

Projective Geometry

Fall 2021 - R. L. Herman



Perspective Drawing

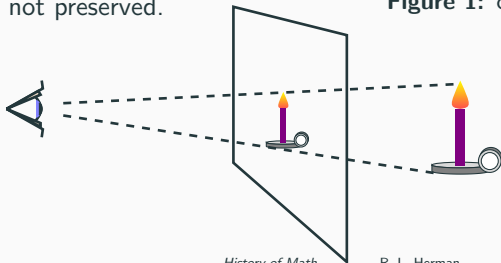
- Art - Perspective Drawing
- Before Renaissance- no illusion of depth and space.
- 13th century Italian masters used shadowing.
- Mathematics of perspective

Lengths not preserved.

Angles not preserved.

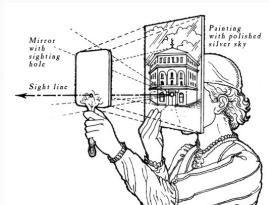
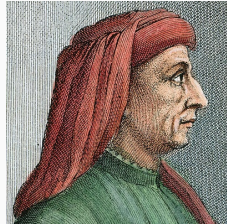


Figure 1: c.1308-1311



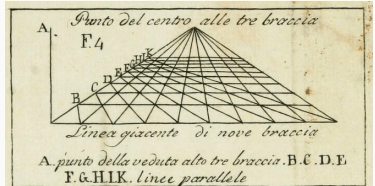
Filippo Brunelleschi (1377-1446)

- Architect
- First to describe linear perspective.
- Experimented (1415-1420) using a panel with a grid of squares and a plaque with a hole at eye level.
- Drawings of the Baptistry in Florence, Place San Giovanni and other Florence landmarks.
- Later his method was studied by Alberti and Da Vinci.



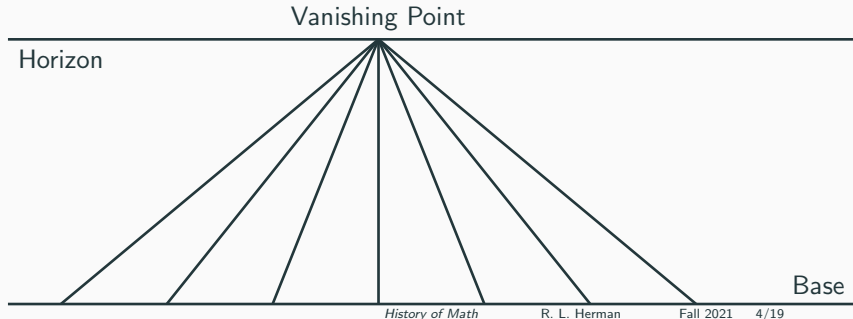
Leon Battista Alberti (1404-1472)

- Alberti's Veil
Transparent cloth on a frame,
Good for actual scenes not
imaginary ones.
- Basic principles:
 1. A straight line in perspective
remains straight.
 2. Parallel lines either remain
parallel or converge to a point.
- Drawing a square-tiled floor, solved
by Alberti (1436).



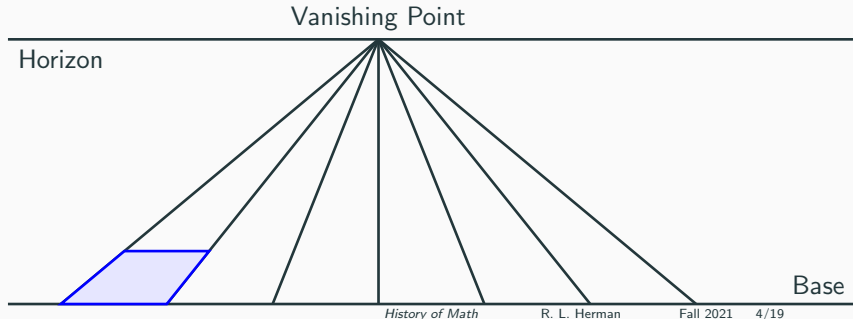
Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.



