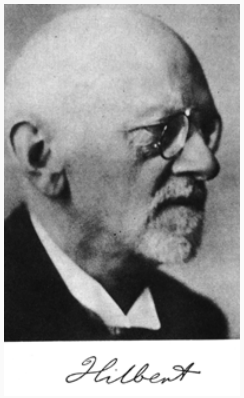
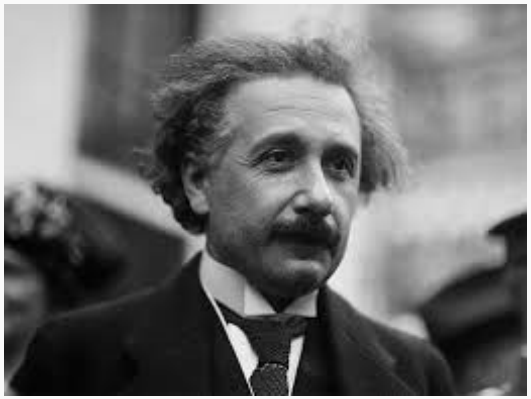
The Early Twentieth Century and Beyond

Fall 2026 - R. L. Herman





Outline

Rigor and Foundations

Nicolas Bourbaki

Mathematics and Physics

Great Problems

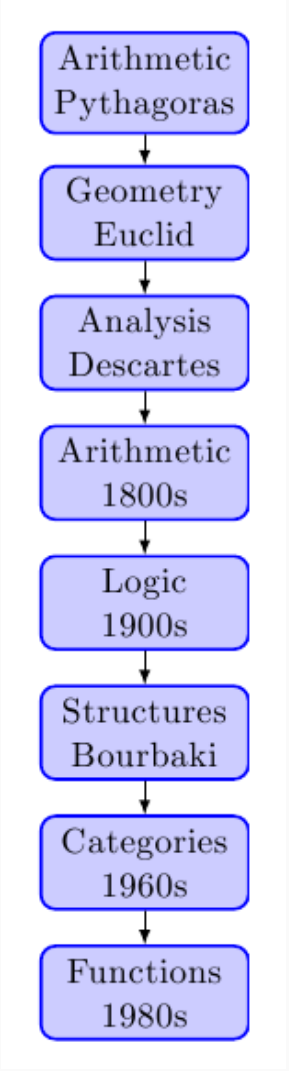
Mathematical Prizes

AI and Mathematical Discovery

Looking Back and Ahead

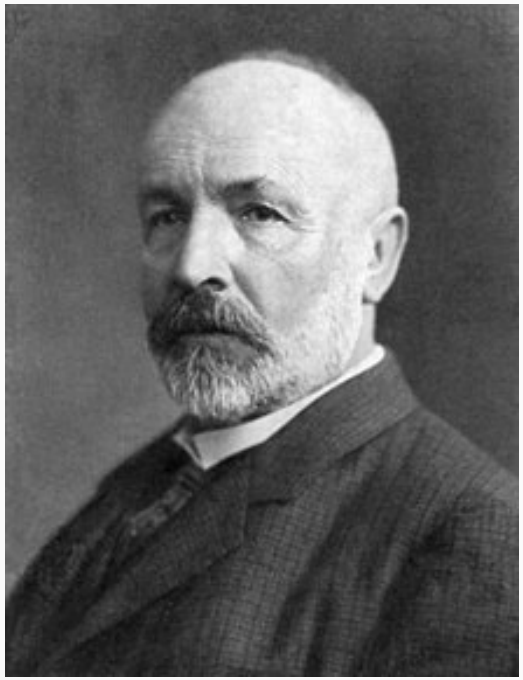
Changing Ideas of Mathematics

- Pythagoreans — arithmetic
- Euclid — geometry
- Descartes — analytic geometry
- Newton — calculus and mechanics
- Euler — number theory and applications
- Gauss — geometry, including non-Euclidean ideas
- Cantor — sets and infinity
- Frege — predicate logic
- Russell and Whitehead — *Principia Mathematica*
- Gödel — limits of formal axiomatic systems



Georg Cantor (1845–1918)

- Set theory grew in part from questions in Fourier series.[a]
- Sets can be compared by *cardinality* (one-to-one correspondence).
- Natural numbers and rational numbers are countable; real numbers are not.
- The *continuum hypothesis* asserts there is no cardinality strictly between $\aleph_0$ and $|\mathbb{R}|$.



A lasting question
Does every infinity have the same size? No.

[a] How did Cantor Discover Set Theory and Topology?
– Connection to Fourier series.

Hilbert's Grand Hotel: A Paradox of Infinity



Imagine infinitely many numbered rooms, all occupied.

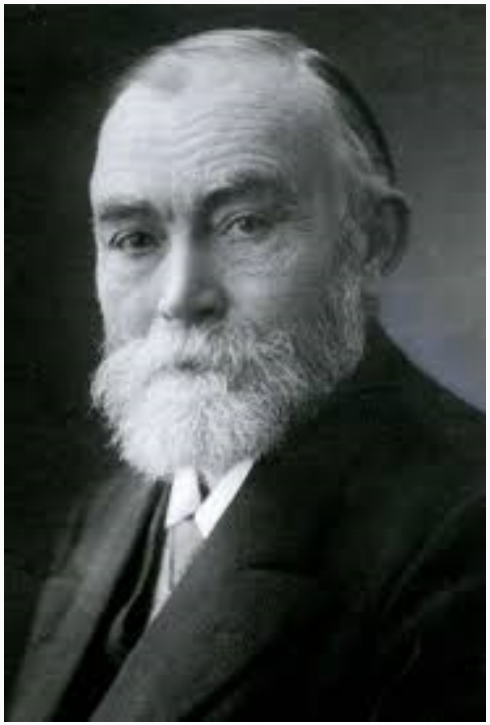
1. Can one additional guest be accommodated?
   Move guest n to room $n+1$.
2. How about infinitely many new guests? Move guest n to room $2n$.
3. How about infinitely many buses, each with infinitely many guests?
   Countably many countable groups are countable.

The hotel was popularized by David Hilbert (1862-1923) in the 1920s.

Gottlob Frege (1848–1925)

Frege developed modern formal predicate logic, foundational to subsequent work in:

- Russell and Whitehead's *Principia Mathematica* (1910–1913),
- The foundations of set theory,
- Kurt Gödel's incompleteness theorems,
- Alfred Tarski's work on formal definitions of truth.



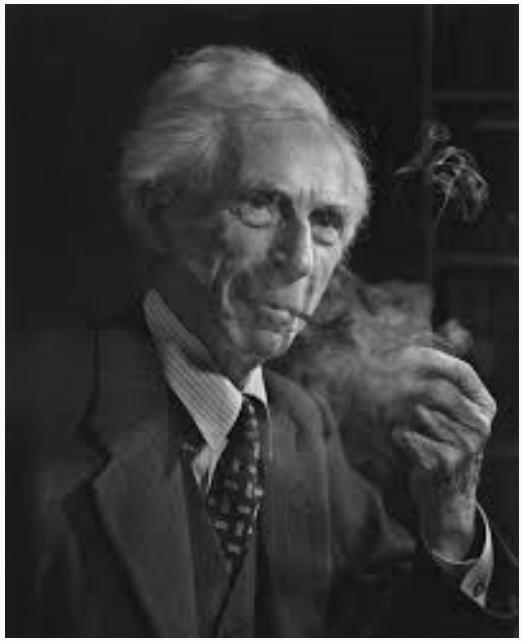
Bertrand Russell (1872–1970): Paradox

The liar paradox
Consider: "This sentence is false."

Russell's paradox
Let $R = \{x : x \notin x\}$, the proposed set of all sets that are not members of themselves.

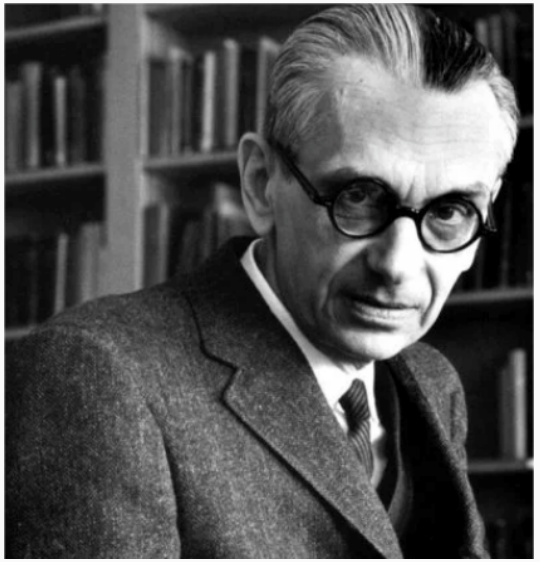
Would $R \in R$? Either answer leads to a contradiction.

