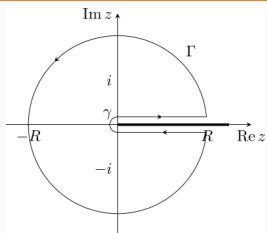


# Complex Analysis

Fall 2021 - R. L. Herman

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# History of Complex Analysis

- Before 1600
  - Cardano 1545, quadratic
  - Bombelli 1572, cubic
  - Harriot 1600, quartic
  - Negative roots - false
  - Complex roots - impossible
- 1600s
  - Descartes, 1637,  $a + b\sqrt{-1}$
  - Wallis 1685
  - Insights from geometry trigonometry, conics - justified
- 1700s
  - Bernoulli - integral transformation
  - Euler - Euler's formula,  $i$
  - Gauss (1799, 1815) FTA, quadratic forms
  - Wessel (1797), Argand (1806) Geometric Visualization
  - Cauchy (1814, 1825) Complex Analysis
  - Riemann (1826-1866) Surfaces

# Complex Numbers, $\mathbb{C}$

- $a + bi \in \mathbb{C}$ ,  $a, b \in \mathbb{R}$ ,  $i = \sqrt{-1}$ .
- Quadratic Equation,  
 $ax^2 + bx + c = 0$ ,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If  $b^2 - 4ac < 0$ ,  
complex conjugate roots.

- Cubics - Role was clearer

$$y^3 = py + q$$

$$y = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}.$$

**Example:**  $x^3 = 15x + 4$

$$\begin{aligned}x &= \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} \\ &= 2 + i + 2 - i = 4.\end{aligned}$$

Bombelli (1572)

$$\begin{aligned}(2 + i)^3 &= (2 + i)(4 + 4i + i^2) \\ &= (2 + i)(3 + 4i) \\ &= 2 + 11i.\end{aligned}$$

# Bernoulli's Transformations

- Johann Bernoulli (1712)

$$\frac{1}{1+z^2} = \frac{1}{(1+iz)(1-iz)}$$
$$= \frac{1}{2} \left( \frac{1}{1-iz} - \frac{1}{1+iz} \right)$$

$$\int \frac{dz}{1+z^2} = \frac{1}{2} \int \left( \frac{1}{1-iz} - \frac{1}{1+iz} \right)$$

- Note:

$$\int \frac{dz}{a+bz} = \frac{1}{b} \ln(a+bz).$$

So, integrating  $(1+t^2)^{-1}$  gives

$$\tan^{-1} z = \frac{1}{2i} [\ln(1+iz) - \ln(1-iz)].$$



## Examples: Tangent Identities

- Bernoulli studied  $y = \tan n\theta$  in terms of  $x = \tan \theta$ .
- *Example:*  $n = 2$   
 $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$ .
- Let  $y = \tan n\theta$ , Then,  
 $n\theta = \tan^{-1} y, \theta = \tan^{-1} x$ .

$$\int \frac{dy}{1+y^2} = n \int \frac{dx}{1+x^2}$$

$$\ln \frac{y+i}{y-i} = n \ln \frac{x+i}{x-i}$$

$$\frac{y+i}{y-i} = A \left( \frac{x+i}{x-i} \right)^n$$

$A = (-1)^{n+1}$ . Solve for  $y$ .

Ex:  $n = 2$  :

$$\tan 2\theta = \frac{2x}{1-x^2}$$

Ex:  $n = 3$  :

$$\tan 3\theta = \frac{x^3 - 3x}{3x^2 - 1}$$

Ex:  $n = 4$  :

$$\tan 4\theta = \frac{4x - 4x^3}{x^4 - 6x^2 + 1}$$

Ex:  $n = 5$  :

