Projective Geometry

Fall 2021 - R. L. Herman

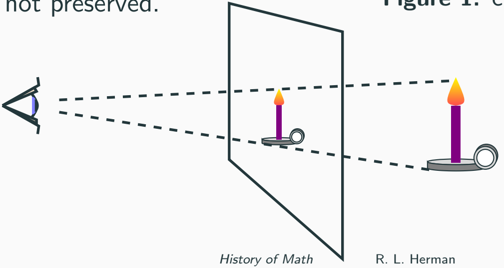
Perspective Drawing

- Art - Perspective Drawing
- Before Renaissance- no illusion of depth and space.
- 13th century Italian masters used shadowing.
- Mathematics of perspective

  Lengths not preserved.

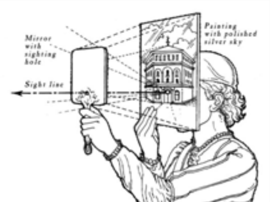
  Angles not preserved.

Figure 1: c.1308-1311



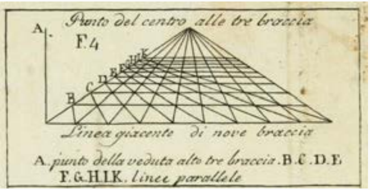
Filippo Brunelleschi (1377-1446)

- Architect
- First to describe linear perspective.
- Experimented (1415-1420) using a panel with a grid of squares and a plaque with a hole at eye level.
- Drawings of the Baptistry in Florence, Place San Giovanni and other Florence landmarks.
- Later his method was studied by Alberti and Da Vinci.



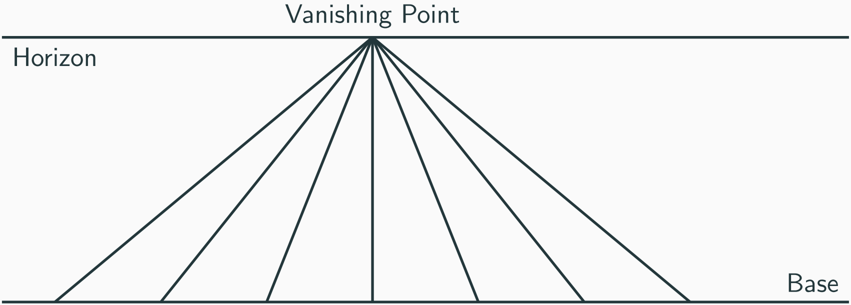
Leon Battista Alberti (1404-1472)

- Alberti's Veil
  Transparent cloth on a frame,
  Good for actual scenes not imaginary ones.
- Basic principles:
  1. A straight line in perspective remains straight.
  2. Parallel lines either remain parallel or converge to a point.
- Drawing a square-tiled floor, solved by Alberti (1436).



Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.



Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.
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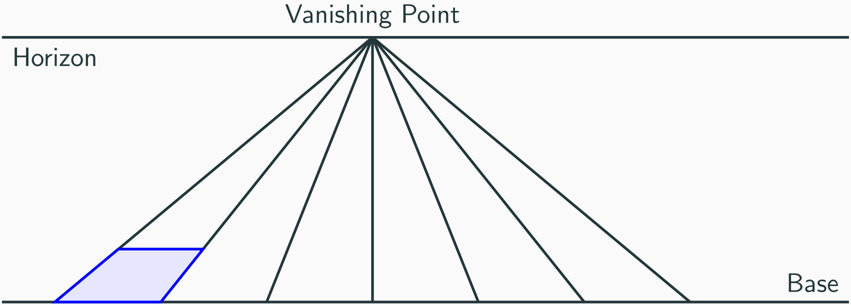

Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.
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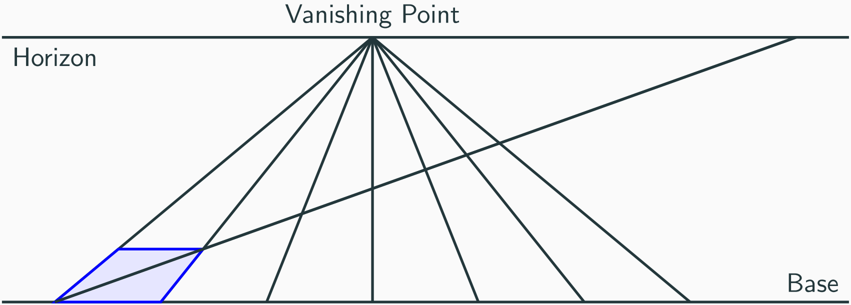
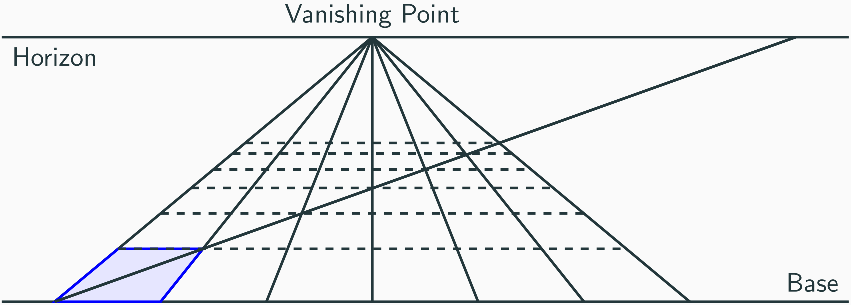
- Extend diagonal.



Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.
- Choose one tile.
- Extend diagonal.
- Intersections determine the horizontals.



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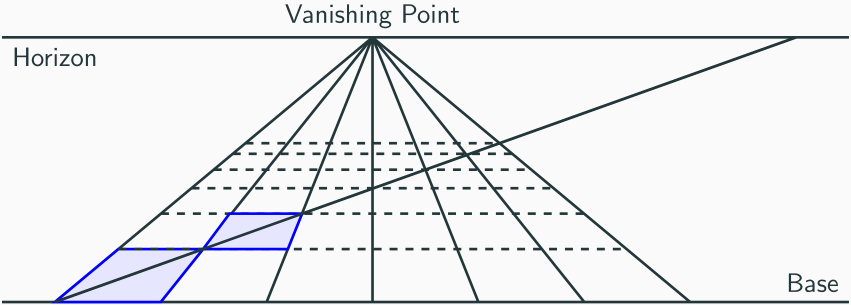

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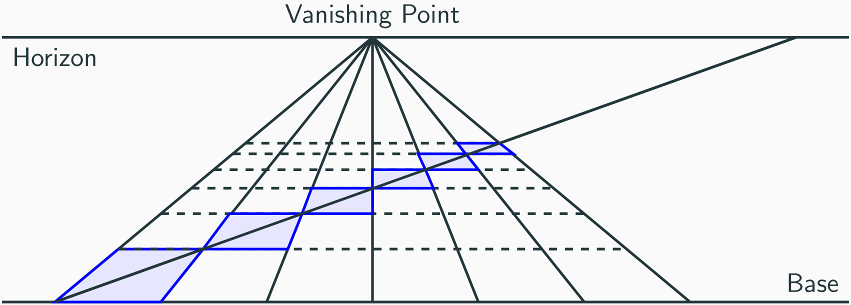

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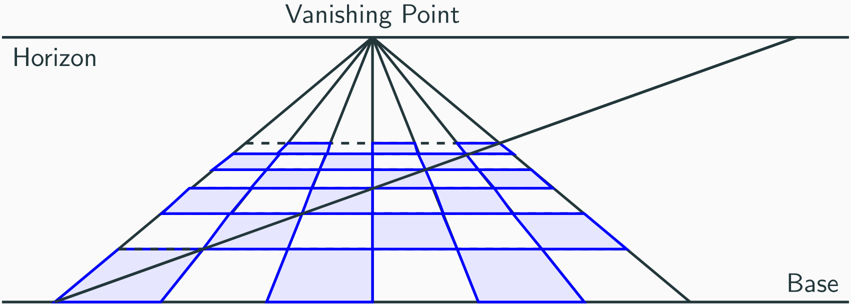

Desargues' Projective Geometry[1]

- Mathematics behind Alberti's Veil: Family of lines (light rays) through a point (eye) plus a plane (veil).
- Recall **Pappus' Theorem**: A_1, A_2, A_3, collinear; B_1, B_2, B_3, collinear; then, so are C_1, C_2, C_3.
- Blaise Pascal (1623-1662) at 16 generalized to conics.
- Desargues (1640) **Projective Geometry** only relies on a straight edge.

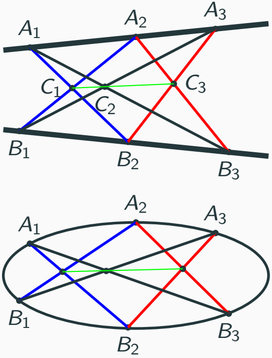

Figure 2: Pappus' and Pascal's Theorems.

[1] Two centuries ahead of his time.

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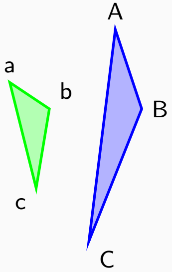

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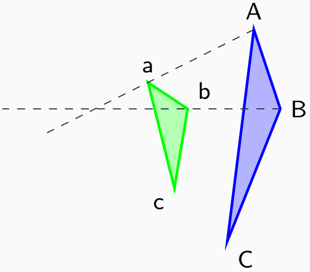

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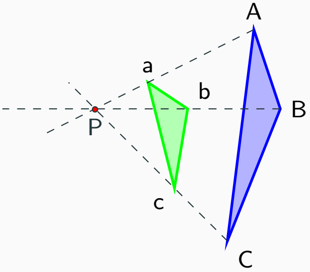

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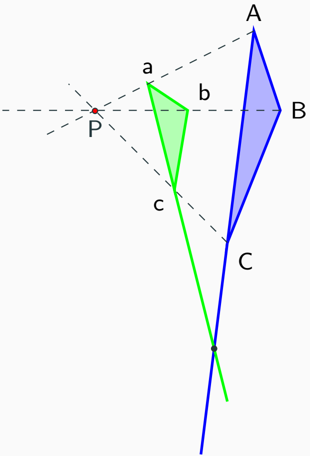

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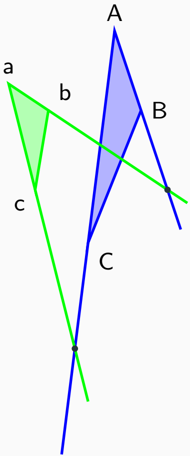

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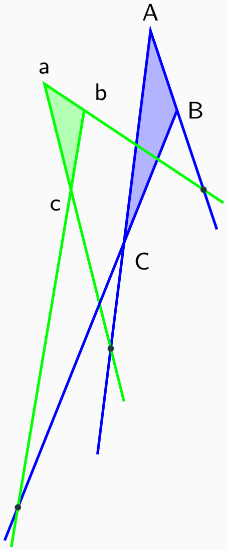

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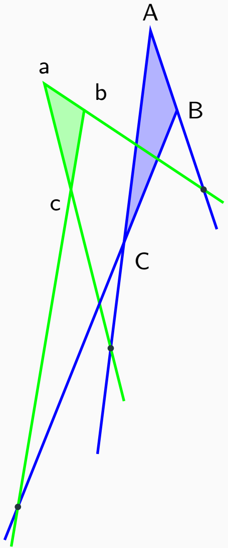

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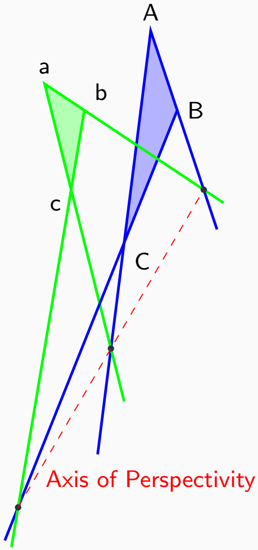
- Points are collinear.



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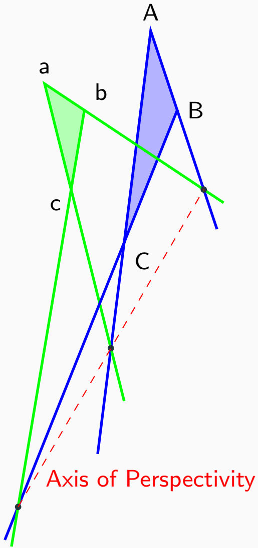
- What if two sides are parallel?



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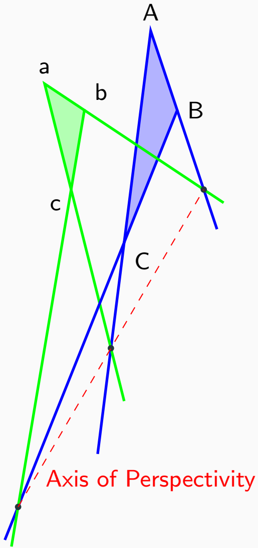
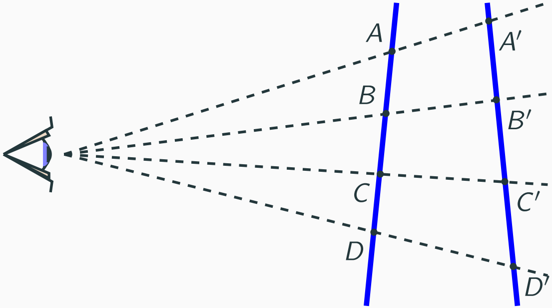
- Points are collinear.
- What if two sides are parallel?
- Need **Projective plane**.



Invariance of the Cross Ratio

Lengths and angles are not preserved.



For any four points on a line, $\frac{\overline{AC}}{\overline{BC}} : \frac{\overline{AD}}{\overline{BD}}$ is invariant, or

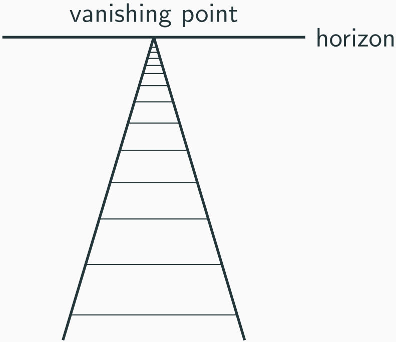
$$\frac{\overline{AC}}{\overline{BC}} : \frac{\overline{AD}}{\overline{BD}} = \frac{\overline{A'C'}}{\overline{B'C'}} : \frac{\overline{A'D'}}{\overline{B'D'}}.$$

Projective Geometry Rebirth in 1800's.

