Euler Equation and Geodesics

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February 2, 2018

Introduction

Newton formulated the laws of motion in his 1687 volumes, collectively called the *Philosophiae Naturalis Principia Mathematica,* or simply the *Principia*. However, Newton's development was geometrical and is not how we see classical dynamics presented when we first learn mechanics. The laws of mechanics are what are now considered analytical mechanics, in which classical dynamics is presented in a more elegant way. It is based upon variational principles, whose foundations began with the work of Euler and Lagrange and have been refined by other now-famous figures in the eighteenth and nineteenth centuries.

Euler coined the term the *calculus of variations* in 1756, though it is also called variational calculus. The goal is to find minima or maxima of functions of the form $f : M \to R$, where M can be a set of numbers, functions, paths, curves, surfaces, etc. Interest in extrema problems in classical mechanics began near the end of the seventeenth century with Newton and Leibniz. In the *Principia*, Newton was interested in the least resistance of a surface of revolution as it moves through a fluid.

Seeking extrema at the time was not new, as the Egyptians knew that the shortest path between two points is a straight line and that a circle encloses the largest area for a given perimeter. Heron, an Alexandrian scholar, determined that light travels along the shortest path. This problem was later taken up by Willibrord Snellius (1580–1626) after whom Snell's law of refraction is named. Pierre de Fermat (1601/7/8-1665), the same for whom Fermat's Last Theorem is named, derived the law of reflection from his principle of least time. Namely, light always travels along a path for which the time taken is a minimum. This led to his principle of geometric optics, which was used to prove Snell's Law of Refraction.

By the seventeenth century, mathematicians were interested in paths of quickest descent. Galileo noted ,"From the preceding it is possible to infer that the path of quickest descent from one point to another is not the shortest path, namely, a straight line, but the arc of a circle." The curves that interested people of that time were the cycloid, the tautochrone (or, isochrone), and the brachistochrone.

Christian Huygens (1629–1695) had shown in 1659 that a particle sliding on a cycloid undergoes simple harmonic motion with a period independent of the starting point. He also showed in 1673 that the solution of the tautochrone problem is the cycloid. Having studied Huygens solutions, Johann Bernoulli (1667–1748), investigated the brachistochrone problem and offered a challenge, specifically aimed at his older brother Jacob (1654–1705). In July 1696, Johann wrote to his mathematical contemporaries that he had solved the problem and challenged them to do so.

> *If two points A and B are given in a vertical plane, to assign to a mobile particle M the path AMB along which, descending under its own weight, it passes from the point A to the point B in the briefest time.*—Smith, D.E. 1929, p. 644.

Johann Bernoulli's solution was published the following year. At that time, five others solved the problem (Jacob Bernoulli, Gottfried Leibniz, Isaac Newton, Ehrenfried Walther von Tschirnhaus, and Guillaume de L'Hopital.) The shape of the curve is that of a cycloid, as we will see later.

It was the study of these problems that led Johann Bernoulli to study variational problems. It turns out that Leonhard Euler was living with his brother Jacob but studied under Johann Bernoulli. He began a systematic study of extreme value problems and was aware of developments by Joseph Louis Lagrange. Euler introduced a condition on the path in the form of differential equations, which we later introduce as Euler's Equation. He took the Principle of Least Action and put it on firm ground. However, it was Lagrange who was to apply the calculus of variations to mechanics as the foundation of analytical mechanics.

Another important player in the development of minimum principles was Pierre Louis Maupertuis (1698–1759). This Principle of Least Action, which he believed was the basis for mechanics, was mostly philosophical but could be summarized as saying that all bodies move with minimum effort. The measure of effort is the action, which has units of energy times time. Maupertuis was a student of Johann Bernoulli, corresponded with Euler, and formulated his Principle of Least Action in 1744. It was based on his thoughts about the reflection and refraction of light. In that same year, Euler added to the ideas that natural phenomena obey laws of maxima or minima. In 1746, Maupertuis went on further to use the Principle of Least Action to support his religious views. This sparked some heated discussions with scientists and philosophers of the day. Also, a dispute as to the originator of the Principle of Least Action emerged as Leibniz was thought by some to be the first to introduce these ideas.

Euler was the first to describe the Principle of Least Action on a firm mathematical basis. Lagrange further developed the principle and published examples of its use in dynamics. He introduced the variation of functions and derived the Euler-Lagrange equations. In 1867 Lagrange generalized the principle of least action basing his work on the conservation of energy and d'Alembert's Principle of Virtual Work. Further modifications by Adrien-Marie Legendre (1752–1833), Carl Gustav Jacob Jacobi (1804–1851), William Rowan Hamilton (1805–1865), and others, led to other formulations that applied to nonstationary constraints and nonconservative forces.

