

Introducing Elliptic Functions via Simulations of Nonlinear Differential Equations

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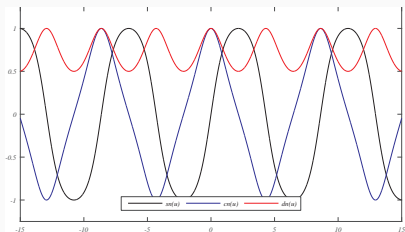


Table of Contents

1. Nonlinear Pendulum
2. Jacobi Elliptic Functions
3. Simulink and ODEs
4. ODE Examples
5. Trig-Elliptic Systems

Nonlinear Pendulum

The Nonlinear Pendulum

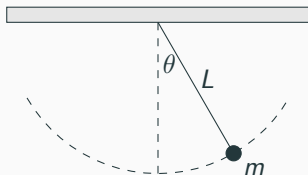


Figure 1: A point mass m is attached to a string of length L and released from rest at $\theta = \theta_0$.

$$\ddot{\theta} + \omega^2 \sin \theta = 0. \quad (1)$$

$$\ddot{\theta} + \omega^2 \sin \theta = 0$$

Multiply Equation (1) by $\dot{\theta}$,

$$\ddot{\theta}\dot{\theta} + \omega^2 \sin \theta \dot{\theta} = 0,$$

and note

$$\frac{d}{dt} \left[\frac{1}{2} \dot{\theta}^2 - \omega^2 \cos \theta \right] = 0.$$

Therefore,

$$\frac{1}{2} \dot{\theta}^2 - \omega^2 \cos \theta = c. \quad (2)$$

Using the initial conditions, $\theta(0) = \theta_0$, $\dot{\theta}(0) = 0$, we have

$$c = -\omega^2 \cos \theta_0 \quad \text{and}$$

$$\dot{\theta}^2 = 2\omega^2(\cos \theta - \cos \theta_0).$$

Second Order ODE from $\dot{\theta}^2 = 2\omega^2(\cos\theta - \cos\theta_0)$

$$\dot{\theta}^2 = 4\omega^2 \left(\sin^2 \frac{\theta}{2} - \sin^2 \frac{\theta_0}{2} \right)$$

Let $kx = \sin \frac{\theta}{2}$, then $2k\dot{x} = \sqrt{1 - k^2x^2} \dot{\theta}$. Then,

$$\dot{x}^2 = \omega^2(1 - x^2)(1 - k^2x^2) \quad (3)$$

Differentiating,

$$2\dot{x}\ddot{x} = \omega^2\dot{x} \left[-2x(1 - k^2x^2) + (1 - x^2)(-2k^2x) \right],$$

yields

$$\ddot{z} = \omega^2 \left[2k^2z^3 - (1 + k^2)z \right] \quad (4)$$

Solution: $x(t) = \text{sn}(\omega t, k)$. - a Jacobi elliptic function

Jacobi Elliptic Functions

Jacobi Elliptic Functions:

$$x(t) = \operatorname{sn}(t, k), \quad y(t) = \operatorname{cn}(t, k) \quad z(t) = \operatorname{dn}(t, k).$$

Solutions of $[\kappa = \sqrt{1 - k^2}]$

$$\dot{x}^2 = (1 - x^2)(1 - k^2x^2), \quad x(0) = 0, \dot{x}(0) = 1,$$

$$\dot{y}^2 = (1 - y^2)(\kappa^2 + k^2y^2), \quad y(0) = 1, \dot{y}(0) = 0,$$

$$\dot{z}^2 = (1 - z^2)(1 - \kappa^2), \quad z(0) = 1, \dot{z}(0) = 0.$$

E.g., integration gives

$$\frac{dx}{dt} = \sqrt{(1 - x^2)(1 - k^2x^2)},$$

or

$$t = \int_0^{x=\operatorname{sn}(t,k)} \frac{d\xi}{\sqrt{(1 - \xi^2)(1 - k^2\xi^2)}}.$$

