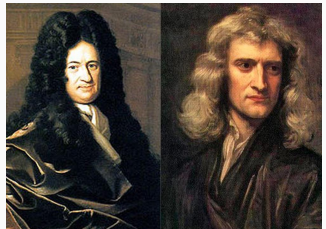


Emergence of Calculus

Fall 2021 - R. L. Herman



The Method of Exhaustion and the Infinite

- Zeno's Paradox of the Arrow
 - “If a body moves from A to B, then before it reaches B it passes through the mid-point, say B_1 of AB. Now to move to B_1 it must first reach the mid-point B_2 of AB_1 . Continue this argument to see that A must move through an infinite number of distances and so cannot move. ” (450 BCE)
- Eudoxus - Method of Exhaustion.
- Archimedes - area of a segment of a parabola is $\frac{4}{3}$ the area of a triangle with the same base and vertex.
- Luca Valerio (1552-1618) published in 1608 *De quadratura parabolae*.
- Kepler (1571-1630): area as sum of lines. Inspired Cavalieri's indivisibles.

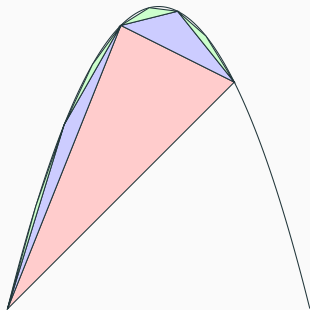


Figure 1: Archimedes: 1st known summation of series.

Developments in the 1600's

Rapid developments first 60 years of 1600's based on Greek geometry, algebra, astronomy (Kepler, Galileo). Led to unification of geometry and algebra.

- Descartes (1596-1650)
- Cavalieri (1598-1647)
- Fermat (1601-1665)
- Roberval (1602-1675)
- Wallis (1616-1703)
- Barrow (1630-1677)
- Gregory (1638-1675)
- Newton (1642-1727)
- Leibniz (1646-1716)

Two main problems

- Tangents
- Areas

Need curves

- Conics
- Archimedean spiral
- Conchoid
- Cissoid
- Cycloid

Seventeenth Century - French, German, English Mathematics

- Galileo Galilei (1564-1642)
- Johannes Kepler (1571-1630)
- 1590 Viète, *The Analytic Art*
- John Napier (1550-1617) and Henry Briggs (1561-1631) - Introduced the logarithm
- French Mathematicians:
René Descartes (1596-1650)
Blaise Pascal (1623-1662)
Pierre de Fermat (1601-1665)



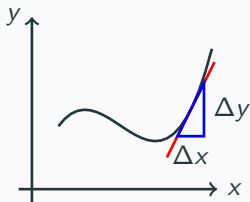
- Descartes
philosopher, mathematician
Discours de la méthode, Marriage of algebra/geometry - analytic geometry
- Pascal
Wrote math before 16
Probability theory
Theology
- Fermat
Created analytic geometry
Contributions to Calculus
Number theory
Scribbled in Diophantus' *Arithmetica*

Tangents

- Pierre de Fermat, René Descartes
- Both studied Apollonius 4-line problem.
- Tangent line approximates the curve at a point.
- Slope $\frac{\Delta y}{\Delta x}$.
- Infinitesimals - increments.
- Fermat
Method for maxima-minima
1636 - Method of Tangents
- 1636 Letter from Descartes to Mersenne
 $dy = f(x + dx) - f(x) = ?dx$.

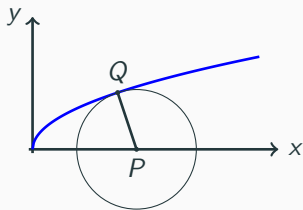
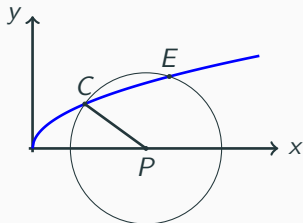


Figure 2: Fermat and Descartes



Descartes vs Fermat - Analytic Geometry, Tangents, Optics

- Descartes published *La Géométrie* - 1637
- Depicted $ax + by = c$ as a line.
- Introduced x, y .
- Fermat introduced analytic geometry earlier.
- Fermat interested in optimization.
- Fermat: lawyer in Toulouse, Math a hobby.
- Descartes denounced him and challenged him to find tangent to folium, $x^3 + y^3 = 3axy$.
- Descartes' Method of Tangents: Find circles tangent to curves.
- Fermat challenged Descartes to explain refraction. Fermat published in 1662.



