# Early Mathematics

Fall 2020 - R. L. Herman

# Early Civilizations

- Egypt - 3100 BCE
- Mesopotamia, or Babylonia - 2100 BCE
- China 1600 BCE
- India 1200 BCE

Arithmetic, Geometry,
No proofs
Problems were practical or recreational

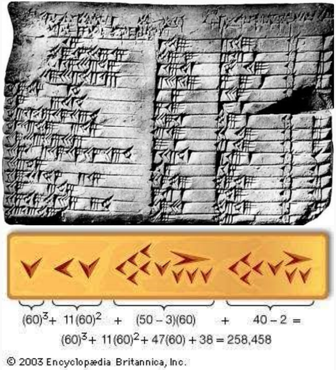



**Figure 1:** Babylonian tablet - Base 60

# Another Sumerian Tablet - YBC 7289


$$1/24/51/10 = 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} \approx 1.414213$$

$$42/25/35 = 42 + \frac{25}{60} + \frac{35}{60^2} \approx 42.426$$

# Egyptian Civilization: 3000 BCE - 300 BCE

- Papyri - scrolls
  - Rhind Papyrus
  - Moscow Papyrus
- Arithmetic - integers, fractions
- Rhind Papyrus
  - Found in Thebe
  - Purchased 1858, A. Henry Rhind
  - $18' \times 13''$
  - Geometry
    - Areas, Volumes
    - Ratios of sides of right triangles
  - Measures - grain
    - 1 hekat $\approx 29224$ in$^3 \geq \frac{1}{2}$ peck
    - 1 ro $= \frac{1}{320}$ hekat

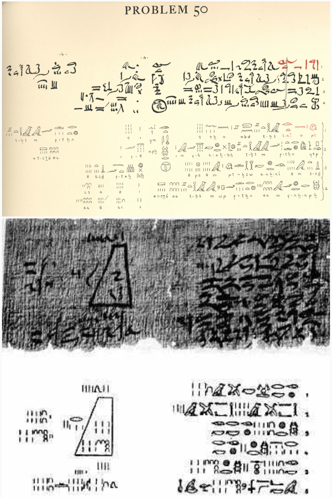


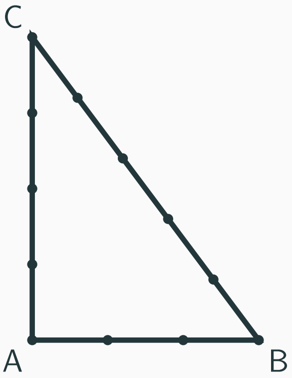

**Figure 2:** Papyri

# Pythagorean Triples

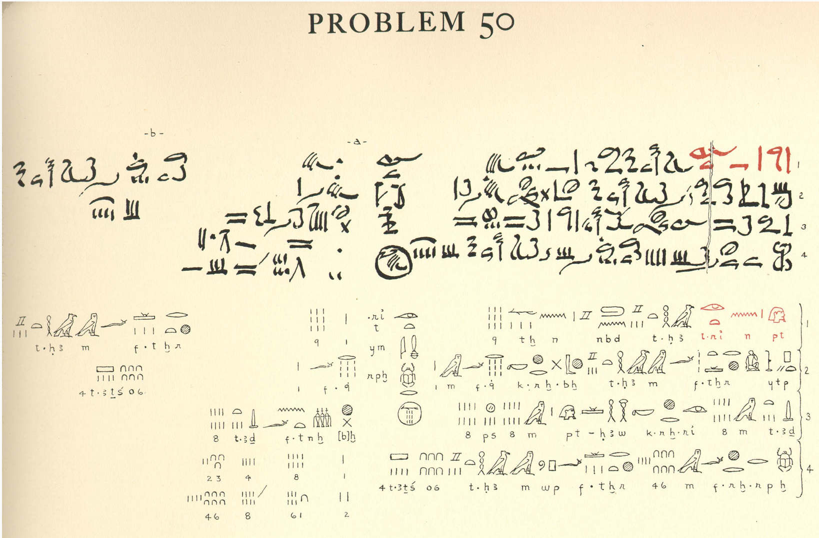
- Pythagorean Theorem
- Triples $(a, b, c)$

$$a^2 + b^2 = c^2$$

  Examples
  - 3-4-5
  - 5-12-13
- Used to Measure Perimeters
- Knotted Ropes
  - Loop with 12 knots



