Introduction to the History of Mathematics

Fall 2020 - R. L. Herman

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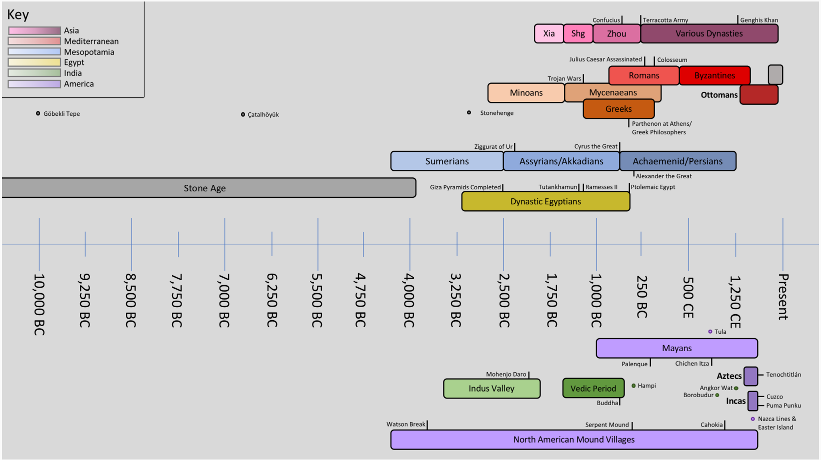
A Civiliation Timeline



Early Civilizations

- Egypt - 3100 BCE
- Mesopotamia (Babylonia, Sumer) - 2100 BCE
- China 1600 BCE
- India 1200 BCE

Arithmetic, Geometry,
No proofs
Problems were practical or recreational

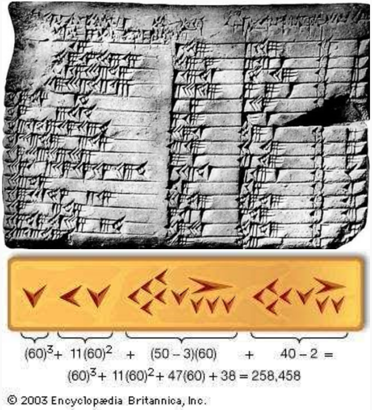


Figure 1: Babylonian tablet - Base 60

Greek Civilization

- Deductive Reasoning
  - Definitions, Axioms
  - Propositions via logic
- Geometry, Trigonometry, Astronomy, Numbers, Conics
- Thales 624-546 BCE
- Pythagoras 6th Century BCE
- Euclid 4th Century BCE *Elements*
- Archimedes 3rd Century BCE
- Appolonius 2nd Century BCE
- Heron 10-70 CE
  Diophantus 200-284 CE,
  Hypatia 400 CE

Figure 2: Euclid

Chinese and Hindu - 700-1200 CE

- Chinese Mathematics 1300 BCE - 1800 CE
  - Pythagorean Thm
  - π estimates
  - Volumes, Applications
  - Pascal's Triangle
  - Chinese Remainder Thm
- Indian Mathematics 1200 BCE, mostly 500-1200 CE
  - Geometry
  - Trigonometry
  - Power sieries
  - Astronomy
  - π estimate
  - Number system, 0
  - Pell's Equation

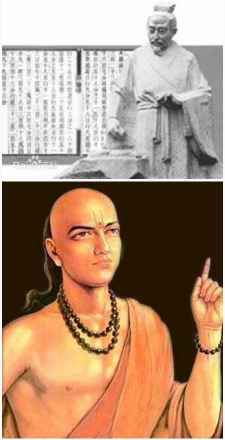


Figure 3: Liu-Hong and Aryabhatta

Middle Eastern Mathematics - 700-1200 CE

In Europe - Dark Ages - 400-1200 CE

- Middle East - Islam 7th Century
- Translations of Greek Mathematics into Arabic
- Arabic Numerals by 1000 CE
- Algebra 825 CE - al-Khwarizmi - called al-Jabr
- Omar Khayyam (1048-1131) - geometric solution of cubic

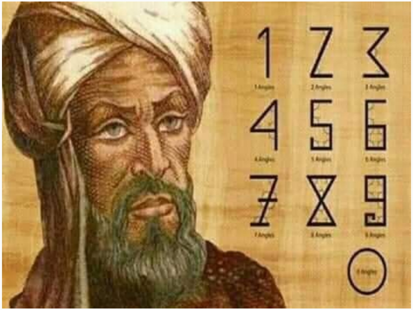


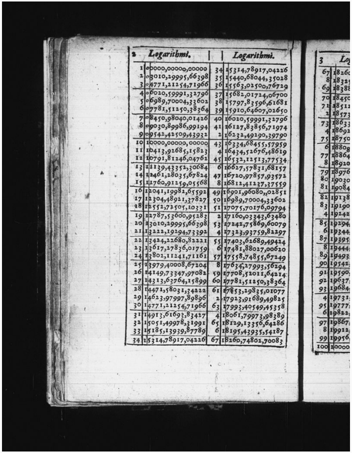
Figure 4: al-Khwarizmi

Around 10th Century - Middle Eastern Mathematics brought to Spain.

It takes 300 years to accept Arabic numerals. - Fibonacci - 1202

Beyond Numerals

- Fractions 4000 years ago
- Sexagesimal (base 60) into 17th century
- Decimal (base 10)
  - al-Uqlidisi - (920-980)
  - al-Kashi (1380-1429)
  - Simon Stevin (1548-1620)
- Logarithms
  - John Napier (1550-1617) used a stange base
  - Henry Briggs (1561-1630) Base 10 Tables
  - 54 square roots of 10 leading to 30 decimal places
  - Tables to 14 decimal places



2	Logarithmi.		Logarithmi.
1	00000,00000,00000	34	15314,78917,04226
2	03010,29995,66398	35	15440,68044,35028
3	04771,21254,71966	36	15563,02500,76729
4	06020,59991,32796	37	15682,01724,06700
5	06989,70004,33602	38	15797,83596,61681
6	07781,51250,38364	39	15910,64607,02650
7	08450,98040,01426	40	16020,59991,32796
8	09030,89986,99194	41	16127,83856,71974
9	09542,42509,43932	2	16232,49290,39790
10	10000,00000,00000	43	16334,68455,57959
11	10413,92685,15823	4	16434,52676,48619
12	10791,81246,04762	45	16532,12513,77534
13	11139,43352,30684	6	16627,57831,68157
14	11461,28035,67824	47	16720,97857,93572
15	11760,91259,05568	8	16812,41237,37559
16	12041,19982,65592	49	16901,96080,,02851
17	12304,48921,37827	50	16989,70004,33602
18	12552,72505,10331	51	17075,70176,09794
19	12787,53600,95283	2	17160,03343,63480
20	13010,29995,66398	53	17242,75869,60079
21	13222,19294,73392	4	17323,93759,82297
22	13424,22680,82221	55	17403,62689,49424
23	13617,27836,01759	6	17481,88027,00620
24	13802,11241,71161	57	17558,74855,67249
25	13979,40008,67204	8	17634,27993,56294
26	14149,73347,97082	59	17708,52011,64214
27	14313,63764,15899	60	17781,51250,38364
28	14471,58031,34222	61	17853,29835,01077
29	14623,97997,89896	2	17923,91689,49825
30	14771,21254,71966	63	17993,40549,45358
31	14913,61693,83427	4	18061,79973,98389
32	15051,49978,31991	65	18129,13356,64286
33	15185,13939,87789	6	18195,43935,54187
34	15314,78917,04226	67	18260,74802,70083

Figure 5: Brigg's Tables

- Fibonacci (Leonardo of Pisa) (1170-1250) *Liber Abaci*
- Equation Solving contests
- Solutions of cubic and quartic
- Depressed cubic
  del Ferro (1465-1526)
- Cubic and quartic equations
  Tartaglia (1500-1557)
  Cardano (1501-1576)
  Ars Magna
  Ferrari (1522-1565)
- Bombelli (1526-1572)
- Viète (1540-1603)
  Adriaan van Roomen Problem

Figure 6: Cardano and Tartaglia

Unification of Geometry and Algebra

- Symbolic Algebra
  - Rhetorical until 15th century
  - Syncopated/abbrev. - 1500
  - Symbolic algebra developed 16-17th century
- Unification
  - Oresme (1320-1382) - Velocity-time graphs, $\sum \frac{1}{n}$
  - Descartes (1596-1650)
    - Rep. curves by equations
    - Coordinate systems - published *The Method*
    - Made use of variables which can vary continuously - lines.
  - Fermat (1607-1665)
    - Rep. equations by curves

Figure 7: Fermat and Descartes

The Rise of Calculus

- Archimedes - 3rd century BCE
- Kepler (1571-1630)
- Cavalieri (1598-1647)
- Fermat (1607-1665)
- Wallis (1616-1673)
- Pascal (1623-1662)
- Barrow (1630-1677)
- Wren (1632-1723)
- Gregory (1638-1675)
- Newton (1642-1726)
  - *Principia* 1687
- Leibniz (1646-1716)
  - *Notation* $\frac{d}{dx}$, $\int$

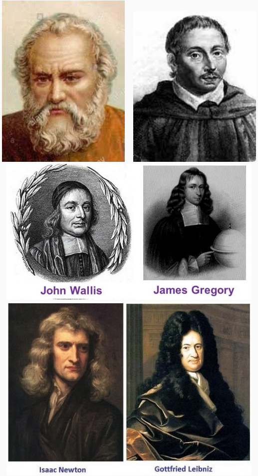
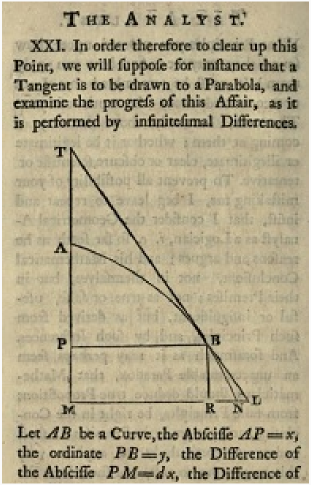


Figure 8: Archimedes, Cavalieri, Wallis, Gregory, Newton, and Leibniz

The Infinitesimal

- Hippasus 500 BCE
    - Pythagorean
    - $\sqrt{2}$ irrational
- Introduction of Infinitesimals
    - Cavilieiri and Torricelli
    - Stevin, Wallis, Harriot
- Critics
    - Jesuits in Italy
    - George Berkeley (1685-1753) *The Analyst*, - *A Discourse Addressed to an Infidel Mathematician*, 1734
    - infinitesimals undermine mathematics and rationality
- Augustin-Louis Cauchy - 1821


THE ANALYST.

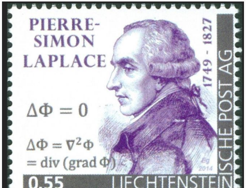
XXI. In order therefore to clear up this Point, we will ſuppoſe for inſtance that a Tangent is to be drawn to a Parabola, and examine the progreſs of this Affair, as it is performed by infiniteſimal Differences.

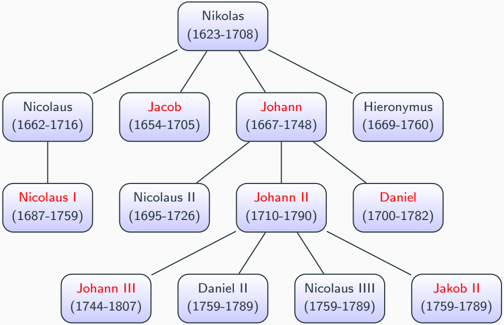
Let AB be a Curve, the Abſciſſe $AP = x$, the ordinate $PB = y$, the Difference of the Abſciſſe $PM = dx$, the Difference of

Figure 9: Berkeley's *The Analyst*

Exploiting Calculus

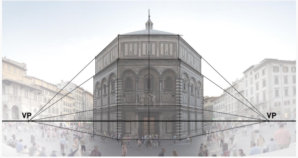
- Bernoulli Family
- Euler (1707-1783)
- Laplace (1749-1827)
- Neptune discovered using math - 1846

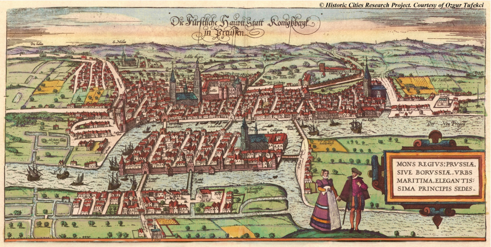




From Geometry to Topology

- Bruneschelli and Perspective
- Projective Geometry
- Euler's geometry without distance
- Birth of Topology
- How smoke rings led to Knot Theory.





The Birth of Rigor - 19th Century

Non-Euclidean Geometry

- Parallel Postulate
- Hyperbolic Geometry
  - Nikolai Lobachevsky (1792-1856)
  - Johann Bolyai (1802-1860)
  - Johann Carl Friedrich Gauss (1777-1855)
- Elliptic Geometry
  - Georg Friedrich Bernhard Riemann (1826-1866)
  - Prince of Mathematicians
- By 1870's doubted Euclid

Figure 11: Gauss, Lobachevsky, Bolyai

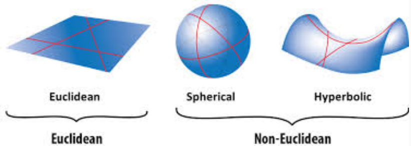


Figure 12: Different Geometries

19th Century Group Theory

- Joseph-Louis Lagrange (1736-1813)
- Johann Carl Friedrich Gauss (1777-1855)
- Augustin Cauchy (1789-1857)
- Niels Henrik Abel (1802-1829)
- Évariste Galois (1811-1832)
- Arthur Cayley (1821-1895)
- Camille Jordan (1838-1922)


Figure 13: Abel and Galois

19th Century Analysis and Set Theory

- Jean-Baptiste Joseph Fourier (1768-1830)
- Johann Carl Friedrich Gauss (1777-1855)
- Augustin Cauchy (1789-1857)
- Karl Weierstrass (1815-1897)
- George Boole (1815-1864)
- Georg Friedrich Bernhard Riemann (1826-1866)
- Richard Dedekind (1831-1916)
- Georg Ferdinand Ludwig Philipp Cantor (1845-1918)
  - Founder of set theory
  - Defined infinite sets

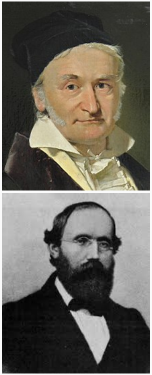


Figure 14: Gauss and Riemann

19th Century Number Theory

- Marie-Sophie Germain (1776-1831)
- Johann Carl Friedrich Gauss (1777-1855)
  Disquisitiones Arithmeticae - 1801
- Adrien-Marie Legendre (1752-1833) and
  Peter Gustav Lejeune Dirichlet (1805-1859) prove Fermat's Last Theorem for $n = 5$ in 1825
  - Dirichlet, $n = 14$ in 1832.
- Riemann hypothesis, distribution of primes - 1832.
- Charles Jean de la Vallée-Poussin and Jacques Hadamard - Prime Number Theorem. 1896
- Hermann Minkowski: Geometry of Numbers, 1896.

Figure 15: Sophie Germain, Adrien-Marie Legendre

The Modern Era

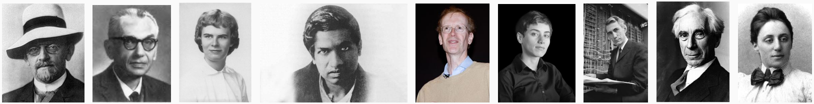


Figure 16: Hilbert, Gödel, Uhlenbeck, Ramanujan, Wiles, Mirzakhani, Shannon, Russell, Noether

- Chronology of 20th Century Mathematicians
- Greatest Mathematicians born between 1860 and 1975
- Pictures of Famous 20th Century Mathematicians
- The Story of Math Website

Early Mathematics

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Early Civilizations

- Egypt - 3100 BCE
- Mesopotamia, or Babylonia - 2100 BCE
- China 1600 BCE
- India 1200 BCE

Arithmetic, Geometry,
No proofs
Problems were practical or recreational

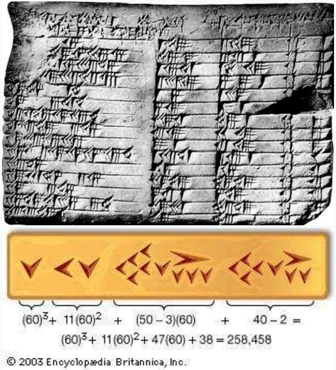


Figure 1: Babylonian tablet - Base 60

Another Sumerian Tablet - YBC 7289

$$1/24/51/10 = 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} \approx 1.414213$$

$$42/25/35 = 42 + \frac{25}{60} + \frac{35}{60^2} \approx 42.426$$

Egyptian Civilization: 3000 BCE - 300 BCE

- Papyri - scrolls
    - Rhind Papyrus
    - Moscow Papyrus
- Arithmetic - integers, fractions
- Rhind Papyrus
    - Found in Thebe
    - Purchased 1858, A. Henry Rhind
    - $18' \times 13''$
    - Geometry
        - Areas, Volumes
        - Ratios of sides of right triangles
    - Measures - grain
        - 1 hekat ≈ 29224 in$^3 \geq \frac{1}{2}$ peck
        - 1 ro $= \frac{1}{320}$ hekat

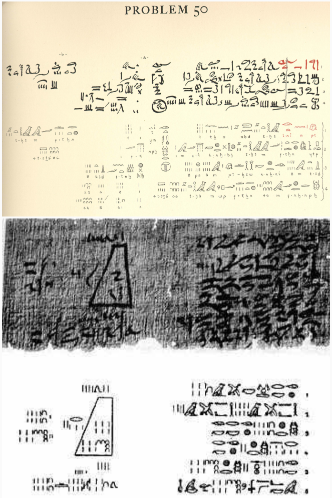


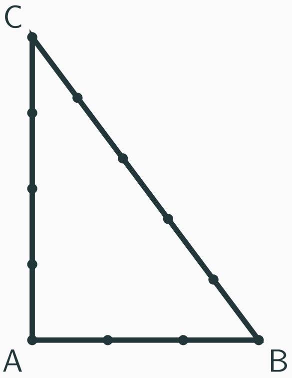
Figure 2: Papyri

Pythagorean Triples

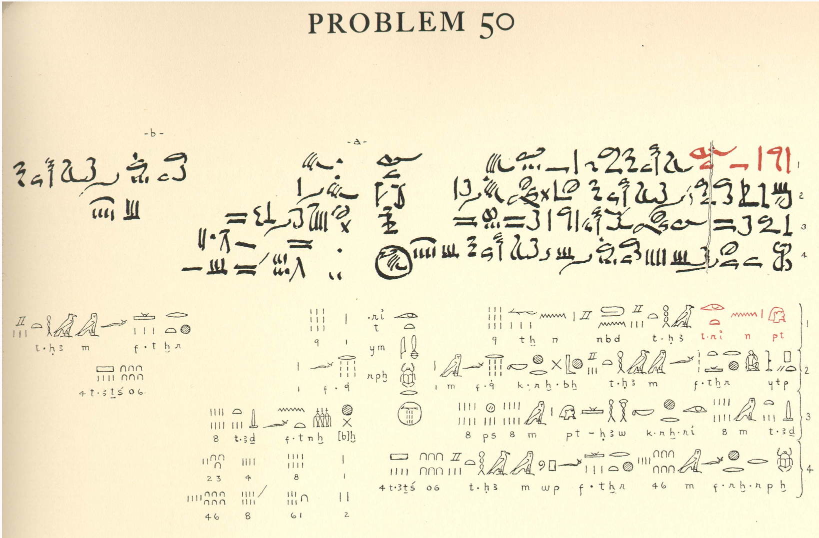
- Pythagorean Theorem
- Triples (a, b, c)

$$a^2 + b^2 = c^2$$

  Examples
  - 3-4-5
  - 5-12-13
- Used to Measure Perimeters
- Knotted Ropes
  - Loop with 12 knots



Rhind Papyrus - Problem 50



Rhind Papyrus - Problem 50

Problem 50

tp n ỉr·t ꜣḥ·t dbn n ḫt·w[1] 9 pty rḫt · f m ꜣḥ·t
Example of making a field round of khet 9. *What is the amount of it in area?*

ḫb · ḫr · k $\dot{9}$ · f m 1 ḏꜣ·t m 8 ỉr·ḫr · k wꜣḥ–tp m 8 sp 8 ḫpr·ḫr · f m 64
Take away thou ⅑ of it, namely, 1; *the remainder is* : 8. *Make thou the multiplication* : 8 *times* 8; *becomes it* : 64;

rḫt · f pw m ꜣḥ·t 60[2] śṯꜣ·t 4
the amount of it, this is, in area, 60 setat 4.

ỉr·t my ḫpr
The doing as it occurs:

1	9
$\dot{9}$ · f *of it*	1.

ḫ[b] ḫnt · f ḏꜣ·t 8
Take away from it; the remainder is 8.

1	8
2	16
4	32
\8	64

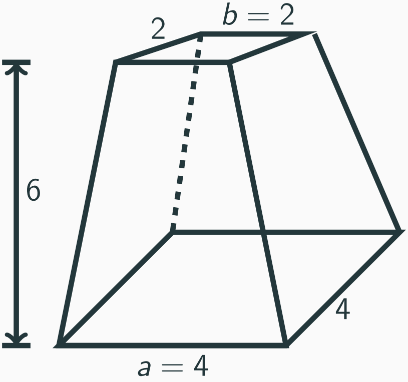
rḫt · f m ꜣḥ·t 60[2] śṯꜣ·t 4
The amount of it in area: 60 setat 4.

[1] The w suggested by the plural strokes has been omitted on the plate. The same omission occurs on the figure in Problem 51, and in Problem 52, line 2.

[2] The scribe has by mistake written here either the number 60 or the special form for 6 used in Problem 48 in writing 6 *setat*. He may have had in his mind the fact that he was actually dealing with 60 *setat* (which, however, would not properly be written in this way), and he had written the abstract number 60 a moment before at the end of the multiplication, or, remembering that 60 *setat* is written with the numeral 6, he did write 6, but used the special sign instead of the ordinary numeral.

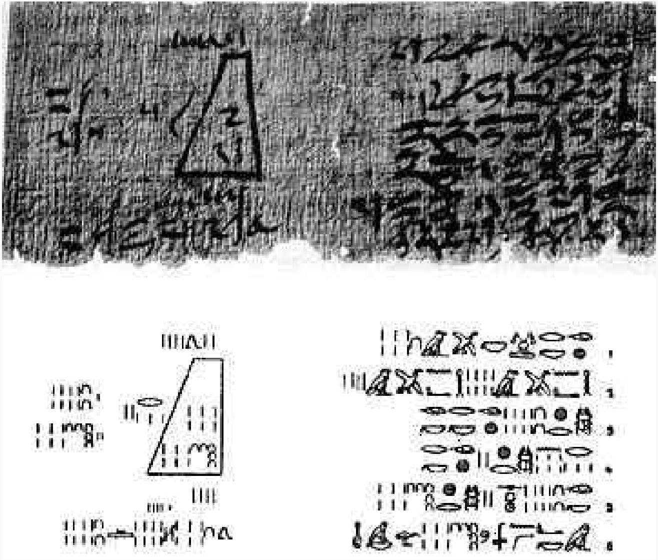
Moscow Papyrus

- 1850 BCE
- Golenishchev bought in 1892 or 1893 in Thebes
- Housed in Moscow
- 25 Problems
- `https://en.wikipedia.org/wiki/Moscow_Mathematical_Papyrus`
- Problem 14 - Frustrum of a Pyramid



$$V = \frac{h}{3}\left(a^2 + ab + b^2\right)$$

Moscow Papyrus - Problem 14 - Frustrum of Pyramid



Tigris and Euphrates Region

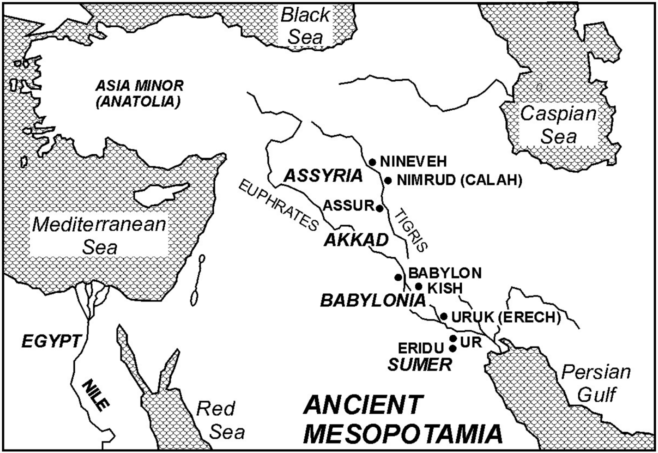


Figure 3: Tigris and Euphrates Rivers

Mesopotamia, or Babylonia - 2100 BCE

- Tigris-Euphrates Region
- More Advanced
- Babylonians, Sumerians
- Clay Tablets
- Base 60 Arithmetic
- Notation $13_{60} = 1\_3 = 1/3$
- Examples

$$
\begin{aligned}
1/3 &= 1(60) + 3 = & 63 \\
1/59 &= 1(60) + 59 = & 119 \\
2/49 &= 2(60) + 49 = & 169 \\
3/31/49 &= 3(60^2) + 31(60) + 49
\end{aligned}
$$

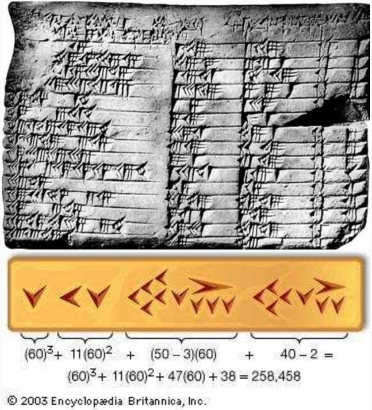
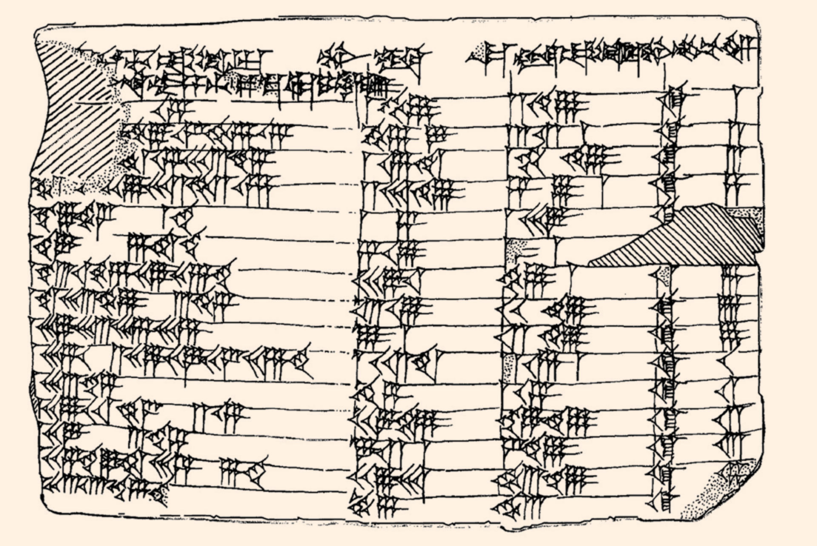


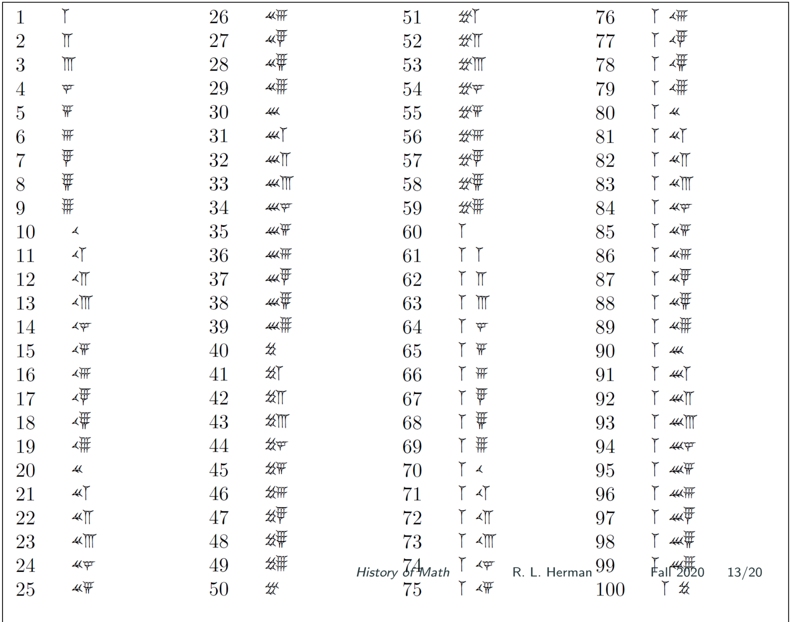
Figure 4: Babylonian tablet - Base 60

Plimpton 322 Clay Tablet - Homework!

Sketch of the Plimpton 322 Tablet



Babylonian Numerals 1-100 (Base 60)



Akkadian Table of 9's

2 Akkadian Tablet (-1700)

In the paper "Sherlock Holmes in Babylon," *Amer. Math. Monthly* 87 (1980), 335-345, C. Buck describes Babylonian mathematics. He begins with a discussion of a clay tablet from 3700 years ago as shown in Table 2. There are four columns. You should convince yourself that this is a table of 9's.