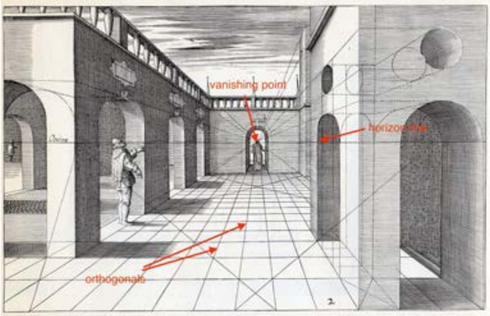
Perspective

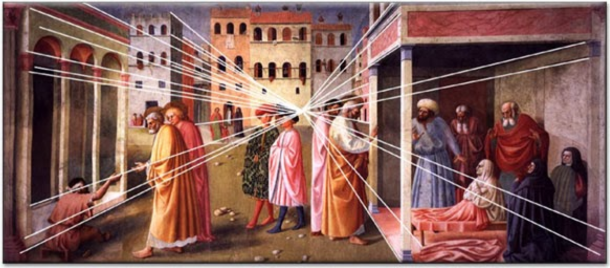
1. Parallel lines meet at a pt.
2. Lines map to lines.
3. Conics map to conics.

Example: Train tracks.



One Point Perspective - Find the Vanishing Points





Two Point Perspective - Find the Vanishing Points

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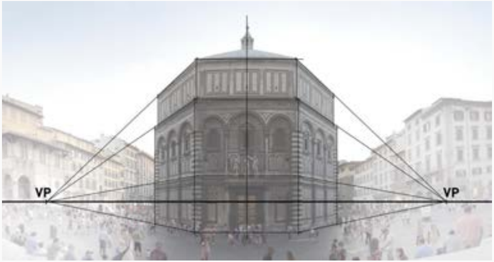

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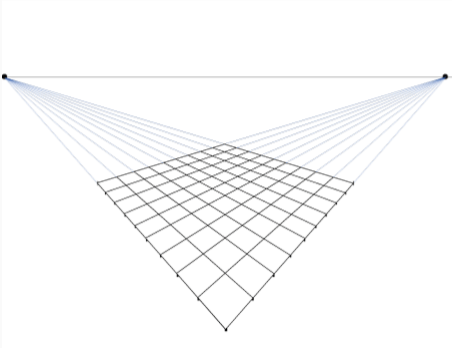
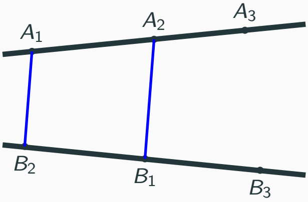


Points at Infinity

- Artists' use vanishing points.
- Pappus' Theorem -
  Consider parallel lines A_1B_2, A_2B_1.
  Does the theorem hold?



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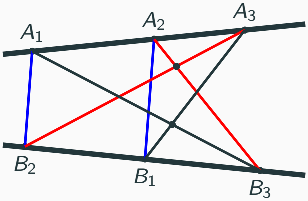

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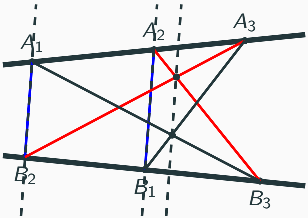

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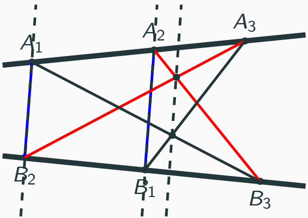

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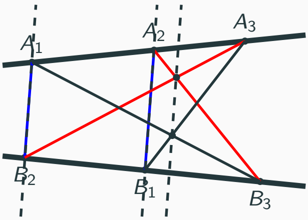

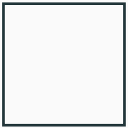

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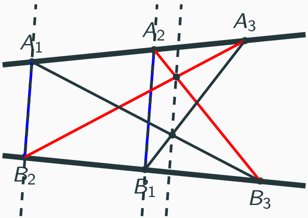

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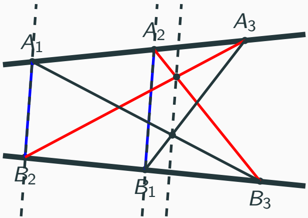
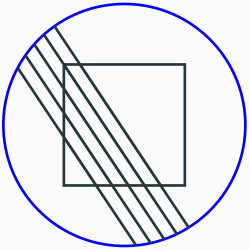
- Look at a plane
- Add parallel lines.
  Where do they go?
- Line at Infinity
- Plane + line at infinity =
  Projective Plane

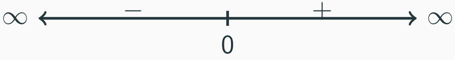




Line at infinity

Projective Line

- Consider the real line, $\mathbb{R}$.



Projective Line

- Consider the real line, $\mathbb{R}$.
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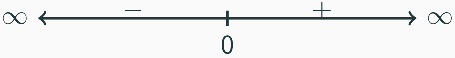

Projective Line

- Consider the real line, $\mathbb{R}$.
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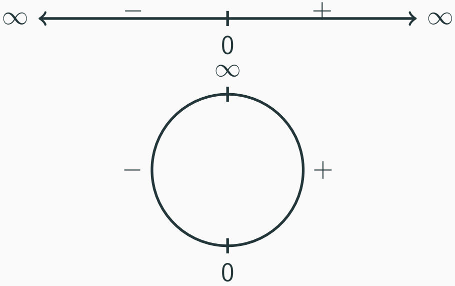
- Topologically a circle!



Projective Line

- Consider the real line, $\mathbb{R}$.
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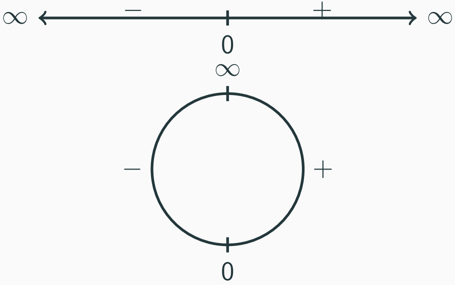
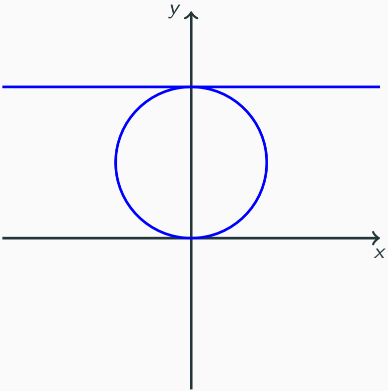
- Topologically a circle!
- We can map the circle to $\mathbb{R}$.





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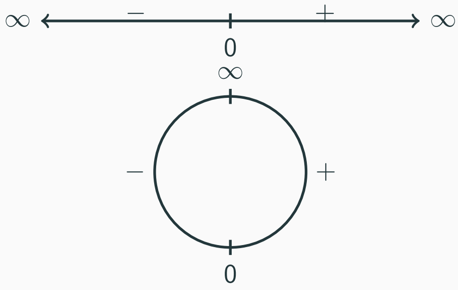

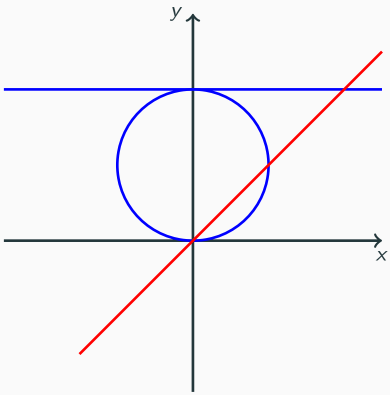

Intersection: $y = b$, $y = mx$:
$x = \frac{b}{m}$.

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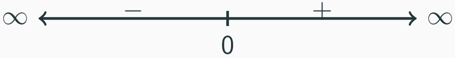

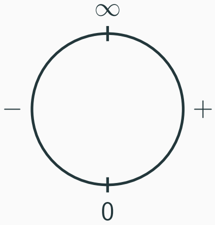

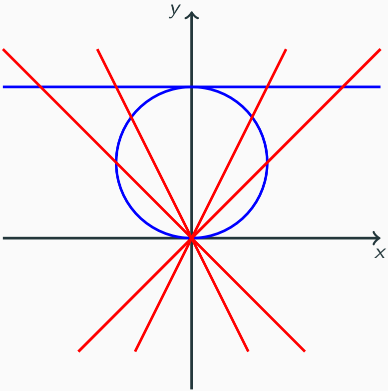

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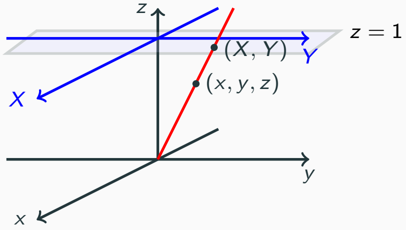
Homogeneous Coordinates

- Point on line: (x, y, z)
- All points on line map to (X, Y) in the plane.
- (X, Y) are called homogeneous coordinates.
- Points on line are multiples, $(x', y', z') = \lambda(x, y, z)$.
- Point on plane: $(\frac{x}{z}, \frac{y}{z}, 1)$, or

$$X = \frac{x}{z}, \quad Y = \frac{y}{z}.$$



Curves

- Curve in plane, $Y = X^2$.
- Translates to

$$\frac{y}{z} = \left(\frac{x}{z}\right)^2.$$

- This is a surface in (x, y, z)-space,

$$x^2 = yz.$$

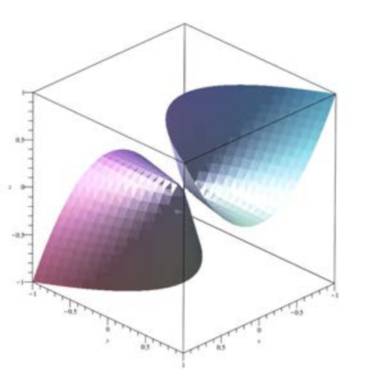


Figure 3: Surface $x^2 - yz$.

Curves

- Curve in plane, $Y = X^2$.
- Translates to

$$\frac{y}{z} = \left(\frac{x}{z}\right)^2.$$

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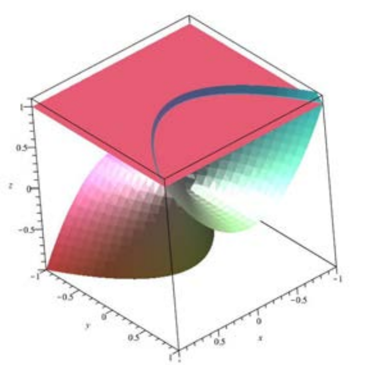
- Slicing with planes, like Alberti's veil, one gets projections of the curve.



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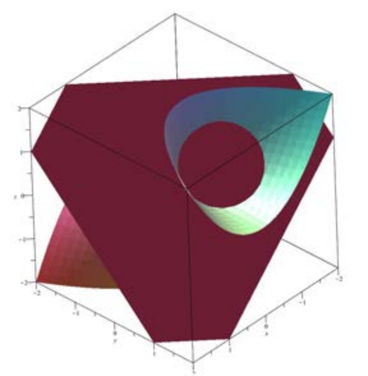


Figure 3: Surface $x^2 - yz$.

Projective Sphere

- Map points on a plane to points on surface of unit sphere, $\mathbb{S}^2$.
- Lines through South Pole uniquely intersect the plane and sphere.
- All points mapped except $(0, 0, 0)$. This point can be mapped to the line at infinity.
- Lines through origin are points of the real projective plane, $\mathbb{RP}^2$.

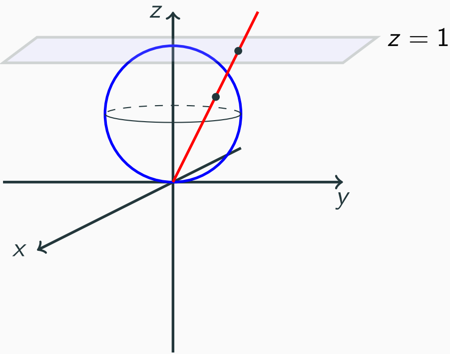
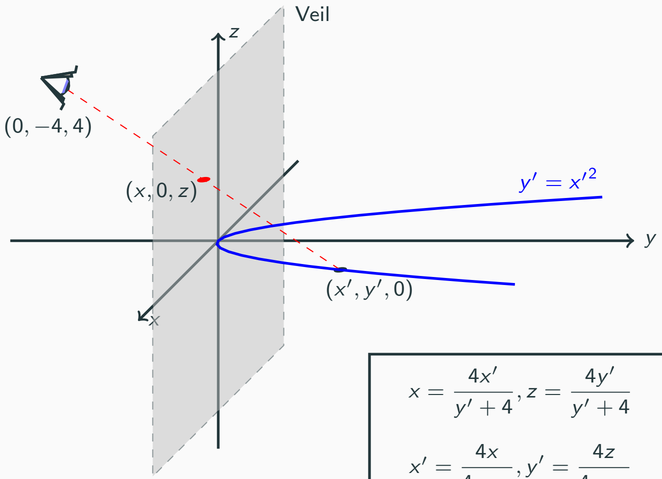


Figure 4: Stereographic Projection



$$x = \frac{4x'}{y'+4}, z = \frac{4y'}{y'+4}$$

$$x' = \frac{4x}{4-z}, y' = \frac{4z}{4-z}$$

Figure 5: Problems 8.4.2-8.4.4

Looking into the Veil - Parabola Projected

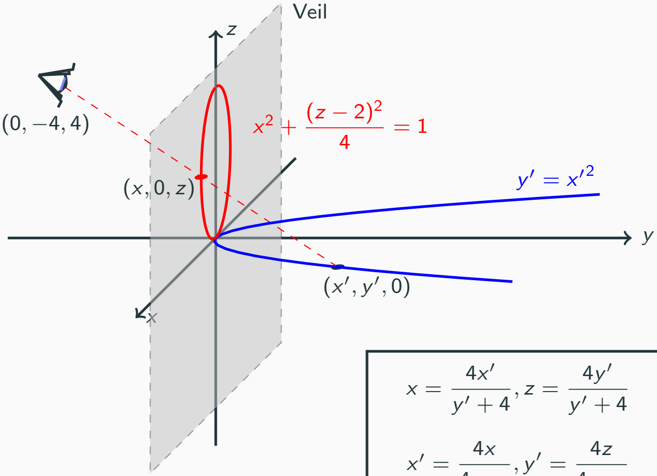


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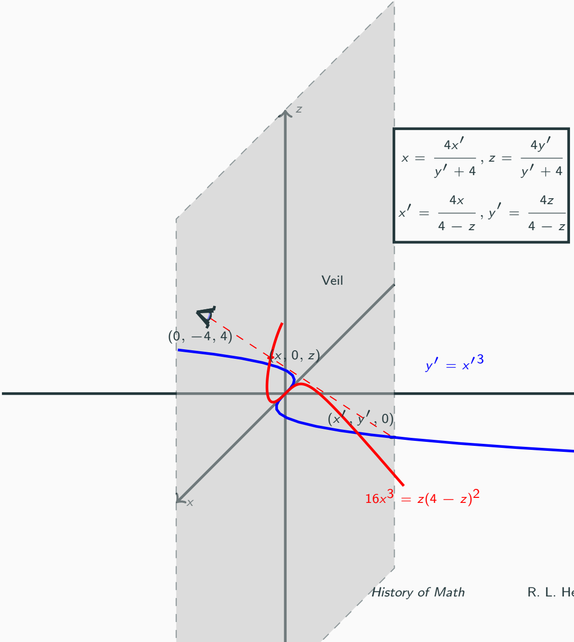
Figure 5: Problems 8.4.2-8.4.4

Viewing A Cubic in the Veil



$$x = \frac{4x'}{y'+4},\ z = \frac{4y'}{y'+4}$$

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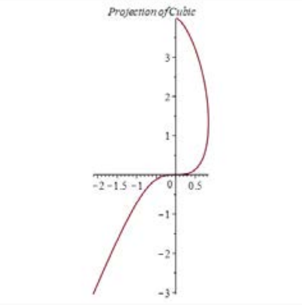
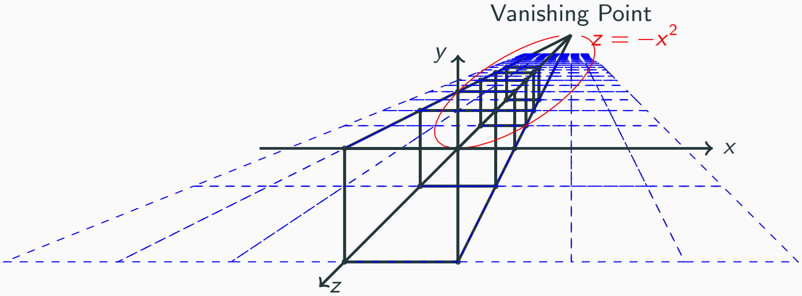


Above: Maple plot of projected cubic to the veil.