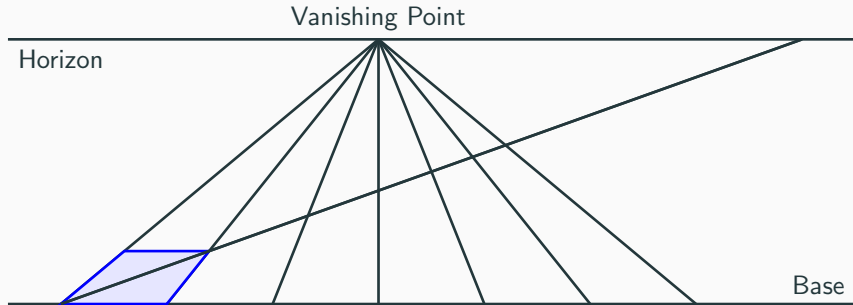
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- Align nonhorizontal lines equally along base, converging to one point on the horizon.
- Choose one tile.



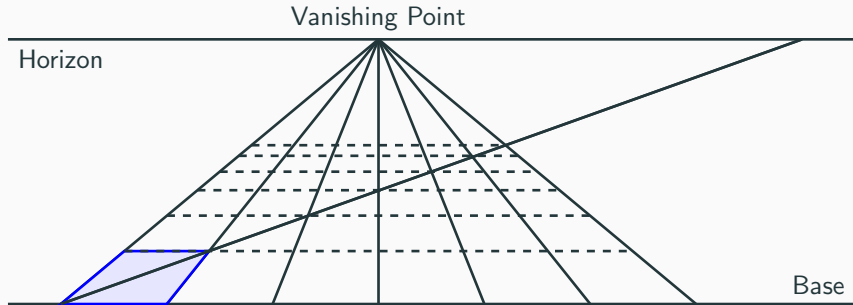
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- Align nonhorizontal lines equally along base, converging to one point on the horizon.
- Choose one tile.
- Extend diagonal.



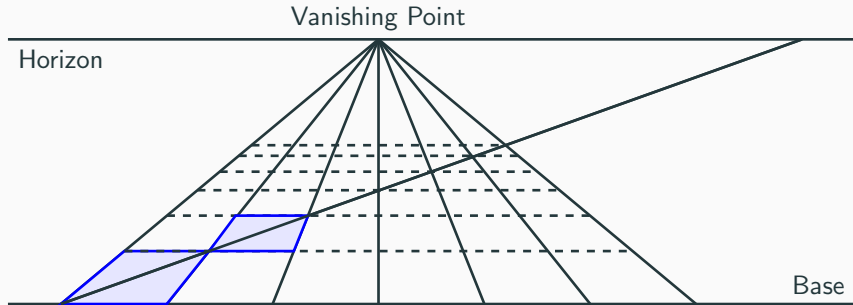
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- Intersections determine the horizontals.



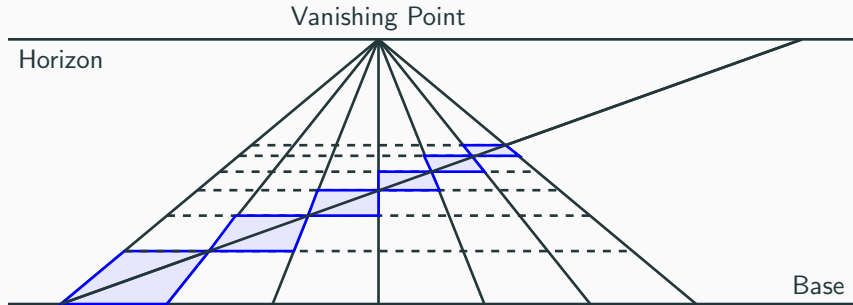
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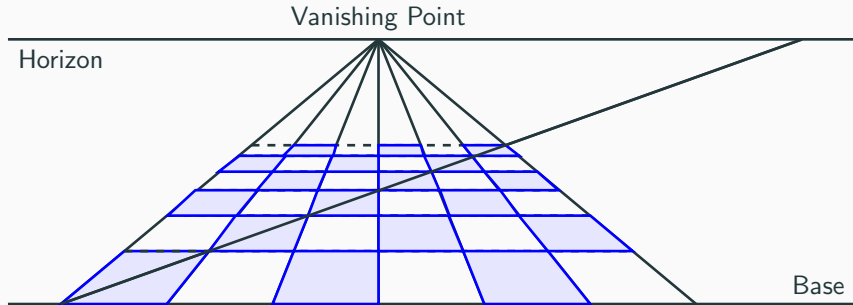
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Desargues' Projective Geometry¹