- Naive unrestricted set formation fails.
- Axiomatic approaches such as Zermelo–Fraenkel set theory avoid this construction.



Kurt Gödel (1906–1978)

- **1931:** first incompleteness theorem. Any consistent, effectively axiomatized formal system strong enough for elementary arithmetic is incomplete.
- Second incompleteness theorem: such a system cannot prove its own consistency, assuming it is consistent.
- Joined the Institute for Advanced Study in Princeton in 1940, where he befriended Einstein.
- In 1949 he investigated rotating cosmological solutions with closed timelike curves.



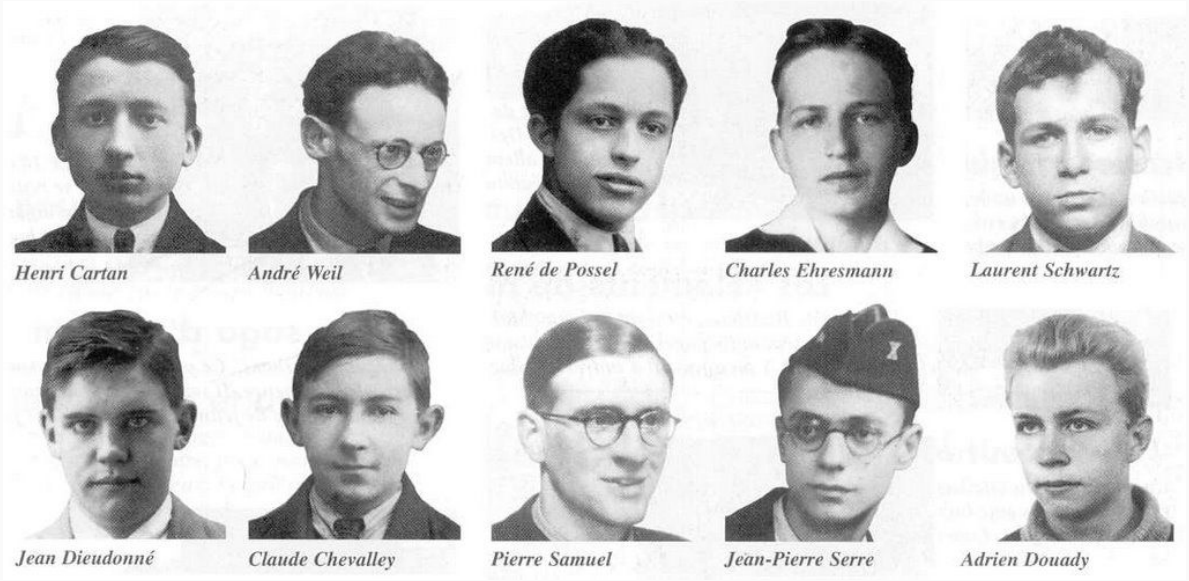


Nicolas Bourbaki (1935 -):

Founded in France in 1935 by a group of young mathematicians including:

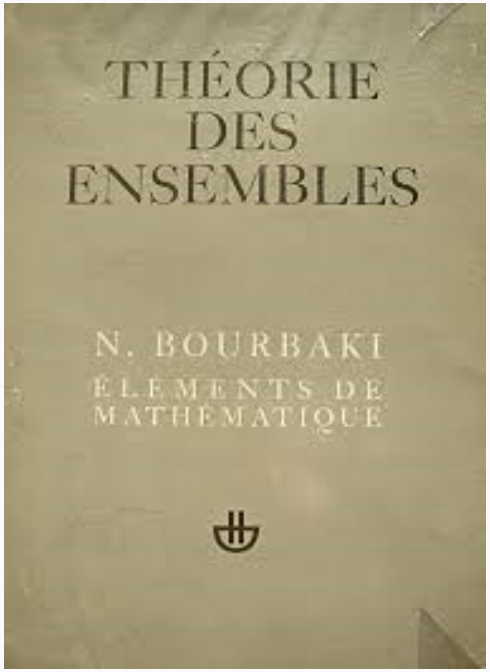
- Henri Cartan,
- Claude Chevalley,
- Jean Delsarte,
- Jean Dieudonné,
- Charles Ehresmann,
- René de Possel,
- Szolem Mandelbrojt, and
- André Weil.

Later contributors and associates included Laurent Schwartz, Jean-Pierre Serre, Alexander Grothendieck, and others.



Nicolas Bourbaki: A Collective Pseudonym

- Began with dissatisfaction over available analysis textbooks. (Henri Cartan complained to Weil).
- Fall of French mathematics - France's losses during World War I had affected an entire generation of mathematicians.
- The rise of German Mathematics (and physicists).
- They sought to write better analysis texts.
- A planned analysis textbook expanded into the multi-volume *Éléments de mathématique*, published beginning in 1939.



The Bourbaki Project

- Bourbaki – secret society, named for past French general (Mathematician who never was).
- Original plan -one volume of 1000 pages, collectively written, – on analysis.
- Continued far beyond its original scope.
- By 1967: 10 books in several volumes, over 60 chapters: Set Theory, Algebra, General Topology, Real Analysis, Topological Vector Spaces, Integration, Commutative Algebra, Varieties, Lie Groups and Algebras, Spectral Theory. index, website
- In 2016 - Algebraic Topology;
- 2019 Revised Spectral Theory.
- 2023 and 2026 ...

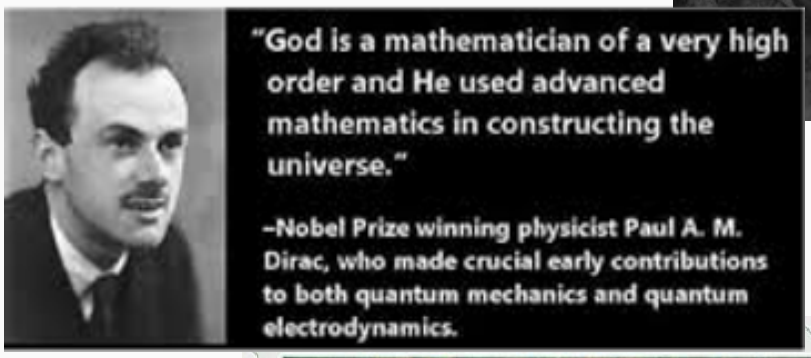


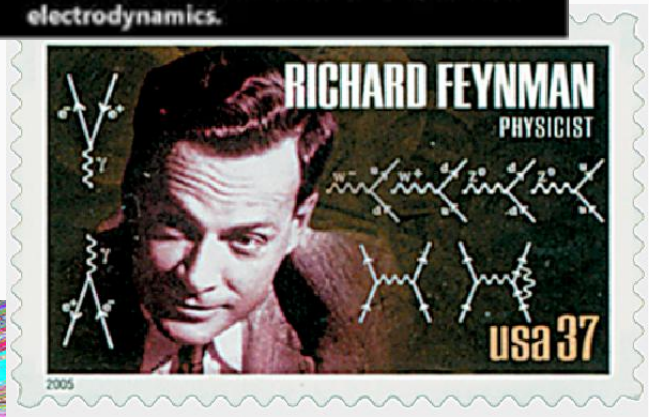
Mathematical and Physical Revolutions

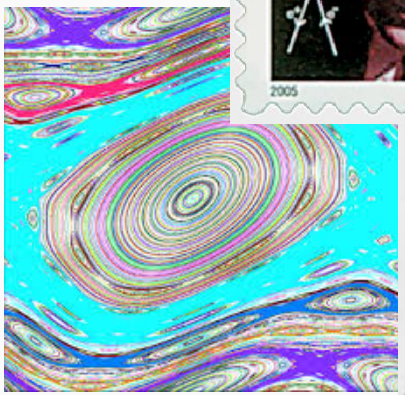
- Quantum theory
- Special and general relativity
- Higher-dimensional theories
- Quantum electrodynamics
- String theory
- Nonlinear dynamics and chaos
- Computation and the information age

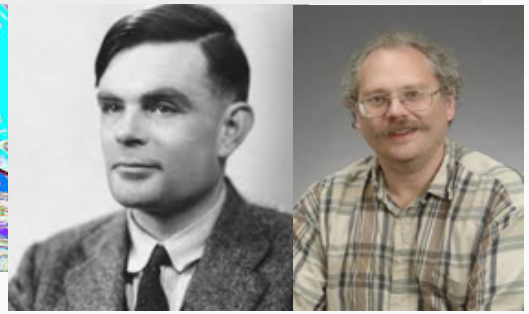
We turn to higher dimensions ...









