri poſſit quadratrix
actenus traditum non fuit. Adeo ut videar
Geometriam hac methodo ultra terminos a
tos in immenſum promoveri. Cum hac ra-
& generalis ad ea porrigatur problemata, quæ
lus, atque adeo Algebraicis æquationibus non

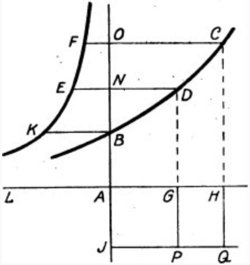
ad problemata Tranſcendentia, ubicunque di-
que occurrunt, calculo tractanda vix quicquam

res in axe ſumtorum, & ad axem applicatorum, æquetur ſemiquadra-
to ordinatæ ultimæ) in cujus executione tamen nonnihil a ſcopo
deflexit, quod in nova methodo non miror; ideo gratiſſimum ipſi
aliisq; fore arbitror, *ſi hoc loco aditum rei, cujus tam late patet utilitas,*
patefecero. Nam inde omnia hujuſmodi theoremata ac problemata,
quæ admirationi merito fuere, ea facilitate fluunt, ut jam non magis
ea diſci teneriq; neceſſe ſit, quam plurima vulgaris Geometriæ theore-
mata illi ediſcenda ſunt, qui ſpecioſam tenet. Sic ergo in caſu præ-
dicto procedo. Sit ordinata x, abſciſſa y, intervallum inter perpen-
dicularem & ordinatam quod dixi ſit p, patet ſtatim methodo mea
fore $p\,dy = x\,dx$ quod & Dn. Craigius ex ea obſervavit; qua æquatione
differentiali verſa in ſummatricem, fit $\int p\,dy = \int x\,dx$. Sed ex iis quæ in
methodo tangentium expoſui, patet eſſe $d, \frac{1}{2}xx = x\,dx$; ergo contra $\frac{1}{2}$
$xx = \int x\,dx$ (ut enim poteſtates & radices in vulgaribus calculis, ſic no-
bis ſummæ & differentiæ ſeu $\int$ & d, reciprocæ ſunt). Habemus ergo
$\int p\,dy = \frac{1}{2}xx$. Quod erat dem. Malo autem dx & ſimilia adhibere,
quam literas pro illis, quia iſtud dx eſt modificatio quædam ipſius x,
& ita ope ejus fit, ut ſola quando id fieri opus eſt litera x, cum ſuis
ſcilicet poteſtatibus & differentialibus calculum ingrediatur, & rela-
tiones tranſcendentes inter x & aliud exprimantur. Qua ratione et-
iam lineas tranſcendentes æquatione explicare licet, verbi grat. Sit ar-
cus a, ſinus verſus x, fiet $a = \int dx : \sqrt{2x - xx}$, & ſi cycloidis ordinata
ſit y, fiet $y = \sqrt{2x - xx} + \int dx : \sqrt{2x - xx}$, quæ æquatio perfecte ex-
primit relationem inter ordinatam y & abſciſſam x, & ex ea omnes
cycloidis proprietates demonſtrari poſſunt; promotuſque eſt hoc
modo calculus analyticus ad eas lineas, quæ non aliam magis ob
cauſam hactenus excluſæ ſunt, quam quod ejus incapaces crederen-
tur: Interpolationes quoque Wallisianæ & alia innumera hinc deri-

Other Problems

- Tautochrone - Time independent of starting point, Huygens, 1659.
- Brachistochrone - Curve of fastest descent, Johann Bernoulli, 1696.
- Isochrone - Connects points of equal time travel, Leibniz 1687, Jacob Bernoulli 1690.
- Cycloid - Huygens pendulum driven clock, 1656.
- Catenary - hanging chain.
- 1691 - Jacob Bernoulli - parabolic spiral.
- Elastica and lemniscate, Sep 1694, Jacob, Oct 1694, Johann.

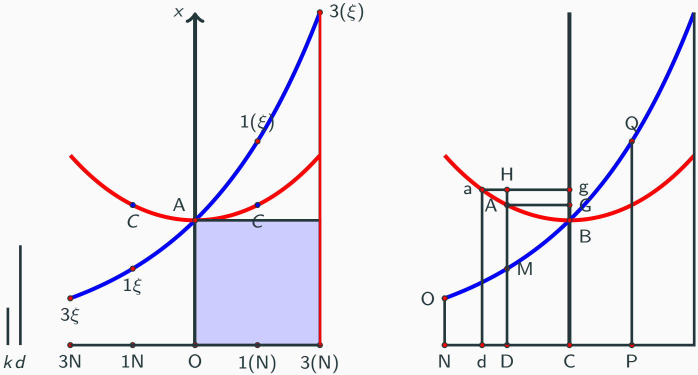
Problems involving *quadrature* and *rectification.* Solution "by quadratures" of the differential equation $a\,dx = a^2\,dy/y$ [Johann Bernoulli, 1692] Lines EN and PG are chosen so that the areas KBNE and AJPG are equal. Intersection D is on the sought curve. Bernoulli was aware that the solution curve was the "Logarithmica" but he considered the geometrical construction more fundamental



The Catenary - Find the curve of a hanging chain.

- Leonardo Da Vinci was the first to consider it.
- Followed by Galileo and others - believed the curve was a parabola.
- Christian Huygens claimed at 17 the curve was not algebraic without any proof.
- May 1690, Jacob Bernoulli posed challenge in the *Acta Eruditorum.*
- Gottfried Leibniz immediately responded. Leibniz,Huygens and Johann Bernoulli separately found the equation for catenaryand published their solutions on 1691.
- While in Paris, Johann Bernoulli discussed catenary in his Lectures on the Integral Calculus, Lecture Thirty-Six, Lecture Twelve, and Lecture Thirty-Seven. The lectures were written out for Guillaume Marquis de L'Hôpital in 1691-1692.
- Compared Leibniz's solution to his own.

The Catenary Solutions of Leibniz (left) and Bernoulli (right)



CB = subtangent = a BG= x, GA=CD=y, DM=z Gg= dx, Dd=Ha=dy

CD=CP $\Rightarrow$ DM:CB = CB:PQ

Therefore, PQ $= a^2/z$ and DA $= \frac{1}{2}$(DM+PQ) $= \dfrac{a^2+z^2}{2z}$ = CB + BG $= a + x$

Solve for z : $z = a + x + \sqrt{2ax + x^2}$, find dz and use $z\,dy = a\,dz$

$$\frac{a\,dx}{\sqrt{2ax+x^2}} = dy$$

Leonhard Euler (1707-1783)

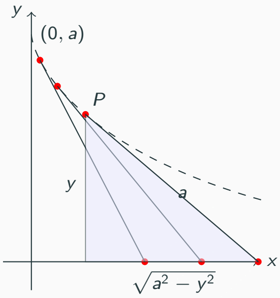
- Connection with Bernoulli family.
- 1748 *Introductio in analysin infinitorum, on precalculus,*
- 1755 *Institutiones calculi differentialis,* on differential calculus,
- 1768 *Institutionum Calculi Integralis* on integral calculus, Euler Archives
  - Vol. I Integration of first order differential equations, E342.
  - Vol II. Resolution of higher order ODEs, E366.
  - Vol III. Resolution of partial differential equations and includes calculus of variations, E385.

- Separable eq.
- Homogeneous eq.
- Reducible to separable eq.
- Linear first order eq
- Bernoulli eq.
- Riccati eq.
- Exact eq.
- Integrating factor
- Particular/General solutions
- Resolution of first order ODEs in Power Series
- First order non-explicit eq.
- Calculus of variations

Tractrix

French physician Claude Perrault (1613-1688), brother of Charles Perrault, who published *Cinderella* and *Little Red Riding Hood,* placed his watch in the middle of the table and pulled the end of the watch chain along the edge of the table. "What is the shape of the curve traced by the watch?" First studied by Christiaan Huygens, gave it the name tractrix (1692).

- Another inverse tangent problem: Find a curve whose tangent has a constant length, a.
- $\dfrac{dy}{dx} = -\dfrac{y}{\sqrt{a^2 - y^2}}$.
- Huygens, 1693, tractional motion and mechanical devices.
- Vincenzo Riccatti (1676-1775) (son of Jacobo) proved all 1st order differential equations could be constructed using tractional motion, 1752. Too late!

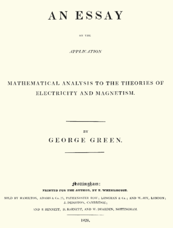


Summary

- Descartes and Fermat - geometric vs algebraic curves.
- Newton - fluxional equations.
- Leibniz (Huygens, Bernoulli's) - inverse tangent problems.
- Euler - sought general theory, no figures, introduction of methods.
- First textbooks
  - 1671 Newton, published 1736.
  - 1694 David Gregory, 1st systematic presentation of the method of fluxions Manuscript 'Isaaci Neutoni Methodus Fluxionum'.
  - 1696 Marquis de Hôpital, 1st differential calculus text. Online version and online translation.
  - 1700 Louis Carré, 1st French book on the integral calculus. online.
  - 1704 Charles Hayes, online, 1st English fluxions text.
  - 1707 Gabriele Manfredi, 1st differential equations text. online.
  - 1737 Thomas Simpson, *A New Treatise of Fluxions* online
  - 1742 Colin Maclaurin, *A Treatise of Fluxions,* online.
  - 1748 Maria Agnesi, *Foundations of Analysis for the Use of Italian Youth* with 200 pages on solving differential equations. online

Why Are Green's Functions Used By Oil Companies and What is a Green's Function?

September 23, 2022 - Dr. R. L. Herman



AN ESSAY

ON THE

APPLICATION

MATHEMATICAL ANALYSIS TO THE THEORIES OF ELECTRICITY AND MAGNETISM.

BY

GEORGE GREEN.

Nottingham:

PRINTED FOR THE AUTHOR, BY T. WHEELHOUSE.

SOLD BY HAMILTON, ADAMS & Co. 33, PATERNOSTER ROW; LONGMAN & Co.; AND W. JOY, LONDON; J. DEIGHTON, CAMBRIDGE;

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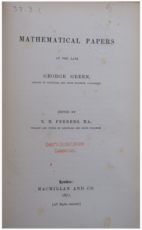
1828.



MATHEMATICAL PAPERS

OF THE LATE

GEORGE GREEN,

FELLOW OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE.

EDITED BY

N. M. FERRERS, M.A.,

FELLOW AND TUTOR OF GONVILLE AND CAIUS COLLEGE.

London:

MACMILLAN AND CO.

1871.

Frame 1

Introduction
George Green (1793-1841)
Green's Theorem
Green's Function
Cylindrical Oil Reservoir

Abstract

Green's functions, named after a relatively unknown grain miller, are used to solve boundary value problems. We will follow their path from an unknown 1928 Essay on the Application of Mathematical Analysis to the Theory of Electricity and Magnetism to a standard technique in mathematical physics in the mid-1900's. We sketch how Green's functions provide solutions to flow problems in reservoirs.

A Lifetime with Green's Functions

- Graduate School
  - Mathematical Physics
  - Electrodynamics
  - Integral Equations
  - Poisson-Boltzmann Equation between two spheres: $\nabla^2\psi = \sinh\psi$.
- Teaching
  - Differential Equations
  - Textbooks and Notes
- Recent - Petroleum Engineering

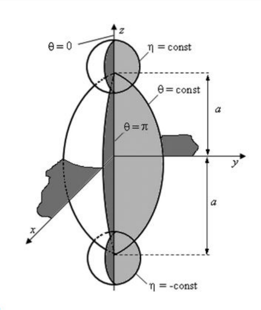


Image from here.

The two-dimensional Laplace operator, ∇^2, in bispherical coordinates is given by:

$$\nabla^2\psi = \frac{(\cosh\eta - \cos\theta)^3}{\beta^2 \sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\cosh\eta - \cos\theta}\frac{\partial\psi}{\partial\theta}\right) + \sin(\theta)\frac{\partial}{\partial\eta}\left(\frac{1}{\cosh\eta - \cos\theta}\frac{\partial\psi}{\partial\eta}\right)\right]$$

Who was Green?

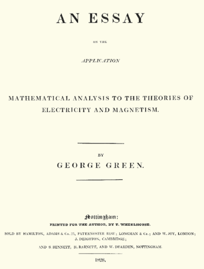
George Green's Life (1793-1841)

- Self-taught, about one year of formal schooling, between 8 and 9.
- Lived most of his life Sneinton, Nottinghamshire.
- His father, George, was a baker who built and owned a brick windmill to grind grain.
- In 1828 he published his famous essay.
  He published privately at his expense.
  Sold to 51 mostly friends.
- Wealthy landowner, mathematician, Edward Bromhead bought a copy and encouraged Green.
- Green did not contact Bromhead for two years.
- 1829, father died wealthy left to son and daughter.
- Younger George had time to pursue mathematics.
- In his final years at Cambridge, Green became rather ill, and in 1840 he returned to Sneinton, only to die a year later.

Discovery of 1828 Paper

- Green's work was not well known during his lifetime.
- In 1833 Robert Murphy (1806–1843) quoted the essay.
- In 1845, Green's essay was rediscovered & popularised by William Thomson (21 yr).
- In 1871 Ferrers assembled *The Mathem. Papers of the Late George Green*[a].
- Other contributions in these papers:

  On the motion of waves in a canal
    Green's pre-WKB approx.
  Green's Theorem and Identities.
  Green's Functions (named by Riemann), and potential functions.



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1828.

[a] His essay: arXiv:0807.0088.

Green's Theorem - Page 23, Green's *Essay*

Let U and V be two continuous functions of the rectangular co-ordinates x, y, z, whose differential co-efficients do not become infinite at any point within a solid body of any form whatever; then will

$$\int dxdydz\, U\delta V + \int d\sigma U \left(\frac{dV}{dw}\right) = \int dxdydz\, V\delta U + \int d\sigma V \left(\frac{dU}{dw}\right);$$

the triple integrals extending over the whole interior of the body, and those relative to $d\sigma$, over its surface, of which $d\sigma$ represents an element: dw being an infinitely small line perpendicular to the surface, and measured from this surface towards the interior of the body.

Modern Notation: Also *Green's Second Identity*

$$\int\int\int U\nabla^2 V\, dv + \int\int U\frac{\partial V}{\partial n}\, d\sigma = \int\int\int V\nabla^2 U\, dv + \int\int V\frac{\partial U}{\partial n}\, d\sigma.$$

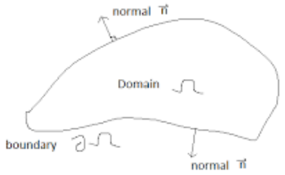
Equivalent to the Divergence Theorem. [Lagrange, 1754; Gauss, 1813 pub 1833,39; Ostrogradsky proved 1831.]