- Mathematics behind Alberti's Veil: Family of lines (light rays) through a point (eye) plus a plane (veil).
- Recall **Pappus' Theorem**: A_1, A_2, A_3 , collinear; B_1, B_2, B_3 , collinear; then, so are C_1, C_2, C_3 .
- Blaise Pascal (1623-1662) at 16 generalized to conics.
- Desargues (1640) **Projective Geometry** only relies on a straight edge.

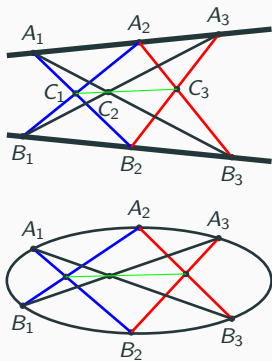


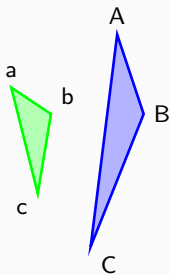
Figure 2: Pappus' and Pascal's Theorems.

¹Two centuries ahead of his time.

Girard Desargues (1591-1661)

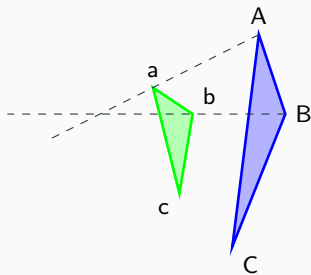
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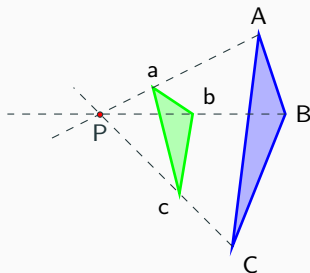
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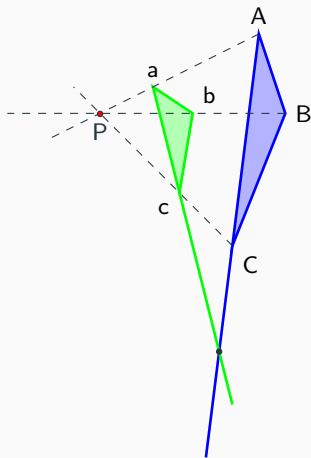


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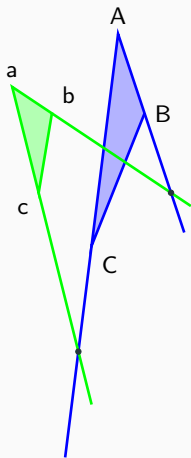
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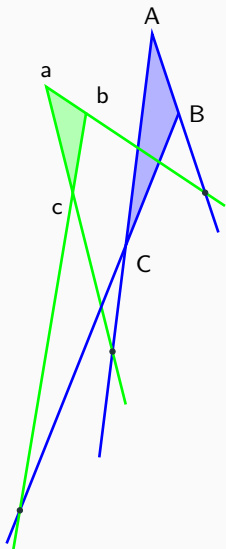


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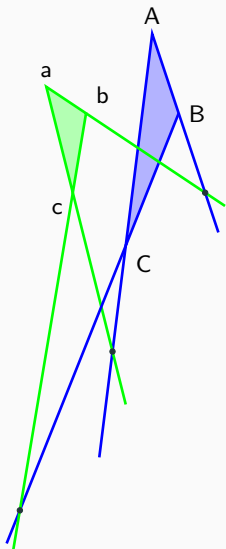


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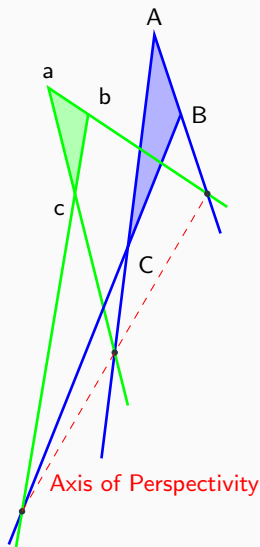


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- Points are collinear.

