Hyperbolic Volumes

The Next Steps

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Outline

1. Thesis Overview
2. Volume of Hyperbolic Tetrahedra
3. New Topics/Questions
4. Content
   - 4.1 Dihedral Angle
   - 4.2 Areas and Volumes
   - 4.3 Area of Hyperbolic Triangle

Thesis Overview - First Draft

Volumes of Hyperbolic Knots

Gauss, Listing, Tait	What is a knot?	Braids, Knots, Links
	↓	
Tait, Little Thurston	Classify Knots	Knot Tables hyperbolic, torus, or satellite
	↓	
Alexander, Jones, Kaufmann	Knot Invariants	Polynomials
	↓	
Lobachevsky, Schläfli	Knot Complements	Hyperbolic Volumes
	↓	
Thurston, Milnor Weeks, Purcell	Triangulation	SnapPea, SnapPy, Regina

Explanation

On the last slide the flow of the thesis leads to computing volumes of hyperbolic knots.

Block

- What is a knot? - Definition, history.
- Classify knots - Tables by Tait and others. More recently, classification of torus, hyperbolic, and satellite knots.
- Knot invariants - General search to distinguish knots.
- Knot complements - Focus on *volumes of knots* and hyperbolic geometry.
- Triangulation - Use software to do survey of volumes.

Alertblock

Need to provide theory for volumes of ideal hyperbolic tetrahedra.

Volume of Ideal Hyperbolic Tetrahedron

Key Functions

1. Lobachevsky function, Λ, л, or Л.
   $л(\theta) = -\int_0^\theta \ln|2\sin u|\, du$.
2. Clausen's integral.
   $\mathrm{Cl}_2(\theta) = \int_0^\theta \ln|2\sin\frac{t}{2}|\, dt$.
3. Dilogarithm,
   $\mathrm{Li}_2(z) = \sum_{k=1}^\infty \frac{z^k}{k^2} = \int_z^0 \frac{\ln(1-t)}{t}\, dt$
4. $л(\theta) = \frac{1}{2}\mathrm{Cl}_2(2\theta)$.
5. $\mathrm{Cl}_2(\theta) = \mathrm{Im}\left(\mathrm{Li}_2(e^{i\theta})\right)$.

Lobachevsky and Schläfli independently determined volume of *orthoschemes* in terms of dihedral angles.

The derivation of the volume of hyperbolic tetrahedra can be found in Thurston's Ch. 7 notes by Milnor or in Purcell.

The Dilogarithm function is also given as $\psi(z)$.

It is useful to answer some basic questions.

$$\mathrm{vol}(T_{ideal}) = \frac{1}{4}\left[л(\gamma+\alpha) - л(\gamma-\alpha) - л(\pi/2+\alpha) + л(\pi/2-\alpha)\right].$$

[1] л and Л are Cyrillic 'L's' for Lobachevsky.

Some Initial Questions

- What are dihedral angles?
- What are the volumes of regular polyhedra?
- What are hyperbolic spaces, $\mathbb{H}^2$ and $\mathbb{D}^2$?
- What are path lengths along geodesics?
- What are the areas of hyperbolic triangles in $\mathbb{H}^2$?
- What properties are needed of $\mathcal{J}(\theta)$ to compute vol(T)?
- What is the volume of a general hyperbolic tetrahedron?
- How are these results connected to the results in SnapPy?

Some of these answers can be found in the links under section Hyperbolic 3-Manifolds on the knot page.

Dihedral Angle = angle between faces of a polygon

For the regular tetrahedron, we seek the angle between the red and yellow faces. This is the same as the angle between the normals to each face.

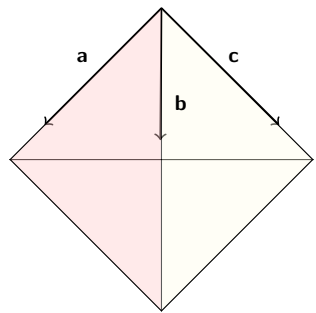
$\mathbf{a} = (-1, 0, -\frac{1}{\sqrt{2}}) - (0, 1, -\frac{1}{\sqrt{2}}) = (-1, -1, 0)$

$\mathbf{b} = (0, -1, -\frac{1}{\sqrt{2}}) - (0, 1, -\frac{1}{\sqrt{2}}) = (0, -2, 0)$

$\mathbf{c} = (1, 0, -\frac{1}{\sqrt{2}}) - (0, 1, -\frac{1}{\sqrt{2}}) = (1, -1, 0)$

$\mathbf{n}_1 = \mathbf{a} \times \mathbf{b}$, $\mathbf{n}_2 = \mathbf{b} \times \mathbf{c}$. Then, $\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$.