Areas Under Curves

- First studied by Eudoxus, Archimedes
- Bonaventura Cavalieri (1598-1647) - indivisibles

Fill area with lines.

But, an infinite number of lines sum to infinity.

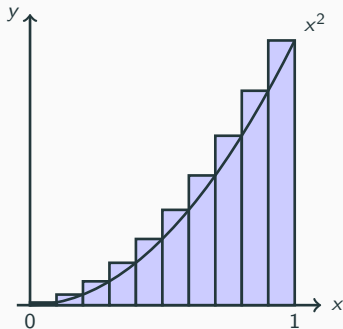
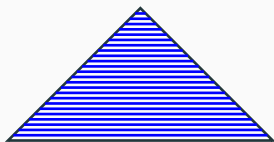
- Archimedes, John Wallis (1616-1703):

$$\int_0^1 x^2 dx.$$

N segments of width $\frac{1}{N}$. and height $\left(\frac{k}{N}\right)^2$, $k = 1, 2, \dots, N$.

$$A \approx \sum_{k=1}^N \frac{1}{N} \left(\frac{k}{N}\right)^2.$$

History of Math



R. L. Herman

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Areas Under Curves (cont'd)

Find the sum

$$\begin{aligned}A &\approx \sum_{k=1}^N \frac{1}{N} \left(\frac{k}{N}\right)^2 \\&= \frac{1}{N^3} \sum_{k=1}^N k^2 \\&= \frac{1}{N^3} \frac{N(N+1)(2N+1)}{6} \\&\sim \frac{2N^3}{6N^3} = \frac{1}{3}.\end{aligned}$$

Wallis showed

$$\int_0^a x^n dx = \frac{x^{n+1}}{n+1} \Big|_0^a = \frac{a^{n+1}}{n+1}$$

for $k = 1, 2, \dots, 9$.

Note:

$$\begin{aligned}\sum_{k=1}^N k &= 1 + 2 + \dots + \underbrace{(N-1) + N}_{2+(N-1)} \\&= \underbrace{\hspace{10em}}_{1+N} \\&= N \frac{N+1}{2}.\end{aligned}$$

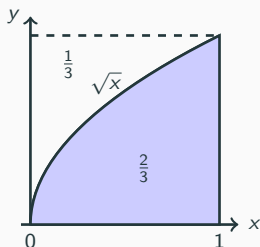
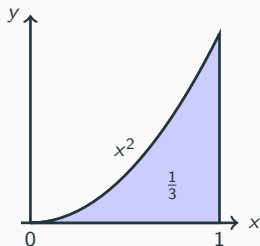


Figure 3: John Wallis

Integrating Powers, $\int x^k dx$,

- al Haytham (965-1039) $k = 1, 2, 3, 4$.
- Cavalieri (1635) knew for k up to 9.
- Proven in general by Fermat, Descartes, Roberval, 1630's.
- Fractional Powers (Fermat)
Ex: $\int_0^1 \sqrt{x} dx$
Use the symmetry in the figures.
- Areas under x^k , need sums
 $1^k + 2^k + \dots + n^k$.
- Volumes - use cylinders, $V = \pi r^2 h$.
Sums needed: $1^{2k} + 2^{2k} + \dots + n^{2k}$.

Note: $1^3 + 2^3 + \dots + k^3 = (1 + 2 + \dots + k)^3$.



