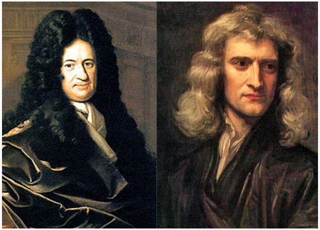
Emergence of Calculus

Fall 2020 - R. L. Herman



Developments in the 1600's

Rapid developments first 60 years of 1600's based on Greek geometry, algebra, astronomy (Kepler, Galileo). Led to unification of geometry and algebra.

- Descartes (1596-1650)
- Cavalieri (1598-1647)
- Fermat (1601-1665)
- Wallis (1616-1703)
- Barrow (1630-1677)
- Gregory (1638-1675)
- Newton (1642-1727)
- Leibniz (1646-1716)

Two main problems

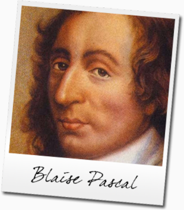
- Tangents
- Areas

Need curves

- Conics
- Archimedean spiral
- Conchoid
- Cissoid
- Cycloid

Seventeenth Century - French, German, English Mathematics

- Galileo Galilei (1564-1642)
- Johannes Kepler (1571-1630)
- 1590 Vièta, *The Analytic Art*
- John Napier (1550-1617) and Henry Briggs (1561-1631) - Introduced the logarithm
- French Mathematicians:
  René Descartes (1596-1650)
  Blaise Pascal (1623-1662)
  Pierre de Fermat (1601-1665)



- Descartes

  philosopher, mathematician
  Discours de la méthode,
  Marriage of algebra/geometry - analytic geometry
- Pascal

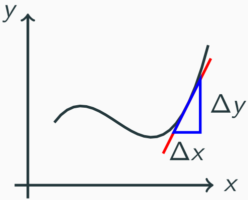
  Wrote math before 16
  Probability theory
  Theology
- Fermat

  Created analytic geometry
  Contributions to Calculus
  Number theory
  Scribbled in Diophantus' *Arithmetica*

Tangents

- Pierre de Fermat, René Descartes
- Studied Apollonius 4-line problem.
- Tangent line approximates the curve at a point.
- Slope $\frac{\Delta y}{\Delta x}$.
- Infinitesimals - increments.
- Fermat
  Method for maxima-minima
  1636 - Method of Tangents
- 1636 Letter from Descartes to Mersenne
  $dy = f(x + dx) - f(x) = ?dx$.

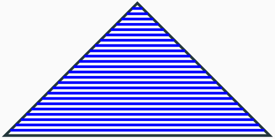
Figure 1: Fermat and Descartes



Areas Under Curves

- First studied by Eudoxus, Archimedes
- Bonaventura Cavalieri (1598-1647) - indivisibles

  Fill area with lines.
  But, an infinite number of lines sum to infinity.

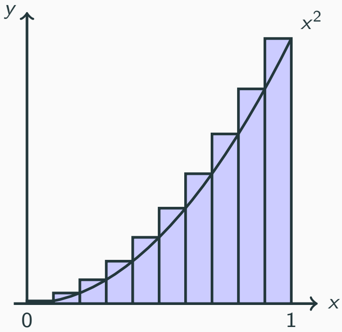


- Archimedes, John Wallis (1616-1703):

$$\int_0^1 x^2\,dx.$$

N segments of width $\frac{1}{N}$. and height $\left(\frac{k}{N}\right)^2$, $k = 1, 2, \ldots, N$.

$$A \approx \sum_{k=1}^{N} \frac{1}{N}\left(\frac{k}{N}\right)^2.$$



Areas Under Curves (cont'd)

Find the sum

$$\begin{aligned} A &\approx \sum_{k=1}^{N} \frac{1}{N}\left(\frac{k}{N}\right)^2 \\ &= \frac{1}{N^3}\sum_{k=1}^{N} k^2 \\ &= \frac{1}{N^3}\frac{N(N+1)(2N+1)}{6} \\ &\sim \frac{2N^3}{6N^3} = \frac{1}{3}. \end{aligned}$$