# Rhind Papyrus - Problem 50



# Rhind Papyrus - Problem 50

*Problem 50*

tp n ỉr·t ꜣḥ·t dbn n ḫt·w[1] 9 pty rḫt · f m ꜣḥ·t
*Example of making a field round of* khet 9. *What is the amount of it in area?*

ḫb · ḫr · k 9̇ · f m 1 ḏꜣ·t m 8 ỉr·ḫr · k wꜣḥ–tp m 8 sp 8 ḫpr·ḫr · f m 64
*Take away thou* ⅑ *of it, namely,* 1; *the remainder is :* 8. *Make thou the multiplication :* 8 *times* 8; *becomes it :* 64;

rḫt · f pw m ꜣḥ·t 60[2] śṯꜣ·t 4
*the amount of it, this is, in area,* 60 setat 4.

ỉr·t my ḫpr
*The doing as it occurs:*

| | | | |
|---|---|---|---|
| 1 | 9 | 1 | 8 |
| 9̇ · f *of it* | 1. | 2 | 16 |
| | | 4 | 32 |
| | | \8 | 64 |

ḫ[b] ḫnt · f ḏꜣ·t 8
*Take away from it; the remainder is* 8.

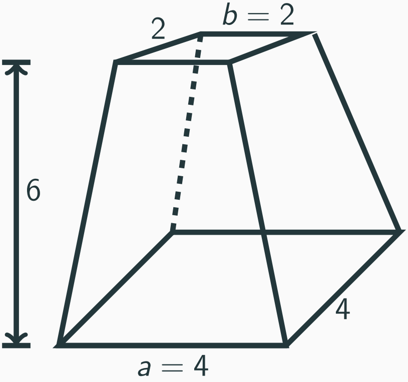
rḫt · f m ꜣḥ·t 60[2] śṯꜣ·t 4
*The amount of it in area:* 60 setat 4.

[1] The w suggested by the plural strokes has been omitted on the plate. The same omission occurs on the figure in Problem 51, and in Problem 52, line 2.

[2] The scribe has by mistake written here either the number 60 or the special form for 6 used in Problem 48 in writing 6 *setat*. He may have had in his mind the fact that he was actually dealing with 60 *setat* (which, however, would not properly be written in this way), and he had written the abstract number 60 a moment before at the end of the multiplication, or, remembering that 60 *setat* is written with the numeral 6, he did write 6, but used the special sign instead of the ordinary numeral.

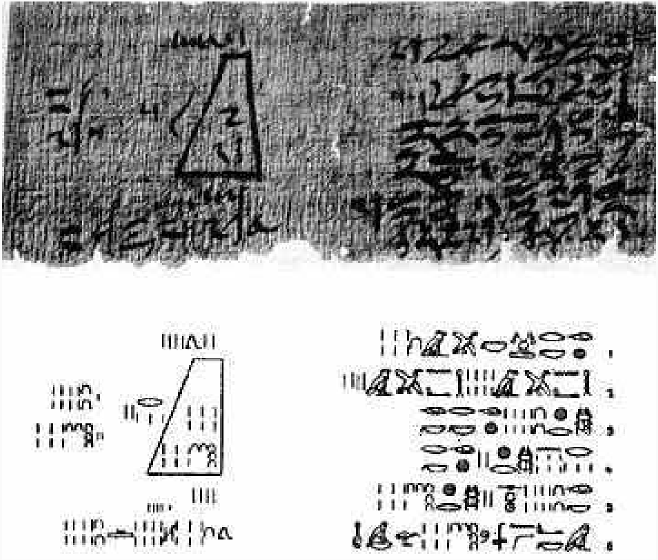
# Moscow Papyrus

- 1850 BCE
- Golenishchev bought in 1892 or 1893 in Thebes
- Housed in Moscow
- 25 Problems
- `https://en.wikipedia.org/wiki/Moscow_Mathematical_Papyrus`
- Problem 14 - Frustrum of a Pyramid




