Introducing Elliptic Functions via Simulations of Nonlinear Differential Equations

MAA-SE 2018, Clemson University

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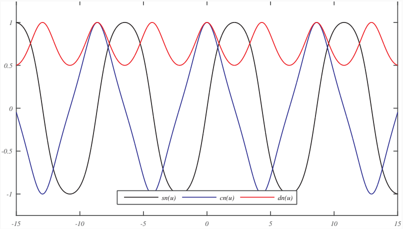

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The Nonlinear Pendulum

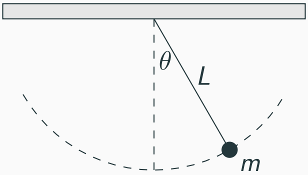


Figure 1: A point mass m is attached to a string of length L and released from rest at $\theta = \theta_0$.

$$\ddot{\theta} + \omega^2 \sin\theta = 0. \tag{1}$$

Nonlinear Pendulum - Quadrature $\ddot{\theta} + \omega^2 \sin\theta = 0$

Multiply Equation (1) by $\dot{\theta}$,

$$\ddot{\theta}\dot{\theta} + \omega^2 \sin\theta\dot{\theta} = 0,$$

and note

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta\right] = 0.$$

Therefore,

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = c. \tag{2}$$

Using the initial conditions, $\theta(0) = \theta_0$, $\dot{\theta}(0) = 0$, we have

$$c = -\omega^2\cos\theta_0 \qquad \text{and}$$

$$\dot{\theta}^2 = 2\omega^2(\cos\theta - \cos\theta_0).$$

Second Order ODE from $\dot{\theta}^2 = 2\omega^2(\cos\theta - \cos\theta_0)$

$$\dot{\theta}^2 = 4\omega^2 \left(\sin^2 \frac{\theta}{2} - \sin^2 \frac{\theta_0}{2} \right)$$

Let $kx = \sin \frac{\theta}{2}$, then $2k\dot{x} = \sqrt{1 - k^2x^2}\,\dot{\theta}$. Then,

$$\dot{x}^2 = \omega^2(1 - x^2)(1 - k^2x^2) \tag{3}$$

Differentiating,

$$2\dot{x}\ddot{x} = \omega^2\dot{x}\left[-2x(1 - k^2x^2) + (1 - x^2)(-2k^2x)\right],$$

yields

$$\ddot{z} = \omega^2\left[2k^2z^3 - (1 + k^2)z\right] \tag{4}$$

Solution: $x(t) = \operatorname{sn}(\omega t, k)$. - a Jacobi elliptic function

Jacobi Elliptic Functions

Jacobi Elliptic Functions: $x(t) = \textbf{sn}\,(t,k)$, $y(t) = \textbf{cn}\,(t,k)$ $z(t) = \textbf{dn}\,(t,k)$.

Solutions of $[\kappa = \sqrt{1-k^2}]$

$$\begin{aligned}
\dot{x}^2 &= (1-x^2)(1-k^2x^2), \quad x(0)=0, \dot{x}(0)=1, \\
\dot{y}^2 &= (1-y^2)(\kappa^2+k^2y^2), \quad y(0)=1, \dot{y}(0)=0, \\
\dot{z}^2 &= (1-z^2)(1-\kappa^2), \quad z(0)=1, \dot{z}(0)=0.
\end{aligned}$$

E.g., integration gives

$$\frac{dx}{dt} = \sqrt{(1-x^2)(1-k^2x^2)},$$

or

$$t = \int_0^{x=\text{sn}\,(t,k)} \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}.$$

Elliptic Integrals - Arclength of Ellipse

Recall that

$$\int_0^x \frac{d\xi}{\sqrt{1-\xi^2}} = \sin^{-1} x \equiv u$$

or

$$u = \int_0^{\sin u} \frac{d\xi}{\sqrt{1-\xi^2}}.$$

Similarly,

$$\begin{aligned} F(\sin\phi, k) &= \int_0^{\sin\phi} \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} \\ &= \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad 0 \le \phi \le \frac{\pi}{2}. \end{aligned} \tag{5}$$

$$\sin\phi = \operatorname{sn}(u,k), \cos\phi = \operatorname{cn}(u,k), \sqrt{1-k^2\sin^2\theta} = dn(u,k).$$

Jacobi Elliptic Functions vs Trigonometric Functions

Trigonometric Functions

$$\begin{aligned} \frac{d}{dt}\sin x &= \cos x, \\ \frac{d}{dt}\cos x &= -\sin x. \end{aligned}$$