Wallis showed

$$\int_0^a x^n\,dx = \frac{x^{n+1}}{n+1}\bigg|_0^a = \frac{a^{n+1}}{n+1}$$

for $k = 1, 2, \ldots, 9.$

Note:

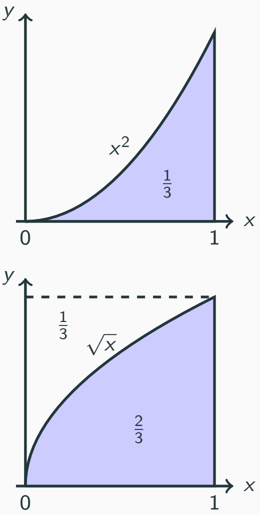
$$\begin{aligned} \sum_{k=1}^{N} k &= \underbrace{1 + \underbrace{2 + \cdots + (N-1)}_{2+(N-1)} + N}_{1+N} \\ &= N\frac{N+1}{2}. \end{aligned}$$

Figure 2: John Wallis

Integrating Powers, $\int x^k\,dx$,

- al Haytham (965-1039) $k = 1, 2, 3, 4$.
- Cavalieri (1635) knew for k up to 9.
- Proven in general by Fermat, Descartes, Roberval, 1630's.
- Fractional Powers (Fermat)
  Ex: $\int_0^1 \sqrt{x}\,dx$
  Use the symmetry in the figures.
- Areas under x^k, need sums $1^k + 2^k + \cdots n^k$.
- Volumes - use cylinders, $V = \pi r^2 h$.
  Sums needed: $1^{2k} + 2^{2k} + \cdots n^{2k}$.

Note: $1^3 + 2^3 + \cdots + k^3 = (1 + 2 + \cdots + k)^3$.



Evangelista Torricelli (1608-1647), barometer inventor

- Inverse Powers, x^{-1}
- Area under $y = \frac{1}{x}$.

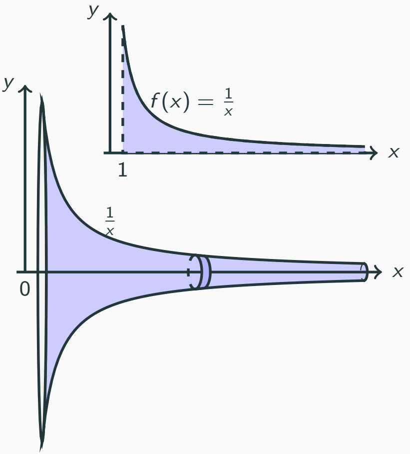
$$\int_1^\infty \frac{1}{x}\, dx = \infty.$$

- Torricelli's trumpet (Gabriel's horn)

$$V = \pi \int_1^\infty \frac{1}{x^2}\, dx = \pi.$$

$$A = 2\pi \int_1^\infty \frac{dx}{x} \sqrt{1 + \left(-\frac{1}{x^2}\right)^2}$$

$$> 2\pi \int_1^\infty \frac{1}{x}\, dx = \infty.$$



What? You cannot paint the surface but can fill the trumpet with paint.

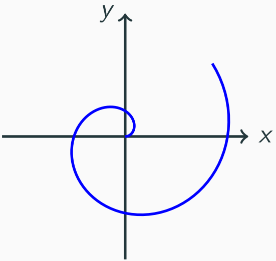
Hobbes - "to understand this for sense, it is not required that a man should be a geometrician or logician, but that he should be mad."

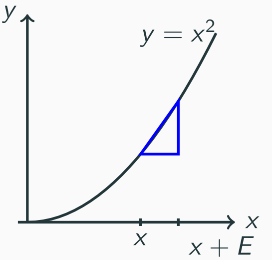
Tangents, Maxima, Minima

- Curves studied like Archimede's spiral, $r = a\theta$
- Fermat - studied polynomials
- Work simpler than Descartes
- Used infinitesimals, E
- **Example:** $y = x^2$

$$\frac{(x+E)^2 = x^2}{E} = 2x + E.$$

- Generalized to polynomials, $p(x, y) = 0$.





