Greek Mathematics

Fall 2020 - R. L. Herman

Thales of Miletus

- Greek Numerals - 450 BCE
  Attic Ionic
- Geometry
  - Thales (ca. 640-546 BCE)
  - Thales' Theorem
  - Intercept Theorem

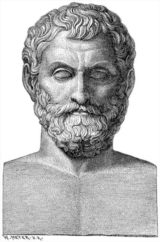


Figure 1: Thales

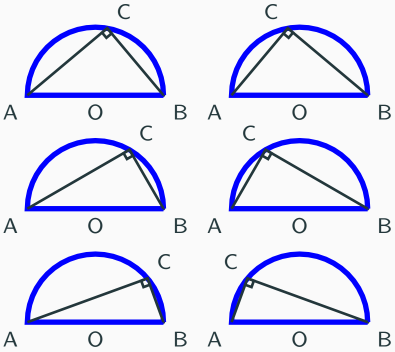


Figure 2: Thales' Theorem: An angle inscribed in a semicircle is a right angle.

Thales' Theorem: Inscribed Angle = 90^o

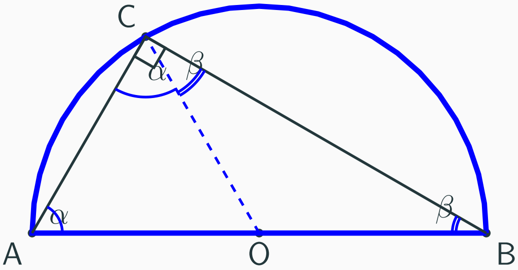


Figure 3: Proof by Picture.

Radii: $\overline{AO} = \overline{OB} = \overline{OC}$.

Isoceles triangles: AOC and OBC.

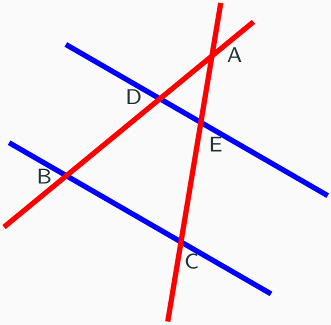
Sum of angles in ABC $= 2\alpha + 2\beta = 180^o$ implies $\alpha + \beta = 90^o$.

Intercept Theorem

If two (or more) parallel lines (blue) are intersected by two self-intersecting lines (red), then the ratios of the line segments of the first intersecting line is equal to the ratio of similar line segments of the second line.

Prove by using similar triangles:

$$\frac{\overline{DE}}{\overline{BC}} = \frac{\overline{AE}}{\overline{AC}} = \frac{\overline{AD}}{\overline{AB}}$$



Pythagoras of Samos (570-495 BCE)

- Known from Philolaus and others
- School in Croton, 530 BCE
  - vegetarian, communal, secret
  - All is number.
- Philosophy - love of wisdom
- Mathematics - that which is learned

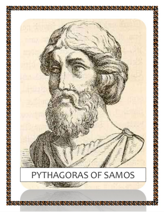


Figure 4: Pythagoras

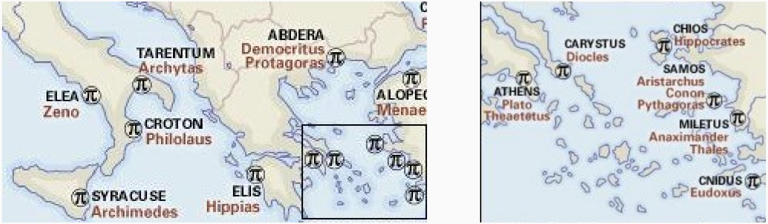


Figure 5: Locate Samos and Croton.

Numerology - Numbers have meanings.

Even is male; Odd is female.

1. = generator
2. = opinion
3. = harmony
4. = justice
5. = marriage
6. = creation
7. = planets

10 is holiest (tetractys, tetrad, decad)

Figure 6: Tetractys

Triangular numbers:

$$1, 3, 6, 10, \ldots$$

Also the four seasons, planetary motions, music, four elements, fourth triangular number, etc.

