Cubic Equations

Fall 2025 - R. L. Herman



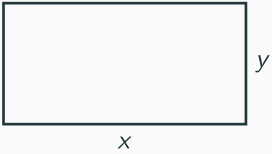
Solutions of Polynomial Equations

- Linear equations, known solutions.
- Chinese - Gaussian elimination: Systems of n linear equations and n unknowns.
- Quadratic equations: Need square roots.
- Cubic equations: Need square roots and cube roots. Solved in 16th century.
- Quintic equation: studied in 1820's. Eventually lead to group theory!

Figure 1: Leonardo da Vinci attempts Delian problem (Doubling cube).

Quadratic Equations

Babylonian Method (Modern Notation)



Find x and y for a given perimeter and area.

$$\begin{aligned} x + y &= p \\ xy &= q. \end{aligned}$$

Eliminate y, $x^2 + q = px$. Then,

$$x, y = \frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}.$$

Method - Compute the following

1. $\frac{x+y}{2}$
2. $\left(\frac{x+y}{2}\right)^2$
3. $\left(\frac{x+y}{2}\right)^2 - xy = \frac{(x+y)^2 - 4xy}{4}$
4. $\sqrt{\frac{(x+y)^2 - 4xy}{4}} = \frac{x-y}{2}$
5. By inspection, get x, y from p and q, since

$$\begin{aligned} \frac{x-y}{2} &= \sqrt{\frac{p^2 - 4q}{4}} \\ &= \sqrt{\left(\frac{p}{2}\right)^2 - q}. \end{aligned}$$

Quadratic Equations (cont'd)

- Brahmagupta (628) - Explicit $ax^2 + bx = c$.

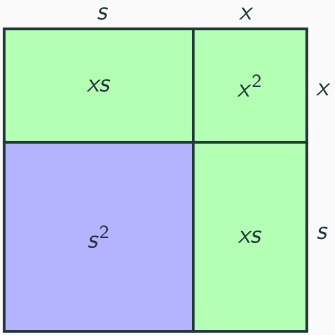
$$x = \frac{\sqrt{4ac + b^2} - b}{2a},$$

- Euclid - Prop. 28
- al'Khwarizimi

$$\begin{aligned} x^2 + 2xs &= n \\ x^2 + 2xs + s^2 &= n + s^2 \\ (x+s)^2 &= n + s^2 \end{aligned}$$

- Quadratic Irrationals

$$\frac{a + \sqrt{b}}{\sqrt{\sqrt{a} + \sqrt{b}}}$$



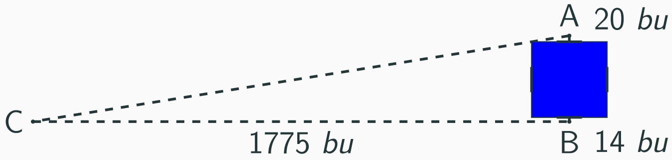
Note from the figure:
Green area $= x^2 + 2xs = n$.
No negative lengths (solutions).

Solving Quadratics Han Dynasty Style

Jiuzhang Suanshu (Gougu 20).

今有邑方不知大小，各中。出北二十步有木。出南十四步，折而西行一千七百七十五步木。：邑方何?

There is a square town of unknown dimensions. There is a gate in the middle of each side. There is a tree located 20 *bu* outside the North Gate. If one leaves the town by the South Gate, walks 14 *bu* due south, then walks due west for 1775 *bu*, the tree will just come into view. What are the dimensions of the town? [1 *bu* = 1.38 m.]



https://math-physics-problems.fandom.com/wiki/Solving_Quadratics_Han_Dynasty_Style

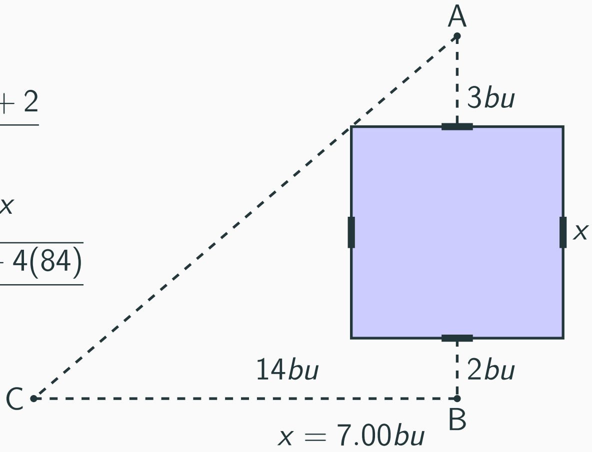
Another Example - Derivation of the Quadratic Equation

