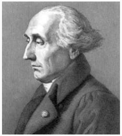
Non-Euclidean Geometry and Group Theory

Fall 2020 - R. L. Herman








Euclidean Geometry

- 300 BCE - Euclid's *Elements*
- Five Postulates.
- 5th Postulate - not needed in first 28 propositions.
- Proclus (410-485) Equivalent postulate.
- Giralomo Saccheri (1667-1733) Assume 5th postulate false and get contradiction.
- Used assumption - lines infinite. Led to contradiction of P1, almost P2.

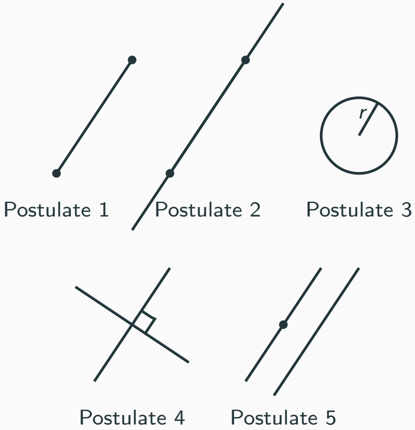


Figure 1: Euclid's 5 Postulates.

Spherical Geometry

- Lines = geodesics, Lie on great circles.
- Euclidean triangles, $a + b + c = \pi$.
- Spherical triangles, $a + b + c > \pi$.
- Thomas Harriot (1560-1621), astronomy, mathematics, and navigation
- Johann Heinrich Lambert (1726-1777)
    - General properties of map projections.
    - hyperbolic functions
    - π is irrational
    - optics

$a + b + c = \pi + \frac{A}{R^2}$.

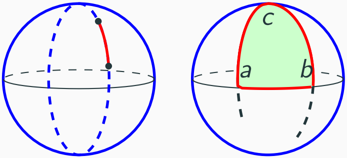


Figure 2: Harriot and Lambert.

Parallel Postulate Revisited

- Carl Friedrich Gauss (1777-1885) started on it in 1799; was convinced it was independent of first 4.
- János Bolyai (1802-1860) - Believed a non-Euclidean geometry existed.
- Nikolai Lobachevsky (1792-1856) - independently 1840 new 5th postulate: There exists two lines parallel to a given line through a given point not on the line. Developed trig identities, hyperbolic geometry.

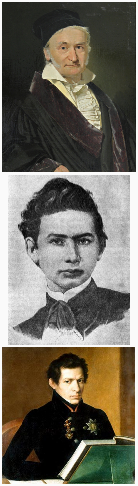
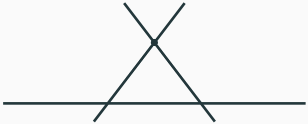
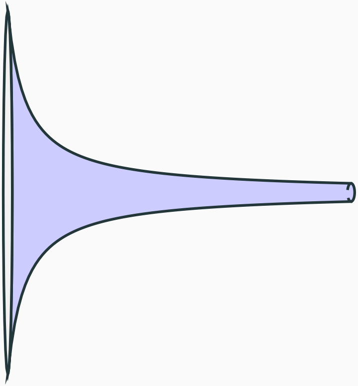


Figure 3: Gauss, Bolyai, Lobachevsky

Riemannian Geometry

- Georg Friedrich Bernhard Riemann (1826-1866)
  - Published in 1868 Lecture
  - Spherical geometry
  - Riemannian geometry → differential geometry
  - Every line through a point not on a given line meets the line.
- Eugenio Beltrami (1835-1900) Published interpretations of non-Euclidean geometry - introduced pseudosphere in 1868.
- Curvature, k.