$$V = \frac{h}{3}\left(a^2 + ab + b^2\right)$$

# Moscow Papyrus - Problem 14 - Frustrum of Pyramid



# Tigris and Euphrates Region

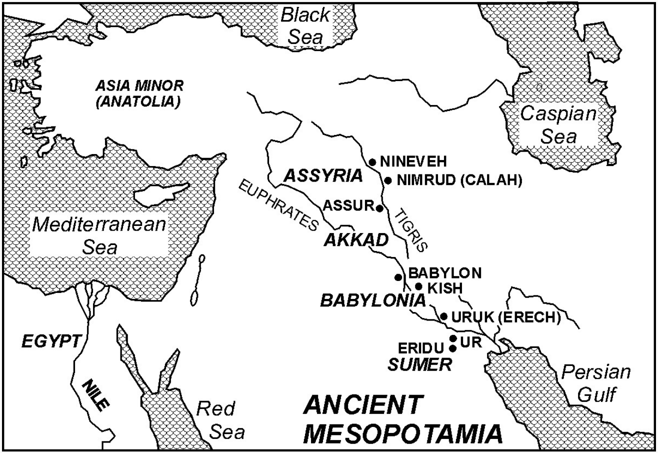



**Figure 3:** Tigris and Euphrates Rivers

# Mesopotamia, or Babylonia - 2100 BCE

- Tigris-Euphrates Region
- More Advanced
- Babylonians, Sumerians
- Clay Tablets
- Base 60 Arithmetic
- Notation $13_{60} = 1\textvisiblespace 3 = 1/3$
- Examples

$$\begin{aligned}
1/3 &= 1(60) + 3 = & 63 \\
1/59 &= 1(60) + 59 = & 119 \\
2/49 &= 2(60) + 49 = & 169 \\
3/31/49 &= 3(60^2) + 31(60) + 49 &
\end{aligned}$$

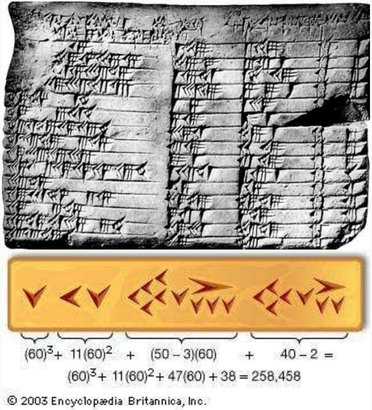
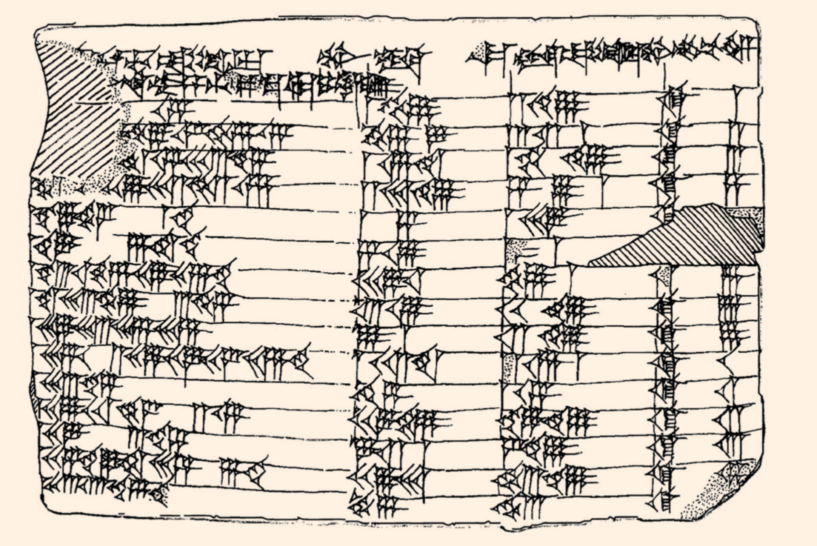