Elliptic Integrals - Arclength of Ellipse

Recall that

$$\int_0^x \frac{d\xi}{\sqrt{1-\xi^2}} = \sin^{-1} x \equiv u$$

or

$$u = \int_0^{\sin u} \frac{d\xi}{\sqrt{1-\xi^2}}.$$

Similarly,

$$\begin{aligned} F(\sin \phi, k) &= \int_0^{\sin \phi} \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} \\ &= \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad 0 \leq \phi \leq \frac{\pi}{2}. \end{aligned} \quad (5)$$

$$\sin \phi = \operatorname{sn}(u, k), \cos \phi = \operatorname{cn}(u, k), \sqrt{1-k^2\sin^2\theta} = \operatorname{dn}(u, k).$$

Jacobi Elliptic Functions vs Trigonometric Functions

Trigonometric Functions

$$\frac{d}{dt} \sin x = \cos x,$$

$$\frac{d}{dt} \cos x = -\sin x.$$

Jacobi Elliptic Functions

$$\frac{d}{dt} \operatorname{sn}(t) = \operatorname{cn}(t) \operatorname{dn}(t),$$

$$\frac{d}{dt} \operatorname{cn}(t) = -\operatorname{sn}(t) \operatorname{dn}(t),$$

$$\frac{d}{dt} \operatorname{dn}(t) = -k^2 \operatorname{cn}(t) \operatorname{sn}(t).$$

Initial Value Problem

$$\dot{x} = y, \quad x(0) = 0,$$

$$\dot{y} = -x. \quad y(0) = 1.$$

Initial Value Problem

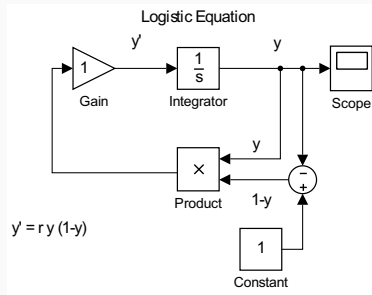
$$\dot{x} = yz, \quad x(0) = 0,$$

$$\dot{y} = -xz, \quad y(0) = 0,$$

$$\dot{z} = -k^2 xy, \quad z(0) = 0.$$

Simulink and ODEs

- What is Simulink
 - Graphical environment for designing simulations
 - Product of Mathworks
 - Select and connect blocks
- Use in Differential Equations
 - Project component of class
 - Modeling applications



Solving a Differential Equation

Consider initial value problem:

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0.$$

Solution

$$x(t) = x_0 + \int_0^t f(x(t)) dt.$$

Think of the solution as

$$x(t) = \int x'(t) dt.$$



Figure 2: Schematic for a general system.

Model of a Differential Equation

Modeling

$$x(t) = \int x'(t) dt = \int f(x(t)) dt.$$

Schematic



Simulink Model

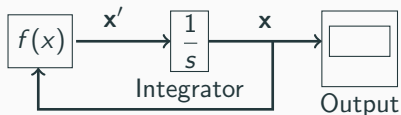
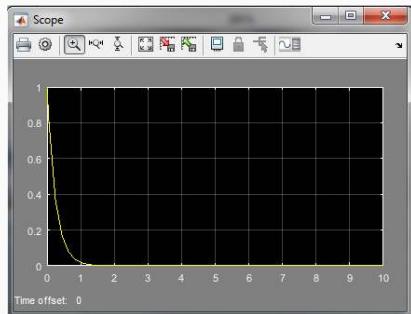
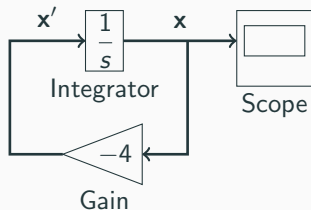


Figure 3: Model for solving $x' = f(x)$.

Simple First Order Differential Equation

Solve $x' = -4x$, $x(0) = 1$.



Simulink Workspace

The screenshot displays the Simulink workspace with the following components:

- Main Diagram:** A feedback loop system. The input signal passes through a Gain block with a value of -4, then an Integrator block (represented by $\frac{1}{s}$), and finally a Scope block.
- Configuration Parameters:** The Solver is set to 'ode45 (Dormand-Prins)'. The simulation time is from 0.0 to 10.0. Solver options include 'Variable-step' type, 'auto' for max/min/initial step sizes, and 'Disable All' for shape preservation.
- Simulink Library Browser:** Shows the 'Commonly Used Blocks' section. The 'Integrator' block is highlighted.
- Scope:** A plot showing the system's response over time. The y-axis ranges from 0 to 1, and the x-axis (Time offset) ranges from 0 to 10. The plot shows a smooth curve starting at 1 and decaying towards 0.