Green's Identities

In modern notation Green's first and second identities are given by the following with ψ and ϕ satisfying appropriate conditions of differentiability on a given domain..

$$\int_\Omega [\phi \nabla^2 \psi + \nabla\psi \cdot \nabla\phi]\, dV = \int_{\partial\Omega} \phi \nabla\psi \cdot \mathbf{n}\, dS. \tag{1}$$

$$\int_\Omega [\phi \nabla^2 \psi - \psi \nabla^2 \phi]\, dV = \int_{\partial\Omega} [\phi \nabla\psi - \psi\nabla\phi] \cdot \mathbf{n}\, dS. \tag{2}$$



Poisson's Equation - Green Introduced in Essay

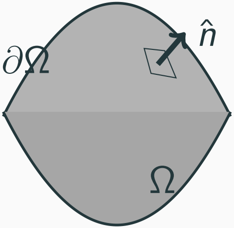
Let Poisson's equation,

$$\nabla^2 u(\mathbf{r}) = f(\mathbf{r}),$$

hold inside a region Ω bounded by the surface $\partial\Omega$.

This is the nonhomogeneous form of Laplace's equation.

$f(\mathbf{r})$, could represent a heat source in a steady-state problem, a charge distribution in an electrostatic problem, or an oil well.



Green's Function - Modern Approach

What is the response of the system to a point source?

The point source at $\mathbf{r}'$ is felt at $\mathbf{r}$. Call the response $G(\mathbf{r}, \mathbf{r}')$.

The response (Green's) function would satisfy

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'),$$

where $\delta(\mathbf{r} - \mathbf{r}')$ is the Dirac delta function satisfying

$$\delta(\mathbf{r}) = 0, \quad \mathbf{r} \neq \mathbf{0},$$

$$\int_\Omega \delta(\mathbf{r})\, dV = 1.$$

$$\int_\Omega \delta(\mathbf{r} - \mathbf{r}') f(\mathbf{r})\, dV = f(\mathbf{r}').$$

Green and others talked about *singularity.* It wasn't until mid-1900's that the Dirac delta function was understood in theory of distributions.

Green's Function and Poisson's Equation, $\nabla^2 u = 0$.

Green's second identity:

$$\int_{\partial\Omega} [\phi\nabla\psi - \psi\nabla\phi] \cdot \mathbf{n}\, dS = \int_{\Omega} [\phi\nabla^2\psi - \psi\nabla^2\phi]\, dV.$$

Letting $\phi = u(\mathbf{r})$ and $\psi = G(\mathbf{r}, \mathbf{r}')$, we have

$$\begin{aligned}
& \int_{\partial\Omega} [u(\mathbf{r})\nabla G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\nabla u(\mathbf{r})] \cdot \mathbf{n}\, dS \\
= & \int_{\Omega} \left[u(\mathbf{r})\nabla^2 G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\nabla^2 u(\mathbf{r})\right]\, dV \\
= & \int_{\Omega} [u(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')f(\mathbf{r})]\, dV \\
= & \; u(\mathbf{r}') - \int_{\Omega} G(\mathbf{r}, \mathbf{r}')f(\mathbf{r})\, dV. \qquad (3)
\end{aligned}$$

Solve for $u(\mathbf{r}')$.

Solution

We have

$$u(\mathbf{r}') = \int_\Omega G(\mathbf{r},\mathbf{r}') f(\mathbf{r})\, dV + \int_{\partial\Omega} [u(\mathbf{r})\nabla G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla u(\mathbf{r})] \cdot \mathbf{n}\, dS. \quad (4)$$

If both $u(\mathbf{r})$ and $G(\mathbf{r},\mathbf{r}')$ satisfied Dirichlet conditions, $u = 0$ on $\partial\Omega$, then the last integral vanishes and we are left with

$$u(\mathbf{r}') = \int_\Omega G(\mathbf{r},\mathbf{r}') f(\mathbf{r})\, dV.$$

In many applications, $G(\mathbf{r},\mathbf{r}') = G(\mathbf{r}',\mathbf{r})$. Then,

$$u(\mathbf{r}) = \int_\Omega G(\mathbf{r},\mathbf{r}') f(\mathbf{r}')\, dV'.$$

If we know the Green's function, we can solve nonhomogeneous differential equations and nonhomogenous boundary value and initial value problems.

Green's Functions for PDE's

The wave, heat, and Laplace's equation are typical homogeneous PDEs.

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right) u &= 0,\\ \left(\frac{\partial}{\partial t} - k\nabla^2\right) u &= 0,\\ \nabla^2 u &= 0. \end{aligned} \tag{5}$$

They can be written in the form

$$\mathcal{L}u(x) = 0,$$

where $\mathcal{L}$ is a differential operator, $x = \{\mathbf{r}, t\}$.

We solve the nonhomogeneous equations, $\mathcal{L}u(x) = f(x)$, by seeking out the Green's function, $\mathcal{L}G(x, x') = \delta(x - x')$.

Diffusion Equation

$$\frac{\partial u(\mathbf{r},t)}{\partial t} - k\nabla^2 u(\mathbf{r},t) = S(\mathbf{r},t).$$

$$\frac{\partial G(\mathbf{r},\mathbf{r}',t,t')}{\partial t} - k\nabla^2 G(\mathbf{r},\mathbf{r}',t,t') = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t').$$

How to find $G(\mathbf{r},\mathbf{r}',t,t')$:

Integral transforms: Laplace to remove t-dependence.

Symmetry to reduce to ODEs.

Often, one obtains Green's functions analytically as infinite series. Problems with convergence.

There are numerical techniques needed for inverse Laplace transforms.

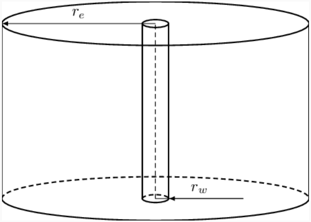
Cylindrical Oil Reservoirs - Diffusion Equation

Let $p = p(r, t)$ be the pressure in the reservior. A general form of the line source problem would be

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) - \frac{\partial p}{\partial t} = S(r,t), \quad (6)$$

$$p(r,0) = p_0(r),$$

$$\lim_{r\to 0} r\frac{\partial p}{\partial r} = b(t), \quad \lim_{r\to\infty} p(r,t) = 0. \quad (7)$$



Here $S(r, t)$ would be a source term and $b(t)$ a lower boundary value.

Cylindrical Oil Reservoirs - Boundary Conditions

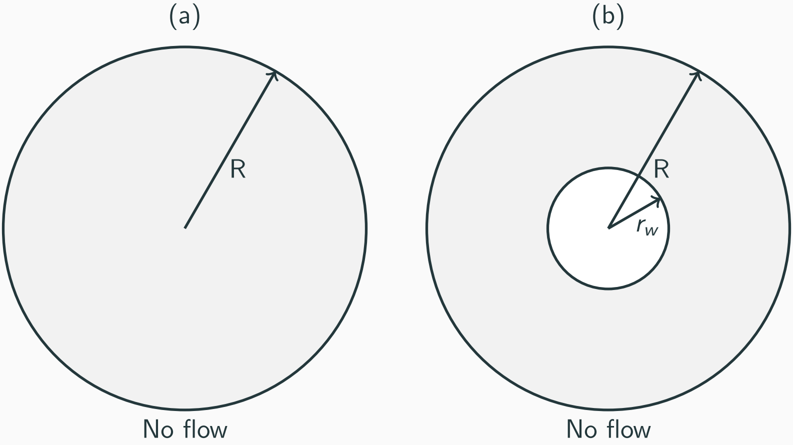


Figure 2: Examples with no flow boundaries. (a) Line source in a finite region of radius R. (b) Finite well source of radius r_w inside a finite region of radius R.

Laplace Transform

For the diffusion equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)-\frac{\partial p}{\partial t}=S(r,t), \tag{8}$$
$$p(r,0)=p_0(r),$$

one can apply the Laplace transform,

$$\hat{p}(r,s)=\int_0^\infty p(r,t)e^{-st}\,dt,$$

to the diffusion equation to obtain

$$r^2\frac{\partial^2\hat{p}}{\partial r^2}+r\frac{\partial\hat{p}}{\partial r}-r^2s\hat{p}=-r^2p_0(r)+r^2\hat{S}(r,s)\equiv F(r,s).$$

The general solution takes the form

$$\hat{p}(r,s)=c_1I_0(r\sqrt{s})+c_2K_0(r\sqrt{s})+P(r,s). \tag{9}$$

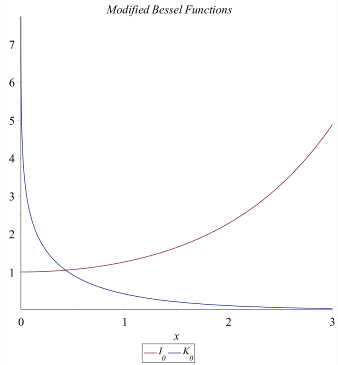
Modified Bessel Functions

$I_n(x)$: Modified Bessel function of the first kind.

$K_n(x)$: Modified Bessel function of the second kind.

These are related to the Bessel functions of the first and second kind,

$$I_n(x) = i^{-n} J_n(ix), \qquad K_n(x) = \frac{\pi}{2} \frac{I_{-n}(x) - I_n(x)}{\sin n\pi}.$$



Examples

At this point one considers different boundary value problems and if one can find exact solutions or numerical solutions.

To make sure numerical techniques work, one needs analytical solutions to toy problems.

- Solve for $\hat{p}(r, s)$.
- Find $p(r, t)$ using table look-up or the Bromwich integral.
  The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt, \quad s > 0. \tag{10}$$

  The inverse Laplace transform is obtained using the Bromwich integral, or Fourier-Mellin integral,

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds. \tag{11}$$

Next - Some rough computations!

Next we go through a few notes indicating the type of functions and mathematics needed.

- Modified Bessel functions.
- Line source infinite reservoir.
- Constant rate at finite radius.
- No-flow boundaries.
- Finite wells.
- Application of Green's functions.
- Inversion of Laplace transform.

During the presentation some research notes were presented. Here we add the important computation of the inverse Laplace transform.

Inverse Laplace Transform

Starting with the Bromwich integral,

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) e^{st}\, ds, \tag{12}$$

we choose c so that all poles are to the left of the contour such as seen in Figure 3. The contour is a closed semicircle enclosing all the poles. One then relies on a generalization of Jordan's Lemma.[1]

We follow up with a seemingly simple problem.

[1] One has a choice to close the contour to the left or right of the contour. Writing the exponential as $e^{st} = e^{(s_R + is_I)t} = e^{s_R t} e^{is_I t}$, we see that the second factor is an oscillating factor. The growth in the exponential can only come from the first factor. In order for the exponential to decay as the radius of the semicircle grows, we need $s_R t < 0$. Since $t > 0$, then $s < 0$ and we close the contour to the left. If $t < 0$, then the enclosed contour to the right would enclose no singularities and preserve the causality of $f(t)$.

Inverse Laplace Transform - Contour

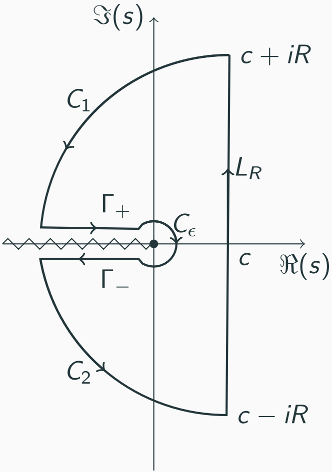


Figure 3: A contour used for applying the Bromwich integral to the Laplace transform $F(s) = K_0(r\sqrt{s})$ with a branch point at $s = 0$.

Inverse Laplace Transform - Example

Example

Let $F(s) = K_0(r\sqrt{s})$. Then, the Bromwich integral is given by

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K_0(r\sqrt{s}) e^{st}\, ds. \tag{13}$$

Since there is a branch point at $s = 0$, we choose $c > 0$.

We consider the complex integral around the contour

$$C = \lim_{R\to\infty} \lim_{\epsilon\to 0} (L_R + C_1 + \Gamma_+ + C_\epsilon + \Gamma_- + C_2)$$

in Figure 3,

$$I = \frac{1}{2\pi i} \oint_C K_0(r\sqrt{z}) e^{tz}\, dz = 0.$$

Inverse Laplace Transform - Example (cont'd)

The integrals over C_1 and C_2 vanish by Jordan's Lemma. So, we need to compute $f(t)$ as

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\left(-\frac{1}{2\pi i}\int_{\Gamma_+} K_0(r\sqrt{z})e^{tz}\,dz-\frac{1}{2\pi i}\int_{C_\epsilon} K_0(r\sqrt{z})e^{tz}\,dz-\frac{1}{2\pi i}\int_{\Gamma_-} K_0(r\sqrt{z})e^{tz}\,dz\right).$$

First we note that

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{C_\epsilon} K_0(r\sqrt{z})e^{tz}\,dz &= -\lim_{\epsilon\to 0}\int_{-\pi}^{\pi} K_0(r\sqrt{\epsilon e^{i\theta}})e^{t\epsilon e^{i\theta}}\,i\epsilon e^{i\theta}\,d\theta\\ &= \lim_{\epsilon\to 0}\frac{i}{2}\epsilon\ln\epsilon\int_{-\pi}^{\pi} d\theta = 0. \qquad (14)\end{aligned}$$

For the integrals over $\Gamma_\pm$ we let $z = xe^{\pm i\pi}$, giving

$$\begin{aligned}\int_{\Gamma_\pm} K_0(r\sqrt{z})e^{tz}\,dz &= \mp\int_0^\infty K_0(r\sqrt{xe^{\pm i\pi}})e^{txe^{\pm i\pi}}e^{\pm i\pi}\,dx\\ &= \pm\int_0^\infty K_0(\pm ir\sqrt{x})e^{-tx}\,dx.\end{aligned}$$

Inverse Laplace Transform - Example (cont'd)

Letting $y = r\sqrt{x}$, or $x = y^2/r^2$ and $dx = 2ydy/r^2$, we have

$$\pm \int_0^\infty K_0(\pm ir\sqrt{x})e^{-tx}\,dx = \pm\frac{2}{r^2}\int_0^\infty K_0(\pm iy)e^{-ty^2/r}y\,dy$$

Combining the integrals, we have

$$\begin{aligned} & \int_{\Gamma_+} K_0(r\sqrt{z})e^{tz}\,dz + \int_{\Gamma_-} K_0(r\sqrt{z})e^{tz}\,dz && (15)\\ = & \int_0^\infty [K_0(ir\sqrt{x}) - K_0(-ir\sqrt{x})]e^{-tx}\,dx \\ = & -i\pi\int_0^\infty J_0(r\sqrt{x})e^{-tx}\,dx, && (16)\end{aligned}$$

where we employed the identity

$$K_0(ir\sqrt{x}) - K_0(-ir\sqrt{x}) = -i\pi I_0(ir\sqrt{x}) = -i\pi J_0(r\sqrt{x}), \quad r > 0.$$