**Figure 4:** Babylonian tablet - Base 60

# Plimpton 322 Clay Tablet - Homework!

# Sketch of the Plimpton 322 Tablet



# Babylonian Numerals 1-100 (Base 60)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 𒁹 | 26 | 𒎙𒐋 | 51 | 𒐐𒁹 | 76 | 𒁹 𒌋𒐋 |
| 2 | 𒈫 | 27 | 𒎙𒐌 | 52 | 𒐐𒈫 | 77 | 𒁹 𒌋𒐌 |
| 3 | 𒐈 | 28 | 𒎙𒐍 | 53 | 𒐐𒐈 | 78 | 𒁹 𒌋𒐍 |
| 4 | 𒐉 | 29 | 𒎙𒐎 | 54 | 𒐐𒐉 | 79 | 𒁹 𒌋𒐎 |
| 5 | 𒐊 | 30 | 𒌍 | 55 | 𒐐𒐊 | 80 | 𒁹 𒎙 |
| 6 | 𒐋 | 31 | 𒌍𒁹 | 56 | 𒐐𒐋 | 81 | 𒁹 𒎙𒁹 |
| 7 | 𒐌 | 32 | 𒌍𒈫 | 57 | 𒐐𒐌 | 82 | 𒁹 𒎙𒈫 |
| 8 | 𒐍 | 33 | 𒌍𒐈 | 58 | 𒐐𒐍 | 83 | 𒁹 𒎙𒐈 |
| 9 | 𒐎 | 34 | 𒌍𒐉 | 59 | 𒐐𒐎 | 84 | 𒁹 𒎙𒐉 |
| 10 | 𒌋 | 35 | 𒌍𒐊 | 60 | 𒁹 | 85 | 𒁹 𒎙𒐊 |
| 11 | 𒌋𒁹 | 36 | 𒌍𒐋 | 61 | 𒁹 𒁹 | 86 | 𒁹 𒎙𒐋 |
| 12 | 𒌋𒈫 | 37 | 𒌍𒐌 | 62 | 𒁹 𒈫 | 87 | 𒁹 𒎙𒐌 |
| 13 | 𒌋𒐈 | 38 | 𒌍𒐍 | 63 | 𒁹 𒐈 | 88 | 𒁹 𒎙𒐍 |
| 14 | 𒌋𒐉 | 39 | 𒌍𒐎 | 64 | 𒁹 𒐉 | 89 | 𒁹 𒎙𒐎 |
| 15 | 𒌋𒐊 | 40 | 𒐏 | 65 | 𒁹 𒐊 | 90 | 𒁹 𒌍 |
| 16 | 𒌋𒐋 | 41 | 𒐏𒁹 | 66 | 𒁹 𒐋 | 91 | 𒁹 𒌍𒁹 |
| 17 | 𒌋𒐌 | 42 | 𒐏𒈫 | 67 | 𒁹 𒐌 | 92 | 𒁹 𒌍𒈫 |
| 18 | 𒌋𒐍 | 43 | 𒐏𒐈 | 68 | 𒁹 𒐍 | 93 | 𒁹 𒌍𒐈 |
| 19 | 𒌋𒐎 | 44 | 𒐏𒐉 | 69 | 𒁹 𒐎 | 94 | 𒁹 𒌍𒐉 |
| 20 | 𒎙 | 45 | 𒐏𒐊 | 70 | 𒁹 𒌋 | 95 | 𒁹 𒌍𒐊 |
| 21 | 𒎙𒁹 | 46 | 𒐏𒐋 | 71 | 𒁹 𒌋𒁹 | 96 | 𒁹 𒌍𒐋 |
| 22 | 𒎙𒈫 | 47 | 𒐏𒐌 | 72 | 𒁹 𒌋𒈫 | 97 | 𒁹 𒌍𒐌 |
| 23 | 𒎙𒐈 | 48 | 𒐏𒐍 | 73 | 𒁹 𒌋𒐈 | 98 | 𒁹 𒌍𒐍 |
| 24 | 𒎙𒐉 | 49 | 𒐏𒐎 | 74 | 𒁹 𒌋𒐉 | 99 | 𒁹 𒌍𒐎 |
| 25 | 𒎙𒐊 | 50 | 𒐐 | 75 | 𒁹 𒌋𒐊 | 100 | 𒁹 𒐏 |