$$\tan 5\theta = \frac{x^5 - 10x^3 + 5x}{5x^4 - 10x^2 + 1}$$

# The Fundamental Theorem of Algebra

- Integration of  $\frac{p(x)}{q(x)}$  for  $p(x), q(x)$  polynomials
- Need Integration by parts Assumes  $q(x)$  can be factored  
- Fundamental Theorem of Algebra (FTA)
- Albert Girard (1629), *L'invention en algèbre*,  
First to claim there are always  $n$  roots of degree  $n$  polynomial.
- By 1750 - Any polynomial with real coefficients can be factored into real linear and quadratic factors.
- Nicolas II Bernoulli (1687-1759) gave a counterexample:  
 $p(x) = x^4 - 4x^3 + 2x^2 + 4x + 4$ .
- Euler found the factors:

$$x^2 - \left(2 \pm \sqrt{4 + 2\sqrt{7}}\right)x + \left(1 \pm \sqrt{4 + 2\sqrt{7} + \sqrt{7}}\right)$$

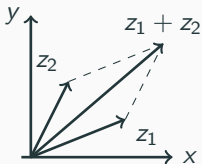
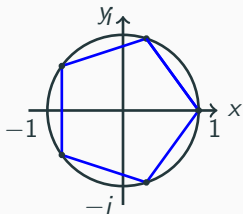
He gave incorrect proof for any quartic.

His was followed by proofs from d'Alembert and Gauss.

# Roots of Unity

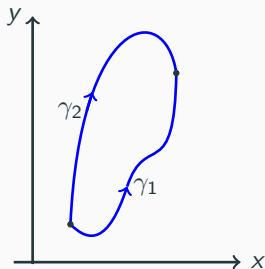
- Cotes, de Moivre, Euler
  - $x^n - 1 = 0$  Seems  $x = \sqrt[n]{1}$ .
  - $x = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$ ,  
 $k = 0, 1, \dots, n - 1$ .
- **Roots of unity.**
  - Geometric Interpretation
- Caspar Wessel, surveyor.
  - Complex number = point in the complex plane, 1797.
  - Also, proposed vectors.
- Argand, 1806, visual representation, operational (translation, rotation, reflection)
- Gauss also rediscovered, 1831.

$$e^{2k\pi/5}, k = 0, 1, \dots, 4$$



# Representing Complex Numbers

- Gauss (1777-1855) adopted “complex number,” used  $i$ .
- Integration in  $\mathbb{C}$ -plane.
- $\int_{\gamma} \phi(z) dz$  is path independent for “nice”  $\phi(z)$ .
- Cauchy proved later, in 1814 talk, published 1827. - Now called *Cauchy's Thm.*



Path Independence

$$\int_{\gamma_1} \phi(z) dz = \int_{\gamma_2} \phi(z) dz$$

Equivalently, for a simple, closed loop  $\Gamma$ ,  $\int_{\Gamma} \phi(z) dz = 0$ .

# Augustin Louis Cauchy (1789-1857)

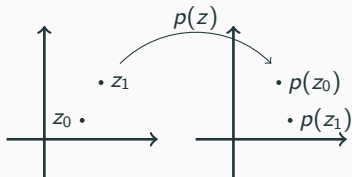
- Father of Complex Analysis
- Does  $f(x) \rightarrow f(z)$  make sense?
- Integration along paths (1814)  
pub 1827.
- Cauchy's Theorem,  
Cauchy-Riemann Equations.
- Calculus of Residues (1826) -  
dealing with singularities.
- Convergence of infinite series.
- Use complex integration to  
integrate real functions.
- Path Independence (1825).
- Complex function of complex  
variable (1828).



# Fundamental Theorem of Algebra I

Every polynomial  $p(x)$  can be written as a product of linear complex factors. (Contains 1750 version)

- d'Alembert (1717-1783)
- **Lemma**  $p(z_0) \neq 0$ ,  $p(z) \neq \text{constant}$ . There exists a  $z_1 = z_0 + w$  such that  $|p(z_1)| < |p(z_0)|$  where  $|a + bi| = \sqrt{a^2 + b^2}$ .