Variational Problems

THE TERM *calculus of variations* WAS FIRST COINED by Euler in 1756 as a description of the method that Joseph Louis Lagrange had introduced the previous year. The method was since expanded and studied by Euler, Hamilton, and others. As noted at the beginning of the chapter, the main idea is to determine which functions $y(x)$ will minimize, or maximize, integrals of the form

$$J[y] = \int_a^b f(x, y(x), y'(x))\, dx, \tag{1}$$

where a, b, $f(a)$, $f(b)$, and $f(x)$ are given. Integrals like $J[y]$ are called functionals. This is a mapping from a function space to a scalar. J takes a function and spits out a number.

An example of a functional is the length of a curve in the plane. As we will recall, the length of a curve $y = y(x)$ from $x = x_1$ to $x = x_2$ is given by

$$L[y] = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

This maps a given curve $y(x)$ to a number, L.

We are interested in finding the extrema of such functionals. We will not formally determine if the extrema are minima or maxima. However, in most cases, it is clear which type of extrema comes out of the analysis. Further analysis of the second variation can be found elsewhere, such as Lanczos' book on variational calculus. For interesting problems, historical principles have led to the formulation of problems in the calculus of variations, such as the Principles of Least Time, Least Action, Least Effort, or the shortest distance between geometric points on a surface. We will explore some standard examples leading to finding the extrema of functionals.

Example 1. Find the functional needed to determine the shortest distance between two given points in the plane.

We have all heard that the shortest distance between points is a straight line. Of course, this statement needs some clarification. We will consider that the possible paths lie in a two-dimensional plane as shown in Figure 1. We then consider the length of a curve connecting two points in the plane. Let the points be given by (x_1, y_1) and (x_2, y_2). We assume that the functional form of the curve is given by $y = f(x)$.

Recall from the previous chapter that for a parametrized path $\mathbf{r}(t)$, the length of a curve is given by

$$L = \int_C ds = \int_{t_1}^{t_2} |\mathbf{r}'(t)|\, dt.$$

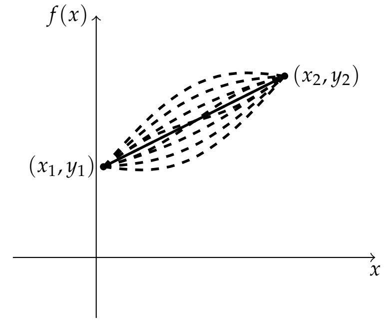

Figure 1: Possible paths between two fixed points.

If we let the parameter be $t = x$, then

$$\mathbf{r}(x) = x\mathbf{i} + y(x)\mathbf{j}$$

and

$$\mathbf{r}'(x) = \mathbf{i} + y'(x)\mathbf{j}.$$

So, the length of the curve $y = y(x)$ from $x = x_1$ to $x = x_2$ is

$$\begin{aligned} L[y] &= \int_{x_1}^{x_2} |\mathbf{r}'(x)|\, dx \\ &= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \end{aligned} \tag{2}$$

Euler Equation

IN THE PREVIOUS EXAMPLES WE HAVE REDUCED THE PROBLEMS to finding functions $y = y(x)$ that extremize functionals of the form

$$J[y] = \int_a^b F(x, y, y')\, dx$$

for F twice continuous (C^2) in all variables. Formally, we say that J is stationary,

Stationary functionals.

$$\delta J[y] = 0,$$

at the function $y = y(x)$, or that the function $y = y(x)$ that extremizes $J[y]$ satisfies

Euler's Equation.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0. \tag{3}$$

This is called the Euler's Equation. We will derive Euler's Equation and then show how it is used for some common examples.

The idea is to consider all paths connected to the two fixed points and finding the path that is an extremum of $J[y]$. In fact, we need only consider parametizing paths near the optimal path and writing the problem in a form that we can use with the methods for local extrema of real functions.

Let's consider the paths $y(x;\epsilon) = u(x) + \epsilon\eta(x)$ near the optimal path $u = u(x)$ with $\eta(a) = \eta(b) = 0$ and η is C^2. Then, we consider the functional

$$J[y] = J[u + \epsilon\eta] \equiv \phi(\epsilon).$$

We note that if $J[u+\epsilon\eta]$ has a local extremum at u, then u is a stationary function for J. This will occur when

$$\left.\frac{d\phi}{d\epsilon}\right|_{\epsilon=0} = 0.$$

We compute this derivative and find

$$\begin{aligned} \frac{d\phi}{d\epsilon} &= \frac{d}{d\epsilon}\int_a^b F(x, u+\epsilon\eta, u'+\epsilon\eta')\,dx \\ &= \int_a^b \frac{\partial}{\partial\epsilon} F(x, u+\epsilon\eta, u'+\epsilon\eta')\,dx \\ &= \int_a^b \left[\frac{\partial F}{\partial y}\frac{\partial y}{\partial \epsilon} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \epsilon}\right] dx \\ &= \int_a^b \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x)\right] dx. \end{aligned} \tag{4}$$