Inverse Laplace Transform - Example (cont'd)

We need the Laplace transform of $J_0(r\sqrt{x})$. Writing $w(x) = J_0(r\sqrt{x})$, $w(x)$ satisfies the initial value problem

$$4xw'' + 4w' + r^2 w = 0, \quad w(0) = 1, w'(0) = 0.$$

Taking the Laplace transform, $W(t) = \int_0^\infty J_0(r\sqrt{x})e^{-tx}\,dx$,

$$\begin{aligned} 0 &= -4\frac{d}{dt}(t^2 W(t) - t) + 4(tW(t) - 1) + r^2 W(t) \\ &= -4t^2 W'(t) - 4tW(t) + r^2 W(t) \end{aligned} \tag{17}$$

The solution of this differential equation is $W(t) = \frac{1}{t}e^{-r^2/4t}$. Therefore, we have that

$$f(t) = \frac{1}{2t}e^{-r^2/4t}.$$

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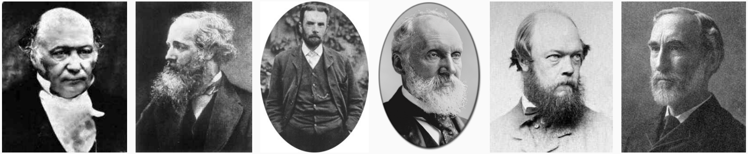
Thanks for listening. For more than above you can check out my course notes, people.uncw.edu/hermanr/books.htm

Heaviside's Operational Calculus, Telegraphy, and the Laplace Transform

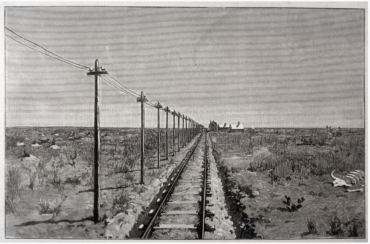
Graduate Seminar, Fall 2023

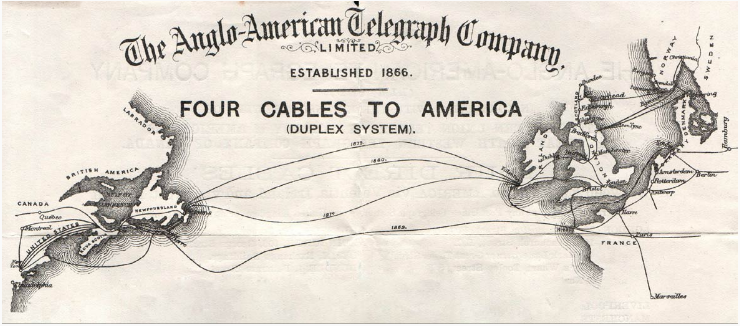
R. L. Herman



Outline

- Telegraphic Cable
- William Thomson
- The Maxwellians
- Oliver Heaviside
- Heaviside's Operational Methods
- Laplace Transforms
- Summary





Rise of the Telegraph

- 1833, Carl Friedrich Gauss (1777-1855) and Eduard Friedrich Weber (1806-1871) recognised that electric signals could be used to pass messages.
- Adapted by Sir William Fothergill Cooke (1806-1879) and Charles Wheatstone (1802-1875),
- The first public electric telegraph was established in 1837 along the Great Western railway from Paddington to West Drayton.
- Adoption of Greenwich Mean Time (GMT).

Figure 1: Wheatstone and Cooke.

Rise of the Telegraph

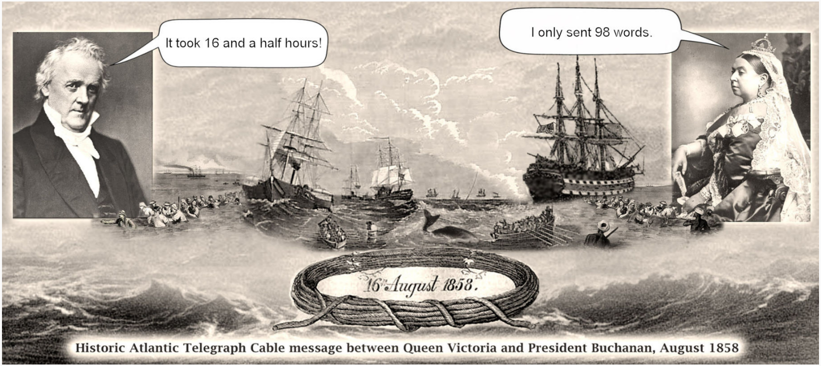
- Transmission wires along railway track supported poles.
- Samuel Morse, Morse code, 1838.
- Dreamed thousands of miles of cable.
- Morse insulated wire with tarred hemp, New York Harbour, 1842,
- He telegraphed through submerged wire.
- 1st commercial line, 1844 in U.S.
- 1850 Great Britain to France.

Transatlantic Telegraph

- America-Nova Scotia-Newfoundland - Largest , Frederick W. Gisbone. - 1853, not profitable.
- 1857 Cyrus West Field - New York, Newfoundland & London Telegraph Co.
- 1857/8 Several attempts.
- Whitehouse vs Thomson.
- HMS Agamemnon and USS Niagara
- Aug, 5 - Iceland, cable broke twice.
- 1858 Ships headed towards each other.
- Cable broke - 6 km, 100 km, 370 km.
- Jul 29, 1858, Got to ports Aug 4/5: Agamemnon to Valentia, Ireland and Niagara to Trinity Bay.

The First Transatlantic Message

- Aug 16, 1858 - Queen Victoria and President Buchanan exchanged messages.
- Two char/min - 1st message, 16 hrs.
- There has to be a better way! - Eventually, July 1866.



- Whitehouse cranked voltage from 600V to 2000V, frying the insulation. Dismissed Aug. 17.

William Thomson (1824-1907)

- Father, James Thomson, taught math in Belfast and Univ. of Glasgow.
- William attended Univ. of Glasgow, 1834 (at 10).
- Read Jean-Baptiste-Joseph Fourier.
- First two articles, at 16-17, defended Fourier.
- Cambridge, 1841-5, earned B.A. with high honours.
- In 1845, obtained George Green's essay and went to Paris next day.
- Chair of natural philosophy (physics) at the Univ. of Glasgow at 22.


WILLIAM THOMSON: THE YOUNG PROFESSOR

Thomson became interested in the telegraphy problem in 1854: Exchanged letters with George Gabriel Stokes (1819-1903) (Thomson 1856).

Thomson's 1855 Paper

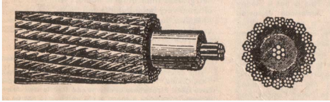
III. "On the Theory of the Electric Telegraph." By Professor WILLIAM THOMSON, F.R.S. Received May 3, 1855.

The following investigation was commenced in consequence of a letter received by the author from Prof. Stokes, dated Oct. 16, 1854. It is now communicated to the Royal Society, although only in an incomplete form, as it may serve to indicate some important practical applications of the theory, especially in estimating the dimensions of telegraph wires and cables required for long distances; and the author reserves a more complete development and illustration of the mathematical parts of the investigation for a paper on the conduction of Electricity and Heat through solids, which he intends to lay before the Royal Society on another occasion.

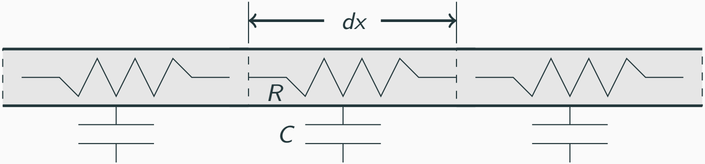
Extract from a letter to Prof. Stokes, *dated Largs, Oct.* 28, 1854.

" Let c be the electro-statical capacity per unit of length of the wire; that is, let c be such that clv is the quantity of electricity

William Thomson's Telegraph Theory - 1855

Treat the coaxial cable as a long, thin conductor, perfectly electrically insulated.



Think of the cable as a network of resistances and electrical capacity (capacitance) and use Kirchoff's laws on an infinitesimal section to derive an equation for the voltage, $v(x, t)$. [Ohm - 1827, Kirchoff - 1845.]

Thomson's Diffusion Equation

This resulted in a diffusion equation:

$$\frac{\partial v^2}{\partial x^2} = RC\frac{\partial v}{\partial t}. \tag{1}$$

It was Fourier's heat equation with solution (Thomson 1856),

$$v = \frac{Q\sqrt{R}}{\sqrt{\pi Ct}} e^{-RCx^2/4t}.$$

The maximum effect is at position x and time $t = \frac{1}{6}RCx^2$. This is **Thomson's law of squares**. Examples in Rayleigh's *Theory of Sound.*

Thomson solved several special cases in his correspondence with Stokes as recalled by Thomson (1856). Thomson's theory had many practical applications.

Further Develpoments

- Thomson had the first teaching laboratory,
- Engaged his students in testing materials and his ideas.
- Used the theory/experiment to understand underwater telegraphy.
- Explained the speed of the current in a telegraph cable,
- Dispersion caused signals of low frequency to diffuse less.

Stokes solved the more general case

$$\begin{aligned} v(x,0) &= 0, \quad 0 < x < \infty \\ v(0,t) &= f(t), \quad 0 < t < \infty, \end{aligned}$$

arriving at the solution (Nahin 2002)

$$v(x,t) = \frac{x}{2\sqrt{\pi}} \int_0^t (t-t')^{-\frac{3}{2}} e^{-x^2/4(t-t')} f(t')\, dt'.$$

William Thomson - a.k.a Lord Kelvin

This was in the backdrop of the Atlantic Cable Project (Nahin 2002).

- Developed the theory, designed experiments, and obtained patents.
- Was instrumental to the success of the trans-Atlantic cable, completed 1866, after disputes with Whitehouse. (Crossland 2008; Flood, McCartney, and Whitaker 2008; Bart and Bart 2008).
- For his work on the trans-Atlantic telegraph project:
    - Knighted by Queen Victoria, becoming Sir William Thomson, 1866.
    - Recognized for achievements in thermodynamics becoming Baron Kelvin, of Largs, 1892. (Crossland 2008)

Thomson's theory of the electric telegraph remained the main theory for decades. It worked fine for long underwater cables, but to transmit human conversation, the diffusion was far too much.

Maxwell's Theory of Electricity and Magnetism

During this time scientists were beginning to move from the mechanical world of Newton and Lagrange to the world of Faraday, Oersted, Ampere, and others.

- James Clerk Maxwell (1831-1879)
- Michael Faraday (1791-1867) encouraged Maxwell.
- "A Dynamical Theory of the Electromagnetic Field," EM waves, Maxwell (1865).
- "A Treatise on Electricity and Magnetism," 1873.
- Promoters of Maxwell's work: G. F. Fitzgerald (1851-1901), O. Heaviside (1850-1925), and O. Lodge (1851-1940). The Maxwellians (O'Hara and Pricha 1987; Hunt 1991).
- The race was on to produce electromagnetic waves, Hertz (1857-1894).
- Maxwell's theory reworked by Heaviside.

Challenge to Thomson's Theory

- The story of the attempts to connect continents with telegraph cables and Thomson's role is described by Hunt (2018, 2021, 2012).
- The subsequent contributions of Heaviside acan be found in (Nahin 2002).
- In 1876 Heaviside derived the telegrapher's equation independently and updated Thomson's diffusion theory by insisting that self-inductance was important.
- This was contrary to what people working on underwater telegraphy believed.
- It led to a few disputes.

Figure 2: Who was Oliver Heaviside?

Oliver Heaviside (1850-1925)

- Heaviside left school at sixteen. (Nahin 2002)
- He studied at home for two years.
- Worked as telegraph operator, Danish-Norwegian-English Telegraph Co., advice from uncle C. Wheatstone, 1868.
- He was transferred to Newcastle-on-Tyne, 1870, and later appointed Chief Operator.
- He left in 1874. Only job he would ever have.
- He spent the next couple of years working on electric theory.
- He studied and reformulated Maxwell's theory.

Note: Heaviside and Josiah Gibbs gave us Vector Analysis and opposed quaternions introduced by Hamilton and promoted by Tait. He gave us Maxwell's Equations.

Oliver Heaviside (1850-1925)

- Heaviside began publishing in 1872.
- He furthered Thomson's theory, 1876.
- Derived the telegraph equation.
- Self-induction is important in telegraphy.
- Others opposed him on this.
- He was asked to stop publishing for *The Electrician* in 1887. (Watson-Watt 1950; Edge 1983; Nahin 1991, 2002; Giorello and Sinigaglia 2005; Hunt 2012; Mahon 2017).
- Heaviside did have some supporters including Thomson and Maxwell.

Heaviside's Theory

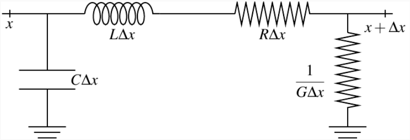


Figure 3: A section of transmission Line to find the potential drop from x to $x + \Delta x$ across an inductor and resistor with leakage.

Using Kirchoff's voltage and current laws, consider the voltage drops from x to $x + \Delta x$ across the resistor and the inductor:

$$\Delta v = -iR\,\Delta x - \frac{\partial i}{\partial t} L\,\Delta x.$$

The current can leak out. The overall change in current is given by

$$\Delta i = -vG\,\Delta x - \frac{\partial v}{\partial t} C\,\Delta x.$$

The Telegrapher's Equation

Dividing by Δx and letting Δx approach zero, we have the two equations

$$\begin{aligned} v_x + iR + Li_t &= 0 \\ Cv_t + Gv + i_x &= 0. \end{aligned} \tag{2}$$

Differentiating the first equation with respect to x and using i_x from the second equation, leads to an equation for the voltage,

$$\frac{1}{LC}\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial^2 t} + \left(\frac{R}{L} + \frac{G}{C}\right)\frac{\partial v}{\partial t} + \frac{GR}{LC}v. \tag{3}$$

A similar equation can be derived for the current.[1]

1. Gray (1923) wrote *The Equation of Telegraphy* comparing known solutions of ,

$$\frac{\partial^2 V}{\partial t^2} + 2\gamma\frac{\partial V}{\partial t} = a^2\frac{\partial^2 V}{\partial x^2},$$

and used the Riemann-Green method.

Heaviside's Operational Calculus

- Used to solve partial differential equations.
- Methods were criticized - not being rigorous and hard to understand.
- First people to publish justifications of Heaviside's methods: Bromwich (1917) and Wagner (1916). (Lützen 1979)
- Both used complex integrals.
- Bromwich - applications from the *Theory of Sound* (Rayleigh 1894) and equation similar to telegrapher's equation using a Green's function.
- Attempted to justify Heaviside's work and eventually the Laplace transform emerged. (Lützen 1979, 2012).