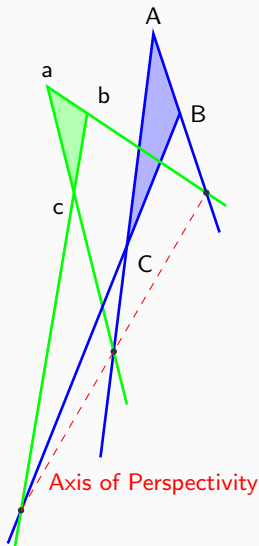


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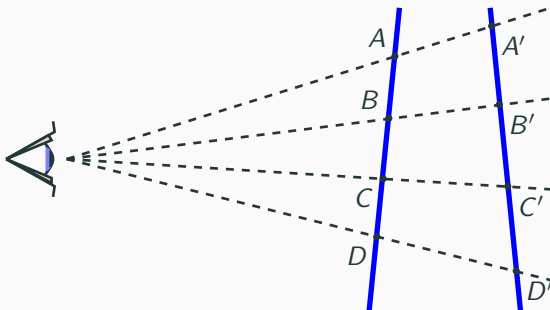
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- Points are collinear.
- What if two sides are parallel?
- Need **Projective plane**.



Invariance of the Cross Ratio

Lengths and angles are not preserved.



For any four points on a line, $\frac{\overline{AC}}{\overline{BC}} : \frac{\overline{AD}}{\overline{BD}}$ is invariant, or

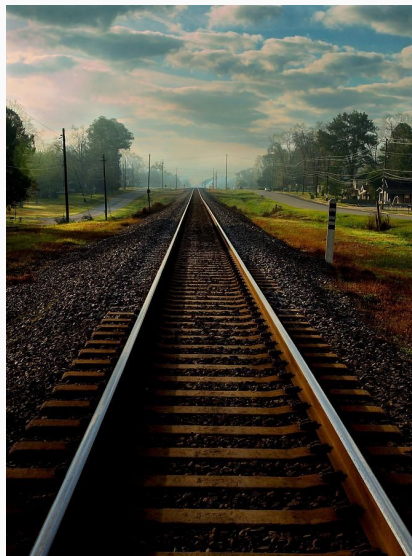
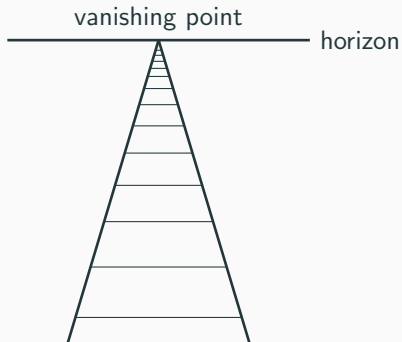
$$\frac{\overline{AC}}{\overline{BC}} : \frac{\overline{AD}}{\overline{BD}} = \frac{\overline{A'C'}}{\overline{B'C'}} : \frac{\overline{A'D'}}{\overline{B'D'}}.$$

Projective Geometry Rebirth in 1800's.

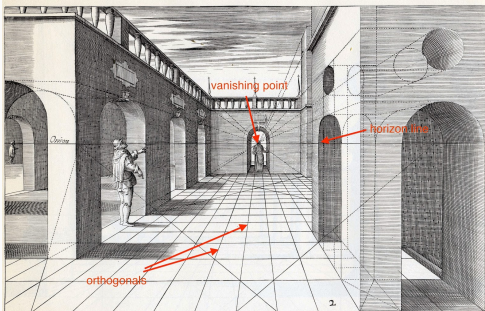
Perspective

1. Parallel lines meet at a pt.
2. Lines map to lines.
3. Conics map to conics.

Example: Train tracks.