We can perform an integration by parts on the second integral in order to move the derivative off of $\eta'(x)$. This is accomplished by setting $u(x) = \frac{\partial F}{\partial y'}$ and $dv = \eta'(x)\,dx$ in the integration by parts formula. Then,

$$\int_a^b \frac{\partial F}{\partial y'}\eta'(x)\,dx = \eta(x)\frac{\partial F}{\partial y'}\Big|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta(x)\,dx.$$

The first terms vanish because $\eta(a) = \eta(b) = 0$.

This leaves

$$\begin{aligned} \frac{d\phi}{d\epsilon} &= \int_a^b \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x)\right] dx \\ &= \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)\,dx. \end{aligned} \tag{5}$$

Evaluation at $\epsilon = 0$ gives

$$\int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)\,dx = 0.$$

Noting that $\eta(x)$ is an arbitrary function and that this integral vanishes for all $\eta(x)$, we can say that the integrand vanishes,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0,$$

for all $x \in [a, b]$. This is Euler's Equation, (3).

Because $F = F(x, y(x), y'(x))$, one can prove a second form of Euler's Equation. We first note for the Chain Rule that

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial x}.$$

Now we insert

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

from Euler's Equation to find

$$\begin{aligned} \frac{dF}{dx} &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial x} \\ &= \frac{\partial F}{\partial x} + \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)y' + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial x} \end{aligned}$$

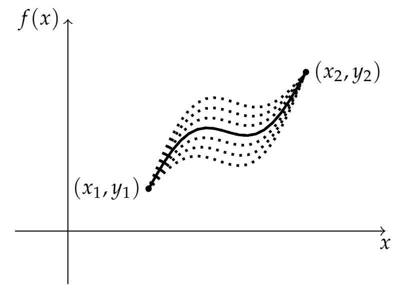

Figure 2: Paths near an optimal path between two fixed points.

$$= \frac{\partial F}{\partial x} + \frac{d}{dx}\left(\frac{\partial F}{\partial y'}y'\right). \tag{6}$$

Rearranging this result, we obtain

The second form of Euler's Equation.

$$\frac{\partial F}{\partial x} - \frac{d}{dx}\left(F - \frac{\partial F}{\partial y'}y'\right) = 0. \tag{7}$$

This is the second form of Euler's Equation.

The second form of Euler's Equation is handy when $\frac{\partial F}{\partial x} = 0$, or when $F = F(y, y')$ is independent of x. In this case, we have

$$F - \frac{\partial F}{\partial y'}y' = c,$$

where c is an arbitrary constant.

There are other special cases. In Euler's Equation if $F = F(x, y')$,

$$\frac{\partial F}{\partial y'} = c,$$

and when $F = F(x, y)$, we have

$$\frac{\partial F}{\partial y} = 0.$$

Finally, when $F = F(y')$, Euler's Second Equation implies

$$y'' = 0,$$

or $y(x) = c_1 + c_2 x$.

Special Cases:

$$F(x,y) \quad \Rightarrow \quad \frac{\partial F}{\partial y} = 0$$

$$F(x,y') \quad \Rightarrow \quad \frac{\partial F}{\partial y'} = c$$

$$F(y,y') \quad \Rightarrow \quad F - \frac{\partial F}{\partial y'}y' = c$$

$$F(y') \quad \Rightarrow \quad y'' = 0$$

Example 2. Find the path with the shortest distance between two points in a plane.

Shortest distance between points.

In Example 1, we derived the functional

$$L[y] = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

which gives the length of the path $y = y(x)$ from $x = a$ to $x = b$. The path that makes this functional stationary satisfies the Euler equation with

$$F(x, y, y') = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Because $F(x, y, y') = F(y')$ is independent of x and y, Euler's second form yields

$$y(x) = c_1 + c_2 x.$$

This is obviously a straight line path, and the constants can be found using the prescribed values of $y(x)$ at $x = a$ and $x = b$.

Another approach to such problems is to work with parametrized paths, such as $\{(x(t), y(t)) | t \in [a, b]\}$. In these cases, one has to find Euler equations based on integrals of the form

$$J[x, y] = \int_a^b F(t, x(t), y(y), \dot{x}(t), \dot{y}(t))\, dt.$$

As before, we look at variations of the path about a path that makes $J[x, y]$ stationary. Let that path be represented by $u(t), v(t)$. Then we consider the variations as functions of t in the form

$$x(t) = u(t) + \epsilon\eta(t),$$

$$y(t) = v(t) + \epsilon\xi(t), \tag{8}$$

where $\eta(a) = \eta(b) = 0$ and $\xi(a) = \xi(b) = 0$. We can now proceed as before.