The diagram looks better if the tree located 3 *bu* outside the North Gate, one leaves the town by the South Gate, walks 2 *bu* due south, then walks due west for 14 *bu*.

$$\frac{3}{x/2} = \frac{x+3+2}{14}$$

$$84 = x^2 + 5x$$

$$x = \frac{-5 \pm \sqrt{25 + 4(84)}}{2}$$

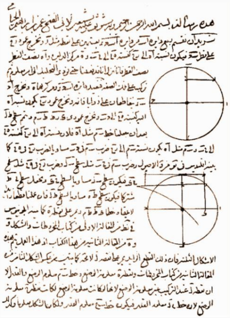


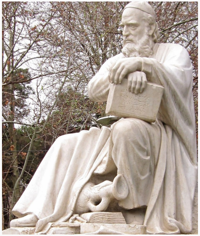
Cubic Equations

- Babylonians - Table of cubes
- Greeks - Geometric Problems
  - Duplicating cube (Delian Prob.).
  - Intersecting conics.
  - Cutting Sphere with plane.
- Omar Khayyam (1048-1131)
  - First general theory of cubics.
  - Provided 19 types of cubic.
  - Example

    $$x^3 + ax^2 = bx + c,$$

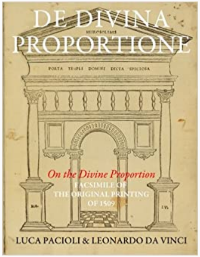
    Cube and square equals side plus number.
  - Geometric: Intersect two hyperbolae.





Pacioli, da Vinci and della Francesca

- Luca Pacioli (1445-1517)
    - Franciscan friar, tutor.
    - *Summa de arithmetica*, 1494. Father of Accounting.
    - *Divina proportione*, 1509.
    - "Solution to cubic is impossible!"
    - On table: slate, chalk, compass, dodecahedron. Hanging: Rhombicuboctahedron half-filled with water.
- Leonardo da Vinci (1452-1519)
- Piero della Francesca's (1420-1492) Painter/mathematician, met Alberti, 1451. Wrote books: algebra, perspective, Archimedean polyhedra. Pacioli used his work.



The Secret Formula

How a Mathematical Duel Inflamed Renaissance Italy and Uncovered the Cubic Equation, by Fabio Toscano, 2020.

- The Abbaco Master

  Italian Wars, Brescia, 1512, Tartaglia.
- The Rule of the Thing

  cosa (thing), *censo* (x^2), *numero*.
- The Venetian Challenge

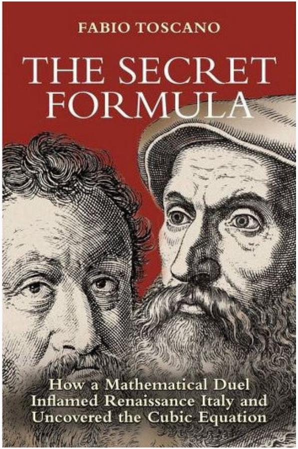
  1535, Fior challenges Tartaglia.
- An Invitation to Milan

  Entrance of Gerolamo Cardano.
- The Old Professor's Notebook

  da Coi vs. Ferrari, del Ferro's priority:

  Solution: things and cube equal to number.
- The Final Dual

  Ars Magna, 1545. Ferrari vs. Tartaglia, 1547.



The Search for Solutions - *Enter del Ferro*

- Scipio del Ferro (1465-1526)
  - University of Bologna, notebooks
  - Printing press - Guttenberg
  - 1506/1514, solution of **depressed cubic**: $x^3 + ax = b$.
  - Public Challenges led to secrecy.
- Gave to Antonio Maria Fior (Florido).
- Tartaglia (Nicolo Fontana) (1499-1557)
  - 1512, French attack - sabre wound led to stammer.
  - Self-educated
  - 1530 da Coi wrote to him $x^3+3x^2 = 5$, $x^3+6x^2+8x = 1000$.