Operational methods for differential equations and the exploration of fractional differentiation had been studied for a number of years going back to the work of Euler and Leibniz (Deakin 1981; Petrova 1987).

Some of this is summarized in Moore's 1921 text (Moore 1971) and from Carslaw and Jaeger's 1941 book on operational methods (Bateman 1942; Carslaw and Jaeger 1941).

Example: Edmund T. Whittaker Obituary for Heaviside

Edmund T. Whittaker (1873-1956) describes how Heaviside would use operational calculus[2] to solve the differential equation (Moore 1971)

$$\frac{d^2y(t)}{dt^2} + k^2 y(t) = 0. \tag{4}$$

Let $D = \frac{d}{dt}$. We write symbolically,

$$(D^2 + k^2)y(t) = 0.$$

Now, manipulate algebraically: Multiply by D^{-2},

$$(1 + k^2 D^{-2})y(t) = D^{-2}(0).$$

What is $D^{-2}(0)$?

2. According to Whittaker, Heaviside was accustomed to using symbolic differential operators. Boole (1872) devoted two chapters in *A Treatise on Differential Equations* to symbolic methods.

Example (cont'd)

$D^{-2}(0)$ is the most general function whose second derivative vanishes.

It is $a+bt$, where a and b are arbitrary constants. So,

$$(1+k^2D^{-2})y(t) = D^{-2}(0) = a+bt.$$

Solve for $y(t)$, using a series expansion,

$$\begin{aligned} y(t) &= (1+k^2D^{-2})^{-1}(a+bt) \\ &= (1-k^2D^{-2}+k^4D^{-4}-\cdots)(a+bt). \end{aligned} \tag{5}$$

We apply D^{-n} to the functions 1 and t, where

$$D^{-1} = \int_0^t d\tau, \qquad D^{-2} = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2, \ldots.$$

So, we have $D^{-1}t = \frac{1}{2}t^2$, etc.

Example (cont'd)

Formally, using series expansions, we have

$$\begin{aligned} y(t) &= (1+k^2D^{-2})^{-1}(a+bt) \\ &= (1-k^2D^{-2}+k^4D^{-4}-\cdots)(a+bt) \\ &= a(1-k^2D^{-2}+k^4D^{-4}-\cdots)(1) \\ &\quad +b(1-k^2D^{-2}+k^4D^{-4}-\cdots)(t). \\ &= a\left(1-\frac{k^2t^2}{2!}+\frac{k^4t^4}{4!}-\cdots\right)+b\left(t-\frac{k^2t^3}{3!}+\frac{k^4t^5}{5!}-\cdots\right) \\ &= a\cos kt+\frac{b}{k}\sin kt. \qquad (6) \end{aligned}$$

Since a and b are arbitrary constants, we have found the general solution to the differential equation $y''+k^2y=0$.

This method works for many linear differential equations, even partial differential equations like the heat and telegrapher's equations.

Heat Equation Example

Heaviside solved Thomson's equation for cables,

$$\frac{\partial^2 V}{\partial x^2} = k \frac{\partial V}{\partial t}, \tag{7}$$

where $k = RC$.

Heaviside let $p = \frac{\partial}{\partial t}$, giving

$$\frac{\partial^2 V}{\partial x^2} = pkV. \tag{8}$$

Treating p as algebraic, solve the second order differential equation:

$$V = e^{-(pk)^{1/2}x} V_0,$$

assuming bounded solutions.

What does $e^{-\sqrt{pk}} = e^{-\sqrt{k \frac{\partial}{\partial t}}}$ mean?

Heat Equation Example (cont'd)

The solution can be used to find the surface gradient at $x = 0$.

First, we note that if $\dfrac{\partial^2}{\partial x^2} = pk$, then $\dfrac{\partial}{\partial x} = (pk)^{1/2}$. Then,

$$\begin{aligned}
\left(\frac{\partial V}{\partial x}\right)_{x=0} &= \left(-(pk)^{1/2} e^{-(pk)^{1/2}x} V_0\right)_{x=0} \\
&= -\sqrt{pk}\,V_0 = -k^{1/2}\left(\frac{\partial}{\partial t}\right)^{1/2} V_0. \qquad (9)
\end{aligned}$$

Heaviside often obtained expressions involving fractional derivatives.

In fact, many before Heaviside spent time trying to make sense out of nonstandard derivatives and integrals. Ross (1977) describes some of the early work on fractional derivatives.

Fractional differentation

- In 1695 Leibniz communicated about fractional derivatives to Johann Bernoulli and l'Hôpital.
- In 1729 Euler communicated to Goldbach the general form

$$\frac{d^n x^p}{dx^n} = \frac{\Gamma(p+1)}{\Gamma(p-n+1)} x^{p-n}, \tag{10}$$

  using the Gamma function, $\Gamma(n) = n!$ for integers n.
- One definition (Riemann-Liouville)

$$f^{(q)}(x) = \frac{1}{\Gamma(k-q)} \frac{d^k}{dx^k} \int_a^x (x-t)^{k-q-1} f(t)\, dt.$$

- From Euler's formula (10) we have for $n = \frac{1}{2}$

$$D^{1/2} \cdot 1 = \frac{\Gamma(1)}{\Gamma(\frac{1}{2})} t^{-1/2} = \frac{1}{\sqrt{\pi t}}.$$

Heat Equation Example 2

Heaviside applied operational methods to the heat equation for more complicated problems.

Consider the heat equation

$$\frac{\partial^2 V}{\partial x^2} = k \frac{\partial V}{\partial t}, \tag{11}$$

with $V = 0$, $t < 0$, and boundary condition

$$V_0 - V = h \frac{\partial V}{\partial x}, \quad x = 0.$$

Find the solution such that $V = V_1$ at $x = 0$.

Defining $D = \frac{d}{dx}$, the operational form of the heat equation is

$$D^2 V = kpV. \tag{12}$$

Heat Equation Example 2 (cont'd)

The boundary condition can be written at $x = 0$, using $DV = \sqrt{kp}V$, as

$$V_0 - V_1 = h\frac{\partial V}{\partial x} = h\sqrt{kp}V_1.$$

Solving for V_1 and defining $a = kh^2$, we have

$$V_1 = \frac{1}{1 + \sqrt{ap}}V_0.$$

So, how do we work with this solution? We symbolically expand $(1 + \sqrt{ap})^{-1}$ as a geometric series, ignoring convergence issues.

This is one place where Heaviside managed to upset mathematicians. Heaviside saw mathematics as an experimental science. Edge (1983) quotes him, "Mathematics is of two kinds, Rigorous and Physical. The former is Narrow: the latter Bold and Broad. To have to stop to formulate rigorous demonstrations would put a stop to most physico-mathematical enquiries. Am I to refuse to eat because I do not fully understand the mechanism of digestion?"

Heat Equation Example 2 (cont'd)

There are two ways to do the expansion. First, we have

$$\begin{aligned} V_1 &= \frac{1}{1+\sqrt{ap}} V_0 \\ &= \left[1-\sqrt{ap}+ap-(ap)^{3/2}+\cdots\right] V_0 \\ &= \left[1-\sqrt{ap}-(ap)^{3/2}-(ap)^{5/2}-\cdots\right] V_0 \\ &= \left[1-\sum_{k=0}^{\infty}(ap)^{k+\frac{1}{2}}\right] V_0. \end{aligned} \tag{13}$$

Note that the positive terms in the expansion simply using

$$p^n V_0 = \begin{cases} V_0, & n=0, \\ 0, & n=1,2,\ldots \end{cases}$$

since derivatives of a constant vanish.

Heat Equation Example 2 (cont'd)

The second way to perform the expansion is in powers of $1/\sqrt{ap}$:

$$\begin{aligned}
V_1 &= \frac{1}{1+\sqrt{ap}} V_0 \\
&= \frac{1}{\sqrt{ap}} \left[\frac{1}{1+\frac{1}{\sqrt{ap}}} \right] V_0 \\
&= \frac{1}{\sqrt{ap}} \left[1 - (ap)^{-1/2} + (ap)^{-1} - (ap)^{-3/2} + \cdots \right] V_0 \\
&= \left[(ap)^{-1/2} - (ap)^{-1} + (ap)^{-3/2} - \cdots \right] V_0 \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} (ap)^{-n/2} V_0. \qquad (14)
\end{aligned}$$

For both series we need to perform fractional differentiation.

Heat Equation Example 2 (cont'd)

Using Euler's derivative formula

$$\frac{d^n x^p}{dx^n} = \frac{\Gamma(p+1)}{\Gamma(p-n+1)} x^{p-n}, \tag{15}$$

we have for the first series (13),

$$p^{k+\frac{1}{2}} \cdot 1 = \frac{t^{-\frac{1}{2}-k}}{\Gamma\left(\frac{1}{2}-k\right)}, \quad k = 0, 1, 2, \ldots.$$

For the second series (14), we need

$$p^{-n/2} \cdot 1 = \frac{1}{\Gamma(\frac{n}{2}+1)} t^{n/2}, \quad n = 1, 2, \ldots.$$

These results can be written out in an explicit form using properties of Gamma functions.

Heat Equation Example 2 (cont'd)

We can show

$$\frac{1}{\Gamma\left(\frac{1}{2}-k\right)} = \frac{(-1)^k}{\pi}\Gamma\left(k+\frac{1}{2}\right) = \frac{(-1)^k}{\sqrt{\pi}}\frac{(2k)!}{2^{2k}k!}. \tag{16}$$

So, the solution in the first case is

$$\begin{aligned} V_1 &= \left[1-\sum_{k=0}^{\infty}(ap)^{k+\frac{1}{2}}\right]V_0 \qquad (17)\\ &= V_0\left[1-\left(\frac{a}{\pi t}\right)^{1/2}\sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{2^{2k}k!}\left(\frac{a}{t}\right)^k\right]\\ &= V_0\left[1-\left(\frac{a}{\pi t}\right)^{1/2}\left(1-\frac{a}{2t}+1\cdot 3\left(\frac{a}{2t}\right)^2-\cdots\right)\right]. \qquad (18)\end{aligned}$$

This is an asymptotic series for large t.

Heat Equation Example 2 (cont'd)

For the second series (14), we can show for small t

$$\begin{aligned} V_1 &= \sum_{n=1}^{\infty}(-1)^{n+1}(ap)^{-n/2}V_0 \\ &= 2V_0\sqrt{\frac{t}{\pi a}}\sum_{k=0}^{\infty}\frac{2^k}{(2k+1)!!}\left(\frac{t}{a}\right)^k - V_0\sum_{k=1}^{\infty}\frac{(t/a)^k}{k!} \\ &= 2V_0\sqrt{\frac{t}{\pi a}}\left[1+\frac{2}{3}\,\frac{t}{a}+\frac{4\,t^2}{15\,a^2}+\frac{8\,t^3}{105\,a^3}+\frac{16\,t^4}{945\,a^4}+\cdots\right]+V_0\left[1-e^{t/a}\right]. \end{aligned} \tag{19}$$

Summing the infinite series, we have the solution

$$\begin{aligned} V_1 &= V_0\left[e^{\frac{t}{a}}\left(1-\operatorname{erfc}\left(\sqrt{\frac{t}{a}}\right)\right)+1-e^{t/a}\right] \\ &= V_0\left[1-e^{\frac{t}{a}}\operatorname{erfc}\left(\sqrt{\frac{t}{a}}\right)\right]. \end{aligned} \tag{20}$$

Watson's Lemma

The large t result can be understood using (Bender and Orszag 1999).

Watson's Lemma - Asymptotic Expansions Let $f(t)$ be continuous on the interval $0 \le t \le b$ and have the asymptotic expansion[3]

$$f(t) \sim t^{\alpha}(a_0 + a_1 t^{\beta} + a_2 t^{2\beta} + \cdots) = t^{\alpha} \sum_{n=0}^{\infty} a_n t^{n\beta}$$

as $t \to 0^+$ and for $\alpha > -1$, $\beta > 0$. Then,

$$\int_0^b f(t) e^{-xt}\, dt \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha+\beta n+1}} \quad \text{as } x \to \infty. \tag{21}$$

3. The power series $\sum_{n=0}^{\infty} a_n (t - t_0)^n$ is asymptotic to $f(t)$ if

$$\left| f(t) - \sum_{n=0}^{N} a_n (t - t_0)^n \right| \ll |t - t_0|^N$$

as $t \to t_0$ for every N.

Application of Watson's Lemma

We apply Watson's Lemma (21) to the complementary error function,

$$\operatorname{erfc}(\lambda) = \frac{2}{\sqrt{\pi}} \int_\lambda^\infty e^{-s^2}\, ds,$$

after the variable substitution, $\tau = 2(s-\lambda)$. Then,

$$\begin{aligned}\operatorname{erfc}(\lambda) &= \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-(\tau/2+\lambda)^2}\, d\tau \\ &= \frac{e^{-\lambda^2}}{\sqrt{\pi}} \int_0^\infty e^{-\tau^2/4} e^{-\lambda\tau}\, d\tau.\end{aligned}$$

Then, we identify

$$f(\tau) = e^{-\tau^2/4} = \sum_{n=0}^\infty \frac{(-1)^n \tau^{2n}}{n! 4^n}$$

in Watson's Lemma.

Application of Watson's Lemma (cont'd)

Furthermore, we have $\alpha = 0$, $\beta = 2$, and $a_n = \frac{(-1)^n}{n!4^n}$. Therefore,

$$\begin{aligned}
\operatorname{erfc}(\lambda) &\sim \frac{e^{-\lambda^2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha + \beta n + 1)}{\lambda^{\alpha+\beta n+1}} \quad \text{as } \lambda \to \infty \\
&= \frac{e^{-\lambda^2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!4^n} \frac{\Gamma(2n+1)}{\lambda^{2n+1}} \\
&= \frac{e^{-\lambda^2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \lambda^{2n+1}} \frac{(2n)!}{2^n n!} \\
&= \frac{e^{-\lambda^2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \lambda^{2n+1}} \frac{(2n+1)!!}{2n+1} \\
&= \frac{e^{-\lambda^2}}{\sqrt{\pi}} \left(\frac{1}{\lambda} - \frac{1}{2\lambda^3} + \frac{3}{4\lambda^5} - \frac{15}{8\lambda^7} + \frac{105}{16\lambda^9} \dots \right). \qquad (22)
\end{aligned}$$

Heat Equation Example 2 (cont'd)

Letting $\lambda = \sqrt{\frac{t}{a}}$, we have for the Heaviside solution (20)

$$\begin{aligned} V_1 &= V_0 \left[1 - e^{\frac{t}{a}} \operatorname{erfc}\left(\sqrt{\frac{t}{a}} \right) \right]. \\ &\sim V_0 \left[1 - \frac{1}{\sqrt{\pi}} \left(\left(\frac{a}{t}\right)^{1/2} - \frac{1}{2}\left(\frac{a}{t}\right)^{3/2} + \frac{3}{4}\left(\frac{a}{t}\right)^{5/2} - \dots \right) \right] \\ &= V_0 \left[1 - \sqrt{\frac{a}{\pi t}} \left(1 - \frac{1}{2}\left(\frac{a}{t}\right) + \frac{3}{4}\left(\frac{a}{t}\right)^2 - \dots \right) \right]. \end{aligned} \tag{23}$$

This is the large t expansion found earlier in Equation (18).