Noting $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d})$,



$$\begin{aligned} \mathbf{n}_1 \cdot \mathbf{n}_2 &= (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c}) \\ &= (\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{b} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c}) \\ &= |\mathbf{a}||\mathbf{b}|^2|\mathbf{c}|(\cos^2 60^o - \cos 60^o) = -\frac{1}{4}|\mathbf{a}||\mathbf{b}|^2|\mathbf{c}|. \end{aligned}$$

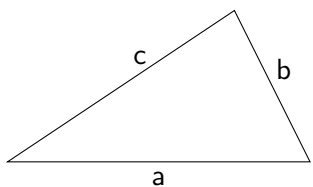
$|\mathbf{n}_1||\mathbf{n}_2| = |\mathbf{a}||\mathbf{b}|^2|\mathbf{c}| \sin^2 60^o = \frac{3}{4}|\mathbf{a}||\mathbf{b}|^2|\mathbf{c}|$.

So, $\cos\theta = \frac{1}{3}$, or $\theta \approx 70.53^o$.

Heron's Formula

- Hero of Alexandria (10-70), $A = \sqrt{s(s-a)(s-b)(s-c)}$, where s is semiperimeter, $s = \frac{1}{2}(a+b+c)$.
- Brahmagupta, cyclic quadrilateral, $A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$, where s is semiperimeter
- Cayley–Menger determinant

$$A^2 = -\frac{1}{16}\begin{vmatrix} 0 & a^2 & b^2 & 1 \\ a^2 & 0 & c^2 & 1 \\ b^2 & c^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

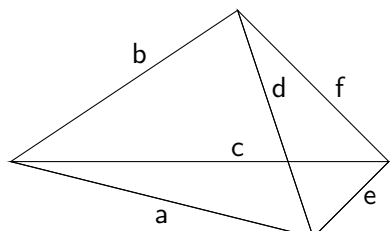


Volume of Tetrahedra

- Generalizing Heron's Formula
  - Piero della Francesca (c. 1415-1492)
  - Tartaglia (c. 1500-1557)
  - Euler
- Cayley–Menger determinant

$$V^2 = \frac{1}{(3!)^2 2^3} \begin{vmatrix} 0 & a^2 & b^2 & c^2 & 1 \\ a^2 & 0 & d^2 & e^2 & 1 \\ b^2 & d^2 & 0 & f^2 & 1 \\ c^2 & e^2 & f^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

- Extension of Heron's Formula to Tetrahedra, June 2022.



Non-Euclidean Areas and Volumes

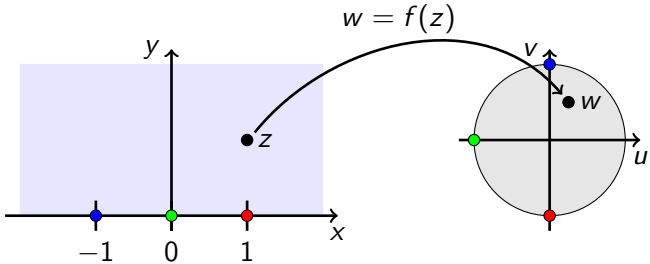
Areas of triangles with angles α, β, γ

- $\mathbb{E}^2$, $A = \frac{s}{2} \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2}$, s - semiperimeter.
- $\mathbb{S}^2$, $A = \alpha + \beta + \gamma - \pi$.
- $\mathbb{H}^2$, $A = \pi - (\alpha + \beta + \gamma)$.
- 1906, G. Sforza[2] - formula for the volume of a non-Euclidean tetrahedron.
- Cayley– Menger formula - multidimensional analogue.
- Schläfli (1858) - the volume of an orthoscheme in $\mathbb{S}^3$
  - volumes in terms of dihedral angles, 1860, Kesner (1936) hyperbolic.
- Lobachevsky, Bolyai - hyperbolic orthoscheme
- Lobachevsky 1835 Milnor 1982 - ideal tetrahedron.
- Abrosimov, Mednykh - Sforza formula for the volume of an arbitrary tetrahedron.