Jacobi Elliptic Functions

$$\begin{aligned} \frac{d}{dt}\operatorname{sn}(t) &= \operatorname{cn}(t)\operatorname{dn}(t), \\ \frac{d}{dt}\operatorname{cn}(t) &= -\operatorname{sn}(t)\operatorname{dn}(t), \\ \frac{d}{dt}\operatorname{dn}(t) &= -k^2\operatorname{cn}(t)\operatorname{sn}(t). \end{aligned}$$

Initial Value Problem

$$\begin{aligned} \dot{x} &= y, \quad x(0) = 0, \\ \dot{y} &= -x. \quad y(0) = 1. \end{aligned}$$

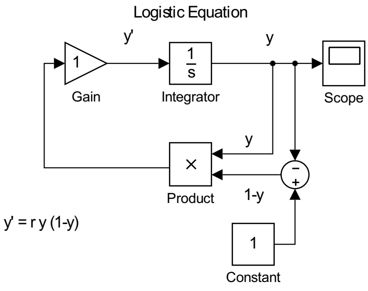
Initial Value Problem

$$\begin{aligned} \dot{x} &= yz, \quad x(0) = 0, \\ \dot{y} &= -xz, \quad y(0) = 0, \\ \dot{z} &= -k^2xy, \quad z(0) = 0. \end{aligned}$$

Simulink and ODEs

Simulink

- What is Simulink
  - Graphical environment for designing simulations
  - Product of Mathworks
  - Select and connect blocks
- Use in Differential Equations
  - Project component of class
  - Modeling applications



Solving a Differential Equation

Consider initial value problem:

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0.$$

Solution

$$x(t) = x_0 + \int_0^t f(x(t))\,dt.$$

Think of the solution as

$$x(t) = \int x'(t)\,dt.$$

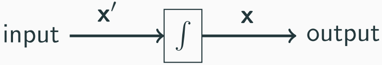


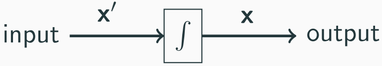
Figure 2: Schematic for a general system.

Model of a Differential Equation

Modeling

$$x(t) = \int x'(t)\,dt = \int f(x(t))\,dt.$$

Schematic



Simulink Model

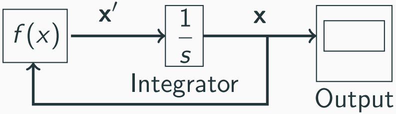
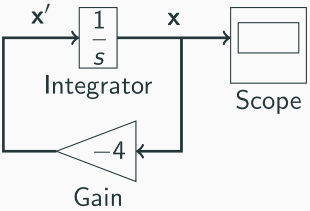
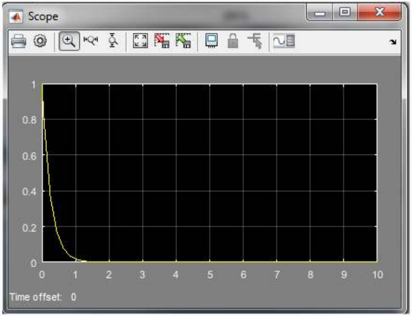


Figure 3: Model for solving $x' = f(x)$.

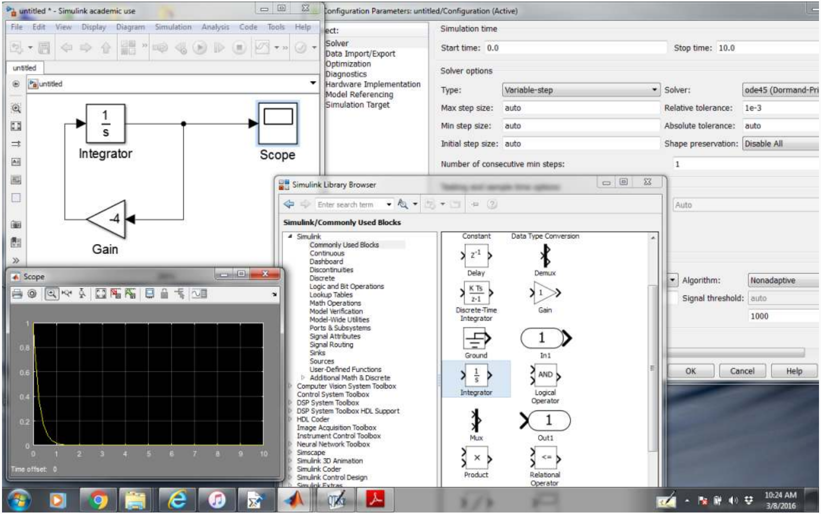
Simple First Order Differential Equation