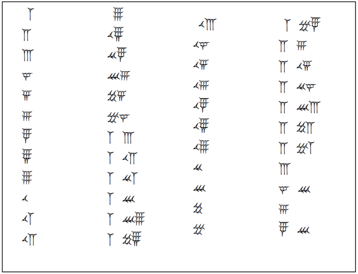


Table 2: Table of 9's.

As an example, the last entry in the first column is $12 =$. Then, $9 \times 12 = 108 =$. Note that in base 60 we have $108 = 1(60) + 48$.

In the second column is a one ( ) and 48 ( ) separated by a space. Buck introduces a slash notation to write this as 1/48.

It is easy to add in base 60. Buck gives the example $14/28/31 + 3/35/45 = 18/4/16$.

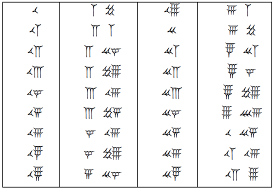
Babylonian Squares

How can a table of squares be useful? In modern notation, we see that

$$ab = \frac{1}{4}\left[(a+b)^2 - (a-b)^2\right]. \qquad (1)$$

Let's find the product 11×14. Using Table 3, the formula gives

$$\begin{aligned} 11(14) &= \frac{1}{4}\left[(11+14)^2 - (11-14)^2\right] \\ &= \frac{1}{4}(25^2 - 3^2) \\ &= \frac{1}{4}(10/25 - 9) \text{ (base 60)} \\ &= \frac{1}{4}(10/16) \text{ (base 60)} \\ &= \frac{1}{4}(10(60)+16) = \frac{616}{4} = 154. \end{aligned} \qquad (2)$$



10	1/40	19	6/1
11	2/1	20	6/40
12	2/24	21	7/21
13	2/49	22	8/4
14	3/16	23	8/49
15	3/45	24	9/36
16	4/16	25	10/25
17	4/49	26	11/16
18	5/24	27	12/9

Table 3: Table of squares with Babylonian numerals in the left table and slash notation on the right side.

4 Pythagorean Triples

Another interesting tablet from the time is the Plimpton 322 tablet shown in Figure 4. This tablet has a listing of Pythagorean triples. The last column has a list of numbers from 1 to 15. Columns two and three seem to be the hypotenuse, C, and one leg, B, of the right triangle shown in Figure 1. Recall from the Pythagorean Theorem that

$$C^2 = B^2 + D^2.$$

The triple (D, B, C) is called a Pythagorean triple.

We now know that these triples are parametrized by the pair (a, b) as follows:

$$B = a^2 - b^2, \quad C = a^2 + b^2, \quad D = 2ab,$$

since

$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2.$$

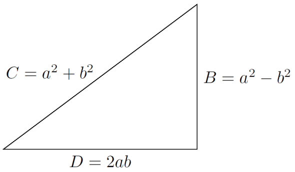


Figure 1: Right triangle with columns two and three as sides B and C, respectively. Pythagorean triples were later found to have a parametrization (a, b).

Transcription - Brackets indicate guesses

[59]/0/15	1/59	2/49	ki	1
[56/56]/58/14/50/6/15	56/7	1/20/25	ki	2
[55/7]/41/15/33/45	1/16/41	1/50/49	ki	3
53/10/29/32/52/16	3/31/49	5/9/1	ki	4
48/51/1/40	1/5	1/37	ki	[5]
47/6/41/40	5/19	8/1	[ki]	[6]
43/11/56/28/26/40	38/11	59/1	ki	7
41/33/45/14/3/45	13/19	20/49	ki	8
38/33/36/36	8/1	12/49	ki	9
35/10/2/28/27/24/26/40	1/22/41	2/16/1	ki	10
33/45	45	1/15	ki	11
29/21/54/2/15	27/59	48/49	ki	12
27/0/3/45	2/41	4/49	ki	13
25/48/51/35/6/40	29/31	53/49	ki	14
23/13/46/40	56	53	ki	[15]

Sketch of the Plimpton 322 Tablet

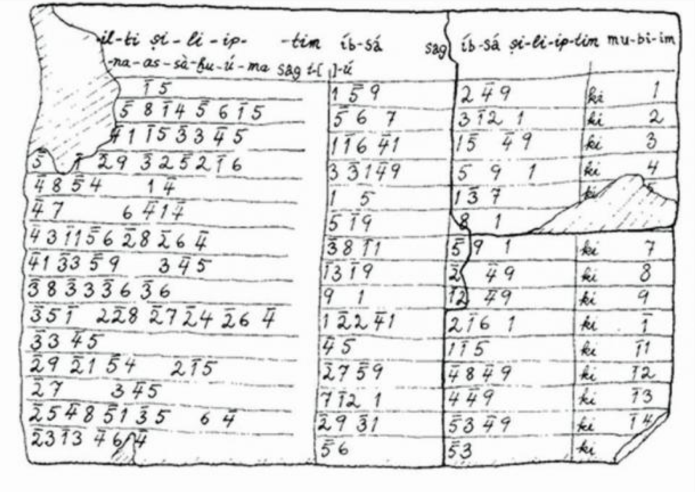


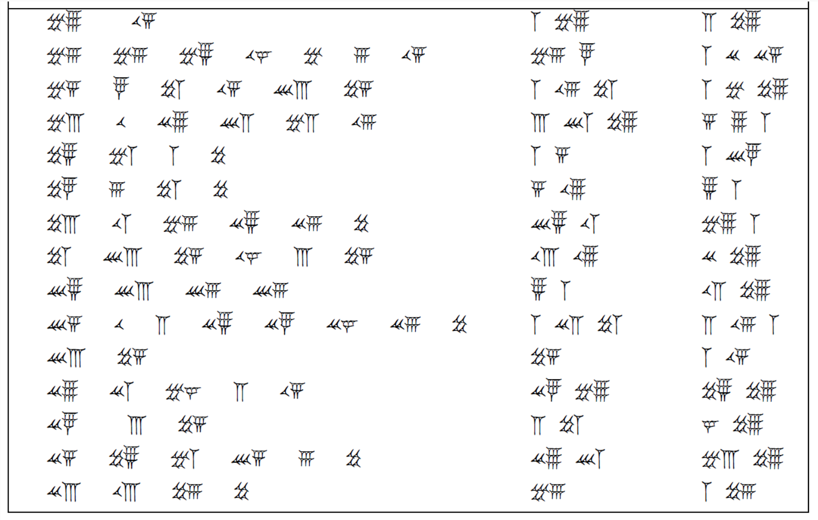
Figure 5: Arabic numerals base 60. The bars designate place holders.

Buck's Corrected Values

Second column - base 60 values for $(B/D)^2$ with $D^2 = C^2 - B^2$.

#	A	B	C	a	b
1	59/0/15	119	169	12	5
2	56/56/58/14/50/6/15	3367	4825	64	27
3	55/7/41/15/33/45	4601	6649	75	32
4	53/10/29/32/52/16	12709	18541	125	54
5	48/51/1/40	65	97	9	4
6	47/6/41/40	319	481	20	9
7	43/11/56/28/26/40	2291	3541	54	25
8	41/33/45/14/3/45	799	1249	32	15
9	38/33/36/36	481	769	25	12
10	35/10/2/28/27/24/26/40	4961	8161	81	40
11	33/45	45	75	1	0.5
12	29/21/54/2/15	1679	2929	48	25
13	27/0/3/45	161	289	15	8
14	25/48/51/35/6/40	1771	3229	50	27
15	23/13/46/40	56	106	9	5

Buck's Corrected Values - Babylonian Numerals



Greek Mathematics

Fall 2020 - R. L. Herman

Thales of Miletus

- Greek Numerals - 450 BCE
  Attic Ionic
- Geometry
  - Thales (ca. 640-546 BCE)
  - Thales' Theorem
  - Intercept Theorem

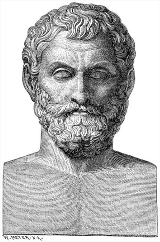


Figure 1: Thales

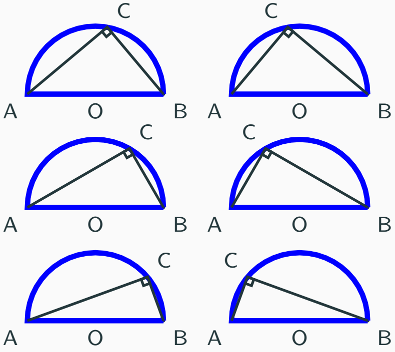


Figure 2: Thales' Theorem: An angle inscribed in a semicircle is a right angle.

Thales' Theorem: Inscribed Angle = 90^o

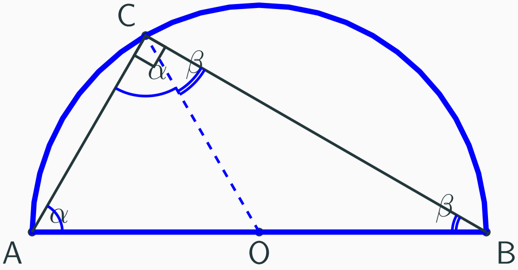


Figure 3: Proof by Picture.

Radii: $\overline{AO} = \overline{OB} = \overline{OC}$.

Isoceles triangles: AOC and OBC.

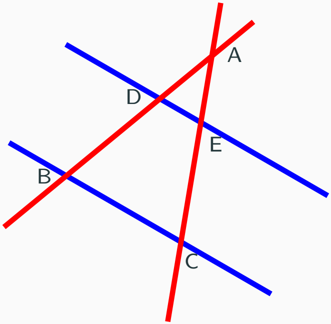
Sum of angles in ABC $= 2\alpha + 2\beta = 180^o$ implies $\alpha + \beta = 90^o$.

Intercept Theorem

If two (or more) parallel lines (blue) are intersected by two self-intersecting lines (red), then the ratios of the line segments of the first intersecting line is equal to the ratio of similar line segments of the second line.

Prove by using similar triangles:

$$\frac{\overline{DE}}{\overline{BC}} = \frac{\overline{AE}}{\overline{AC}} = \frac{\overline{AD}}{\overline{AB}}$$



Pythagoras of Samos (570-495 BCE)

- Known from Philolaus and others
- School in Croton, 530 BCE
  - vegetarian, communal, secret
  - All is number.
- Philosophy - love of wisdom
- Mathematics - that which is learned

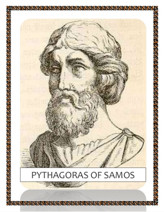


Figure 4: Pythagoras

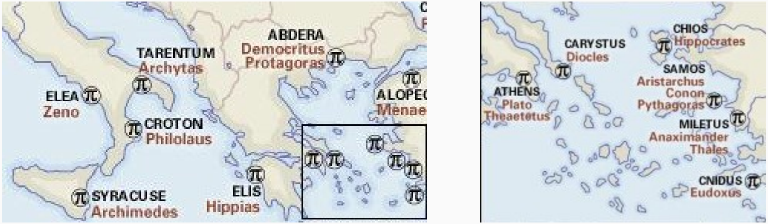


Figure 5: Locate Samos and Croton.

Numerology - Numbers have meanings.

Even is male; Odd is female.

1. = generator
2. = opinion
3. = harmony
4. = justice
5. = marriage
6. = creation
7. = planets

10 is holiest (tetractys, tetrad, decad)

Figure 6: Tetractys

Triangular numbers:

$$1, 3, 6, 10, \dots$$

Also the four seasons, planetary motions, music, four elements, fourth triangular number, etc.

Number Theory

- Triangular Numbers

$$1, 3, 6, 10, \ldots$$

- Perfect Numbers [Sum factors $< n$.]

$$\begin{aligned} 6 &= 1+2+3 \\ 10 &\neq 1+2+5 \\ 28 &= 1+2+4+7+14 \end{aligned}$$

- Amicable Numbers,

$$\begin{aligned} 220 &: 1+2+4+5+10+11+20+22+44+55+110 = \\ 284 &: 1+2+4+71+142 = \end{aligned}$$

Pythagorean Theorem, $a^2 + b^2 = c^2$

- Known by Babylonians and Egyptians
- Many Proofs over the years
- Attributed to Pythagoras
- Pythagorean Triples (a, b, c)

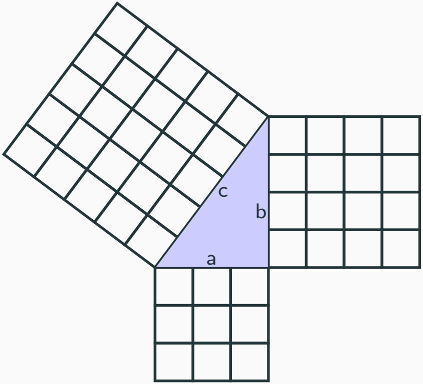


Figure 7: Euclid's Proof

Ratios

Segments are **commensurable** if there exist a segment EF such that $\overline{AB} = p\overline{EF}$ and $\overline{CD} = q\overline{EF}$, where p and q are integers.

Therefore,

$$\frac{\overline{AB}}{\overline{CD}} = \frac{p}{q}.$$

Sometimes written as $p : q$.

Led to *Music of the Spheres*.

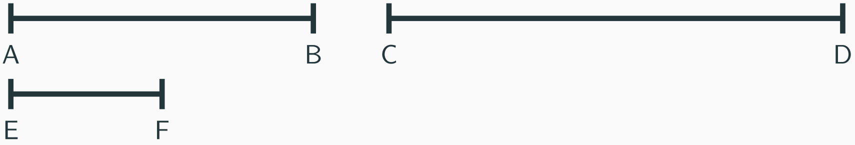


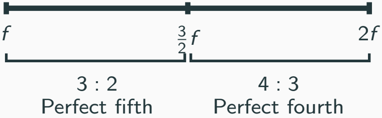
Figure 8: Commensurate Segments

Pythagorean Scale - Series of Musical Notes

Goal - To produce a music scale.

Want sounds that are pleasing when played together. Need simple ratios.

- **Octave**: From f to $2f$ (2^{nd} Harmonic).
  Ex: D goes to D, an octave higher.
- Next Notes?
  Go up by **perfect fifth**. 3 : 2
  Gives an A.
  Go down by **perfect fifth**. 2 : 3
  Gives and G (wrong octave).
  Double: 4:3 **perfect fourth.**
- From A go up perfect fifth to E.
- From G go down to C and adjust.
- Pentatonic scale: D, E, G, A, C, D.



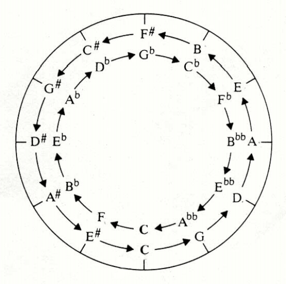
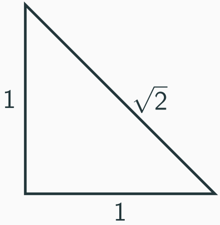


Figure 9: Circle of fifths. `https://www.phys.uconn.edu/~gibson/Notes/Section3_4/Sec3_4.htm`

Irrational Numbers

- Hippasus of Metapontum (c. 530 - c. 450 BCE).
- Credited proving $\sqrt{2}$ is irrational.
- Drowned - possibly not an accident.
- Plato wrote Theodorus of Cyrene (c. 400 BC) proved the irrationality of $\sqrt{3}$ to $\sqrt{17}$.
- Greeks knew sum of angles of triangle $= 2(90^o) = 180^o$.
- Construction of figures with compass and straight edge.



Classical Construction Problems

- Squaring the Circle (Quadrature)
- Doubling the Cube (Volume)
- Trisecting a Angle
  Impossibility Proof: 1857, Pierre Wantzel, needs Modern Algebra

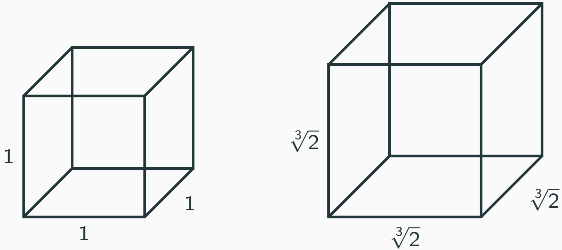


Figure 10: Doubling the Cube.

Hippocrates of Chios (c. 470 - c. 410 BCE)

- Not Hippocrates of Kos (c. 460 - c. 370 BCE),
  of the Hippocratic Oath
  Father of Medicine
- Mathematician, geometer, and astronomer.
- Went to Athens.
- Used *reductio ad absurdum* arguments (proof by contradiction).
- Wrote geometry textbook, *Elements*
- Sought Quadrature of Circle.
- Quadrature of Lune.

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.

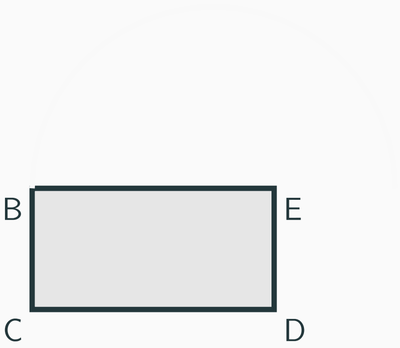


Figure 11: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.

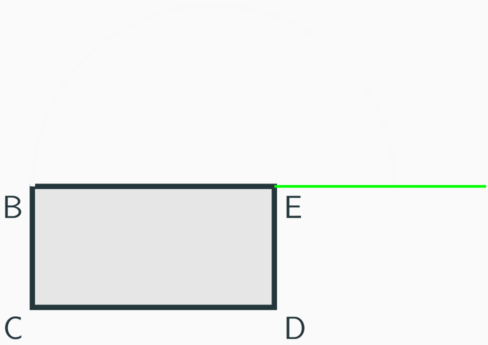


Figure 11: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.
- Get F such that $\overline{EF} = \overline{ED}$.

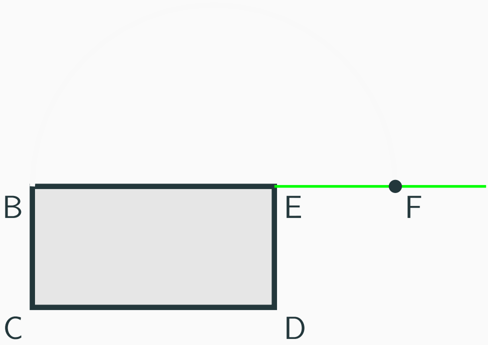


Figure 11: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.
- Get F such that $\overline{EF} = \overline{ED}$.
- How do you bisect BF?

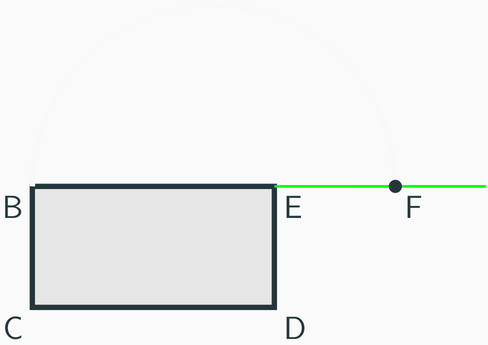


Figure 11: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
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- Get F such that $\overline{EF} = \overline{ED}$.
- How do you bisect BF?
- Bisect segment BF.

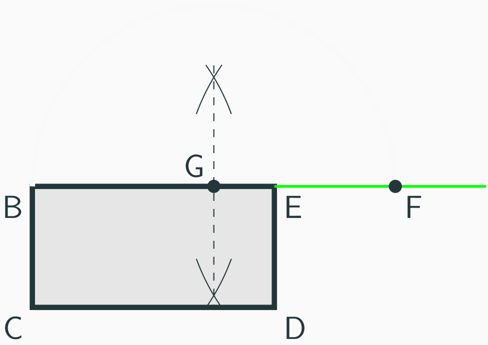


Figure 11: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.
- Get F such that $\overline{EF} = \overline{ED}$.
- How do you bisect BF?
- Bisect segment BF.
- Draw semicircle about G.

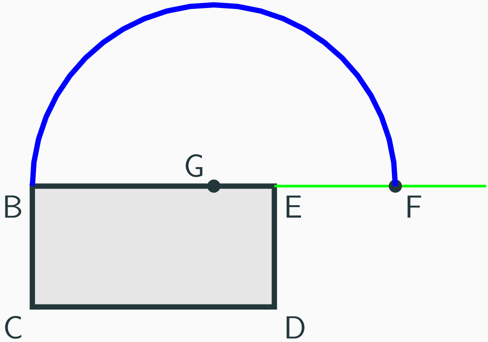


Figure 11: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.
- Get F such that $\overline{EF} = \overline{ED}$.
- How do you bisect BF?
- Bisect segment BF.
- Draw semicircle about G.
- Get point H.

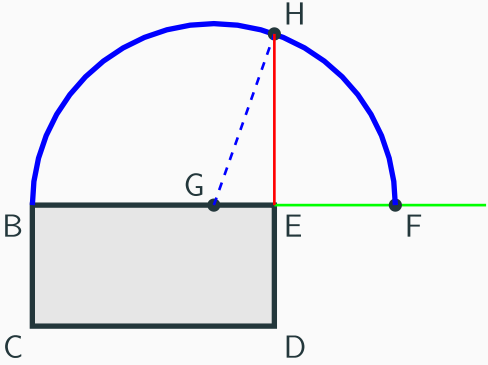


Figure 11: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.
- Get F such that $\overline{EF} = \overline{ED}$.
- How do you bisect BF?
- Bisect segment BF.
- Draw semicircle about G.
- Get point H.
- Construct square EKLH.
- Prove the areas are equal.

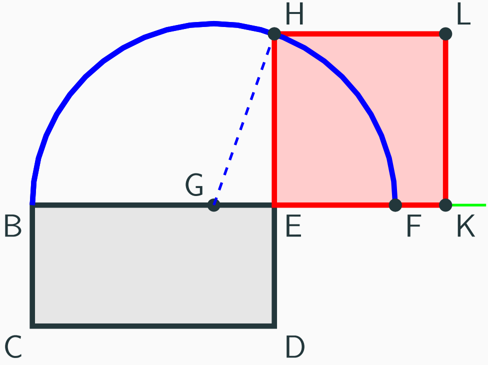


Figure 11: Quadrature of a Rectangle

Proof of Equal Areas

Label lengths a, b, c.

Area of Gray Rectangle BCDE:

$$\begin{aligned} A &= (a+b)\overline{ED} \\ &= (a+b)\overline{EF} \\ &= (a+b)(a-b) \\ &= a^2 - b^2. \end{aligned}$$

Area of Red Square EKLH:

Use Pythagorean Theorem:

$$A = c^2 = a^2 - b^2.$$

Thus, the area of the square is the same as the given rectangle; i.e., we **squared the rectangle**.

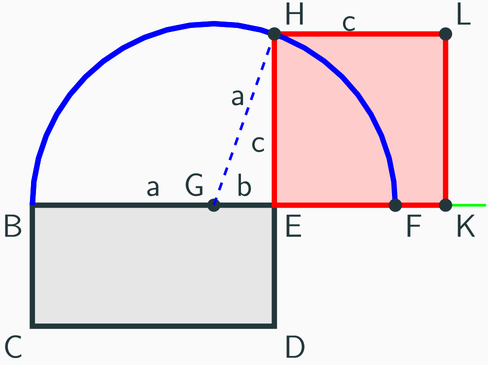
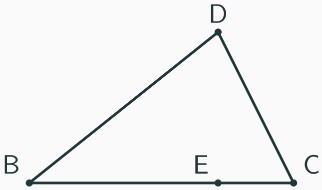


Figure 12: Quadrature of a Rectangle

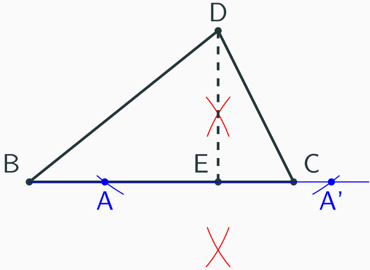
Quadrature of a Triangle

- Start with a triangle.



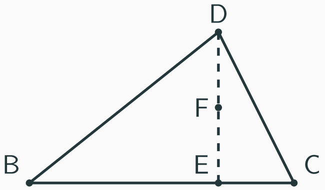
Quadrature of a Triangle

- Start with a triangle.
- Construct perpendicular measuring height.
    1. Draw blue arcs about D.
    2. Bisect AA' using red arcs..



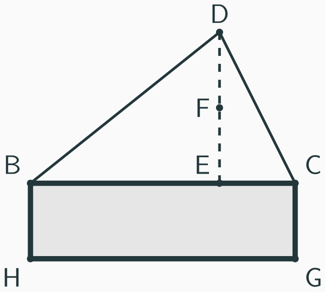
Quadrature of a Triangle

- Start with a triangle.
- Construct perpendicular measuring height.
  1. Draw blue arcs about D.
  2. Bisect AA' using red arcs..
- Bisect perpendicular.



Quadrature of a Triangle

- Start with a triangle.
- Construct perpendicular measuring height.
  1. Draw blue arcs about D.
  2. Bisect AA' using red arcs..
- Bisect perpendicular.
- Construct a rectangle with height CG = EF.
- Square this rectangle.



Quadrature of a Lune

- Lune is the figure bounded by two circular arcs.
- Hippocrates squared a special lune.
- Based on
  - Pythagorean Theorem.
  - Angle inscribed in semicircle is right.
  - Ratio of Areas of circles

$$\frac{A_1}{A_2} = \frac{D_1^2}{D_2^2}.$$

  - Triangles are quadrable.
- Hippocrates proof not valid.

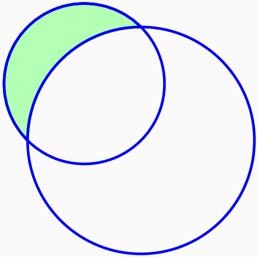


Figure 13: Lune or Crescent.

Hippocrates' Quadrature of a Lune

- $\overline{AB}^2 = \overline{AC}^2 + \overline{CB}^2 = 2\overline{AC}^2$
- Semicircle areas

$$\frac{A(AEC)}{A(ACB)} = \frac{\overline{AC}^2}{\overline{AB}^2} = \frac{1}{2}.$$

- Area of Lune = Area of ΔAOC.
- ΔAOC quadrable, so is the lune.

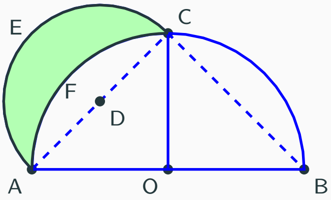


Figure 14: Lune AECF is quadrable.

Can one Square the circle?

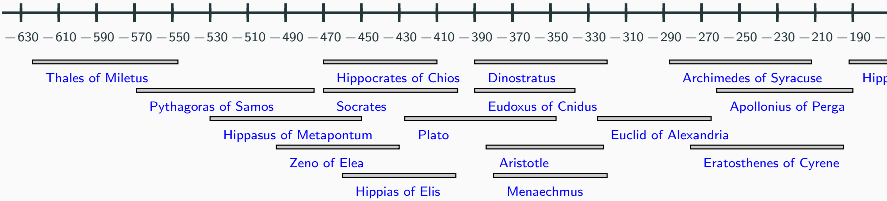
Unsolved until Ferdinand Lindemann (1852-1939).

Algebraic Numbers, solutions of polynomial equations with integer coefficients.

Ex: $x^2 - 2 = 0$ has solution $\pm\sqrt{2}$.

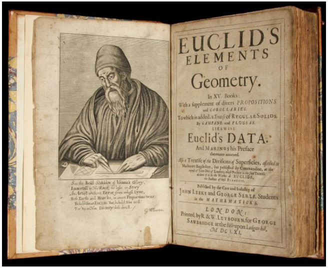
Transcendental Numbers, numbers that aren't algebraic.

Timeline of Greek Mathematicians



Euclid's Elements

Fall 2020 - R. L. Herman



Euclid of Alexandria (c. 325 - c. 265 BCE)

- Founder of Geometry
- Active in Alexandria, Egypt during reign of Ptolemy I (323–283 BC).
- *Elements of Geometry*
    - Most famous mathematical work of classical antiquity.
    - World's oldest continuously used mathematical textbook.
    - Geometry, proportion, and number theory.
    - 13 Books
    - 465 Propositions
    - 23 Definitions (point, line, straight line, . . .)
    - 5 Postulates
    - 5 Axioms

Figure 1: Euclid

The Thirteen Books

Book 1 Fundamental propositions of plane geometry.

- Congruent triangles.
- Theorems on parallel lines.
- Sum of the angles of a triangle.
- The Pythagorean theorem.

Book 2 Geometric algebra.

Book 3 Properties of circles.

- Theorems on tangents and inscribed angles.

Book 4 Inscribed and circumscribed regular polygons around circles.

Book 5 Arithmetic theory of proportion.

Book 6 Theory of proportion in plane geometry.

Book 7 Elementary number theory.

- prime numbers, greatest common denominators, etc.

Book 8 Geometric series.

Book 9 Applications and theorems on the infinitude of prime numbers, and the sum of a geometric series.

The Thirteen Books

Book 10 Incommensurable (irrational) magnitudes using the so-called "Method of Exhaustion."

Book 11 Propositions of three-dimensional geometry.

Book 12 Relative volumes of cones, pyramids, cylinders, and spheres using the Method of Exhaustion.

Book 13 The five Platonic solids.

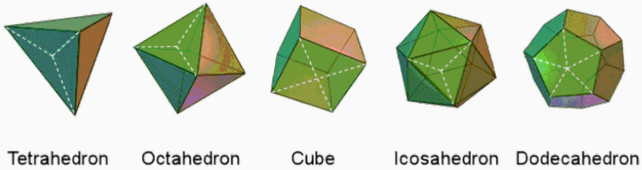


Figure 2: Platonic Solids

Definitions i

Def 1. A point is that which has no part.