Figure 2: Tartaglia

Sidenotes

Pacioli, *Summa de arithmetica* 1494, Not solvable:

$$\begin{aligned} n &= ax + bx^3 \\ n &= ax^2 + bx^3 \\ n &= ax^3 + bx^4 \end{aligned}$$

Tartaglia, Abaco teacher/master but engaged in other activities.

Challenges not uncommon, but expect challenger to know solutions to their questions.

Not happy with da Coi questions since da Coi did not know answers.

The Plot Thickens

- Tartaglia boasted he could solve $x^3 + ax^2 = c$.
- Florido challenged Tartaglia
  - Each posed 30 problems
  - Florido mostly gave problems of form $x^3 + ax^2 = c$.
  - Tartaglia won by solving depressed cubic 1535, but didn't publish.
- Girolamo Cardano (1501-1576)
  - Gambler, astronomer, physician, astrologer, heretic, father of murderer.
  - Begged for solution from Tartaglia. Finally, they met in Milan.
  - Tartaglia eventually gave solution in 1539 as a Poem if it was kept secret.
  - It was not in Cardano's book.

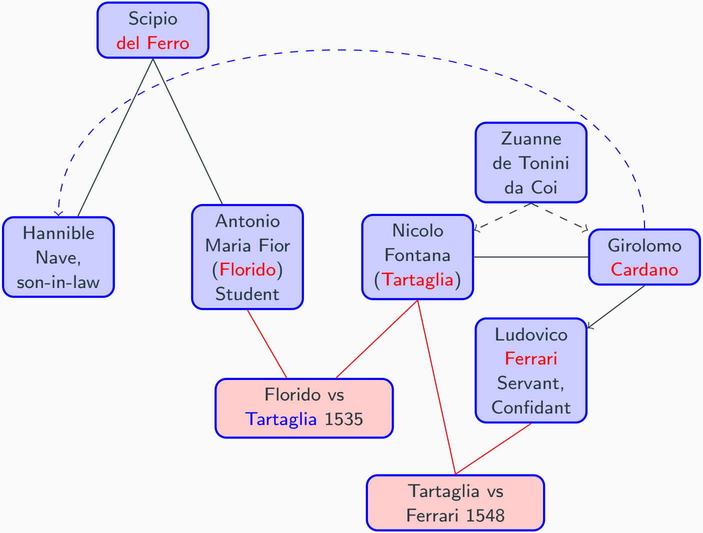
Figure 3: Cardano, *Ars Magna*.

Enter Ludovico Ferrari

- Ludovico Ferrari (1522-1565)
  - Servant at 14
  - Secretary, confidant
  - Worked on problems with Cardano
  - Cubic and biquadratic equations

- da Coi $\rightarrow$ Cardano $\rightarrow$ Ferrari
  - 4th degree polynomial
  - Ferrari solution involved solving cubic
  - Publishing was a problem.
- 1543 Trip to Florence, stopped in Bologna on the way.
  - Visitied Hannible Nave, del Ferro's son-in-law.
  - Saw del Ferro's notes.
  - Cardano believed he could publish in his *Ars Magna*, 1545.
- Barrage of letters from Tartaglia!

The Players in the Cubic Story



Tartaglia vs Ferrari - 1548

- Public debate in Milan, Ferrari's hometown.
- Cardano was absent.
- Tartaglia lost, blamed crowd.
- Tartaglia worked on arithmetic.
- Ferrari became professor in Bologna, 1565.
  Was poisoned 1565, white arsenic, possibly by sister.
- Cardano predicted exact date of his own death in 1576.


