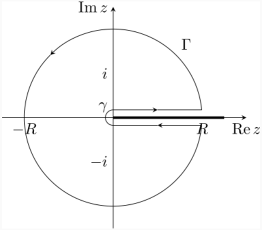
Complex Analysis

Fall 2025 - R. L. Herman



History of Complex Analysis

- Before 1600
  - Cardano 1545, quadratic
  - Bombelli 1572, cubic
  - Harriot 1600, quartic
  - Negative roots - false
  - Complex roots - impossible
- 1600s
  - Descartes, 1637, $a + b\sqrt{-1}$
  - Wallis 1685
  - Insights from geometry trigonometry, conics - justified
- 1700s
  - Bernoulli - integral transformation
  - Euler - Euler's formula, i
  - Gauss (1799, 1815) FTA, quadratic forms
  - Wessel (1797), Argand (1806) Geometric Visualization
  - Cauchy (1814, 1825) Complex Analysis
  - Riemann (1826-1866) Surfaces

Complex Numbers, $\mathbb{C}$

- $a + bi \in \mathbb{C}$, $a, b \in \mathbb{R}$, $i = \sqrt{-1}$.
- Quadratic Equation,
  $ax^2 + bx + c = 0$,
  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
  If $b^2 - 4ac < 0$,
  complex conjugate roots.
- Cubics - Role was clearer
  $y^3 = py + q$
  $y = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} +$
  $\sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}.$

Example: $x^3 = 15x + 4$

$$\begin{aligned} x &= \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} \\ &= 2 + i + 2 - i = 4. \end{aligned}$$

Bombelli (1572)

$$\begin{aligned} (2 + i)^3 &= (2 + i)(4 + 4i + i^2) \\ &= (2 + i)(3 + 4i) \\ &= 2 + 11i. \end{aligned}$$

Bernoulli's Transformations

- Johann Bernoulli (1712)

$$
\begin{aligned}
\frac{1}{1+z^2} &= \frac{1}{(1+iz)(1-iz)} \\
&= \frac{1}{2}\left(\frac{1}{1-iz}+\frac{1}{1+iz}\right) \\
\int \frac{dz}{1+z^2} &= \frac{1}{2}\int \left(\frac{1}{1-iz}+\frac{1}{1+iz}\right)
\end{aligned}
$$

- Note:

$$\int \frac{dz}{a+bz} = \frac{1}{b}\ln(a+bz).$$

So, integrating $(1+t^2)^{-1}$ gives

$$\tan^{-1} z = \frac{1}{2i}\left[\ln(1+iz) - \ln(1-iz)\right].$$

Examples: Tangent Identities

- Bernoulli studied $y = \tan n\theta$ in terms of $x = \tan\theta$.
- *Example:* $n = 2$
  $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$.
- Let $y = \tan n\theta$, Then,
  $n\theta = \tan^{-1} y, \theta = \tan^{-1} x$.

$$\begin{aligned} \int \frac{dy}{1+y^2} &= n\int \frac{dx}{1+x^2} \\ \ln\frac{y+i}{y-i} &= n\ln\frac{x+i}{x-i} \\ \frac{y+i}{y-i} &= A\left(\frac{x+i}{x-i}\right)^n \end{aligned}$$

$A = (-1)^{n+1}$. Solve for y.

Ex: $n = 2$:

$$\tan 2\theta = \frac{2x}{1-x^2}.$$

Ex: $n = 3$:

$$\tan 3\theta = \frac{x^3-3x}{3x^2-1}.$$

Ex: $n = 4$:

$$\tan 4\theta = \frac{4x-4x^3}{x^4-6x^2+1}.$$

Ex: $n = 5$:

$$\tan 5\theta = \frac{x^5-10x^3+5x}{5x^4-10x^2+1}.$$

The Fundamental Theorem of Algebra

- Integration of $\frac{p(x)}{q(x)}$ for $p(x), q(x)$ polynomials
- Need Integration by parts. Assumes $q(x)$ can be factored
  - Fundamental Theorem of Algebra (FTA)
- Albert Girard (1629), *L'invention en algèbre*,
  First to claim there are always n roots of degree n polynomial.
- By 1750 - Any polynomial with real coefficients can be factored into real linear and quadratic factors.
- Nicolas II Bernoulli (1687-1759) gave a counterexample:
  $p(x) = x^4 - 4x^3 + 2x^2 + 4x + 4.$
- Euler found the factors:

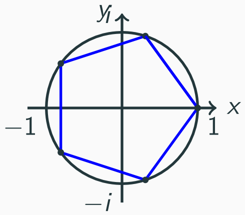
$$x^2 - \left(2 \pm \sqrt{4 + 2\sqrt{7}}\right) x + \left(1 \pm \sqrt{4 + 2\sqrt{7}} + \sqrt{7}\right)$$

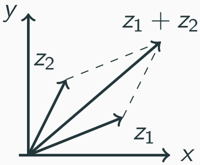
He gave incorrect proof for any quartic.
His was followed by proofs from d'Alembert and Gauss.

Roots of Unity

- Cotes, de Moivre, Euler
  - $x^n - 1 = 0$. Seems $x = \sqrt[n]{1}$.
  - $x = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$, $k = 0, 1, \ldots, n-1$. **Roots of unity.**
  - Geometric Interpretation
- Caspar Wessel, surveyor.
  - Complex number = point in the complex plane, 1797.
  - Also, proposed vectors.
- Argand, 1806, visual representation, operational (translation, rotation, reflection)
- Gauss also rediscovered, 1831.

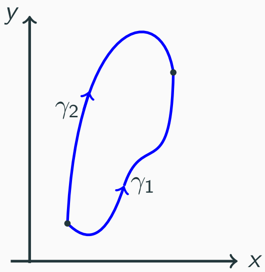
$e^{2k\pi i/5}, k = 0, 1, \ldots, 4$





Representing Complex Numbers

- Gauss (1777-1855) adopted "complex number," used i.
- Integration in $\mathbb{C}$-plane.
- $\int_\gamma \phi(z)\,dz$ is path independent for "nice" $\phi(z)$.
- Cauchy proved later, in 1814 talk, published 1827. - Now called *Cauchy's Theorem.*



Path Independence

$$\int_{\gamma_1} \phi(z)\,dz = \int_{\gamma_2} \phi(z)\,dz$$

Equivalently, for a simple, closed loop Γ, $\int_\Gamma \phi(z)\,dz = 0$.

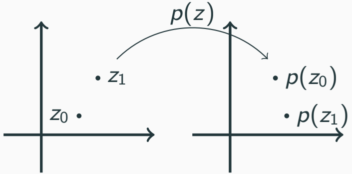
Augustin Louis Cauchy (1789-1857)

- Father of Complex Analysis
- Does $f(x) \to f(z)$ make sense?
- Integration along paths (1814) pub 1827.
- Cauchy's Theorem, Cauchy-Riemann Equations.
- Calculus of Residues (1826) - dealing with singularities.
- Convergence of infinite series.
- Use complex integration to integrate real functions.
- Path Independence (1825).
- Complex function of complex variable (1828).

Fundamental Theorem of Algebra I

Every polynomial $p(x)$ can be written as a product of linear complex factors. (Contains 1750 version)

- d'Alembert (1717-1783)
- **Lemma** $p(z_0) \neq 0$, $p(z) \neq$ constant. There exists a $z_1 = z_0 + w$ such that $|p(z_1)| < |p(z_0)|$ where $|a + bi| = \sqrt{a^2 + b^2}$.