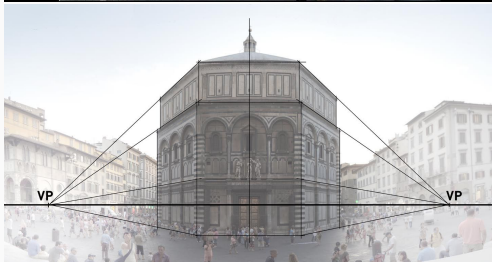
One Point Perspective - Find the Vanishing Points



Two Point Perspective - Find the Vanishing Points



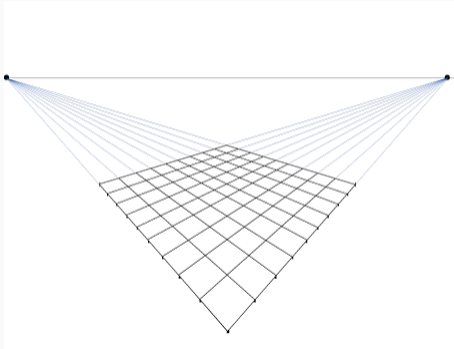
Two Point Perspective - Find the Vanishing Points



Two Point Perspective Vanishing Points

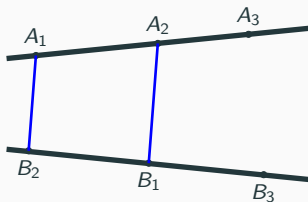


Two Point Perspective Vanishing Points



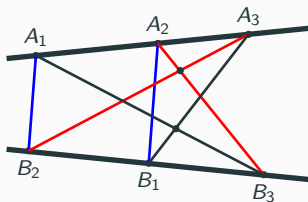
Points at Infinity

- Artists' use vanishing points.
- Pappus' Theorem -
Consider parallel lines A_1B_2 , A_2B_1 .
Does the theorem hold?



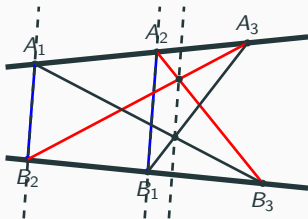
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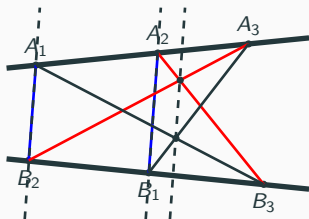
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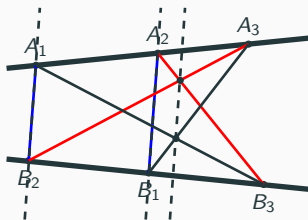
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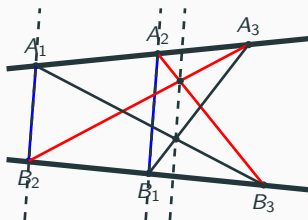
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- Look at a plane



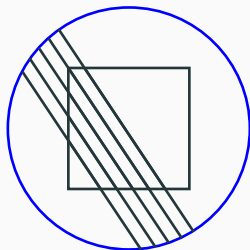
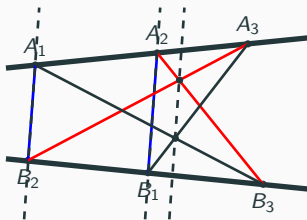
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- Look at a plane
- Add parallel lines.
Where do they go?
- Line at Infinity
- Plane + line at infinity =
Projective Plane



Line at infinity

Projective Line

- Consider the real line, \mathbb{R} .



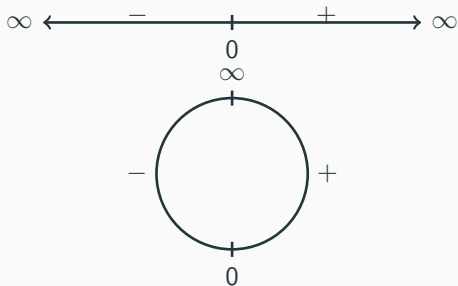
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- Consider the real line, \mathbb{R} .
- Add point at infinity, real projective line, \mathbb{RP}^1 .