Proof

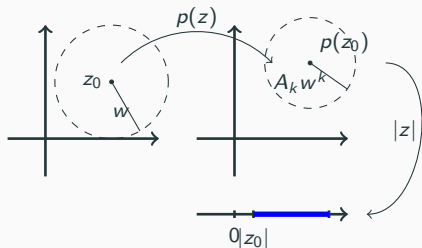
$$p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n.$$

$$p(z_0 + w) = a_0 z_0^n + a_1 z_0^{n-1} + \cdots + a_n + A_1 w + A_2 w^2 + \cdots + A_n w^n.$$

# Fundamental Theorem of Algebra II

$$p(z_0 + w) = a_0 z_0^n + a_1 z_0^{n-1} + \cdots + a_n + A_k w^k + \epsilon$$

Here  $A_k w^k$  is the first nonzero, lowest power of  $w$  term and  $\epsilon$  contains the higher powers terms in  $w$  and is small for large  $|z|$ .



$\exists w$  such that  $p(z_0) + A_k w^k$  is closer to the origin.

Let  $p(z) \neq 0$ . By the lemma,  $\exists$  a point closer than  $z_0$  to the origin.

$\therefore$  there exists a zero of  $p(z)$ .

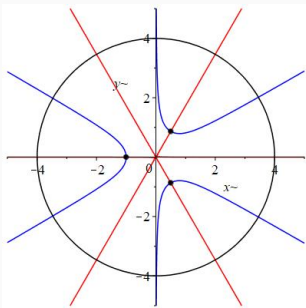
# Fundamental Theorem of Algebra III

- Gauss attempted several proofs.
- Karl Weierstrauss (1815-1897) - continuous functions on closed, bounded regions which assume maximum and minimum values.
- Gauss (1799 Thesis) considered curves  $\operatorname{Re}(p(z)) = 0$ ,  $\operatorname{Im}(p(z)) = 0$ ,  $z = x + iy$ .
- For  $|z|$  large,  $\operatorname{Re}(a_0 z^n) = 0$ ,  $\operatorname{Im}(a_0 z^n) = 0$ , curves are asymptotic to lines through the origin.
- Curves  $\operatorname{Re}(p(z)) = 0$ ,  $\operatorname{Im}(p(z)) = 0$ , entering  $|z| = r$  must come out and intersect inside disk. [See examples.]

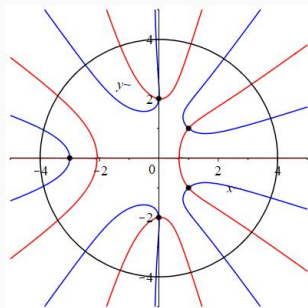


# Examples

Plotting  $Re(p(z))$  and  $Im(p(z))$ , outside a large circle one gets alternating lines. Inside the circle they must intersect for  $p(z) = 0$ .



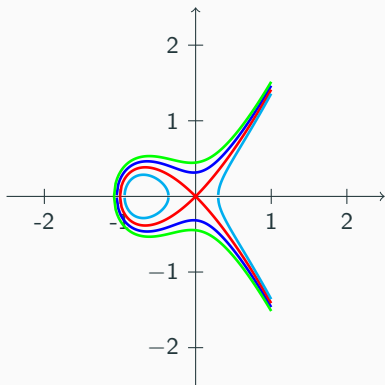
**Figure 1:**  $p(z) = z^3 + 1$ .



**Figure 2:**  
 $p(z) = z^5 + z^4 + 10z^2 - 16z + 24 = (z - 1 - i)(z - 1 + i)(z^2 + 4)(z + 3)$ .

# Theory of Curves, $p(x, y) = 0$

- Descartes - linear/lines
  - quadratic/conics
- Newton - cubics
- Recall Bezout's Intersection Thm
  - ◁ Count multiplicities.
  - ◁ Intersection with  $\infty$ .
- 19th Century
  - Projective Geometry
    - homogeneous coordinates
    - Möbius, Plücker - 1830
  - Complex Numbers
    - Gauss - FTA
  - Topological ideas
    - Riemann surfaces, 1850's



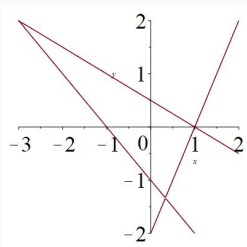
**Figure 3:** Cubic curves of form  $y^2 = x^3 + x^2 + bx + 2b$