Scilab's Xcos Workspace

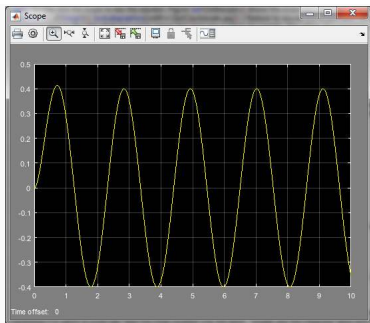
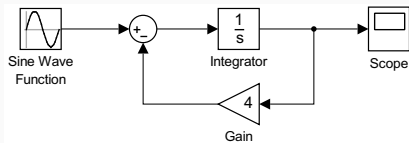
The screenshot displays the Scilab Xcos workspace with four main windows:

- Palette browser - Xcos:** Shows a tree view of block categories on the left and a grid of blocks on the right, including ANDBLK, BIGSOM_f, CMSCOPE, and CONST_m.
- Graphic window number 20004:** A plot window showing a sine wave oscillating between -2 and 4 over a time interval from 0 to 10.
- *Untitled - 8:14:42 AM - Xcos:** A block diagram window showing a feedback loop. It includes a sine wave input, a summing junction (Σ), a 1/s integrator block, a gain block (-4), and a scope block (CMSCOPE) monitoring the signal.
- Scilab 5.5.2 Console:** A console window showing the startup execution process, including "loading initial environment" and a command prompt "-->".

ODE Examples

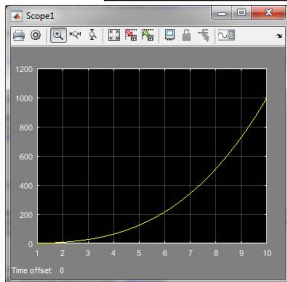
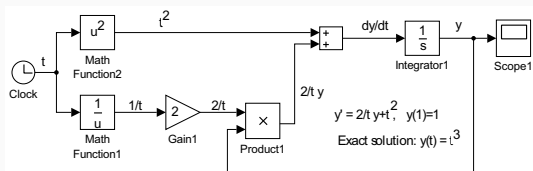
First Order Differential Equation - Example 1

$$\text{Solve } x' = 2 \sin 3t - 4x, \quad x(0) = 0.$$



First Order Differential Equation - Example 2

$$\text{Solve } y' = \frac{2}{t}y + t^2, \quad y(1) = 1.$$

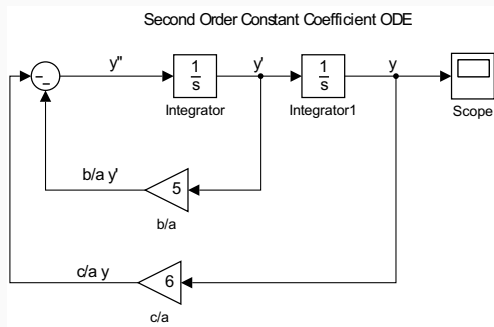


Second Order ODEs - Example 3

Solve $ay'' + by' + cy = 0$, $y(0) = y_0, y'(0) = v_0$.

$$y = \int y' dx, \quad y' = \int y'' dx,$$

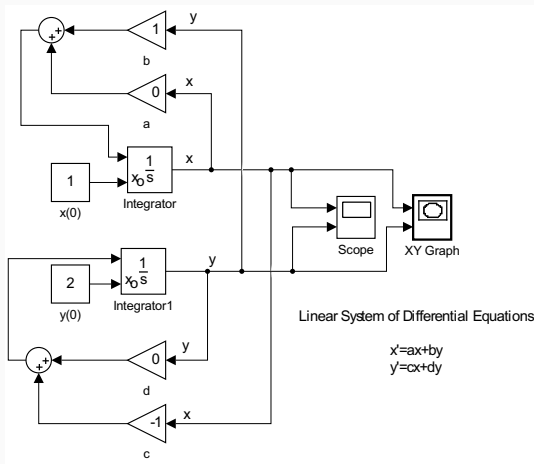
$$y'' = -\frac{b}{a}y' - \frac{c}{a}y.$$