Perspective Drawing

Looking at conics from a different perspective: The parabola $z = -x^2$ looks like an ellipse.



In the 1600's mathematicians had other mathematics to attend to. So, we return to geometry in the 1800's.

Emergence of Calculus

Fall 2021 - R. L. Herman

The Method of Exhaustion and the Infinite

- Zeno's Paradox of the Arrow

  "If a body moves from A to B, then before it reaches B it passes through the mid-point, say B_1 of AB. Now to move to B_1 it must first reach the mid-point B_2 of AB_1. Continue this argument to see that A must move through an infinite number of distances and so cannot move. " (450 BCE)

- Eudoxus - Method of Exhaustion.
- Archimedes - area of a segment of a parabola is $\frac{4}{3}$ the area of a triangle with the same base and vertex.
- Luca Valerio (1552-1618) published in 1608 *De quadratura parabolae.*
- Kepler (1571-1630): area as sum of lines. Inpsired Cavalieri's indivisibles.

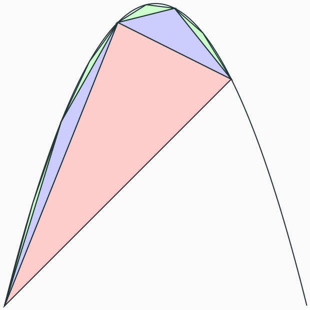


Figure 1: Archimedes: 1st known summation of series.

Developments in the 1600's

Rapid developments first 60 years of 1600's based on Greek geometry, algebra, astronomy (Kepler, Galileo). Led to unification of geometry and algebra.

- Descartes (1596-1650)
- Cavalieri (1598-1647)
- Fermat (1601-1665)
- Roberval (1602-1675)
- Wallis (1616-1703)
- Barrow (1630-1677)
- Gregory (1638-1675)
- Newton (1642-1727)
- Leibniz (1646-1716)

Two main problems

- Tangents
- Areas

Need curves

- Conics
- Archimedean spiral
- Conchoid
- Cissoid
- Cycloid

Seventeenth Century - French, German, English Mathematics

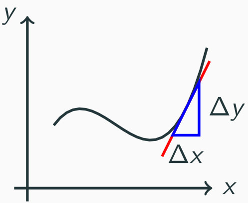
- Galileo Galilei (1564-1642)
- Johannes Kepler (1571-1630)
- 1590 Vièta, *The Analytic Art*
- John Napier (1550-1617) and Henry Briggs (1561-1631) - Introduced the logarithm
- French Mathematicians:
  René Descartes (1596-1650)
  Blaise Pascal (1623-1662)
  Pierre de Fermat (1601-1665)

- Descartes
  philosopher, mathematician
  Discours de la méthode, Marriage of algebra/geometry - analytic geometry
- Pascal
  Wrote math before 16
  Probability theory
  Theology
- Fermat
  Created analytic geometry
  Contributions to Calculus
  Number theory
  Scribbled in Diophantus' *Arithmetica*

Tangents

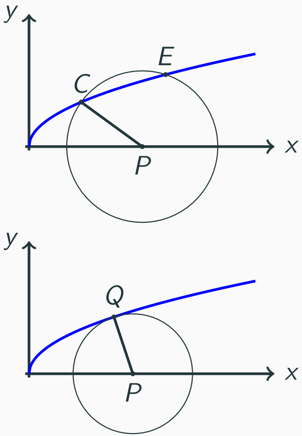
- Pierre de Fermat, René Descartes
- Both studied Apollonius 4-line problem.
- Tangent line approximates the curve at a point.
- Slope $\frac{\Delta y}{\Delta x}$.
- Infinitesimals - increments.
- Fermat
  Method for maxima-minima
  1636 - Method of Tangents
- 1636 Letter from Descartes to Mersenne
  $dy = f(x + dx) - f(x) = ?dx.$

Figure 2: Fermat and Descartes



Descartes vs Fermat - Analytic Geometry, Tangents, Optics

- Decartes published *La Géométrie* - 1637
- Depicted $ax + by = c$ as a line.
- Introduced x, y.
- Fermat introduced analytic geometry earlier.
- Fermat interested in optimization.
- Fermat: lawyer in Toulouse, Math a hobby.
- Descartes denounced him and challenged him to find tangent to folium, $x^3 + y^3 = 3axy$.
- Descartes' Method of Tangents: Find circles tangent to curves.
- Fermat challenged Descartes to explain refraction. Fermat published in 1662.



Areas Under Curves

- First studied by Eudoxus, Archimedes
- Bonaventura Cavalieri (1598-1647) - indivisibles

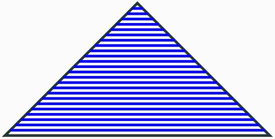
    Fill area with lines.
    But, an infinite number of lines sum to infinity.

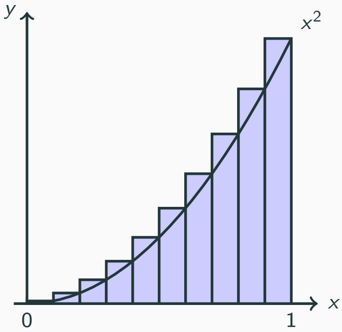
- Archimedes, John Wallis (1616-1703):

$$\int_0^1 x^2\, dx.$$

N segments of width $\frac{1}{N}$. and height $\left(\frac{k}{N}\right)^2$, $k = 1, 2, \ldots, N$.

$$A \approx \sum_{k=1}^{N} \frac{1}{N} \left(\frac{k}{N}\right)^2.$$





Areas Under Curves (cont'd)

Find the sum

$$\begin{aligned} A &\approx \sum_{k=1}^{N} \frac{1}{N}\left(\frac{k}{N}\right)^2 \\ &= \frac{1}{N^3}\sum_{k=1}^{N} k^2 \\ &= \frac{1}{N^3}\frac{N(N+1)(2N+1)}{6} \\ &\sim \frac{2N^3}{6N^3} = \frac{1}{3}. \end{aligned}$$

Wallis showed

$$\boxed{\int_0^a x^n\,dx = \left.\frac{x^{n+1}}{n+1}\right|_0^a = \frac{a^{n+1}}{n+1}}$$

for $k = 1, 2, \ldots, 9$.

Note:

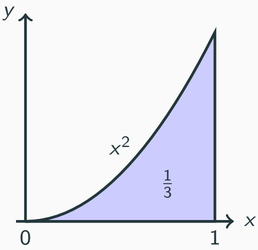
$$\begin{aligned} \sum_{k=1}^{N} k &= \underbrace{1 + \underbrace{2 + \cdots + (N-1)}_{2+(N-1)} + N}_{1+N} \\ &= N\frac{N+1}{2}. \end{aligned}$$

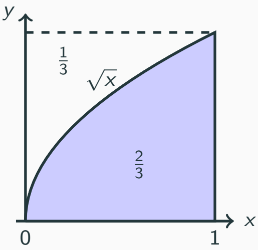
Figure 3: John Wallis

Integrating Powers, $\int x^k \, dx$,

- al Haytham (965-1039) $k = 1, 2, 3, 4$.
- Cavalieri (1635) knew for k up to 9.
- Proven in general by Fermat, Descartes, Roberval, 1630's.
- Fractional Powers (Fermat)
  Ex: $\int_0^1 \sqrt{x} \, dx$
  Use the symmetry in the figures.
- Areas under x^k, need sums $1^k + 2^k + \cdots n^k$.
- Volumes - use cylinders, $V = \pi r^2 h$.
  Sums needed: $1^{2k} + 2^{2k} + \cdots n^{2k}$.

Note: $1^3 + 2^3 + \cdots + k^3 = (1 + 2 + \cdots + k)^3$.





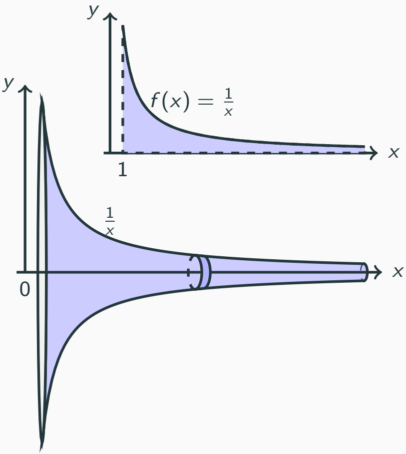
Evangelista Torricelli (1608-1647), barometer inventor

- Inverse Powers, x^{-1}
- Area under $y = \frac{1}{x}$.

$$\int_1^\infty \frac{1}{x}\, dx = \infty.$$

- 1641 Torricelli's trumpet (Gabriel's horn)

$$\begin{aligned} V &= \pi \int_1^\infty \frac{1}{x^2}\, dx = \pi. \\ A &= 2\pi \int_1^\infty \frac{dx}{x} \sqrt{1 + \left(-\frac{1}{x^2}\right)^2} \\ &> 2\pi \int_1^\infty \frac{1}{x}\, dx = \infty. \end{aligned}$$



What? You cannot paint the surface but can fill the trumpet with paint.

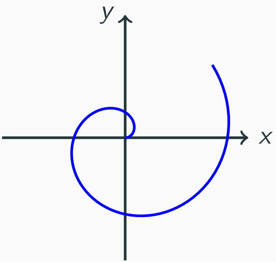
Hobbes - "to understand this for sense, it is not required that a man should be a geometrician or logician, but that he should be mad."

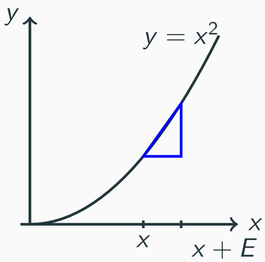
Tangents, Maxima, Minima

- Curves studied like Archimede's spiral, $r = a\theta$
- Fermat - studied polynomials
- Work simpler than Descartes
- Used infinitesimals, E
- **Example:** $y = x^2$

$$\frac{(x+E)^2 - x^2}{E} = 2x + E.$$

- Generalized to polynomials, $p(x, y) = 0$.





John Wallis' (1655) *Arithmetica Infinitorium*

- Combined Descartes' analytic geometry and Cavalieri's indivisibles.
- Some results already known.
- New approach to fractional powers.
- Ambivalent use of infinitesimals - attacked by Thomas Hobbes (1588-1679).
- Formulae for π known by
  - Gregory, Newton, Leibniz
- Madhava (1350-1425) found π to 13 decimal places using series,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

Wallis' Formulae:

$$\frac{\pi}{4} = \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \cdots$$

$$\frac{\pi}{2} = \left(\frac{2}{1} \cdot \frac{2}{3}\right) \cdot \left(\frac{4}{3} \cdot \frac{4}{5}\right) \cdot \cdots$$

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cdots}}}$$

Already known formula:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Isaac Newton (1642-1727)

- Major use of infinite series
- *A Treatise of the Methods of Series and Fluxions*
- *Quadrature of the Hyperbola* Written in 1665, 1st publication in 1668 by Mercator
- Akin to decimal expansions - powers of $\frac{1}{10}$ replaced by x^n.
- Example:

$$\log(1+x) = \int_0^x \frac{dt}{1+t}$$

[Note: Here $\log x = \ln x$.]

Note: Geometric series

$$\begin{aligned} 1 + t + t^2 + \cdots &= \frac{1}{1-t}, |t| < 1. \\ 1 - t + t^2 - \cdots &= \frac{1}{1+t}, |t| < 1. \end{aligned}$$

Then,

$$\begin{aligned} y &= \log(1+x) \\ &= \int_0^x (1 - t + t^2 - \cdots)\, dt \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots . \end{aligned}$$

Invert Power Series

We have for $y = \log(1+x)$,

$$y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots .$$

In order to invert the series, let $x = a_0 + a_1 y + a_2 y^2 + \cdots$. Then,

$$\begin{aligned} y &= \left(a_0 + a_1 y + a_2 y^2 + \cdots\right) - \frac{1}{2}\left(a_0 + a_1 y + a_2 y^2 + \cdots\right)^2 + \cdots \\ &= a_0 - \frac{1}{2}a_0^3 + \frac{1}{3}a_0^3 + a_1\left(a_0^2 - a_0 + 1\right) y \\ &\quad + \left[a_2\left(a_0^2 - a_0 + 1\right) + \left(a_0 - \frac{1}{2}\right)\right] y^2 \\ &\quad + \left[\frac{a_1^3}{3} + a_1 a_2 (2a_0 - 1) + a_3\left(a_0^2 - a_0 + 1\right)\right] y^3 + \ldots . \end{aligned}$$

Equate coefficients of powers of y, then ...