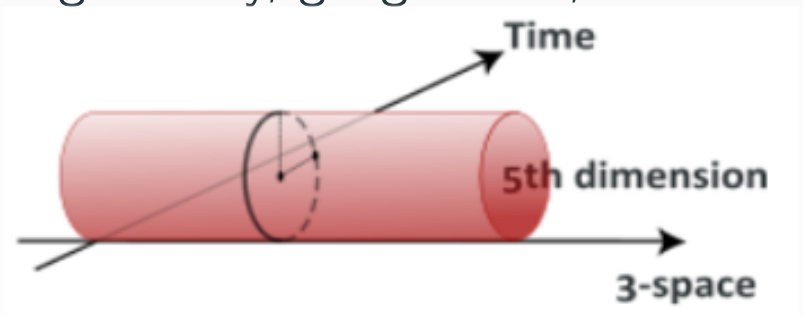


Kaluza–Klein: The Search for Extra Dimensions

- Einstein's special relativity (1905) and Minkowski spacetime (1908): four-dimensional spacetime.
- General relativity (1915): gravity as spacetime geometry.
- Theodor Kaluza (1921): five-dimensional extension aimed at unifying gravity and electromagnetism.
- Oskar Klein (1926): the fifth dimension may be compact and microscopic.
- Later connections: geometry, gauge fields, and fiber bundles.


Theodor Kaluza

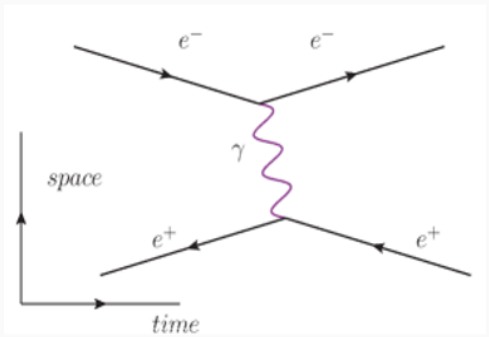

Oskar Klein



Feynman Diagrams and Quantum Electrodynamics

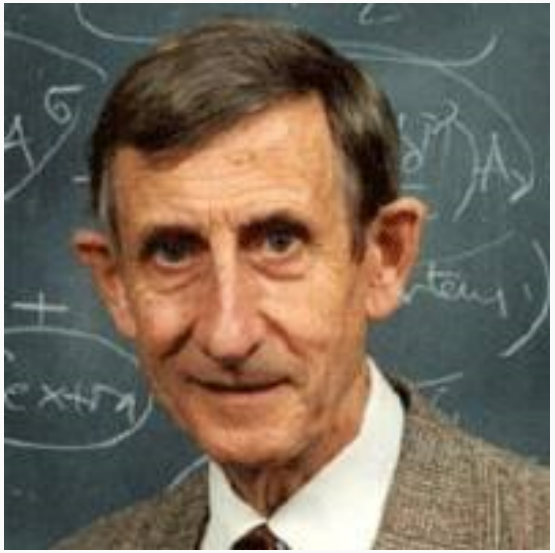
- Richard Feynman (1918–1988) developed diagrammatic methods for calculations in quantum electrodynamics (QED).
- QED describes the interaction of charged particles and electromagnetic radiation.
- Nobel Prize in Physics (1965): Feynman, Julian Schwinger, and Sin-Itiro Tomonaga.
- Freeman Dyson showed the equivalence of major formulations; he did not share the prize.





Freeman Dyson: A Mathematical "Divorce"?

> *I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has ended in divorce.*
>
> — Freeman Dyson, Gibbs Lecture (1972)



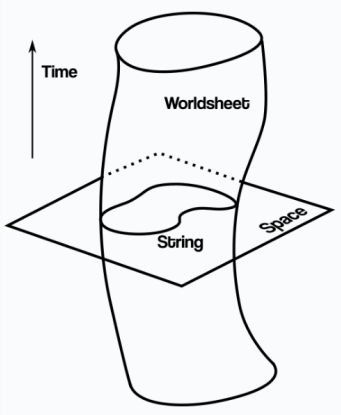
Question

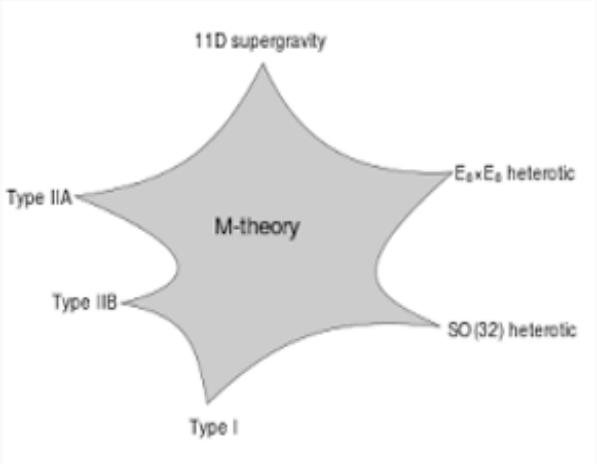
Did later developments restore the dialogue between mathematics and physics?

String Theory

- Gabriele Veneziano's 1968 amplitude used Euler's beta function.
- Fundamental objects modeled as one- dimensional strings rather than point particles.
- Bosonic string theory: 26 spacetime dimensions.
- Superstring theories: 10 spacetime dimensions.
- M-theory: an 11-dimensional framework.
- Extra dimensions are often proposed to be compactified.

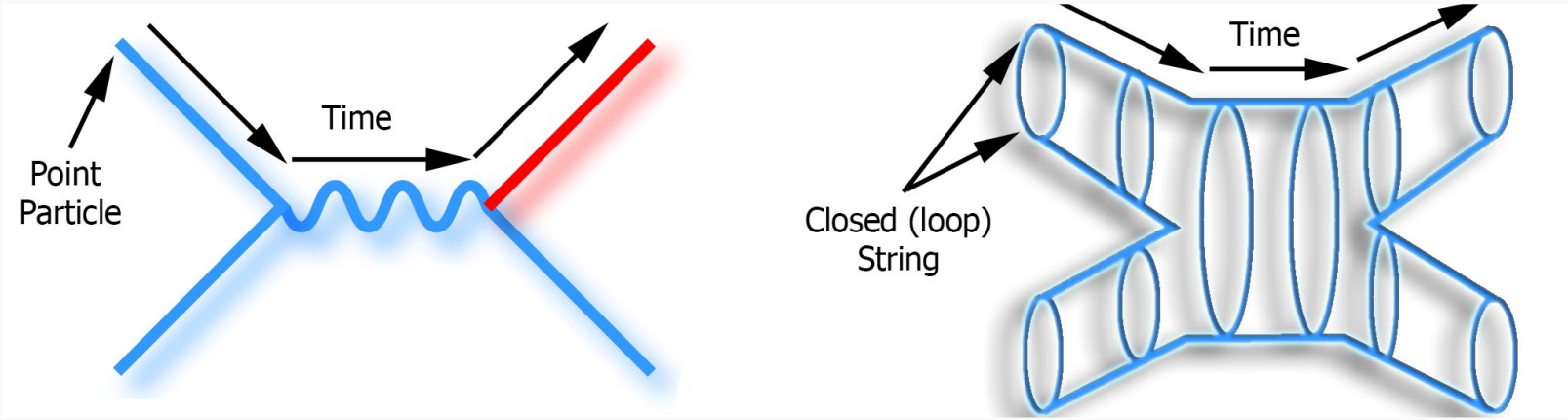
$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$