Figure 4: Tartaglia and Ferrari

Solution of the Quadratic $x^2 + ax + b = 0$

- Completing the square:
  $\left(x + \frac{a}{2}\right)^2 + b - \frac{a^2}{4} = 0.$
- Solution: $x + \frac{a}{2} = \pm\sqrt{\frac{a^2}{4} - b}.$
- Graph of parabola
  $y = x^2 + ax + b$
  Vertex $\left(-\frac{a}{2}, b - \frac{a^2}{4}\right)$
- Number of real solutions?
- Substitute $x = u - \frac{a}{2}$:
  $0 = x^2 + ax + b$
  $0 = (u - \frac{a}{2})^2 + a(u - \frac{a}{2}) + b$
  $0 = u^2 + b - \frac{a^2}{4}.$
- Solve for u.

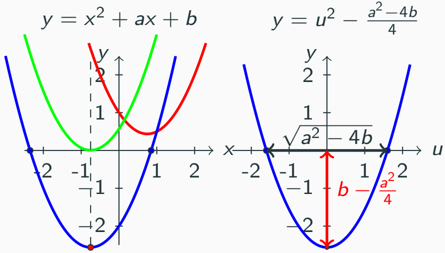


Figure 5: Plots of parabolae. Translating blue parabola on left by $\frac{a}{2}$ results in that on the right.

Plotting a Cubic Function $y = x^3 + ax^2 + bx + c$

- Set $y = 0$, then
  $x^3 + ax^2 + bx + c = 0.$
- Solutions are black points.
  Always have a real solution.
- $y' = 3x^2 + 2ax + b$ and
  $y'' = 6x + a.$
- Inflection point: $y'' = 0$ for
  $x_0 = -\frac{a}{3}$, $y_0 = c - \frac{ab}{3} + \frac{2a^3}{27}.$
- Slope of tangent at $(x_0, y_0 = q)$
  is $p = b - \frac{a^2}{3}.$
- Extrema at
  $x_\pm = -\frac{a}{3} \pm \frac{\sqrt{a^2 - 3b}}{3}.$

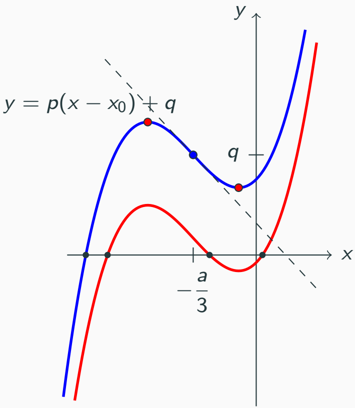


Figure 6: Plots of the cubic function exhibiting either one or three real roots.

Can you complete the cube: $(x + \alpha)^3 = x^3 + 3x^2\alpha + 3x\alpha^2 + \alpha^3$?

Solution of the Cubic $x^3 + ax^2 + bx + c = 0$.

Let $x = y - \frac{a}{3}$.

Then, $y^3 + py + q = 0$, where

$$\begin{aligned} p &= b - \frac{a^2}{3}, \\ q &= c - \frac{ab}{3} + \frac{2a^3}{27}. \end{aligned}$$

Let $y = u + v$:

$$u^3 + v^3 + (p + 3uv)(u + v) + q = 0.$$

Let $p + 3uv = 0$, then

$$\begin{aligned} u^3 v^3 &= -\frac{p^3}{27}, \\ u^3 + v^3 &= -q. \end{aligned}$$

Now, define $X = u^3$. $Y = v^3$.

We obtain

$$\begin{aligned} X + Y &= -q. \\ XY &= -\frac{p^3}{27}, \end{aligned}$$

Does this look familiar?

The solution of Cubic:

$$X, Y = u^3, v^3 = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \frac{p^3}{27}},$$

$y = u + v$,

$x = y - \frac{a}{3}$.

Example: $2x^3 - 30x^2 + 162x - 350 = 0.$

Let $x = y - \frac{b}{3a} = y + \frac{30}{6} = y + 5.$

We obtain a depressed cubic (del Ferro), $y^3 + 6y - 20 = 0.$

Letting $y = u + v$, $X, Y = u^3, v^3$, we solve

$$X + Y = -q = 20, \quad XY = -\frac{p^3}{27} = -\frac{6^3}{27} = -8.$$

Eliminating Y, $X^2 - 20X - 8 = 0.$

Solving, leads to $X = 10 \pm \sqrt{108}, \qquad Y = 20 - X = 10 \mp \sqrt{108}.$

So, let $u = \sqrt[3]{10 + 6\sqrt{3}} \qquad v = \sqrt[3]{10 - 6\sqrt{3}}$ and

$$y = u + v = \sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}}$$

$$x = \sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}} + 5$$

Depressed Cubic $y^3 + 6y - 20 = 0$.