Evangelista Torricelli (1608-1647), barometer inventor

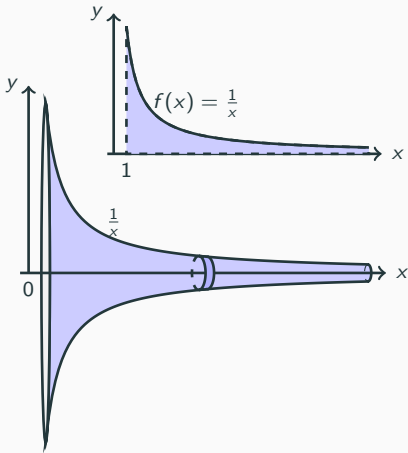
- Inverse Powers, x^{-1}
- Area under $y = \frac{1}{x}$.

$$\int_1^{\infty} \frac{1}{x} dx = \infty.$$

- 1641 Torricelli's trumpet (Gabriel's horn)

$$V = \pi \int_1^{\infty} \frac{1}{x^2} dx = \pi.$$

$$A = 2\pi \int_1^{\infty} \frac{dx}{x} \sqrt{1 + \left(-\frac{1}{x^2}\right)^2}$$
$$> 2\pi \int_1^{\infty} \frac{1}{x} dx = \infty.$$



What? You cannot paint the surface but can fill the trumpet with paint.

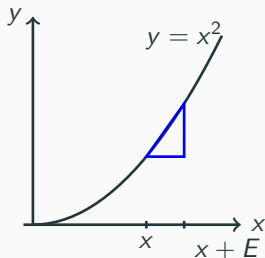
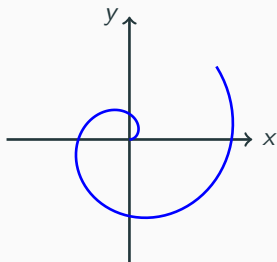
Hobbes - "to understand this for sense, it is not required that a Historiographer be a geometrical or a Mathematician, but that he will build by words."

Tangents, Maxima, Minima

- Curves studied like Archimede's spiral, $r = a\theta$
- Fermat - studied polynomials
- Work simpler than Descartes
- Used infinitesimals, E
- **Example:** $y = x^2$

$$\frac{(x + E)^2 - x^2}{E} = 2x + E.$$

- Generalized to polynomials, $p(x, y) = 0$.



John Wallis' (1655) *Arithmetica Infinitorum*

- Combined Descartes' analytic geometry and Cavalieri's indivisibles.
- Some results already known.
- New approach to fractional powers.
- Ambivalent use of infinitesimals - attacked by Thomas Hobbes (1588-1679).
- Formulae for π known by - Gregory, Newton, Leibniz
- Madhava (1350-1425) found π to 13 decimal places using series,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

Wallis' Formulae:

$$\begin{aligned}\frac{\pi}{4} &= \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \\ \frac{\pi}{2} &= \left(\frac{2}{1} \cdot \frac{2}{3}\right) \cdot \left(\frac{4}{3} \cdot \frac{4}{5}\right) \cdots \\ \frac{4}{\pi} &= 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \dots}}}\end{aligned}$$

Already known formula:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Isaac Newton (1642-1727)

- Major use of infinite series
- *A Treatise of the Methods of Series and Fluxions*
- *Quadrature of the Hyperbola*
Written in 1665,
1st publication in 1668 by
Mercator
- Akin to decimal expansions -
powers of $\frac{1}{10}$ replaced by x^n .
- Example:

$$\log(1+x) = \int_0^x \frac{dt}{1+t}$$

[Note: Here $\log x = \ln x$.]

Note: Geometric series

$$1 + t + t^2 + \dots = \frac{1}{1-t}, |t| < 1.$$

$$1 - t + t^2 - \dots = \frac{1}{1+t}, |t| < 1.$$

Then,

$$\begin{aligned} y &= \log(1+x) \\ &= \int_0^x (1 - t + t^2 - \dots) dt \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots \end{aligned}$$

Invert Power Series

We have for $y = \log(1 + x)$,

$$y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$$

In order to invert the series, let $x = a_0 + a_1y + a_2y^2 + \dots$. Then,

$$\begin{aligned}y &= (a_0 + a_1y + a_2y^2 + \dots) - \frac{1}{2}(a_0 + a_1y + a_2y^2 + \dots)^2 + \dots \\&= a_0 - \frac{1}{2}a_0^2 + \frac{1}{3}a_0^3 + a_1(a_0^2 - a_0 + 1)y \\&\quad + \left[a_2(a_0^2 - a_0 + 1) + \left(a_0 - \frac{1}{2} \right) \right] y^2 \\&\quad + \left[\frac{a_1^3}{3} + a_1a_2(2a_0 - 1) + a_3(a_0^2 - a_0 + 1) \right] y^3 + \dots\end{aligned}$$

Equate coefficients of powers of y , then ...