From Feynman Diagrams to Strings

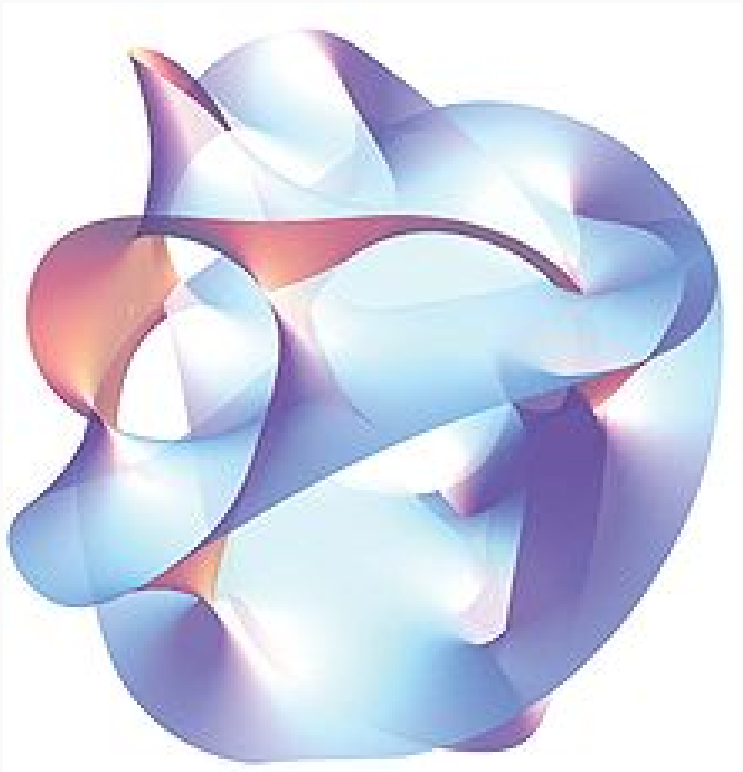
- Worldlines become worldsheets.
- Geometry and topology enter particle interactions.
- Knot theory, surfaces, and differential geometry become important mathematical tools.



Calabi–Yau Manifolds

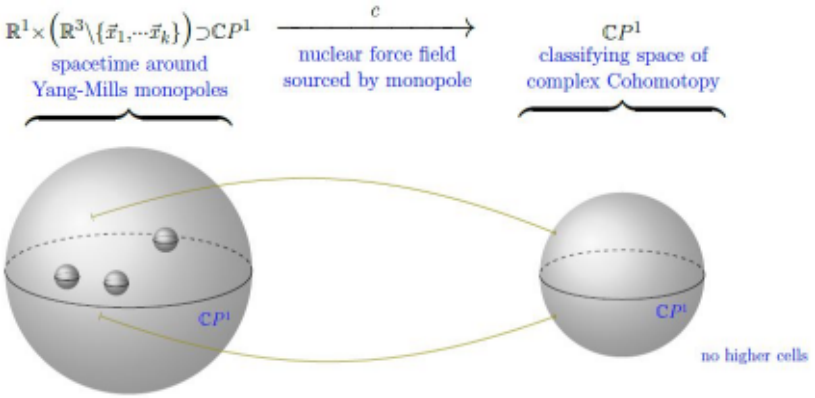
- Extra dimensions may be curled into compact geometric spaces.
- Calabi–Yau manifolds are among the geometries studied in superstring compactification.
- Named for Eugenio Calabi and Shing-Tung Yau.
- A striking meeting point of geometry and theoretical physics.

$$x_1^5+x_2^5+x_3^5+x_4^5+\psi x_1x_2x_3x_4 = 1.$$

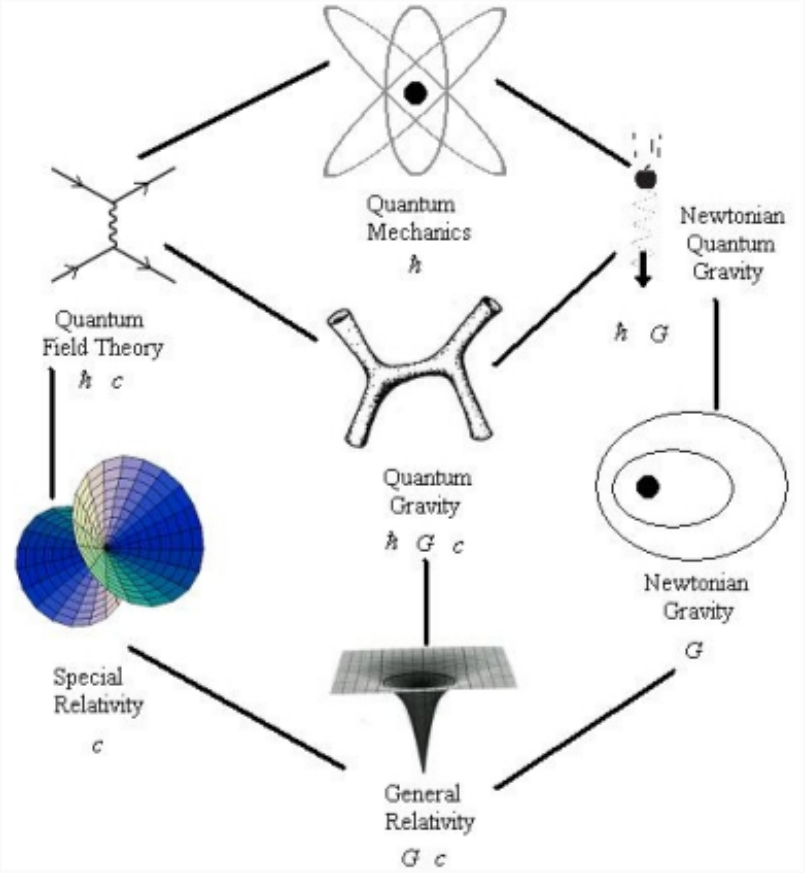


Mathematics and Physics: Not Divorced

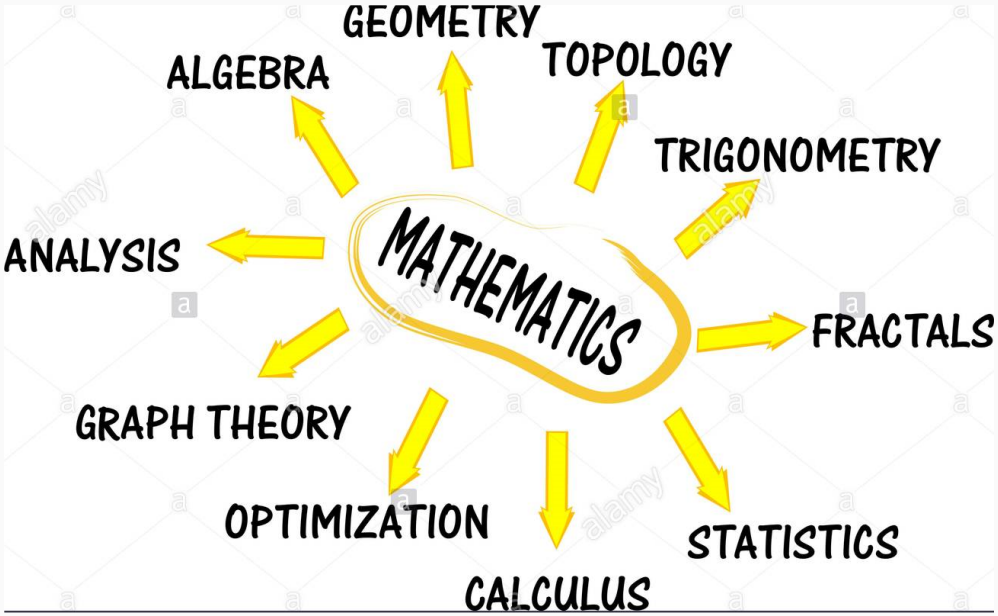
- Michael Atiyah (1929–2019) helped develop connections among geometry, topology, and mathematical physics.
- Gauge theory and the Atiyah–Singer index theorem exemplify the interaction.
- The relationship between mathematics and physics continued to generate new questions and structures.



Atiyah-Hitchin charge quantization – The moduli space of SU(2) Yang-Mills monopoles is the cocycle space of complex-rational Cohomotopy of any sphere enclosing them.