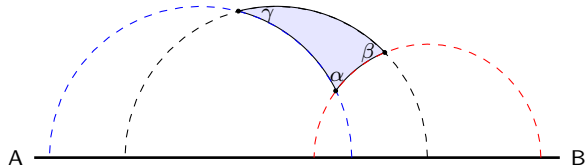
[2] Giuseppe Sforza (-1927) not Bernardino Gaetano Scorza (1876-1939)

Transformation from Half-Plane to Poincaré Disk

The Möbius, or linear fractional, transformation taking $-1, 0, 1$ to $i, -1, -i$ is given by $w = \dfrac{z-i}{z+i}$. Letting $z = x + iy$ and $w = u + iv$, then $u = \frac{x^2+y^2-1}{x^2+(1+y)^2}$, and $v = -\frac{2x}{x^2+(1+y)^2}$ by seeking real and imaginary parts of $w = \dfrac{z-i}{z+i} \cdot \dfrac{\bar{z}-i}{\bar{z}-i} = \dfrac{|z|^2 - 1 - i(z+\bar{z})}{|z|^2 + 1 - i(z-\bar{z})}$.

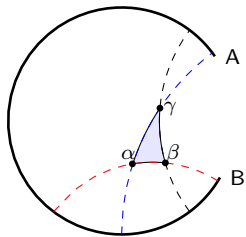


Area of Hyperbolic Triangles

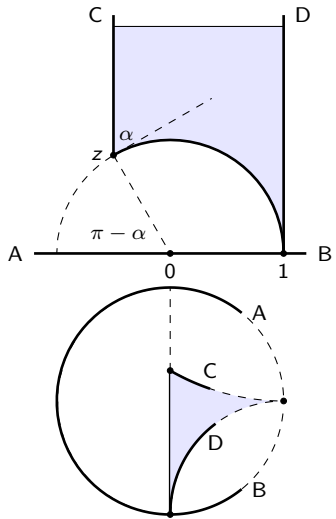


The area is given by the defect: $A = \pi - (\alpha + \beta + \gamma)$.

Map the Half-Plane to the Poincaré disk: $f(x, y) = \left(\frac{x^2+y^2-1}{x^2+(1+y)^2}, -\frac{2x}{x^2+(1+y)^2} \right)$



$\frac{2}{3}$-Ideal Hyperbolic Triangle Area



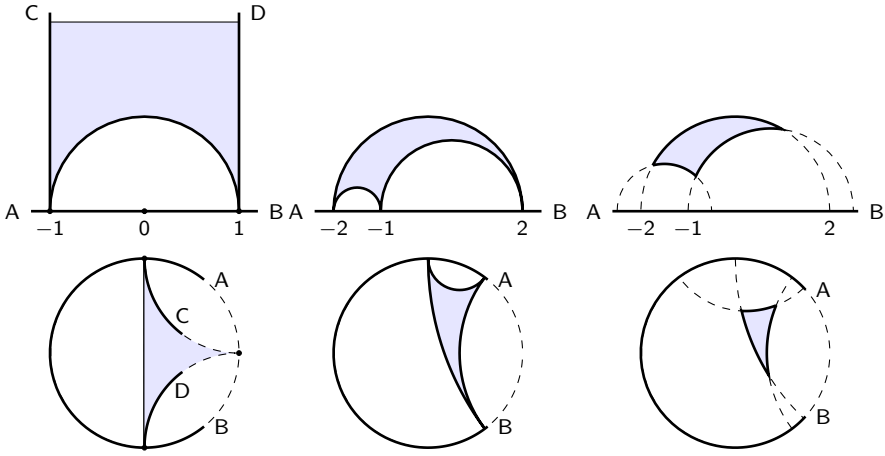
$\frac{2}{3}$-ideal triangles have 2 vertices at infinity. Using the metric for the half-plane model,

$$ds^2 = y^{-2}(dx^2 + dy^2),$$

and area element $dA = y^{-2}\,dxdy$, the area of a $\frac{2}{3}$-ideal hyperbolic triangle is

$$\begin{aligned} A &= \int_{\cos(\pi-\alpha)}^{1} \int_{\sqrt{1-x^2}}^{\infty} \frac{dydx}{y^2} \\ &= \int_{\cos(\pi-\alpha)}^{1} \frac{dx}{\sqrt{1-x^2}} \\ &= \pi - \alpha. \end{aligned}$$