Series Inversion (cont'd)

We solve the resulting system of equations:

$$0 = a_0 - \frac{1}{2}a_0^3 + \frac{1}{3}a_0^3$$

$$1 = a_1 (a_0^2 - a_0 + 1)$$

$$0 = a_2 (a_0^2 - a_0 + 1) + \left(a_0 - \frac{1}{2} \right)$$

$$0 = \frac{a_1^3}{3} + a_1 a_2 (2a_0 - 1) + a_3 (a_0^2 - a_0 + 1).$$

The first equation gives $a_0 = 0$. The next two give $a_1 = 1$ and $a_2 = \frac{1}{2}$.

Continuing Newton found that

$$a_0 = 0, a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{6}, a_4 = \frac{1}{24}, \dots, a_n = \frac{1}{n!}.$$

Newton's Series for Exponential

So far, inversion of

$$\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots .$$

led to

$$x = y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \dots .$$

However,

$$y = \log(1 + x) \Rightarrow x = e^y - 1.$$

So, we found the series expansion

$$e^y = 1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \dots .$$

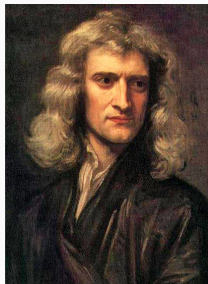


Figure 4: Newton

Newton's Series for Sine

$$\text{Newton knew } \sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}.$$

Recall binomial series: $(a+b)^n = \sum_{k=0}^n C_{n,k} a^{n-k} b^k$, where the coefficients are $C_{n,k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$. Then,

$$(1+a)^p = 1 + pa + \frac{p(p-1)}{2!}a^2 + \frac{p(p-1)(p-2)}{3!}a^3 + \dots$$

$$\begin{aligned} \sin^{-1} x &= \int_0^x \frac{dt}{\sqrt{1-t^2}}, \quad a = -t^2, p = -\frac{1}{2}, \\ &= \int_0^x \left(1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \dots + \frac{-\frac{1}{2}(-\frac{3}{2})\cdots(\frac{1}{2}-k)}{k!}(-t^2)^k + \dots \right) \\ &= x + \frac{1}{2} \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^5}{5} + \dots + \frac{1 \cdot 3 \cdots (2k-1)}{2 \cdot 4 \cdots 2k} \frac{x^{2k+1}}{2k+1} + \dots \end{aligned}$$

Inverting, Newton found $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$.

Gottfried Wilhelm Leibniz (1646-1716)

- Librarian, philosopher, diplomat, doctorate in law.
- First papers in calculus (1684).
- Led to long dispute.
- Better notation, $\frac{dy}{dx}$, $\int dx$.
- Sum, product, quotient rules.
- Proved Fundamental Theorem of Calculus,
 $\frac{d}{dx} \int f(x) dx = f(x)$.



Figure 5: Leibniz

Infinite Series

- Geometric series,
Known to Euclid (Zeno's paradox)
Leonhard Euler (1707-1783)

$$a + ar + ar^2 + \dots + ar^n + \dots = \frac{a}{1-r}, |r| < 1.$$

- Harmonic Series - Oresme (1350)

$$\begin{aligned} & 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \dots \\ &= (1) + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots \\ &\geq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots = \infty. \end{aligned}$$

- Power series - 17th Century,
Gregory, Wallis, Taylor, Mclaurin, . . .



Figure 6: Euler

Basel Problem (1644)

- Posed by Pietro Mengoli (1626-1686).