Great Problems and Prizes



David Hilbert (1862–1943)

- At the 1900 International Congress of Mathematicians in Paris, Hilbert presented an influential selection of problems.
- His complete published list contained 23 problems.
- They became a research agenda for mathematics in the twentieth century.
- Some have been resolved, some remain open, and some have proven too broad or ambiguous for a simple yes/no answer.



Hilbert's Problems 1–6

1. The continuum hypothesis
2. Consistency of arithmetic
3. Equality of volumes of certain tetrahedra
4. Straight line as shortest distance
5. Continuous transformation groups and differentiability
6. Axiomatization of physics

Their modern status is nuanced: several questions require qualification.

Hilbert's Problems 7–14

7. Irrationality and transcendence
8. Distribution of prime numbers
9. General reciprocity laws
10. Algorithm for Diophantine equations
11. Quadratic forms over number fields
12. Generalizing Kronecker's theorem
13. Solutions of the general seventh-degree equation
14. Finiteness of systems of algebraic functions

Hilbert's Problems 15–23

15. Rigorous foundation for Schubert's calculus
16. Topology of algebraic curves and surfaces
17. Nonnegative forms as sums of squares
18. Space-filling with congruent polyhedra
19. Analyticity of regular variational solutions
20. General boundary-value problems
21. Differential equations with prescribed monodromy
22. Uniformization by automorphic functions
23. Further development of the calculus of variations

Fields Medal

- Named for John Charles Fields.
- Normally awarded every four years at the International Congress of Mathematicians.
- Awarded to mathematicians who have not reached their 40th birthday before January 1 of the award year.
- Typically two to four medals awarded.
- Motto: *Transire suum pectus mundoque potiri* ("Rise above oneself and grasp the world").



See Fields Medalists

Abel Prize





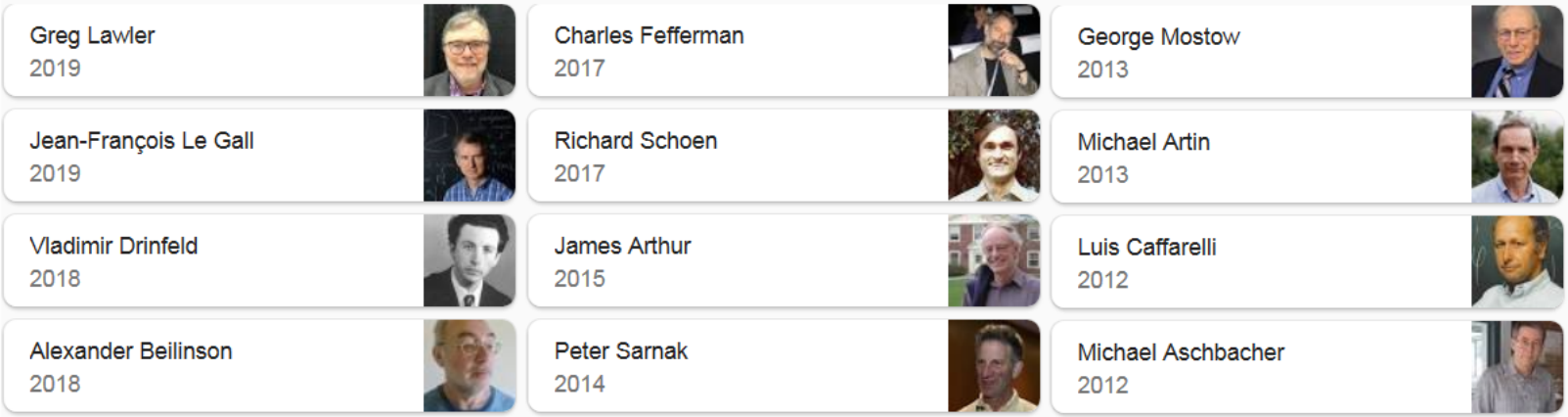
- Named for Niels Henrik Abel (1802–1829).
- A prize in Abel's honor was advocated around 1899, notably by Sophus Lie.
- First awarded in 2003.
- An annual international prize recognizing outstanding mathematical achievements.



See Laureates

Wolf Prize in Mathematics

- Established by the Wolf Foundation in Israel.
- First awarded in 1978.
- Recognizes outstanding scientific and artistic achievements.
- Mathematics is one of the Wolf Prize fields.
  - Agriculture, Chemistry, Medicine, Physics and Arts
- See Winners



The Millennium Prize Problems

The Clay Mathematics Institute announced seven problems in 2000, each carrying a $1 million prize.



- Birch and Swinnerton-Dyer conjecture
- Hodge conjecture
- Navier–Stokes existence and smoothness
- P versus NP
- Poincaré conjecture — **proved** by Grigori Perelman
- Riemann hypothesis
- Yang–Mills existence and mass gap

From Hilbert's Problems to AI-Assisted Research

1900: Hilbert
Twenty-three problems help set the agenda for twentieth-century mathematics.

2000: Millennium Problems
Seven problems highlight major questions still open at the turn of the century.

2026: A change in method
AI systems contribute conjectures, counterexamples, and proposed proofs to active research.

New question
How should mathematicians verify, interpret, and credit AI-generated arguments?

The examples that follow are a historical snapshot: November 21, 2026.

AI and Mathematical Research: May–October 2026

May An OpenAI system produced a counterexample to an Erdős unit-distance conjecture.

August OpenAI described ten advances in mathematics and theoretical computer science, with proofs prepared for independent scrutiny.

September An OpenAI system announced a finite-time singularity construction for *forced* three-dimensional Navier–Stokes equations, accompanied by Lean formalization.

October AI-generated papers included arguments for a *quasi-Riemann hypothesis*, weaker than the Riemann hypothesis.

Sources: OpenAI (Aug. 1, Sept. 8, Oct. 2026); *Scientific American* (Sept. 21, 2026).

Why the Navier–Stokes Qualification Matters

The Navier–Stokes equations describe viscous fluid motion:

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + f, \qquad \nabla \cdot u = 0.$$

- In September 2026, an AI-generated construction exhibited finite-time blowup with a specially chosen *external force* f.
- This relates to a permitted formulation of the Clay problem, but is not a resolution of the central *unforced* regularity question $f = 0$.
- The distinction illustrates why the *precise statement* of a theorem matters as much as a machine-checked proof.

Sources: OpenAI (Sept. 8, 2026); *Scientific American* (Sept. 21, 2026).

The Riemann Hypothesis: A Remarkable Claim

Riemann's 1859 conjecture asserts that every nontrivial zero of the analytically continued zeta function satisfies

$$\boxed{\zeta(s) = 0 \quad \Longrightarrow \quad \Re(s) = \tfrac{1}{2} \quad \text{(nontrivial zeros).}}$$

- **November 20, 2026:** Levent Alpöge announced that he and Claude had discovered a proof, extending to Dirichlet L-functions (a generalized Riemann hypothesis).
- He released roughly **17 million lines of Lean code**, rather than a conventional mathematical exposition.
- As of November 21: The announcement remains a claim while independent checking and mathematical interpretation proceed.

Sources: L. Alpöge, *Remark* (Nov. 20, 2026); *Quanta Magazine* (Nov. 20, 2026).

Formal Verification Is Not the Whole Story

What Lean can check

Given a precise formal statement, definitions, axioms, and a complete proof term, Lean's kernel checks that the conclusion follows by its logical rules.

What humans must still assess

Does the formal statement match the intended conjecture?

Are the definitions and assumptions appropriate?

What mathematical ideas explain the result?

Historical echo

Russell and Whitehead sought formal foundations in *Principia Mathematica*. A century later, some proposed proofs are so large that computers, rather than readers, check their details.