Solve $x' = -4x, \quad x(0) = 1.$

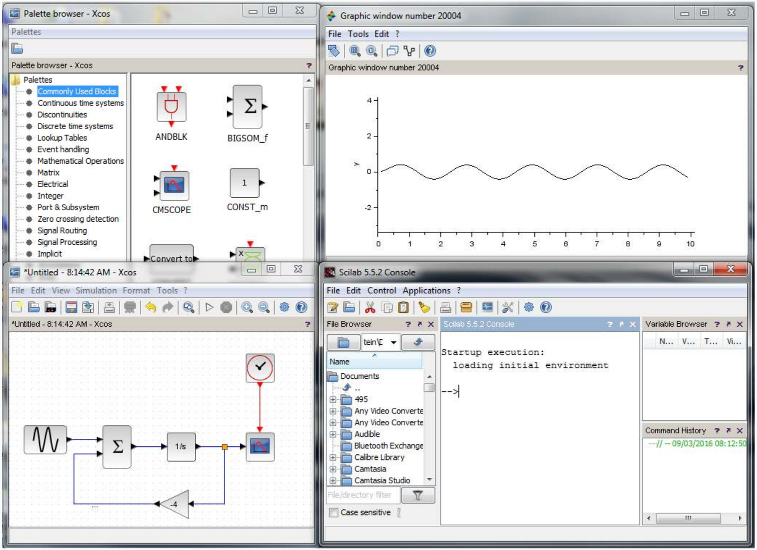




Simulink Workspace



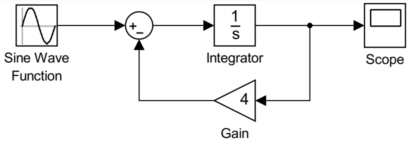
Scilab's Xcos Workspace

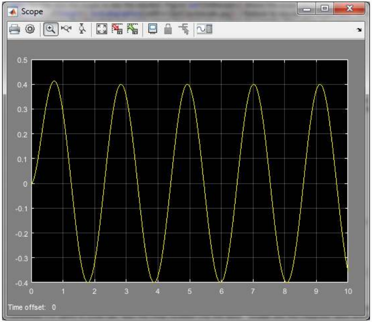


ODE Examples

First Order Differential Equation - Example 1

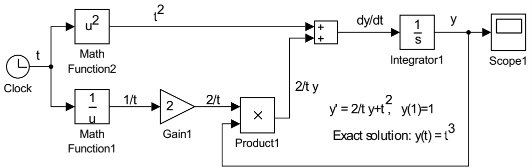
Solve $x' = 2\sin 3t - 4x, \quad x(0) = 0.$

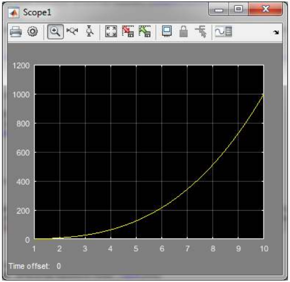




First Order Differential Equation - Example 2

Solve $y' = \frac{2}{t}y + t^2, \quad y(1) = 1.$



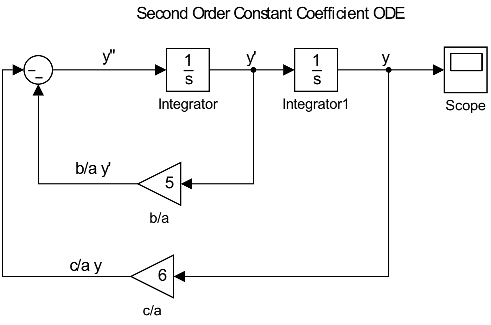


Second Order ODEs - Example 3

Solve $ay'' + by' + cy = 0, \quad y(0) = y_0, y'(0) = v_0.$

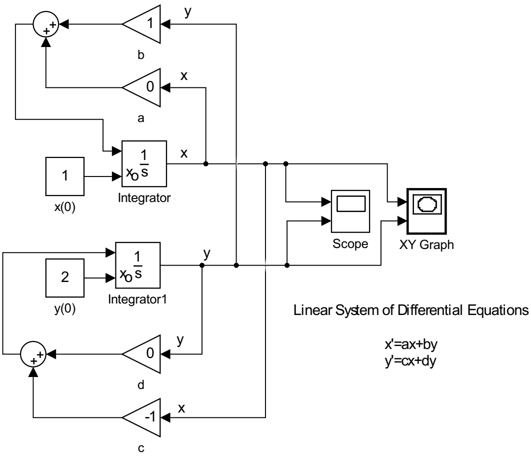
$$y = \int y'\, dx, \quad y' = \int y''\, dx,$$