Series Inversion (cont'd)

We solve the resulting system of equations:

$$\begin{aligned} 0 &= a_0 - \frac{1}{2}a_0^3 + \frac{1}{3}a_0^3 \\ 1 &= a_1\left(a_0^2 - a_0 + 1\right) \\ 0 &= a_2\left(a_0^2 - a_0 + 1\right) + \left(a_0 - \frac{1}{2}\right) \\ 0 &= \frac{a_1^3}{3} + a_1 a_2(2a_0 - 1) + a_3\left(a_0^2 - a_0 + 1\right). \end{aligned}$$

The first equation gives $a_0 = 0$. The next two give $a_1 = 1$ and $a_2 = \frac{1}{2}$.

Continuing Newton found that

$$a_0 = 0, a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{6}, a_4 = \frac{1}{24}, \ldots, a_n = \frac{1}{n!}.$$

Newton's Series for Exponential

So far, inversion of

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots .$$

led to

$$x = y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots .$$

However,

$$y = \log(1+x) \Rightarrow x = e^y - 1.$$

So, we found the series expansion

$$e^y = 1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots .$$

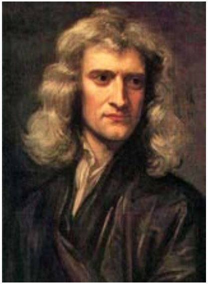


Figure 4: Newton

Newton's Series for Sine

Newton knew $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$.

Recall binomial series: $(a+b)^n = \sum_{k=0}^n C_{n,k} a^{n-k} b^k$, where the coefficients are $C_{n,k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$. Then,

$$\begin{aligned}
(1+a)^p &= 1 + pa + \frac{p(p-1)}{2!}a + \frac{p(p-1)(p-2)}{3!}a^2 + \cdots . \\
\sin^{-1} x &= \int_0^x \frac{dt}{\sqrt{1-t^2}}, \quad a = -t^2, p = -\frac{1}{2}, \\
&= \int_0^x \left(1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \cdots + \frac{-\frac{1}{2}(-\frac{3}{2})\cdots(\frac{1}{2}-k)}{k!}(-t^2)^k + \cdots\right) \\
&= x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \cdots + \frac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots 2k}\frac{x^{2k+1}}{2k+1} + \cdots
\end{aligned}$$

Inverting, Newton found $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$.

Gottfried Wilhelm Leibniz (1646-1716)

- Librarian, philosopher, diplomat, doctorate in law.
- First papers in calculus (1684).
- Led to long dispute.
- Better notation, $\frac{dy}{dx}, \int\, dx$.
- Sum, product, quotient rules.
- Proved Fundamental Theorem of Calculus, $\frac{d}{dx}\int f(x)\,dx = f(x)$.

Figure 5: Leibniz

- Geometric series,
  Known to Euclid (Zeno's paradox)
  Leonhard Euler (1707-1783)

  $$a+ar+ar^2+\cdots+ar^n+\cdots = \frac{a}{1-r}, |r| < 1.$$

Figure 6: Euler

- Harmonic Series - Oresme (1350)

$$\begin{aligned} & 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+\frac{1}{4}+\cdots \\ = \quad & (1)+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots \\ \geq \quad & \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots=\infty. \end{aligned}$$

- Power series - 17th Century,
  Gregory, Wallis, Taylor, Mclaurin, ...

Basel Problem (1644)

- Posed by Pietro Mengoli (1626-1686).

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

- Jacob and Johann Bernoulli (1704) tackled. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=?$

Figure 7: Jacob and Johann

$$\begin{aligned}\sum_{n=1}^{N}\frac{1}{n(n+1)} &= \sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots\left(\frac{1}{N}-\frac{1}{N+1}\right)\\ &= \left(1-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cdots\left(\cancel{\frac{1}{N}}-\frac{1}{N+1}\right)\\ &= 1-\frac{1}{N+1}\underset{N\to\infty}{\Longrightarrow}1.\end{aligned}$$

Euler's Solution of Basel Problem - 1734

- Descartes' Factor Theorem
- $p(x)$ - polynomial
- $p(r) = 0$ implies $p(x) = (x - r)q(x)$, $q(x)$ - polynomial

Proof:

$$\begin{aligned} p(x) &= a_0 + a_1 x + \cdots + a_n x^n \\ p(y) &= a_0 + a_1 y + \cdots + a_n y^n \\ p(x) - p(x) &= a_1(x - y) + \cdots + a_n(x^n - y^n) \end{aligned}$$

$$x^n - y^n = (x - y)\left(x^{n-1} + x^{n-2}y + \cdots + y^{n-1}\right)$$

Let $y = r$,

$$\begin{aligned} p(x) &= (x - r)[a_1 + a_2(x + r) + \cdots + a_n(x^{n-1} + x^{n-2}r + \cdots + r^{n-1})] \\ &= (x - r)q(x). \end{aligned}$$

Leonhard Euler's Solution of Basel Problem

$\sin x$ has roots $n\pi,\ n = 0, \pm 1, \pm 2, \ldots$. - Generalize Factor Theorem:

$$\begin{aligned}
\sin x &= Ax\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\cdots \\
&= Ax\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots \\
&= A\left[x - \frac{x^3}{\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) + x^5(\cdots) - \cdots\right].
\end{aligned}$$

Compare to

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots .$$

Then $A = 1$, and

$$\frac{1}{3!} = \frac{1}{\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

What are the next coefficients? - Needs proofreading

We need the x^5 terms in the expansion

$$\begin{aligned} \sin x &= x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right) \\ &= x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\cdots\left(1-\frac{x^2}{m^2\pi^2}\right)\cdots \end{aligned}$$

These are found by multiplying $\frac{x^2}{m^2\pi^2}$ times one of the other factors $\frac{x^2}{n^2\pi^2}$, $n \neq m$, and summing:

$$x\sum_{m=1}^{\infty}\frac{x^2}{m^2\pi^2}\sum_{n=1,n\neq m}^{\infty}\frac{x^2}{n^2\pi^2} = \frac{x^5}{\pi^4}\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{n=1,n\neq m}^{\infty}\frac{1}{n^2} = \frac{x^5}{5!}.$$

$$\frac{\pi^4}{5!} = \sum_{m=1}^{\infty}\frac{1}{m^2}\left[\zeta(2)-\frac{1}{m^2}\right] = \zeta(2)^2-\zeta(4).$$

Another Approach to Obtain $\zeta(4)$

Differentiate $\ln \sin x = \cot x$ and use the known series expansion for $x \cot x$ in terms of Bernoulli numbers.

$$\begin{aligned}
\sin x &= x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right) \\
\ln \sin x &= \ln x + \sum_{n=1}^{\infty} \ln \left(1 - \frac{x^2}{n^2\pi^2}\right) \\
x \cot x &= 1 - \sum_{n=1}^{\infty} \frac{2x^2}{n^2\pi^2 - x^2} \\
&= 1 - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{x^2}{n^2} \sum_{k=0}^{\infty} \left(\frac{x^2}{n^2\pi^2}\right)^k \\
1 - \frac{x^2}{3} - \frac{x^4}{45} - \frac{2x^6}{945} + \ldots &= 1 - \frac{2}{\pi^2} \sum_{k=0}^{\infty} \left(\frac{x}{\pi}\right)^{2k+2} \zeta(2k+2) \\
x \cot x &= 1 + \sum_{k=0}^{\infty} (-1)^k B_{2k} (2x)^{2k}
\end{aligned}$$

Results for the Riemann Zeta Function, $\zeta(s)$

Therefore, we have

$$\begin{aligned} \zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots &= \frac{\pi^4}{90}, \\ \zeta(2n) = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots &= (-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n}, \end{aligned}$$

where B_{2n} are Bernoulli numbers, $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, \ldots,$

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Euler (1748) - Zeta function can be defined as

$$\begin{aligned} \zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \\ &= \left(1 - \frac{1}{2^s}\right)^{-1}\left(1 - \frac{1}{3^s}\right)^{-1}\left(1 - \frac{1}{5^s}\right)^{-1}\cdots\left(1 - \frac{1}{p^s}\right)^{-1}\cdots, \end{aligned}$$

where p is prime.

Bernhard Riemann's Contribution

- Bernhard Riemann (1826-1866)
- Riemann zeta function

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots .$$

- Values

  $\zeta(1) = \infty$, harmonic series
  $\zeta(2) = \frac{\pi^2}{6}$
  $\zeta(3)$ irrational, proved by Apery (1981)

- Zeros

  $\zeta(-2n) = 0$, n integer > 0.
  Riemann Hypothesis:
  $\zeta(\sigma + it) = 0$ when $\sigma = \frac{1}{2}$.

- Connection to primes?

Figure 8: Georg Friedrich Bernhard Riemann

$$\zeta(s) = \frac{1}{\Gamma(2s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx$$

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}$$

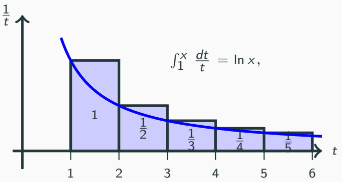
$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(s)$$

Connection to Primes and Other Tidbits

$$\begin{aligned}\zeta(s) &= \left(1-\frac{1}{2^s}\right)^{-1}\left(1-\frac{1}{3^s}\right)^{-1}\cdots\left(1-\frac{1}{p^s}\right)^{-1}\cdots \\ &= \prod_{p=\text{prime}}\left(1-\frac{1}{p^s}\right)^{-1} \\ &= \prod_{p=\text{prime}}\left[1+\frac{1}{p^s}+\left(\frac{1}{p^s}\right)^2+\cdots\right]\end{aligned}$$

- Primes less than $x \sim \int_2^x \frac{dt}{\log t}$
- Euler-Mascheroni Constant $\gamma \approx 0.577218\ldots.$
- Generalizing $n!$

$$\Gamma(n+1) = n\Gamma(n), \Gamma(0) = 1.$$



$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right) = \gamma.$$

Leonhard Euler (1707-1783)

Euler (at 14) studied under Johann Bernoulli, graduated in 1723.

Went to St. Petersburg in 1727, Berlin in 1741, and back to St. Petersburg in 1766.

By 1730's - lost vision in right eye and blind by 1771.

866 books and papers - 228 after death. *Opera Omnia* - over 25,000 pgs

First appearance of e - letter to Goldbach in 1731.

Euler published *Introductio in Analysin infinitorum* - 1748

Euler's Formula, $e^{ix} = \cos x + i \sin x$.

Euler's Identity, $e^{i\pi} + 1 = 0$.

Euler's constant, γ

Euler's Polyhedral Formula, $V + F = E + 2$.

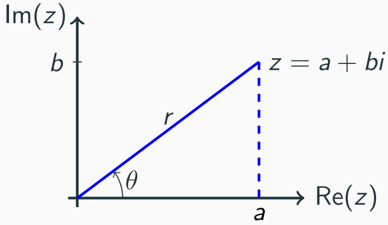
Amicable Numbers

- Recall Greeks knew 220 and 284; i.e., sum of proper factors of 220 $=$ 284 and vice versa.
- Thabit ibn Qurra (836-901) discovered the next amicable pair, 17296, 18416.
- Pierre Fermat rediscovered this pair in 1636.
- René Descartes discovered in 1638 9,363,584 and 9,437,056.
- 1747, Euler published [E100] giving 30 amicable pairs.
- By 1750 - Euler found 61 pairs!

Euler's Formula

- Complex numbers, polar form.
  $z = a + bi,\ a = r\cos\theta, b = r\sin\theta$

$$\begin{aligned} z &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta). \end{aligned}$$



- Exponential of imaginary number

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\ &= 1 + i\theta - \frac{(\theta)^2}{2!} - i\frac{(\theta)^3}{3!} + \cdots \\ &= \left(1 - \frac{(\theta)^2}{2!} + \cdots\right) + i\left(\theta - \frac{(\theta)^3}{3!} + \cdots\right) \\ e^{i\theta} &= \cos\theta + i\sin\theta. \end{aligned}$$

Euler's Formula Applications $e^{i\theta} = \cos\theta + i\sin\theta.$

- $\theta = \pi,\ e^{i\pi} = -1$, or $e^{i\pi} + 1 = 0.$
- $\left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n\ e^{in\theta} = \cos n\theta + i\sin n\theta$ implies **de Moivre's Theorem**

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n.$$

- **Example:** $n = 2$

$$\begin{aligned} \cos 2\theta + i\sin 2\theta &= (\cos\theta + i\sin\theta)^2 \\ &= \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta. \end{aligned}$$

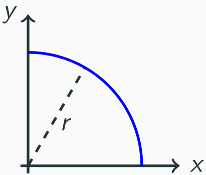
$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta. \end{aligned}$$

Rectification of an Ellipse

- Rectification = Finding arclengths
- $y = y(x)$

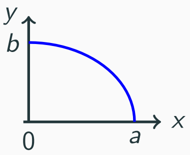
$$L = \int_a^b \sqrt{1 + y'^2}\, dx.$$

- **Example:** Circle



$$\begin{aligned} L &= 4\int_0^r \sqrt{1 + {\frac{x^2}{y^2}}^2}\, dx \\ &= 4\int_0^r \sqrt{1 + {\frac{x^2}{r^2 - x^2}}^2}\, dx \\ &= 4r\int_0^r \frac{dx}{\sqrt{r^2 - x^2}} = 4r\sin^{-1} 1 = 4r\left(\frac{\pi}{2}\right) = 2\pi r. \end{aligned}$$

Arclength of an Ellipse



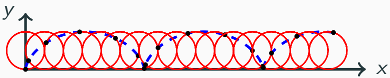
- **Example:** Ellipse
  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad x \geq 0, y \geq 0.$

$$
\begin{aligned}
y &= \frac{b}{a}\sqrt{a^2 - x^2} \\
y' &= \frac{bx}{a\sqrt{a^2 - x^2}} \\
1 + y'^2 &= \frac{a^2 - k^2x^2}{a^2 - x^2}, \quad k = \frac{a^2 - b^2}{a^2} \\
L &= 4\int_0^a \sqrt{\frac{a^2 - k^2x^2}{a^2 - x^2}}\, dx.
\end{aligned}
$$

Historical Curves

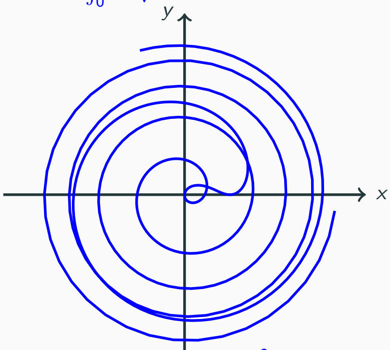
- 1609 - Kepler - Mars' orbit is an ellipse.
- 1659 - Pascal *Dimensions des lignes courbes de toutes les Roulettes.*
- 1658 Proof by Wren published by Wallis in 1659 - proof of the rectification of the cycloid.
- 1676 - Newton - infinite series.
- 1742 - Maclaurin - expansion in eccentricities.
- 1691 - Jacob Bernoulli - parabolic spiral.