Infinitesimals

- Wallis, *Arithmatica Infinitorium*
- Some results already known.
- New approach to fractional powers.
- Ambivalent use of infinitesimals attacked by Thomas Hobbes (1588-1679).
- Formulae for π known by
  - Gregory, Newton, Leibniz
- Madhava (1350-1425) found π to 13 decimal places using series,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

Wallis' Formulae:

$$\pi = \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \cdots$$

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cdots}}}$$

Already known formula:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Isaac Newton (1642-1727)

- Major use of infinite series
- *A Treatise of the Methods of Series and Fluxions*
- *Quadrature of the Hyperbola*
  Written in 1665,
  1st publication in 1668 by Mercator
- Akin to decimal expansions - powers of $\frac{1}{10}$
- Example:

$$\log(1+x) = \int_0^x \frac{dt}{1+t}$$

Note: Geometric series

$$\begin{aligned} 1+t+t^2+\cdots &= \frac{1}{1-t}, |t|<1. \\ 1-t+t^2-\cdots &= \frac{1}{1+t}, |t|<1. \end{aligned}$$

Then,

$$\begin{aligned} y &= \log(1+x) \\ &= \int_0^x (1-t+t^2-\cdots)\, dt \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots . \end{aligned}$$

Invert Power Series

We have for $y = \log(1 + x)$,

$$y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots .$$

In order to invert the series, let $x = a_0 + a_1 y + a_2 y^2 + \cdots$. Then,

$$\begin{aligned} y &= \left(a_0 + a_1 y + a_2 y^2 + \cdots\right) - \frac{1}{2}\left(a_0 + a_1 y + a_2 y^2 + \cdots\right)^2 + \cdots \\ &= a_0 - \frac{1}{2}a_0^3 + \frac{1}{3}a_0^3 + a_1\left(a_0^2 - a_0 + 1\right) y \\ &\quad + \left[a_2\left(a_0^2 - a_0 + 1\right) + \left(a_0 - \frac{1}{2}\right)\right] y^2 \\ &\quad + \left[\frac{a_1^3}{3} + a_1 a_2 (2a_0 - 1) + a_3\left(a_0^2 - a_0 + 1\right)\right] y^3 + \ldots . \end{aligned}$$

Equate coefficients of powers of y, then ...

Series Inversion (cont'd)

We solve the resulting system of equations:

$$\begin{aligned}
0 &= a_0 - \frac{1}{2}a_0^3 + \frac{1}{3}a_0^3 \\
1 &= a_1\left(a_0^2 - a_0 + 1\right) \\
0 &= a_2\left(a_0^2 - a_0 + 1\right) + \left(a_0 - \frac{1}{2}\right) \\
0 &= \frac{a_1^3}{3} + a_1 a_2(2a_0 - 1) + a_3\left(a_0^2 - a_0 + 1\right).
\end{aligned}$$

The first equation gives $a_0 = 0$. The next two give $a_1 = 1$ and $a_2 = \frac{1}{2}$.

Continuing Newton found that

$$a_0 = 0, a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{6}, a_4 = \frac{1}{24}, \ldots, a_n = \frac{1}{n!}.$$

Newton's Series for Exponential

So far, inversion of

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots .$$

led to

$$x = y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots .$$

Recalling series expansion for e^x,

$$y = \log(1+x) \Rightarrow x = e^y - 1.$$

So,

$$e^y = 1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots .$$

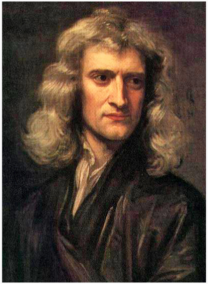


Figure 3: Newton

Newton's Series for Sine

Newton knew $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$.