Therefore, the solution of the second Heaviside example Equation (20),

$$V_1 = V_0 \left[1 - e^{\frac{t}{a}} \operatorname{erfc}\left(\sqrt{\frac{t}{a}} \right) \right],$$

agrees with the solution obtained using operational calculus.

Age of the Earth

Thomson was interested in problems about the age of the Earth and Sun.

When he was sixteen he wrote that measuring the rate of heat loss from the surface of the Earth could put a bound on the age of the Earth (England, Molnar, and Richter 2007). This interest might have been sparked by reading Fourier's works.

Some of the first quantitative studies of the heat equation were by Fourier (Fourier 1808, 1820, 1822). Fourier had written on the temperature of the Earth and the diffusion of heat in a spherical solid (Godard 2017).

He later wrote a general paper about terrestrial temperatures (Fourier 1824b), which was reprinted (Fourier 1827) and translated in 1837 (Fourier 1824a). This has led to some misconceptions about his role in the origins of the greenhouse effect (Fleming 1999).

Age of the Earth (cont'd)

Naturally Thomson (1862) would use Fourier's work and in 1862 he predicted the age of the Earth based on the heat equation.

In the mid-1800's estimates of the age of the Earth went from a few thousand years to hundreds of millions based on geological estimates. Also, Darwin's theory of evolution came out in 1859.

Assuming an initial high temperature and constant diffusivity, Thomson asked how long it would take to reach the current temperature gradient at the Earth's surface of 1°F/50 ft. He came up with 98 million years (England, Molnar, and Richter 2007; Nahin 1985; Harrison 1987).

This was not long enough according to the geologists. A debate between physicists and geologists ensued based on Thomson's estimates (Jackson 2008).

Age of the Earth (cont'd)

Thomson's theory was accepted by the physics community for decades until in 1895 John Perry (1850-1920), a former assistant of Thomson, challenged Lord Kelvin (Perry 1895; England, Molnar, and Richter 2007; Shipley 2001).

Perry challenged Kelvin's assumptions: the thermal conductivity may not be constant. He found an increase in the age estimate.

This led to a debate amongst supporters of Thomson vs those of Perry.

Peter Guthrie Tait (1831 - 1901) sided with Kelvin and Heaviside took up the problem using his operational mathematics, deriving both Kelvin's and Perry's estimates.

Age of the Earth (cont'd)

Heaviside even opened the second volume of his *Electromagnetic Theory* (Heaviside 1922) with a chapter on the Age of the Earth.

It is interesting that Heaviside used a similar analysis of the diffusion equation to arrive at the age of the Earth using Thomson's data. Then, he took Perry's idea of a nonconstant diffusivity leading to an equation of the form $V_1 = \frac{1}{1+\sqrt{ap}} V_0$ as described in more detail in (Nahin 1985).

This allowed Perry (1895) to obtain a value for the age of the Earth of more than three times Thomson's estimate of 100 million years (Nahin 1985; Shipley 2001).

The debate continued for many years later (Jeffreys 1916, 1927).

The Evolution of Operational Calculus

Heaviside was not the first to use symbolic methods (Cooper 1952). However, he did propel its use, especially amongst those who choose to leave the rigor to others.

There were several efforts to either explain or bring rigor and prove that there was more to Heaviside's methods that might be more palatable to the mathematicians of the day.

Several definitions of Laplace transforms emerged along with contour integral methods such as the Bromwich integral. Van der Pol and Bremmer (1950) attribute the complex integral to Riemann in 1859. They also uses a two-sided Laplace transform throughout the book.

Early papers on the subject were written by Bromwich (1917), Carson Carson (1922), Van der Pol (1929), and Bateman (Bateman 1904). For example Bateman (1904) refers to Pincherle's book (Pincherle and Amaldi 1901) and his inversion formula.

Origins of Laplace Transforms

There are many papers attempting to trace the origins of the Laplace transform.

Deakin (1981, 1982) wrote two in-depth papers tracing the use of integrals to solve differential equations.

These include the appearance of integrals similar to Laplace and Mellin types in the works of Euler, Lagrange, and Laplace.

The origins of solving differential equations using integrals dates back to to Euler (1707-1783). Euler (1768) considered solutions in the form $y(u) = \int [K(u) + Q(x)]^m P(x)\, dx$ and Euler (1744) used the form $\int e^{ax} X(x)\, dx$.

The Laplace transform is named after Pierre-Simon, Marquis de Laplace (1749-1827) based on his work on probability theory (Laplace 1782).

Laplace Transforms

Eventually, the definition of Laplace transforms became standardized and Tables of Laplace Transforms became common such as the Bateman Project (Erdélyi et al. 1954).

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt, \quad s > 0, \tag{24}$$

where $\lim_{t\to\infty} f(t)e^{-st} = 0$ to guarantee convergence of the integral.

The inverse Laplace transform is obtained using the Bromwich integral (Herman 2016), or Fourier-Mellin integral,

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds. \tag{25}$$

Laplace Transforms and The Heat Equation

We solve Heaviside's heat equation examples using the Laplace transform.

We seek a solution of the heat equation on a semi-infinite interval with either a fixed or a mixed boundary condition at $x = 0$:

$$\frac{\partial^2 u}{\partial x^2} = k\frac{\partial u}{\partial t}, \quad x > 0, t > 0, \quad u(x, 0) = 0, \tag{26}$$

$u(x, t) \to 0$ as $x \to \infty$ and satisfies one of the boundary conditions

(a) $u = u_0$ at $x = 0$.

(b) $u_0 - u = h\dfrac{\partial u}{\partial x}$ at $x = 0$.

We want to determine either

(a) What is the temperature gradient at the origin, $u_x(0, t)$?

(b) What is the temperature at the origin, $u(0, t) = u_1(t)$?

Laplace Transforms: PDE to ODE

Defining the Laplace transform,

$$U(x,s) = \int_0^\infty u(x,t)e^{-st}\,dt,$$

and transforming the heat equation, we have

$$U_{xx}(x,s) - ksU(x,s) = 0.$$

Bound solutions of this differential equation on $x \in [0,\infty)$ are given in the form

$$U(x,s) = A(s)e^{-\sqrt{ks}x}. \tag{27}$$

Laplace Transforms: Boundary Condition (a)

For boundary condition $u = u_0$ at $x = 0$ find $A(s)$.

We have $U(0,s) = \dfrac{u_0}{s} = A(s)$. Therefore,

$$U(x,s) = \frac{u_0}{s} e^{-\sqrt{ks}x}.$$

Since we want the gradient of the temperature at the origin,

$$U_x(0,s) = -\frac{u_0}{s}\sqrt{ks} = -\frac{u_0\sqrt{k}}{\sqrt{s}}.$$

The inverse Laplace transform of $s^{-1/2}$ from a table or inverse transform, we have

$$u_x(0,t) = -\frac{u_o\sqrt{k}}{\sqrt{\pi t}}.$$

Laplace Transforms: Boundary Condition (b)

For boundary condition $u_0 - u = h\frac{\partial u}{\partial x}$ at $x = 0$, its Laplace transform is

$$\frac{u_0}{s} - U(x,s) = h\frac{\partial U(x,s)}{\partial x}.$$

Inserting the solution $U(x,s) = A(s)e^{-\sqrt{ks}x}$ from Equation (27), we have

$$\frac{u_0}{s} - A(s)e^{-\sqrt{ks}x} = -h\sqrt{ks}A(s)e^{-\sqrt{ks}x}.$$

We now set $x = 0$ and solve for $A(s)$, to find

$$A(s) = \frac{u_0}{s(1 - h\sqrt{ks})}.$$

Laplace Transforms: Boundary Condition (b)

The Laplace transform of the solution to the boundary value problem is

$$U(x,s) = \frac{u_0}{s(1-h\sqrt{ks})} e^{-\sqrt{ks}x}$$

and the Laplace transform of the solution at $x = 0$ is

$$U(0,s) = \frac{u_0}{s(1-h\sqrt{ks})}. \tag{28}$$

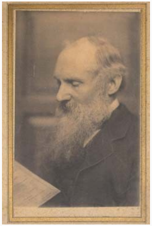
We can use a computer algebra system (CAS) like Maple or Mathematica. Doing so yields

$$u(0,t) = u_0 \left(1 - e^{\frac{t}{h^2 k}} \operatorname{erfc}\left(-\frac{\sqrt{t}}{h\sqrt{k}}\right)\right). \tag{29}$$

This solution agrees with Heaviside's solution (20) for $a = kh^2$.

Summary - Lessons from the Past

- Underwater Telegraphy
- Fourier's Heat Equation
- William Thomson/Lord Kelvin
- Oliver Heaviside
- Operational Calculus
- Laplace Transforms
- Part of upcoming book on *Applications of the Laplace Transform*



Thanks for Listening

Thank you for your attention.

References are provided on the remaining slides.

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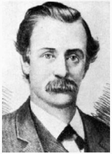
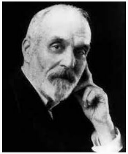
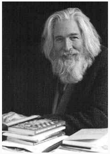
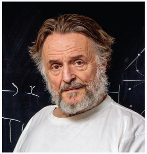
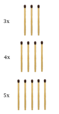
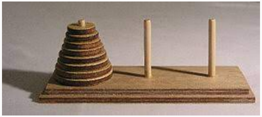
     

The Joy of Mathematical Puzzles and Games

Part I. The Logic Puzzles of Raymond M. Smullyan

Fall 2024 - R. L. Herman

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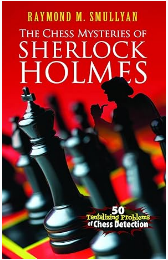
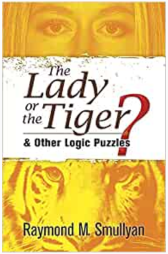


RAYMOND M. SMULLYAN
THE CHESS MYSTERIES OF
SHERLOCK HOLMES
50 Tantalizing Problems of Chess Detection



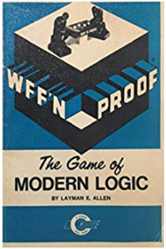
The Lady or the Tiger?
& Other Logic Puzzles
Raymond M. Smullyan

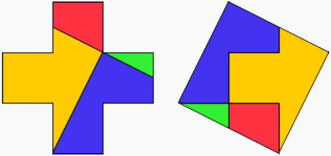
Puzzles and Games

- Logic puzzles,
- Number games,
- Geometrical puzzles,
- Games of chance
- Network problems, and
- Combinatorial problems.

How is a raven

like a writing desk?



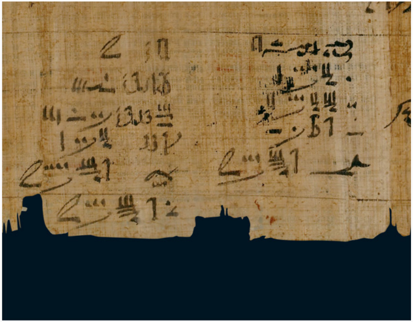


1		31		33		63	
	51		3		19		
		49		15			
			45		61	36	13
		25				21	
28		8	41	24			
43	6	55			10		
				58		38	

```
   S E N D
+  M O R E
-----------
 M O N E Y
```

- Rhind Papyrus - 1650 BCE
  Long scroll $18' \times 13''$.
  Purchased 1858.
- Problem 79
- Fibonacci's *Liber Abaci*

Seven houses contain seven cats.
Each cat kills seven mice.
Each mouse had eaten seven ears of grain.
Each ear of grain would have produced seven hekats of wheat.
What is the total of all of these?



houses	7
cats	49
mice	343
spelt	2401
hekat	16807
Total	19607

Historical Puzzles

- 1256, Ibn Kallikan, total grains of wheat placed on a chess board: $18,446,744,073,709,551,615$.
- Archimedes 287-212 BCE

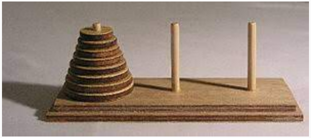
  Cattle Problem

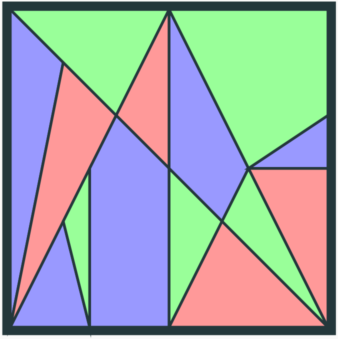
  Ostomachion

  (*loculus Archimedius*)
- Leonardo of Pisa 1100 (Fibonacci Sequence - Rabbits)
- Tower of Hanoi, Édouard Lucas, 1883.
- Magic squares

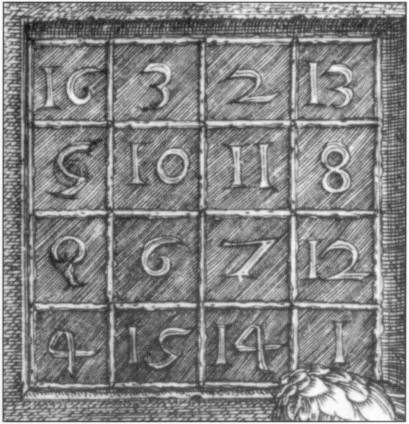
  2500 BCE, Chinese, *Lo-shu.*

  Dürer's *Melancolia*, 1514 - next.
- See more here.





Dürer's *Melancolia*, 1514



Charles L. Dodgson (1832-1898)

- Mathematical logician, Author.
- Christ Church College Oxford.

  First Class honours in math.
  Mathematics lecturer.
- Works - Euclid, Determinants, Logic, Elections.
- Photography
- Alice Liddell, daughter of Dean Henry George Liddell.
- Charles Lutwidge in Latin: Carolus Lodovicus, Anglicise and reverse order - pen name Lewis Carroll.
- Published puzzles and stories.