Cycloid, Parabolic Spiral, and Lemniscate



$x = r(t - \sin t),\ y = r(1 - \cos t)$

$L = \int_0^{2\pi} r\sqrt{2 - 2\cos t}\, dt = 8r$

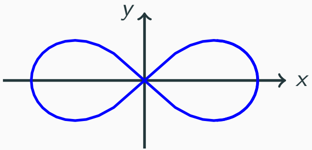


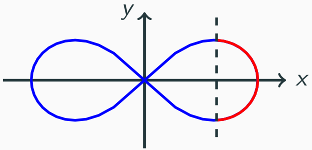
$(a - r)^2 = 2ab\theta$

$s = \int \sqrt{1 + \frac{r^2(a-r)^2}{a^2b^2}}\, dr$

- **Example:** Lemniscate, $r^2 = \cos 2\theta$

$$L = 4\int_0^1 \frac{dr}{\sqrt{1 - r^4}}$$





Elastica

Sep 1694, Jacob Bernoulli

Oct 1694, Johann Bernoulli

Elliptic Functions

- Lemniscate integral leads to new functions, $u = \int_0^x \frac{dt}{\sqrt{1-t^4}}$.
- Compare to $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$.
- Elliptic Integrals: $\int R(t, \sqrt{p(t)})\, dt$, R is rational function, $p(t)$ is polynomial of degree 3 or 4.
- Bernoulli (1694) - geometry, mechanics.
- Fagnano (1682-1766) - Doubling arc of lemniscate.
- Carl Friedrich Gauss (1777-1855) ∼1800 studied inverse $x = sl(u)$ Doubly periodic

$$\begin{aligned} sl(u + 2\bar{u}) &= sl(u) \\ sl(u + 2i\bar{u}) &= sl(u) \end{aligned}$$

- Rediscovered by Niel Henrik Abel (1802-1829) and Carl Gustav Jacobi (1804-1851) in 1820's.

Elliptic Integral Addition Theorems

- **Example** Circle

$$\begin{aligned} \sin 2u &= 2 \sin u \cos u \\ &= 2 \sin u \sqrt{1 - \sin^2 u} \end{aligned}$$

- Let $u = \sin^{-1} x$. Then,

$$\begin{aligned} 2u &= 2 \int_0^x \frac{dt}{\sqrt{1-t^2}} \\ &= \sin^{-1}\left(2 \sin u \sqrt{1 - \sin^2 u}\right) \\ &= \sin^{-1}\left(2x\sqrt{1-x^2}\right) \\ 2\int_0^x \frac{dt}{\sqrt{1-t^2}} &= \int_0^{2x\sqrt{1-x^2}} \frac{dt}{\sqrt{1-t^2}}. \end{aligned}$$

Elliptic Integrals

- Study of Inversions
  Gauss 1790s
  Abel 1823 (pub 1827)
  Jacobi 1829 book
- 1786 (40 yrs later)
  Legendre classified elliptic integrals into 3 cases, book 1825.
  Examples:

$$F(\phi) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}, \quad E(\phi) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2\theta}\, d\theta.$$

- Riemann placed in geometric setting - torus.

Gauss' *AGM* - Arithmetic-geometric mean

- Gauss's constant $G = \frac{1}{AGM(1,\sqrt{2})} = \frac{2}{\pi}\int_0^1 \frac{dx}{\sqrt{1-x^4}} = 0.8346268\ldots$
- Between 1 and $\sqrt{2}$ is $\dfrac{\pi}{\bar{\omega}} = \dfrac{1}{G}$.
- Arithmetic mean $\dfrac{a+b}{2}$.
- Geometric mean $\dfrac{a}{g} = \dfrac{g}{b} \Rightarrow g = \sqrt{ab}$.
- $AGM(a,b)$ algorithm: Start with $a_0 = a, \quad b_0 = b$,

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad n = 0, 1, \ldots.$$

- Gauss - $AGM(1,\sqrt{2}) = \dfrac{\pi}{\bar{\omega}}$ to 11 decimal places.
- Led to study of general theory, modular functions, theta functions - Ramanujan (early 1900s).

Application of $AGM(a, b)$

Example: $AGM(1, 2)$. Start with $a_0 = 1, \quad b_0 = 2,$

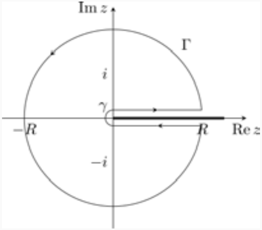
$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad n = 0, 1, \ldots.$$

a_n	b_n
1.0000	2.0000
1.5000	1.4142
1.4571	1.4565
1.4568	1.4568
$\vdots$	$\vdots$

$$AGM(a, b) = \frac{\pi}{4} \frac{a+b}{K\left(\frac{a-b}{a+b}\right)}, \quad K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

Complex Analysis

Fall 2021 - R. L. Herman



History of Complex Analysis

- Before 1600
  - Cardano 1545, quadratic
  - Bombelli 1572, cubic
  - Harriot 1600, quartic
  - Negative roots - false
  - Complex roots - impossible
- 1600s
  - Descartes, 1637, $a+b\sqrt{-1}$
  - Wallis 1685
  - Insights from geometry trigonometry, conics - justified
- 1700s
  - Bernoulli - integral transformation
  - Euler - Euler's formula, i
  - Gauss (1799, 1815) FTA, quadratic forms
  - Wessel (1797), Argand (1806) Geometric Visualization
  - Cauchy (1814, 1825) Complex Analysis
  - Riemann (1826-1866) Surfaces

Complex Numbers, $\mathbb{C}$

- $a + bi \in \mathbb{C}$, $a, b \in \mathbb{R}$, $i = \sqrt{-1}$.
- Quadratic Equation,
  $ax^2 + bx + c = 0$,
  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
  If $b^2 - 4ac < 0$,
  complex conjugate roots.
- Cubics - Role was clearer
  $y^3 = py + q$
  $y = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}.$

Example: $x^3 = 15x + 4$

$$\begin{aligned} x &= \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} \\ &= 2 + i + 2 - i = 4. \end{aligned}$$

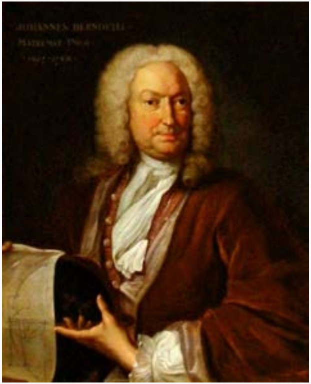
Bombelli (1572)

$$\begin{aligned} (2+i)^3 &= (2+i)(4 + 4i + i^2) \\ &= (2+i)(3+4i) \\ &= 2 + 11i. \end{aligned}$$

Bernoulli's Transformations

- Johann Bernoulli (1712)

$$\begin{aligned}\frac{1}{1+z^2} &= \frac{1}{(1+iz)(1-iz)}\\ &= \frac{1}{2}\left(\frac{1}{1-iz}-\frac{1}{1+iz}\right)\\ \int\frac{dz}{1+z^2} &= \frac{1}{2}\int\left(\frac{1}{1-iz}-\frac{1}{1+iz}\right)\end{aligned}$$



- Note:

$$\int\frac{dz}{a+bz}=\frac{1}{b}\ln(a+bz).$$

So, integrating $(1+t^2)^{-1}$ gives

$$\tan^{-1} z=\frac{1}{2i}\left[\ln(1+iz)-\ln(1-iz)\right].$$

Examples: Tangent Identities

- Bernoulli studied $y = \tan n\theta$ in terms of $x = \tan\theta$.
- *Example:* $n = 2$
  $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$.
- Let $y = \tan n\theta$, Then, $n\theta = \tan^{-1} y, \theta = \tan^{-1} x$.

$$\begin{aligned} \int \frac{dy}{1+y^2} &= n\int \frac{dx}{1+x^2} \\ \ln\frac{y+i}{y-i} &= n\ln\frac{x+i}{x-i} \\ \frac{y+i}{y-i} &= A\left(\frac{x+i}{x-i}\right)^n \end{aligned}$$

$A = (-1)^{n+1}$. Solve for y.

Ex: $n = 2$:

$$\tan 2\theta = \frac{2x}{1-x^2}.$$

Ex: $n = 3$:

$$\tan 3\theta = \frac{x^3-3x}{3x^2-1}.$$

Ex: $n = 4$:

$$\tan 4\theta = \frac{4x-4x^3}{x^4-6x^2+1}.$$

Ex: $n = 5$:

$$\tan 5\theta = \frac{x^5-10x^3+5x}{5x^4-10x^2+1}.$$

The Fundamental Theorem of Algebra

- Integration of $\frac{p(x)}{q(x)}$ for $p(x), q(x)$ polynomials
- Need Integration by parts Assumes $q(x)$ can be factored
  - Fundamental Theorem of Algebra (FTA)
- Albert Girard (1629), *L'invention en algèbre*,
  First to claim there are always n roots of degree n polynomial.
- By 1750 - Any polynomial with real coefficients can be factored into real linear and quadratic factors.
- Nicolas II Bernoulli (1687-1759) gave a counterexample:
  $p(x) = x^4 - 4x^3 + 2x^2 + 4x + 4$.
- Euler found the factors:

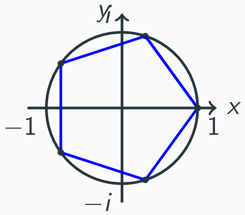
$$x^2 - \left(2 \pm \sqrt{4 + 2\sqrt{7}}\right) x + \left(1 \pm \sqrt{4 + 2\sqrt{7}} + \sqrt{7}\right)$$

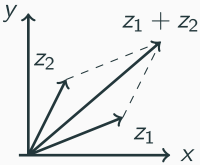
He gave incorrect proof for any quartic.
His was followed by proofs from d'Alembert and Gauss.

Roots of Unity

- Cotes, de Moivre, Euler
  - $x^n - 1 = 0$ Seems $x = \sqrt[n]{1}$.
  - $x = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$, $k = 0, 1, \ldots, n-1$. **Roots of unity.**
  - Geometric Interpretation
- Caspar Wessel, surveyor.
  - Complex number = point in the complex plane, 1797.
  - Also, proposed vectors.
- Argand, 1806, visual representation, operational (translation, rotation, reflection)
- Gauss also rediscovered, 1831.

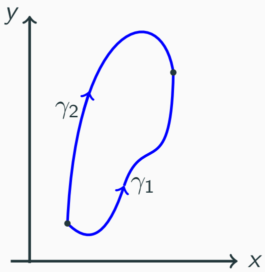
$e^{2k\pi/5}, k = 0, 1, \ldots, 4$





Representing Complex Numbers

- Gauss (1777-1855) adopted "complex number," used i.
- Integration in $\mathbb{C}$-plane.
- $\int_\gamma \phi(z)\,dz$ is path independent for "nice" $\phi(z)$.
- Cauchy proved later, in 1814 talk, published 1827. - Now called *Cauchy's Thm.*



Path Independence

$$\int_{\gamma_1} \phi(z)\,dz = \int_{\gamma_2} \phi(z)\,dz$$

Equivalently, for a simple, closed loop Γ, $\int_\Gamma \phi(z)\,dz = 0$.

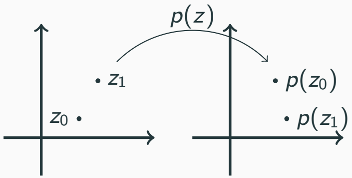
Augustin Louis Cauchy (1789-1857)

- Father of Complex Analysis
- Does $f(x) \to f(z)$ make sense?
- Integration along paths (1814) pub 1827.
- Cauchy's Theorem, Cauchy-Riemann Equations.
- Calculus of Residues (1826) - dealing with singularities.
- Convergence of infinite series.
- Use complex integration to integrate real functions.
- Path Independence (1825).
- Complex function of complex variable (1828).

Fundamental Theorem of Algebra I

Every polynomial $p(x)$ can be written as a product of linear complex factors. (Contains 1750 version)

- d'Alembert (1717-1783)
- **Lemma** $p(z_0) \neq 0$, $p(z) \neq$ constant. There exists a $z_1 = z_0 + w$ such that $|p(z_1)| < |p(z_0)|$ where $|a + bi| = \sqrt{a^2 + b^2}$.



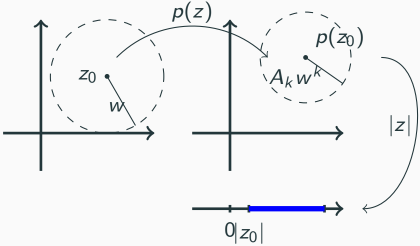
Proof

$$
\begin{aligned}
p(z) &= a_0 z^n + a_1 z^{n-1} + \cdots + a_n. \\
p(z_0 + w) &= a_0 z_0^n + a_1 z_0^{n-1} + \cdots + a_n + A_1 w + A_2 w^2 + \cdots + A_n w^n.
\end{aligned}
$$

Fundamental Theorem of Algebra II

$$p(z_0 + w) = a_0 z_0^n + a_1 z_0^{n-1} + \cdots + a_n + A_k w^k + \epsilon$$

Here $A_k w^k$ is the first nonzero, lowest power of w term and ϵ contains the higher powers terms in w and is small for large $|z|$.



$\exists w$ such that $p(z_0) + A_k w^k$ is closer to the origin.

Let $p(z) \neq 0$. By the lemma, $\exists$ a point closer than z_0 to the origin.
$\therefore$ there exists a zero of $p(z)$.

Fundamental Theorem of Algebra III

- Gauss attempted several proofs.
- Karl Weierstrauss (1815-1897) - continuous functions on closed, bounded regions which assume maximum and minimum values.
- Gauss (1799 Thesis) considered curves $\text{Re}(p(z)) = 0$, $\text{Im}(p(z)) = 0$, $z = x + iy$.
- For $|z|$ large, $\text{Re}(a_0 z^n) = 0$, $\text{Im}(a_0 z^n) = 0$, curves are asymptotic to lines through the origin.
- Curves $\text{Re}(p(z)) = 0$, $\text{Im}(p(z)) = 0$, entering $|z| = r$ must come out and intersect inside disk. [See examples.]

Examples

Plotting $Re(p(z)$ and $Im(p(z))$, outside a large circle one gets alternating lines. Inside the circle they must intersect for $p(z) = 0$.

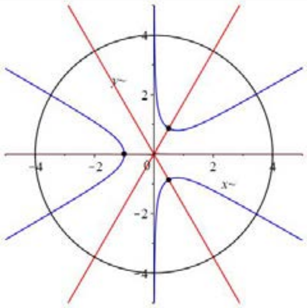


Figure 1: $p(z) = z^3 + 1$.