Curvature

- $k = 0,\ k > 0,\ k < 0.$
- sums of angles of triangles $a + b + c - \pi = kA.$

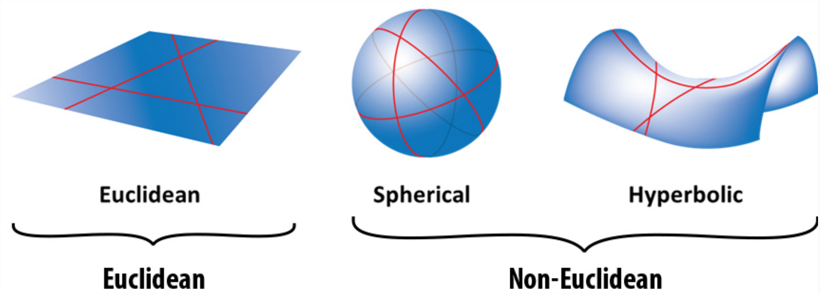


Figure 4: Surfaces of Constant Curvature.

Hyperbolic Geometry

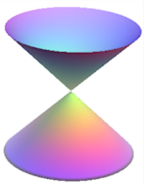
- Sphere

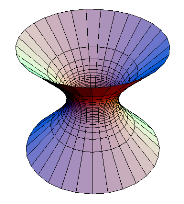
$$x^2 + y^2 + z^2 = \text{const}$$

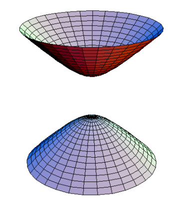
- Modify

$$x^2 + y^2 - z^2 = K$$

- $K = 0$, $z^2 = x^2 + y^2$. Cones.
- $K = 1$, $x^2 + y^2 - z^2 = 1$.
  Hyperboloid of one sheet
- $K = 1$, $z^2 - x^2 - y^2 = 1$.
  Hyperboloid of two sheets.





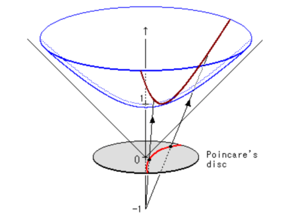


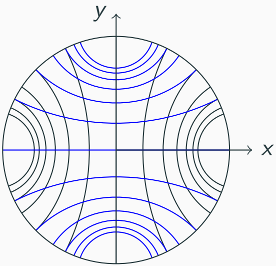
Beltrami-Poincaré Model

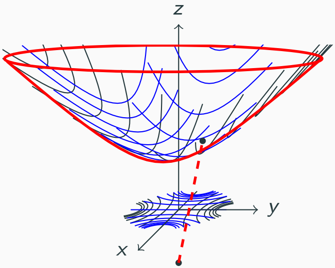
- Poincaré's Disks

  $$(x, y, z) = (c \cosh t, \sinh t, \sqrt{1 + c^2} \cosh t)$$

- Stereographic Projection thru $(0, 0, -1)$ to $z = 0$: $(x, y, z) \to \frac{(x,y)}{1+z}$.
- Hyperbolic geometry.



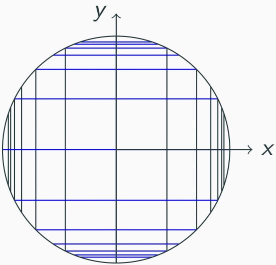


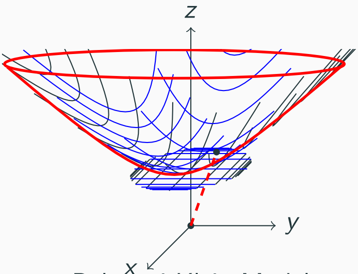


Beltrami-Poincaré Model

Beltrami-Klein Model

- Stereographic Projection thru $(0,0,0)$ to $z = 1 : (x,y,z) \to \frac{(x,y)}{z}$.
- Klein's Disks
  Projection to $(0,0,1)$

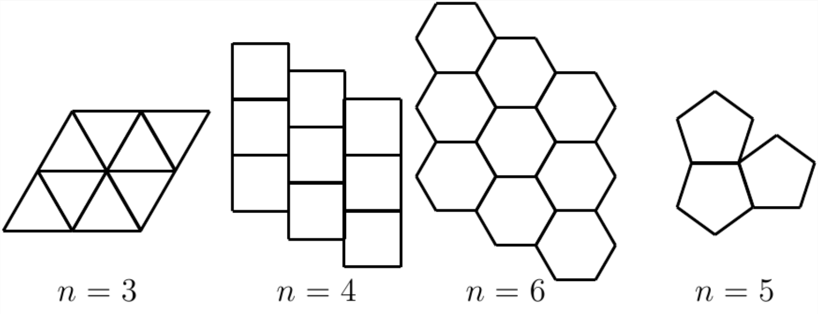




Beltrami-Klein Model

Tiling the Plane

One can tile the plane with a single polygon with sides 3, 4, and 6. However, one cannot fit pentagons together. As seen below, the angles do not allow for a fit. For large *n*, the interior angles are too small.