Recall binomial series: $(a+b)^n = \sum_{n=0}^n C_{n,k} a^{n-k} b^k$, where the coefficients are $C_{n,k} = \frac{n!}{k!(n-k)!}$. Then,

$$\begin{aligned}
(1+a)^p &= 1 + pa + \frac{p(p-1)}{2!} a + \frac{p(p-1)(p-2)}{3!} a^2 + \cdots . \\
\sin^{-1} x &= \int_0^x \frac{dt}{\sqrt{1-t^2}}, \quad a = -t^2, p = -\frac{1}{2}, \\
&= x - \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5}
\end{aligned}$$

Inverting, Newton found

$$\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots .$$

Gottfried Wilhelm Leibniz (1646-1716)

- Librarian, philosopher, diplomat, doctorate in law.
- First papers in calculus (1684).
- Led to long dispute.
- Better notation, $\frac{dy}{dx}, \int \, dx$.
- Sum, product, quotient rules.
- Proved Fundamental Theorem of Calculus, $\frac{d}{dx} \int f(x)\, dx = f(x)$.

Figure 4: Leibniz

Infinite Series

- Geometric series,
  Known to Euclid (Zeno's paradox)
  Leonhard Euler (1707-1783)

  $$a+ar+ar^2+\cdots+ar^n+\cdots = \frac{a}{1-r}, |r|<1.$$

- Harmonic Series - Oresme (1350)

$$\begin{aligned} & 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+\frac{1}{4}+\cdots \\ = & (1)+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots \\ \geq & \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots=\infty. \end{aligned}$$

- Power series - 17th Century,
  Gregory, Wallis, Taylor, Mclaurin, ...

Figure 5: Euler

Basel Problem (1644)

- Posed by Pietro Mengoli (1626-1686).

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

- Jacob and Johann Bernoulli (1704) tackled. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = ?$

Figure 6: Jacob and Johann

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n(n+1)} &= \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \left(\frac{1}{N} - \frac{1}{N+1}\right) \\
&= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots \left(\cancel{\frac{1}{N}} - \frac{1}{N+1}\right) \\
&= 1 - \frac{1}{N+1} \underset{N \to \infty}{\Longrightarrow} 1.
\end{aligned}$$

Euler's Solution of Basel Problem - 1734

- Descartes' Factor Theorem
- $p(x)$ - polynomial
- $p(r) = 0$ implies $p(x) = (x - r)q(x)$, $q(x)$ - polynomial

Proof:

$$\begin{aligned} p(x) &= a_0 + a_1 x + \cdots + a_n x^n \\ p(y) &= a_0 + a_1 y + \cdots + a_n y^n \\ p(x) - p(x) &= a_1(x - y) + \cdots + a_n(x^n - y^n) \end{aligned}$$

$$x^n - y^n = (x - y)\left(x^{n-1} + x^{n-2}y + \cdots + y^{n-1}\right)$$

Let $y = r$,

$$\begin{aligned} p(x) &= (x - r)[a_1 + a_2(x + r) + \cdots + a_n(x^{n-1} + x^{n-2}r + \cdots + r^{n-1})] \\ &= (x - r)q(x). \end{aligned}$$

Leonhard Euler's Solution of Basel Problem

$\sin x$ has roots $n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. - Generalize Factor Theorem:

$$\begin{aligned}
\sin x &= Ax\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\cdots \\
&= Ax\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots \\
&= A\left[x - \frac{x^3}{\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) + x^5(\cdots) - \cdots\right].
\end{aligned}$$

Compare to

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots .$$

Then $A = 1$, and

$$\frac{1}{3!} = \frac{1}{\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Other Results

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^4}{90},$$

$$1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\cdots = (-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n},$$

where B_{2n} are Bernoulli numbers, $B_2=\frac{1}{6}, B_4=-\frac{1}{30}, B_6=\frac{1}{42},\ldots,$

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}.$$

Euler (1748) - Zeta function

$$\begin{aligned}\zeta(s) &= 1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots \\ &= \left(1-\frac{1}{2^s}\right)^{-1}\left(1-\frac{1}{3^s}\right)^{-1}\left(1-\frac{1}{5^s}\right)^{-1}\cdots\left(1-\frac{1}{p^s}\right)^{-1}\cdots,\end{aligned}$$

where p is prime.

Bernhard Riemann's Contribution

- Bernhard Riemann (1826-1866)
- Riemann zeta function

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots .$$

- Values

  $\zeta(1) = \infty$, harmonic series
  $\zeta(2) = \frac{\pi^2}{6}$
  $\zeta(3)$ irrational, proved by Apery (1981)

- Zeros

  $\zeta(s) = 0$, s integer < 0.
  Riemann Hypothesis:
  $\zeta(\sigma + it) = 0$ when $\sigma = \frac{1}{2}$.