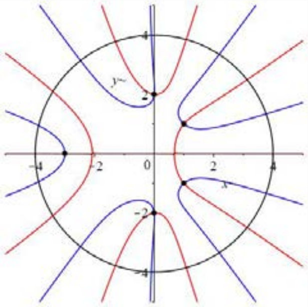


Figure 2: $p(z) = z^5 + z^4 + 10z^2 - 16z + 24 = (z - 1 - i)(z - 1 + i)(z^2 + 4)(z + 3)$.

Theory of Curves, $p(x, y) = 0$

- Descartes - linear/lines
  - quadratic/conics
- Newton - cubics
- Recall Bezout's Intersection Thm
  - ◁ Count multiplicities.
  - ◁ Intersection with ∞.
- 19th Century
  - Projective Geometry
    homogeneus coordinates
    Möbius, Plücker - 1830
  - Complex Numbers
    Gauss - FTA
  - Topological ideas
    - Riemann surfaces, 1850's

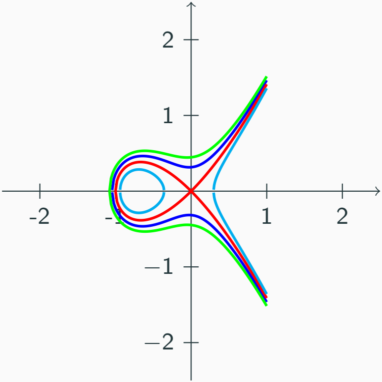
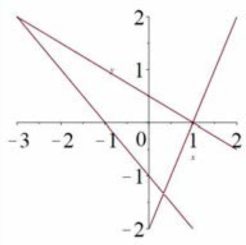


Figure 3: Cubic curves of form $y^2 = x^3 + x^2 + bx + 2b$

Cubic Curves

Consider products of linear factors or lines
$p(x, y) = (a_1x + b_1y + c_1)(a_2x + b_2y + c_2)(a_3x + b_3y + c_3)$

- Ex: $p(x, y) = (x + y + 1)(x + 2y - 1)(-2x + y + 2)$



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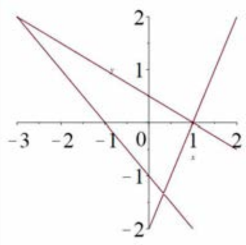

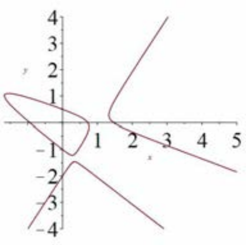

Cubic Curves

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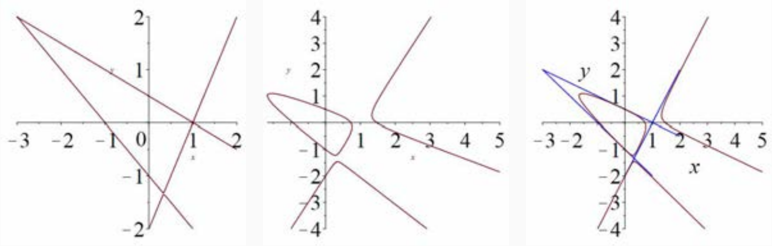
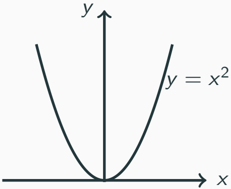
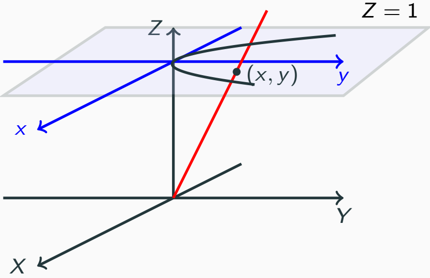
- Branches go to **points at infinity**. Consider projective geometry.



Projective Geometry

- Homogeneous coordinates:
  $x = \frac{X}{Z}, y = \frac{Y}{Z}$.
- Introduced by Möbius, Pücker.
- **Example:** $y = x^2$ gives $X^2 = YZ$





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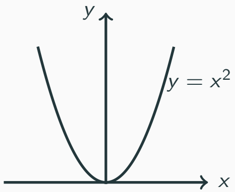
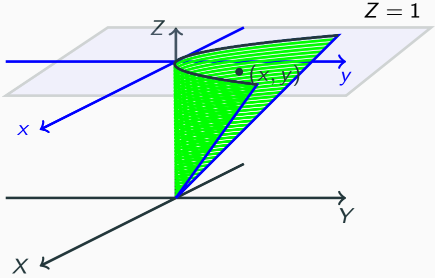
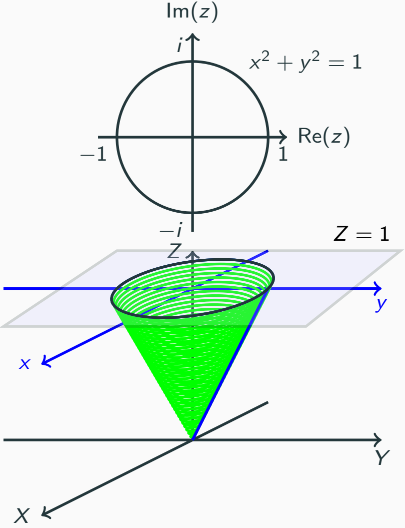
- Introduced by Möbius, Pücker.
- **Example:** $y = x^2$ gives $X^2 = YZ$
- Lines thru origin (projective plane).
- $X^2 = YZ$ is a "cone"
- Points at Infinity:
  $Z = 0 \Rightarrow X = 0$,
- These points, $[0, Y, 0]$, lie on horizon.





Projective Plane and Complex Numbers

- **Example:** $x^2 + y^2 = 1$
- Projective curve: $X^2 + Y^2 = Z^2$
- Pts at infinity,
  $Z = 0 \Rightarrow X^2 + Y^2 = 0.$
- In $\mathbb{C}$, Circular pts at infinity.
  $X = 1, Y = i : I_1 = (1, i, 0)$
  $X = 1, Y = -i : I_2 = (1, -i, 0)$



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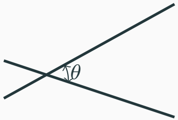
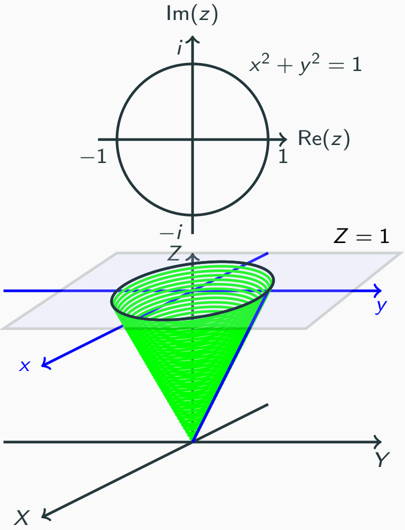
- Edmund Laguerre (1834-1886)
  - Angles, $\theta = i \log R$.
- R - Cross ratio





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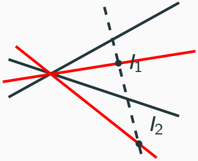
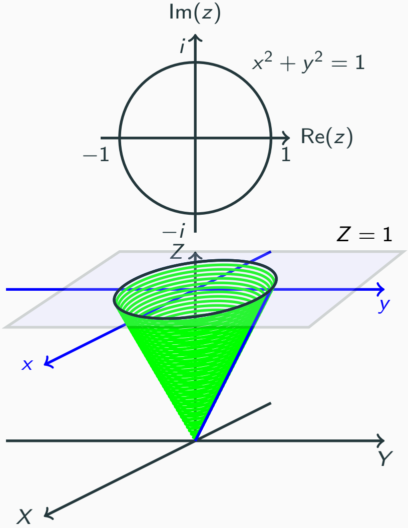
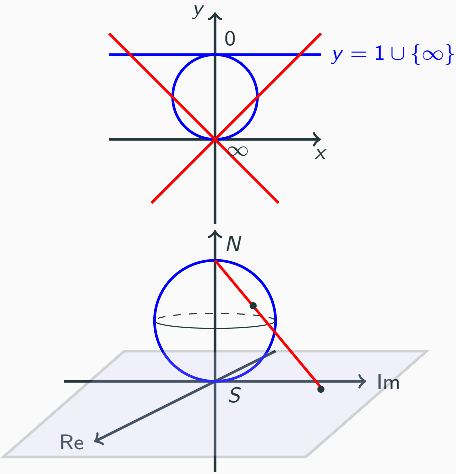




Stereographic Projection

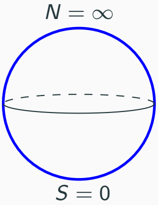
What do complex curves look like?

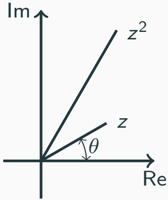
- Projective lines:
  Lines thru origin
  Topologically looks like a circle, S^1, after adding point at infinity
- Extend to $\mathbb{C}$ - topologically, S^2
- Stereographic Projection
  Connect pts in $\mathbb{C}$ to North Pole.
- N mapped to pt at ∞.
- Möbius (1790-1868) Image of circle = circle.



Mapping Functions onto Surfaces

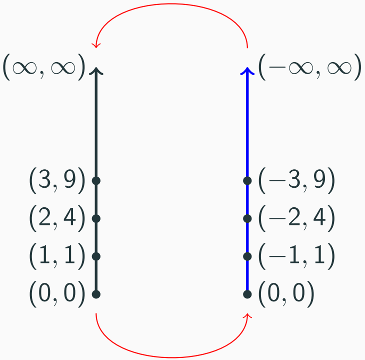
- Riemann (1826-1866)
  - Riemannian manifolds
  - Curved spaces - Gauss
  - Complex Analysis - Riemann surfaces [Cauchy (1788-1857)]
  - Number theory - $\zeta(s)$
- Start with a Sphere
- Extend $f : \mathbb{C} \to \mathbb{C}$ to $g : S^2 \to S^2$.
- Complex function, $f(z) = z^2$
  Let $z = re^{i\theta}$.
  [$\theta =$ argument, $r =$ modulus, $|z|$.]
  Then, $f(z) = r^2 e^{2i\theta}$.

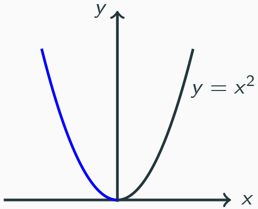


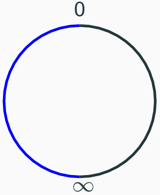


Real Function, $y = f(x)$, Mapped to S^1

- **Example** $f(x) = x^2$.

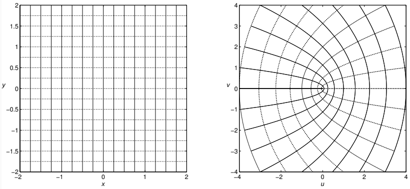


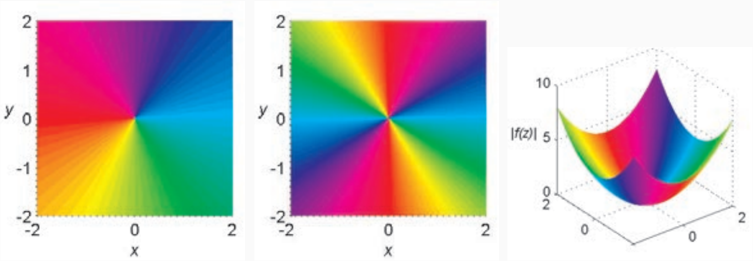




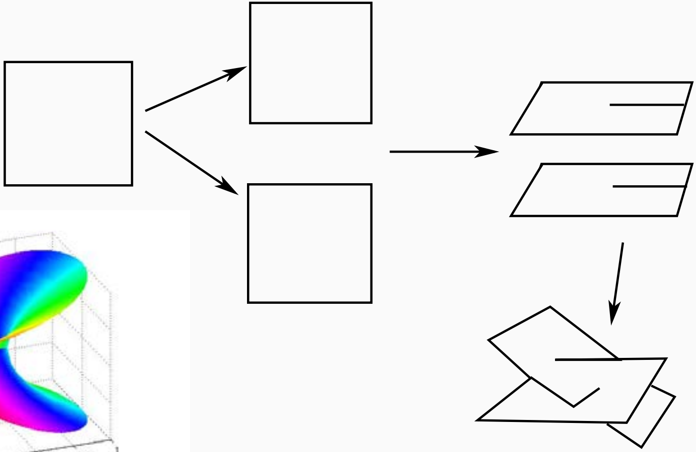
Visualizing Complex Functions: $w = f(z) = u(x, y) + iv(x, y)$

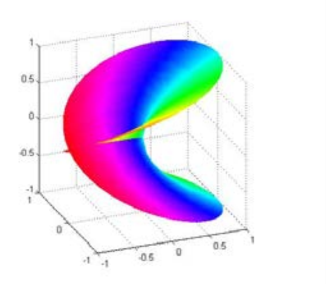
- What is $f(z) = z^2$?
- Map xy-plane to uv-plane.
- $(x + iy)^2 = x^2 - y^2 + 2ixy$.
- $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$
- Domain Coloring





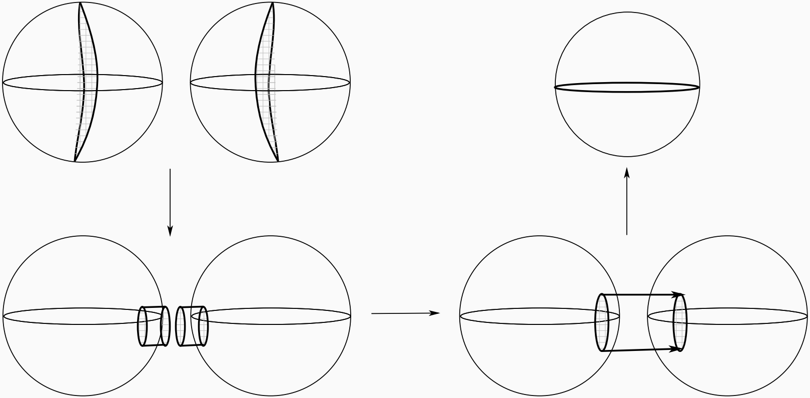
- Riemann Sheets - Two copies of $\mathbb{C}$. Riemann's Dissertation, 1851.
- **Example:** $w = \sqrt{z}$.