Other Tilings

- Johannes Kepler (1571-1630)
  - Studied Tilings
  - *Harmonicae Mundi* (Harmony of the World).
  - Planned in 1599.
  - Published 1619 - delay by Tycho Brahe to look as orbit of Mars.
- Roger Penrose (1931-)
  - 2020 Nobel Prize
  - 70's Inspired by Tilings - Penrose tilings. In 80's found in nature.
  - and M. C. Escher (1889-1972)
  - Circle Limit - Tiling Hyperbolic Plane.
- Others - Polyominoes and Pentominoes.

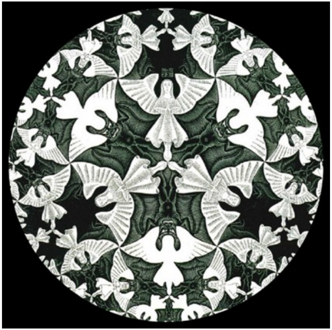


Figure 5: Circle Limit IV

Group Theory

1843 - Joseph Louiville (1809-1882) reviewed manuscript, published 1846. - introduction of groups and fields.

- Group Theory Origin - Galois
- Group of Substitutions.
- Euler - Fermat's Little Theorem
  p prime, $(a, p) = 1$,
  $$a^{p-1} \equiv 1 (\text{mod } p).$$
  $\phi(n) = \#\{k \in \{1, 2, \ldots, n-1\} | (k, n) = 1\}.$
  $\phi(5) = 4,\ \{1, 2, 3, 4\}$
  $\phi(8) = 4,\ \{1, 3, 5, 7\}.$
- Group Properties:

  closed, identity,
  Inverse, associative

Example: $n = 5$

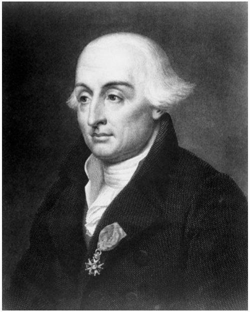
x	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Carl Friedrich Gauss (1777-1855)

- *Disquisitions Arithmeticae* - 1801
- Summary and Extension on Number Theory.
- Initiated finite Abelian groups.
- Proved Fermat's Little Theorem. Represented integers as quadratic forms, like Fermat Primes ($4n + 1 = x^2 + y^2$.) for x and y integers.
- Binary quadratic forms - $ax^2 + bxy + cy^2$ - for a, b, c integers.
  - composition has properties of an abelian group.
- Did not have a general theory of groups.

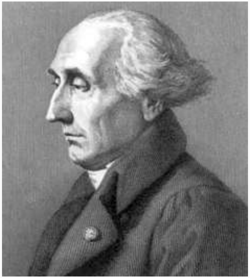
Joseph Louis Lagrange (1736-1813)

- Born in Turin, Italy.
- Professor at 19 (artillery school).
- 1766 Frederick the Great sought great mathematician.
- Lagrange replaced Euler in Berlin for 20 yrs.
- Invited by Louis XVI to Paris.
- 1795 - established dept. École Normal.
- 1797 - established dept. École Polytechnique.
- Napoleon made him senator, count, and he received many other honors.
- Sought solution of quintic by studying cubic and quartic.
- Made many other contributions.