More Hyperbolic Triangles



Two examples of ideal hyperbolic triangles with three points at ∞.

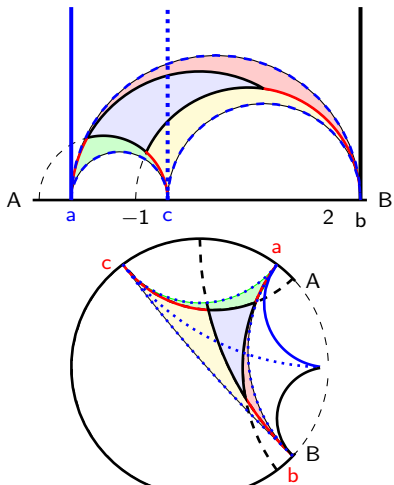
Example of general hyperbolic triangle.

Area of General Hyperbolic Triangles

Begin with the blue hyperbolic triangle with interior angles α, β, and γ. Once can surround the triangle with three $\frac{2}{3}$-ideal triangles with angles $\pi - \alpha$, $\pi - \beta$, and $\pi - \gamma$. The total region covered by the four triangles gives an ideal triangle, whose area is π.

Therefore,

$$\begin{aligned} \pi &= A + (\pi - (\pi - \alpha)) + (\pi - (\pi - \beta)) \\ &\quad + (\pi - (\pi - \gamma)) \\ &= A + (\alpha + \beta + \gamma) \\ A &= \pi - (\alpha + \beta + \gamma). \end{aligned}$$



Lobachevsky Function

The Lobachevsky function, $\Lambda(\theta) = -\int_0^\theta \ln|2\sin u|\, du$, can be related to the dilogarithm function,

$$\psi(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = -\int_0^z \frac{\ln(1-t)}{t}\, dt.$$

Let $z = e^{2i\theta}$, $t = e^{2i\phi}$, then $dt = 2it\, d\phi$. Also,

$$\begin{aligned}
\ln(1-t) &= \ln(1-e^{2i\phi}) \\
&= \ln(1 - cos2\phi - i\sin 2\phi) \\
&= \frac{1}{2}\ln((1-\cos 2\phi)^2 + \sin^2 2\phi) - i\tan^{-1}\left(\frac{\sin 2\phi}{1-cos2\phi}\right) \\
&= \frac{1}{2}\ln(2-2\cos 2\phi) - i\tan^{-1}\cot\phi \\
&= \ln(2\sin\phi) - i(\frac{\pi}{2} - \phi). \qquad (1)
\end{aligned}$$

Fourier Sine Series

$$
\begin{aligned}
\psi(e^{2i\theta}) - \psi(1) &= -\int_0^z \frac{\ln(1-t)}{t}\,dt \\
\psi(e^{2i\theta}) - \sum_{k=1}^{\infty}\frac{1}{k^2} &= -\int_0^{\theta}\left[\ln(2\sin\phi) - i(\frac{\pi}{2}-\phi)\right] 2i\,d\phi \\
\psi(e^{2i\theta}) - \zeta(2) &= -\pi\theta + \theta^2 + 2i\,\text{Л}(\theta) \\
\psi(e^{2i\theta}) = \sum_{k=1}^{\infty}\frac{e^{2ik\theta}}{k^2} &= \frac{\pi^2}{6} - \pi\theta + \theta^2 + 2i\,\text{Л}(\theta), \quad 0 \le \theta \le \pi. \qquad (2)
\end{aligned}
$$

Taking the imaginary part of each side gives the Fourier sine series.

$$
\text{Л}(\theta) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{\sin(2k\theta)}{k^2}
$$

Milnor's Convergent Series

Milnor claims that a better series is given by

$$\text{л}(\theta) = \theta \left(1 - \ln|2\theta| + \sum_{n=1}^{\infty} \frac{B_n (2\theta)^{2n}}{2n(2n+1)!} \right).$$

Here the B_n are the Bernoulli numbers, defined by Milnor as $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, etc. However, there is another convention using the generator

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \tag{3}$$

Then, $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}, \ldots$, and the remaining $B_n = 0$ for n odd. We'll use the latter which is also the definition in Maple.