- Connection to primes?

Figure 7: Georg Friedrich Bernhard Riemann

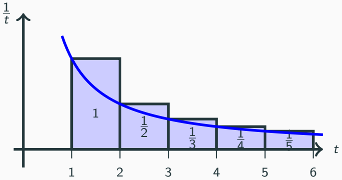
Connection to Primes

$$\zeta(s) = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \cdots \left(1 - \frac{1}{p^s}\right)^{-1} \cdots$$

$$\left(1 - \frac{1}{2^s}\right)^{-1} = 1 + \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 + \cdots$$

- Primes less than $x \sim \int_2^x \frac{dt}{\log t}$
- Euler-Mascheroni Constant
  $\int_1^x \frac{dt}{t} = \ln x$, $\gamma \approx 0.577218\ldots.$

$$\lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right) = \gamma.$$



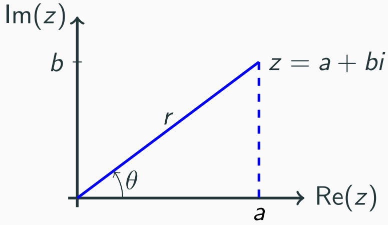
- Generalizing $n!$

$$\Gamma(n+1) = n\Gamma(n), \Gamma(0) = 1.$$

Euler's Formula

- Complex numbers, polar form.
  $z = a + bi,\ a = r\cos\theta, b = r\sin\theta$

$$\begin{aligned} z &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta). \end{aligned}$$



- Exponential of imaginary number

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\ &= 1 + i\theta - \frac{(\theta)^2}{2!} - i\frac{(\theta)^3}{3!} + \cdots \\ &= \left(1 - \frac{(\theta)^2}{2!} + \cdots\right) + i\left(\theta - \frac{(\theta)^3}{3!} + \cdots\right) \\ e^{i\theta} &= \cos\theta + i\sin\theta. \end{aligned}$$

Euler's Formula Applications $e^{i\theta} = \cos\theta + i\sin\theta$.

- $\theta = \pi,\ e^{i\pi} = -1$, or $e^{i\pi} + 1 = 0$.
- $\left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n\ e^{in\theta} = \cos n\theta + i\sin n\theta$ implies **de Moivre's Theorem**

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n.$$

- **Example:** $n = 2$

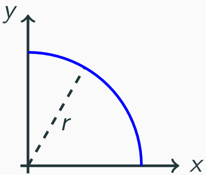
$$\begin{aligned}\cos 2\theta + i\sin 2\theta &= (\cos\theta + i\sin\theta)^2 \\ &= \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta.\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta.\end{aligned}$$

Rectification of an Ellipse

- Rectification = Finding arclengths
- $y = y(x)$

$$L = \int_a^b \sqrt{1 + y'^2}\, dx.$$



- **Example:** Circle

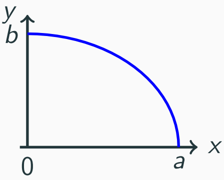
$$\begin{aligned} L &= 4\int_0^r \sqrt{1 + {\frac{x^2}{y^2}}^2}\, dx \\ &= 4\int_0^r \sqrt{1 + {\frac{x^2}{r^2 - x^2}}^2}\, dx \\ &= 4r\int_0^r \frac{dx}{\sqrt{r^2 - x^2}} = 4r\sin^{-1} 1 = 4r\left(\frac{\pi}{2}\right) = 2\pi r. \end{aligned}$$

Other Arclengths

- **Example:** Ellipse

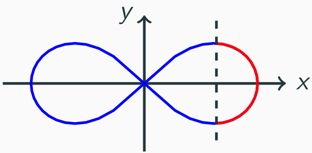
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad x \geq 0, y \geq 0.$

$$\begin{aligned} y &= \frac{b}{a}\sqrt{a^2 - x^2} \\ y' &= \frac{bx}{a\sqrt{a^2 - x^2}} \\ 1 + y'^2 &= \frac{a^2 - k^2x^2}{a^2 - x^2}, \quad k = \frac{a^2 - b^2}{a^2} \\ L &= 4\int_0^a \sqrt{\frac{a^2 - k^2x^2}{a^2 - x^2}}\, dx. \end{aligned}$$