Number Theory

- Triangular Numbers

$$1, 3, 6, 10, \ldots$$

- Perfect Numbers [Sum factors $< n$.]

$$\begin{aligned} 6 &= 1+2+3 \\ 10 &\neq 1+2+5 \\ 28 &= 1+2+4+7+14 \end{aligned}$$

- Amicable Numbers,

$$\begin{aligned} 220 &: 1+2+4+5+10+11+20+22+44+55+110 = \\ 284 &: 1+2+4+71+142 = \end{aligned}$$

Pythagorean Theorem, $a^2 + b^2 = c^2$

- Known by Babylonians and Egyptians
- Many Proofs over the years
- Attributed to Pythagoras
- Pythagorean Triples (a, b, c)

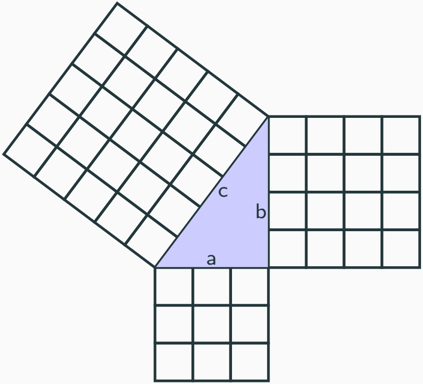


Figure 7: Euclid's Proof

Ratios

Segments are **commensurable** if there exist a segment EF such that $\overline{AB} = p\overline{EF}$ and $\overline{CD} = q\overline{EF}$, where p and q are integers.

Therefore,

$$\frac{\overline{AB}}{\overline{CD}} = \frac{p}{q}.$$

Sometimes written as $p : q$.

Led to *Music of the Spheres*.

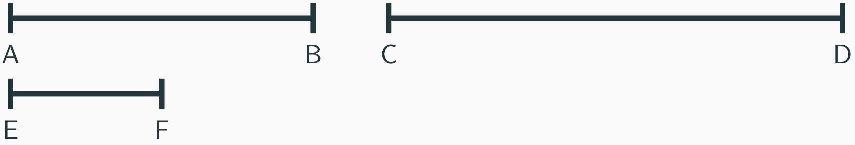


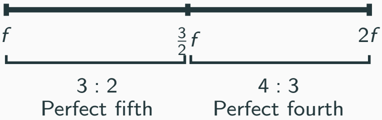
Figure 8: Commensurate Segments

Pythagorean Scale - Series of Musical Notes

Goal - To produce a music scale.

Want sounds that are pleasing when played together. Need simple ratios.

- **Octave**: From f to $2f$ (2^{nd} Harmonic).
  Ex: D goes to D, an octave higher.
- Next Notes?
  Go up by **perfect fifth**. 3 : 2
  Gives an A.
  Go down by **perfect fifth**. 2 : 3
  Gives and G (wrong octave).
  Double: 4:3 **perfect fourth.**
- From A go up perfect fifth to E.
- From G go down to C and adjust.
- Pentatonic scale: D, E, G, A, C, D.



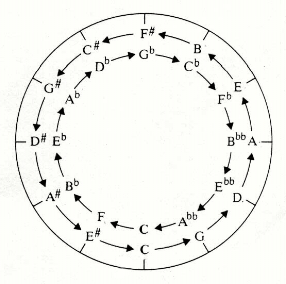
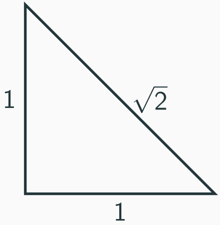


Figure 9: Circle of fifths. `https://www.phys.uconn.edu/~gibson/Notes/Section3_4/Sec3_4.htm`

Irrational Numbers

- Hippasus of Metapontum (c. 530 - c. 450 BCE).
- Credited proving $\sqrt{2}$ is irrational.
- Drowned - possibly not an accident.
- Plato wrote Theodorus of Cyrene (c. 400 BC) proved the irrationality of $\sqrt{3}$ to $\sqrt{17}$.
- Greeks knew sum of angles of triangle $= 2(90^o) = 180^o$.
- Construction of figures with compass and straight edge.