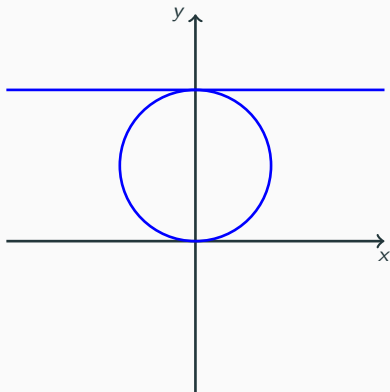
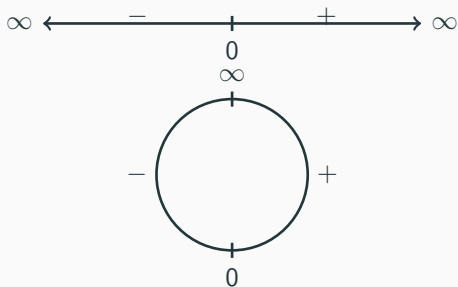
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- Topologically a circle!



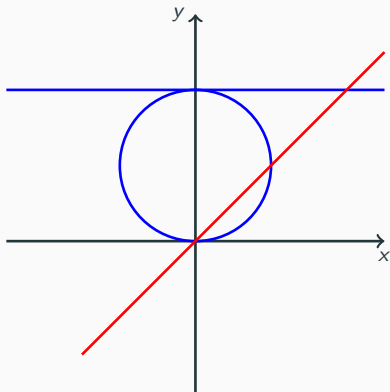
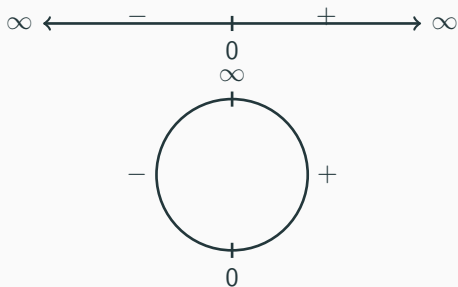
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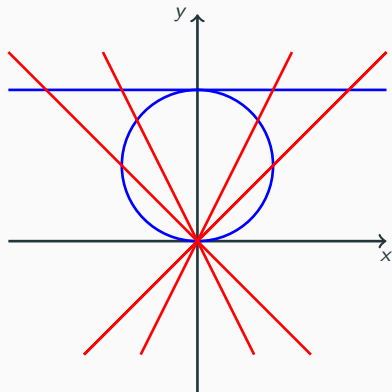
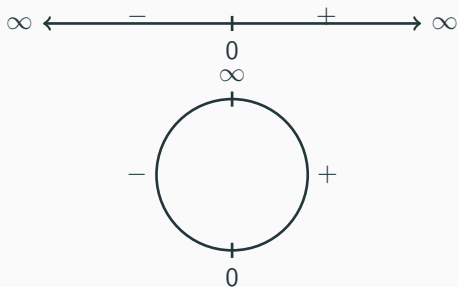
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Intersection: $y = b, y = mx :$
 $x = \frac{b}{m}.$

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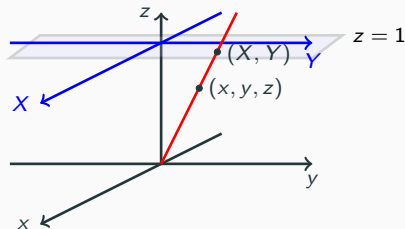