Note: $y = 2$ is a solution. Therefore, $y^3 + 6y - 20 = (y-2)(y^2 + \alpha y + \beta)$.

Method 1: Expand and match coefficients.

$y^3 + 6y - 20 = y^3 + (\alpha - 2)y^2 + (\beta - 2\alpha)y - 2\beta$

$\alpha - 2 = 0, \qquad \beta - 2\alpha = 6, \qquad 2\beta = 20 \quad \Rightarrow \quad \alpha = 2, \beta = 10.$

Method 2: Long division.

$$
\begin{array}{rl}
 & \quad\; y^2 + 2y + 10 \\
y - 2 \,\big) & y^3 \qquad + 6y - 20 \\
 & -y^3 + 2y^2 \\
 & \qquad 2y^2 + 6y \\
 & \quad -2y^2 + 4y \\
 & \qquad\qquad 10y - 20 \\
 & \qquad\quad -10y + 20 \\
 & \qquad\qquad\qquad 0
\end{array}
$$

Find other roots:

$$y^2 + 2y + 10 = 0$$

$$
\begin{aligned}
y &= \frac{-2 \pm \sqrt{4 - 40}}{2} \\
&= -1 \pm 3i.
\end{aligned}
$$

Nested Radicals

We have found that $y = \sqrt[3]{10+6\sqrt{3}} + \sqrt[3]{10-6\sqrt{3}}$. Can one simplify this?
We consider $\sqrt[3]{10 \pm 6\sqrt{3}} = \sqrt{a} \pm \sqrt{b}$. Note:

$$\begin{aligned}
(\sqrt{a} \pm \sqrt{b})^3 &= a\sqrt{a} \pm 3a\sqrt{b} + 3b\sqrt{a} \pm b\sqrt{b} \\
&= (a+3b)\sqrt{a} \pm (3a+b)\sqrt{b} \\
&= 10 \pm 6\sqrt{3}. \qquad (1)
\end{aligned}$$

From (1) $b = 3$, $a + 9 = 10$, $\quad 3a + 3 = 6$.

Then, $a = 1$ and $\sqrt[3]{10 \pm 6\sqrt{3}} = 1 \pm \sqrt{3}$.

So,

$$\begin{aligned}
y &= \sqrt[3]{10+6\sqrt{3}} + \sqrt[3]{10-6\sqrt{3}} \\
&= 1 + \sqrt{3} + 1 - \sqrt{3} \\
&= 2. \qquad (2)
\end{aligned}$$

Complex Solutions $y^2 + 2y + 10 = 0.$

We found $y^3 + 6y - 20 = (y-2)(y^2 + 2y + 10) = 0.$

we solved the quadratic: $y = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm 3\sqrt{-1}.$

- Cardano, complex numbers
  "as subtle as they are useless."
- Raphael Bombelli (1526-1572)
  First to take seriously.
- Ex: $x^3 = 15x + 4$
  $x = \sqrt[3]{2 + 11\sqrt{-11}} + \sqrt[3]{2 - 11\sqrt{-11}}$
- But, $x = 4$ is a solution!
- Complex numbers, $a + bi$, $i = \sqrt{-1}$.
- $x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}$

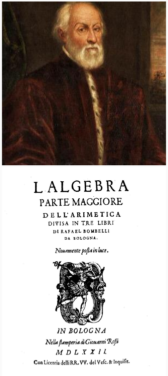


Figure 7: Bombelli

Cube Root of Complex Numbers

- Last Example: $x^3 = 15x + 4$
  $x = \sqrt[3]{2+11i} + \sqrt[3]{2-11i}$
- Seek: $\sqrt[3]{2+11i} = c + di$.

$$\begin{aligned} \sqrt[3]{2+11i} &= c + di \\ 2 + 11i &= (c+di)^3 \\ &= c^3 + 3c^2 di + 3c(di)^2 + (di)^3 \\ &= c^3 - 3cd^2 + i(3c^2 d - d^3). \end{aligned}$$

Then

$$\begin{aligned} 2 &= c^3 - 3cd^2 = c(c^2 - 3d^2), \\ 11 &= 3c^2 d - d^3 = d(3c^2 - d^2). \end{aligned}$$

- Bombelli: c, d, positive integers.