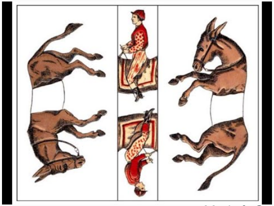
Sam Loyd (1841-1911) - “America’s Greatest Puzzlist”

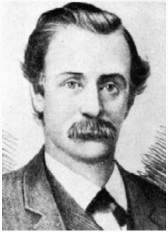
- 1st chess problem published: NY Saturday Courier 4/14/1855.
- Problem Ed., *Chess Monthly*, 16.
- *Chess Strategy*, 1878.
- Worked with Dudeney, scandal.
- Over 10,000 puzzles, 15 puzzle.
- Popularized tangrams.
- Sam Jr. carried on.

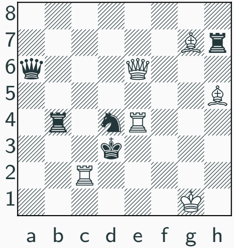
1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

$\Longrightarrow$ $1000

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	



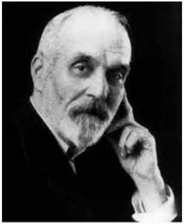


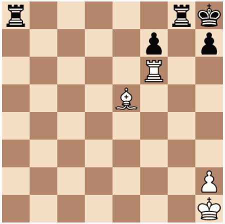


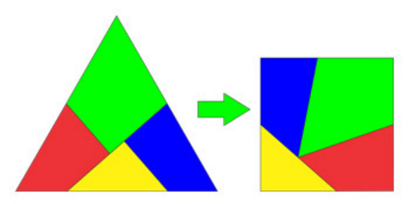
White to play and mate in four.

Henry Dudeney (1857-1930)

- Clerk in the Civil Service
- No college, Family of educators.
- Started with chess, puzzles.
- Pseudonym ‘Sphinx’.
- 1893, corresponded with Loyd.
- Contributed to the Strand Magazine for over 30 years,
- British Chess Problem Society, Founding member, 1918.
- Puzzle collections.
- Haberdasher’s problem.



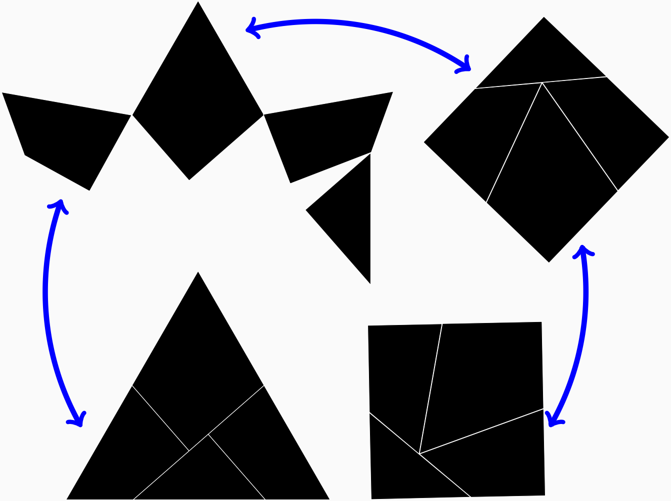




Future Talk on Dissection Problems

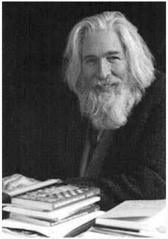
Haberdasher's Problem - Dudeney, 1902

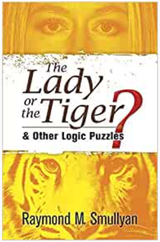
With three cuts, dissect an equilateral triangle into a square.

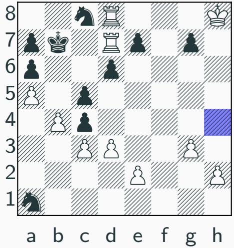


Raymond Smullyan (1919-2017)

- Mathematician, magician, concert pianist, logician and philosopher.
- From Far Rockaway, NYC.
- Composed 1st chess puzzle at 16.
- Retrograde analysis in chess.
- Unusal education.
  - Piano lessons in CA.
  - Studied Logic, Chicago.
  - Taught -Dartmouth, 1954-1956, B.S. from Univ. of Chicago, 1955.
  - Ph.D. Princeton, 1959, advisor: Alonzo Church.







Find the missing piece in the blue space.

Smullyan on Johnny Carson

Figure 1: `https://www.youtube.com/watch?v=E27v83WWiGo` (12:21)

Liar's Paradox: "This statement is false."

The key issue is **self-reference**, i.e., it refers to itself leading to a loop of contradictory reasoning.

In formal systems of mathematics or logic, self-reference can introduce similar challenges, like Russell's Paradox.

- Discovered by Bertrand Russell in 1901.
- Exposed a fundamental problem in set theory.
  The paradox: Is the "set of all sets that do not contain themselves" itself a member of this set?
  - If it is, it leads to a contradiction (it must contain itself).
  - If it isn't, it leads to a contradiction (it must not contain itself).
- Led to changes in formal systems and axiomatic set theory
- Hilbert's Program sought a complete and consistent foundation for all of mathematics.

Gödel's Incompleteness Theorems

- Gödel responded Hilbert's efforts, in 1931.
- Results revealed limitations of formal systems (in arithmetic).
- In any formal system, there are statements that:
  - Cannot be proven true or false within the system (First Theorem).
  - If the system is consistent (free of contradictions), it cannot prove its own consistency (Second Theorem).
- Gödel did this by constructing a self-referential statement within the formal system — similar to the Liar Paradox.

Smullyan used accessible logical puzzles to introduce the notions of self-reference, paradox, and formal systems — all central ideas in Gödel's work.

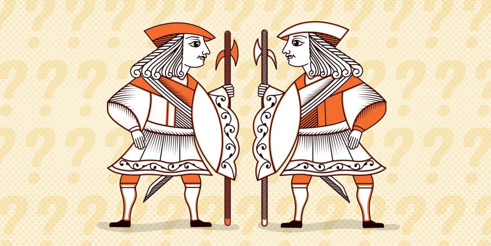
Let the Games Begin

- Ready to try some puzzles?
- From Smullyan's *The Lady or the Tiger?*

    Knights and Knaves
    The Lady or the Tiger
    The Mystery of the Monte Carlo Lock
- The Hardest Logic Puzzle Ever

Knights and Knaves

A special island is inhabited only by knights and knaves. Knights always tell the truth and knaves always lie.



You meet two inhabitants: A and B. A tells you that B is a knave. B says, "Neither A nor I are knaves." Who is a knight and who is a knave?

For more problems: Ch 3 *What Is the Name of This Book?*

The Lady or the Tiger - Frank Stockton

online

A King heard a story where a prisoner had to choose between two doors: behind one there was a princess, behind the other a tiger. If the prisoner chose the princess, he could marry her; if he chose the tiger, he would probably be eaten.

The Challenge

The King liked the idea but didn't want to leave things to chance. So, he decided to post signs on the doors giving clues as to what was behind the doors. Also, it would be possible that there are princesses behind both doors or tigers behind both doors. A clever prisoner who can reason logically would be able to save his own life.

Trial 1 - One sign is true and the other is false. Pick a door!

Trial 1 - One sign is true and the other is false. Pick a door!

Did you really wait?

Trial 1 - Solution?

Trial 1 - Informal reasoning.

Room 1 In this room there is a lady and in the other is a lion.

Room 2 In one of these rooms is a lady and in one there is a tiger.

One of the doors is true. If Door 1 is true, then Door 2 is also true.

Therefore, Door 1 is false and Door 2 is true.

Conclusion: A tiger is behind Door 1 and a Lady behind Door 2. Truth Table

---

Trial 2 The signs are changed. They are either both true or both false. Which room would you pick?

The Hardest Logic Puzzle Ever - George Boolos, 1996

Three gods A, B, and C are called, in no particular order, True, False, and Random. True always speaks truly, False always speaks falsely, but whether Random speaks truly or falsely is a completely random matter. Your task is to determine the identities of A, B, and C by asking three yes–no questions; each question must be put to exactly one god. The gods understand English, but will answer all questions in their own language, in which the words for yes and no are da and ja, in some order. You do not know which word means which.

Figure 2: Some of Smullyan's many books.

The Monte Carlo Lock Puzzle

From Smullyan's *The Lady of the Tiger & Other Logic Puzzles.*

- Inspector Craig is called to a case.
- The safe combination (on one card) was lost.
- Martin Farkus, a mathematician, left clues about code.
- The safe has to be opened by June 1, or it is blown.
- The code has capital letters of any length and may be repeated.
- Entering a code will

    Open the lock.
    Jam the lock, or
    It does nothing (neutral).

Goal Use Farkus' clues to open the lock.

Farkas' Clues

Denoting xy as the concatenation of the codes x and y, then for any letter combinations x and y, there are "special relations" satisfying the properties below.

Property Q: For any combination x, the combination QxQ is specially related to x.

Property L: If x is specially related to y, then Lx is specially related to Qy.

Property V (the reversal property: If x is specially related to y, then Vx is specially related to the reverse of y.

Property R (the repetition property: If x is specially related to y, then Rx is specially related to yy.

Property Sp: If x is specially related to y, then if x jams the lock, y is neutral, and if x is neutral, then y jams the lock.

Symbolic Summary of Clues

We write $y \to x$ to mean "x is specially related to y." Then,

- The code contains the letters L, Q, R, V.
- Denote xy as the concatenation of the codes x and y.
- Use $\overleftarrow{y}$ for the reverse order of the code y.

For codes x and y, the following are true:

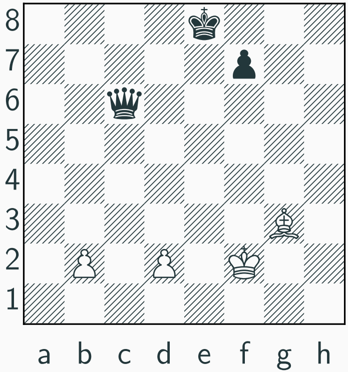
- Q: $x \to QxQ$.
- L: If $y \to x$, then $Qy \to Lx$,
- V: If $y \to x$, then $\overleftarrow{y} \to Vx$,
- R: If $y \to x$, then $yy \to Rx$.

Craig surmised that a code x opens the safe if and only if $x \to x$.*
For example: $RVLVQRVLVQ \to RVLVQRVLVQ$,

*From Property Sp, if $x \to x$, then x both jams the lock and is neutral. This contradiction implies that x must open the lock.

The Chess Mysteries of Sherlock Holmes

Raymond Smullyan opens his 1979 book, *The Chess Mysteries of Sherlock Holmes*, with the following chessboard position. He then asks, "Suppose I told you that in the following position no pawn has ever reached the eight square. Would you believe me?"



Solutions

Go no further!

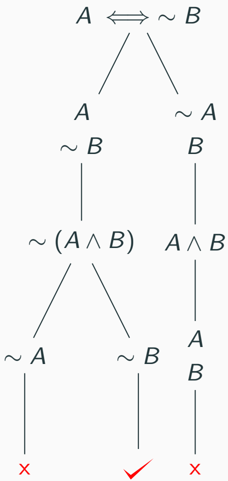
Knights and Knaves Analysis: A and $\sim B$

back

Start with the statements (A: "B is a knave." B: "Neither A nor I are knaves.") $A \iff \sim B$ and $B \iff A \wedge B$.

A	B	$\sim B$	$A \iff \sim B$	$A \wedge B$	$B \iff A \wedge B$
T	T	F	F	T	T
T	F	T	T	F	T
F	T	F	T	F	F
F	F	T	F	F	T

Table 1: Truth table.



A	B	$\sim A$	$\sim B$	$A \wedge B$	$\sim (A \wedge B)$	$\sim A \vee \sim B$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Table 2: de Morgan's Law $\sim (A \wedge B) \iff \sim A \vee \sim B$.

Lady or Lion - a Truth Table.

back

L_1: Lion behind door 1.

L_2: Lion behind door 2.

P_1: Lady behind door 1.

P_2: Lady behind door 2.

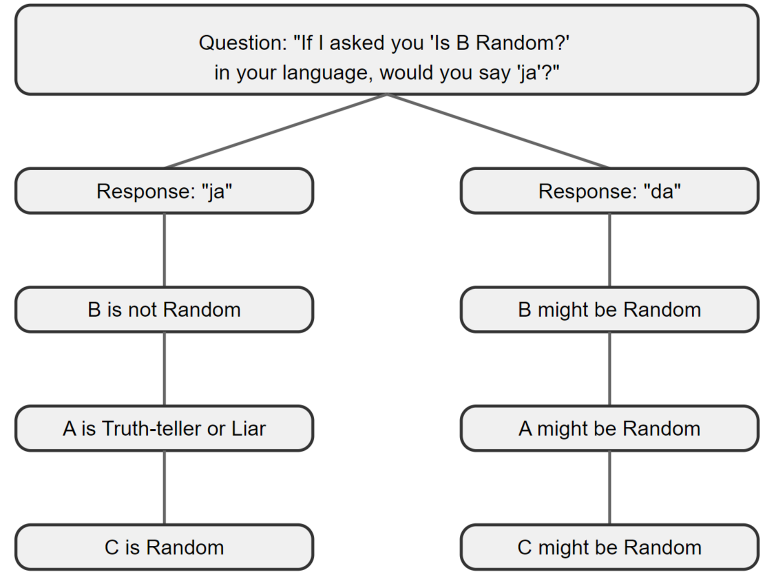
Door 1: P_1 and L_2.

Door 2: P_1 and L_2, or P_2 and L_1.

L_1	L_2	P_1	P_2	$A = P_1 \wedge L_2$	$P_2 \wedge L_1$	$B = A \vee (P_2 \wedge L_1)$	$A \wedge \sim B$	$\sim A \wedge B$
T	T	T	T	T	T	T	F	F
T	T	T	F	T	F	T	F	F
T	T	F	T	F	T	T	F	T
T	T	F	F	F	F	F	F	F
T	F	T	T	F	T	T	F	T
T	F	T	F	F	F	F	F	F
T	F	F	T	F	T	T	F	T
T	F	F	F	F	F	F	F	F
F	T	T	T	T	F	T	F	F
F	T	T	F	T	F	T	F	F
F	T	F	T	F	F	F	F	F
F	T	F	F	F	F	F	F	F
F	F	T	T	F	F	F	F	F
F	F	T	F	F	F	F	F	F
F	F	F	T	F	F	F	F	F
F	F	F	F	F	F	F	F	F

The Hardest Logic Question

back



Calculus Texts, Etc. Jan 12, 2024
Dr. R. L. Herman Revised: October 7, 2025

Old Calculus Texts, Stokes' Theorem, etc.

The following is a list of sources collected over the past couple of years. The first section is an attempt to look at the oldest calculus textbooks with links when available. Other sections relate to topics that have come up needing a trail of sources to answer some question related to origins of some topics. At the end is a collection of references.