Def 2. A line is breadthless length.

Def 3. The ends of a line are points.

Def 4. A straight line is a line which lies evenly with the points on itself.

Def 5. A surface is that which has length and breadth only.

Def 6. The edges of a surface are lines.

Def 7. A plane surface is a surface which lies evenly with the straight lines on itself.

Def 8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

Def 9. And when the lines containing the angle are straight, the angle is called rectilinear.

Definitions ii

Def 10. When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.

Def 11. An obtuse angle is an angle greater than a right angle.

Def 12. An acute angle is an angle less than a right angle.

Def 13. A boundary is that which is an extremity of anything.

Def 14. A figure is that which is contained by any boundary or boundaries.

Def 15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

Def 16. And the point is called the center of the circle.

Definitions iii

Def 17. A diameter of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

Def 18. A semicircle is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.

Def 19. Rectilinear figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.

Def 20. Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.

Definitions iv

Def 21. Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.

Def 22. Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.

Def 23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Postulates

Postulate 1. To draw a straight line from any point to any point.

Postulate 2. To produce a finite straight line continuously in a straight line.

Postulate 3. To describe a circle with any center and radius.

Postulate 4. That all right angles equal one another.

Postulate 5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Common Notions

Notion 1. Things which equal the same thing also equal one another.

Notion 2. If equals are added to equals, then the wholes are equal.

Notion 3. If equals are subtracted from equals, then the remainders are equal.

Notion 4. Things which coincide with one another equal one another.

Notion 5. The whole is greater than the part.

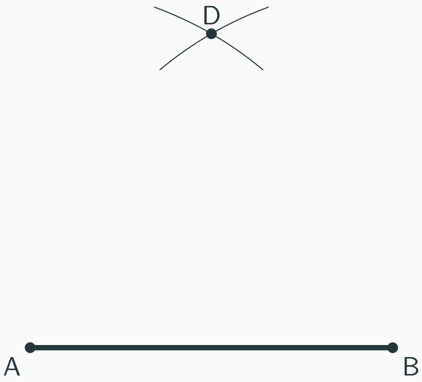
Proposition 1

To construct an equilateral triangle on a given finite straight line.

- Start with segment AB.

Proposition 1

To construct an equilateral triangle on a given finite straight line.

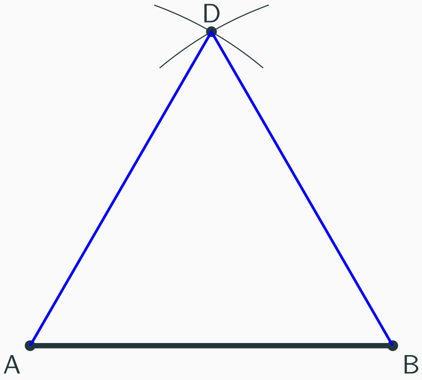


- Start with segment AB.
- Draw circular arcs about A, B of radius AB.

Proposition 1

To construct an equilateral triangle on a given finite straight line.

- Start with segment AB.
- Draw circular arcs about A, B of radius AB.
- Draw line segments AD, BD



Regular Polygons

Construct using straight edge and compass.

- Triangles (Euclid I.1)
- Squares (Euclid I.46)
- Pentagons (Euclid IV.11)
- Hexagons (Euclid IV.15)
- Septagon (heptagon) (no)
- Octagon (Euclid III.30)
- Nonagon (no)
- 15-gon (Euclid IV.16)
- Double the number of sides of a given regular polygon, 8, 10, 12, 16, 20, 24, etc. (Euclid III.30)

Constructible regular n-gons

Is it possible to construct all regular polygons with compass and straightedge? If not, which n-gons (that is polygons with n edges) are constructible and which are not?

- Carl Friedrich Gauss, 1976: the regular 17-gon is constructible.
- Theory of Gaussian periods in his *Disquisitiones Arithmeticae*. 1801.
- Gave sufficient condition for the constructibility.
- Proof of necessity - Pierre Wantzel in 1837.
- Gauss–Wantzel theorem:
  A regular n-gon can be constructed with compass and straightedge if and only if n is the product of a power of 2 and any number of distinct Fermat primes:

$$n = 2^m p_1 p_2 \cdots p_k, \qquad p = 2^{2^\ell} + 1.$$

Book 13 - Platonic Solids

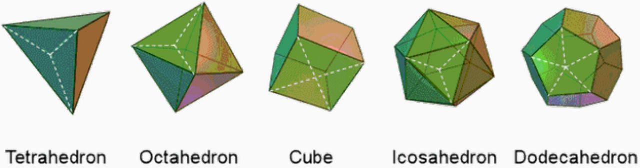
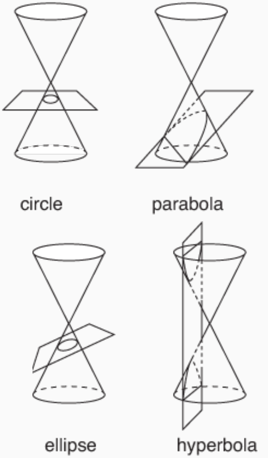


Figure 3: Platonic Solids

- Octahedron - 8 $\Delta's$
- Dodecahedron - 12 pentagons
- Icosahedron - 20 $\Delta's$

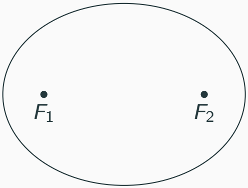
Conic Sections

- Possibly defined by **Menaechmus** (380–320 BCE) (Duplicating the cube)
- **Euclid** - four lost books on conics.
- **Archimedes** of Syracuse (287-212 BCE) studied conics, area bounded by a parabola and a chord in *Quadrature of the Parabola*.
- **Apollonius** of Perga (262-190 BCE), eight-volume *Conics*. Terms: parabola, ellipse, hyperbola
- **Pappus** of Alexandria (290 – 350 BCE) - focus directrix.



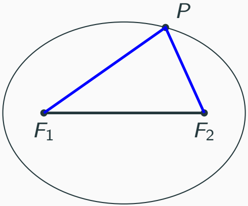
Ellipse

- Focal points: F_1, F_2



Ellipse

- Focal points: F_1, F_2
- $\overline{F_1 P} + \overline{F_2 P} = 2a$



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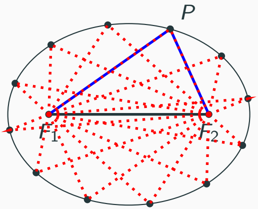
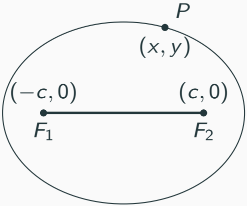


Ellipse

- Focal points: F_1, F_2
- $\overline{F_1P} + \overline{F_2P} = 2a$
- Algebra leads to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

a, $b =$ semimajor/semiminor axes with $c = \sqrt{a^2 - b^2}$, $a > b$.



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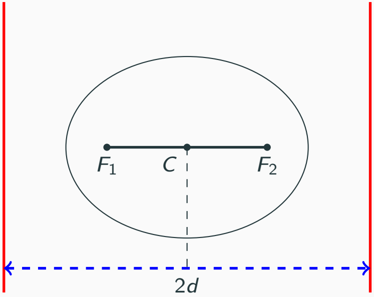

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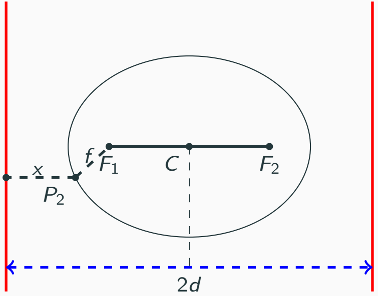
  a, $b =$ semimajor/seminor axes with $c = \sqrt{a^2 - b^2}$, $a > b$.
- Directrix $d = \frac{a^2}{c}$,
- Eccentricity $\epsilon = \frac{f}{x} = \frac{c}{a}$.



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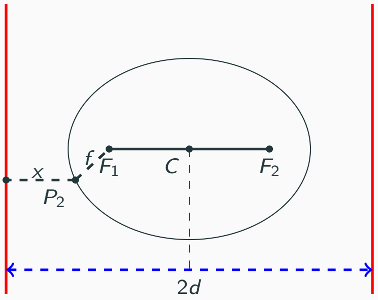
  a, b = semimajor/semiminor axes with $c = \sqrt{a^2 - b^2}$, $a > b$.
- Directrix $d = \frac{a^2}{c}$,
- Eccentricity $\epsilon = \frac{f}{x} = \frac{c}{a}$.
- Eccentricty of Conics

  $\epsilon = 0$, circle.

  $\epsilon = 1$, parabola.

  $\epsilon = 0$, hyperbola.



Dandelin Sphere - Germinal Pierre Dandelin (1822)

- Inscribed spheres tangent to cone and intersecting plane.
- Intersection is a conic.
- Tangent pts to sphere are focal points.
- Used to prove theorems of Apollonius.
    - Conic section is the set points such that the sum of the distances to two fixed points is constant.
    - The distance from the focus is proportional to the distance from a fixed line (directrix).
    - The constant of proportionality is the eccentricity

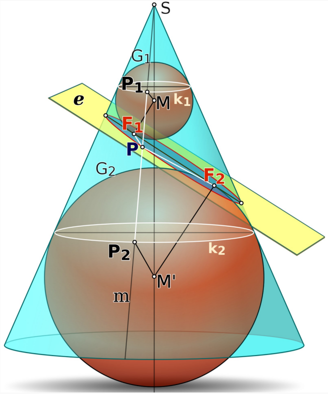
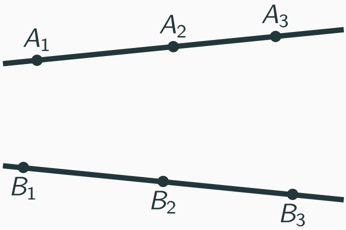


Figure 4: Dandelin Spheres

Other Geometry Gems

- Pappas' Theorem (290-350)
  - Connect 6 pts on two lines
  - A_1-B_2, B_2-A_1, etc.



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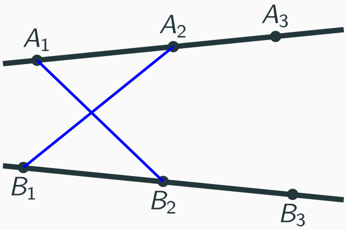

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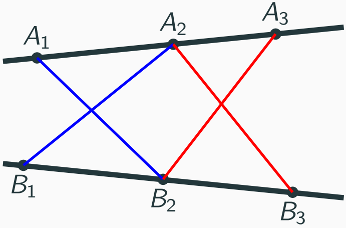

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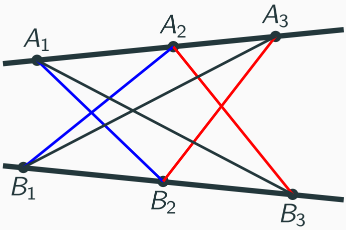

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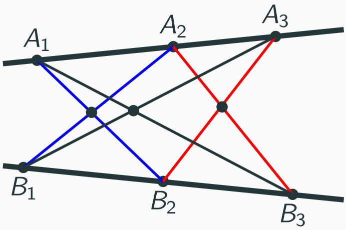
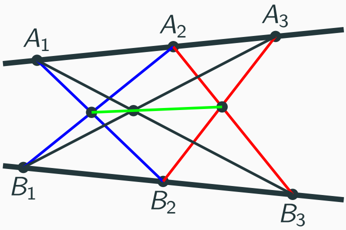


Other Geometry Gems

- Pappas' Theorem (290-350)
  - Connect 6 pts on two lines
  - A_1-B_2, B_2-A_1, etc.
- The three points are collinear.
- The beginning of projective geometry.



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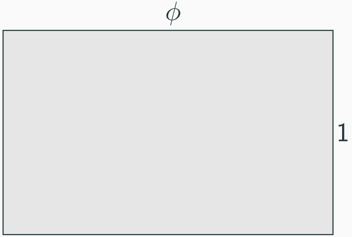

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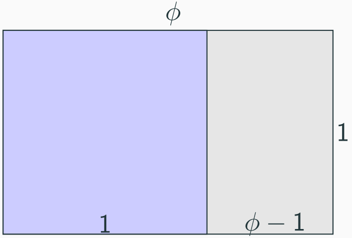
- Golden Rectangle Ratios: $\frac{\phi}{1} = \frac{1}{\phi-1}$



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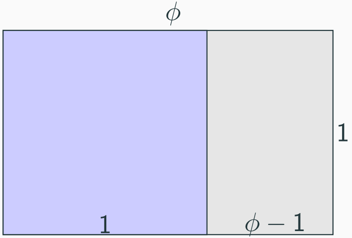
- Golden Rectangle Ratios: $\frac{\phi}{1} = \frac{1}{\phi - 1}$
- Solution $\phi^2 = \phi + 1$



$$\phi = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1+\sqrt{5}}{2} \approx 1.61803\ldots.$$

The Parthenon

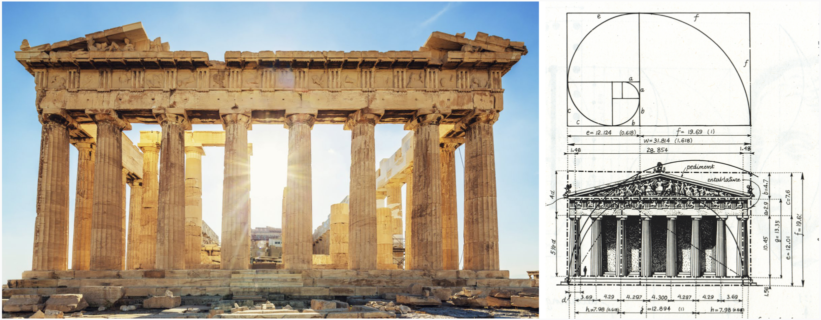


Figure 5: The Parthenon and the Golden Rectangle

Greek Mathematics II

Fall 2020 - R. L. Herman

Greek Number Theory

- **Pythagorean Theorem**
  $x^2 + y^2 = z^2$, x, y, z integers.
- **Diophantine Equations**
  Solve $3x + 5y = 1$, x, y integers.
- Euclid
  - Proved **# primes infinite**, Book IX, Prop 20
  - **Perfect Numbers**, Book VII, Def 22, Book IX, Prop 36
    Euclid proves:
    If $2^n - 1$ is prime, then $(2^n - 1)\,2^{n-1}$ is perfect.
    Mersenne prime: $2^n - 1$.
  - **Polygonal Numbers**

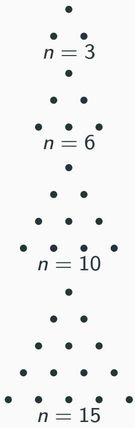


Figure 1: Polygonal Numbers

Euclidean Algorithm - Book VII, Prop 1

- $\gcd(a, b)$: **Greatest common divisor** of a and b.
- Algorithm

$$\begin{aligned} a_1 &= \max(a, b) - \min(a, b) \\ b_1 &= \min(a, b) \\ &\text{repeat} \end{aligned} \quad (1)$$

Terminates when $a_{i+1} = b_{i+1}$.

Example: Find $\gcd(210, 45)$.

$$\begin{aligned} a_1 &= 210 - 45 = 165 \\ b_1 &= 45 \end{aligned}$$

Continuing the computation:

a_i	b_i
210	45
165	45
120	45
75	45
30	45
15	30
15	15

We find $\gcd(210, 45) = 15$.

Euclidean Algorithm - Another Approach

Find the greatest common divisor of positive integers, a and b.

- If $a < b$, exchange a and b.
- Divide a by b and get the remainder, r. Thus,

$$a = qb + r.$$

  - If $r \neq 0$, replace a by b and b by r. Repeat the division.
  - If $r = 0$, report $gcd(a, b) = b$.

Example: Find $\gcd(210, 45)$.

$$\begin{aligned} 210 &= 4 \cdot 45 + 30 \\ 45 &= 1 \cdot 30 + 15 \\ 30 &= 2 \cdot 15 \end{aligned}$$

Thus, $\gcd(210, 45) = 15$.

Pell's Equation (1611-1685)

- $x^2 - Ny^2 = 1$, N is a nonsquare integer and x, y are integer solutions.
- Example of a Diophantine equation.
- Relation to $\sqrt{2}$: $x^2 - 2y^2 = 0$, $y = 1 \Rightarrow x = \sqrt{2}$.
- $x^2 - 2y^2 = 1$, If x, y large, then $\frac{x}{y} \approx \sqrt{2}$.
- Known to Pythagoreans
- Archimedes' Cattle Problem

From *The New York Times*
January 18, 1931, p. 54

CATTLE PROBLEM SOLVED

Moreover, Final Conditions Set by Archimedes Can Be Worked Out

To the Editor of The New York Times:

Frank G. Nelson, whose interesting letter regarding his solution of the cattle problem of Archimedes appeared in THE TIMES, would feel flattered if he had the translation of this problem which is possessed by me, for, according to Archimedes, he is no mere "novice in numbers," since no such person could be expected to arrive at a correct solution—as has Mr. Nelson—of the first seven equations presented by the problem.

But Mr. Nelson's conclusion that the final conditions set by the problem cannot be solved is erroneous—at least according to a large number of mathematicians who have worked on it. As far back as 1860, Amthor showed that the total of the cattle would be represented by a number containing 206,545 figures, the printing of which would require about two full pages of THE NEW YORK TIMES. Since it has been calculated that it would take the work of a thousand men for a thousand years to determine the complete number, it is obvious that the world will never have a complete solution, which should relieve the mind of any linotype operator who fears that he might be called on to set it. However, the first thirty-one figures have been computed, as have the last twelve, and the solution, for those who are interested, is

7,760,271 081,800

in which the line of dots represents thirty solved and 206,502 unsolved numbers.

The above solution was worked out by the Hillsboro Mathematical Club of Hillsboro, Ill., which was formed by A. H. Bell in 1889 to labor on the problem. Nearly four years were spent by the three club members on the work, and the results were published in the American Mathematical Monthly in 1895. An interesting summary of the mathematical steps involved in the determination of these enormous numbers—there are ten altogether, each containing 206,544 or 206,545 figures—is to be found in Recreations in Mathematics, by H. E. Licks (Van Nostrand, 1917).

Archimedes was evidently fond of problems involving enormous numbers, as his book "Arenarius" discusses the solution of the problem of determining the number of grains of sand in a sphere the size of the earth. This number is, however, of insignificant size in comparison with that representing the solution of the cattle problem. In fact, it has been calculated that if the cattle represented by this number were reduced to the size of the smallest bacterium, they could not be contained in a sphere having the diameter of the Milky Way, across which astronomers calculate that it takes light, traveling at about 186,000 miles a second, 10,000 years to travel.

NORMAN MERRIMAN,
New York, Jan. 12, 1931.

Pell's Equation General Solution

- $x^2 - Ny^2 = 1$,
  N is a nonsquare integer and
  x, y are integer solutions.
- Let $z = x + y\sqrt{n}$, $x, y \in \mathbb{Z}$
  and $\bar{z} = x - y\sqrt{n}$.
- $\text{Norm}(z) = z\bar{z} = x^2 - Ny^2 = 1$.
- $\text{Norm}(zw) = \text{Norm}(z)\text{Norm}(w)$.

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Let

$$\begin{aligned} z &= x_1 + y_1\sqrt{n}, \\ w &= x_2 + y_2\sqrt{n}, \\ zw &= x_3 + y_3\sqrt{n}, \end{aligned}$$

Then

$$\begin{aligned} x_3 &= x_1x_2 + ny_1y_2, \\ y_3 &= x_1y_2 + x_2y_1. \end{aligned}$$

Since $\text{Norm}(zw) = 1$,
(x_3, y_3) is a solution.

Pell's Equation General Solution

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- $\text{Norm}(z) = z\bar{z} = x^2 - Ny^2 = 1$.
- $\text{Norm}(zw) = \text{Norm}(z)\text{Norm}(w)$.
- Example: $x^2 - 3y^2 = 1$
- Guess $(2, 1)$.
  So, $z = 2 + \sqrt{3} = w$.

$$\begin{aligned} zw &= \left(2 + \sqrt{3}\right)^2 \\ &= 7 + 4\sqrt{3}. \end{aligned}$$

Let

$$\begin{aligned} z &= x_1 + y_1\sqrt{n}, \\ w &= x_2 + y_2\sqrt{n}, \\ zw &= x_3 + y_3\sqrt{n}, \end{aligned}$$

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Let

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Then

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Since $\text{Norm}(zw) = 1$,
(x_3, y_3) is a solution.

Then, $(7, 4)$ is a solution.

Eudoxus of Cnidus (c.390 – c. 337 BCE)

- Studied under Plato
- Taught Aristotle
- Astronomer, Mathematician
- Theory of Proportions

  Circles: $A \propto r^2$,
  Spheres: $V \propto r^3$,
  Volume of a pyramid
  Volume of a cone
- Studied reals, continuous quantities
- Method of Exhaustion
  Due to Antiphon (480–411 BCE)
  Area from a sequence of inscribed polygons

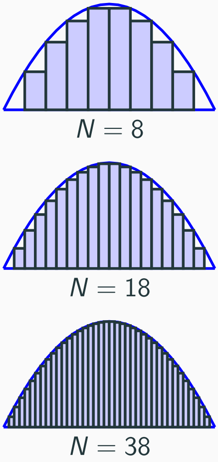
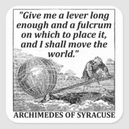
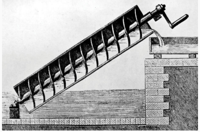


Figure 2: Method of Exhaustion

Archimedes of Syracuse (287-212 BCE)

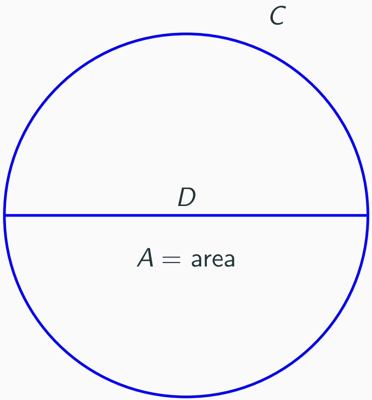
- Went to Alexandria, Egypt
  then, back to Syracuse, Sicily
- Greatest Mathematician of Antiquity
- Mathematician, Engineer, Inventor
  - Archimedean screw, lever, pulley
- King Heiro II's crown - Eureka
  Archimedes Principle of Bouyancy
- According to Plutarch (46-120)
  - Marcellus - Syracuse 212 BCE
  - Claw of Archimedes
  - Heat Ray
  - Prone to intense concentration
  - Death of Archimedes





Archimedes' Mathematics

- Mastered Euclid and Eudoxus' (c. 390-337 BCE) Method of Exhaustion
- *Measurement of a Circle*
  $\frac{C}{D} = \text{const.}, \quad \frac{A}{D^2} = \text{const.}$



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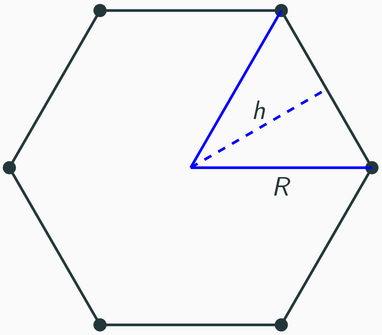
- *Measurement of a Circle*
  $\frac{C}{D} = \text{const.}, \quad \frac{A}{D^2} = \text{const.}$
- Regular Polygons (n-gon)
  $A_p = \frac{1}{2}hQ$, $Q = \text{Perimeter}$



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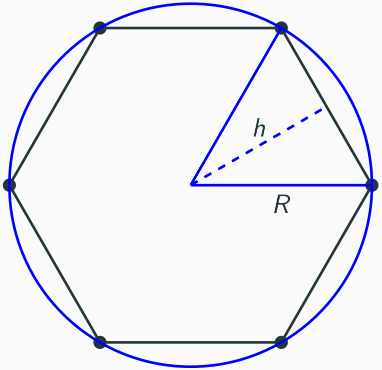
- Regular Polygons (n-gon)
  $A_p = \frac{1}{2}hQ$, $Q = \text{Perimeter}$
- Inscribed Polygons
  $A_p = \frac{1}{2}anh < \text{area of circle}$



Archimedes' Mathematics

- Mastered Euclid and Eudoxus' (c. 390-337 BCE) Method of Exhaustion
- *Measurement of a Circle*
  $\frac{C}{D} = \text{const.}, \quad \frac{A}{D^2} = \text{const.}$
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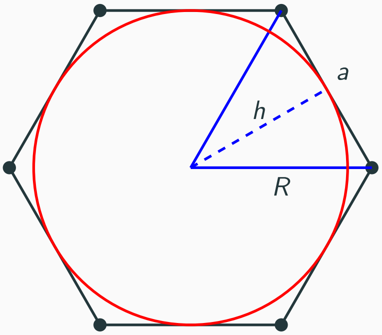
  $A_p = \frac{1}{2} anh < \text{area of circle}$
- Circumscribed Polygon
  $a = 2R \sin \frac{180}{n}, h = \sqrt{R^2 - \frac{a^2}{4}}.$



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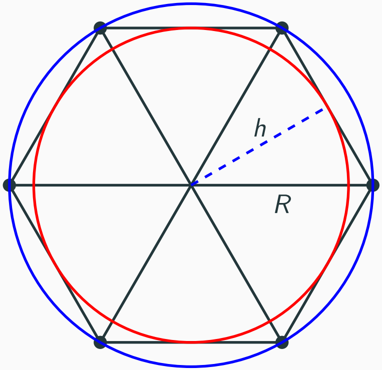
- Circumscribed Polygon $a = 2R\sin\frac{180}{n}, h = \sqrt{R^2 - \frac{a^2}{4}}.$
- Approximation of π, $\frac{A_p}{R^2} < \pi < \frac{A_p}{h^2}$

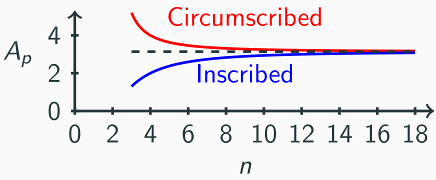


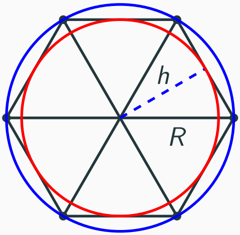
Estimating π

- Approximation of π,
  $\frac{A_p}{R^2} < \pi < \frac{A_p}{h^2}$,
- Recall
  $a = 2R \sin \frac{180}{n}$,
  $h = \sqrt{R^2 - \frac{a^2}{4}} = R \cos \frac{180}{n}$,
  $A_p = \frac{1}{2} anh = nhR \sin \frac{180}{n}$.
- Therefore,

$$\frac{n}{2} \sin \frac{360}{n} < \pi < n \tan \frac{180}{n},$$

- Hexagon ($n = 6$),
  $2.598 < \pi < 3.464$.
- Archimedes - up to 96-gon
  $3.1394 < \pi < 3.1427$.





Archimedes' Inscribed and Circumscribed n-gons

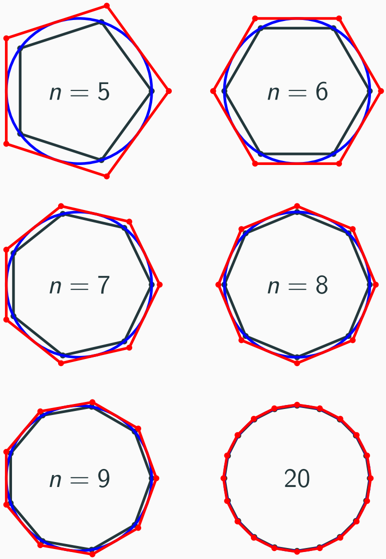
Consider a fixed circle of radius R.

- Inscribed n-gon: $h = R\cos\frac{180}{n}$, $A_i = nhR\sin\frac{180}{n}$.
- Circumscribed n-gon: $r = \frac{R}{\cos\frac{180}{n}}$, $A_c = nHr\sin\frac{180}{n}$.
- Thus,

$$\begin{aligned} A_i &= nR^2\tan\frac{180}{n}, \\ A_c &= \frac{n}{2}R^2\sin\frac{360}{n}. \end{aligned}$$

- This gives

$$\frac{n}{2}\sin\frac{360}{n} < \pi < n\tan\frac{180}{n},$$

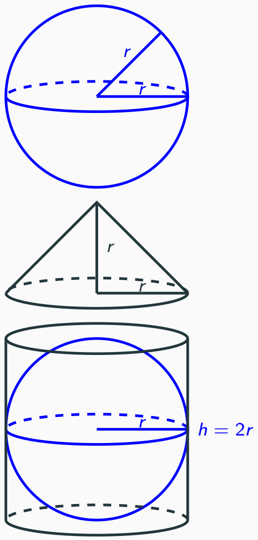


Early Approximations of π

- Bible, $\pi \approx 3$.
- Egyptians, $\left(\frac{4}{3}\right)^4 = \frac{256}{81} \approx 3.1604938$
- Ptolemy (150), 360-gon, 3.1416
- Chinese (430-501) $\frac{355}{113} \approx 3.14159292$
- Hindu (1100) $\frac{3927}{1250} \approx 3.1416$
- Viete', 393,216-gon, π to 9 places.
- Dutch, 17^{th} Century, 35 places.
- Lambert (1728-1777) - irrationality proof.
- William Jones (1706) π
- Euler popularized notation

- Leibniz-Madhava
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$
- Euler
$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$
- Ramanujan
$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!}{k!^4} \frac{1103 + 26390k}{396^{4k}}$$

- Spirals, Area of Parabolae
- Volumes and Surface Areas of 3D Objects
- A(Sphere) = 4 A(Great circle) $A = 4(\pi r^2)$.
- V(Sphere) = 4 V(Cone) $V = 4\left(\frac{1}{3}\pi r^3\right)$.
- Sphere inside Cylinder
  Cylinder Area =
  $2\pi r(2r) + 2(\pi r^2) = 6\pi r^2$
  $= \frac{3}{2}$ Sphere Area.
  Volume $= (\pi r^2)h = 2\pi r^3$
  $= \frac{3}{2}$ Sphere Volume.