## 2 Akkadian Tablet (-1700)

In the paper "Sherlock Holmes in Babylon," *Amer. Math. Monthly* 87 (1980), 335-345, C. Buck describes Babylonian mathematics. He begins with a discussion of a clay tablet from 3700 years ago as shown in Table 2. There are four columns. You should convince yourself that this is a table of 9's.

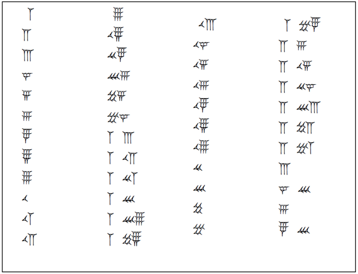


Table 2: Table of 9's.

As an example, the last entry in the first column is $12 =$ . Then, $9 \times 12 = 108 =$ . Note that in base 60 we have $108 = 1(60) + 48$.

In the second column is a one ( ) and 48 ( ) separated by a space. Buck introduces a slash notation to write this as 1/48.

It is easy to add in base 60. Buck gives the example $14/28/31 + 3/35/45 = 18/4/16$.

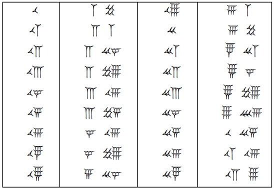
# Babylonian Squares

How can a table of squares be useful? In modern notation, we see that

$$ab = \frac{1}{4}\left[(a+b)^2 - (a-b)^2\right]. \tag{1}$$

Let's find the product $11 \times 14$. Using Table 3, the formula gives

$$\begin{aligned} 11(14) &= \frac{1}{4}\left[(11+14)^2 - (11-14)^2\right] \\ &= \frac{1}{4}(25^2 - 3^2) \\ &= \frac{1}{4}(10/25 - 9) \text{ (base 60)} \\ &= \frac{1}{4}(10/16) \text{ (base 60)} \\ &= \frac{1}{4}(10(60)+16) = \frac{616}{4} = 154. \end{aligned} \tag{2}$$



| | | | |
|---|---|---|---|
| 10 | 1/40 | 19 | 6/1 |
| 11 | 2/1 | 20 | 6/40 |
| 12 | 2/24 | 21 | 7/21 |
| 13 | 2/49 | 22 | 8/4 |
| 14 | 3/16 | 23 | 8/49 |
| 15 | 3/45 | 24 | 9/36 |
| 16 | 4/16 | 25 | 10/25 |
| 17 | 4/49 | 26 | 11/16 |
| 18 | 5/24 | 27 | 12/9 |

Table 3: Table of squares with Babylonian numerals in the left table and slash notation on the right side.

# 4 Pythagorean Triples

Another interesting tablet from the time is the Plimpton 322 tablet shown in Figure 4. This tablet has a listing of Pythagorean triples. The last column has a list of numbers from 1 to 15. Columns two and three seem to be the hypotenuse, $C$, and one leg, $B$, of the right triangle shown in Figure 1. Recall from the Pythagorean Theorem that

$$C^2 = B^2 + D^2.$$

The triple $(D, B, C)$ is called a Pythagorean triple.

We now know that these triples are parametrized by the pair $(a, b)$ as follows:

$$B = a^2 - b^2, \quad C = a^2 + b^2, \quad D = 2ab,$$

since

$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2.$$

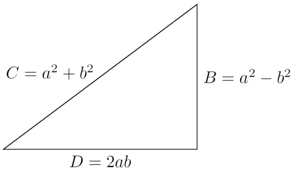



Figure 1: Right triangle with columns two and three as sides B and C, respectively. Pythagorean triples were later found to have a parametrization $(a, b)$.