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

- Jacob and Johann Bernoulli (1704) tackled. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = ?$

$$\begin{aligned} \sum_{n=1}^N \frac{1}{n(n+1)} &= \sum_{n=1}^N \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{N} - \frac{1}{N+1} \right) \\ &= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{N} - \frac{1}{N+1} \right) \\ &= 1 - \frac{1}{N+1} \xrightarrow{N \rightarrow \infty} 1. \end{aligned}$$



Figure 7: Jacob and Johann

Euler's Solution of Basel Problem - 1734

- Descartes' Factor Theorem
- $p(x)$ - polynomial
- $p(r) = 0$ implies
- $p(x) = (x - r)q(x)$,
- $q(x)$ - polynomial

Proof:

$$\begin{aligned}p(x) &= a_0 + a_1x + \cdots + a_nx^n \\p(y) &= a_0 + a_1y + \cdots + a_ny^n \\p(x) - p(y) &= a_1(x - y) + \cdots + a_n(x^n - y^n) \\x^n - y^n &= (x - y)(x^{n-1} + x^{n-2}y + \cdots + y^{n-1})\end{aligned}$$

Let $y = r$,

$$\begin{aligned}p(x) &= (x - r)[a_1 + a_2(x + r) + \cdots + a_n(x^{n-1} + x^{n-2}r + \cdots + r^{n-1})] \\&= (x - r)q(x).\end{aligned}$$

Leonhard Euler's Solution of Basel Problem

$\sin x$ has roots $n\pi$, $n = 0, \pm 1, \pm 2, \dots$ - Generalize Factor Theorem:

$$\begin{aligned}\sin x &= Ax \left(1 - \frac{x}{\pi}\right) \left(1 + \frac{x}{\pi}\right) \cdots \\ &= Ax \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{2^2\pi^2}\right) \left(1 - \frac{x^2}{3^2\pi^2}\right) \cdots \\ &= A \left[x - \frac{x^3}{\pi^2} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) + x^5(\cdots) - \cdots \right].\end{aligned}$$

Compare to

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots.$$

Then $A = 1$, and

$$\frac{1}{3!} = \frac{1}{\pi^2} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

What are the next coefficients? - Needs proofreading

We need the x^5 terms in the expansion

$$\begin{aligned}\sin x &= x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right) \\ &= x \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{2^2\pi^2}\right) \left(1 - \frac{x^2}{3^2\pi^2}\right) \cdots \left(1 - \frac{x^2}{m^2\pi^2}\right) \cdots\end{aligned}$$

These are found by multiplying $\frac{x^2}{m^2\pi^2}$ times one of the other factors $\frac{x^2}{n^2\pi^2}$, $n \neq m$, and summing:

$$x \sum_{m=1}^{\infty} \frac{x^2}{m^2\pi^2} \sum_{n=1, n \neq m}^{\infty} \frac{x^2}{n^2\pi^2} = \frac{x^5}{\pi^4} \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{n=1, n \neq m}^{\infty} \frac{1}{n^2} = \frac{x^5}{5!}.$$

$$\frac{\pi^4}{5!} = \sum_{m=1}^{\infty} \frac{1}{m^2} \left[\zeta(2) - \frac{1}{m^2} \right] = \zeta(2)^2 - \zeta(4).$$

Another Approach to Obtain $\zeta(4)$

Differentiate $\ln \sin x = \cot x$ and use the known series expansion for $x \cot x$ in terms of Bernoulli numbers.

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2 \pi^2}\right)$$

$$\ln \sin x = \ln x + \sum_{n=1}^{\infty} \ln \left(1 - \frac{x^2}{n^2 \pi^2}\right)$$

$$x \cot x = 1 - \sum_{n=1}^{\infty} \frac{2x^2}{n^2 \pi^2 - x^2}$$

$$= 1 - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{x^2}{n^2} \sum_{k=0}^{\infty} \left(\frac{x^2}{n^2 \pi^2}\right)^k$$

$$1 - \frac{x^2}{3} - \frac{x^4}{45} - \frac{2x^6}{945} + \dots = 1 - \frac{2}{\pi^2} \sum_{k=0}^{\infty} \left(\frac{x}{\pi}\right)^{2k+2} \zeta(2k+2)$$

$$x \cot x = 1 + \sum_{k=0}^{\infty} (-1)^k B_{2k} (2x)^{2k}$$

Results for the Riemann Zeta Function, $\zeta(s)$

Therefore, we have

$$\begin{aligned}\zeta(2) &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^4}{90}, \\ \zeta(2n) &= 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots = (-1)^{n-1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n},\end{aligned}$$

where B_{2n} are Bernoulli numbers, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, \dots ,

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Euler (1748) - Zeta function can be defined as

$$\begin{aligned}\zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \\ &= \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \cdots \left(1 - \frac{1}{p^s}\right)^{-1} \cdots,\end{aligned}$$

where p is prime.