# Cubic Curves

Consider products of linear factors or lines

$$p(x, y) = (a_1x + b_1y + c_1)(a_2x + b_2y + c_2)(a_3x + b_3y + c_3)$$

- Ex:  $p(x, y) = (x + y + 1)(x + 2y - 1)(-2x + y + 2)$

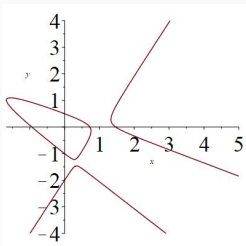
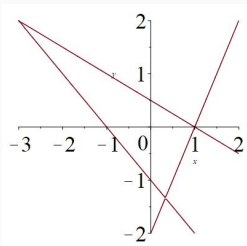


# Cubic Curves

Consider products of linear factors or lines

$$p(x, y) = (a_1x + b_1y + c_1)(a_2x + b_2y + c_2)(a_3x + b_3y + c_3)$$

- Ex:  $p(x, y) = (x + y + 1)(x + 2y - 1)(-2x + y + 2)$
- Modify:  $p(x, y) = (x + y + 1)(x + 2y - 1)(-2x + y + 2) + \frac{x^2}{2}$

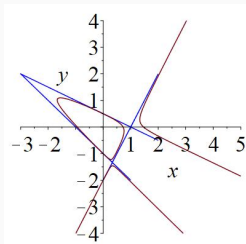
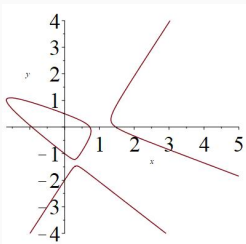
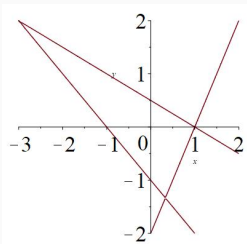


# Cubic Curves

Consider products of linear factors or lines

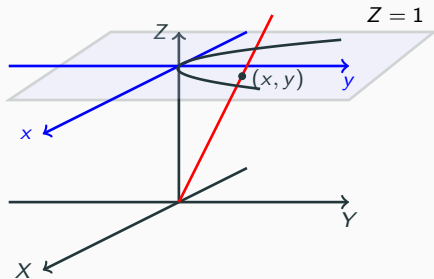
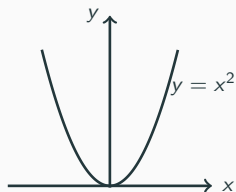
$$p(x, y) = (a_1x + b_1y + c_1)(a_2x + b_2y + c_2)(a_3x + b_3y + c_3)$$

- Ex:  $p(x, y) = (x + y + 1)(x + 2y - 1)(-2x + y + 2)$
- Modify:  $p(x, y) = (x + y + 1)(x + 2y - 1)(-2x + y + 2) + \frac{x^2}{2}$
- Branches go to **points at infinity**. Consider projective geometry.



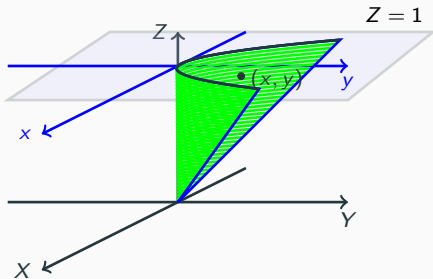
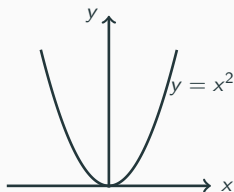
# Projective Geometry

- Homogeneous coordinates:  
 $x = \frac{X}{Z}, y = \frac{Y}{Z}$ .
- Introduced by Möbius, Pücker.
- **Example:**  $y = x^2$  gives  $X^2 = YZ$