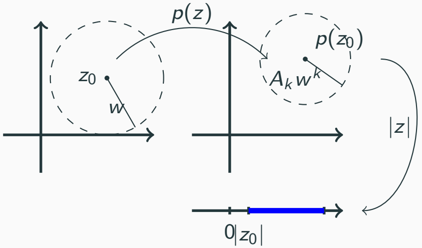
Proof

$$
\begin{aligned}
p(z) &= a_0 z^n + a_1 z^{n-1} + \cdots + a_n. \\
p(z_0 + w) &= a_0 z_0^n + a_1 z_0^{n-1} + \cdots + a_n + A_1 w + A_2 w^2 + \cdots + A_n w^n.
\end{aligned}
$$

Fundamental Theorem of Algebra II

$$p(z_0 + w) \quad = \quad a_0 z_0^n + a_1 z_0^{n-1} + \cdots + a_n + A_k w^k + \epsilon$$

Here $A_k w^k$ is the first nonzero, lowest power of w term and ϵ contains the higher powers terms in w and is small for large $|z|$.

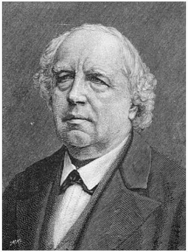


$\exists w$ such that $p(z_0) + A_k w^k$ is closer to the origin.

Let $p(z) \neq 0$. By the lemma, $\exists$ a point closer than z_0 to the origin. $\therefore$ there exists a zero of $p(z)$.

Fundamental Theorem of Algebra III

- Gauss attempted several proofs.
- Karl Weierstrauss (1815-1897) - continuous functions on closed, bounded regions which assume maximum and minimum values.
- Gauss (1799 Thesis) considered curves $\mathrm{Re}(p(z)) = 0$, $\mathrm{Im}(p(z)) = 0$, $z = x + iy$.
- For $|z|$ large, $\mathrm{Re}(a_0 z^n) = 0$, $\mathrm{Im}(a_0 z^n) = 0$, curves are asymptotic to lines through the origin.
- Curves $\mathrm{Re}(p(z)) = 0$, $\mathrm{Im}(p(z)) = 0$, entering $|z| = r$ must come out and intersect inside disk. [See examples.]



Examples

Plotting $Re(p(z)$ and $Im(p(z))$, outside a large circle one gets alternating lines. Inside the circle they must intersect for $p(z) = 0$.

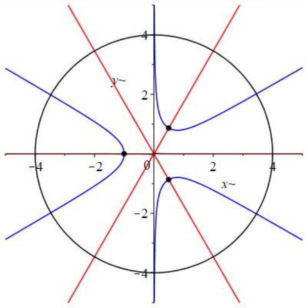


Figure 1: $p(z) = z^3 + 1$.

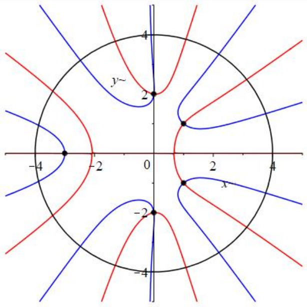


Figure 2:
$p(z) = z^5 + z^4 + 10z^2 - 16z + 24 = (z - 1 - i)(z - 1 + i)(z^2 + 4)(z + 3)$.

Theory of Curves, $p(x, y) = 0$

- Descartes - linear/lines
  - quadratic/conics
- Newton - cubics
- Recall Bezout's Intersection Thm
  - ◁ Count multiplicities.
  - ◁ Intersection with ∞.
- 19th Century
  - Projective Geometry
    homogeneus coordinates
    Möbius, Plücker - 1830
  - Complex Numbers
    Gauss - FTA
  - Topological ideas
    - Riemann surfaces, 1850's

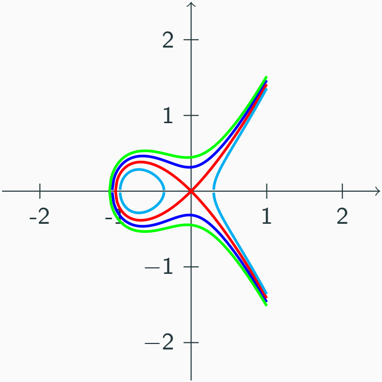
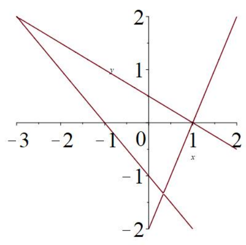