Classical Construction Problems

- Squaring the Circle (Quadrature)
- Doubling the Cube (Volume)
- Trisecting a Angle
  Impossibility Proof: 1857, Pierre Wantzel, needs Modern Algebra

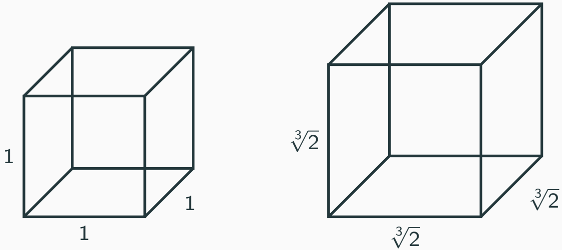


Figure 10: Doubling the Cube.

Hippocrates of Chios (c. 470 - c. 410 BCE)

- Not Hippocrates of Kos (c. 460 - c. 370 BCE),
  of the Hippocratic Oath
  Father of Medicine
- Mathematician, geometer, and astronomer.
- Went to Athens.
- Used *reductio ad absurdum* arguments (proof by contradiction).
- Wrote geometry textbook, *Elements*
- Sought Quadrature of Circle.
- Quadrature of Lune.

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.

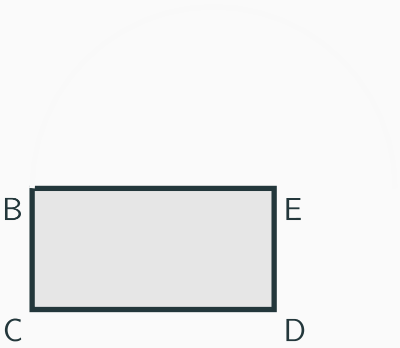
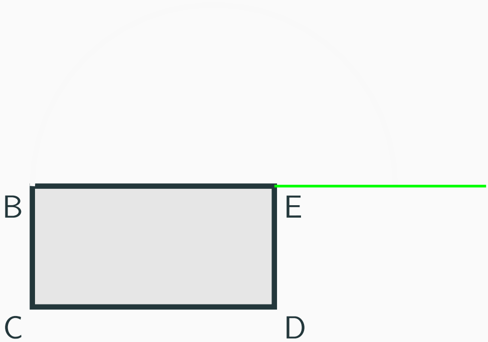


Figure 11: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.



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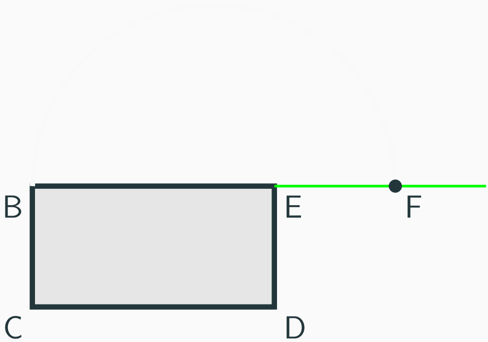
Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.
- Get F such that $\overline{EF} = \overline{ED}$.



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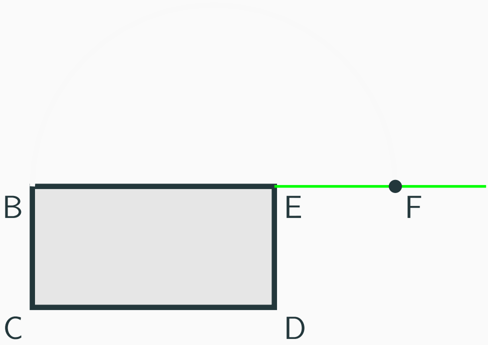
- Get F such that $\overline{EF} = \overline{ED}$.
- How do you bisect BF?