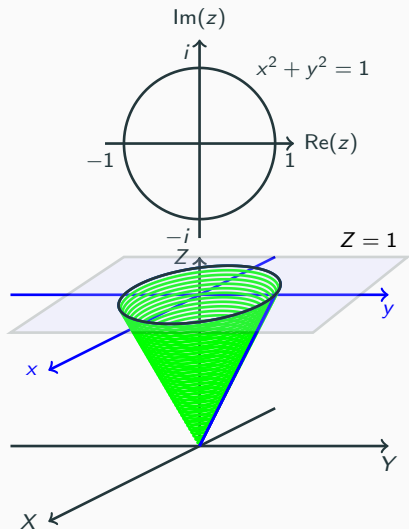
# Projective Geometry

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- Introduced by Möbius, Pücker.
- **Example:**  $y = x^2$  gives  $X^2 = YZ$
- Lines thru origin (projective plane).
- $X^2 = YZ$  is a “cone”
- Points at Infinity:  
 $Z = 0 \Rightarrow X = 0$ ,
- These points,  $[0, Y, 0]$ , lie on horizon.



# Projective Plane and Complex Numbers

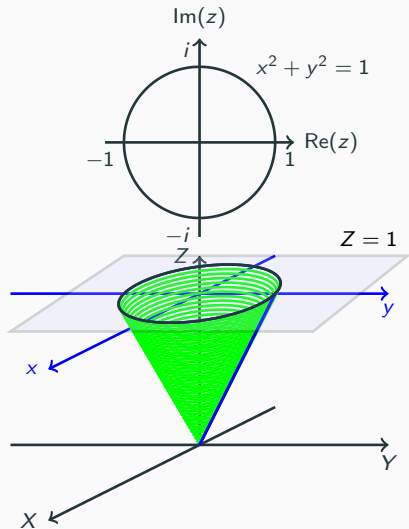
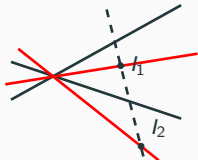
- **Example:**  $x^2 + y^2 = 1$
- Projective curve:  $X^2 + Y^2 = Z^2$
- Pts at infinity,  
 $Z = 0 \Rightarrow X^2 + Y^2 = 0$ .
- In  $\mathbb{C}$ , Circular pts at infinity.  
 $X = 1, Y = i : l_1 = (1, i, 0)$   
 $X = 1, Y = -i : l_2 = (1, -i, 0)$





# Projective Plane and Complex Numbers

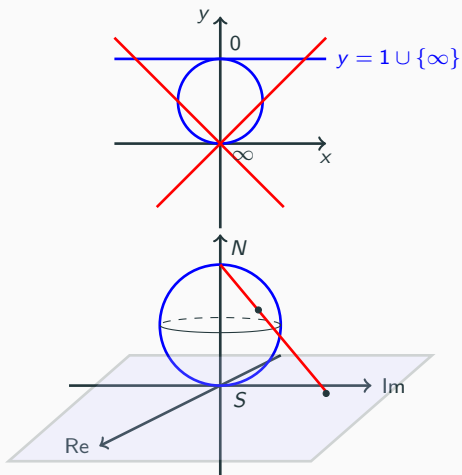
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 $X = 1, Y = i : l_1 = (1, i, 0)$   
 $X = 1, Y = -i : l_2 = (1, -i, 0)$
- Edmund Laguerre (1834-1886)  
- Angles,  $\theta = i \log R$ .
- $R$  - Cross ratio



# Stereographic Projection

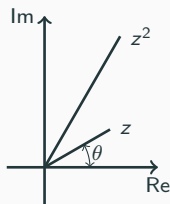
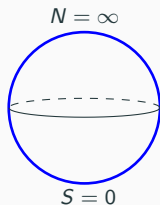
What do complex curves look like?

- Projective lines:
  - Lines thru origin
  - Topologically looks like a circle,  $S^1$ , after adding point at infinity
- Extend to  $\mathbb{C}$  - topologically,  $S^2$
- Stereographic Projection
  - Connect pts in  $\mathbb{C}$  to North Pole.
- N mapped to pt at  $\infty$ .
- Möbius (1790-1868) Image of circle = circle.