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Homogeneous Coordinates

- Point on line: (x, y, z)
- All points on line map to (X, Y) in the plane.
- (X, Y) are called homogeneous coordinates.
- Points on line are multiples, $(x', y', z') = \lambda(x, y, z)$.
- Point on plane: $(\frac{x}{z}, \frac{y}{z}, 1)$, or

$$X = \frac{x}{z}, \quad Y = \frac{y}{z}.$$



Curves

- Curve in plane, $Y = X^2$.
- Translates to

$$\frac{y}{z} = \left(\frac{x}{z}\right)^2.$$

- This is a surface in (x, y, z) -space,

$$x^2 = yz.$$

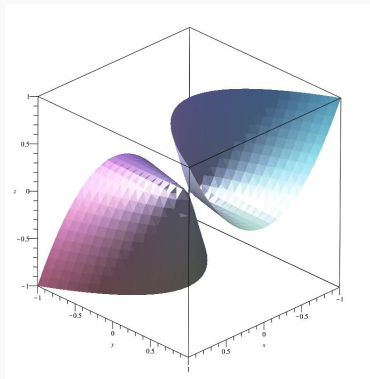


Figure 3: Surface $x^2 = yz$.

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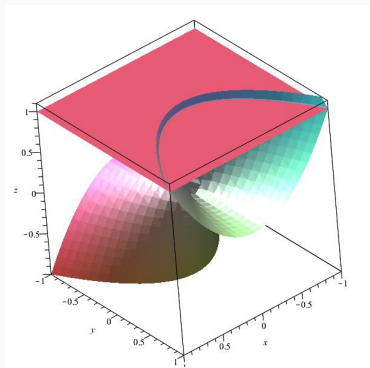


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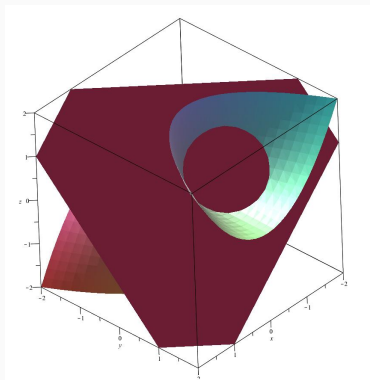


Figure 3: Surface $x^2 = yz$.

Projective Sphere

- Map points on a plane to points on surface of unit sphere, \mathbb{S}^2 .
- Lines through South Pole uniquely intersect the plane and sphere.
- All points mapped except $(0, 0, 0)$. This point can be mapped to the line at infinity.
- Lines through origin are points of the real projective plane, \mathbb{RP}^2 .