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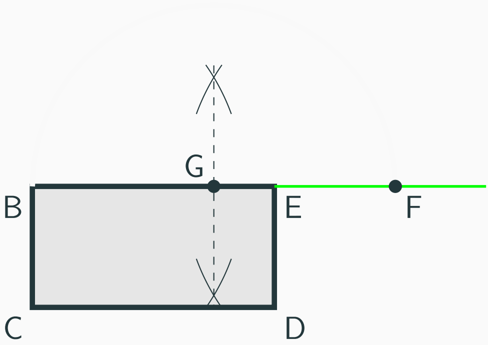

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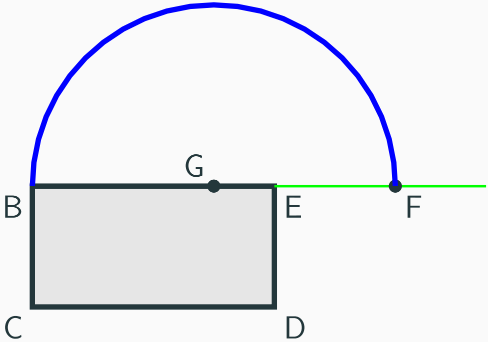
- How do you bisect BF?
- Bisect segment BF.
- Draw semicircle about G.



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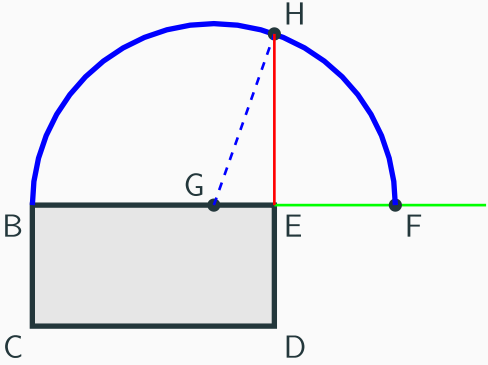
- How do you bisect BF?
- Bisect segment BF.
- Draw semicircle about G.
- Get point H.



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Quadrature of Rectangle

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- Extend segment BE.
- Get F such that $\overline{EF} = \overline{ED}$.
- How do you bisect BF?
- Bisect segment BF.
- Draw semicircle about G.
- Get point H.
- Construct square EKLH.
- Prove the areas are equal.

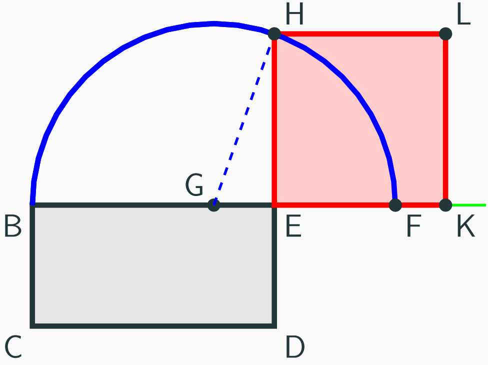


Figure 11: Quadrature of a Rectangle

Proof of Equal Areas

Label lengths a, b, c.

Area of Gray Rectangle BCDE:

$$\begin{aligned} A &= (a+b)\overline{ED} \\ &= (a+b)\overline{EF} \\ &= (a+b)(a-b) \\ &= a^2 - b^2. \end{aligned}$$

Area of Red Square EKLH:

Use Pythagorean Theorem:

$$A = c^2 = a^2 - b^2.$$

Thus, the area of the square is the same as the given rectangle; i.e., we **squared the rectangle**.