Figure 3: Cubic curves of form $y^2 = x^3 + x^2 + bx + 2b$

Cubic Curves

Consider products of linear factors or lines
$p(x, y) = (a_1x + b_1y + c_1)(a_2x + b_2y + c_2)(a_3x + b_3y + c_3)$

- Ex: $p(x, y) = (x + y + 1)(x + 2y - 1)(-2x + y + 2)$



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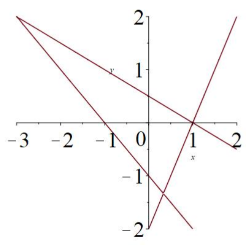

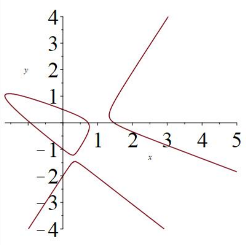

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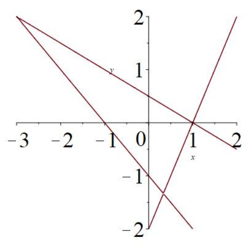
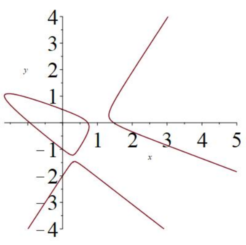
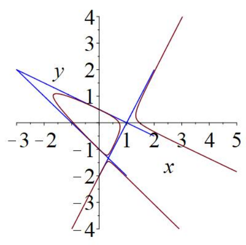
- Branches go to **points at infinity**. Consider projective geometry.

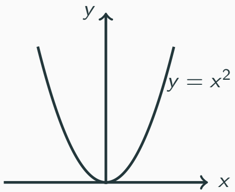


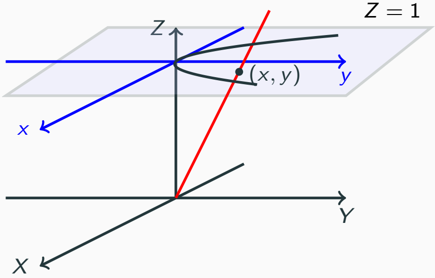




Projective Geometry

- Homogeneous coordinates:
  $x = \frac{X}{Z}, y = \frac{Y}{Z}$.
- Introduced by Möbius, Pücker.
- **Example:** $y = x^2$ gives $X^2 = YZ$





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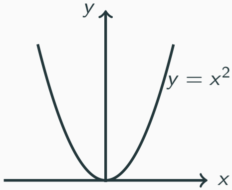
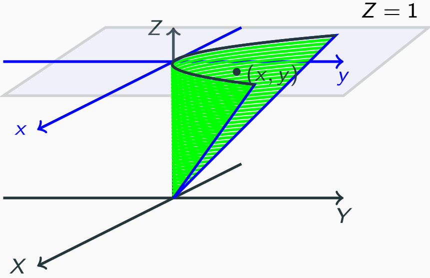
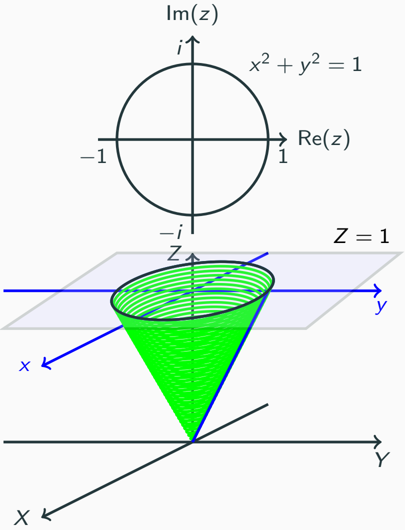
- **Example:** $y = x^2$ gives $X^2 = YZ$
- Lines thru origin (projective plane).
- $X^2 = YZ$ is a "cone"
- Points at Infinity: $Z = 0 \Rightarrow X = 0$,
- These points, $[0, Y, 0]$, lie on horizon.