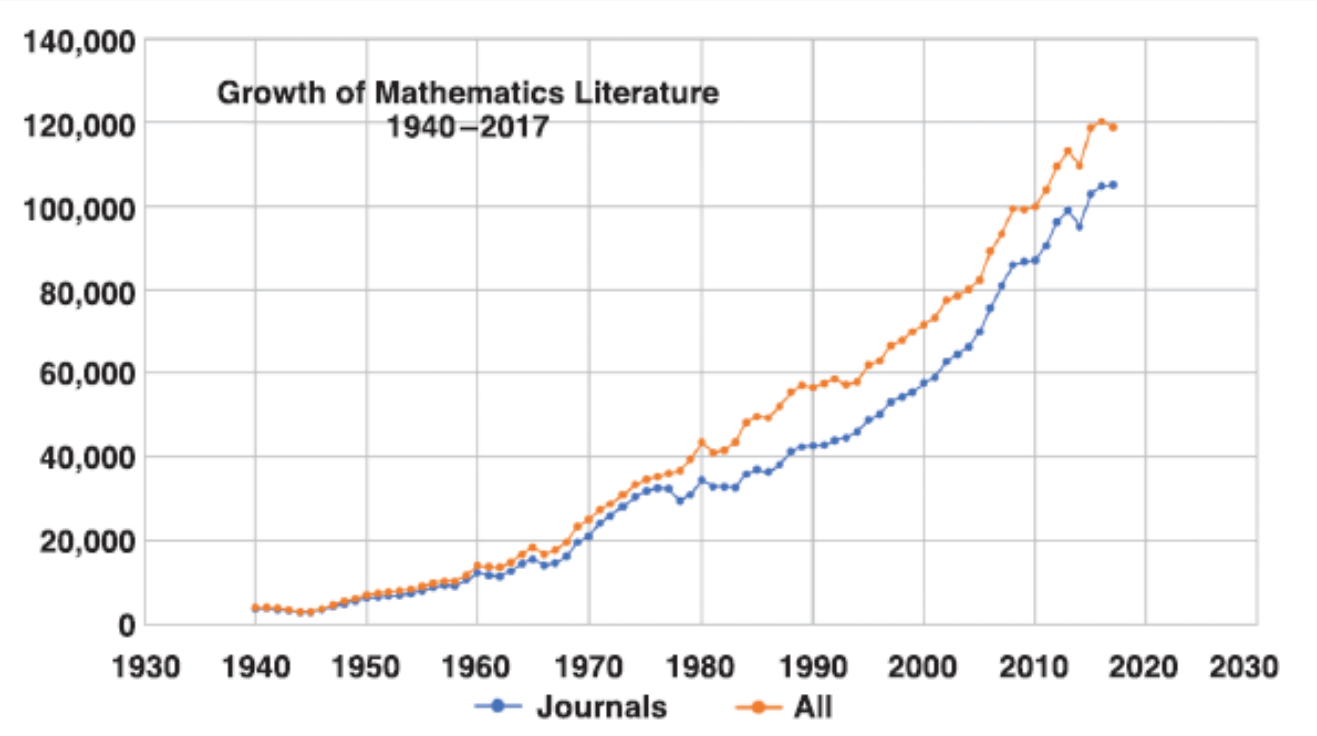
A Question for the Next Generation

Is the purpose of mathematics merely to establish that something is true, or also to understand *why* it is true?

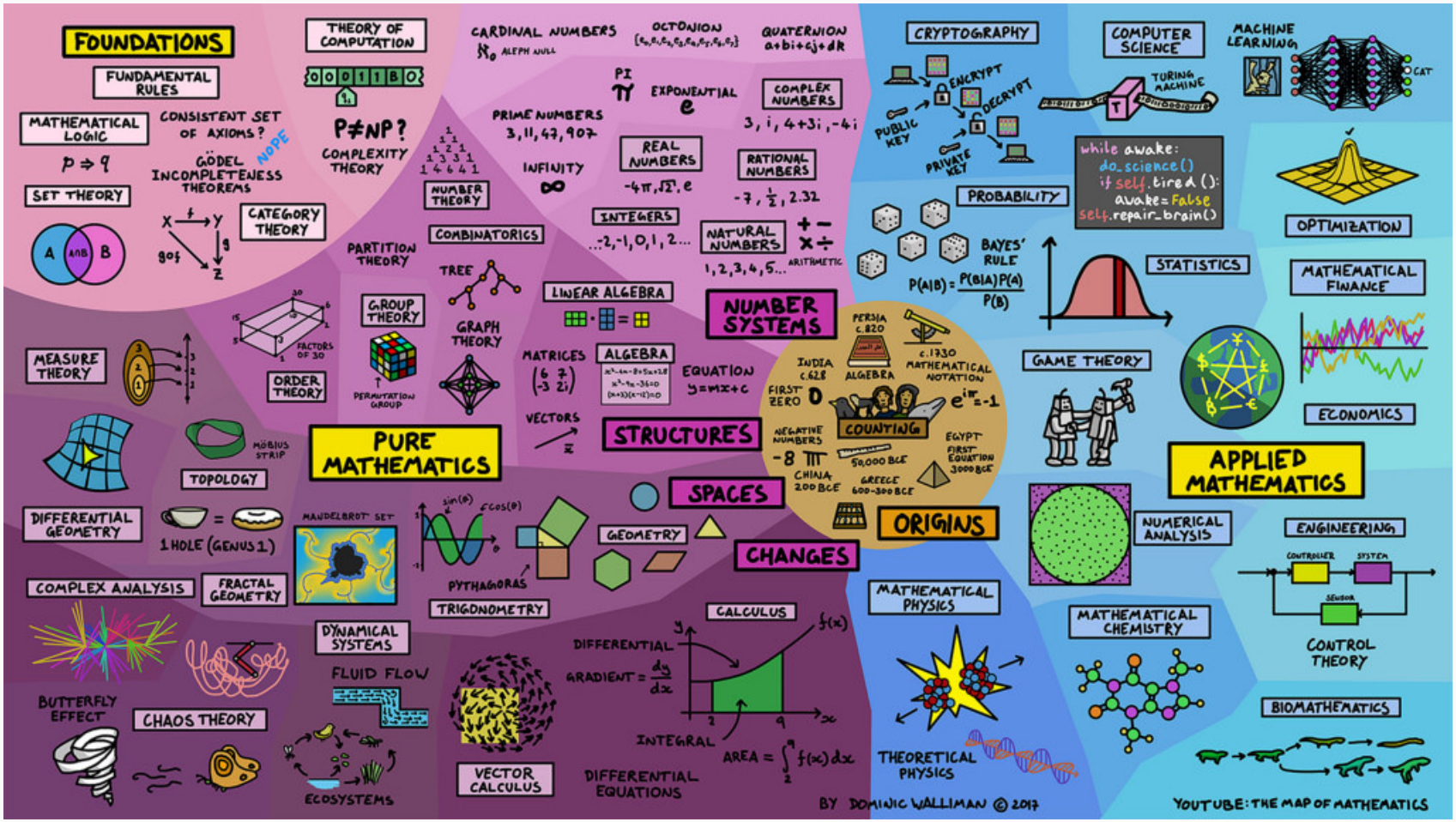
- Discovery, verification, and understanding are related but distinct.
- AI changes how arguments may be found; it does not eliminate the need for mathematical judgment.
- Hilbert's questions remain relevant—even when the tools used to investigate them change radically.