Archimedes' Manuscripts

- What we know from 3 books
- Codex A Lost in 1564
- Codex B Lost in 1311
- Codex C Discovered 1906
  - 4th century Parchment bound
  - 10th Century Book, Constantinople housed great texts
  - 1204 4th Crusade destroyed books
  - 87 Sheets (43.5 goat skins)
  - 1229 Century taken apart, scraped, cut in half, written over with Christian prayer
  - Moved to Palestine, 400 yrs

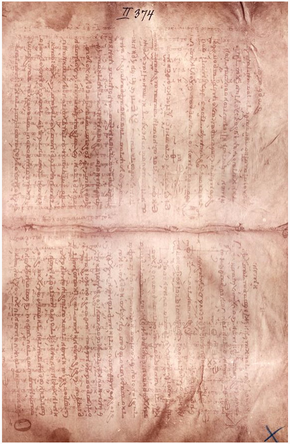


Figure 3: Codex C Page

The Walters Museum - `http://www.archimedespalimpsest.net`

Palimpsest, http://www.archimedespalimpsest.org/about/

- 1846 in Istanbul, leaf removed to Cambridge.
- 1906 Johan Heiberg took pictures and translated.
- 1922 It went missing
- 1998 Sold for $2,000,000 - Christies of NY auction. Moldy, Charred,
- 7 Manuscripts

  The Equilibrium of Planes, Spiral Lines, The Measurement of the Circle, Sphere and Cylinder, On Floating Bodies, The Method of Mechanical Theorems, and the Stomachion.

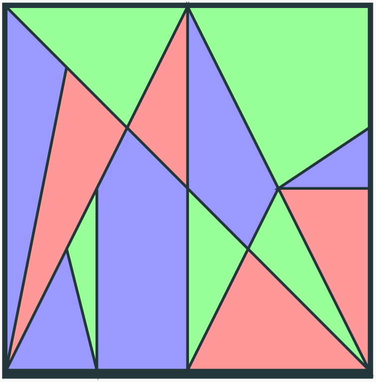


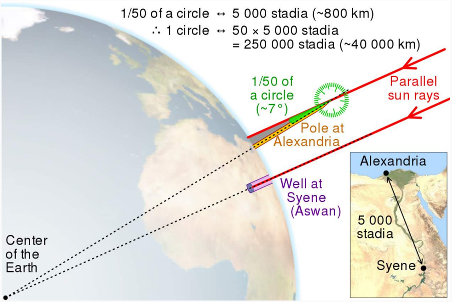
Figure 4: Number of different arrangements of the Stomachion, 17,152.

Eratosthenes of Cyrene (276 - 194 BCE)

- Chief Librarian at Library of Alexandria (300-529 BCE)
  Burned 641 by Arabs
- Sieve of Eratosthenes
- Finding primes

$$\cancel{1}, 2, 3, \cancel{4}, 5, \cancel{6}, 7, \cancel{8}, \cancel{9}, \cancel{10}, 11, \ldots$$

- Circumference of Earth
  - Syrene - 1st day summer
    Sun directly overhead
  - Alexandria - small shadow
  - 250,000 stadia
    1 stade ≈ 526.37 ft
    Equals $24,466$ mi.
    Current - $24,860$ mi.

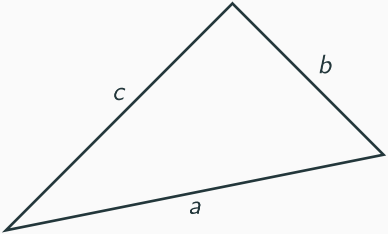


Heron of Alexandria (c. 10-70)

- Inventor
- aeolipile,rocket-like reaction engine
- first-recorded steam engine
- Hero's wind-powered organ
- The first vending machine
- A wind-wheel operating an organ,
- The force pump
- A syringe-like device
- the principle of the shortest path of light:
- standalone fountain
- A programmable cart

Heron's Formula

$$A = \sqrt{(s-a)(s-b)(s-c)}$$
$$S = \frac{1}{2}(a+b+c)$$



Last of the Ancient Greek Mathematicians

- Ptolemy (100-170)
  - Astronomy
  - Copernicus (1300)
  - Heliocentric vs geocentric
- Diophantus (200's)
  - Equations with integer solutions.
- Hypatia (370-415)
  - Father - Theon
  - Martyr
  - Movie - *Agora*

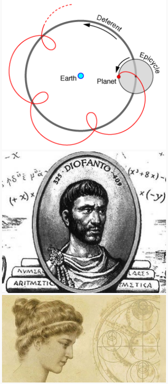
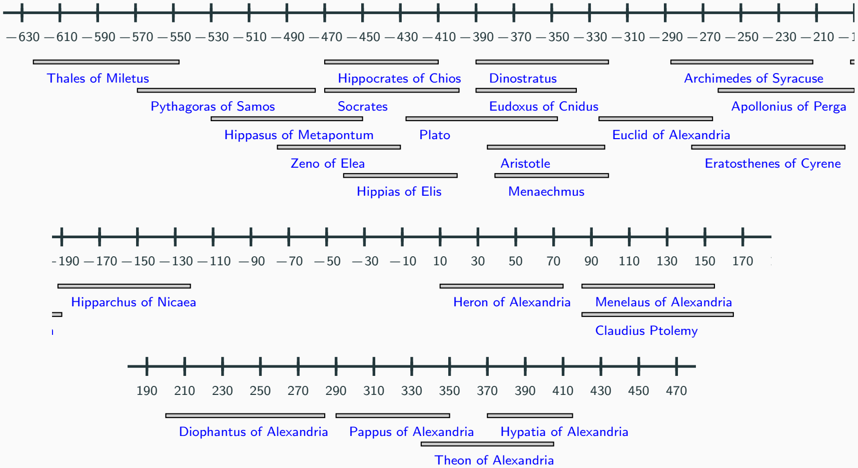


Figure 5: Epicycles, Diophantus, and Hypatia

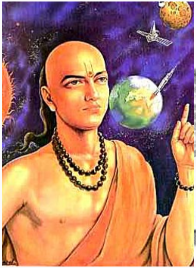
Romans - Little contribution to mathematics.

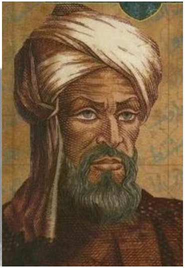
Timeline of Greek Mathematicians



Early Asian Mathematics

Fall 2020 - R. L. Herman





Overview

China

- Unique development
- *Zhoubi Suanjing* - c. 300 BCE
- *Tsinghua Bamboo Slips*, - decimal times table. 305 BCE
- Chinese abacus ($<$190 CE)
- After book burning (212 BCE), Han dynasty (202 BCE–220) produced mathematics works.

India

- *Pingala* (3rd–1st cent. BCE) - binary numeral system, binomial theorem, Fibonacci numbers.
- *Siddhantas*, 4-5 cent. astronomical treatises, trigonometry.

Arabian-Islamic (330-1450)

- Preserved Greek texts
- 7-14th cent. Development of algebra, etc.
- Hindu-Arabic numerals

Chinese Mathematics

- *The Nine Chapters on the Mathematical Art.*
  - 246 word problems on agriculture, business, geometry, engineering, surveying.
  - Proof for the Pythagorean theorem.
  - Formula for Gaussian elimination.
  - Provides values of π. [They had approximated as 3.]
- Computing π
  - Liu Xin (d. 23 AD), $\pi \approx 3.1457$.
  - Zhang Heng (78–139), $\pi \approx 3.1724, 3.162$ using $\sqrt{10}$.
  - Liu Hui (3rd century), commented on the Nine Chapters, $\pi = 3.14159$.
  - Zu Chongzhi (5th century), $\pi = 3.141592$ remained the most accurate value almost 1000 years.
- Zu Chongzhi gave a method (Cavalieri's principle) for volume of a sphere.

Pascal's Triangle

Known by early Chinese mathematicians:

- Systems of linear equations
- Chinese Remainder Theorem
- Square roots
- Pythagorean Theorem
- Euclidean algorithm
- Pascal's Triangle

$$\begin{aligned}
(a+b)^0 &= 1\\
(a+b)^1 &= a+b\\
(a+b)^2 &= a^2+2ab+b^2\\
(a+b)^3 &=\\
(a+b)^4 &=
\end{aligned}$$

Figure 1: Binomial Expansions, $(a+b)^n \qquad n = 0, 1, \ldots.$

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- Typical term, $a^{n-k}b^k$, $k = 0, 1, \ldots, n$.

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\end{aligned}
$$

Figure 1: Binomial Expansions, $(a+b)^n = \sum_{k=0}^{n} C_{n,k} a^{n-k} b^k, \; n = 0, 1, \ldots.$

Pascal's Triangle

Known by early Chinese mathematicians:

- Systems of linear equations
- Chinese Remainder Theorem
- Square roots
- Pythagorean Theorem
- Euclidean algorithm
- Pascal's Triangle
- Typical term, $a^{n-k}b^k$, $k = 0, 1, \ldots, n$.
- What is the coefficient?

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a + b \\
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\end{aligned}$$

Figure 1: Binomial Expansions, $(a+b)^n = \sum_{k=0}^{n} C_{n,k} a^{n-k} b^k, \; n = 0, 1, \ldots.$

Pascal's Triangle

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Figure 2: Pascal's Triangle, $C_{n,k} = \frac{n!}{(n-k)!k!}$

Pascal's Triangle

$$
\begin{array}{c}
1 \\
1 + 1 \\
1 + 2 + 1 \\
1 + 3 + 3 + 1 \\
1 + 4 + 6 + 4 + 1 \\
1 + 5 + 10 + 10 + 5 + 1
\end{array}
$$

Sum each row:

Figure 2: Pascal's Triangle

Pascal's Triangle

Sum each row:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & = 1 \\
 & & & & & 1 & + & 1 & & & & & = 2 \\
 & & & & 1 & + & 2 & + & 1 & & & & = 4 \\
 & & & 1 & + & 3 & + & 3 & + & 1 & & & = 8 \\
 & & 1 & + & 4 & + & 6 & + & 4 & + & 1 & & = 16 \\
1 & + & 5 & + & 10 & + & 10 & + & 5 & + & 1 & & = 32
\end{array}
$$

Figure 2: Pascal's Triangle

Pascal's Triangle

$$1 = 1$$

$$1 + 1 = 2$$

Sum each row: $$1 + 2 + 1 = 4$$

Sum $= 2^n$. $$1 + 3 + 3 + 1 = 8$$

$$1 + 4 + 6 + 4 + 1 = 16$$

$$1 + 5 + 10 + 10 + 5 + 1 = 32$$

Figure 2: Pascal's Triangle

Euclidean Algorithm Example

Example
One can only go 23 units or 79 units left or right. How many of each would it take to go from 0 to 1?

Euclidean Algorithm Example

Example
One can only go 23 units or 79 units left or right. How many of each would it take to go from 0 to 1?

- $79m + 23n = 1.$

Euclidean Algorithm Example

Example
One can only go 23 units or 79 units left or right. How many of each would it take to go from 0 to 1?

- $79m + 23n = 1.$
- Use Euclidean Algorithm

$$79 = 3 \cdot 23 + 10$$

Euclidean Algorithm Example

Example
One can only go 23 units or 79 units left or right. How many of each would it take to go from 0 to 1?

- $79m + 23n = 1.$
- Use Euclidean Algorithm

$$\begin{aligned} 79 &= 3 \cdot 23 + 10 \\ 23 &= 2 \cdot 10 + 3 \end{aligned}$$

Example
One can only go 23 units or 79 units left or right. How many of each would it take to go from 0 to 1?

- $79m + 23n = 1.$
- Use Euclidean Algorithm

$$
\begin{aligned}
79 &= 3 \cdot 23 + 10 \\
23 &= 2 \cdot 10 + 3 \\
10 &= 3 \cdot 3 + 1 \\
3 &= 3 \cdot 1
\end{aligned}
$$

Euclidean Algorithm Example

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One can only go 23 units or 79 units left or right. How many of each would it take to go from 0 to 1?

- $79m + 23n = 1.$
- Use Euclidean Algorithm

$$\begin{aligned} 79 &= 3 \cdot 23 + 10 \\ 23 &= 2 \cdot 10 + 3 \\ 10 &= 3 \cdot 3 + 1 \\ 3 &= 3 \cdot 1 \end{aligned}$$

In Reverse:

$$1 = 10 - 3 \cdot 3$$

Euclidean Algorithm Example

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One can only go 23 units or 79 units left or right. How many of each would it take to go from 0 to 1?

- $79m + 23n = 1.$
- Use Euclidean Algorithm

$$\begin{aligned} 79 &= 3 \cdot 23 + 10 \\ 23 &= 2 \cdot 10 + 3 \\ 10 &= 3 \cdot 3 + 1 \\ 3 &= 3 \cdot 1 \end{aligned}$$

In Reverse:

$$\begin{aligned} 1 &= 10 - 3 \cdot 3 \\ &= 10 - 3(23 - 2 \cdot 10) \end{aligned}$$

Euclidean Algorithm Example

Example
One can only go 23 units or 79 units left or right. How many of each would it take to go from 0 to 1?

- $79m + 23n = 1.$
- Use Euclidean Algorithm

$$
\begin{aligned}
79 &= 3 \cdot 23 + 10 \\
23 &= 2 \cdot 10 + 3 \\
10 &= 3 \cdot 3 + 1 \\
3 &= 3 \cdot 1
\end{aligned}
$$

In Reverse:

$$
\begin{aligned}
1 &= 10 - 3 \cdot 3 \\
&= 10 - 3(23 - 2 \cdot 10) \\
&= 7 \cdot 10 - 3 \cdot 23 \\
&= 7 \cdot (79 - 3 \cdot 23) - 3 \cdot 23 \\
&= 7 \cdot 79 - 24 \cdot 23
\end{aligned}
$$

Euclidean Algorithm Example

Example
One can only go 23 units or 79 units left or right. How many of each would it take to go from 0 to 1?

- $79m + 23n = 1.$
- Use Euclidean Algorithm

$$\begin{aligned} 79 &= 3 \cdot 23 + 10 \\ 23 &= 2 \cdot 10 + 3 \\ 10 &= 3 \cdot 3 + 1 \\ 3 &= 3 \cdot 1 \end{aligned}$$

In Reverse:

$$\begin{aligned} 1 &= 10 - 3 \cdot 3 \\ &= 10 - 3(23 - 2 \cdot 10) \\ &= 7 \cdot 10 - 3 \cdot 23 \\ &= 7 \cdot (79 - 3 \cdot 23) - 3 \cdot 23 \\ &= 7 \cdot 79 - 24 \cdot 23 \end{aligned}$$

Thus, $m = 7, n = -24.$

Chinese Remainder Theorem

The Chinese remainder theorem: If one knows the remainders of the Euclidean division of an integer x by several integers, then one can determine uniquely the remainder of the division of x by the product of these integers, assuming the divisors are pairwise coprime. Earliest - Sun-tzu in *Sun-tzu Suan-ching*.

If $p_1, p_2, \ldots, p_n$ are relatively prime, then

$$\begin{aligned} x &= r_1 \pmod{p_1} \\ x &= r_2 \pmod{p_2} \\ &\vdots \\ x &= r_n \pmod{p_n} \end{aligned}$$

always has a solution.

Chinese Remainder Theorem Example

Example

$$\begin{aligned} x &= 2 \pmod 3 \\ x &= 3 \pmod 5 \\ x &= 2 \pmod 7 \end{aligned}$$

First equation means $x = 3n + 2$. Insert into second:

$$\begin{aligned} 3n + 2 &= 3 \pmod 5 \\ 3n &= 1 \pmod 5 \\ 3n &= 6 \pmod 5 \\ n &= 2 \pmod 5 \end{aligned}$$

So,

$$\begin{aligned} x &= 3n + 2 \\ &= 3(5m + 2) \\ &= 15m + 8. \end{aligned}$$

From third equation

$$\begin{aligned} 15m + 8 &= 2 \pmod 7 \\ 15m + 1 &= 2 \pmod 7 \\ 15m &= 1 \pmod 7 \\ 15m &= 15 \pmod 7 \\ m &= 1 \pmod 7 \end{aligned}$$

Therefore, $m = 7k + 1$ and $x = 105k + 23$.

Indian Mathematics (500-1200)

- Major mathematicians
  - Aryabhata (476-550?)
  - Bhaskara I (600-680)
  - Brahmagupta (598-668)
  - Bhaskara II (1114-1185)
  - Madhava (1350-1425)
- Contributions
  - Algebra
  - Geometry
  - Trigonometry
  - Spherical trigonometry
  - Diophantine Equations
  - Mathematical astronomy
  - Place-value decimal system

Brahmagupta:
$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$
$s = \frac{1}{2}(a+b+c+d)$ is semiperimeter

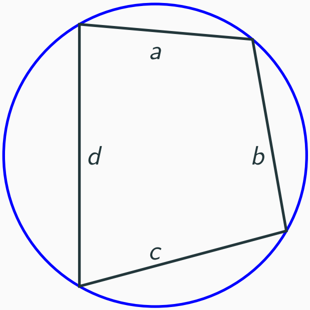


Figure 3: Cyclic Quadrilaterals

Aryabhata (476-550)

- Major work, *Aryabhatiya*, mathematics and astronomy,

    arithmetic, algebra,
    plane trigonometry,
    spherical trigonometry.
    continued fractions,
    quadratic equations,
    sums of power series, and
    table of sines.

- 108 verses, 13 introductory verses
- Relativity of motion
- *Arya-siddhanta*,

    Astronomical computations
    Astronomical instruments

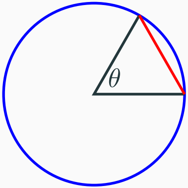
Figure 4: Aryabhata on the grounds of IUCAA, Pune.

Table of Sines

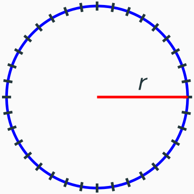
- Introduction of sine
- Aryabhata's sine table
- Based on half chords
  vs Hipparchus, Menlaus, Ptolemy.
- Also, provided differences
- From Babylonians, base 60
  degrees, minutes, seconds
- Circumference $= 21600'$.
- Aryabhata, $\pi = 3.1416$
- Bhaskara I approximation

$$\sin\theta \approx \frac{4\theta(180-\theta)}{40500-\theta(180-\theta)}.$$

- Mādhava's - more accurate.



Chord of θ



$360^o = 21600'$

$$r = \frac{21600}{3.1416} \approx 3438$$

Pell's Equation, $x^2 - Ny^2 = 1$, N Nonsquare

Brahmagupta:

$$(x_1^2 - Ny_1^2)(x_2^2 - Ny_2^2) = (x_1x_2 + Ny_1y_2)^2 - N(x_1y_2 + x_2y_1)^2$$

If $x_1^2 - Ny_1^2 = k_1$ and $x_2^2 - Ny_2^2 = k_2$, then

$$\begin{aligned} x &= x_1x_2 + Ny_1y_2 \\ y &= x_1y_2 + x_2y_1 \end{aligned}$$

solves $x^2 - Ny^2 = k_1k_2$.

This gives a composition of triples, (x_1, y_1, k_1) and (x_2, y_2, k_2) to give (x, y, k_1k_2).

Example (Brahmagupta) $x^2 - 92y^2 = 1$.

Note: $10^2 - 92(1)^2 = 8$. Thus, triple $= (10, 1, 8)$.

Pell's Equation (cont'd)

- $10^2 - 92(1)^2 = 8 \to (10, 1, 8)$.
- Compose $(10, 1, 8)$ with itself.
  $(10 \cdot 10 + 92 \cdot 1 \cdot 1, 10 \cdot 1 + 1 \cdot 10, 8 \cdot 8) = (192, 20, 64)$
- or, $192^2 - 92(20)^2 = 64$
  $24^2 - 92\left(\frac{5}{2}\right) = 1$
- Compose $(24, \frac{5}{2}, 1)$ with itself: $(1151, 120, 1)$.
- Bhaskara II (1150) - cyclic process always works.

Pell's Equation (cont'd)

- $10^2 - 92(1)^2 = 8 \to (10, 1, 8)$.
- Compose $(10, 1, 8)$ with itself. $(10 \cdot 10 + 92 \cdot 1 \cdot 1, 10 \cdot 1 + 1 \cdot 10, 8 \cdot 8) = (192, 20, 64)$
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- Compose $(24, \frac{5}{2}, 1)$ with itself: $(1151, 120, 1)$.
- Bhaskara II (1150) - cyclic process always works.
- Proved by Lagrange (1768) $gcd(a, b) = 1,\ a^2 - Nb^2 = k$.
- Compose (a, b, k) with $(m, 1, m^2 - N)$ gives $(am + Nb, a + bm, k(m^2 - N))$.
- Rescale

$$\left(\frac{am + Nb}{k}, \frac{a + bm}{k}, m^2 - N\right)$$

Pell's Equation (cont'd)

- $10^2 - 92(1)^2 = 8 \to (10, 1, 8)$.
- Compose $(10, 1, 8)$ with itself. $(10 \cdot 10 + 92 \cdot 1 \cdot 1, 10 \cdot 1 + 1 \cdot 10, 8 \cdot 8) = (192, 20, 64)$
- or, $192^2 - 92(20)^2 = 64$ $24^2 - 92\left(\frac{5}{2}\right) = 1$
- Compose $(24, \frac{5}{2}, 1)$ with itself: $(1151, 120, 1)$.
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- Rescale

$$\left(\frac{am + Nb}{k}, \frac{a + bm}{k}, m^2 - N\right)$$

- Fermat (1657), $x^2 - 61y^2 = 1$, $x = 1766319049$, $y = 226153980$.

Number Systems

- Egyptian - hieroglyphics
- Greek - letters
- Roman - I, V, X, L, C, D, M, I, III, IV, ...
- Chinese - counting boards
- Hindu

  $\sim$ 600: 1,2,...,9
  800: placed digits ,powers of 10, zero.
- Arabic - Adopted Hindu for Hindu–Arabic numeral system.

Arabian-Islamic Mathematics

- Absorbed Greek and Hindu math
- Contributed to

  Algebra, geometry, astronomy,
  plane and spherical trigonometry.
- Al'Khwarizmi (780-850)
  - al Jabr and algorithm
  Solved equations with variables.
- Omar Khyayyam (1048-1131) studied cubic equations, and the intersection of conics, Khyayyam's triangle, parallel postulate.
- al-Kashi (1380-1429), decimal and sexagesimal fractions, good approximation to π. Law of Cosines.

Rise of European Mathematics

- Fall of the Roman Empire
- Middle Ages, Medieval Period, 5th to the 15th century.
- Byzantine Empire (330-1453) - Church split, preservation of Greek works.
- Al'Khwarizmi's work and Euclid translated.
- Crusades (1095-1291), The Plague (1347 to 1351), Church - monasteries
- Al'Khwarizmi's work and Euclid translated.
- Johannes Gutenberg' printing press, 1440.
- Renaissance (1400-1600) and the Age of Discovery.
- Questioning of Aristotle
- Church of England (1534), Protestant vs Roman Catholic

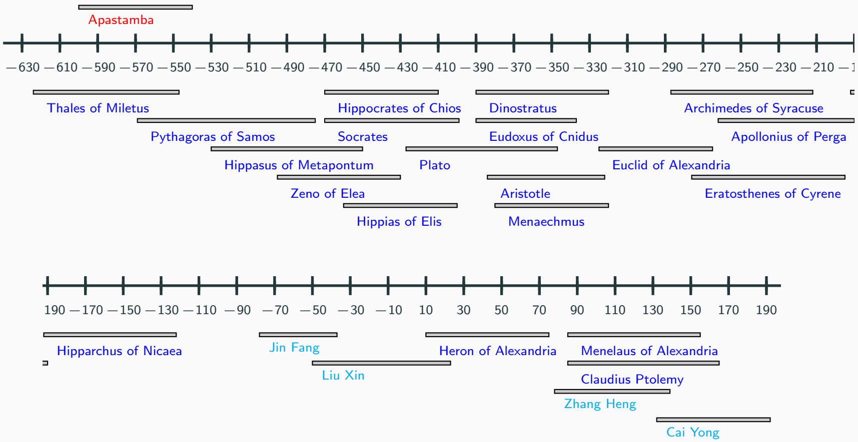
Medieval Mathematicians

- Leonardo of Pisa (Fibonacci) (1200).
- Nicole Oresme (1323-1382), coordinate geometry, fractional exponents, infinite series.
- Johann Müller Regiomontatus (1436-1476), separated trigonometry from astronomy.
- And others:

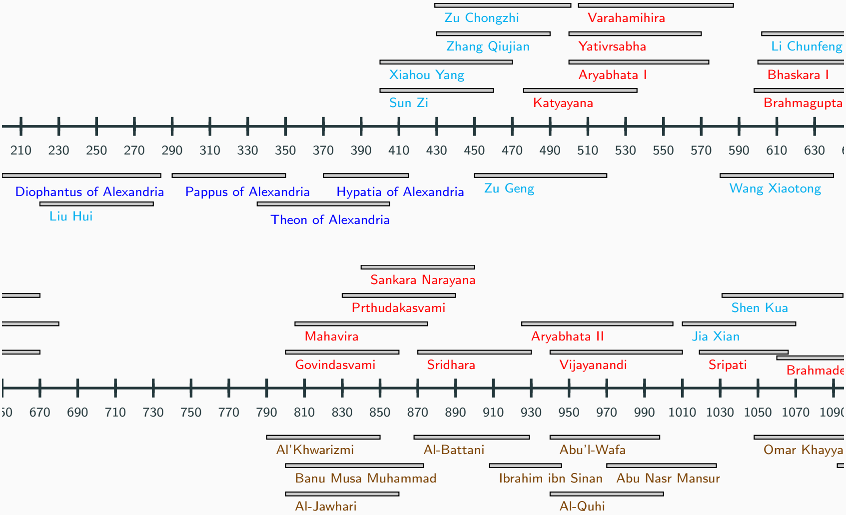
Roger Bacon (1214-1292)	William of Ockham (1288-1348)
Filippo Brunelleschi (1377-1446)	Leone Battista Alberti (1404-1472)
Nicholas of Cusa (1401-1464)	Luca Pacioli (1445-1517)
Leonardo da Vinci (1452-1519)	Scipione del Ferro (1465-1526)
Nicolaus Copernicus (1473-1543)	

- Rise of European Mathematics .. beginning in Italy.

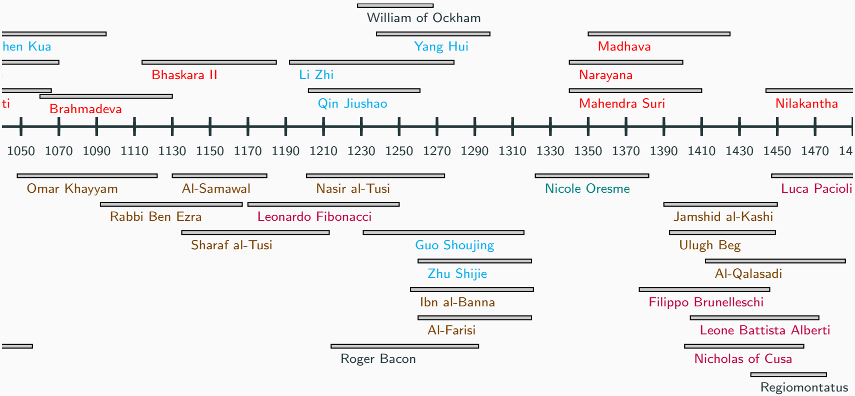
Timeline of Ancient Mathematicians i



Timeline of Ancient Mathematicians ii



Timeline of Ancient Mathematicians iii



Cubic Equations

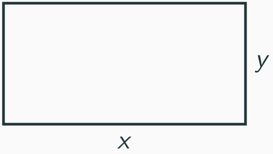
Fall 2020 - R. L. Herman

Solutions of Polynomial Equations

- Linear equations
- Chinese - Gaussian elimination
  Systems of n linear equations and n unknowns
- Quadratic equations
  Need square roots
- Cubic equations
  Need square roots and cube roots
  16th century
- Quintic equation - 1820's
  Eventually lead to group theory!