Projective Plane and Complex Numbers

- **Example:** $x^2 + y^2 = 1$
- Projective curve: $X^2 + Y^2 = Z^2$
- Pts at infinity,
  $Z = 0 \Rightarrow X^2 + Y^2 = 0.$
- In $\mathbb{C}$, Circular pts at infinity.
  $X = 1, Y = i : I_1 = (1, i, 0)$
  $X = 1, Y = -i : I_2 = (1, -i, 0)$



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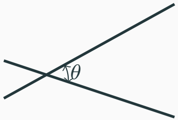
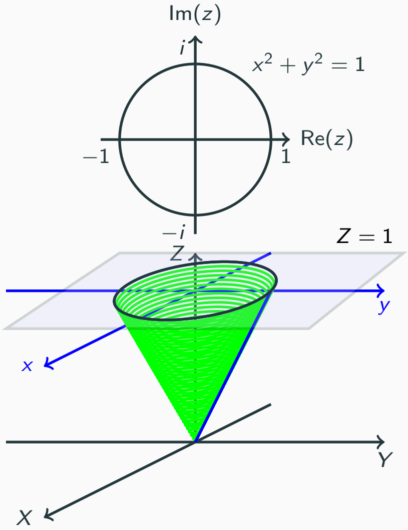
- Edmund Laguerre (1834-1886)
  - Angles, $\theta = i \log R$.
- R - Cross ratio





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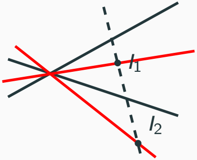
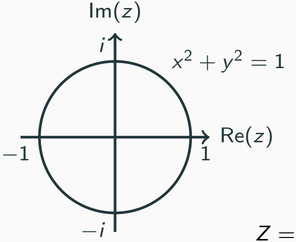
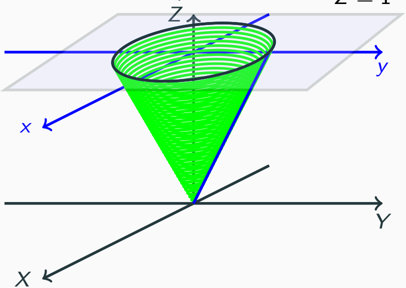


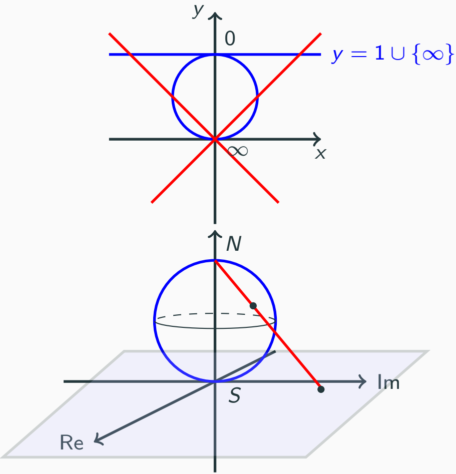




Stereographic Projection

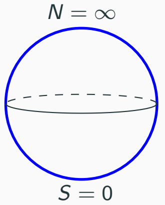
What do complex curves look like?

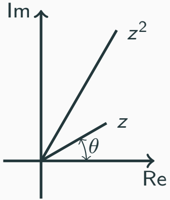
- Projective lines:
  Lines thru origin
  Topologically looks like a circle,
  S^1, after adding point at infinity
- Extend to $\mathbb{C}$ - topologically, S^2
- Stereographic Projection
  Connect pts in $\mathbb{C}$ to North Pole.
- N mapped to pt at ∞.
- Möbius (1790-1868) Image of
  circle = circle.