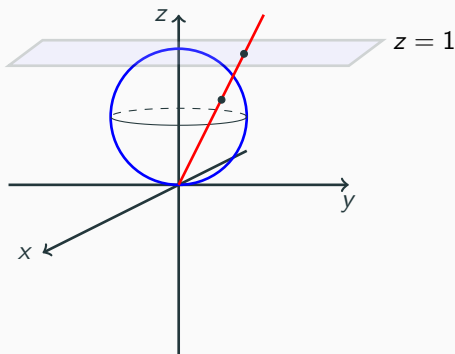


Figure 4: Stereographic Projection

Looking into the Veil - Parabola Projected

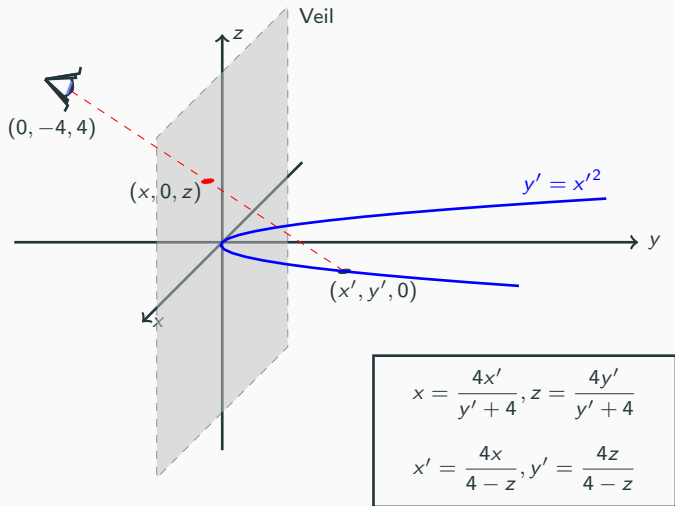


Figure 5: Problems 8.4.2-8.4.4

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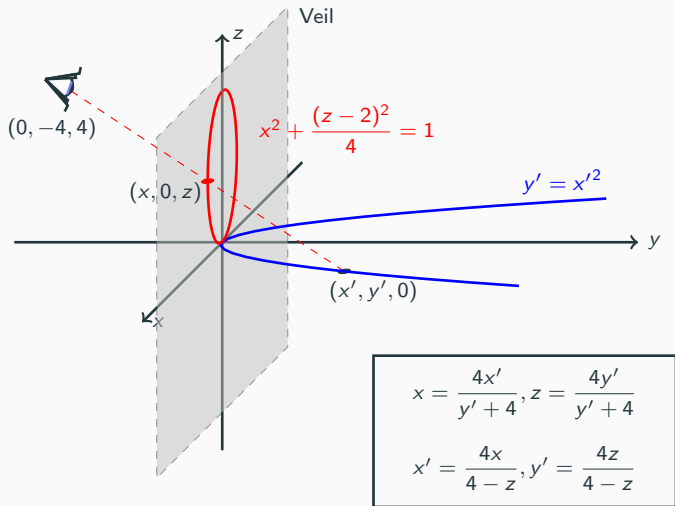
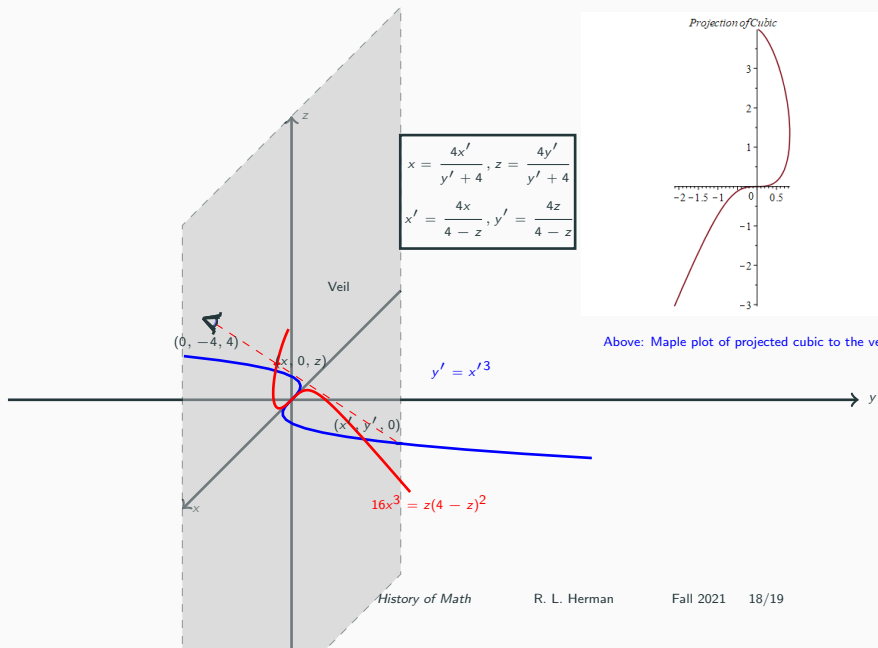


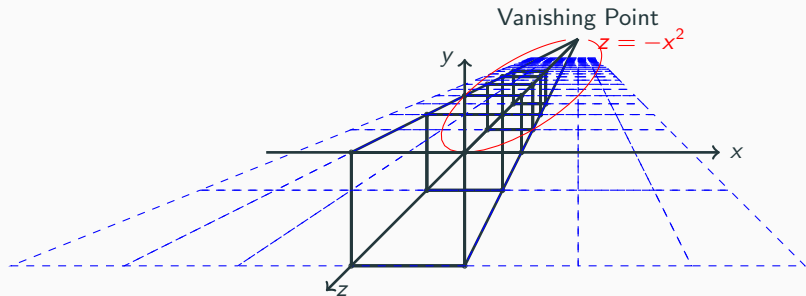
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Viewing A Cubic in the Veil



Perspective Drawing

Looking at conics from a different perspective: The parabola $z = -x^2$ looks like an ellipse.



In the 1600's mathematicians had other mathematics to attend to. So, we return to geometry in the 1800's.