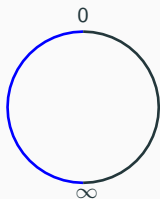
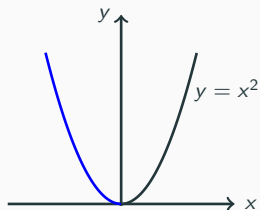
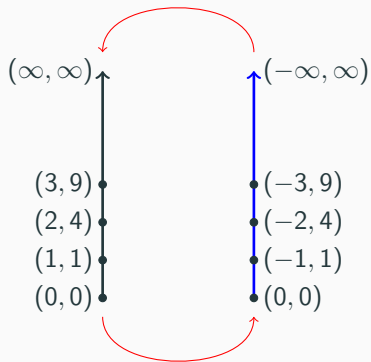
# Mapping Functions onto Surfaces

- Riemann (1826-1866)
  - Riemannian manifolds
  - Curved spaces - Gauss
  - Complex Analysis - Riemann surfaces [Cauchy (1788-1857)]
  - Number theory -  $\zeta(s)$
- Start with a Sphere
- Extend  $f : \mathbb{C} \rightarrow \mathbb{C}$  to  $g : S^2 \rightarrow S^2$ .
- Complex function,  $f(z) = z^2$   
Let  $z = re^{i\theta}$ .  
[ $\theta =$  argument,  $r =$  modulus,  $|z|$ .]  
Then,  $f(z) = r^2 e^{2i\theta}$ .



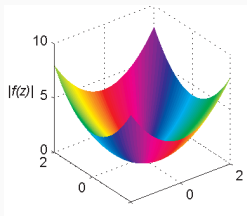
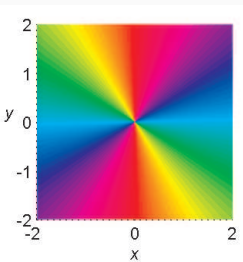
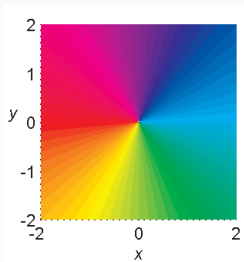
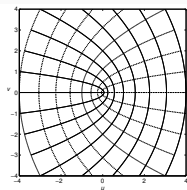
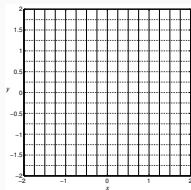
# Real Function, $y = f(x)$ , Mapped to $S^1$

- Example  $f(x) = x^2$ .



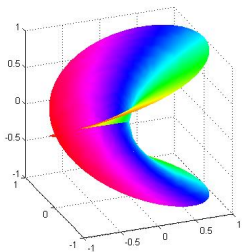
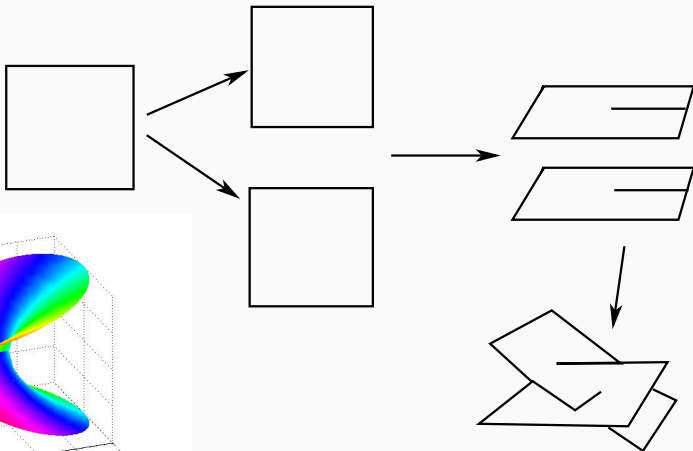
# Visualizing Complex Functions: $w = f(z) = u(x, y) + iv(x, y)$

- What is  $f(z) = z^2$ ?
- Map  $xy$ -plane to  $uv$ -plane.
- $(x + iy)^2 = x^2 - y^2 + 2ixy$ .
- $u(x, y) = x^2 - y^2$ ,  $v(x, y) = 2xy$
- Domain Coloring