$$y'' = -\frac{b}{a}y' - \frac{c}{a}y.$$



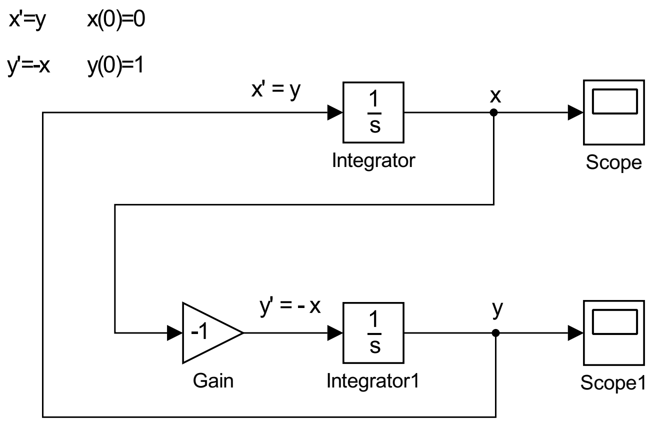
Linear Systems of Differential Equations - Example 4

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy. \end{aligned}$$

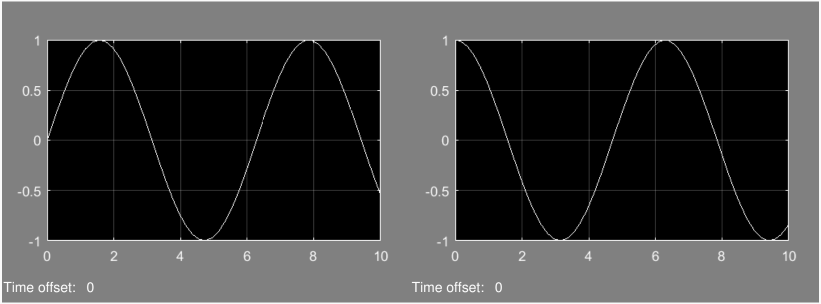


Trig-Elliptic Systems

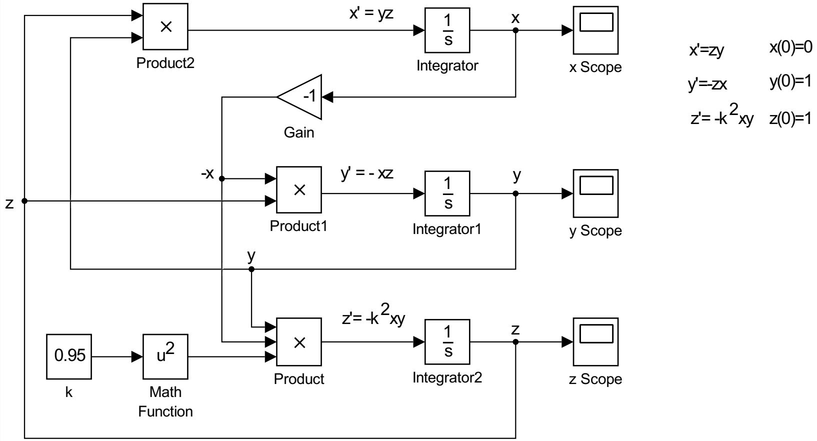
Trigonometric Simulink Model



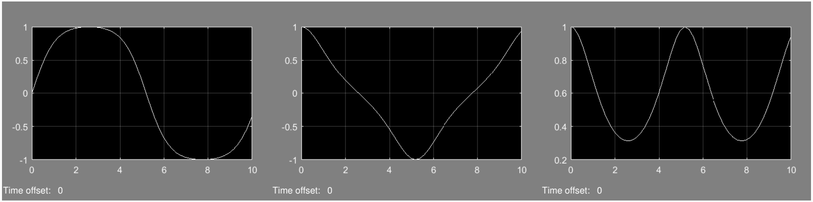
Trigonometric Simulink Model - Results

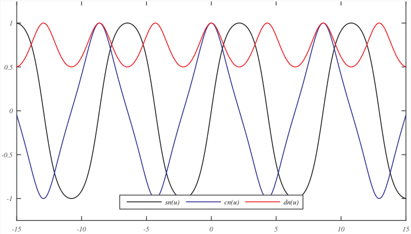


Elliptic Function Simulink Model



Elliptic Function Simulink Model - Results





Conclusion

Nonlinear Pendulum

Jacobi Elliptic Functions

Simulink and ODEs

ODE Examples

Trig-Elliptic Systems

The End!

Thank you! Dr. R.L. Herman, hermanr@uncw.edu

Herman, R. L., *Solving Differential Equations Using Simulink* , `http://people.uncw.edu/hermanr/MAT361/Simulink/index.htm`

Meyer, K., M., Jacobi Elliptic Functions from a Dynamical Systems Point of View, *Amer. Math. Monthly* **108** (2001) p. 729.