Contents

Calculus Textbooks

- (Cajori 1919) A history of fluxions. online. This has a list of early publications on fluxions. Also, Guicciardini et al. 2019 discusses the reception of Newton's Method of Series and Fluxions. online. There is an Online Books Page for Calculus before 1800 and Guicciardini (1989), *The development of Newtonian calculus in Britain 1700-1800*. See list in Section
- Isaac Newton (1642–1727)
  - Newton (1693), *Tractatus de quadratura curvarum*, published in Newton (1704) *Optiks.*
  - Newton (1686) *Philosophiæ naturalis principia mathematica.*
  - 1671 *The Method of Fluxions and Infinite Series; with its Applications to the Geometry of Curved-Lines* published Newton and Colson (1736), Google Books.
  - 1711 *Analysis per quantitatum series, fluxiones, ac differentias* online, Mathematical Treasures.
  - Newton classification of differential equations in terms of fluxions and fluents.

  - Guicciardini (1999), Reading the *Principia*, Guicciardini 2022, and David Gregory's Manuscript 'Isaaci Neutoni Methodus Fluxionum' (1694).

- Gottfried Wilhelm Leibniz (1646-1716)
  - Leibniz (1684) was first publication on calculus in *Acta Eruditorum.* online and a translation. Here he also gives his version of Debeaune's problem.
  - Leibniz 1686, online, translation.
  - See also Roero (2005), Leibniz's first three papers on the calculus (1684, 1686, 1693), online.
  - Leibniz (1920) papers, translated by Child, are online and here. See GoogleBooks. Also, Child (1916, 1917) reviews the early papers in Part I, 1916 and Part II, 1917.

- Jacob and Johann Bernoulli - 1680s
  - Bernoulli Volumes
  - Bernoulli and Ferguson (2004) Lectures on The Integral Calculus, Johann Bernoulli, online. Also, check Bos (2023) on tractional motion.
  - Jacob Bernoulli, teacher and rival ...
  - Selected Letters from the Correspondence Between the Marquis de L'Hôpital and Johann Bernoulli

- l'Hôpital (1696) - The first differential calculus textbook, *Analyse des Infiniment Petits, Pour l'intelligence des lignes courbes* (Analysis of the infinitely small to understand curved lines), was written by Guillaume François Marquis de Hôpital (1661-1704) (l'Hôpital 1696). Other editions appeared posthumously from 1716 to 1781. Online version and (Stone 1730) online translation Wikipedia summary. He was a gardener (or son of the gardner) of John Campbell, 2nd Duke of Argyll (1680-1743).

- Carré (1700) - Louis Carré (1663-1711) published the first French book on the integral calculus. online.

- Harris (1702) - *A New Short Treatise of Algebra.* John Harris (1666-1719) wrote twenty pages devoted to fluxions. This was the first introduction to fluxions. See "An Introduction to the Doctrine of Fluxions" (2003) and Guicciardini (1989).

- Harris (1704) - In *Lexicon Technicum: or, an Universal English Dictionary of Arts and Sciences* see the Fluxion entry. Huge scientific dictionary in two volumes including articles on series, algebraic equations, trigonometry and conics. In the second volume (1710) was an English translation of Newton's *De quadratura* (Newton 1693) as *Quadrature of curves.*

- Hayes (1704) - Charles Hayes (1678-1760) wrote the first English text on Newton's method of fluxions, *A Treatise of Fluxions* online, dedicated to the Director of the Royal African Company.

- Manfredi (1707) - Gabriele Manfredi (1681-1761) published first book on differential equations.

- Reyneau (1708) - Charles René Reyneau (1656-1728) wrote a text to provide instruction in the new mathematics developed at the beginning of the 18th century, *Analyse démontrée, ou la Méthode de résoudre les problèmes des mathématiques, et d'apprendre facilement ces sciences...*, Volume I, Volume II - Second Edition, (Reyneau 1738). Other volumes at HathiTrust

- Reyneau (1714) - He wrote a lesser known two volume work, *La science du calcul des grandeurs en général, ou les élémens des mathématiques* online, Second Edition, 1739. Other volumes at HathiTrust

- Varignon (1725), *Eclaircissemens sur l'Analyse des infiniment petits* online is a "Collection of meditations and research by Mr. Varignon on the Analysis of the Infinitely Small by the Marquis de l'Hôpital" published three years after his death. See story about Pierre Varignon (1654–1722).

- Simpson (1737) - Thomas Simpson (1710-1761) *A New Treatise of Fluxions* online. Also, Simpson (1750) expanded this to *The doctrine and application of fluxions.* online. More at HathiTrust

- Deidier and Deidier (1740) - L'Abbé Deidier (1698 1746), *Le calcul différentiel et le calcul intégral: expliqués et appliqués à la géométrie*

- La Caille (1741) - Nicolas-Louis de La Caille (1713 - 1762), *Leçons élémentaires de mathématiques, ou, Elemens d'algebre et de géométrie*, 1772 Edition, has had a small section (pgs. 264-277) on differential and integral calculus since 1756.

- Maclaurin (1742) - Colin Maclaurin (1698-1746) wrote a two volume work, *A Treatise of Fluxions,* online and 1801 Edition.

- Agnesi (1748)- Maria Agnesi (1718-1799) wrote *Instituzioni analitche ad uso della gioventu italiana (Foundations of Analysis for the Use of Italian Youth)* online. See Mathematical Treasures Mingari Scarpello and Ritelli (2019) notes that more than 200 pages are devoted to solving differential equations.

- Euler (1748) - Leonhard Euler (1707-1783), *Introductio in analysin infinitorum*, online and translation by Ian Bruce.

- Bougainville et al. (1754) - Louis-Antoine de Bougainville (1729-1811), according to MacTutor, was"influenced by d'Alembert and he wrote *Traité du calcul -intégral* in 1752, See Google Books or Gallica. This extended de l'Hôpital's book ... to cover the integral calculus." De Bougainville et al. 1754. This was noted at David Zitarelli's site, based on Struik (1956) [See article], that he was one of four mathematicians at Ticonderoga. Struik (1956) noted that ". In 1756 ... the second volume of the integral calculus appeared, probably the first book exclusively devoted to differential equations."online.

- Euler (1755) - *Institutiones calculi differentialis*, online and translation by Ian Bruce.

- Euler (1768) - *Institutiones calculi integrals volumen-Primum*, Vol I, online. A three-volume textbook on integral calculus and contained many of Euler's discoveries about differential equations. translation by Ian Bruce.

- Waring (1770) - Edward Waring (1736-1798) wrote *Meditationes Analyticæ* online, as an expansion of part of Waring (1762), *Miscellanea Analytica*, which was on the theory of numbers and algebraic equations, online.

- Cousin (1777) - Jacques Antoine Joseph Cousin (1739-1800), *Leçons de calcul différentiel et de calcul intégral*, online. He provides an interesting history. Google.

- Bézout 1764 - *Cours de mathématiques, à l'usage du corps de l'artillerie ...: à l'usage du corps de l'artillerie* by Étienne Bézout (1730-1783) Vol. 1, Vol. 2, Vol. 3 is on calculus. Domingues (2004) mentions the use of Bezout (1774) in Portugal as well as other writings, online.

- Cunha (1790) - *Principios Mathematicos*, a 21 part encyclopedia of mathematics in Portuguese by José Anastácio da Cunha (1744-1787), online. Cunha (1811), *Principes Mathématiques*, was a French translation by João Manuel d'Abreu (1757–1815), online. For more see Domingues (2004).

- Lacroix (1797) - 1797-1800 - *Traité du Calcul Différentiel et du Calcul Intégral*, Volume 1, Volume 2, Volume 3. "... the most comprehensive work of its kind for that time."
- Lacroix (1802) - Sylvestre François Lacroix (1765-1843), *Traité élémentaire de calcul différentiel et de calcul intégral* Hathi Trust online.
- Ivory, James (1809) In papers 1809–1824, he was the first to use continental analytical techniques, 'The Most Unlucky Person That Ever Existed', Craik (2000) online and his Obituary
- Hirsch (1810) - Meier (Meyer) gave a set of integral tables and methods, *Integraltafeln: Oder, Sammlung Von Integralformeln,* online. An English translation, Hirsch (1823) *Integral Tables: Or, A Collection of Integral Formulas*, can be found online.
- Other Early Integral Tables
  - Newton (1671) had tables in his work, *The Method of Fluxions and Infinite Series; with its Applications to the Geometry of Curved-Lines* published Newton and Colson (1736).
  - Legendre (1811) - Adrien Marie Legendre (1752-1833) wrote several volumes on Tables of Elliptic functions, Gallica. These did not serve the same purpose as Hirsh's tables.
  - Peirce (1902) - Benjamin Osgood Peirce (1854-1914) gave a short table of integrals 1910, 2nd ed. A 1903 version is online, and Google shows the earlier edition, (Peirce 1893). Byerly (1889) includes an 1889 edition of Peirce's tables which is online. This is a later edition of Byerly's 1882 text, (Byerly 1881).
  - De Haan (1858) published *Tables d'intégrales définies*, online and 1867 version, supplemented by *Supplément aux tables d'intégrales définies* in 1864 online. The supplement gave references to where integrals were found.
  - Abramowitz and Stegun (1948), *Handbook of Mathematical Functions ...* , Google, 1964
- Dealtry (1816) - William Dealtry, (1775-1847), *The Principles of Fluxions*, online.
- Lacroix (1816) - *An Elementary Treatise on the Differential and Integral Calculus*, Lacroix translation, 1816. online. from 1797 online. First English edition, responsible for the introduction of Continental methods of analysis and notation. Herschel, Babbage and Peacock translated and edited it.
- Jephson (1826), *The Fluxional Calculus: An Elementary Treatise* blends Lagrangian and Newtonian methods. Volume 1, and Jephson (1830) Volume 2 He might be the Thomas Jephson at https://www.wikitree.com/wiki/Jephson-257 or in the trial. Also mentioned at Bidder Bio.
- Boucharlat (1820) - Jean-Louis Boucharlat (1775-1848) was the second edition, online. Boucharlat (1828) was a translation into English by Ralph Blakelock found here. Alcouffe (2020) provides a little more on the history of the text and use by Marx. Open Source Access. Another mention of the first edition being in 1813 was by Zerner (1989), This paper is in French and notes "the elementary treatise of Lacroix (1802) and the elements of Boucharlat (1813) were republished until 1881 and 1891, respectively, having each had nine editions."
- (De Morgan 1836) *The Differential and Integral Calculus*, by Augustus De Morgan (1806-1871), online. and (De Morgan 1842) The differential and integral calculus, containing . . . online.
- Ottley (1838), William Campbell Ottley (1808–1843), *A Treatise on the Differential Calculus: With a Collection of Examples*, online. Autographed Note.

- Ritchie (1836) William Ritchie (1790-1837), *Principles of the Differential and Integral Calculus*,online and Joseph Anthony Spencer edited the 2nd ed., online, Ritchie and Spencer (1847). Possibly (1823-1873) link to genealogy.
- Woolhouse (1854), Wesley Stoker Barker Woolhouse (1809-1893), *Elements of the Differential Calculus*, online.
- Charles Davies (1855) first commercially successful book by an American author online, described by (Rickey and Shell-Gellasch 2010), Introduction to America at West Point. There is also Davies (1838) edition. here and a second edition here.
- Duhamel (1856) wrote a two volume French calculus text of all most a thousand pages in four parts. The first three chapters are in Volume 1 (585 pgs.) The second volume is on differential equations (375 pgs.). See the online two volume set.

  1. Des Quantités Considérées Comme Limites.
  2. Calcul Des Dérivées et des Différentielles des Fonctions. Calcul Inverse Où Intégration des Différentielles.
  3. Des Limites de Sommes. Calcul Inverse du Calcul Differentiel.
  4. Intégration des Équations Différentielles.

  At EUDML Vol 1, (1860) Vol 2, (1861). Second Volume (1856) is also at Gallica. Still might need to check Dhombres (1985) about
- *A Treatise on the Integral Calculus*, by I Todhunter, 1863. Has spherical polar coordinates! Pg 225 online.
- (Todhunter 1864)*A Treatise on the Differential Calculus and the Elements of the Integral Calculus: With Numerous Examples*, Todhunter, 1863, online, 1868
- (Todhunter 1873) *A History of the Mathematical Theories of Attraction and the Figure of the Earth from the Time of Newton to that of Laplace*. Volume 2, Todhunter, 1873. Attributes to Legendre form of spherical coords. online.
- Byerly (1879) - William Elwood Byerly (1849–1935) wrote *Elements of the differential calculus, with examples and applications*, online.
- Byerly (1881) - William Elwood Byerly (1849–1935) wrote *Elements of the integral calculus*, online. Later revised as (Byerly 1889), which includes an 1889 edition of Peirce's tables. [See earlier.] (Greenhill 1896) *Differential and Integral Calculus, with Applications*, by Greenhill, G. (George), Sir, 1886. online.
- (Newcomb 1887)*Elements of Differential and Integral Calculus*, by Newcomb, Simon, 1887. online.
- Greenhill 2nd ed, 1891 online. Has spherical polar coordinates!
- (McMahon and Snyder 1898)*Elements of the Differential Calculus*, by McMahon, James, and Snyder, Virgil, 1898 online.
- (Lamb 1924) *An Elementary Course of Infinitesimal Calculus*, by Lamb, Horace, Sir, 1897. online.
- *An Elementary Treatise on the Integral Calculus*, by Johnson, William Woolsey, 1898. online. Page 168-9 Spherical (polar coordinates in space) Uses rho! Pg 168

- (De Morgan 1909) *Elementary Illustrations of the Differential and Integral Calculus*, by De Morgan, Augustus, 1899, online. This was a reprint of (earlier) 1842 bound version of two volume set from numbers 135 and 140 of the Library of Useful Knowledge (1832).
- Edmond Maillet's Notes: 1910-11, 1913-14
- (Townsend and Goodenough 1910) *Essentials of Calculus* by Townsend, Edgar Jerome, Goodenough, George Alfred, pg 265 uses rho online. - Cajori's copy
- Edwards (1921, 1922) *A Treatise On The Integral Calculus* Vol. I, Vol. II, and here. Edwards (1886) *Differential Calculus with Applications and Numerous Examples: An Elementary Treatise...* Edwards (1893), *Differential Calculus for Beginners,* online. Theorems of Stokes and Green. Harmonic Analysis - Chapter XXXIX.