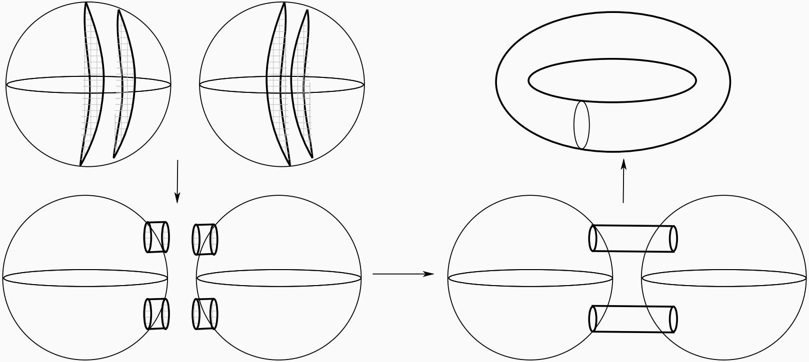
Mapping $f(z) = z^2$ to S^2.

- **Example:** $f(z) = z^2$.

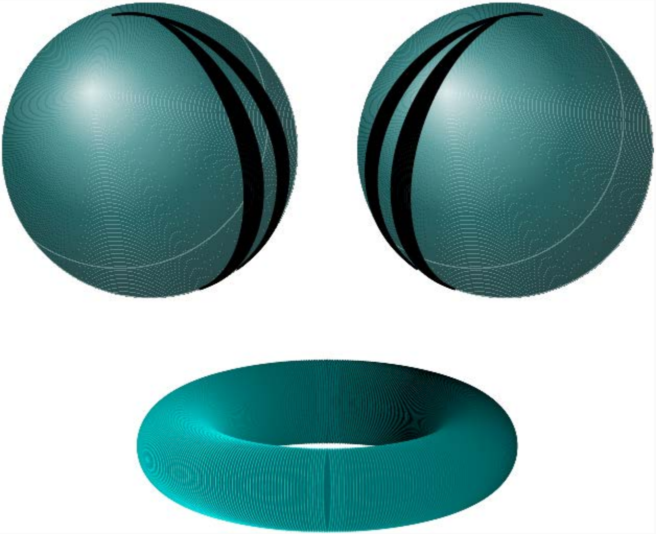


Riemann Surfaces and Elliptic Integrals

- **Example:** $\int_0^x \frac{dz}{\sqrt{z(z-a)(z-b)(z-c)}}$.
- $w^2 = z(z-a)(z-b)(z-c)$
- Beginning of topology.



Merging Two Cut Riemann Spheres



Beginnings of Topology ...

Figure 4: Genus $g = 1, 2, 3$.

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Women in Mathematics in the 1800s

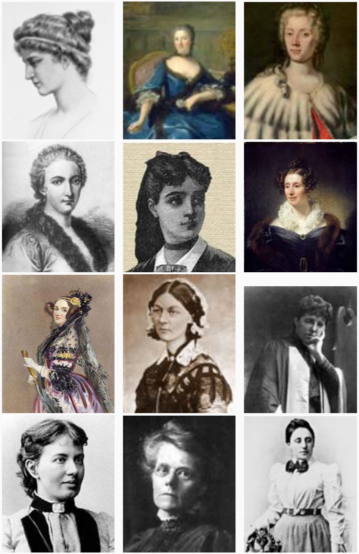
Fall 2021 - R. L. Herman





Famous Women Mathematicians Before 1900

- Hypatia of Alexandria (c. 350-415)
- Émilie du Châtelet (1706-1749)
- Laura Bassi (1711-1788)
- Maria Agnesi (1718-1799)
- Sophie Germain (1776-1831)
- Mary Fairfax Somerville (1780-1872)
- Ada Lovelace (1815-1852)
  (Augusta Byron, Countess of Lovelace)
- Florence Nightingale (1820-1910)
- Charlotte Angas Scott (1848-1931)
- Sofia Kovalevskaya (1850-1891)
- Alicia Boole Stott (1860-1940)
- Amalie 'Emmy' Noether (1882-1935)



Émilie du Châtelet (1706-1749)

- Gabrielle-Émilie Le Tonnelier de Breteuil
- Father - official at the Court of Louis XIV at Versailles.
- Husband - Marquis Florent-Claude Chastellet, military man, governor of Semur-en-Auxois in Burgundy.
- Lovers: Pierre Louis Moreau de Maupertuis (1698-1759), Alexis Clairaut (1713-1765) and François-Marie Arouet (Voltaire) (1694-1778).
- Wrote on Newton, Leibniz, and the propagation of fire.
- Translation of the *Principia* into French.
- Debated Euler and others over *vis viva*, "living force," or kinetic energy Σmv^2.

Figure 1: Émilie du Châtelet

Laura Bassi and Marie Agnessi

- Laura Bassi (1711-1788)
  - 1st female physics professor. Studied Newton, electricity.
  - Second in the world: Ph.D., 1732.

    1st - philosopher Elena Cornaro Piscopia, 1678.
  - First woman: doctorate in science.
- Maria Agnessi (1718-99).
  - First woman: mathematics handbook.
  - First woman appointed: mathematics professor.
  - First book on both differential and integral calculus
  - Witch of Agnesi curve.

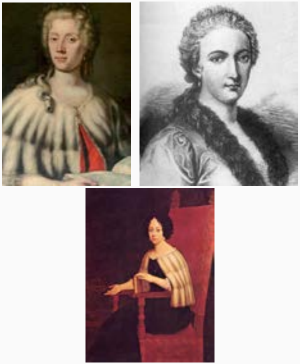


Figure 2: Bassi, Agnessi, Piscopia.

Marie Sophie Germain (1776-1831)

- Self-taught, French revolution
- 1794 - École Polytechnique - for men
  Signed HW - Monsieur Le Blanc
- Joseph-Louis Lagrange (1736-1813)
- Adrien-Marie Legendre (1752-1833)
- Gauss (1777-1855) - letters 1804-12; saved his life.
- Germain Primes - If p is prime, then so is $2p+1$ Ex: $5 = 2(2)+1$, $7 = 2(3)+1$, ~~$9 = 2(4)+1$~~
- Elasticity work did not get her name on Eiffel Tower.

Figure 3: Sophie Germain

- Fermat's Last Theorem
- Chladni Plates, elasticity.
- Competitions 1811, 1813, 1815.

Mary Fairfax Somerville (1780-1872)

- Mathematics and astronomy
- Wrote books
- Jointly - the first female member of the Royal Astronomical Society with Caroline Herschel.
- First to sign petition to Parliament to give women the right to vote.
- experiments to explore the relationship between light and magnetism
- Translated/expanded Laplace's work, 1831, *The Mechanism of the Heavens*.
- First Geography text, 1848.

Figure 4: Mary Sommerville.

Ada Lovelace (1815-1852)

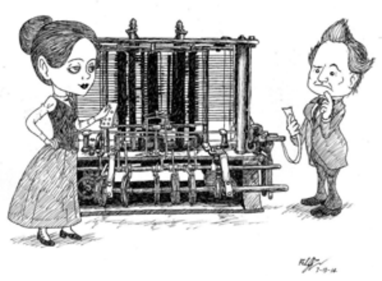
- Daughter of Lord Byron, (poet, died 1824) and
- Mathematician Anne Isabelle Milbanke, self-named as "princess of parallelograms."
- She wrote papers and first computer programs.

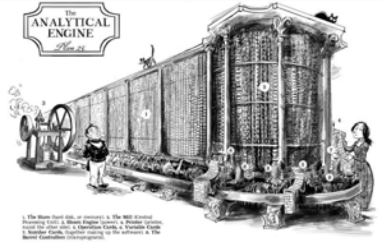
Figure 5: Charles Babbage

Figure 6: Augusta Ada Byron

Ada Lovelace and Charles Babbage

- Charles Babbage (1791-1881)
  - English mathematician, philosopher, engineer.
  - 1833 Difference Engine.
  - 1844 Analytical Engine.
  - Designed, never Built.
- Lovelace first algorithm for a machine.
- 1842-1843, Translated an article Luigi Menabrea on the engine. added notes containing first computer program.
- Loops, recursion - Bernoulli numbers, systems of linear equations.
- 1980's - Ada, programming language.





Florence Nightingale (1820-1910)

- Crimean War (1853-1856)
- Supervised nurses.
- Studied under famous mathematicians.
- Used statistics - mortality rates
- Pioneer in data visualization, polar area diagrams.
- National heroine, 1883 recipient of the Royal Red Cross, and later others.

Figure 7: Florence Nightingale

Sofia Kovalevskaya (1850-1891)

- Born Sofya Vasilyevna Korvin-Krukovskaya in Moscow.
- Education in Europe.
- Teachers - Hermann von Helmholtz, Gustav Kirchhoff and Robert Bunsen.
- Advisor - Weierstrass (1874) - 3 papers PDEs, elliptic integrals, Saturn's rings.
- 1st woman to get doctorate in math outside Italy. - not enrolled! 1874.
- 1883 Teaching position, U. Stockholm.
- 1889 1st to hold chair in European university since Laura Bassi and Maria Agnessi.

Figure 8: Sofia Kovalevskaya and Karl Weierstrass (1815-1897)

Sofia Kovalevskaya (1850-1891)

- Light waves, tops, wrote books.
- 1886 - French Competition - spinning tops.
- 1889 - Swedish Academy of Science Prize Chebyshev got her membership in Imperial Academy of Sciences
- 1891 - On vacation, Influenza - pneumonia.
- Cauchy–Kovalevskya Theorem: local existence and uniqueness theorem for Cauchy problem in PDEs.
- Kowalevski top - a symmetric top with a particular ratio of the moments of inertia: $I_1 = I_2 = 2I_3$.

Figure 9: Sofia Kovalevskaya

Turn of Century - Charlotte Scott and Alicia Stott

Charlotte Angas Scott (1848-1931)

- One of 1st woman to obtain a doctorate in England.
- Studied under Arthur Cayley.
- Algebraic curves of degree higher than two.
- 1885 - 1st mathematician at Bryn Mawr College, dept head.
- A founder of AMS.

Alicia Boole Stott (1860-1940)

- Parents: George Boole (1815-1864) and Mary Everest Boole (1832-1916).
- Four-dimensional polytopes.
- Exactly six regular polytopes in four dimensions
- Worked with Harold Coxeter,(1907–2003).

Amalie ‘Emmy’ Noether (1882-1935)

- German mathematician
- Abstract algebra - theories of rings, fields, and algebras.
- Noether’s theorem - connects symmetry and conservation laws.
- Mathematical Institute of Erlangen, 1908–1915 - without pay.
- University of Göttingen, 1915-1933, First four years lecturing under Hilbert’s name.
- Bryn Mawr - 1933-5.
- Lectured at Institute for Advanced Study in Princeton.

Figure 10: Emmy Noether

That’s *Not* All Folks! (Click on the Links)

- Ellen Amanda Hayes (1851-1930)
- Christine Ladd-Franklin (1847-1930)
- Elizaveta Fedorovna Litvinova (1845-1919)
- Ada Isabell Maddison (1869-1950)
- Helen Abbot Merrill (1864-1949)
- Mary Frances Winston Newson (1869-1959)
- Mary Emily Sinclair (1878-1955)
- Pauline Sperry (1885-1967)
- Anna Johnson Pell Wheeler (1883-1966)
- Grace Chisholm Young (1868-1944)







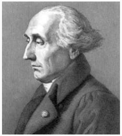



Non-Euclidean Geometry and Group Theory

Fall 2021 - R. L. Herman








Euclidean Geometry

- 300 BCE - Euclid's *Elements*
- Five Postulates.
- 5th Postulate - not needed in first 28 propositions.

If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.

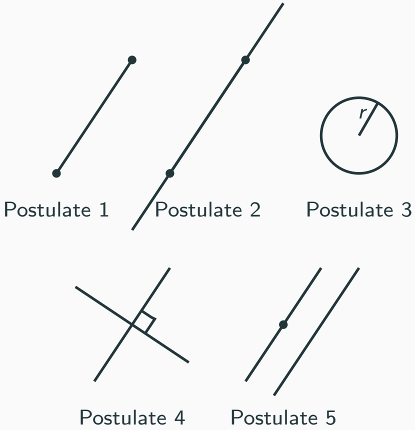


Figure 1: Euclid's 5 Postulates.

Statements of Parallel Axiom in Text

$\mathbf{P}_1$ For each straight line L and point P outside L there is exactly one line through P that does not meet L.

Equivalent statements

The angle sum of a triangle $= \pi$. - Euclid.

The locus of points equidistant from a straight line is a straight line. - al-Haytham.

Similar triangles of different sizes exist. - Wallis

Saccheri (1733) - provided two alternatives to arrive at proof by contradiction.

$\mathbf{P}_0$ There is not line through P that does not meet L.

$\mathbf{P}_2$ There are at least two lines through P that do not meet L.

Parallel Postulate

- Proclus (410-485) Equivalent postulate. Revived as Playfair axiom.
- William Ludlam (1785):
  Two straight lines, meeting at a point, are not both parallel to a third line.
- John Playfair, *Elements of Geometry* (1795):
  Playfair's axiom: Two straight lines which intersect one another cannot be both parallel to the same straight line.
- Many false attempts to prove based on other four postulates.
- 1663 John Wallis "To each triangle, there exists a similar triangle of arbitrary magnitude."
- Giralomo Saccheri (1667-1733) Assume 5th postulate false and get contradiction.
- Used assumption - lines are infinite. Led to contradiction of $\mathbf{P}_1$, almost $\mathbf{P}_2$.
- d'Alembert, 1767 - "The scandal of elementary geometry."

Spherical Geometry

- Lines = geodesics,
  Lie on great circles.
- Euclidean triangles, $a + b + c = \pi$.
- Spherical triangles, $a + b + c > \pi$.
- Thomas Harriot (1560-1621), astronomy, mathematics, and navigation
- Johann Heinrich Lambert (1726-1777)
  - General properties of map projections.
  - hyperbolic functions
  - π is irrational
  - optics

$a + b + c = \pi + \frac{A}{R^2}$.

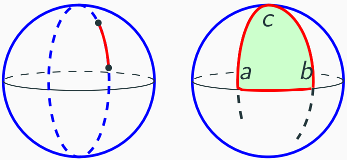


Figure 2: Harriot and Lambert.

Other Geometries

- Ferdinand Karl Schweikart (1780) Astral geometry, sum of three angles of a triangle is less than two right angles.
- Wrote to Gauss, 1818, via student Christian Ludwig Gerling (1788-1864).
- Franz Taurinus (1784-1854), Schweikart's nephew. Proposed geometry on a sphere of imaginary radius, logarithmic-spherical geometry.
- 1826, hyperbolic law of cosines in *Geometriae prima elementa.*
- Wrote to Gauss. after being encouraged, he sent copies of his works with no reply.
- Later he burned copies of his book.