Since 2 is prime, $c = 1, 2$.

If $c = 1$, $2 = 1 - 3d^2$. No!

If $c = 2$, then $d = 1$.

$$\begin{aligned} 2 &= 8 - 6d^2 \\ 11 &= 12d - d^3 \end{aligned}$$

Then,

$$\begin{aligned} x &= \sqrt[3]{2+11i} + \sqrt[3]{2-11i} \\ &= (2+i) + (2-i) = 4. \end{aligned}$$

Françios Viète (1540-1603)

- Counselor to Henry III, IV, France
- French Wars of Religion 1562-1598
- Tutored Catherine de Parthenay (1554-1631), noblewoman, mathematician
- 1596 - Adriaan van Roomen
  "No French mathematician could solve the 45th degree polynomial."
  $x^{45} - 45x^{43} + 945x^{41} + \cdots - 3795x^3 + 45x = A.$
- Viète solved quickly:
  $2\sin(45\alpha) = A, \quad x = 2\sin\alpha.$
- Trig identity: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$
  Let $y = \cos\theta$. $4y^3 - 3y = c, |c| \leq 1$. $c = \cos 3\theta.$
  Solve for θ given c. Solution, $y = \cos\theta.$
- Use identities to rewrite $2\sin(45\alpha) = A$ in terms of $2\sin\alpha$.

Figure 8: Viète, Henry IV, and van Roomen.

Viète's Solution

Define the quantities

$$\begin{aligned} c &= 2\sin 45\theta, & y &= 2\sin 15\theta, \\ z &= 2\sin 5\theta, & x &= 2\sin\theta. \end{aligned} \tag{3}$$

Problem: Find x, given c.

Use the identities:

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha \tag{4}$$

$$\sin 5\alpha = 5\sin\alpha - 20\sin^3\alpha + 16\sin^5\alpha \tag{5}$$

Then,

$$c = 2\sin 45\theta = 6\sin 15\theta - 8\sin^3 15\theta = 3y - y^3. \tag{6}$$

$$y = 2\sin 15\theta = 6\sin 5\theta - 8\sin^3 5\theta = 3z - z^3. \tag{7}$$

$$z = 2\sin 5\theta = 10\sin\theta - 40\sin^3\theta + 32\sin^5\theta = 5x - 5x^3 + x^5. \tag{8}$$

Viète's Solution (cont'd)

Since $z = 5x - 5x^3 + x^5$., we write c in terms of x :

$$\begin{aligned}
y &= 3z - z^3 \\
&= 3[5x - 5x^3 + x^5] - [5x - 5x^3 + x^5]^3 \\
&= -x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x. \qquad (9) \\
c &= 3y - y^3 \\
&= 3[-x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x] \\
&\quad -[-x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x]^3 \\
&= x^{45} - 45x^{43} + 945x^{41} - 12300x^{39} + 111150x^{37} - 740259x^{35} + 3764565x^{33} \\
&\quad -14945040x^{31} + 46955700x^{29} - 117679100x^{27} + 236030652x^{25} - 378658800x^{23} \\
&\quad +483841800x^{21} - 488494125x^{19} + 384942375x^{17} - 232676280x^{15} + 105306075x^{13} \\
&\quad -34512075x^{11} + 7811375x^9 - 1138500x^7 + 95634x^5 - 3795x^3 + 45x \\
&= P_{45}(x). \qquad (10)
\end{aligned}$$

Math Symbols

- $+$ Oresme (1300)
- $-$ Widman (1400)
- $=$ Recorde (1500)
- $\times$ Outred (1500)
- Letters Viète (1500)
- Descartes (1500-1600)
  unknowns, constants a, b, c
  variables x, y, z
- $<>$ Harriot (1600)
- ∞ Wallis (1700)
- imaginary, Descartes
- $x^{3/2}, x^{-1}$, Newton 1600
- $x^2 \rightarrow xx$, Gauss 1800
- π, i, Σ. Euler 1700
- $f(x)$
- $\frac{df}{dx}, \int$ Leibniz 1600