Integrating the Laurent Series Expansion for $\cot z$

Next, we have the Laurent series expansion [See Equation (6).]

$$\cot z = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} z^{2n-1}. \tag{4}$$

Since $\int \cot z \, dz = \ln |\sin z| + C$, then

$$\ln |\sin z| = \ln |z| + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} \frac{z^{2n}}{2n} + c_1.$$

Integrating again

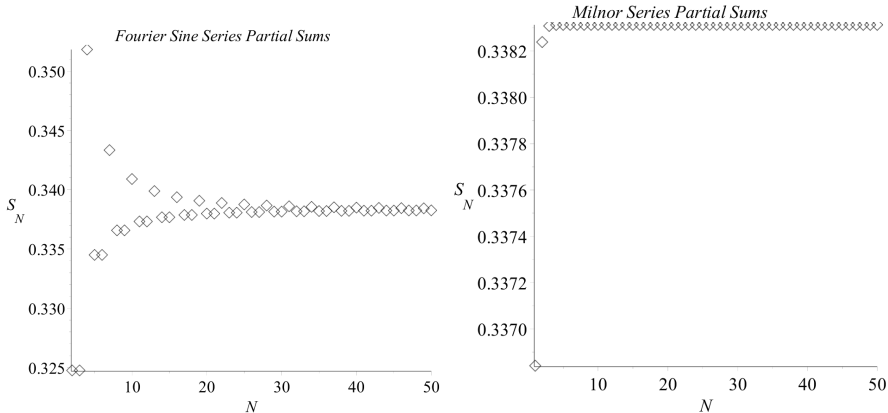
$$\int \ln |\sin z| \, dz = z \ln |z| - z + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n+1)!} \frac{z^{2n+1}}{2n} + c_1 z + c_2.$$

The Final Expansion

$$
\begin{aligned}
л(\theta) &= -\int_0^{\theta} \ln|2\sin z|\, dz \\
&= -\theta(\ln 2) - \int_0^{\theta} \ln|\sin z|\, dz \\
&= -\theta(\ln 2) - \theta\left[\ln|\theta| - 1 + \sum_{n=1}^{\infty}(-1)^n \frac{2^{2n}B_{2n}}{(2n+1)!}\frac{\theta^{2n}}{2n}.\right] \\
л(\theta) &= \theta\left(1 - \ln|2\theta| - \sum_{n=1}^{\infty}(-1)^n \frac{B_{2n}(2\theta)^{2n}}{2n(2n+1)!}\right).
\end{aligned}
$$

Here we have used $\lim_{z\to 0} z\ln|z| = 0$ and $\lim_{z\to 0}[\ln|\sin z| - \ln|z|] = 0$ in order to fix the integration constants $c_1 = c_2 = 0$.

Convergence of Series for $\theta = \frac{\pi}{3}$



Sum of first 50 terms in sum: Fourier series: 0.3382584185, Milnor series: 0.3383138686. Maple's computation using polylogarithm: 0.3383138689 and integration: 0.3383138692.