Bernhard Riemann's Contribution

- Bernhard Riemann (1826-1866)
- Riemann zeta function

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

- Values

$$\zeta(1) = \infty, \text{ harmonic series}$$

$$\zeta(2) = \frac{\pi^2}{6}$$

$$\zeta(3) \text{ irrational, proved by Apery (1981)}$$

- Zeros

$$\zeta(-2n) = 0, n \text{ integer} > 0.$$

Riemann Hypothesis:

$$\zeta(\sigma + it) = 0 \text{ when } \sigma = \frac{1}{2}.$$

- Connection to primes?



Figure 8: Georg Friedrich Bernhard Riemann

$$\zeta(s) = \frac{1}{\Gamma(2s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx$$

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}$$

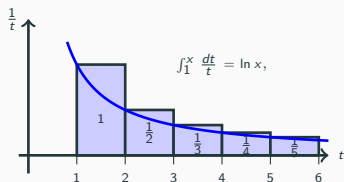
$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(s)$$

Connection to Primes and Other Tidbits

$$\begin{aligned}\zeta(s) &= \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \cdots \left(1 - \frac{1}{p^s}\right)^{-1} \cdots \\ &= \prod_{p=\text{prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \\ &= \prod_{p=\text{prime}} \left[1 + \frac{1}{p^s} + \left(\frac{1}{p^s}\right)^2 + \cdots\right]\end{aligned}$$

- Primes less than $x \sim \int_2^x \frac{dt}{\log t}$
- Euler-Mascheroni Constant
 $\gamma \approx 0.577218 \dots$
- Generalizing $n!$

$$\Gamma(n+1) = n\Gamma(n), \Gamma(0) = 1.$$



$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right) = \gamma.$$

Leonhard Euler (1707-1783)

Euler (at 14) studied under Johann Bernoulli, graduated in 1723.

Went to St. Petersburg in 1727, Berlin in 1741, and back to St. Petersburg in 1766.

By 1730's - lost vision in right eye and blind by 1771.

866 books and papers - 228 after death. *Opera Omnia* - over 25,000 pgs

First appearance of e - letter to Goldbach in 1731.

Euler published *Introductio in Analysin infinitorum* - 1748

Euler's Formula, $e^{ix} = \cos x + i \sin x$.

Euler's Identity, $e^{i\pi} + 1 = 0$.

Euler's constant, γ

Euler's Polyhedral Formula, $V + F = E + 2$.

Amicable Numbers

- Recall Greeks knew 220 and 284;
i.e., sum of proper factors of 220 =
284 and vice versa.
- Thabit ibn Qurra (836-901)
discovered the next amicable pair,
17296, 18416.
- Pierre Fermat rediscovered this pair
in 1636.
- René Descartes discovered in 1638
9,363,584 and 9,437,056.
- 1747, Euler published [E100] giving
30 amicable pairs.
- By 1750 - Euler found 61 pairs!



Euler's Formula

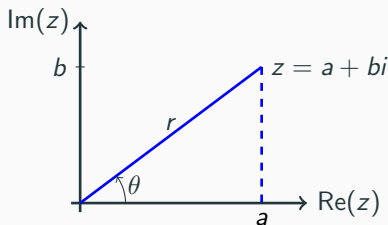
- Complex numbers, polar form.

$$z = a + bi, \quad a = r \cos \theta, \quad b = r \sin \theta$$

$$\begin{aligned} z &= r \cos \theta + ir \sin \theta \\ &= r(\cos \theta + i \sin \theta). \end{aligned}$$

- Exponential of imaginary number

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots \\ &= 1 + i\theta - \frac{(\theta)^2}{2!} - i\frac{(\theta)^3}{3!} + \dots \\ &= \left(1 - \frac{(\theta)^2}{2!} + \dots\right) + i\left(\theta - \frac{(\theta)^3}{3!} + \dots\right) \\ e^{i\theta} &= \cos \theta + i \sin \theta. \end{aligned}$$



Euler's Formula Applications $e^{i\theta} = \cos \theta + i \sin \theta$.