- **Example:** Lemniscate, $r^2 = \cos 2\theta$

$$L = 4\int_0^1 \frac{dr}{\sqrt{1 - r^4}}$$



Elastica
Sep 1694, Jacob Bernoulli
Oct 1694, Johann Bernoulli

Elliptic Functions

- Lemniscate integral leads to new functions, $u = \int_0^x \frac{dt}{\sqrt{1-t^4}}$.
- Compare to $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$.
- Elliptic Integrals: $\int R(t, \sqrt{p(t)})\, dt$, R is rational function, $p(t)$ is polynomial of degree 3 or 4.
- Bernoulli (1694) - geometry, mechanics.
- Fagnano (1682-1766) - Doubling arc of lemniscate.
- Carl Friedrich Gauss (1777-1855) ∼1800 studied inverse $x = sl(u)$ Doubly periodic

$$\begin{aligned} sl(u + 2\bar{u}) &= sl(u) \\ sl(u + 2i\bar{u}) &= sl(u) \end{aligned}$$

- Rediscovered by Niel Henrik Abel (1802-1829) and Carl Gustav Jacobi (1804-1851) in 1820's.

Elliptic Integral Addition Theorems

- **Example** Circle

$$\begin{aligned} \sin 2u &= 2\sin u \cos u \\ &= 2\sin u\sqrt{1-\sin^2 u} \end{aligned}$$

- Let $u = \sin^{-1} x$. Then,

$$\begin{aligned} 2u &= 2\int_0^x \frac{dt}{\sqrt{1-t^2}} \\ &= \sin^{-1}\left(2\sin u\sqrt{1-\sin^2 u}\right) \\ &= \sin^{-1}\left(2x\sqrt{1-x^2}\right) \\ 2\int_0^x \frac{dt}{\sqrt{1-t^2}} &= \int_0^{2x\sqrt{1-x^2}} \frac{dt}{\sqrt{1-t^2}}. \end{aligned}$$

Elliptic Integrals

- Study of Inversions
  Gauss 1790s
  Abel 1823 (pub 1827)
  Jacobi 1829 book
- 1786 (40 yrs later)
  Legendre classified elliptic integrals into 3 cases, book 1825.
  Examples:

$$F(\phi) = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad E(\phi) = \int_0^\phi \sqrt{1-k^2\sin^2\theta}\,d\theta.$$

- Riemann placed in geometric setting - torus.

Gauss' *AGM* - Arithmetic-geometric mean

- Gauss's constant $G = \frac{1}{AGM(1,\sqrt{2})} = \frac{2}{\pi}\int_0^1 \frac{dx}{\sqrt{1-x^4}} = 0.8346268\ldots$
- Between 1 and $\sqrt{2}$ is $\dfrac{\pi}{\bar{\omega}} = \dfrac{1}{G}$.
- Arithmetic mean $\dfrac{a+b}{2}$.
- Geometric mean $\dfrac{a}{g} = \dfrac{g}{b} \Rightarrow g = \sqrt{ab}$.
- $AGM(a,b)$ algorithm: Start with $a_0 = a, \quad b_0 = b,$

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad n = 0, 1, \ldots.$$

- Gauss - $AGM(1,\sqrt{2}) = \dfrac{\pi}{\bar{\omega}}$ to 11 decimal places.
- Led to study of general theory, modular functions, theta functions - Ramanujan (early 1900s).

Application of $AGM(a, b)$

Example: $AGM(1,2)$. Start with $a_0 = 1, \quad b_0 = 2,$

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad n = 0, 1, \ldots .$$

a_n	b_n
1.0000	2.0000
1.5000	1.4142
1.4571	1.4565
1.4568	1.4568
$\vdots$	$\vdots$

$$AGM(a,b) = \frac{\pi}{4}\frac{a+b}{K\left(\frac{a-b}{a+b}\right)}, \quad K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}$$