Quadratic Equations

Babylonians



Find x and y for a given perimeter and area.

$$\begin{aligned} x + y &= p \\ xy &= q. \end{aligned}$$

Eliminate y, $x^2 + q = px$. Then,

$$x, y = \frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}.$$

Method - Compute the following

1. $\frac{x+y}{2}$
2. $\left(\frac{x+y}{2}\right)^2$
3. $\left(\frac{x+y}{2}\right)^2 - xy = \frac{(x+y)^2 - 4xy}{4}$
4. $\sqrt{\frac{(x+y)^2-4xy}{4}} = \frac{x-y}{2}$
5. By inspection, get x, y, since

$$\begin{aligned} \frac{x-y}{2} &= \sqrt{\frac{p^2 - 4q}{4}} \\ &= \sqrt{\left(\frac{p}{2}\right)^2 - q}. \end{aligned}$$

Quadratic Equations (cont'd)

- Brahmagupta (628) - Explicit $ax^2 + bx = c$.

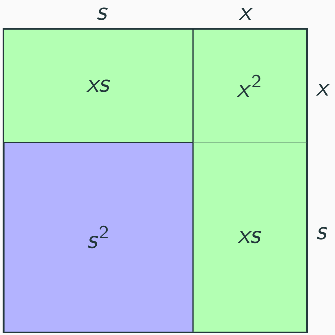
$$x = \frac{\sqrt{4ac + b^2} - b}{2a},$$

- Euclid - Prop. 28
- al'Khwarizimi

$$\begin{aligned} x^2 + 2xs &= n \\ x^2 + 2xs + s^2 &= n + s^2 \\ (x+s)^2 &= n + s^2 \end{aligned}$$

- Quadratic Irrationals

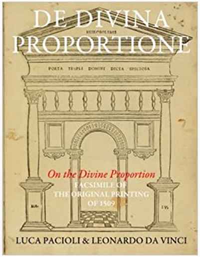
$$\frac{a + \sqrt{b}}{\sqrt{\sqrt{a} + \sqrt{b}}}$$



Note from the figure:
Green area $= x^2 + 2xs = n$.

Cubic Equations

- Babylonians - Table of cubes
- Greeks - Geometric Problems
  - Duplicating cube
  - Intersecting conics
  - Cutting Sphere with plane
- Omar Khayyam (1048-1131)
  - First general theory of cubics
  - Provided several cases
- Luca Pacioli (1445-1517)
  - Franciscan friar, tutor
  - *Summa de arithmetica,...*, 1494. Father of Accounting
  - *Divina proportione*, 1509.
  - "Solution to cubic is impossible!"



The Search for Solutions

- Scipio del Ferro (1465-1526)
  - University of Bologna, notebooks
  - Printing press - Guttenberg
  - 1506/1514, solution of **depressed cubic**: $x^3 + ax = b$.
  - Public Challenges led to secrecy.
- Gave to Antonio Maria Fior (Florido).
- Tartaglia (Nicolo Fontana) (1499-1557)
  - 1512, French attack - sabre wound led to stammer.
  - Self-educated
  - 1530 da Coi wrote to him $x^3+3x^2 = 5$, $x^3+6x^2+8x = 1000$.

Figure 1: Tartaglia

The Plot Thickens

- Tartaglia boasted he could solve $x^3 + ax^2 = c$.
- Florido challenged Tartaglia
  - Each posed 30 problems
  - Florido mostly gave problems of form $x^3 + ax^2 = c$.
  - Tartaglia won by solving depressed cubic 1535, but didn't publish.
- Girolamo Cardano (1501-1576)
  - Gambler, astronomer, physician, astrologer, heretic, father of murderer.
  - Begged for solution from Tartaglia. Finally, they met in Milan.
  - Tartaglia eventually gave solution in 1539 as a Poem if it was kept secret.
  - It was not in Cardano's book.

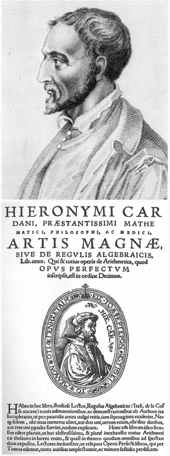
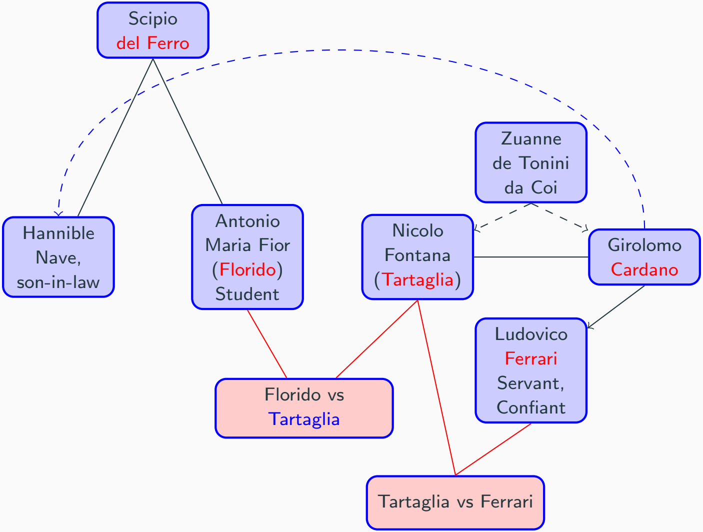


Figure 2: Cardano, *Ars Magna*.

Enter Ludovico Ferrari

- Ludovico Ferrari (1522-1565)
  - Servant at 14
  - Secretary, confidant
  - Worked on problems with Cardano
  - Cubic and biquadratic equations
- da Coi $\rightarrow$ Cardano $\rightarrow$ Ferrari
  - 4th degree polynomial
  - Ferrari solution involved solving cubic
  - Publishing was a problem.
- 1543 Trip to Florence, stopped in Bologna on the way.
  - Visitied Hannible Nave, del Ferro's son-in-law.
  - Saw del Ferro's notes.
  - Cardano believed he could publish in his *Ars Magna*, 1545.
- Barrage of letters from Tartaglia!

The Players in the Cubic Story



Tartaglia vs Ferrari - 1548

- Public debate in Milan, Ferrari's hometown.
- Cardano was absent.
- Tartaglia lost, blamed crowd.
- Tartaglia worked on arithmetic.
- Ferrari became professor in Bologna, 1565.
  Was poisoned 1565, white arsenic, possibly by sister.
- Cardano predicted exact date of his own death in 1576.

Figure 3: Tartaglia and Ferrari

Solution of Cubic $x^3 + ax^2 + bx + c = 0.$

Let $x = y - \frac{a}{3}$.
Then, $y^3 + py + q = 0$, where

$$\begin{aligned} p &= b - \frac{a^3}{3}, \\ q &= c - \frac{ab}{3} + \frac{2a^3}{27}. \end{aligned}$$

Let $y = u + v$:

$$u^3 + v^3 + (p + 3uv)(u + v) + q = 0.$$

Let $p + 3uv = 0$, then

$$\begin{aligned} u^3 v^3 &= -\frac{p^3}{27}, \\ u^3 + v^3 &= -q. \end{aligned}$$

Now, define $X = u^3$. $Y = v^3$.
We obtain

$$\begin{aligned} X + Y &= -q. \\ XY &= -\frac{p^3}{27}, \end{aligned}$$

Does this look familiar?

The solution of Cubic:

$$X, Y = u^3, v^3 = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \frac{p^3}{27}},$$

$y = u + v,$
$x = y - \frac{a}{3}.$

Example: $2x^3 - 20x^2 + 162x - 350 = 0.$

Tartaglia: Let $x = y - \frac{b}{3a} = y + 5.$

We obtain a depressed cubic (del Ferro), $y^3 + 6y - 20 = 0.$

Letting $y = s - t$, where

$$st = \frac{6}{3} = 2, \qquad s^3 - t^3 = 20,$$

and $u = t^3$, gives

$$u^2 + 20u - 2 = 0,$$

Solving, leads to $u = -10 \pm \sqrt{108}, \qquad t = \sqrt[3]{-10 \pm \sqrt{108}}$

$s^3 = 20 + t^3 = 10 \pm \sqrt{108}$

$y = s - t = \sqrt[3]{10 \pm \sqrt{108}} - \sqrt[3]{-10 \pm \sqrt{108}}$

$$x = \sqrt[3]{10 \pm \sqrt{108}} - \sqrt[3]{-10 \pm \sqrt{108}} + 5$$

Depressed Cubic $y^3 + 6y - 20 = 0$.

$y = 2$ is a solution. Therefore, $y^3 + 6y - 20 = (y-2)(y^2 + \alpha y + \beta)$.

Method 1: Expand and match coefficients.

$y^3 + 6y - 20 = y^3 + (\alpha - 2)y^2 + (\beta - 2\alpha)y - 2\beta$

$\alpha - 2 = 0, \qquad \beta - 2\alpha = 6, \qquad 2\beta = 20 \quad \Rightarrow \quad \alpha = 2, \beta = 10.$

Method 2: Long division.

$$
\begin{array}{r}
y^2 \quad + 2y + 10 \\
y - 2 \,\big)\; y^3 \qquad\qquad + 6y - 20 \\
- y^3 + 2y^2 \qquad\qquad\quad \\
\hline
2y^2 \quad + 6y \qquad \\
- 2y^2 \quad + 4y \qquad \\
\hline
10y - 20 \\
- 10y + 20 \\
\hline
0
\end{array}
$$

Complex Solutions $y^2 + 2y + 10 = 0$.

We found $y^3 + 6y - 20 = (y-2)(y^2 + 2y + 10) = 0$.

Solve quadratic: $y = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm 3\sqrt{-1}$.

- Cardano, complex numbers
  "as subtle as they are useless."
- Raphael Bombelli (1526-1572)
  First to take seriously.
- Ex: $x^3 = 15x + 4$
  $x = \sqrt[3]{2 + 11\sqrt{-11}} + \sqrt[3]{2 - 11\sqrt{-11}}$
- But, $x = 4$ is a solution!
- Complex numbers, $a + bi$, $i = \sqrt{-1}$.
- $x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}$

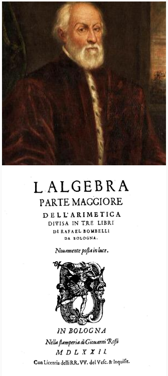


Figure 4: Bombelli

Cube Root of Complex Numbers

- Last Example: $x^3 = 15x + 4$
  $x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}$
- Seek: $\sqrt[3]{2 + 11i} = c + di$.

$$\begin{aligned} \sqrt[3]{2 + 11i} &= c + di \\ 2 + 11i &= (c + di)^3 \\ &= c^3 + 3c^2 di + 3c(di)^2 + (di)^3 \\ &= c^3 - 3cd^2 + i(3c^2 d - d^3). \end{aligned}$$

Then

$$\begin{aligned} 2 &= c^3 - 3cd^2 = c(c^2 - 3d^2), \\ 11 &= 3c^2 d - d^3 = d(3c^2 - d^2). \end{aligned}$$

- Bombelli: c, d, positive integers.

Since 2 is prime, $c = 1, 2$.

If $c = 1$, $2 = 1 - 3d^2$. No!

If $c = 2$, then $d = 1$.

$$\begin{aligned} 2 &= 8 - 6d^2 \\ 11 &= 12d - d^3 \end{aligned}$$

Then,

$$\begin{aligned} x &= \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} \\ &= (2 + i) + (2 - i) = 4. \end{aligned}$$

Françios Viète (1540-1603)

- Counselor to Henry III, IV, France
- French Wars of Religion 1562-1598
- Tutored Catherine de Pathenay (1554-1631), noblewoman, mathematician
- 1596 - Adriaan van Roomen
  "No French mathematician could solve the 45th degree polynomial."
  $x^{45} - 45x^{43} + 945x^{41} + \cdots - 3795x^3 + 45x = A.$
- Viète solved quickly:
  $2\sin(45\alpha) = A, \quad x = 2\sin\alpha.$
- Trig identity: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$
  Let $y = \cos\theta$. $4y^3 - 3y = c, |c| \le 1$. $c = \cos 3\theta$.
  Solve for θ given c. Solution, $y = \cos\theta$.
- Use identities to rewrite $2\sin(45\alpha) = A$ in terms of $2\sin\alpha$.

Figure 5: Viète

Viète's Solution

Define the quantities

$$c = 2\sin 45\theta, \qquad y = 2\sin 15\theta,$$
$$z = 2\sin 5\theta, \qquad x = 2\sin\theta. \tag{1}$$

Problem: Find x, given c.

Use the identities:

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha \tag{2}$$
$$\sin 5\alpha = 5\sin\alpha - 20\sin^3\alpha + 16\sin^5\alpha \tag{3}$$

Then,

$$c = 2\sin 45\theta = 6\sin 15\theta - 8\sin^3 15\theta = 3y - y^3. \tag{4}$$
$$y = 2\sin 15\theta = 6\sin 5\theta - 8\sin^3 5\theta = 3z - z^3. \tag{5}$$
$$z = 2\sin 5\theta = 10\sin\theta - 40\sin^3\theta + 32\sin^5\theta = 5x - 5x^3 + x^5. \tag{6}$$

Viète's Solution (cont'd)

Since $z = 5x - 5x^3 + x^5$., we write c in terms of x :

$$\begin{aligned}
y &= 3z - z^3 \\
&= 3[5x - 5x^3 + x^5] - [5x - 5x^3 + x^5]^3 \\
&= -x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x. \qquad (7) \\
c &= 3y - y^3 \\
&= 3[-x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x] \\
&\quad -[-x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x]^3 \\
&= x^{45} - 45x^{43} + 945x^{41} - 12300x^{39} + 111150x^{37} - 740259x^{35} + 3764565x^{33} \\
&\quad -14945040x^{31} + 46955700x^{29} - 117679100x^{27} + 236030652x^{25} - 378658800x^{23} \\
&\quad +483841800x^{21} - 488494125x^{19} + 384942375x^{17} - 232676280x^{15} + 105306075x^{13} \\
&\quad -34512075x^{11} + 7811375x^9 - 1138500x^7 + 95634x^5 - 3795x^3 + 45x \\
&= P_{45}(x). \qquad (8)
\end{aligned}$$

Math Symbols

- Oresme $+$ 1300
- Widman $-$ 1400
- Recorde $=$ 1500
- Outred $\times$ 1500
- Viète letters 1500
- Descartes 1500-1600 unknowns, constants a, b, c variables x, y, z
- Harriot $<>$ 1600
- Wallis ∞ 1700

- imaginary, Descartes
- $x^{3/2}, x^{-1}$, Newton 1600
- $x^2 \to xx$, Gauss 1800
- π, i, Σ. Euler 1700
- $f(x)$
- $\frac{df}{dx}, \int$ Leibniz 1600

Analytic Geometry

- Fermat (1601-1665)
- Descartes (1596-1650)
- Newton (1642-1727)

Coordinates

- Hipparchus - sky
- Apollonius - conics
- Oresme (1300s) - position, velocity plots
- Fermat-Descartes described curves in coordinate systems
- Degree 1, Linear relations

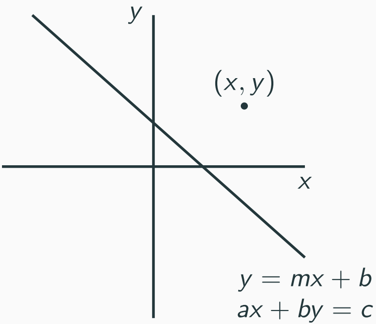


Figure 6: Cartesian system.

Curves of Degree 2 - Quadratics

$$ax^2 + \underbrace{bxy}_{rotation} + cy^2 + \underbrace{dx + ey}_{translation} + f = 0$$

- Describes Conics
- $b \neq 0$, rotation
- $d \neq 0$ or $e \neq 0$, translation
- Classification

$$D = \begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{vmatrix}$$

$D > 0$ ellipse
$D < 0$ hyperbola
$D = 0$ parabola

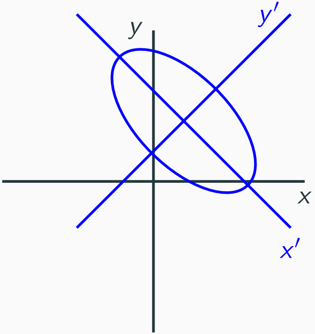


Figure 7: Rotated, translated ellipse.

Curves of Degree 3: $ax^3 + bx^2y + cxy^2 + dy^3 + \cdots = 0$, Cubics

- Newton classified cubic curves, 1710, 72 types (missed 6)
- $y = x^3$ and other types.
- Descarte's folium (leaf) $x^3 + y^3 = 3axy$
- Parametric Solutions

$$x = \frac{3at}{1+t^3}, \quad y = \frac{3at^2}{1+t^3}.$$

- Rational Points
  Ex. $x^3 + y^3 = 1$.
  Let $x = \frac{n}{p}$, $y = \frac{m}{p}$. $n^3 + m^3 = p^3$.
- Fermat - only trivial $(0,1), (1,0)$.
- Fermat's Last Theorem, 1637

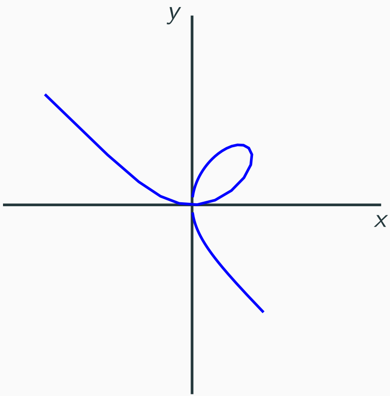


Figure 8: Descarte's Folium.

Fermat's and Bezout's Theorems

- Fermat's Last Theorem, 1637
  $x^n + y^n = z^n$
  Wiles proved in 1995.
- Bezout's Theorem. Let

$$p(x, y) = 0, \qquad \text{degree } n.$$
$$q(x, y) = 0, \qquad \text{degree } m.$$

  Then, p and q intersect in nm points.
- Elimination gives eq of degree nm.
- Need complex numbers, point at infinity.
  Next - Projective Geometry.

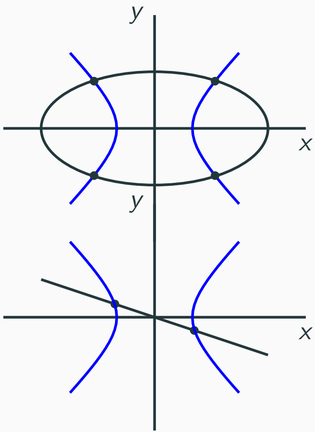


Figure 9: Intersecting Degree 2 curve (blue) with Degree 2 or 1 (black).

Projective Geometry

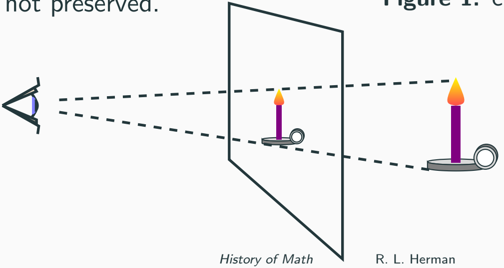
Fall 2020 - R. L. Herman

Perspective Drawing

- Art - Perspective Drawing
- Before Renaissance- no illusion of depth and space.
- 13th century Italian masters used shadowing.
- Mathematics of perspective

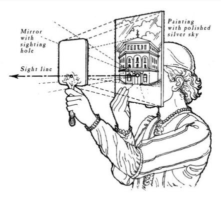
  Lengths not preserved.
  Angles not preserved.

Figure 1: c.1308-1311



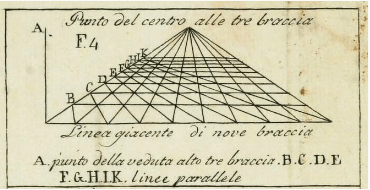
Filippo Brunelleschi (1377-1446)

- Architect
- First to describe linear perspective.
- Experimented (1415-1420) using a panel with a grid of squares and a plaque with a hole at eye level.
- Drawings of the Baptistry in Florence, Place San Giovanni and other Florence landmarks.
- Later his method was studied by Alberti and Da Vinci.



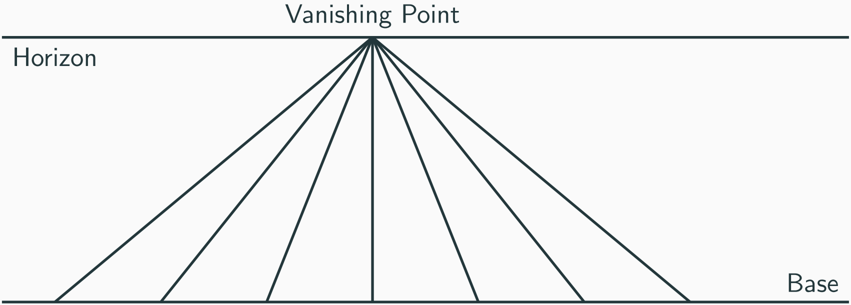
Leon Battista Alberti (1404-1472)

- Alberti's Veil
  Transparent cloth on a frame,
  Good for actual scenes not
  imaginary ones.
- Basic principles:
  1. A straight line in perspective
  remains straight.
  2. Parallel lines either remain
  parallel or converge to a point.
- Drawing a square-tiled floor, solved
  by Alberti (1436).



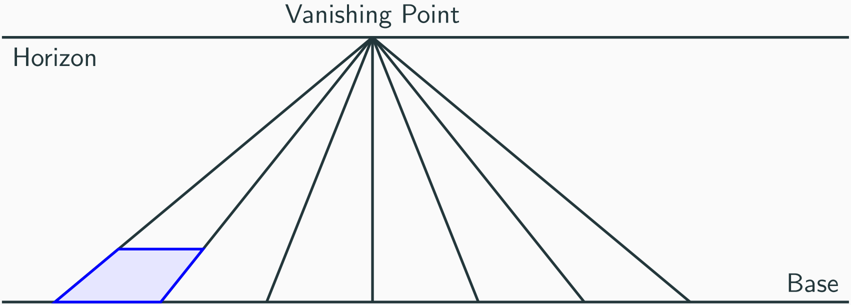
Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.



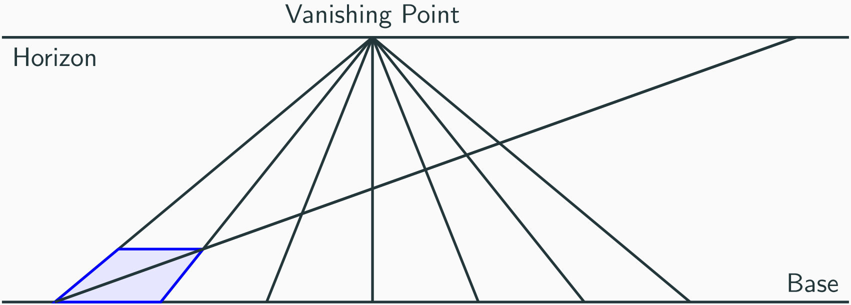
Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.
- Choose one tile.



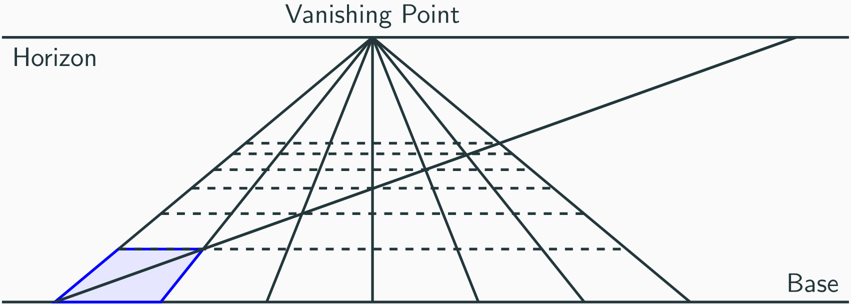
Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.
- Choose one tile.
- Extend diagonal.



Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.
- Choose one tile.
- Extend diagonal.
- Intersections determine the horizontals.



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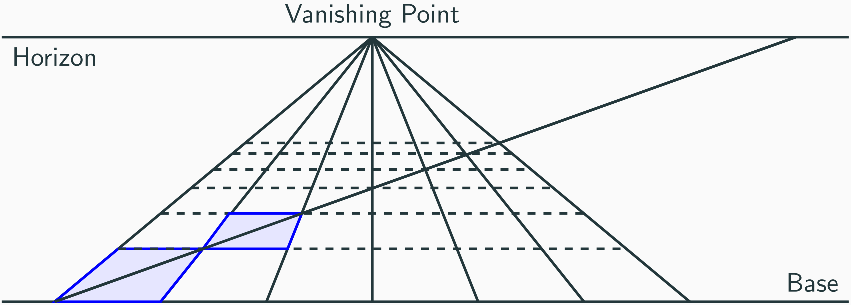

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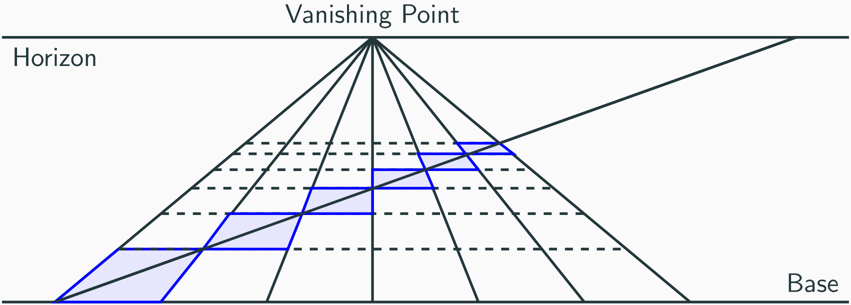

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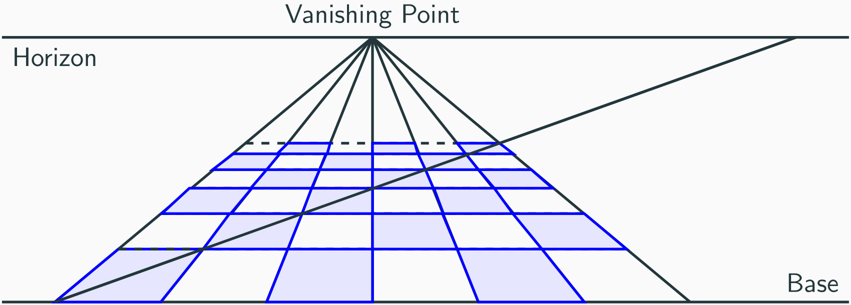


Desargues' Projective Geometry[1]

- Mathematics behind Alberti's Veil: Family of lines (light rays) through a point (eye) plus a plane (veil).
- Recall **Pappus' Theorem**: A_1, A_2, A_3, collinear; B_1, B_2, B_3, collinear; then, so are C_1, C_2, C_3.
- Blaise Pascal (1623-1662) at 16 generalized to conics.
- Desargues (1640) **Projective Geometry** only relies on a straight edge.

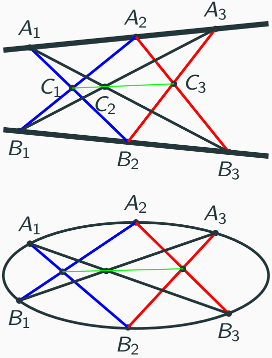
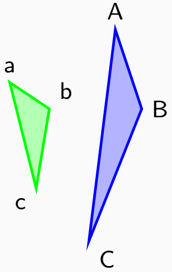


Figure 2: Pappus' and Pascal's Theorems.

[1] Two centuries ahead of his time.

Girard Desargues (1591-1661)

- Architect in Paris, Lyon and engineer.
- Desargues' Theorem in appendix of book on perspective, by friend Abraham Bosse (1602–1676).
  Two triangles are a) in ***perspective axially*** *if and only if they are b) in* ***perspective centrally****.*



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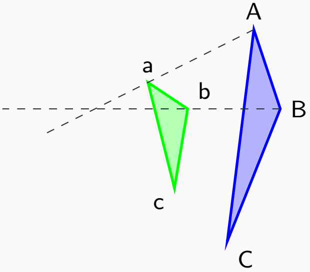
- b) Extend Aa, Bb, Cc **center of perspectivity** P.