Mapping Functions onto Surfaces

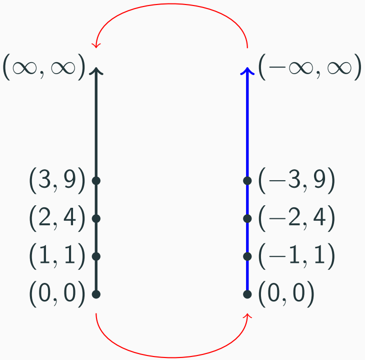
- Riemann (1826-1866)
    - Riemannian manifolds
    - Curved spaces - Gauss
    - Complex Analysis - Riemann surfaces [Cauchy (1788-1857)]
    - Number theory - $\zeta(s)$
- Start with a Sphere
- Extend $f : \mathbb{C} \to \mathbb{C}$ to $g : S^2 \to S^2$.
- Complex function, $f(z) = z^2$
  Let $z = re^{i\theta}$.
  [$\theta =$ argument, $r =$ modulus, $|z|$.]
  Then, $f(z) = r^2 e^{2i\theta}$.

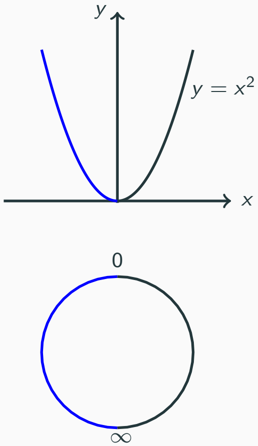




Real Function, $y = f(x)$, Mapped to S^1

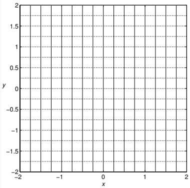
- **Example** $f(x) = x^2$.

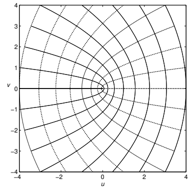


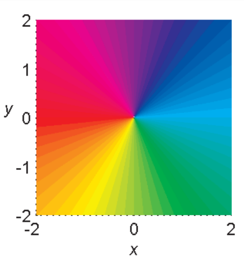


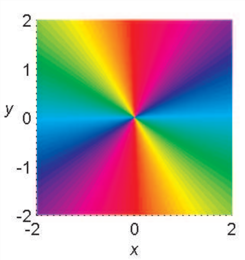
Visualizing Complex Functions: $w = f(z) = u(x, y) + iv(x, y)$

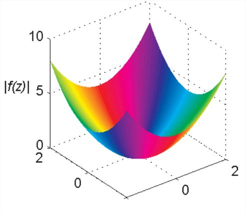
- What is $f(z) = z^2$?
- Map xy-plane to uv-plane.
- $(x + iy)^2 = x^2 - y^2 + 2ixy$.
- $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$
- Domain Coloring