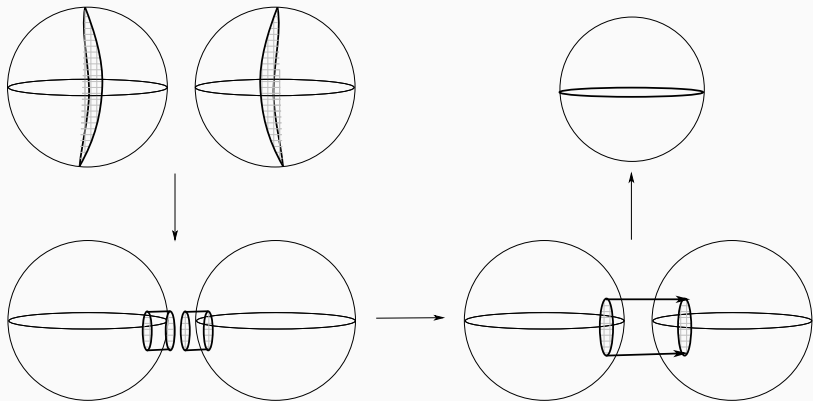
# Riemann Surfaces and the Square Root Function

- Riemann Sheets - Two copies of  $\mathbb{C}$ . Riemann's Dissertation, 1851.
- **Example:**  $w = \sqrt{z}$ .



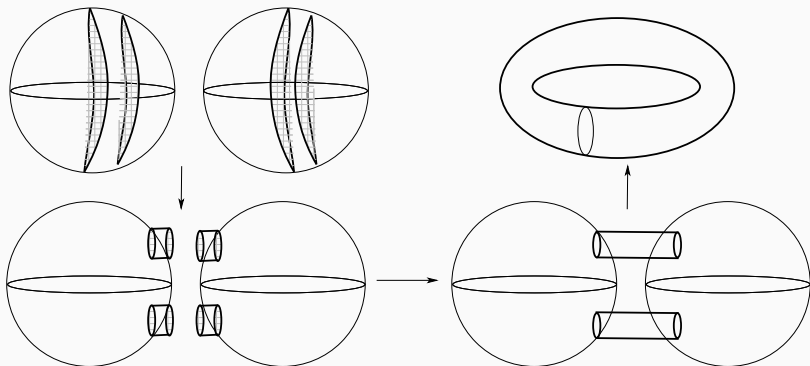
# Mapping $f(z) = z^2$ to $S^2$ .

- **Example:**  $f(z) = z^2$ .

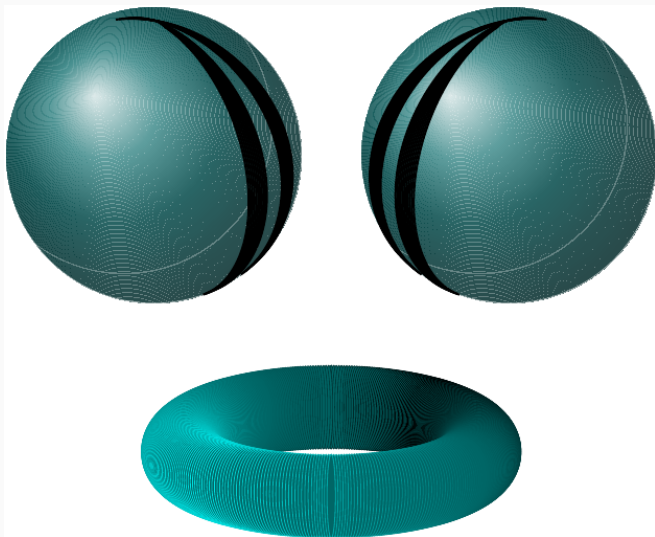


# Riemann Surfaces and Elliptic Integrals

- **Example:**  $\int_0^x \frac{dz}{\sqrt{z(z-a)(z-b)(z-c)}}$ .
- $w^2 = z(z-a)(z-b)(z-c)$
- Beginning of topology.



# Merging Two Cut Riemann Spheres



# Beginnings of Topology ...



Figure 4: Genus  $g = 1, 2, 3$ .



# Beginnings of Topology ...



Figure 4: Genus  $g = 1, 2, 3$ .