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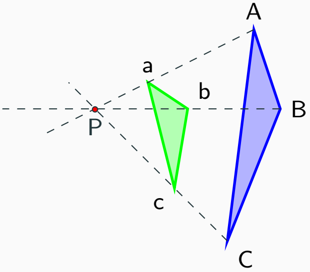

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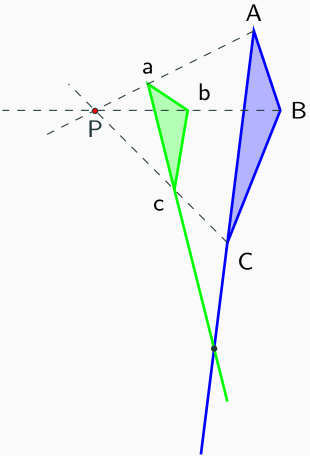

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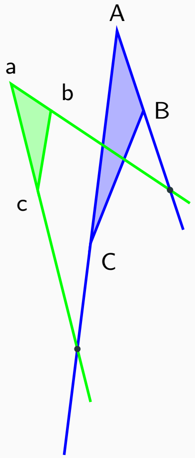

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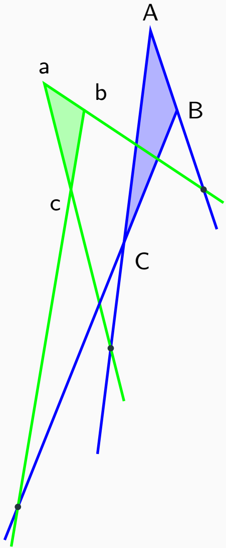

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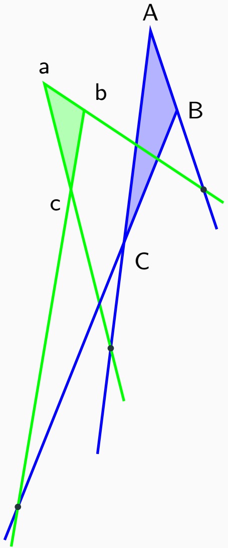

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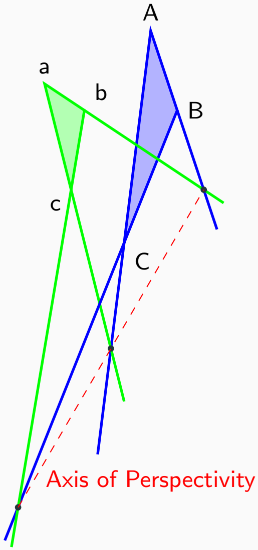

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- Points are collinear.
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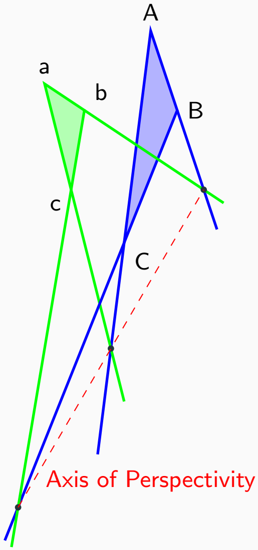

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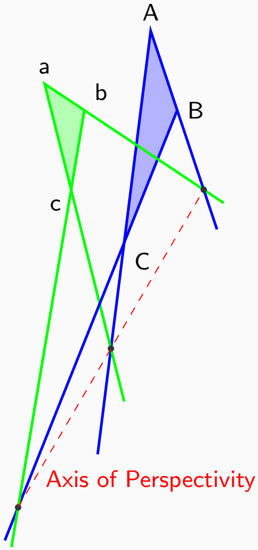
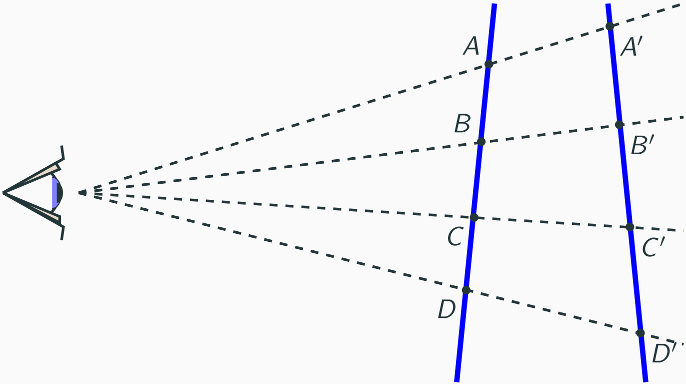
- a) Extend pairs, AC-ac, AB-ab, etc.
- Points are collinear.
- What if two sides are parallel?
- Need **Projective plane**.



Invariance of the Cross Ratio



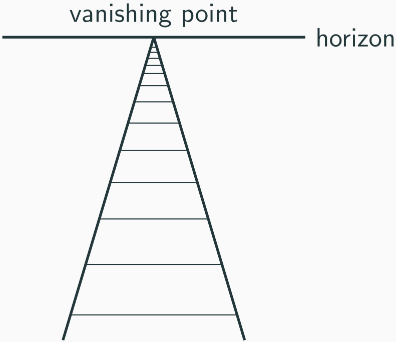
$$\frac{\overline{AC}}{\overline{BC}} : \frac{\overline{AD}}{\overline{BD}} = \frac{\overline{A'C'}}{\overline{B'C'}} : \frac{\overline{A'D'}}{\overline{B'D'}}.$$

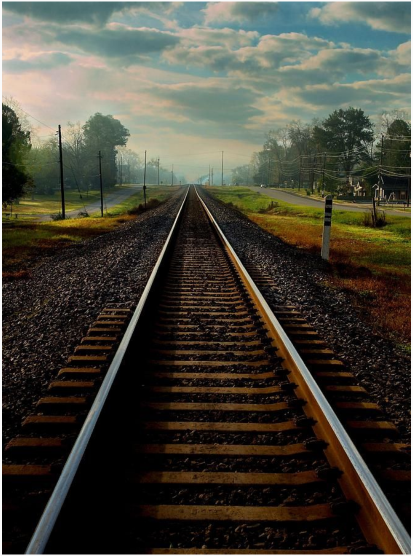
Projective Geometry Rebirth in 1800's.

Perspective

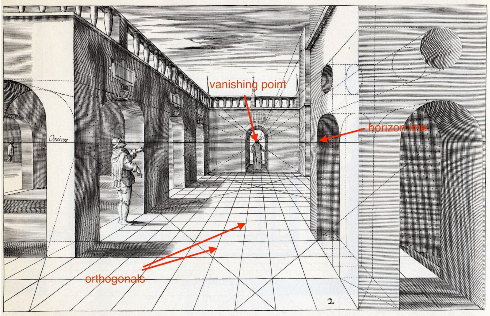
1. Parallel lines meet at a pt.
2. Lines map to lines.
3. Conics map to conics.

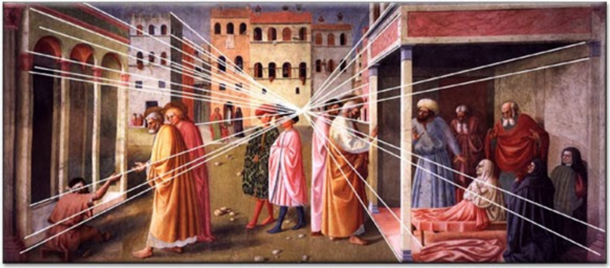
Example: Train tracks.





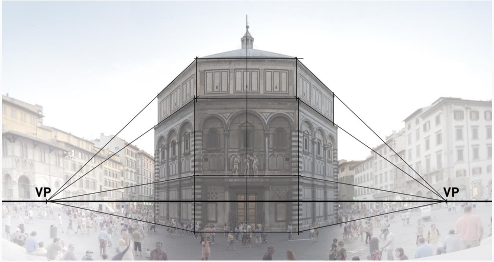
One Point Perspective





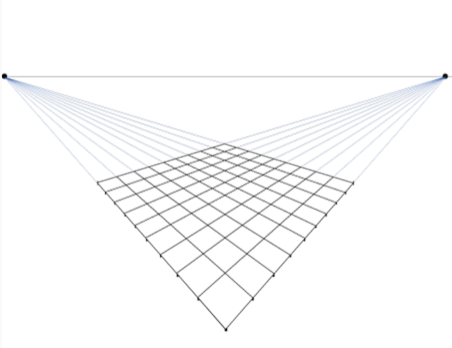
Two Point Perspective

Two Point Perspective



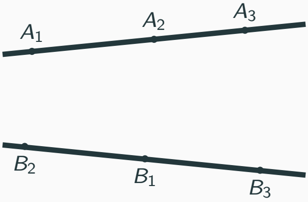
Two Point Perspective Vanishing Points

Two Point Perspective Vanishing Points



Points at Infinity

- Artists' use vanishing points.
- Pappus' Theorem -
  Consider parallel lines A_1B_2, A_2B_1.
  Does the theorem hold?



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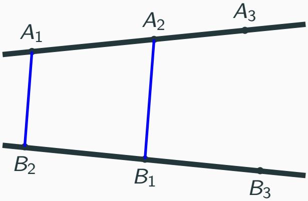

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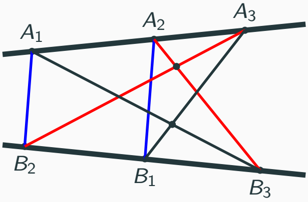

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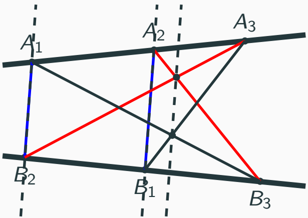

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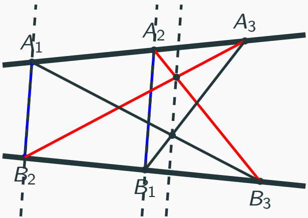
- Desargues - **line at infinity**.



Points at Infinity

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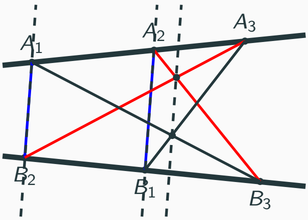
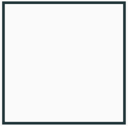
- Pappus' Theorem -
  Consider parallel lines A_1B_2, A_2B_1.
  Does the theorem hold?
- Desargues - **line at infinity**.
- Look at a plane





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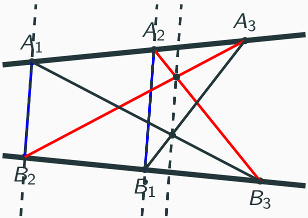
- Look at a plane
- Add parallel lines.
  Where do they go?



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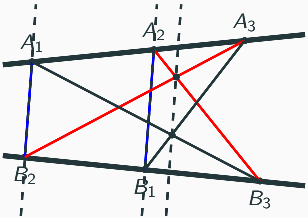
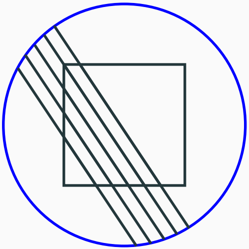
- Add parallel lines.
  Where do they go?
- Line at Infinity
- Plane + line at infinity =
  Projective Plane

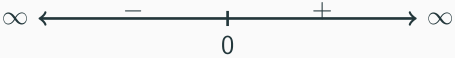




Line at infinity

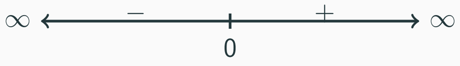
Projective Line

- Consider the real line, $\mathbb{R}$.



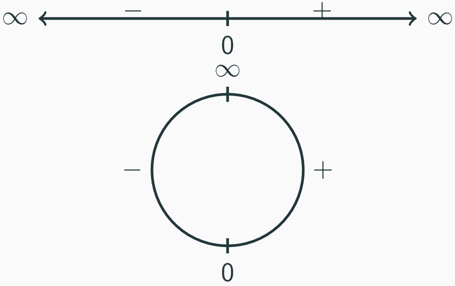
Projective Line

- Consider the real line, $\mathbb{R}$.
- Add point at infinity, projective line.



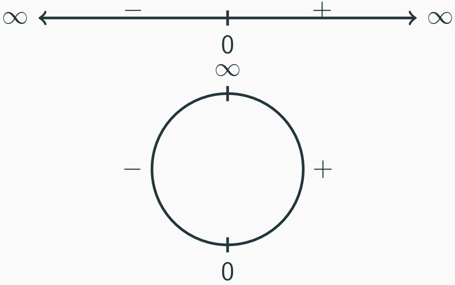
Projective Line

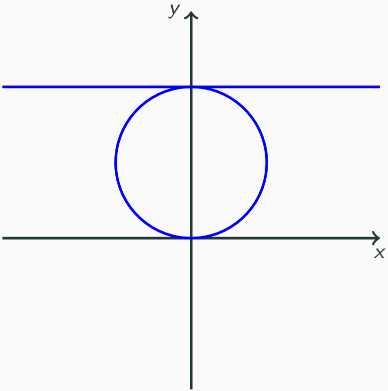
- Consider the real line, $\mathbb{R}$.
- Add point at infinity, projective line.
- Topologically a circle!



Projective Line

- Consider the real line, $\mathbb{R}$.
- Add point at infinity, projective line.
- Topologically a circle!
- We can map the circle to $\mathbb{R}$.





Projective Line

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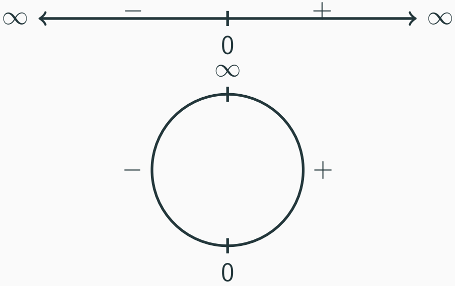

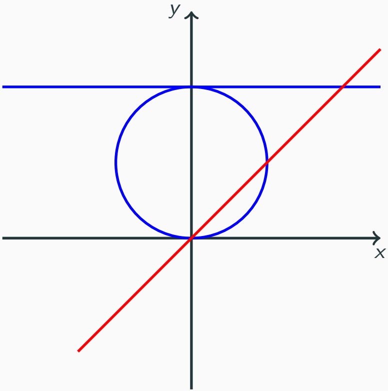

Intersection: $y = b$, $y = mx$: $x = \frac{b}{m}$.

Projective Line

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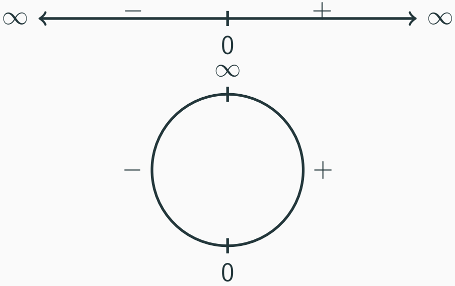
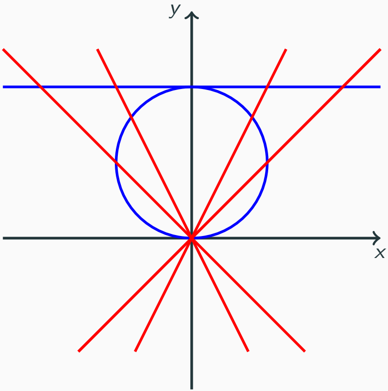


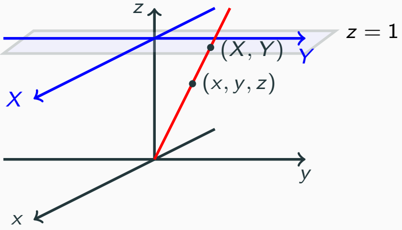


Intersection: $y = b$, $y = mx$: $x = \frac{b}{m}$.

Homogeneous Coordinates

- Point on line: (x, y, z)
- All points on line map to (X, Y) in the plane.
- (X, Y) are called homogeneous coordinates.
- Points on line are multiples, $(x', y', z') = \lambda(x, y, z)$.
- Point on plane: $(\frac{x}{z}, \frac{y}{z}, 1)$, or

$$X = \frac{x}{z}, \quad Y = \frac{y}{z}.$$



Curves

- Curve in plane, $Y = X^2$.
- Translates to

$$\frac{y}{x} = \left(\frac{z}{x}\right)^2.$$

- This is a surface in (x, y, z)-space,

$$x^2 = yz.$$

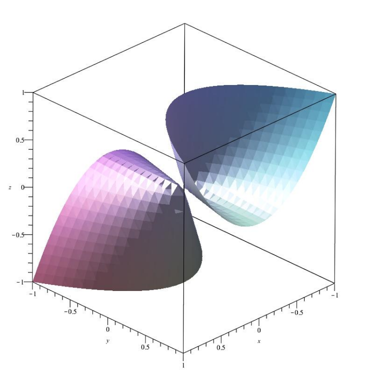


Figure 3: Surface $x^2 - yz$.

Curves

- Curve in plane, $Y = X^2$.
- Translates to

$$\frac{y}{x} = \left(\frac{z}{x}\right)^2.$$

- This is a surface in (x, y, z)-space,

$$x^2 = yz.$$

- Slicing with a plane, like Alberti's veil, one gets a project of the curve.

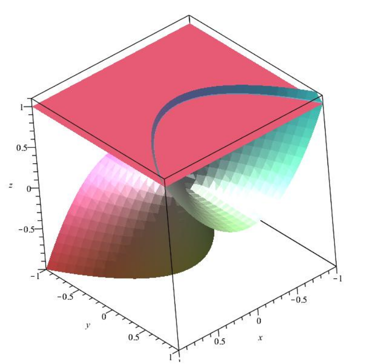


Figure 3: Surface $x^2 - yz$.

Curves

- Curve in plane, $Y = X^2$.
- Translates to

$$\frac{y}{x} = \left(\frac{z}{x}\right)^2.$$

- This is a surface in (x, y, z)-space,

$$x^2 = yz.$$

- Slicing with a plane, like Alberti's veil, one gets a project of the curve.

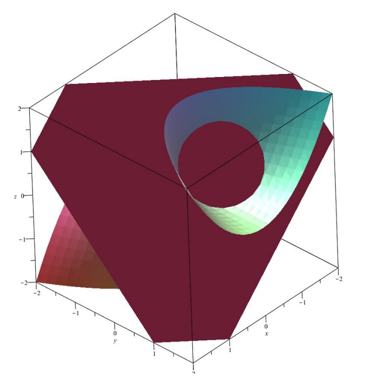


Figure 3: Surface $x^2 - yz$.

Projective Sphere

- Map points on a plane to points on surface of unit sphere, $\mathbb{S}^2$.
- Lines through South Pole uniquely intersect the plane and sphere.
- All points mapped except $(0, 0, 0)$. This point can be mapped to the line at infinity.
-

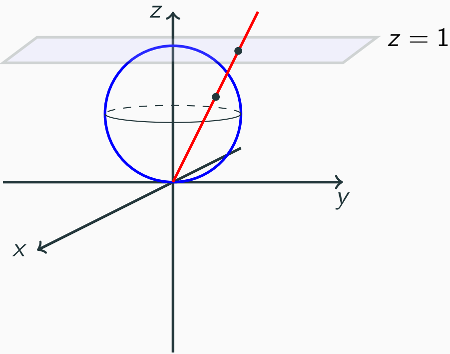
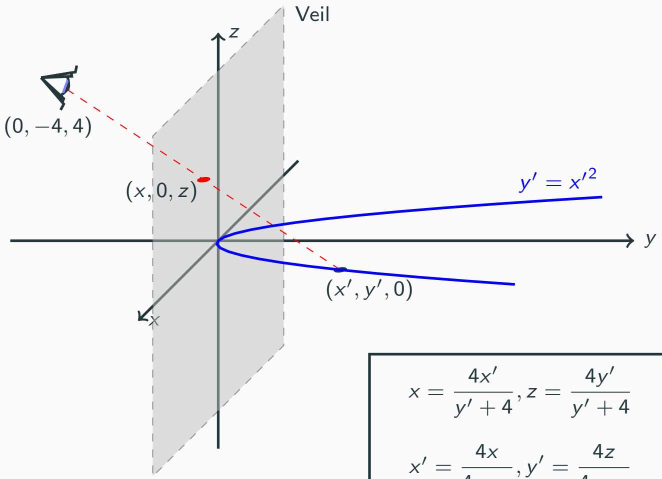


Figure 4: Stereographic Projection

Looking into the Veil

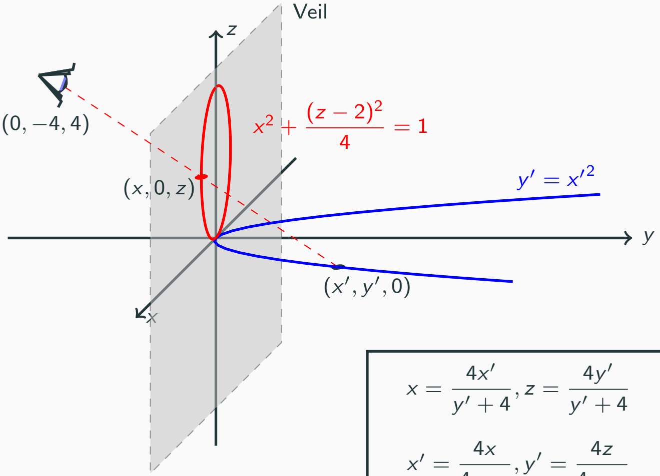


$$x = \frac{4x'}{y'+4}, z = \frac{4y'}{y'+4}$$

$$x' = \frac{4x}{4-z}, y' = \frac{4z}{4-z}$$

Figure 5: Problems 8.4.2-8.4.4

Looking into the Veil



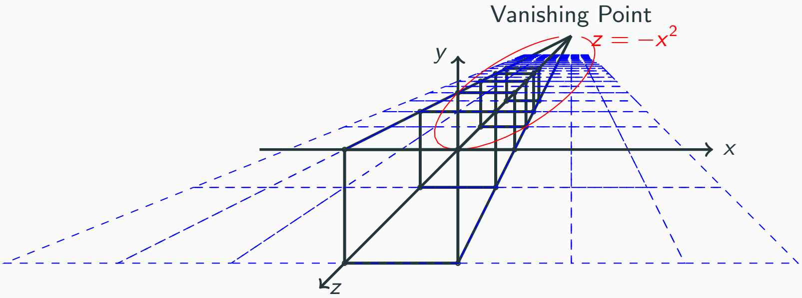
$$x = \frac{4x'}{y'+4}, z = \frac{4y'}{y'+4}$$

$$x' = \frac{4x}{4-z}, y' = \frac{4z}{4-z}$$

Figure 5: Problems 8.4.2-8.4.4

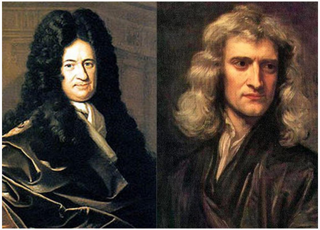
Perspective Drawing

Looking at conics from a different perspective: The parabola $z = -x^2$ looks like an ellipse.



Emergence of Calculus

Fall 2020 - R. L. Herman



Developments in the 1600's

Rapid developments first 60 years of 1600's based on Greek geometry, algebra, astronomy (Kepler, Galileo). Led to unification of geometry and algebra.

- Descartes (1596-1650)
- Cavalieri (1598-1647)
- Fermat (1601-1665)
- Wallis (1616-1703)
- Barrow (1630-1677)
- Gregory (1638-1675)
- Newton (1642-1727)
- Leibniz (1646-1716)

Two main problems

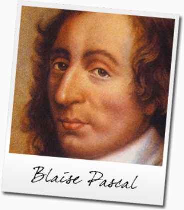
- Tangents
- Areas

Need curves

- Conics
- Archimedean spiral
- Conchoid
- Cissoid
- Cycloid

Seventeenth Century - French, German, English Mathematics

- Galileo Galilei (1564-1642)
- Johannes Kepler (1571-1630)
- 1590 Vièta, *The Analytic Art*
- John Napier (1550-1617) and Henry Briggs (1561-1631) - Introduced the logarithm
- French Mathematicians:
  René Descartes (1596-1650)
  Blaise Pascal (1623-1662)
  Pierre de Fermat (1601-1665)



- Descartes

  philosopher, mathematician
  Discours de la méthode,
  Marriage of algebra/geometry - analytic geometry
- Pascal

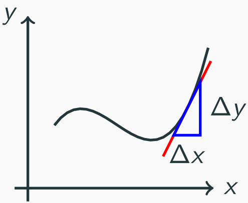
  Wrote math before 16
  Probability theory
  Theology
- Fermat

  Created analytic geometry
  Contributions to Calculus
  Number theory
  Scribbled in Diophantus' *Arithmetica*

Tangents

- Pierre de Fermat, René Descartes
- Studied Apollonius 4-line problem.
- Tangent line approximates the curve at a point.
- Slope $\frac{\Delta y}{\Delta x}$.
- Infinitesimals - increments.
- Fermat
  Method for maxima-minima
  1636 - Method of Tangents
- 1636 Letter from Descartes to Mersenne
  $dy = f(x + dx) - f(x) = ?dx.$

Figure 1: Fermat and Descartes



Areas Under Curves

- First studied by Eudoxus, Archimedes
- Bonaventura Cavalieri (1598-1647) - indivisibles

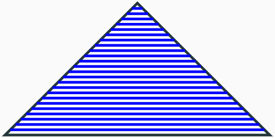
  Fill area with lines.
  But, an infinite number of lines sum to infinity.

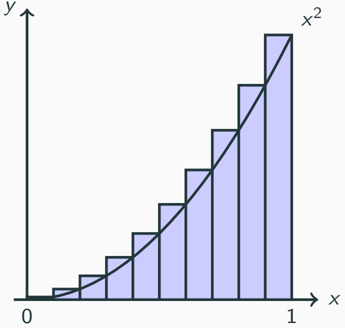
- Archimedes, John Wallis (1616-1703):

$$\int_0^1 x^2\, dx.$$

N segments of width $\frac{1}{N}$. and height $\left(\frac{k}{N}\right)^2$, $k = 1, 2, \ldots, N$.

$$A \approx \sum_{k=1}^{N} \frac{1}{N}\left(\frac{k}{N}\right)^2.$$





Areas Under Curves (cont'd)

Find the sum

$$
\begin{aligned}
A &\approx \sum_{k=1}^{N} \frac{1}{N}\left(\frac{k}{N}\right)^2 \\
&= \frac{1}{N^3}\sum_{k=1}^{N} k^2 \\
&= \frac{1}{N^3}\frac{N(N+1)(2N+1)}{6} \\
&\sim \frac{2N^3}{6N^3} = \frac{1}{3}.
\end{aligned}
$$

Wallis showed

$$\int_0^a x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_0^a = \frac{a^{n+1}}{n+1}$$

for $k = 1, 2, \ldots, 9.$

Note:

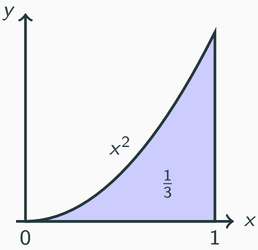
$$
\begin{aligned}
\sum_{k=1}^{N} k &= \underbrace{1 + \underbrace{2 + \cdots + (N-1)}_{2+(N-1)} + N}_{1+N} \\
&= N\frac{N+1}{2}.
\end{aligned}
$$

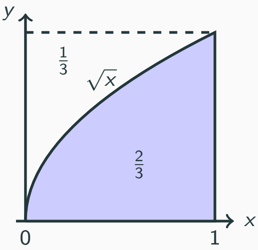
Figure 2: John Wallis

Integrating Powers, $\int x^k\,dx$,

- al Haytham (965-1039) $k = 1, 2, 3, 4$.
- Cavalieri (1635) knew for k up to 9.
- Proven in general by Fermat, Descartes, Roberval, 1630's.
- Fractional Powers (Fermat)
  Ex: $\int_0^1 \sqrt{x}\,dx$
  Use the symmetry in the figures.
- Areas under x^k, need sums $1^k + 2^k + \cdots n^k$.
- Volumes - use cylinders, $V = \pi r^2 h$.
  Sums needed: $1^{2k} + 2^{2k} + \cdots n^{2k}$.

Note: $1^3 + 2^3 + \cdots + k^3 = (1 + 2 + \cdots + k)^3$.





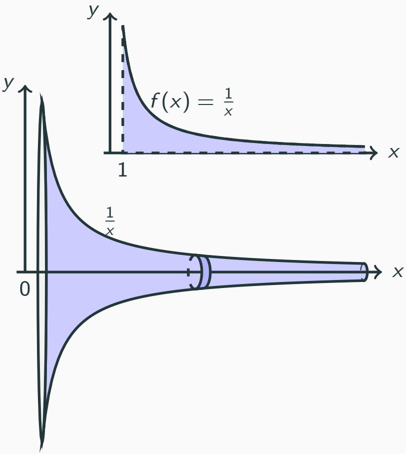
Evangelista Torricelli (1608-1647), barometer inventor

- Inverse Powers, x^{-1}
- Area under $y = \frac{1}{x}$.

$$\int_1^\infty \frac{1}{x}\, dx = \infty.$$

- Torricelli's trumpet (Gabriel's horn)

$$\begin{aligned} V &= \pi \int_1^\infty \frac{1}{x^2}\, dx = \pi. \\ A &= 2\pi \int_1^\infty \frac{dx}{x}\sqrt{1+\left(-\frac{1}{x^2}\right)^2} \\ &> 2\pi \int_1^\infty \frac{1}{x}\, dx = \infty. \end{aligned}$$



What? You cannot paint the surface but can fill the trumpet with paint.

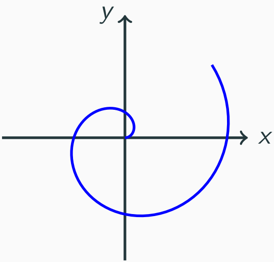
Hobbes - "to understand this for sense, it is not required that a man should be a geometrician or logician, but that he should be mad."

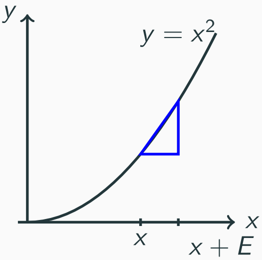
Tangents, Maxima, Minima

- Curves studied like Archimede's spiral, $r = a\theta$
- Fermat - studied polynomials
- Work simpler than Descartes
- Used infinitesimals, E
- **Example:** $y = x^2$

$$\frac{(x+E)^2 = x^2}{E} = 2x + E.$$

- Generalized to polynomials, $p(x, y) = 0$.