# Transcription - Brackets indicate guesses

| | | | | |
|---|---|---|---|---|
| [59]/0/15 | 1/59 | 2/49 | ki | 1 |
| [56/56]/58/14/50/6/15 | 56/7 | 1/20/25 | ki | 2 |
| [55/7]/41/15/33/45 | 1/16/41 | 1/50/49 | ki | 3 |
| 53/10/29/32/52/16 | 3/31/49 | 5/9/1 | ki | 4 |
| 48/51/1/40 | 1/5 | 1/37 | ki | [5] |
| 47/6/41/40 | 5/19 | 8/1 | [ki] | [6] |
| 43/11/56/28/26/40 | 38/11 | 59/1 | ki | 7 |
| 41/33/45/14/3/45 | 13/19 | 20/49 | ki | 8 |
| 38/33/36/36 | 8/1 | 12/49 | ki | 9 |
| 35/10/2/28/27/24/26/40 | 1/22/41 | 2/16/1 | ki | 10 |
| 33/45 | 45 | 1/15 | ki | 11 |
| 29/21/54/2/15 | 27/59 | 48/49 | ki | 12 |
| 27/0/3/45 | 2/41 | 4/49 | ki | 13 |
| 25/48/51/35/6/40 | 29/31 | 53/49 | ki | 14 |
| 23/13/46/40 | 56 | 53 | ki | [15] |

# Sketch of the Plimpton 322 Tablet

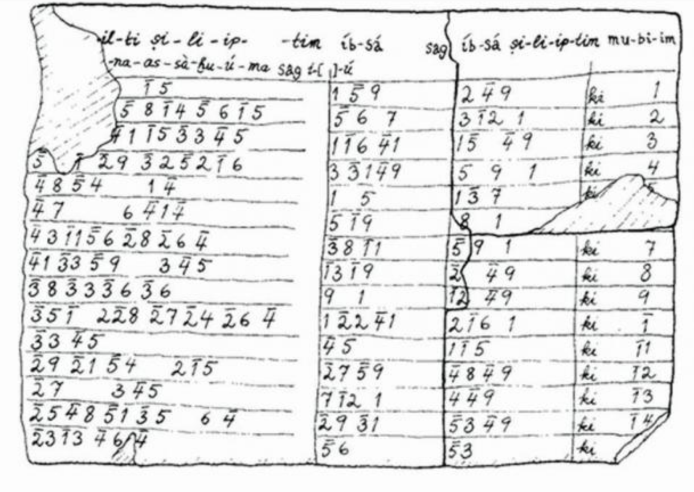


**Figure 5:** Arabic numerals base 60. The bars designate place holders.

# Buck's Corrected Values

Second column - base 60 values for $(B/D)^2$ with $D^2 = C^2 - B^2$.

| # | $A$ | $B$ | $C$ | a | b |
|---|---|---|---|---|---|
| 1 | 59/0/15 | 119 | 169 | 12 | 5 |
| 2 | 56/56/58/14/50/6/15 | 3367 | 4825 | 64 | 27 |
| 3 | 55/7/41/15/33/45 | 4601 | 6649 | 75 | 32 |
| 4 | 53/10/29/32/52/16 | 12709 | 18541 | 125 | 54 |
| 5 | 48/51/1/40 | 65 | 97 | 9 | 4 |
| 6 | 47/6/41/40 | 319 | 481 | 20 | 9 |
| 7 | 43/11/56/28/26/40 | 2291 | 3541 | 54 | 25 |
| 8 | 41/33/45/14/3/45 | 799 | 1249 | 32 | 15 |
| 9 | 38/33/36/36 | 481 | 769 | 25 | 12 |
| 10 | 35/10/2/28/27/24/26/40 | 4961 | 8161 | 81 | 40 |
| 11 | 33/45 | 45 | 75 | 1 | 0.5 |
| 12 | 29/21/54/2/15 | 1679 | 2929 | 48 | 25 |
| 13 | 27/0/3/45 | 161 | 289 | 15 | 8 |
| 14 | 25/48/51/35/6/40 | 1771 | 3229 | 50 | 27 |
| 15 | 23/13/46/40 | 56 | 106 | 9 | 5 |

# Buck's Corrected Values - Babylonian Numerals