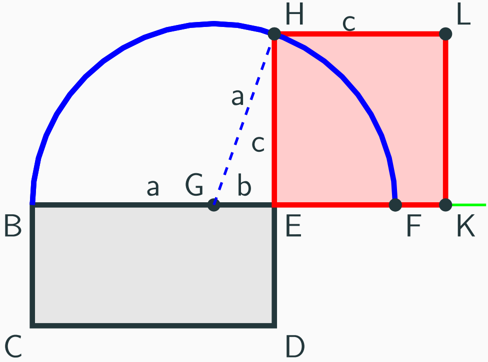
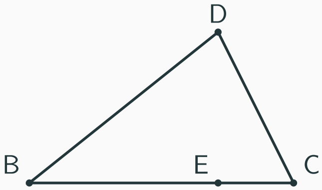


Figure 12: Quadrature of a Rectangle

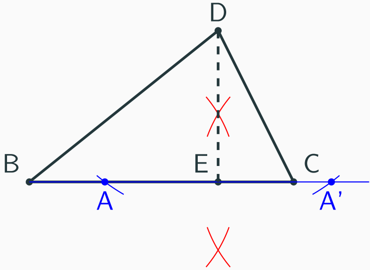
Quadrature of a Triangle

- Start with a triangle.



Quadrature of a Triangle

- Start with a triangle.
- Construct perpendicular measuring height.
    1. Draw blue arcs about D.
    2. Bisect AA' using red arcs..



Quadrature of a Triangle

- Start with a triangle.
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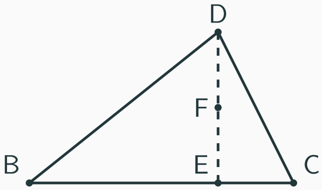
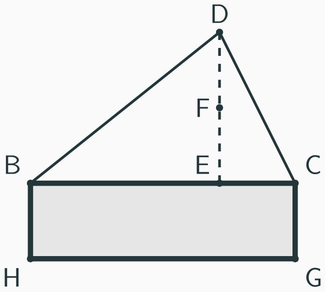
- Bisect perpendicular.



Quadrature of a Triangle

- Start with a triangle.
- Construct perpendicular measuring height.
  1. Draw blue arcs about D.
  2. Bisect AA' using red arcs..
- Bisect perpendicular.
- Construct a rectangle with height CG = EF.
- Square this rectangle.



Quadrature of a Lune

- Lune is the figure bounded by two circular arcs.
- Hippocrates squared a special lune.
- Based on
  - Pythagorean Theorem.
  - Angle inscribed in semicircle is right.
  - Ratio of Areas of circles

$$\frac{A_1}{A_2} = \frac{D_1^2}{D_2^2}.$$

  - Triangles are quadrable.
- Hippocrates proof not valid.

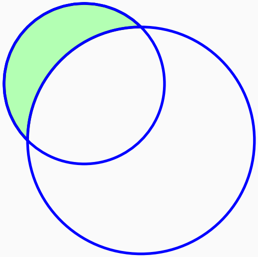


Figure 13: Lune or Crescent.

Hippocrates' Quadrature of a Lune

- $\overline{AB}^2 = \overline{AC}^2 + \overline{CB}^2 = 2\overline{AC}^2$
- Semicircle areas

$$\frac{A(AEC)}{A(ACB)} = \frac{\overline{AC}^2}{\overline{AB}^2} = \frac{1}{2}.$$

- Area of Lune = Area of ΔAOC.
- ΔAOC quadrable, so is the lune.

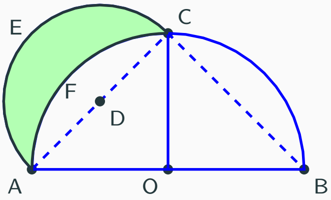


Figure 14: Lune AECF is quadrable.

Can one Square the circle?

Unsolved until Ferdinand Lindemann (1852-1939).

Algebraic Numbers, solutions of polynomial equations with integer coefficients.

Ex: $x^2 - 2 = 0$ has solution $\pm\sqrt{2}$.

Transcendental Numbers, numbers that aren't algebraic.

Timeline of Greek Mathematicians