Infinitesimals

- Wallis, *Arithmatica Infinitorium*
- Some results already known.
- New approach to fractional powers.
- Ambivalent use of infinitesimals attacked by Thomas Hobbes (1588-1679).
- Formulae for π known by
  - Gregory, Newton, Leibniz
- Madhava (1350-1425) found π to 13 decimal places using series,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

Wallis' Formulae:

$$\pi = \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \cdots.$$

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cdots}}}$$

Already known formula:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Isaac Newton (1642-1727)

- Major use of infinite series
- *A Treatise of the Methods of Series and Fluxions*
- *Quadrature of the Hyperbola*
  Written in 1665,
  1st publication in 1668 by Mercator
- Akin to decimal expansions - powers of $\frac{1}{10}$
- Example:

$$\log(1+x) = \int_0^x \frac{dt}{1+t}$$

Note: Geometric series

$$\begin{aligned} 1 + t + t^2 + \cdots &= \frac{1}{1-t}, |t| < 1. \\ 1 - t + t^2 - \cdots &= \frac{1}{1+t}, |t| < 1. \end{aligned}$$

Then,

$$\begin{aligned} y &= \log(1+x) \\ &= \int_0^x (1 - t + t^2 - \cdots)\, dt \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots . \end{aligned}$$

Invert Power Series

We have for $y = \log(1+x)$,

$$y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots .$$

In order to invert the series, let $x = a_0 + a_1 y + a_2 y^2 + \cdots$. Then,

$$\begin{aligned} y &= \left(a_0 + a_1 y + a_2 y^2 + \cdots\right) - \frac{1}{2}\left(a_0 + a_1 y + a_2 y^2 + \cdots\right)^2 + \cdots \\ &= a_0 - \frac{1}{2}a_0^3 + \frac{1}{3}a_0^3 + a_1\left(a_0^2 - a_0 + 1\right) y \\ &\quad + \left[a_2\left(a_0^2 - a_0 + 1\right) + \left(a_0 - \frac{1}{2}\right)\right] y^2 \\ &\quad + \left[\frac{a_1^3}{3} + a_1 a_2 (2a_0 - 1) + a_3\left(a_0^2 - a_0 + 1\right)\right] y^3 + \ldots . \end{aligned}$$

Equate coefficients of powers of y, then ...

Series Inversion (cont'd)

We solve the resulting system of equations:

$$\begin{aligned} 0 &= a_0 - \frac{1}{2}a_0^3 + \frac{1}{3}a_0^3 \\ 1 &= a_1\left(a_0^2 - a_0 + 1\right) \\ 0 &= a_2\left(a_0^2 - a_0 + 1\right) + \left(a_0 - \frac{1}{2}\right) \\ 0 &= \frac{a_1^3}{3} + a_1 a_2(2a_0 - 1) + a_3\left(a_0^2 - a_0 + 1\right). \end{aligned}$$

The first equation gives $a_0 = 0$. The next two give $a_1 = 1$ and $a_2 = \frac{1}{2}$.

Continuing Newton found that

$$a_0 = 0, a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{6}, a_4 = \frac{1}{24}, \ldots, a_n = \frac{1}{n!}.$$

Newton's Series for Exponential

So far, inversion of

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots .$$

led to

$$x = y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots .$$

Recalling series expansion for e^x,

$$y = \log(1+x) \Rightarrow x = e^y - 1.$$

So,

$$e^y = 1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots .$$

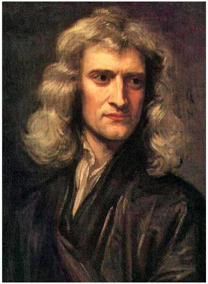


Figure 3: Newton

Newton's Series for Sine

Newton knew $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$.

Recall binomial series: $(a+b)^n = \sum_{n=0}^n C_{n,k} a^{n-k} b^k$, where the coefficients are $C_{n,k} = \frac{n!}{k!(n-k)!}$. Then,

$$\begin{aligned}
(1+a)^p &= 1 + pa + \frac{p(p-1)}{2!} a + \frac{p(p-1)(p-2)}{3!} a^2 + \cdots . \\
\sin^{-1} x &= \int_0^x \frac{dt}{\sqrt{1-t^2}}, \quad a = -t^2, p = -\frac{1}{2}, \\
&= x - \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5}
\end{aligned}$$

Inverting, Newton found

$$\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots .$$

Gottfried Wilhelm Leibniz (1646-1716)

- Librarian, philosopher, diplomat, doctorate in law.
- First papers in calculus (1684).
- Led to long dispute.
- Better notation, $\frac{dy}{dx}, \int\, dx$.
- Sum, product, quotient rules.
- Proved Fundamental Theorem of Calculus, $\frac{d}{dx}\int f(x)\,dx = f(x)$.

Figure 4: Leibniz

Infinite Series

- Geometric series,
  Known to Euclid (Zeno's paradox)
  Leonhard Euler (1707-1783)

  $$a+ar+ar^2+\cdots+ar^n+\cdots = \frac{a}{1-r}, |r| < 1.$$

- Harmonic Series - Oresme (1350)

$$\begin{aligned} & 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+\frac{1}{4}+\cdots \\ = & (1)+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots \\ \geq & \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots=\infty. \end{aligned}$$

- Power series - 17th Century,
  Gregory, Wallis, Taylor, Mclaurin, ...

Figure 5: Euler

Basel Problem (1644)

- Posed by Pietro Mengoli (1626-1686).

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

- Jacob and Johann Bernoulli (1704) tackled. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = ?$

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n(n+1)} &= \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \left(\frac{1}{N} - \frac{1}{N+1}\right) \\
&= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots \left(\cancel{\frac{1}{N}} - \frac{1}{N+1}\right) \\
&= 1 - \frac{1}{N+1} \underset{N \to \infty}{\Longrightarrow} 1.
\end{aligned}$$

Figure 6: Jacob and Johann

Euler's Solution of Basel Problem - 1734

- Descartes' Factor Theorem
- $p(x)$ - polynomial
- $p(r) = 0$ implies $p(x) = (x - r)q(x)$, $q(x)$ - polynomial

Proof:

$$\begin{aligned} p(x) &= a_0 + a_1 x + \cdots + a_n x^n \\ p(y) &= a_0 + a_1 y + \cdots + a_n y^n \\ p(x) - p(x) &= a_1(x - y) + \cdots + a_n(x^n - y^n) \end{aligned}$$

$$x^n - y^n = (x - y)\left(x^{n-1} + x^{n-2}y + \cdots + y^{n-1}\right)$$

Let $y = r$,

$$\begin{aligned} p(x) &= (x - r)[a_1 + a_2(x + r) + \cdots + a_n(x^{n-1} + x^{n-2}r + \cdots + r^{n-1})] \\ &= (x - r)q(x). \end{aligned}$$

Leonhard Euler's Solution of Basel Problem

$\sin x$ has roots $n\pi,\ n = 0, \pm 1, \pm 2, \ldots$. - Generalize Factor Theorem:

$$\begin{aligned}
\sin x &= Ax\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\cdots \\
&= Ax\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\cdots \\
&= A\left[x-\frac{x^3}{\pi^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)+x^5(\cdots)-\cdots\right].
\end{aligned}$$

Compare to

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots .$$

Then $A = 1$, and

$$\frac{1}{3!} = \frac{1}{\pi^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right) \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Other Results

$$
\begin{aligned}
1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots &= \frac{\pi^4}{90}, \\
1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots &= (-1)^{n-1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n},
\end{aligned}
$$

where B_{2n} are Bernoulli numbers, $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, \ldots,$

$$
\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.
$$

Euler (1748) - Zeta function

$$
\begin{aligned}
\zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \\
&= \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \cdots \left(1 - \frac{1}{p^s}\right)^{-1} \cdots,
\end{aligned}
$$

where p is prime.

Bernhard Riemann's Contribution

- Bernhard Riemann (1826-1866)
- Riemann zeta function

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots .$$

- Values

  $\zeta(1) = \infty$, harmonic series
  $\zeta(2) = \frac{\pi^2}{6}$
  $\zeta(3)$ irrational, proved by Apery (1981)

- Zeros

  $\zeta(s) = 0$, s integer < 0.
  Riemann Hypothesis:
  $\zeta(\sigma + it) = 0$ when $\sigma = \frac{1}{2}$.

- Connection to primes?

Figure 7: Georg Friedrich Bernhard Riemann

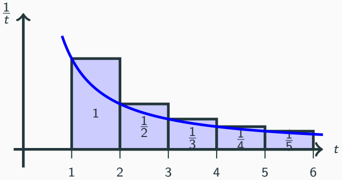
Connection to Primes

$$\zeta(s) = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \cdots \left(1 - \frac{1}{p^s}\right)^{-1} \cdots$$

$$\left(1 - \frac{1}{2^s}\right)^{-1} = 1 + \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 + \cdots$$

- Primes less than $x \sim \int_2^x \frac{dt}{\log t}$
- Euler-Mascheroni Constant
  $\int_1^x \frac{dt}{t} = \ln x$, $\gamma \approx 0.577218\ldots.$

$$\lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right) = \gamma.$$



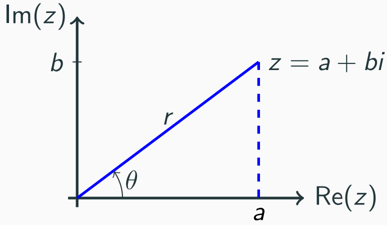
- Generalizing $n!$

$$\Gamma(n+1) = n\Gamma(n), \Gamma(0) = 1.$$

Euler's Formula

- Complex numbers, polar form.
  $z = a + bi,\ a = r\cos\theta, b = r\sin\theta$

$$\begin{aligned} z &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta). \end{aligned}$$



- Exponential of imaginary number

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\ &= 1 + i\theta - \frac{(\theta)^2}{2!} - i\frac{(\theta)^3}{3!} + \cdots \\ &= \left(1 - \frac{(\theta)^2}{2!} + \cdots\right) + i\left(\theta - \frac{(\theta)^3}{3!} + \cdots\right) \\ e^{i\theta} &= \cos\theta + i\sin\theta. \end{aligned}$$

Euler's Formula Applications $e^{i\theta} = \cos\theta + i\sin\theta$.

- $\theta = \pi$, $e^{i\pi} = -1$, or $e^{i\pi} + 1 = 0$.
- $\left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n$ $e^{in\theta} = \cos n\theta + i\sin n\theta$ implies **de Moivre's Theorem**

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n .$$

- **Example:** $n = 2$

$$\begin{aligned}\cos 2\theta + i\sin 2\theta &= (\cos\theta + i\sin\theta)^2\\ &= \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta.\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ \sin 2\theta &= 2\sin\theta\cos\theta.\end{aligned}$$

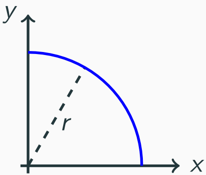
Rectification of an Ellipse

- Rectification = Finding arclengths
- $y = y(x)$

$$L = \int_a^b \sqrt{1 + y'^2}\, dx.$$

- **Example:** Circle

$$\begin{aligned} L &= 4\int_0^r \sqrt{1 + \frac{x^2}{y^2}^2}\, dx \\ &= 4\int_0^r \sqrt{1 + \frac{x^2}{r^2 - x^2}^2}\, dx \\ &= 4r\int_0^r \frac{dx}{\sqrt{r^2 - x^2}} = 4r\sin^{-1} 1 = 4r\left(\frac{\pi}{2}\right) = 2\pi r. \end{aligned}$$

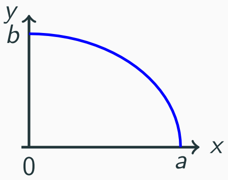


Other Arclengths

- **Example:** Ellipse

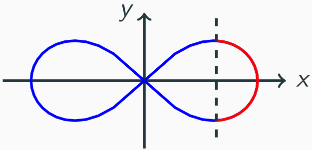
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad x \geq 0, y \geq 0.$

$$\begin{aligned} y &= \frac{b}{a}\sqrt{a^2 - x^2} \\ y' &= \frac{bx}{a\sqrt{a^2 - x^2}} \\ 1 + y'^2 &= \frac{a^2 - k^2x^2}{a^2 - x^2}, \quad k = \frac{a^2 - b^2}{a^2} \\ L &= 4\int_0^a \sqrt{\frac{a^2 - k^2x^2}{a^2 - x^2}}\, dx. \end{aligned}$$



- **Example:** Lemniscate, $r^2 = \cos 2\theta$

$$L = 4\int_0^1 \frac{dr}{\sqrt{1 - r^4}}$$



Elastica
Sep 1694, Jacob Bernoulli
Oct 1694, Johann Bernoulli

Elliptic Functions

- Lemniscate integral leads to new functions, $u = \int_0^x \frac{dt}{\sqrt{1-t^4}}$.
- Compare to $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$.
- Elliptic Integrals: $\int R(t, \sqrt{p(t)})\, dt$, R is rational function, $p(t)$ is polynomial of degree 3 or 4.
- Bernoulli (1694) - geometry, mechanics.
- Fagnano (1682-1766) - Doubling arc of lemniscate.
- Carl Friedrich Gauss (1777-1855) ~1800 studied inverse $x = sl(u)$ Doubly periodic

$$\begin{aligned} sl(u+2\bar{u}) &= sl(u) \\ sl(u+2i\bar{u}) &= sl(u) \end{aligned}$$

- Rediscovered by Niel Henrik Abel (1802-1829) and Carl Gustav Jacobi (1804-1851) in 1820's.

Elliptic Integral Addition Theorems

- **Example** Circle

$$\begin{aligned} \sin 2u &= 2\sin u\cos u \\ &= 2\sin u\sqrt{1-\sin^2 u} \end{aligned}$$

- Let $u = \sin^{-1} x$. Then,

$$\begin{aligned} 2u &= 2\int_0^x \frac{dt}{\sqrt{1-t^2}} \\ &= \sin^{-1}\left(2\sin u\sqrt{1-\sin^2 u}\right) \\ &= \sin^{-1}\left(2x\sqrt{1-x^2}\right) \\ 2\int_0^x \frac{dt}{\sqrt{1-t^2}} &= \int_0^{2x\sqrt{1-x^2}} \frac{dt}{\sqrt{1-t^2}}. \end{aligned}$$

Elliptic Integrals

- Study of Inversions
  Gauss 1790s
  Abel 1823 (pub 1827)
  Jacobi 1829 book
- 1786 (40 yrs later)
  Legendre classified elliptic integrals into 3 cases, book 1825.
  Examples:

$$F(\phi) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad E(\phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \theta}\, d\theta.$$

- Riemann placed in geometric setting - torus.

Gauss' *AGM* - Arithmetic-geometric mean

- Gauss's constant $G = \frac{1}{AGM(1,\sqrt{2})} = \frac{2}{\pi}\int_0^1 \frac{dx}{\sqrt{1-x^4}} = 0.8346268\ldots$
- Between 1 and $\sqrt{2}$ is $\dfrac{\pi}{\bar{\omega}} = \dfrac{1}{G}$.
- Arithmetic mean $\dfrac{a+b}{2}$.
- Geometric mean $\dfrac{a}{g} = \dfrac{g}{b} \Rightarrow g = \sqrt{ab}$.
- $AGM(a,b)$ algorithm: Start with $a_0 = a, \quad b_0 = b,$

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad n = 0, 1, \ldots.$$

- Gauss - $AGM(1,\sqrt{2}) = \dfrac{\pi}{\bar{\omega}}$ to 11 decimal places.
- Led to study of general theory, modular functions, theta functions - Ramanujan (early 1900s).

Application of $AGM(a, b)$

Example: $AGM(1, 2)$. Start with $a_0 = 1, \quad b_0 = 2,$

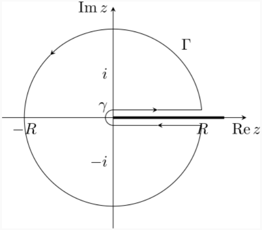
$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad n = 0, 1, \ldots .$$

a_n	b_n
1.0000	2.0000
1.5000	1.4142
1.4571	1.4565
1.4568	1.4568
⋮	⋮

$$AGM(a, b) = \frac{\pi}{4} \frac{a + b}{K\left(\frac{a-b}{a+b}\right)}, \quad K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

Complex Analysis

Fall 2020 - R. L. Herman



History of Complex Analysis

- Before 1600
  - Cardano 1545, quadratic
  - Bombelli 1572, cubic
  - Harriot 1600, quartic
  - Negative roots - false
  - Complex roots - impossible
- 1600s
  - Descartes, 1637, $a+b\sqrt{-1}$
  - Wallis 1685
  - - insights from geometry trigonometry, conics - justified
- 1700s
  - Bernoulli - integral transformation
  - Euler - Euler's formula, i
  - Gauss (1799, 1815) FTA, quadratic forms
  - Wessel (1797), Argand (1806) Geometric Visualization
  - Cauchy (1814) Complex Analysis
  - Riemann (1826-1866) Surfaces

Complex Numbers, $\mathbb{C}$

- $a + bi \in \mathbb{C}$, $a, b \in \mathbb{R}$, $i = \sqrt{-1}$.
- Quadratic Equation,
  $ax^2 + bx + c = 0$,
  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
  If $b^2 - 4ac < 0$,
  complex conjugate roots.
- Cubics - Role was clearer
  $y^3 = py + q$
  $y = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}.$

Example: $x^3 = 15x + 4$

$$\begin{aligned} x &= \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} \\ &= 2 + i + 2 - i = 4. \end{aligned}$$

Bombelli (1572)

$$\begin{aligned} (2+i)^3 &= (2+i)(4 + 4i + i^2) \\ &= (2+i)(3+4i) \\ &= 2 + 11i. \end{aligned}$$

Bernoulli's Transformations

- Johann Bernoulli (1712)

$$
\begin{aligned}
\frac{1}{1+z^2} &= \frac{1}{(1+iz)(1-iz)} \\
&= \frac{1}{2}\left(\frac{1}{1-iz} - \frac{1}{1+iz}\right) \\
\int \frac{dz}{1+z^2} &= \frac{1}{2}\int \left(\frac{1}{1-iz} - \frac{1}{1+iz}\right)
\end{aligned}
$$

- Note:

$$\int \frac{dz}{a+bz} = \frac{1}{b}\ln(a+bz).$$

So,

$$\tan^{-1} z = \frac{1}{2i}\left(\ln(1+iz) - \ln(1-iz)\right).$$

Example

- Bernoulli studied $y = \tan n\theta$ in terms of $x = \tan\theta$.
- *Example:* $n = 2$
  $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$.
- Let $y = \tan n\theta$, Then, $n\theta = \tan^{-1} y, \theta = \tan^{-1} x$.

$$\begin{aligned}\int \frac{dy}{1+y^2} &= n\int \frac{dx}{1+x^2}\\ \ln\frac{y+i}{y-i} &= n\ln\frac{x+i}{x-i}\\ \frac{y+i}{y-i} &= A\left(\frac{x+i}{x-i}\right)^n\end{aligned}$$

$A = (-1)^{n+1}$. Solve for y.

Ex: $n = 2:$

$$\tan 2\theta = \frac{2x}{1-x^2}.$$

Ex: $n = 3:$

$$\tan 3\theta = \frac{x^3-3x}{3x^2-1}.$$

Ex: $n = 4:$

$$\tan 4\theta = \frac{4x-4x^3}{x^4-6x^2+1}.$$

Ex: $n = 5:$

$$\tan 5\theta = \frac{x^5-10x^3+5x}{5x^4-10x^2+1}.$$

Fundamental Theorem of Algebra

- Integration of $\frac{p(x)}{q(x)}$ for $p(x), q(x)$ polynomials
- Need Integration by parts.
- Assumes $q(x)$ can be factored
  - Fundamental Theorem of Algebra (FTA)
- By 1750 - Any polynomial with real coefficients can be factored into real linear and quadratic factors.
- Nicholas Bernoulli (1687-1759) gave a counterexample:
  $p(x) = x^4 - 4x^3 - 2x^2 + 4x + 4.$
- Euler factored it as

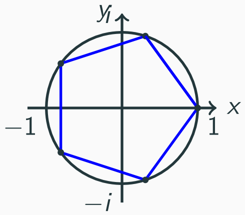
$$x^2 - \left(2 \pm \sqrt{4 + 2\sqrt{2}}\right) x + \left(1 \pm \sqrt{4 + 2\sqrt{7}} + \sqrt{7}\right)$$

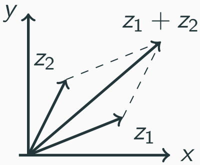
He gave incorrect proof for any quartic
followed by proofs from d'Alembert and Gauss.

Roots of Unity

- Cotes, de Moivre, Euler
  - $x^n - 1 = 0$ Seems $x = \sqrt[n]{1}$.
  - $x = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$,
    $k = 0, 1, \ldots, n-1$.
    Roots of unity.
  - Geometric Interpretation
- Caspar Wessel, surveyor.
  - Complex number = point in the complex plane, 1797.
  - Also, proposed vectors.
- Argand, 1806, visual representation, operational (translation, rotation, reflection)
- Gauss also rediscovered, 1831.

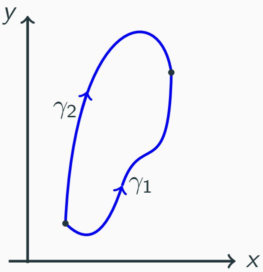
$e^{2k\pi/5}, k = 0, 1, \ldots, 4$





Representing Complex Numbers

- Gauss (1777-1855) adopted "complex number," used i.
- Integration in $\mathbb{C}$-plane.
- $\int_\gamma \phi(z)\, dz$ is path independent for "nice" $\phi(z)$.
- Cauchy proved later, in 1814 talk, published 1827. - Now called *Cauchy's Thm.*



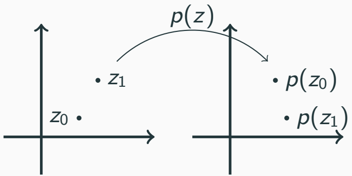
Path Independence

$$\int_{\gamma_1} \phi(z)\, dz = \int_{\gamma_2} \phi(z)\, dz$$

Fundamental Theorem of Algebra I

Every polynomial $p(x)$ can be written as a product of linear complex factors. (Contains 1750 version)

- d'Alembert (1717-1783)
- **Lemma** $p(z_0) \neq 0$, $p(z) \neq$ constant. There exists a z_1 such that $|p(z_1)| < |p(z_0)|$ where $|a + bi| = \sqrt{a^2 + b^2}$.



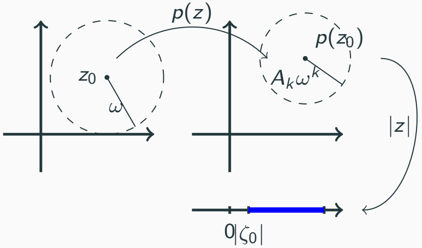
Proof

$$
\begin{aligned}
p(z) &= a_0 z^n + a_1 z^{n-1} + \cdots + a_n. \\
p(z_0 + w) &= a_0 z_0^n + a_1 z_0^{n-1} + \cdots + a_n + A_1 \omega + A_2 \omega^2 + \cdots + A_n \omega^n.
\end{aligned}
$$

Fundamental Theorem of Algebra II

$$p(z_0 + w) = a_0 z_0^n + a_1 z_0^{n-1} + \cdots + a_n + A_k \omega^k + \epsilon$$

Here $A_k\omega^k$ is the first nonzero, lowest power of ω term and ϵ contains the higher powers terms in ω and is small for large $|z|$.



$\exists \omega : \ p(z_0) + A_k\omega^k$ is closer to the origin.

Let $p(z) \neq 0$. By the lemma, $\exists$ point closer than ζ_0 to the origin. $\therefore$ there exists a zero of $p(z)$.

Fundamental Theorem of Algebra III

- Gauss attempted several proofs.
- Karl Weierstrauss (1815-1897) - continuous functions on closed, bounded regions which assume maximum and minimum values.
- Gauss considered curves $\text{Re}(p(z)) = 0$, $\text{Im}(p(z)) = 0$, $z = x + iy$.
- For $|z|$ large, $\text{Re}(a_0 z^n) = 0$, $\text{Im}(a_0 z^n) = 0$, are curves asymptotic to lines through the origin.

Examples

Plotting $Re(p(z)$ and $Im(p(z))$, outide a large circle one gets alternating lines. Inside the circle they must intersect for $p(z) = 0$.

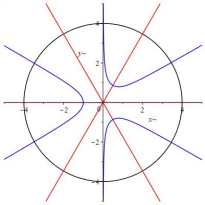


Figure 1: $p(z) = z^3 + 1$.

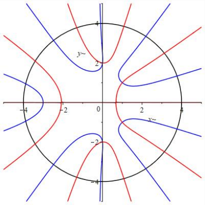


Figure 2: $p(z) = z^5 + z^4 + 10 * z^2 - 16 * z + 24 = (z - 1 - i)(z - 1 + i)(z^2 + 4) * (z + 3)$.

Theory of Curves, $p(x, y) = 0$

- Descartes - linear/lines
  - quadratic/conics
- Newton - conics
- General curves?
- 19th Century
  - Projective Geometry
    homogeneus coordinates
    Möbius, Plücker - 1830
  - Complex Numbers
    Gauss - FTA
    Bezout's Intersection Thm
  - Topological ideas
    - Riemann surfaces, 1850's

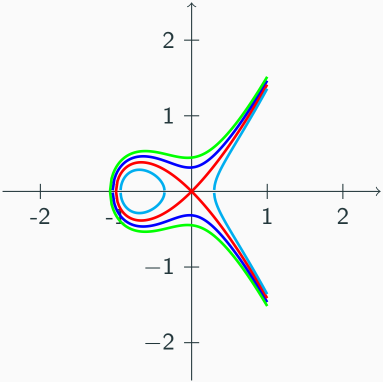
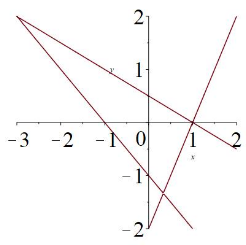


Figure 3: Cubic curves of form $y^2 = x^3 + x^2 + bx + 2b$

Cubic Curves

Consider products of linear factors or lines
$p(x, y) = (a_1x + b_1y + c_1)(a_2x + b_2y + c_2)(a_3x + b_3y + c_3)$

- Ex: $p(x, y) = (x + y + 1)(x + 2y - 1)(-2x + y + 2)$



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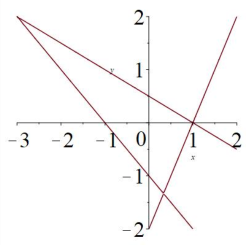
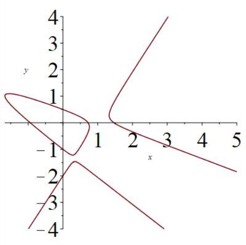
- Modify: $p(x,y) = (x + y + 1)(x + 2y - 1)(-2x + y + 2) + \frac{x^2}{2}$





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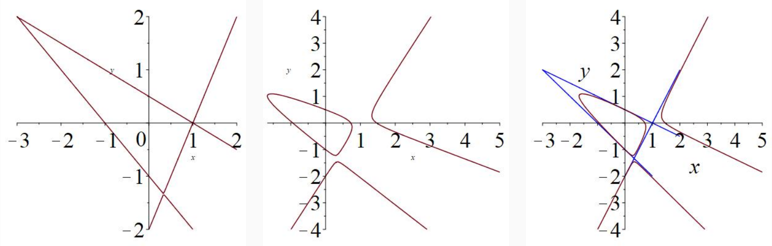
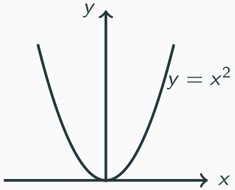
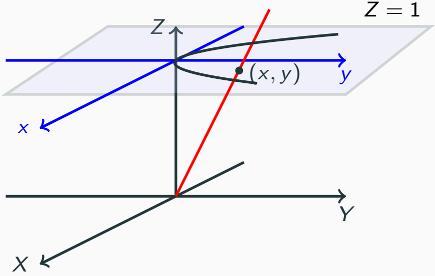
- Modify: $p(x, y) = (x + y + 1)(x + 2y - 1)(-2x + y + 2) + \frac{x^2}{2}$
- Branches go to **points at infinity**. Consider projective geometry.



Projective Geometry

- Homogeneous coordinates: $x = \frac{X}{Z}, y = \frac{Y}{Z}$.
- Introduced by Möbius, Pücker.
- **Example:** $y = x^2$ gives $X^2 = YZ$





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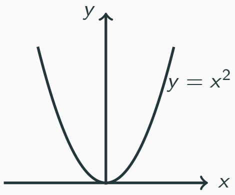
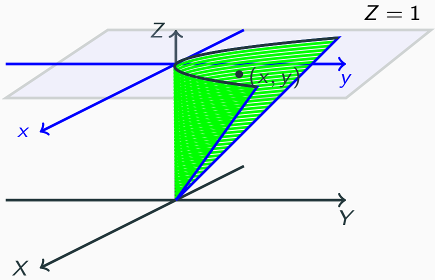
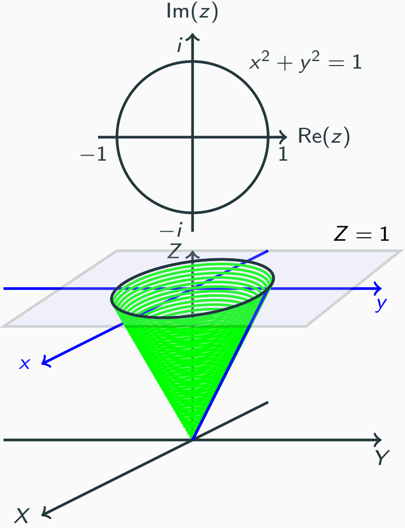
- **Example:** $y = x^2$ gives $X^2 = YZ$
- Lines thru origin (projective plane).
- $X^2 = YZ$ is a "cone"
- Points at Infinity:
  $Z = 0 \Rightarrow X = 0$,
- These points, $[0, Y, 0]$, lie on horizon.