Laurent Series Expansion Derivation

$$\begin{aligned} \cot z = \frac{\cos z}{\sin z} &= i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}} \\ = i\frac{e^{2iz}-1+2}{e^{2iz}-1} &= i+\frac{2i}{e^{2iz}-1}. \end{aligned} \tag{5}$$

Multiplying by z, to obtain $z\cot z = iz + \frac{2iz}{e^{2iz}-1}$. Letting $x = 2iz$, the second term is the generator of Bernoulli numbers given by Equation (3). Therefore,

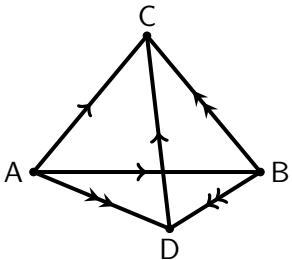
$$\begin{aligned} z\cot z &= iz + \sum_{k=0}^{\infty} B_k \frac{(2iz)^k}{k!} \\ &= B_0 + \sum_{n=1}^{\infty} B_{2n} \frac{(2iz)^{2n}}{(2n)!} \end{aligned} \tag{6}$$

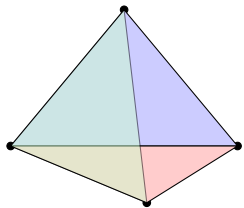
Dividing by z, we obtain Equation (4).

Maple Code to Compute Lobachevsky Function

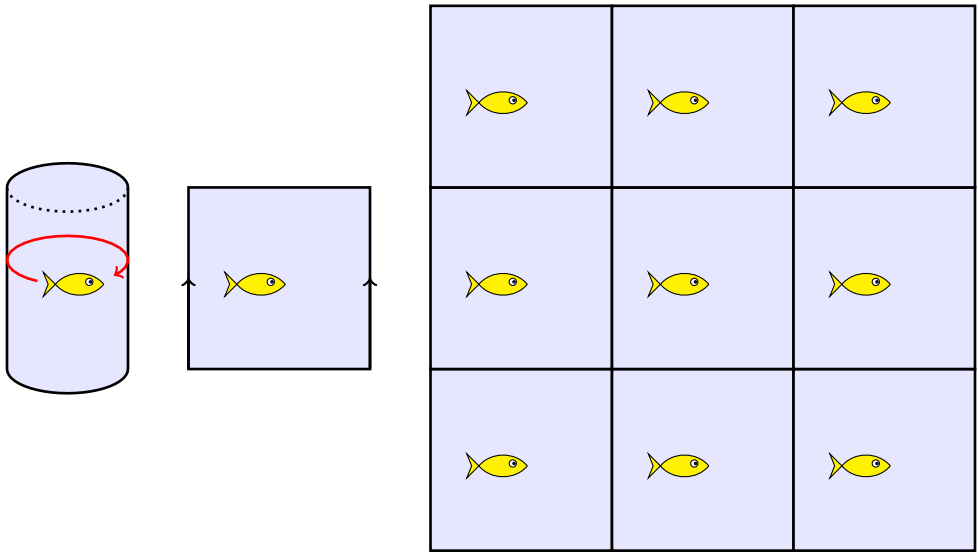
```
restart; with(plots);
% Sine Series
Lambda := (theta, N) -> (1/2)*(sum(sin(2*n*theta)/n^2, n = 1 .. N));
pointplot([seq([k, Lambda((1/3)*Pi, k)], k = 2 .. 50)], symbolsize = 20,
font = [times, roman, 30], title = 'Fourier Sine Series Partial Sums',
labelfont = [times, roman, 30],labels = ['N', 'S[N]']);

% Change digits for more accuracy
Digits := 10;
% Milnor Series
L := (theta, N) -> theta*(1-ln(2*theta)+sum((1/2)*(-1)^(n-1)*bernoulli(2*n)*(2*theta)^(2*n)
/(n*factorial(2*n+1)), n = 1 .. N));
pointplot([seq([k, L((1/3)*Pi, k)], k = 1 .. 50)], symbolsize = 20, symbolsize = 20,
font = [times, roman, 30], title = 'Milnor Series Partial Sums', labelfont = [times, roman, 30],
labels = ['N', 'S[N]']);

% Volumes
V1 := 6*evalf(L((1/3)*Pi, 50));    V2 := 6*evalf(Lambda((1/3)*Pi, 50));
V3 := 6*Im(.5*evalf(polylog(2, exp((1/3)*(2*I)*Pi))-(1/6)*Pi^2+(1/3)*Pi*(Pi-(1/3)*Pi)));
-(int(ln(2*sin(u)), u = 0 .. (1/3)*Pi)); V4 := Re(evalf(6*\%));
V4-V1;    V4-V2;
```

Sandbox





Flat Cylinder



Flat Torus