- $\theta = \pi$, $e^{i\pi} = -1$, or $e^{i\pi} + 1 = 0$.
- $(e^{i\theta})^n = (\cos \theta + i \sin \theta)^n e^{in\theta} = \cos n\theta + i \sin n\theta$ implies
de Moivre's Theorem

$$\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n.$$

- **Example:** $n = 2$

$$\begin{aligned}\cos 2\theta + i \sin 2\theta &= (\cos \theta + i \sin \theta)^2 \\ &= \cos^2 \theta - \sin^2 \theta + 2i \sin \theta \cos \theta.\end{aligned}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\sin 2\theta = 2 \sin \theta \cos \theta.$$

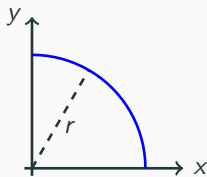
Rectification of an Ellipse

- Rectification = Finding arclengths
- $y = y(x)$

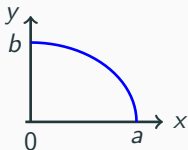
$$L = \int_a^b \sqrt{1 + y'^2} dx.$$

- **Example:** Circle

$$\begin{aligned} L &= 4 \int_0^r \sqrt{1 + \frac{x^2}{y^2}} dx \\ &= 4 \int_0^r \sqrt{1 + \frac{x^2}{r^2 - x^2}} dx \\ &= 4r \int_0^r \frac{dx}{\sqrt{r^2 - x^2}} = 4r \sin^{-1} 1 = 4r \left(\frac{\pi}{2} \right) = 2\pi r. \end{aligned}$$



Arclength of an Ellipse



- **Example:** Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad x \geq 0, y \geq 0.$$

$$y = \frac{b}{a} \sqrt{a^2 - x^2}$$

$$y' = \frac{bx}{a\sqrt{a^2 - x^2}}$$

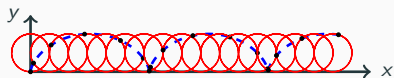
$$1 + y'^2 = \frac{a^2 - k^2x^2}{a^2 - x^2}, \quad k = \frac{a^2 - b^2}{a^2}$$

$$L = 4 \int_0^a \sqrt{\frac{a^2 - k^2x^2}{a^2 - x^2}} dx.$$

Historical Curves

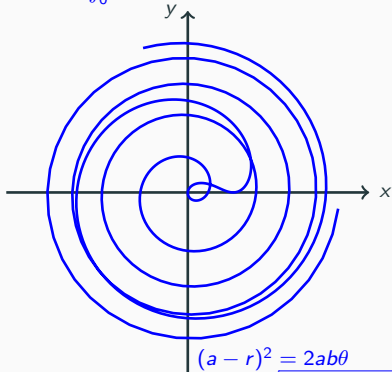
- 1609 - Kepler - Mars' orbit is an ellipse.
- 1659 - Pascal *Dimensions des lignes courbes de toutes les Roulettes.*
- 1658 Proof by Wren published by Wallis in 1659 - proof of the rectification of the cycloid.
- 1676 - Newton - infinite series.
- 1742 - Maclaurin - expansion in eccentricities.
- 1691 - Jacob Bernoulli - parabolic spiral.