Projective Plane and Complex Numbers

- **Example:** $x^2 + y^2 = 1$
- Projective curve: $X^2 + Y^2 = Z^2$
- Pts at infinity,
  $Z = 0 \Rightarrow X^2 + Y^2 = 0.$
- In $\mathbb{C}$, Circular pts at infinity.
  $X = 1, Y = i : I_1 = (1, i, 0)$
  $X = 1, Y = -i : I_2 = (1, -i, 0)$



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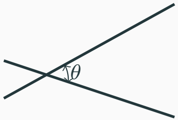
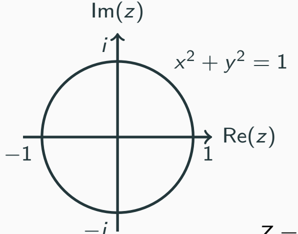
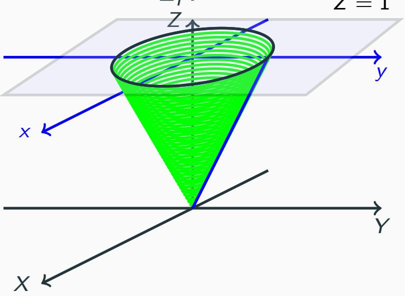
- Edmund Laguerre (1834-1886)
  - Angles, $\theta = i \log R$.
- R - Cross ratio







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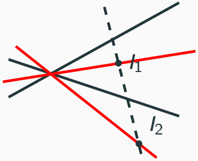
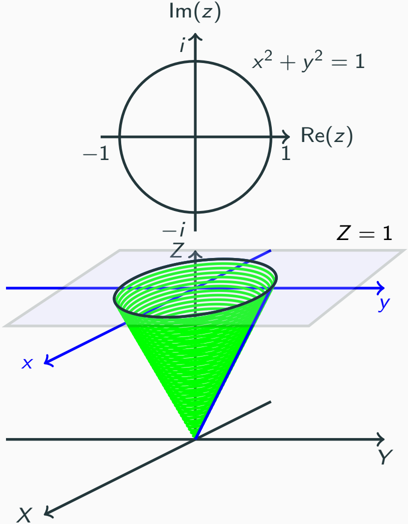
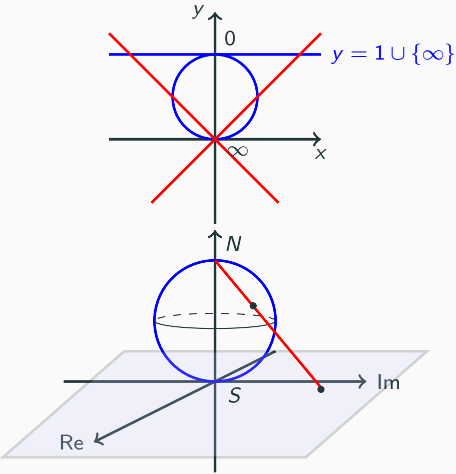




Stereographic Projection

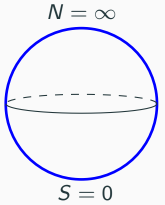
What do complex curves look like?

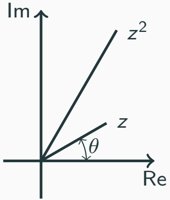
- Projective lines:
  Lines thru origin
  Topologically looks like a circle,
  S^1, after adding point at infinity
- Extend to $\mathbb{C}$ - topologically, S^2
- Stereographic Projection
  Connect pts in $\mathbb{C}$ to North Pole.
- N mapped to pt at ∞.
- Möbius (1790-1868) Image of
  circle = circle.



Riemann Surfaces

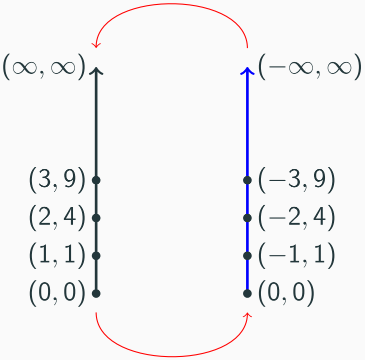
- Riemann (1826-1866)
  - Riemannian manifolds
  - Curved spaces - Gauss
  - Complex Analysis - Riemann surfaces [Cauchy (1788-1857)]
  - Number theory - $\zeta(s)$
- Start with a Sphere
- Extend $f : \mathbb{C} \to \mathbb{C}$ to $g : S^2 \to S^2$.
- Complex function, $f(z) = z^2$
  Let $z = re^{i\theta}$.
  [$\theta =$ argument, $r =$ modulus, $|z|$.]
  Then, $f(z) = r^2 e^{2i\theta}$.

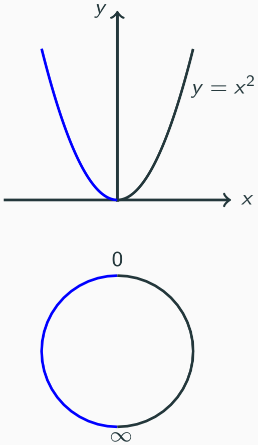




Riemann Surfaces

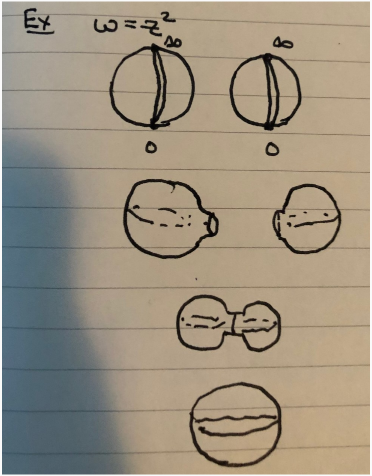
- **Example** $f(x) = x^2$.





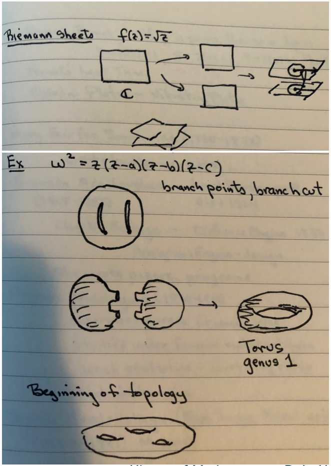
Riemann Surfaces

- **Example:** $\omega = z^2$.



Riemann Surfaces

- Riemann Sheets
- **Example:** $\omega = z^2$.



Women in Mathematics in the 1800s

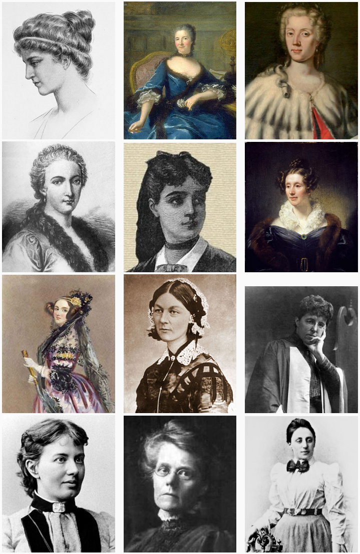
Fall 2020 - R. L. Herman





Famous Women Mathematicians Before 1900

- Hypatia of Alexandria (c. 350-415)
- Émilie du Châtelet (1706-1749)
- Laura Bassi (1711-1788)
- Maria Agnesi (1718-1799)
- Sophie Germain (1776-1831)
- Mary Fairfax Somerville (1780-1872)
- Ada Lovelace (1815-1852) (Augusta Byron, Countess of Lovelace)
- Florence Nightingale (1820-1910)
- Charlotte Angas Scott (1848-1931)
- Sofia Kovalevskaya (1850-1891)
- Alicia Boole Stott (1860-1940)
- Amalie 'Emmy' Noether (1882-1935)



Émilie du Châtelet (1706-1749)

- Gabrielle-Émilie Le Tonnelier de Breteuil
- Father - official at the Court of Louis XIV at Versaille
- Husband - Marquis Florent-Claude Chastellet, military man, governor of Semur-en-Auxois in Burgundy.
- Lovers: Pierre Louis Moreau de Maupertuis (1698-1759), Alexis Clairaut (1713-1765) and François-Marie Arouet (Voltaire) (1694-1778).
- Wrote on Newton, Leibniz, propagation of fire.
- Translation of the *Principia* into French.

Figure 1: Émilie du Châtelet

Marie Sophie Germain (1776-1831)

- Self-taught, French revolution
- 1794 - École Polytechnique - for men
  Signed HW - Monsieur Le Blanc
- Joseph-Louis Lagrange (1736-1813)
- Adrien-Marie Legendre (1752-1833)
- Gauss (1777-1855) - letters 1804-12; saved his life.
- Germain Primes - If p is prime, then so is $2p+1$ Ex: $5 = 2(2)+1$, $7 = 2(3)+1$, ~~$9 = 2(4)+1$~~

Figure 2: Sophie Germain

- Fermat's Last Theorem
- Chladni Plates.

Mary Fairfax Somerville (1780-1872)

- Mathematics and astronomy
- Wrote books
- Jointly - the first female member of the Royal Astronomical Society with Caroline Herschel.
- First to sign petition to Parliament to give women the right to vote.
- experiments to explore the relationship between light and magnetism
- Translated/expanded Laplace's work, 1831, *The Mechanism of the Heavens*.
- First Geography text, 1848.

Figure 3: Mary Sommerville.

Ada Lovelace (1815-1852)

- Daughter of Lord Byron, (poet, died 1824) and
- Mathematician Anne Isabelle Milbanke, self-named as "princess of parallelograms."
- She wrote papers and first computer programs.

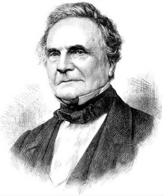
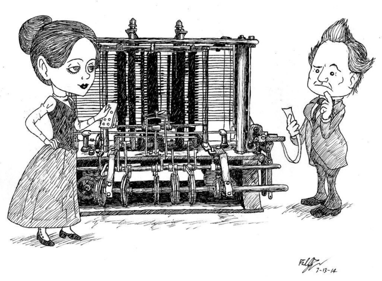


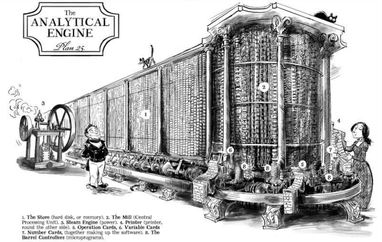
Figure 4: Charles Babbage

Figure 5: Augusta Ada Byron

Ada Lovelace and Charles Babbage

- Charles Babbage (1791-1881)
    - English mathematician, philosopher, engineer.
    - 1833 Difference Engine.
    - 1844 Analytical Engine.
    - Designed, never Built.
- Lovelace first algorithm for a machine.
- 1842-1843, Translated an article Luigi Menabrea on the engine. added notes containing first computer program.
- Ada, programming language.





Florence Nightingale (1820-1910)

- Crimean War (1853-1856)
- Supervised nurses.
- Studied under famous mathematicians.
- Used statistics - mortality rates
- Pioneer in data visualization.
- National heroine, 1883 recipient of the Royal Red Cross, and later others.

Figure 6: Florence Nightingale

Sofia Kovalevskaya (1850-1891)

- Born Sofya Vasilyevna Korvin-Krukovskaya in Moscow.
- Education in Europe.
- Weierstrauss (1874)
  PDEs, elliptic integrals, Saturn's rings.
- 1st woman to get doctorate in math outside Italy. - not enrolled! 1874.
- 1883 Teaching position, U. Stockholm.
- 1889 1st to hold chair in European university since Laura Bassi and Maria Agnessi.

Figure 7: Sofia Kovalevskaya

Laura Bassi and Marie Agnessi

- Laura Bassi (1711-1788)
  - 1st female physics professor. Studied Newton, electricity.
  - Second in the world: Ph.D., 1732.

    1st - philosopher Elena Cornaro Piscopia, 1678.
  - First woman: doctorate in science.
- Maria Agnessi (1718-99).
  - First woman: mathematics handbook.
  - First woman appointed: mathematics professor.
  - First book on both differential and integral calculus
  - Witch of Agnesi curve.

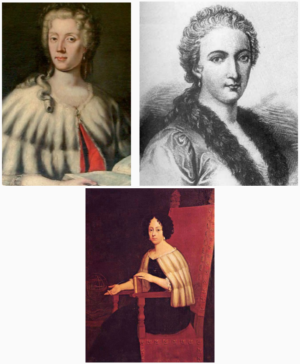


Figure 8: Bassi, Agnessi, Piscopia.

Back to Sofia Kovalevskaya (1850-1891)

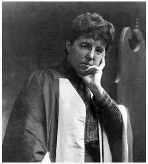
- Light waves, tops, wrote books.
- 1886 - French Competition - spinning tops.
- 1889 - Swedish Academy of Science Prize
  Chebyshev got her membership in Imperial Academy of Sciences
- 1891 - On vacation, Influenza - pneumonia.
- Cauchy–Kowalevski theorem
- Kowalevski top

Figure 9: Sofia Kovalevskaya

Turn of Century - Charlotte Scott and Alicia Stott

Charlotte Angas Scott (1848-1931)

- One of 1st woman to obtain a doctorate in England.
- Studied under Arthur Cayley.
- Algebraic curves of degree higher than two.
- 1885 - 1st mathematician at Bryn Mawr College, dept head.
- A founder of AMS.



Alicia Boole Stott (1860-1940)

- Parents: George Boole (1815-1864) and Mary Everest Boole (1832-1916).
- Four-dimensional polytopes.
- Exactly six regular polytopes in four dimensions
- Worked with Harold Coxeter,(1907–2003).

Amalie 'Emmy' Noether (1882-1935)

- German mathematician
- Abstract algebra - theories of rings, fields, and algebras.
- Noether's theorem - connects symmetry and conservation laws.
- Mathematical Institute of Erlangen, 1908–1915 - without pay.
- University of Göttingen, 1915-1933, First four years lecturing under Hilbert's name.
- Bryn Mawr - 1933-5.
- Lectured at Institute for Advanced Study in Princeton.

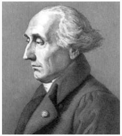
Figure 10: Emmy Noether

Non-Euclidean Geometry and Group Theory

Fall 2020 - R. L. Herman








Euclidean Geometry

- 300 BCE - Euclid's *Elements*
- Five Postulates.
- 5th Postulate - not needed in first 28 propositions.
- Proclus (410-485) Equivalent postulate.
- Giralomo Saccheri (1667-1733) Assume 5th postulate false and get contradiction.
- Used assumption - lines infinite. Led to contradiction of P1, almost P2.

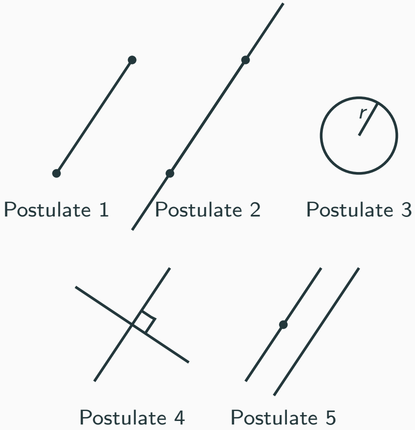


Figure 1: Euclid's 5 Postulates.

Spherical Geometry

- Lines = geodesics, Lie on great circles.
- Euclidean triangles, $a + b + c = \pi$.
- Spherical triangles, $a + b + c > \pi$.
- Thomas Harriot (1560-1621), astronomy, mathematics, and navigation
- Johann Heinrich Lambert (1726-1777)
    - General properties of map projections.
    - hyperbolic functions
    - π is irrational
    - optics

$a + b + c = \pi + \frac{A}{R^2}$.

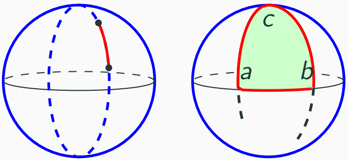


Figure 2: Harriot and Lambert.

Parallel Postulate Revisited

- Carl Friedrich Gauss (1777-1885) started on it in 1799; was convinced it was independent of first 4.
- János Bolyai (1802-1860) - Believed a non-Euclidean geometry existed.
- Nikolai Lobachevsky (1792-1856) - independently 1840 new 5th postulate: There exists two lines parallel to a given line through a given point not on the line. Developed trig identities, hyperbolic geometry.

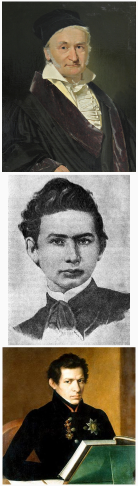
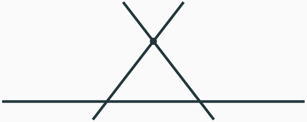
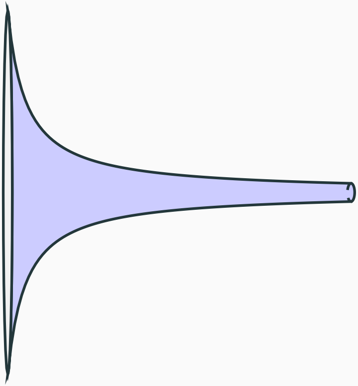


Figure 3: Gauss, Bolyai, Lobachevsky

Riemannian Geometry

- Georg Friedrich Bernhard Riemann (1826-1866)
  - Published in 1868 Lecture
  - Spherical geometry
  - Riemannian geometry $\rightarrow$ differential geometry
  - Every line through a point not on a given line meets the line.
- Eugenio Beltrami (1835-1900) Published interpretations of non-Euclidean geometry - introduced pseudosphere in 1868.
- Curvature, k.





Curvature

- $k = 0,\ k > 0,\ k < 0.$
- sums of angles of triangles $a + b + c - \pi = kA$.

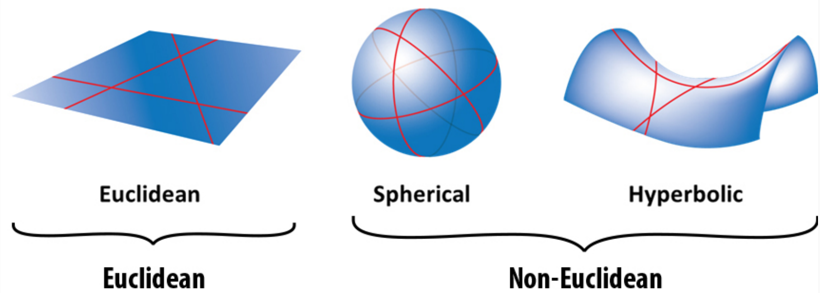


Figure 4: Surfaces of Constant Curvature.

Hyperbolic Geometry

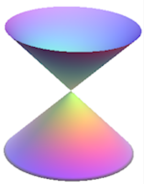
- Sphere

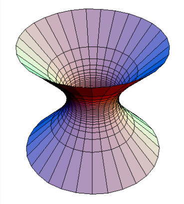
$$x^2 + y^2 + z^2 = \text{const}$$

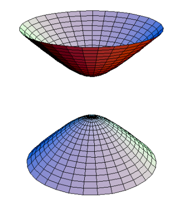
- Modify

$$x^2 + y^2 - z^2 = K$$

- $K = 0$, $z^2 = x^2 + y^2$. Cones.
- $K = 1$, $x^2 + y^2 - z^2 = 1$.
  Hyperboloid of one sheet
- $K = 1$, $z^2 - x^2 - y^2 = 1$.
  Hyperboloid of two sheets.





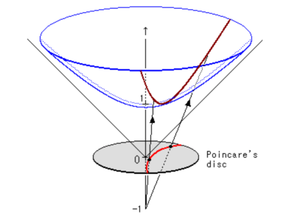


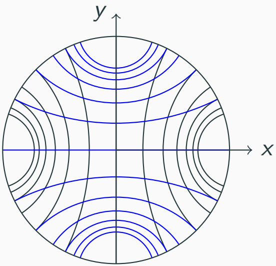
Beltrami-Poincaré Model

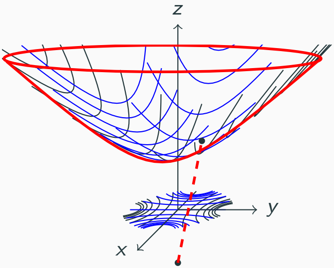
- Poincaré's Disks

$$(x, y, z) = (c \cosh t, \sinh t, \sqrt{1 + c^2} \cosh t)$$

- Stereographic Projection thru $(0, 0, -1)$ to $z = 0 : (x, y, z) \to \frac{(x,y)}{1+z}$.
- Hyperbolic geometry.



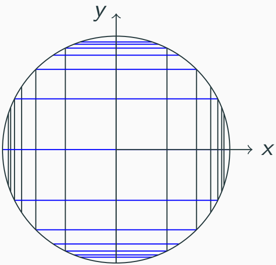


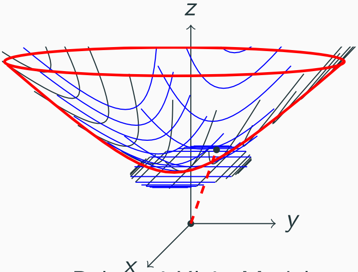


Beltrami-Poincaré Model

Beltrami-Klein Model

- Stereographic Projection thru $(0,0,0)$ to $z=1: (x,y,z) \to \frac{(x,y)}{z}$.
- Klein's Disks
  Projection to $(0,0,1)$

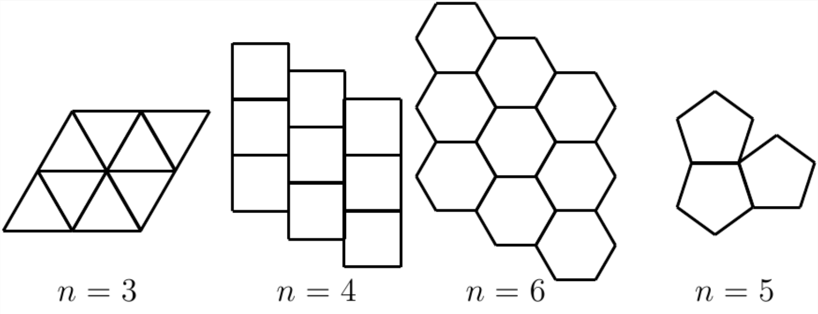




Beltrami-Klein Model

Tiling the Plane

One can tile the plane with a single polygon with sides 3, 4, and 6. However, one cannot fit pentagons together. As seen below, the angles do not allow for a fit. For large *n*, the interior angles are too small.



Other Tilings

- Johannes Kepler (1571-1630)
  - Studied Tilings
  - *Harmonicae Mundi* (Harmony of the World).
  - Planned in 1599.
  - Published 1619 - delay by Tycho Brahe to look as orbit of Mars.
- Roger Penrose (1931-)
  - 2020 Nobel Prize
  - 70's Inspired by Tilings - Penrose tilings. In 80's found in nature.
  - and M. C. Escher (1889-1972)
  - Circle Limit - Tiling Hyperbolic Plane.
- Others - Polyominoes and Pentominoes.

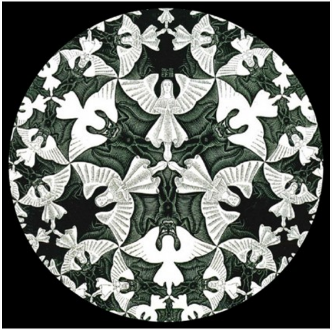


Figure 5: Circle Limit IV

Group Theory

1843 - Joseph Louiville (1809-1882) reviewed manuscript, published 1846. - introduction of groups and fields.

- Group Theory Origin - Galois
- Group of Substitutions.
- Euler - Fermat's Little Theorem
  p prime, $(a, p) = 1$,

  $$a^{p-1} \equiv 1 (\text{mod } p).$$

  $$\phi(n) = \#\{k \in \{1, 2, \ldots, n-1\} \| (k, n) = 1\}.$$

  $\phi(5) = 4$, $\{1, 2, 3, 4\}$
  $\phi(8) = 4$, $\{1, 3, 5, 7\}$.
- Group Properties:

  closed, identity,
  Inverse, associative

Example: $n = 5$

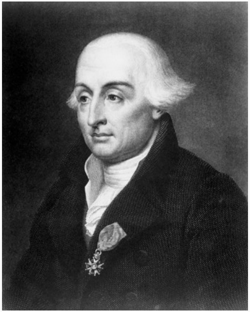
x	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Carl Friedrich Gauss (1777-1855)

- *Disquisitions Arithmeticae* - 1801
- Summary and Extension on Number Theory.
- Initiated finite Abelian groups.
- Proved Fermat's Little Theorem. Represented integers as quadratic forms, like Fermat Primes ($4n+1 = x^2 + y^2$.) for x and y integers.
- Binary quadratic forms - $ax^2 + bxy + cy^2$ - for a, b, c integers.
  - composition has properties of an abelian group.
- Did not have a general theory of groups.

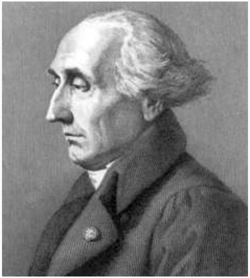
Joseph Louis Lagrange (1736-1813)

- Born in Turin, Italy.
- Professor at 19 (artillery school).
- 1766 Frederick the Great sought great mathematician.
- Lagrange replaced Euler in Berlin for 20 yrs.
- Invited by Louis XVI to Paris.
- 1795 - established dept. École Normal.
- 1797 - established dept. École Polytechnique.
- Napoleon made him senator, count, and he received many other honors.
- Sought solution of quintic by studying cubic and quartic.
- Made many other contributions.



Resolvents

- Consider $x^3 + nx + p = 0$. Let $x = y - \frac{n}{3y}$.
- Yields 6^{th} degree polynomial, $y^6 + py^3 - \frac{n^3}{27} = 0$, the resolvent.
- Let $r = y^3$, $r^2 + pr - \frac{n^3}{27} = 0$.
- Has roots r_1, r_2, where $r_2 = -\left(\frac{n}{3}\right)^3 \frac{1}{r_1}$.
- Then, $x = \sqrt[3]{r_1} + \sqrt[3]{r_2}$
- Cardano got this real root but did not seek complex solutions.
- Lagrange knew there should be 3 roots to a cubic. $\sqrt[3]{r}, \omega\sqrt[3]{r}, \omega^2\sqrt[3]{r}$, where ω is a solution of $x^3 - 1 = 0$. Then,



$$
\begin{aligned}
x_1 &= \sqrt[3]{r_1} + \sqrt[3]{r_2} \\
x_2 &= \omega\sqrt[3]{r_1} + \omega\sqrt[3]{r_2} \\
x_2 &= \omega^2\sqrt[3]{r_1} + \omega^2\sqrt[3]{r_2}
\end{aligned}
$$

Permutation of Roots

- Lagrange then wrote roots of the resolvent
  $y = x_i + \omega x_j + \omega^2 x_k, \quad i, j, k = 1, 2, 3, \quad i \neq j \neq k.$
- 3!=6 permutations of cubic roots.
- In $y^6 + py^3 - \frac{n^3}{27} = 0$, the coefficients of y^5, y^4, y^2, y are
  $x_1 + x_2 + x_3$, $p = x_1x_2x_3$, and $\dfrac{n^3}{27} = \dfrac{(x_1x_2 + x_1x_3 + x_2x_3)^3}{27}$.
- Resolvent coefficients are rational functions of the cubic roots.
- Lagrange obtained similar results for the quartic.
- Lagrange sought solutions of higher order equations using symmetric functions of the roots and permutations.
- Paola Ruffini (1765 – 1822) - 1802, 1805, 1813 - proofs that quintic can't be solved. Not understood.

Niels Henrick Abel (1802-1829)

- Born in Norway into poverty and had a pulmonary condition.
- Mathematical ability discovered by his teacher.
- Toured Europe after college and published 5 papers in *Journal für die reine und angewandte Mathematik*.
- Studied convergence of infinite series, the theory of doubly periodic functions, elliptic functions, elliptic functions and the theory of equations.
- Could not get employment, so tutored.
- At university, thought he had solution of quintic. Then, proved no solution existed.
- Died of tuberculosis before completing his work.

Évariste Galois (1811-1832)

- Born Oct 25, 1811
- Interest in math at 14.
- Read Adrien-Marie Legendre (1752-1833).
- 1828 Failed to get into École Polytechnique.
- 1829 Paper on continued fractions.
- Studied polynomial equations.
- Wrote two papers.
  Reviewed by Arthur Cayley (1821-1895)
  Entered competition.
- 1830 Submitted to Joseph Fourier (1768–1830) - got lost.
  Winners - Niels Henrik Abel (1802-1829) and Carl Gustav Jacobi (1804-1851).
- Published 3 papers.

Figure 6: Évariste Galois

Évariste Galois (cont'd)

- Political turmoil in France.
- Student uprising - Galois left school.
- He was arrested and acquitted.
- Arrested Oct 1831 - April 1832.
- Siméon Denis Poisson (1781-1840) asked him to submit work 1831.
- July 4 - declared work incomprehensible.
- Galois found out in October.
- Auguste Chevalier - privately published manuscript.
- Stayed up all night; wrote letters and note to Chevalier.
- On May 30, fought in duel and lost.

Figure 7: Legendre, Cayley, Fourier, Jacobi, Poisson, Liouville

Symmetry Groups

- Levi ben Gorshun (1321)
  Number of permutations of n objects $= n!$
- Leads to Symmetric Group.
- Felix Klein (1872) extended groups to geometry - studied invariants of groups of transformations.
- Sophus Lie (1842-1899) continuous groups of transformations, applied to differential equations.
- Emmy Noether (1882-1935) related symmetries to constants of motion in physics.

Figure 8: Sophus Lie and Emmy Noether.