Early Differential Equations

- Newton classification of differential equations in terms of fluxions and fluents.
- Debeaune problem
  - Johannes Bernoulli to L'Hôpital on Debeaune problem.
  - Inverse tangent project.
  - Correspondence with Descartes.
  - Pedersen (1978) Bartholin and the Debeaune problem.
  - L'Hôspital's rectification of Debeaune's curve, Ferreira (2011), online.
  - Scriba (1961), Debeauneschen Problems durch Descartes, JSTOR.
- Bittanti (1996), History of Riccati equation. online.
- Duhamel (1856) wrote a two volume French calculus text of all most a thousand pages in four parts. The second volume is on differential equations (375 pgs.). See the online two volume set.
- Boole's Differential Equations (Boole 1872) Treatise, 1872, 3rd Ed. and First Edition, 1859. "Symbolical methods" in the last chapters. Has exercises.

Stokes Theorem and Vector Analysis

- (Stokes 1905) online Mathematical And Physical Papers, Vol. 5. Includes Obituary by Lord Rayleigh.
- Stokes' Theorem. References to the Smith Prize question, 1854. (Stokes 1905), page 320-1. A footnote there tries to explain why Stokes should be credited with the theorem.
  - References were made in 1912 Edition, pg 143 of Thomson and Tait pg 124 Section 190 (j). (Kelvin and Tait 1867)
  - In 1912 edition, page 167 is Appendix A - Extension of Green's Theorem.
  - Thomson (1868) on vortex motion.

  - He references Helmholtz (Helmholtz 1858, 1867) online
  - According to (Maxwell 1873) pg 26, Section 24,

    "This theorem was given by Professor Stokes. Smith's Prize Examination, 1854, question 8. It is proved in Thomson and Tait's Natural Philosophy, § 190 (f)." 1873 Ed. Note: Maxwell sat for the Smith Prize in 1854 and took first place with Edward Routh.
  - Link to the 1850 letter from Thomson to Stokes: link .

- Maxwell's 1873 Edition - link mentions Stokes Smith's Prize question. He also uses nabla on that page. On the next page he discusses convergence (opposite of divergence) and uses word curl. "I propose (with great diffidence) to call the vector part of ... the curl." He is still trying to rewrite his theory using quaternions and credits Tait with notation referencing his paper, "On Green's and other allied Theorems", at link . Tait credits that Stokes' Theorem was "first given by Thomson (Thomson & Tait'S 'Natural Philosophy,' § 190 (j); Thomson on Vortex Motion, Trans. R.S.E., 1868-9,Sec 60 (q)." Even Thomson does not claim it in a later edition of their book. link to (Knott 1911).

- (Katz 1979) *The History of Stokes' Theorem*, Victor J. Katz, JSTOR, mentions textbooks.

- (Tait 1870) relates Stokes' Theorem to Green's Theorem using quaternions and the ∇ operator. Also in (Tait 1898), the first volume of Tait's collection of papers, online.

- In letters between Tait and Maxwell there is a discussion as to what to call certain vector operators. You can find this on page 143 and the following of https://www.maths.ed.ac.uk/~v1ranick/papers/taitbio.pdf.

- (Gibbs 1884) *Elements of Vector Analysis*, Gibbs, 1881-4 notes. online.

- (Wilson and Gibbs 1901) *Vector Analysis*, Gibbs and Wilson, 1901, Gives Theorems of Gauss, Stokes and Green. online.. Gibbs defense online.

- (Bucherer 1904) Elemente der Vektor-Analyse by A. H Bucherer. 1903 Edition, online.

  A 1904 review by Henry Crew (Crew 1904), online. - Has derivations of Gauss, Stokes, Greens Theorems.

- (August 1894) Einführung in die Maxwell'sche Theorie der Elektricität: Mit einem, August Föppl 1894. Lists integral theorems in appendix. online.

- (Coffin 1911) *Vector Analysis: an introduction to vector-methods and their various applications to physics and mathematics,* 1911, online

- Other references:

  - (Katz 1979), *The History of Stokes' Theorem*,
  - (Markvorsen 2008), *The Classical Version of Stokes' Theorem Revisited*,
  - (Crowe 1967), *A History of Vector Analysis.*

- (Stokes 1847), On the theory of oscillatory waves, *Trans. Cam. Philos. Soc.*, 8, 441–455

History of the Slide Rule and Logarithms

- John Napier (1550-1617), 8th Laird of Merchiston, married Elizabeth, daughter of James Stirling, the 4th Laird of Keir and of Cadder. [John is fourth cousin 15 times removed from author; wife is 13th great aunt of author and a distant cousin of James Stirling, the Venetian, who is a third cousin 11 times removed from author. Finally, James Stirling is the 2nd great grand nephew of Elizabeth Stirling, her mother being his third grat granmother.]
  - The Description of the Wonderful Canon of Logarithms, translation by Ian Bruce at http://www.17centurymaths.com/contents/napiercontents.html
  - The Construction of the Wonderful Canon of LOGARITHMS
  - Open University Course
  - 1889 translation Constructio
  - Hobson Lecture, 1914
- (Roegel 2010) *Bürgi's Progress Tabulen (1620)* online.
- Newton's Polynomial Solver - Mathematica Demo
- On the Evolution of K&E Vector Slide Rules paper at the Oughtred Society
- Otis King Calculator Manual.
- Quadrature of the Hyperbola and Circle
  - Crippa (2019) Leibniz's Arithmetical Quadrature of the Circle, online.
  - Edgar and Richeson (2019), Gregory's Theorem, online.
  - O'Hara (1996), Huygens, Leibniz and the 'petit demon', online.
- Hofmann (1939), On the Discovery of the Logarithmic Series ..., online.
- (Cajori 1908) *Notes on the History of the Slide Rule* online.
- Chamberlain (1999) *Long-Scale Slide Rules Revisited,* online.

Historical Topics to be Sorted

- Cotes (1722) *Harmonia Mensurarum.* Cotes (1714) was the first to note the identity $\ln\big(\cos\theta + \sqrt{-1}\sin\theta\big) = \sqrt{-1}\theta$. Although it was presented geometrically. Cajori 1913 [JSTOR] describes the history including when Euler (1748) printed his formula.
- Use of symbols - In 1800, Arbogast (1800) uses the operator D in his *Du calcul des dérivations* at Google Books. Other histories are found at Earliest Uses of Symbols of Calculus.
- Coolidge (1949) History of Binomial Theorem, online.
- Arithmetical books from the invention of printing to the present time, de Morgan, 1847, online.
- Airy 1896 Airy autobiography. online

- Fractional Calculus, book.
- Article: Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus?

  On the history of epsilontics, Sinkevich, 2015 and J. Polish Math Soc 2016.
- Partial Fraction Decomposition is investigated in Newberry (n.d.) with little history. Wikipedia states "Johann Bernoulli and Gottfried Leibniz independently discovered the concept in 1702." This seems to be noted in Laugwitz (1997) [JSTOR].

  Todhunter (1864) describes partial fractions in Chapter II.

  Sandifer (2007) decribes Euler's approach to partial fractions. " we get to chapter 18, the last chapter of the second part of *Calculi differentialis*, titled *De usu calculi differentialis in resolutione fractionum*, 'On the use of differential calculus in the resolution of fractions.' Translation. By this, Euler means what we now call 'partial fractions.' Euler (1748) introduced partial fractions in this masterpiece, *Introductio in analysin infinitorum,* 'Introduction to the analysis of the infinities,' [E101, E102]"
- (Ball 1889) *A History of the Study of Mathematics at Cambridge,* Ball. online.
- (Hamilton 1853) *Lectures on Quaternions: Containing a Systematic Statement of a New Mathematical Method* online
- *An Elementary Treatise on Quaternions,* P. G. Tait, 1890. online.
- (Wu and Yang 2006) *Evolution Of The Concept Of The Vector Potential In The Description Of Fundamental Interactions*, online.
- Sommerfeld, *Partial Differential Equations in Physics* online, 1949.
- (Grabiner 1997) Was Newton's Calculus a Dead End? online.
- History of Gaussian Integrals. Poisson (1834) evaluated $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ using the standard trick as seen in 1835 edition. Sturm (1859) references Poisson in 1859, online. Laplace used another method and people usually refer to (Gauss 1809), at Google Books or in English. Least Squares, or Method of Gauss is discussed in (Celmiņš 1998) highlighting dispute between Gauss and Laplace. Conrad (2016) gives some proofs in the paper. Also, see the History of Statistics page.

  Laplace's *Memoir on the Probability of the Causes of Events* is a translation (Laplace 1986), JSTOR of *Mémoire sur la probabilité de causes par les évenements* (Laplace 1774). Example is shown at YouTube.
- Moivre (1730) added to *Miscellanea Analytica* and supplement discussed in (Deming 1933) referencing (Pearson 1924), JSTOR ref. Archibald (1926) critiques Pearson's paper as to when deMoivre wrote on the probability integral. Then, Daw and Pearson (1972) continue the discussion as found in JSTOR.
- Sturm and Liouville (1837) work on differential equations. Lützen (1984) gives a history online
- Mathematics Genealogy
  - Math Genealogy Visualizer
  - hrefhttp://magjac.com/graphviz-visual-editor/Graphviz Visual Editor
  - My Math Genealogy
- Tikz Online

- Gallica - Isaac Barrow's Lectiones Geometricae (1669); which Morris Kline says is "one of the great contributions to the calculus".
- Important publications in mathematics.
- History of Teaching Calculus.
- Derivative Notation: Lagrange, Leibniz, Euler, and Newton: Video.
- Calculus Made Easy, by Silvanus Thompson.
- Cours De Mathématiques, A L'Usage Des Gardes Du Pavillon Et De La Marine Contenant Le Traité De Navigation, Volume 6, Étienne Bézout, 1769.
- Vincenzo Viviani and Leibniz
  - Roero 1990 - Leibniz and the Temple of Viviani online.
  - *L'intérêt intérnational d'un problème d'analyse proposé par V. Viviani*, 1988, Roero et al. (1988), online.
  - Lanier (1987) Leibniz, la nouvelle analyse et la géométrie ou enquête sur la fenêtre de Viviani, online.
  - Nature, Geoghegan (1888), online He refers to Lacroix.
  - Caddeo, Montaldo, Piu, et al. (2001), The Möbius strip and Viviani's windows, online.
- Francesco Siacci's Theorem
  - Based on papers by Siacci 1879b, 1879a. These can be found here and here
  - In E. T. Whittaker 1917, *A treatise on the analytical dynamics of particles and rigid bodies; with an introduction to the problem of three bodies.* is discussed Siacci's result on pages 21-22 in the Second Edition (1917) [See https://archive.org/details/treatisanalytdyn00whitrich/page/22/mode/2up.]
  - Casey (2011) gave a modern derivation.
- The recent paper on Siacci's Theorem mentions Rowell (1922b) as a curve rederived, online. This short note of June 03 in *Nature* was taken up on June 19 by Wright (1922), online. He mentioned that the result was already in Besant (1914), with no mention of the edition. On page 100, Section 91 of the 1893 second edition, 2nd Ed. online, Besant, William Henry, (1828-1917) considers, the *Motion of a particle in the same medium under the action of a force to a fixed point varying as the distance from that point.*

  On August 12, Rowell (1922a) noted, "it appears to have been recognised in connexion with the spherical pendulum. Prof. Lamb in his 'Dynamics,' p. 288, as I now find, refers to the curve as 'a kind of elliptic spiral,' and Dr. Besant describes it as 'an ellipse gradually shrinking in size."' He goes on to mention, "The problem of the spherical pendulum is identical with that of the motion of a particle in a spherical bowl" and proceeds to look at a spinning bowl. *Perhaps we need a list of textbooks on dynamics.*

  Lamb (1914) has a section on page 287 on the effect of friction on the small oscillations of the spherical pendulum (Art. 29)," online 1914. The spherical pendulum equations are given as

$$\begin{aligned} \frac{d^2x}{dt^2} + k\frac{dx}{dt} + \mu x &= 0. \\ \frac{d^2y}{dt^2} + k\frac{dy}{dt} + \mu y &= 0. \end{aligned} \tag{1}$$

With specific boundary conditions, he wrote the solution $x(t) = ae^{-kt/2}\cos nt, \quad y(t) = be^{-kt/2}\sin nt$. Besant solves the same equations but has a more complicated shrinking ellipse due to different initial conditions.

- Pierce (1955) published a letter and responses Tchebycheff or Chebyshev? and responses. On the previous page, Papoulis (1955) discusses series of orthogonal polynomials.

Early Calculus Authors to be checked

- John Craig, De Moivre, David Gregory, Patio de Duillier, Cotes Ditton. Cheyne.
- John Harris, 1702, 1705, 1710
- Charles Hayts, 1704 .
- William Jones, 1706 .
- Humphry Ditton, 1706
- Commerciiim Epistolicum D. Johannis Collins, 1712
- Joseph Raphson, 1715
- Brook Taylor, 1715 .
- James Stirling, 1717, 1730 .
- Edmund Stone, 1730
- 1734 - Berkeley Critique
- John Colson, 1736
- James Hodgson, 1736
- Thomas Bayes, 1736
- John Muller, 1736
- Anonymous translation of Newton's Method of Fluxions, 1737
- James Smith, 1737
- Thomas Simpson, 1737
- Benjamin Martin, 1739, 1759
- An anonymous text, 1741
- John Rowe, 1741, 1757, 1767
- Maclaurin, "Treatise of Fluxions, 1742"
- John Stewart, 1745 ....

- William Emerson, 1743 (?), 1757, 1768
- Thomas Simpson, 1750
- Nicholas Saunderson, 1756
- John Rowning, 1756
- Israel Lyons, 1758
- William West, 1762
- James Wilson, 1761

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Agnesi, Maria Gaetana. 1748. *Instituzioni analitiche ad uso della gioventu italiana.* Vol. 1. Nella Regia-ducal corte.

Airy, George Biddell. 1896. *Autobiography of Sir George Biddell Airy.* University Press.

Alcouffe, Alain. 2020. "The differential calculus, mathematicians and economists in the nineteenth century: K. Marx and H. Laurent, readers of JL Boucharlat." In *Karl Marx and Mathematics,* 76–92. Routledge.

"An Introduction to the Doctrine of Fluxions." 2003. In *Most Honourable Remembrance: The Life and Work of Thomas Bayes,* 182–252. New York, NY: Springer New York. ISBN: 978-0-387-21561-7. https://doi.org/10.1007/0-387-21561-1_5. https://doi.org/10.1007/0-387-21561-1_5.

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Archibald, Raymond C. 1926. "A rare pamphlet of Moivre and some of his discoveries." *Isis* 8 (4): 671–683.

August, Foeppl. 1894. *Einführung in die Maxwell'sche Theorie der Elektricität.*

Ball, Walter William Rouse. 1889. *A History of the Study of Mathematics at Cambridge.* University Press.

Bernoulli, Johann, and WA Ferguson. 2004. "Lectures on the integral calculus." *21ST CENTURY SCIENCE AND TECHNOLOGY* 17 (1): 34–42.

Besant, William Henry. 1914. *A treatise on dynamics.* G. Bell.

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