Resolvents

- Consider $x^3 + nx + p = 0$. Let $x = y - \frac{n}{3y}$.
- Yields 6^{th} degree polynomial,
  $y^6 + py^3 - \frac{n^3}{27} = 0$, the resolvent.
- Let $r = y^3$, $r^2 + pr - \frac{n^3}{27} = 0$.
- Has roots r_1, r_2, where $r_2 = -\left(\frac{n}{3}\right)^3 \frac{1}{r_1}$.
- Then, $x = \sqrt[3]{r_1} + \sqrt[3]{r_2}$
- Cardano got this real root but did not seek complex solutions.
- Lagrange knew there should be 3 roots to a cubic. $\sqrt[3]{r}, \omega\sqrt[3]{r}, \omega^2\sqrt[3]{r}$, where ω is a solution of $x^3 - 1 = 0$. Then,



$$
\begin{aligned}
x_1 &= \sqrt[3]{r_1} + \sqrt[3]{r_2} \\
x_2 &= \omega\sqrt[3]{r_1} + \omega\sqrt[3]{r_2} \\
x_2 &= \omega^2\sqrt[3]{r_1} + \omega^2\sqrt[3]{r_2}
\end{aligned}
$$

Permutation of Roots

- Lagrange then wrote roots of the resolvent
  $y = x_i + \omega x_j + \omega^2 x_k, \quad i, j, k = 1, 2, 3, \quad i \neq j \neq k.$
- 3!=6 permutations of cubic roots.
- In $y^6 + py^3 - \frac{n^3}{27} = 0$, the coefficients of y^5, y^4, y^2, y are
  $x_1 + x_2 + x_3$, $p = x_1 x_2 x_3$, and $\dfrac{n^3}{27} = \dfrac{(x_1 x_2 + x_1 x_3 + x_2 x_3)^3}{27}$.
- Resolvent coefficients are rational functions of the cubic roots.
- Lagrange obtained similar results for the quartic.
- Lagrange sought solutions of higher order equations using symmetric functions of the roots and permutations.
- Paola Ruffini (1765 – 1822) - 1802, 1805, 1813 - proofs that quintic can't be solved. Not understood.

Niels Henrick Abel (1802-1829)

- Born in Norway into poverty and had a pulmonary condition.
- Mathematical ability discovered by his teacher.
- Toured Europe after college and published 5 papers in *Journal für die reine und angewandte Mathematik*.
- Studied convergence of infinite series, the theory of doubly periodic functions, elliptic functions, elliptic functions and the theory of equations.
- Could not get employment, so tutored.
- At university, thought he had solution of quintic. Then, proved no solution existed.
- Died of tuberculosis before completing his work.

Évariste Galois (1811-1832)

- Born Oct 25, 1811
- Interest in math at 14.
- Read Adrien-Marie Legendre (1752-1833).
- 1828 Failed to get into École Polytechnique.
- 1829 Paper on continued fractions.
- Studied polynomial equations.
- Wrote two papers.
  Reviewed by Arthur Cayley (1821-1895)
  Entered competition.
- 1830 Submitted to Joseph Fourier (1768–1830) - got lost.
  Winners - Niels Henrik Abel (1802-1829) and Carl Gustav Jacobi (1804-1851).
- Published 3 papers.

Figure 6: Évariste Galois

Évariste Galois (cont'd)

- Political turmoil in France.
- Student uprising - Galois left school.
- He was arrested and acquitted.
- Arrested Oct 1831 - April 1832.
- Siméon Denis Poisson (1781-1840) asked him to submit work 1831.
- July 4 - declared work incomprehensible.
- Galois found out in October.
- Auguste Chevalier - privately published manuscript.
- Stayed up all night; wrote letters and note to Chevalier.
- On May 30, fought in duel and lost.

Figure 7: Legendre, Cayley, Fourier, Jacobi, Poisson, Liouville

Symmetry Groups

- Levi ben Gorshun (1321) Number of permutations of n objects $= n!$
- Leads to Symmetric Group.
- Felix Klein (1872) extended groups to geometry - studied invariants of groups of transformations.
- Sophus Lie (1842-1899) continuous groups of transformations, applied to differential equations.
- Emmy Noether (1882-1935) related symmetries to constants of motion in physics.

Figure 8: Sophus Lie and Emmy Noether.