Linear Systems of Differential Equations - Example 4

$$x' = ax + by$$

$$y' = cx + dy$$

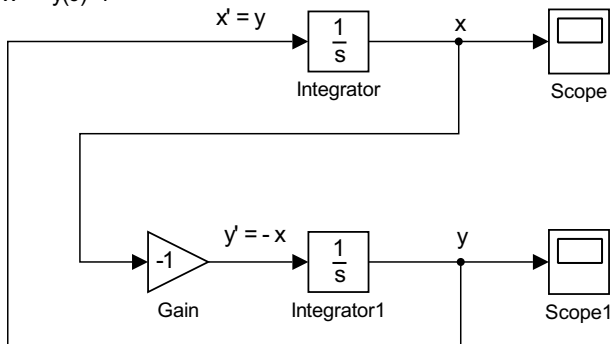


Trig-Elliptic Systems

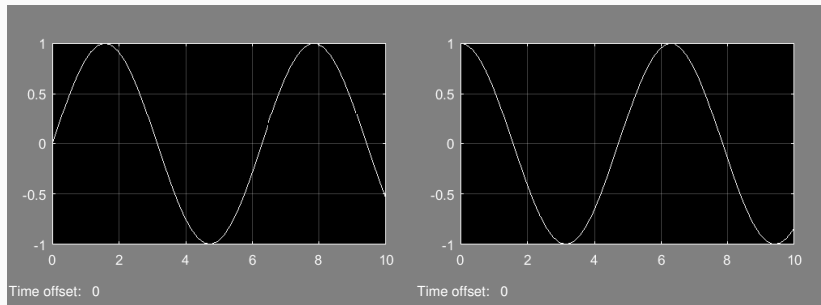
Trigonometric Simulink Model

$$x' = y \quad x(0) = 0$$

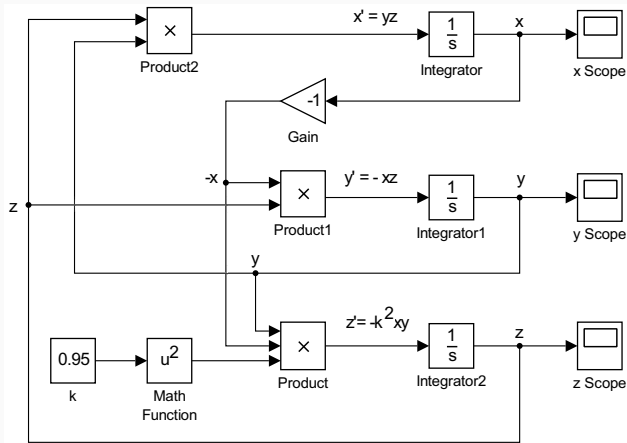
$$y' = -x \quad y(0) = 1$$



Trigonometric Simulink Model - Results

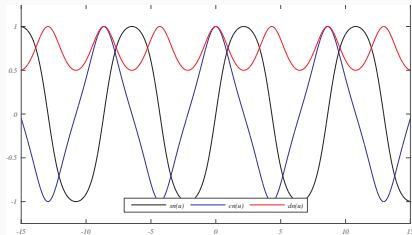
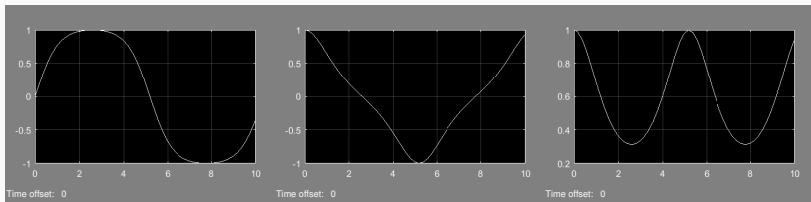


Elliptic Function Simulink Model



$$\begin{aligned}x' &= yz & x(0) &= 0 \\ y' &= -xz & y(0) &= 1 \\ z' &= -k^2 xy & z(0) &= 1\end{aligned}$$

Elliptic Function Simulink Model - Results



Conclusion

Nonlinear Pendulum

Jacobi Elliptic Functions

Simulink and ODEs

ODE Examples

Trig-Elliptic Systems

The End!

Thank you! Dr. R.L. Herman, hermanr@uncw.edu

Herman, R. L., *Solving Differential Equations Using Simulink* ,
<http://people.uncw.edu/hermanr/MAT361/Simulink/index.htm>

Meyer, K., M., Jacobi Elliptic Functions from a Dynamical Systems
Point of View, *Amer. Math. Monthly* **108** (2001) p. 729.