Figure 3: Gerling and Schweikart

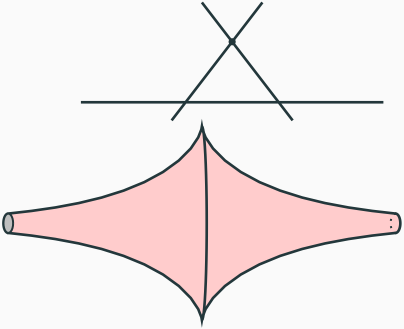
Parallel Postulate Revisited

- Carl Friedrich Gauss (1777-1885) started on it in 1799; was convinced it was independent of first 4.
- Discussed with Farkas Bolyai (1775 - 1856) - told his son no to waste his time.
- János Bolyai (1802-1860) - Believed a non-Euclidean geometry existed.
- Nikolai Lobachevsky (1792-1856) - independently 1840 new 5th postulate: There exists two lines parallel to a given line through a given point not on the line.
  Developed trig identities, hyperbolic geometry.

Figure 4: Gauss, Bolyai, Lobachevsky

Riemannian Geometry

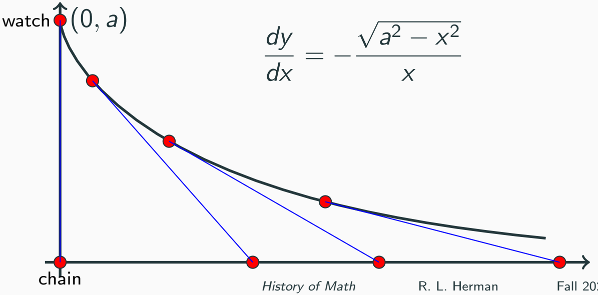
- Georg Friedrich Bernhard Riemann (1826-1866)
  - Published in 1868 Lecture
  - Spherical geometry
  - Riemannian geometry → differential geometry
  - Every line through a point not on a given line meets the line.
- Eugenio Beltrami (1835-1900) Published interpretations of non-Euclidean geometry - introduced pseudosphere in 1868 using a **tractrix**.



tractrix $(a(t - \tanh t), a\operatorname{sech} t)$

Aside: The Tractrix

- Claude Perrault [brother Charles author of *Cinderella*, *Puss-in-Boots*] in 1693, Paris, placed a watch in the middle of a table and pulled its chain along the edge of the table. What was the curve traced out ?
- Studied by Newton (1676), Huygens (1692) and Leibniz (1693). Euler gave complete theory in 1788. [Am. Math. Monthly, 72(10) (1965), 1065-1071.]
- Huygens coined name from Latin, *tractus*.



$$\frac{dy}{dx} = -\frac{\sqrt{a^2 - x^2}}{x}$$

Curvature

- $k = 0,\ k > 0,\ k < 0.$
- sums of angles of triangles $a + b + c - \pi = kA.$

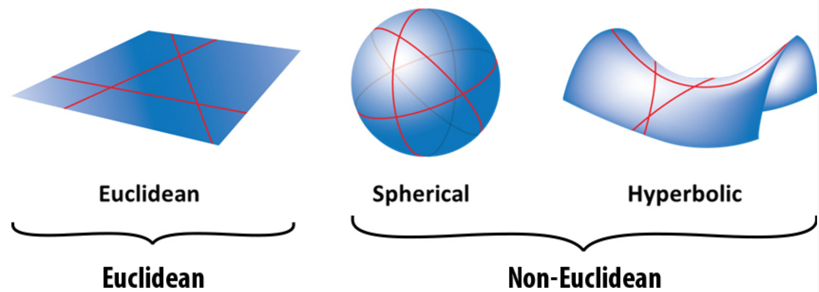


Figure 5: Surfaces of Constant Curvature.

Hyperbolic Geometry

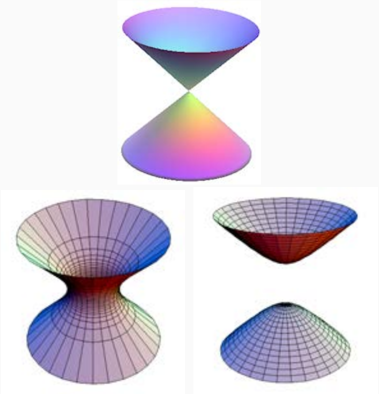
- Sphere

$$x^2 + y^2 + z^2 = \text{const}$$

- Modify

$$x^2 + y^2 - z^2 = K$$

- $K = 0$, $z^2 = x^2 + y^2$. Cones.
- $K = 1$, $x^2 + y^2 - z^2 = 1$.
  Hyperboloid of one sheet
- $K = 1$, $z^2 - x^2 - y^2 = 1$.
  Hyperboloid of two sheets.



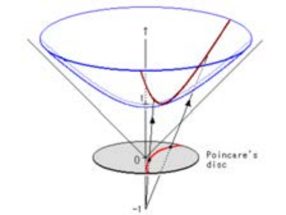
Beltrami-Poincaré Model

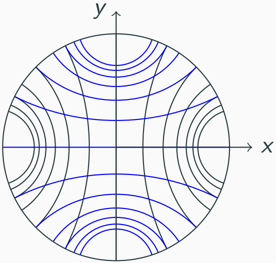
- Poincaré's Disks

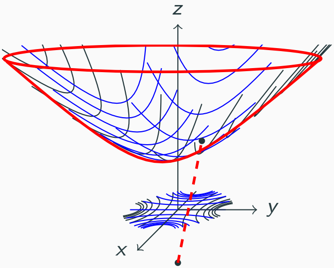
  $$(x, y, z) = (c\cosh t, \sinh t, \sqrt{1+c^2}\cosh t)$$

- Stereographic Projection thru $(0, 0, -1)$ to $z = 0$: $(x, y, z) \to \frac{(x,y)}{1+z}$.

- Hyperbolic geometry.



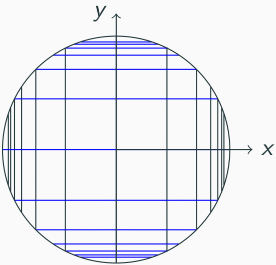


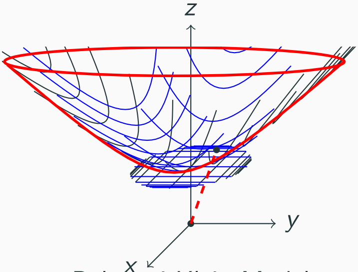


Beltrami-Poincaré Model

Beltrami-Klein Model

- Stereographic Projection thru $(0, 0, 0)$ to $z = 1 : (x, y, z) \to \frac{(x,y)}{z}$.
- Klein's Disks
  Projection to $(0, 0, 1)$

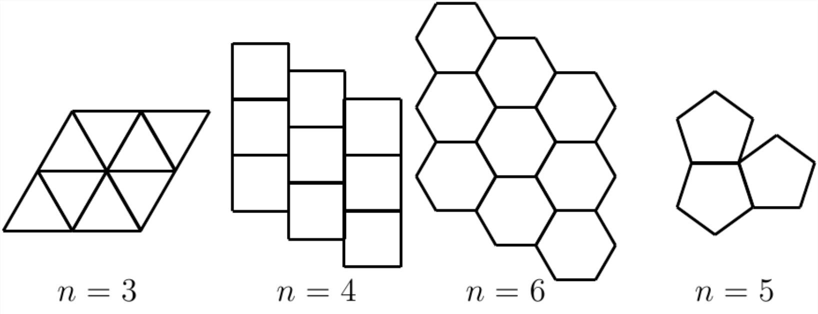




Beltrami-Klein Model

Tiling the Plane

One can tile the plane with a single polygon with sides 3, 4, and 6. However, one cannot fit pentagons together. As seen below, the angles do not allow for a fit. For large *n*, the interior angles are too small.



Other Tilings

- Johannes Kepler (1571-1630)
  - Studied Tilings
  - *Harmonicae Mundi* (Harmony of the World).
  - Planned in 1599.
  - Published 1619 - delay by Tycho Brahe to look as orbit of Mars.
- Roger Penrose (1931-)
  - 2020 Nobel Prize
  - 70's Inspired by Tilings - Penrose tilings. In 80's found in nature.
  - and M. C. Escher (1889-1972)
  - Circle Limit - Tiling Hyperbolic Plane.
- Others - Polyominoes and Pentominoes.

Figure 6: Circle Limit IV

Hyperbolic Tessellations

- Tessellation = cover plane by tiles, no tiles overlap and no space empty.
- Schläfli symbol: $\{n, m\}$,
  $n =$ number of sides on the tile,
  $m =$ number of tiles that meet at a vertex.
  Euclidean: $\dfrac{1}{n} + \dfrac{1}{m} = \dfrac{1}{2}$,
  Hyperbolic: $\dfrac{1}{n} + \dfrac{1}{m} < \dfrac{1}{2}$.

Figure 7: Circle Limits I-IV.

Group Theory

1843 - Joseph Louiville (1809-1882) reviewed Galois' delayed manuscript, published 1846. - introduction of groups and fields.

- Multiplicative group modulo n.
- Euler - Fermat's Little Theorem
  p prime, $(a, p) = 1$,
  $$a^{p-1} \equiv 1 (\text{mod } p).$$
- Euler's ϕ function:
  $$\phi(n) = \#\{k \in \{1, 2, \ldots, n-1\} | (k, n) = 1\}.$$
  $\phi(5) = 4$, $\{1, 2, 3, 4\}$,
  $\phi(8) = 4$, $\{1, 3, 5, 7\}$.
- Group Properties:

  closed, identity, Inverse, associative

Examples: Mod 5 and 8.

x	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

x	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Carl Friedrich Gauss (1777-1855)

- *Disquisitions Arithmeticae* - 1801
- Summary and Extension on Number Theory.
- Initiated finite Abelian groups.
- Proved Fermat's Little Theorem. Represented integers as quadratic forms, like Fermat Primes ($4n + 1 = x^2 + y^2$.) for x and y integers.
- Binary quadratic forms - $ax^2 + bxy + cy^2$ - for a, b, c integers.
  - composition has properties of an abelian group.
- Did not have a general theory of groups.

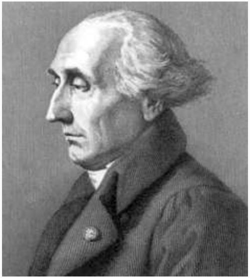
Joseph Louis Lagrange (1736-1813)

- Born in Turin, Italy.
- Professor at 19 (artillery school).
- 1766 Frederick the Great sought great mathematician.
- Lagrange replaced Euler in Berlin for 20 yrs.
- Invited by Louis XVI to Paris.
- 1795 - established dept. at École Normal.
- 1797 - established dept. at École Polytechnique.
- Napoleon made him senator, count, and he received many other honors.
- Sought solution of quintic by studying cubic and quartic.
- Made many other contributions.

Resolvents

- Consider $x^3 + nx + p = 0$. Let $x = y - \frac{n}{3y}$.
- Yields 6^{th} degree polynomial,
  $y^6 + py^3 - \frac{n^3}{27} = 0$, the resolvent.
- Let $r = y^3$, $r^2 + pr - \frac{n^3}{27} = 0$.
- Has roots r_1, r_2, where $r_2 = -\left(\frac{n}{3}\right)^3 \frac{1}{r_1}$.
- Then, $x = \sqrt[3]{r_1} + \sqrt[3]{r_2}$
- Cardano got this real root but did not seek complex solutions.
- Lagrange knew there should be 3 roots.
  $\sqrt[3]{r}, \omega\sqrt[3]{r}, \omega^2\sqrt[3]{r}$, where ω is a cube root of unity, $\omega^3 = 1$. Then,

$$\begin{aligned} x_1 &= \sqrt[3]{r_1} + \sqrt[3]{r_2} \\ x_2 &= \omega\sqrt[3]{r_1} + \omega\sqrt[3]{r_2} \\ x_2 &= \omega^2\sqrt[3]{r_1} + \omega^2\sqrt[3]{r_2} \end{aligned}$$



Permutation of Roots

- Lagrange then wrote roots of the resolvent
  $y = x_i + \omega x_j + \omega^2 x_k, \quad i, j, k = 1, 2, 3, \quad i \neq j \neq k.$
- 3!=6 permutations of cubic roots.
- In $y^6 + py^3 - \frac{n^3}{27} = 0$, the coefficients of y^5, y^4, y^2, y are $x_1 + x_2 + x_3$, $p = x_1 x_2 x_3$, and $\dfrac{n^3}{27} = \dfrac{(x_1 x_2 + x_1 x_3 + x_2 x_3)^3}{27}$.
- Resolvent coefficients are rational functions of the cubic roots.
- Lagrange obtained similar results for the quartic.
- Lagrange sought solutions of higher order equations using symmetric functions of the roots and permutations.
- Paola Ruffini (1765 – 1822) - 1802, 1805, 1813 - gave proofs that quintic can't be solved. Proofs not understood.

Niels Henrick Abel (1802-1829)

- Born in Norway into poverty and had a pulmonary condition.
- Mathematical ability discovered by his teacher.
- Toured Europe after college and published 5 papers in *Journal für die reine und angewandte Mathematik*.
- Studied convergence of infinite series, the theory of doubly periodic functions, elliptic functions, elliptic functions and the theory of equations.
- Could not get employment, so tutored.
- At university, thought he had solution of quintic. Then, proved no solution existed.
- Died of tuberculosis before completing his work.

Évariste Galois (1811-1832)

- Born Oct 25, 1811
- Interest in math at 14.
- Read Adrien-Marie Legendre (1752-1833).
- 1828 Failed to get into École Polytechnique.
- 1829 Paper on continued fractions.
- Studied polynomial equations.
- Wrote two papers.
  Reviewed by Arthur Cayley (1821-1895)
  Entered competition.
- 1830 Submitted to Joseph Fourier (1768–1830) - got lost.
  Winners - Niels Henrik Abel (1802-1829) and Carl Gustav Jacobi (1804-1851).
- Published 3 papers.

Figure 8: Évariste Galois

Évariste Galois (cont'd)

- Political turmoil in France.
- Student uprising - Galois left school.
- He was arrested and acquitted.
- Arrested Jul 1831 - April 29, 1832.
- Siméon Denis Poisson (1781-1840) asked him to submit work 1831.
- July 4 - declared work incomprehensible.
- Galois found out in October.
- Stayed up all night; wrote letters and note to Auguste Chevalier.
- On May 30, fought in duel and lost.
- Chevalier forwarded papers for publication by Joseph Liouville.

Figure 9: Legendre, Cayley, Fourier, Jacobi, Poisson, Liouville

Symmetry Groups

- Levi ben Gorshun (1321) Number of permutations of n objects $= n!$
- Leads to Symmetric Group.
- Felix Klein (1872) extended groups to geometry - studied invariants of groups of transformations.
- Sophus Lie (1842-1899) continuous groups of transformations, applied to differential equations.
- Emmy Noether (1882-1935) related symmetries to constants of motion in physics.


Figure 10: Sophus Lie and Emmy Noether.