Historical snapshot: November 21, 2026

Mathematics Publications



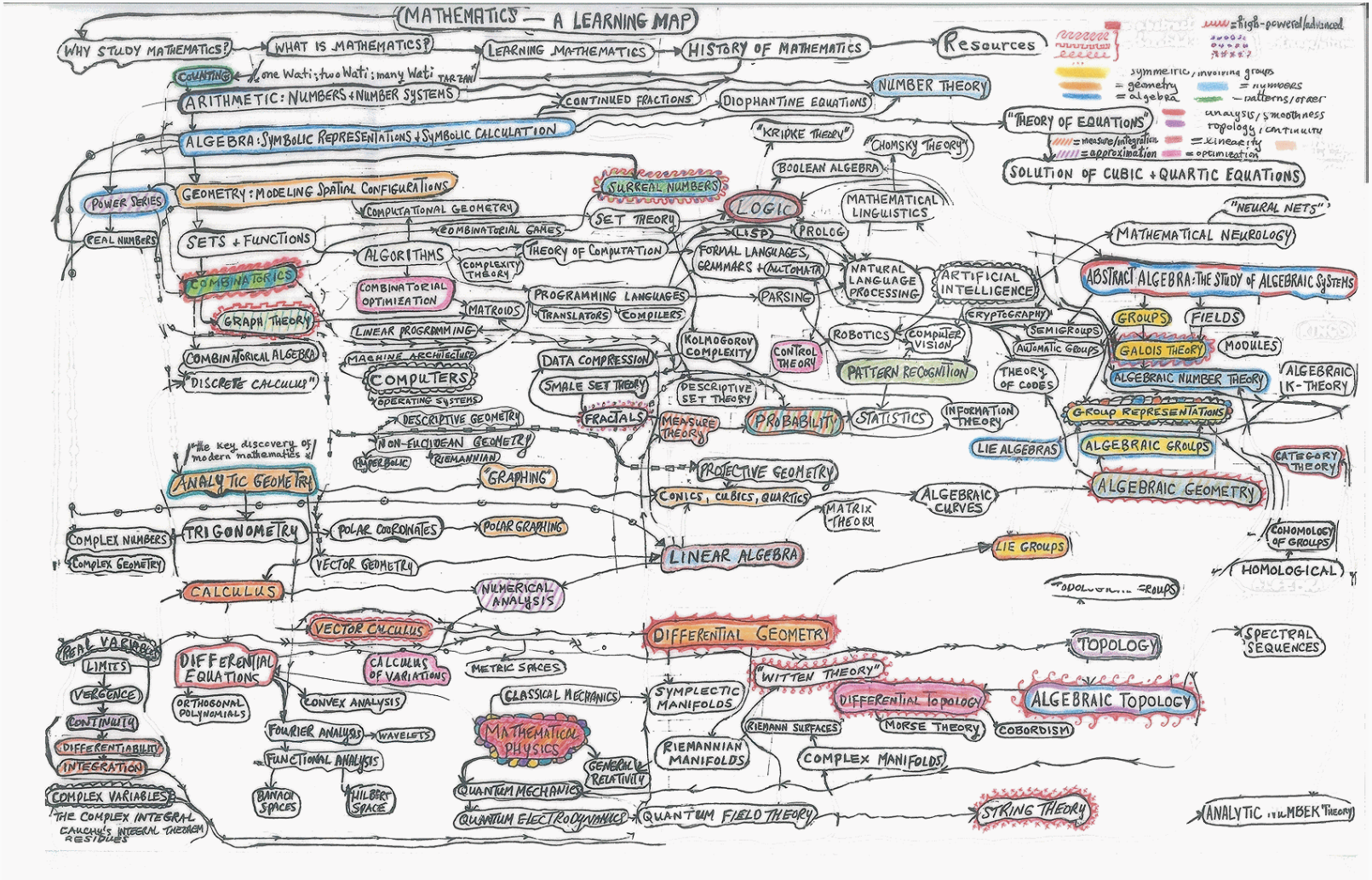
Mind Maps of Mathematics: One Perspective



https://www.sciencealert.com/

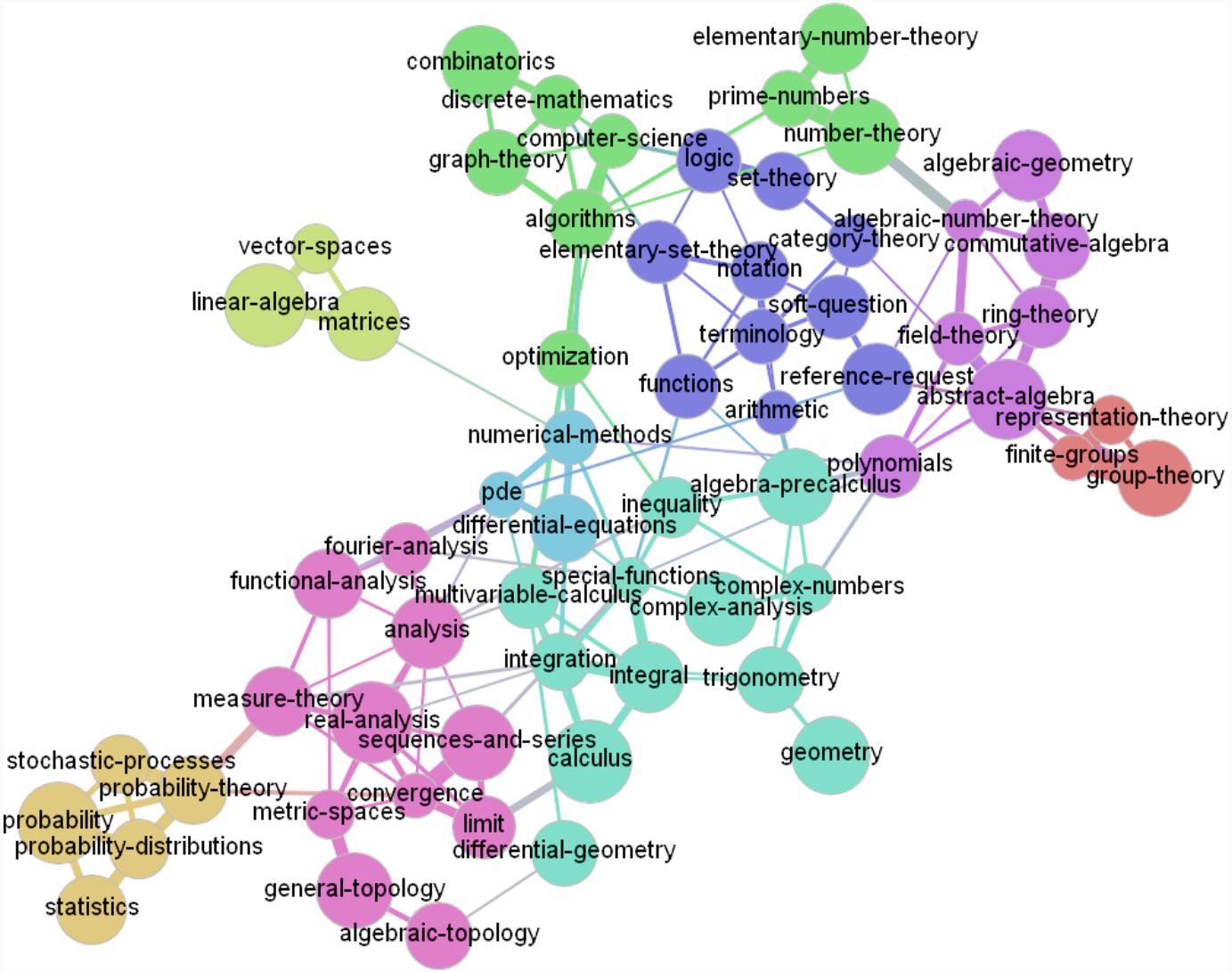
this-mind-boggling-map-explains-how-everything-in-mathematics-is-connected-3