Analytic Geometry

- Fermat (1601-1665)
- Descartes (1596-1650)
- Newton (1642-1727)

Coordinates

- Hipparchus - sky
- Apollonius - conics
- Oresme (1300s) - position, velocity plots
- Fermat-Descartes described curves in coordinate systems
- Degree 1, Linear relations

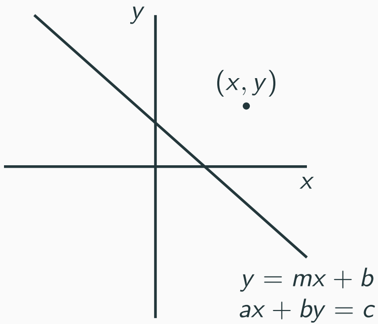


Figure 9: Cartesian system.

Curves of Degree 2 - Quadratics

$$ax^2 + \underbrace{bxy}_{rotation} + cy^2 + \underbrace{dx + ey}_{translation} + f = 0$$

- Describes Conics
- $b \neq 0$, rotation
- $d \neq 0$ or $e \neq 0$, translation
- Classification

$$D = \begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{vmatrix}$$

$D > 0$ ellipse
$D < 0$ hyperbola
$D = 0$ parabola

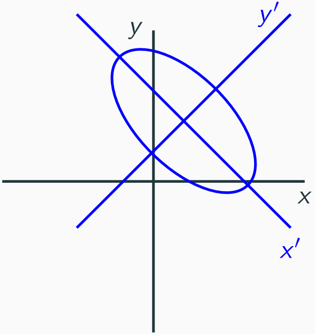


Figure 10: Rotated, translated ellipse.

Curves of Degree 3: $ax^3 + bx^2y + cxy^2 + dy^3 + \cdots = 0$, Cubics

- Newton classified cubic curves, 1710, 72 types (missed 6)
- $y = x^3$ and other types.
- Descarte's folium (leaf) $x^3 + y^3 = 3axy$
- Parametric Solutions

$$x = \frac{3at}{1+t^3}, \quad y = \frac{3at^2}{1+t^3}.$$

- Rational Points
  Ex. $x^3 + y^3 = 1$.
  Let $x = \frac{n}{p}$, $y = \frac{m}{p}$. $n^3 + m^3 = p^3$.
- Fermat - only trivial $(0,1), (1,0)$.
- Fermat's Last Theorem, 1637

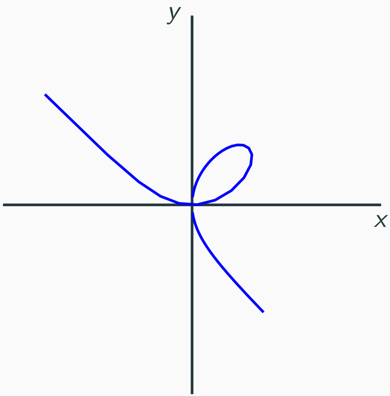


Figure 11: Descarte's Folium.

Fermat's and Bezout's Theorems

- Fermat's Last Theorem, 1637
  $x^n + y^n = z^n$
  Wiles proved in 1995.
- Bezout's Theorem. Let

$$p(x,y) = 0, \qquad \text{degree } n.$$
$$q(x,y) = 0, \qquad \text{degree } m.$$

  Then, p and q intersect in nm points.
- Elimination gives eq of degree nm.
- Need complex numbers,
  point at infinity.
  Next - Projective Geometry.

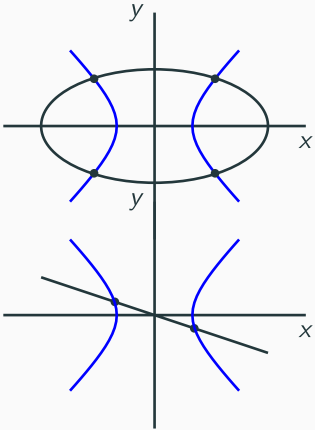


Figure 12: Intersecting Degree 2 curve (blue) with Degree 2 or 1 (black).