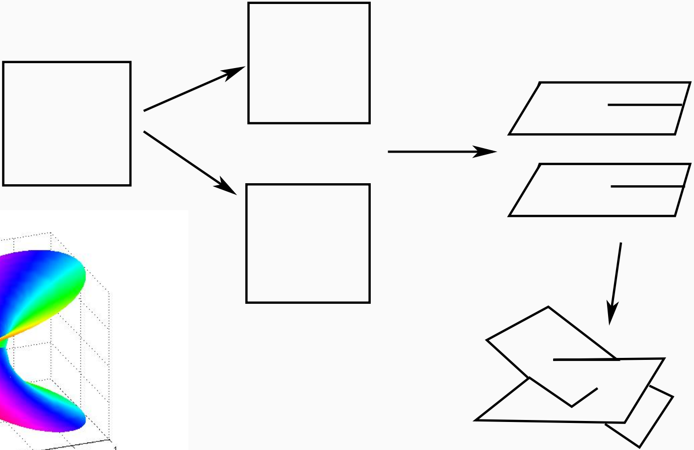


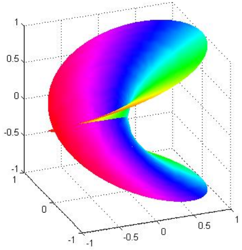




Riemann Surfaces and the Square Root Function

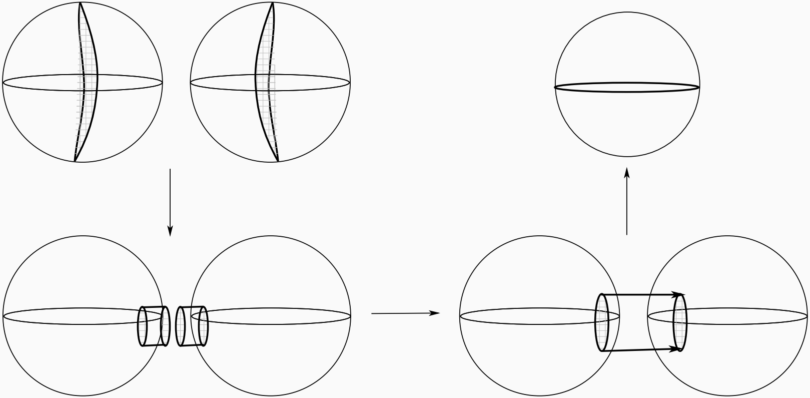
- Riemann Sheets - Two copies of $\mathbb{C}$. Riemann's Dissertation, 1851.
- **Example:** $w = \sqrt{z}$.





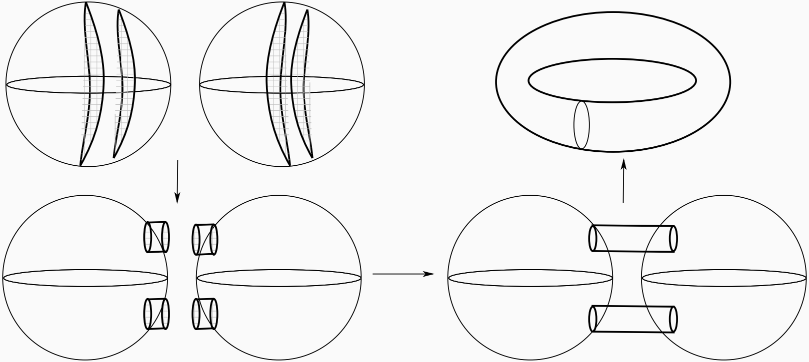
Mapping $f(z) = z^2$ to S^2.

- **Example:** $f(z) = z^2$.

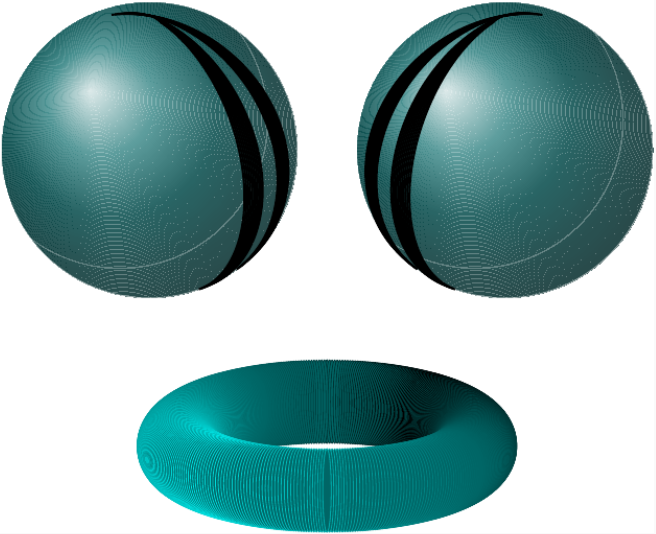


Riemann Surfaces and Elliptic Integrals

- **Example:** $\int_0^x \frac{dz}{\sqrt{z(z-a)(z-b)(z-c)}}$.
- $w^2 = z(z-a)(z-b)(z-c)$
- Beginning of topology.

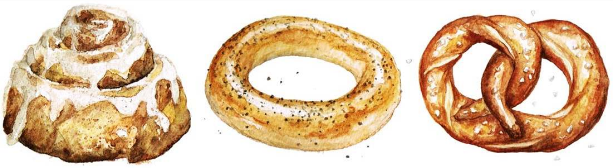


Merging Two Cut Riemann Spheres



Beginnings of Topology ...

Figure 4: Genus $g = 1, 2, 3$.



Beginnings of Topology ...

Figure 4: Genus $g = 1, 2, 3$.