Maps of Mathematical Fields: Another Perspective

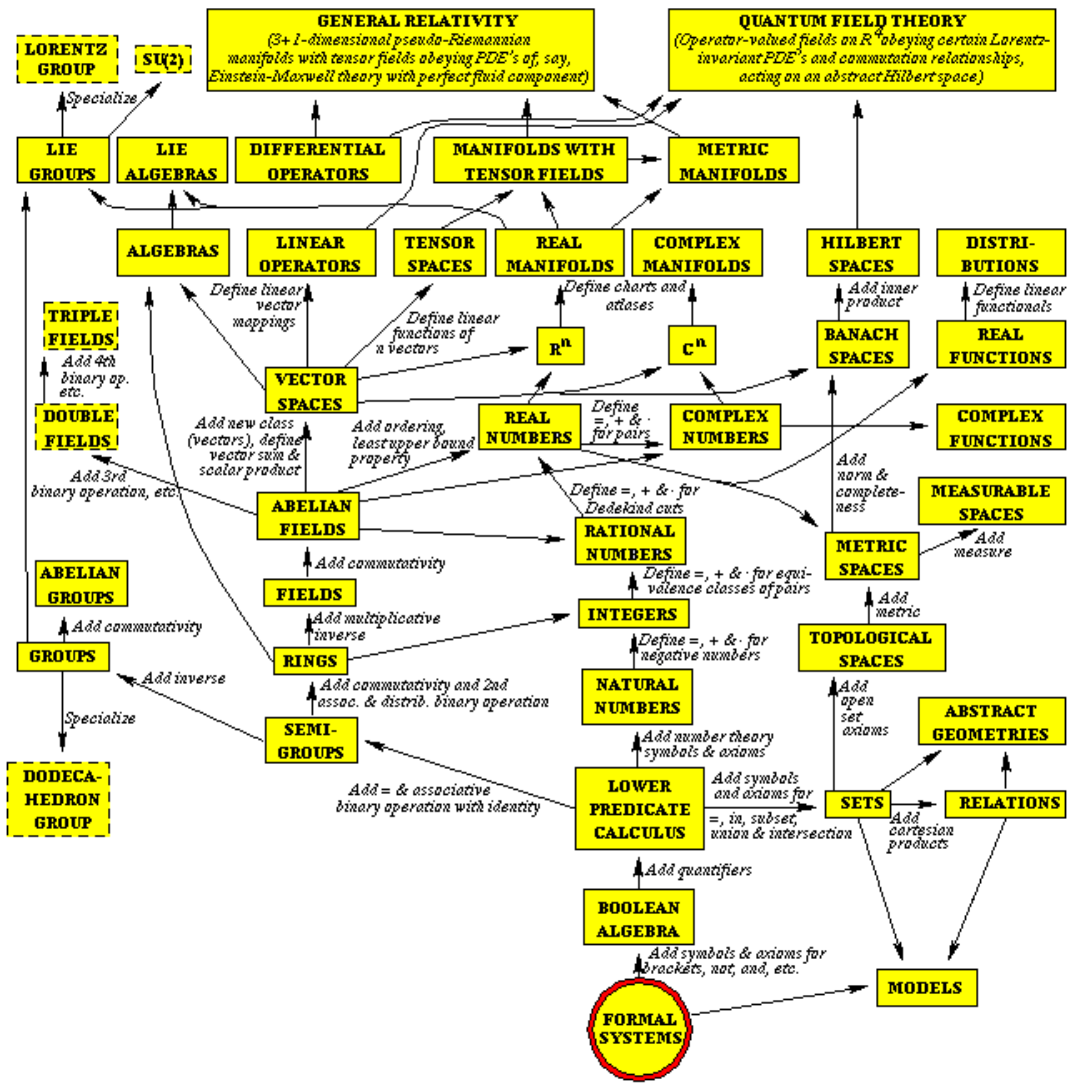


https://math.stackexchange.com/questions/166862/detailed-diagram-with-mathematical-fields-of-study

The Landscape of Mathematics



More Connections



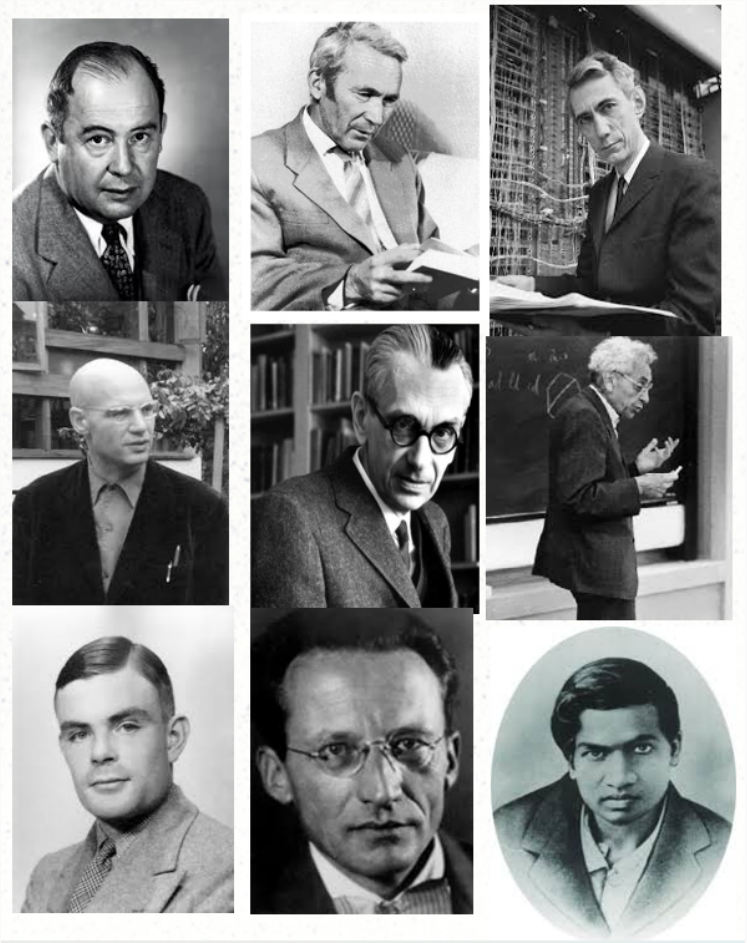
Numerical Algorithms that Changed Computing

1. Newton and quasi-Newton methods
2. Matrix factorizations: LU, Cholesky, QR
3. SVD, QR, and QZ algorithms
4. Monte Carlo methods
5. Fast Fourier transform
6. Krylov subspace methods
7. Image compression (JPEG)
8. PageRank
9. Simplex algorithm
10. Kalman filter

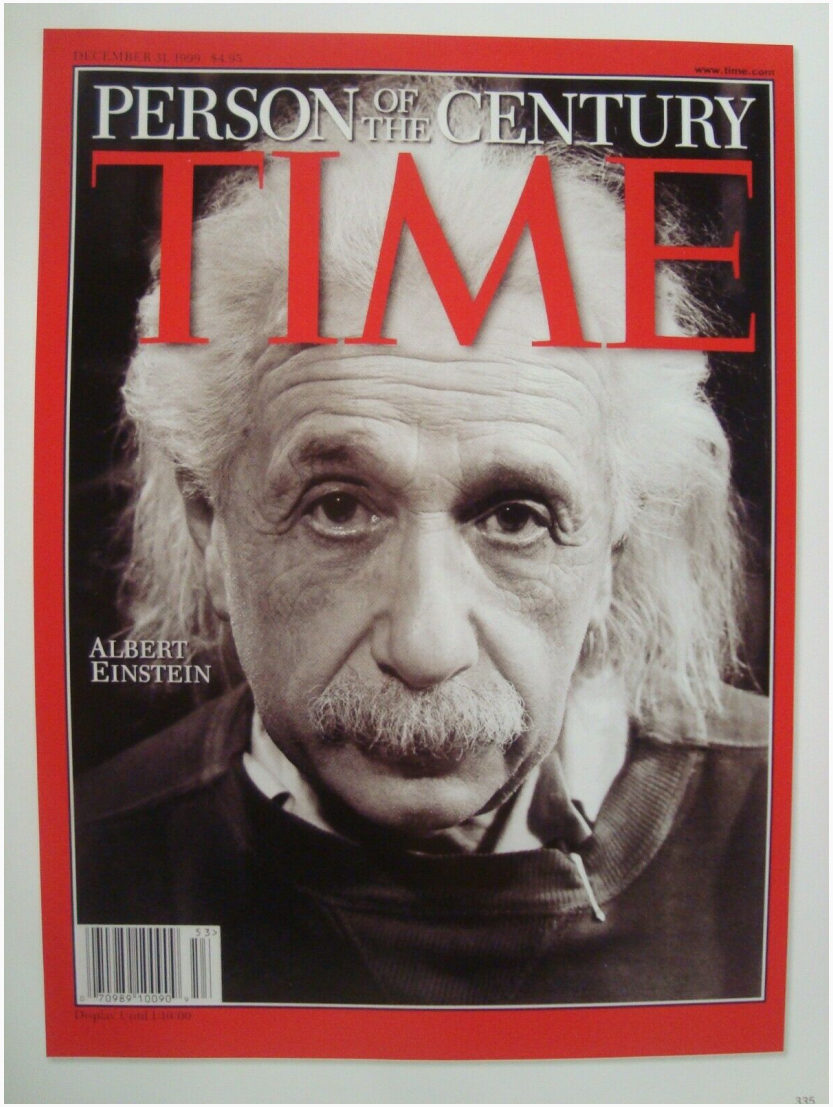
Illustrative list, drawing on discussions of influential numerical algorithms (2000 and 2015), not a definitive ranking.

Great Mathematicians of the Twentieth Century?

- John von Neumann
- Andrey Kolmogorov
- Claude Shannon
- Alexander Grothendieck
- Kurt Gödel
- Paul Erdős
- Alan Turing
- Hermann Weyl
- Srinivasa Ramanujan



Person of the Century?



Good Luck on Your Finals!