Cycloid, Parabolic Spiral, and Lemniscate



$$x = r(t - \sin t), y = r(1 - \cos t)$$

$$L = \int_0^{2\pi} r\sqrt{2 - 2\cos t} dt = 8r$$



$$(a - r)^2 = 2ab\theta$$

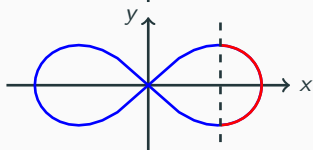
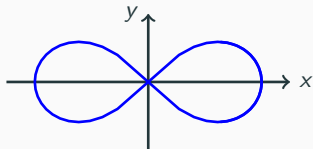
$$s = \int \sqrt{1 + \frac{r^2(a-r)^2}{a^2b^2}} dr$$

History of Math

- **Example:** Lemniscate,

$$r^2 = \cos 2\theta$$

$$L = 4 \int_0^1 \frac{dr}{\sqrt{1 - r^4}}$$



Elastica

Sep 1694, Jacob Bernoulli

Oct 1694, Johann Bernoulli

Elliptic Functions

- Lemniscate integral leads to new functions, $u = \int_0^x \frac{dt}{\sqrt{1-t^4}}$.
- Compare to $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$.
- Elliptic Integrals: $\int R(t, \sqrt{p(t)}) dt$, R is rational function, $p(t)$ is polynomial of degree 3 or 4.
- Bernoulli (1694) - geometry, mechanics.
- Fagnano (1682-1766) - Doubling arc of lemniscate.
- Carl Friedrich Gauss (1777-1855) \sim 1800 studied inverse $x = sl(u)$
Doubly periodic

$$sl(u + 2\bar{u}) = sl(u)$$

$$sl(u + 2i\bar{u}) = sl(u)$$

- Rediscovered by Niel Henrik Abel (1802-1829)
and Carl Gustav Jacobi (1804-1851) in 1820's.

Elliptic Integral Addition Theorems

- **Example Circle**

$$\begin{aligned}\sin 2u &= 2 \sin u \cos u \\ &= 2 \sin u \sqrt{1 - \sin^2 u}\end{aligned}$$

- Let $u = \sin^{-1} x$. Then,

$$\begin{aligned}2u &= 2 \int_0^x \frac{dt}{\sqrt{1-t^2}} \\ &= \sin^{-1} \left(2 \sin u \sqrt{1 - \sin^2 u} \right) \\ &= \sin^{-1} \left(2x \sqrt{1 - x^2} \right) \\ 2 \int_0^x \frac{dt}{\sqrt{1-t^2}} &= \int_0^{2x\sqrt{1-x^2}} \frac{dt}{\sqrt{1-t^2}}.\end{aligned}$$

Elliptic Integrals

- Study of Inversions
Gauss 1790s
Abel 1823 (pub 1827)
Jacobi 1829 book
- 1786 (40 yrs later)
Legendre classified elliptic integrals into 3 cases, book 1825.
Examples:

$$F(\phi) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad E(\phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \theta} d\theta.$$

- Riemann placed in geometric setting - torus.



Gauss' AGM - Arithmetic-geometric mean

- Gauss's constant $G = \frac{1}{AGM(1, \sqrt{2})} = \frac{2}{\pi} \int_0^1 \frac{dx}{\sqrt{1-x^4}} = 0.8346268\dots$
- Between 1 and $\sqrt{2}$ is $\frac{\pi}{\bar{\omega}} = \frac{1}{G}$.
- Arithmetic mean $\frac{a+b}{2}$.
- Geometric mean $\frac{a}{g} = \frac{g}{b} \Rightarrow g = \sqrt{ab}$.
- AGM(a, b) algorithm: Start with $a_0 = a, b_0 = b,$

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}, \quad n = 0, 1, \dots$$

- Gauss - $AGM(1, \sqrt{2}) = \frac{\pi}{\bar{\omega}}$ to 11 decimal places.
- Led to study of general theory, modular functions, theta functions - Ramanujan (early 1900s).

Application of $AGM(a, b)$

Example: $AGM(1, 2)$. Start with $a_0 = 1$, $b_0 = 2$,

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}, \quad n = 0, 1, \dots$$

a_n	b_n
1.0000	2.0000
1.5000	1.4142
1.4571	1.4565
1.4568	1.4568
\vdots	\vdots

$$AGM(a, b) = \frac{\pi}{4} \frac{a + b}{K\left(\frac{a-b}{a+b}\right)}, \quad K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$