We consider the function

$$\phi(\epsilon) = \int_a^b F(t, u + \epsilon\eta, v + \epsilon\xi, \dot{u} + \epsilon\dot{\eta}, \dot{v} + \epsilon\dot{\xi})\, dt$$

and seek conditions such that $\frac{d\phi}{d\epsilon}\big|_{\epsilon=0}$. Computing the derivative, we have

$$\begin{aligned}
\frac{d\phi}{d\epsilon} &= \frac{d}{d\epsilon}\int_a^b F(t, u + \epsilon\eta, v + \epsilon\xi, \dot{u} + \epsilon\dot{\eta}, \dot{v} + \epsilon\dot{\xi})\, dt \\
&= \int_a^b \frac{\partial}{\partial\epsilon} F(t, u + \epsilon\eta, v + \epsilon\xi, \dot{u} + \epsilon\dot{\eta}, \dot{v} + \epsilon\dot{\xi})\, dt \\
&= \int_a^b \left[\frac{\partial f}{\partial x}\frac{dx}{d\epsilon} + \frac{\partial f}{\partial \dot{x}}\frac{d\dot{x}}{d\epsilon} + \frac{\partial f}{\partial y}\frac{dy}{d\epsilon} + \frac{\partial f}{\partial \dot{y}}\frac{d\dot{y}}{d\epsilon}\right] dt \\
&= \int_a^b \left[\frac{\partial f}{\partial x}\eta + \frac{\partial f}{\partial \dot{x}}\dot{\eta} + \frac{\partial f}{\partial y}\xi + \frac{\partial f}{\partial \dot{y}}\dot{\xi}\right] dt. \qquad (9)
\end{aligned}$$

As before, we perform integration by parts to remove the derivatives from η and ξ using the boundary conditions on η and ξ to obtain

$$\int_a^b \left[\left(\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)\right)\eta + \left(\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right)\right)\xi\right] dt = 0.$$

Because $\eta(t)$ and $\xi(t)$ are arbitrary functions, their coefficients in the integral have to vanish. For example, we consider the cases where $\xi(t) = 0$. This leaves

$$\int_a^b \left[\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)\right]\eta\, dt = 0.$$

Then, for $\eta(t)$ arbitrary, we conclude that

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0.$$

Similarly, we can look at cases when $\eta(t) = 0$ and obtain a second Euler Equation,

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0.$$

Thus, we have obtained a pair of Euler equations for extremizing $J[x, y]$ along a parametrized path. This is summarized below:

The path $\{(x(t), y(t)) | t \in [a, b]\}$ for which

$$J[x, y] = \int_a^b F(t, x(t), y(y), \dot{x}(t), \dot{y}(t))\, dt$$

is an extremum that satisfies the two Euler equations

$$\begin{aligned}
\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) &= 0, \\
\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right) &= 0. \qquad (10)
\end{aligned}$$

This is easily generalized to higher-dimensional paths, $\mathbf{r}(t) = (x(t), y(t), z(t))$.

Example 3. Determine the closed curve with a given fixed length that encloses the largest possible area.

As seen in the previous chapter, the area enclosed by curve C is given by

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

This was verified as an example of Green's Theorem in the Plane, as

$$\oint_C x\,dy - y\,dx = \int_S \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) dxdy = 2\int_S dxdy.$$

If the curve is parametrized as $x = x(t)$ and $y = y(t)$, then

$$A = \frac{1}{2}\oint_C (x\dot{y} - y\dot{x})\,dt.$$

Also, the length of the curve is given as

$$L = \oint_C \sqrt{\dot{x}^2 + \dot{y}^2}\,dt.$$

The problem can be mathematically stated as finding the maximum of $A[x,y]$ subject to the constraint $L[x,y]$. This is equivalent to finding the path for which $I[x,y] = A[x,y] + \lambda L[x,y]$ is stationary for some constant λ. Thus, we consider the integral

$$I[x,y] = \oint_C \left(\frac{1}{2}(x\dot{y} - y\dot{x}) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2}\right) dt.$$

We define the integrand

$$F(t,x,\dot{x},y,\dot{y}) = \frac{1}{2}(x\dot{y} - y\dot{x}) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2}.$$

The variational problem states that F satisfies a pair of Euler-Lagrange equations:

$$\begin{aligned}\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) &= 0,\\ \frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) &= 0.\end{aligned} \tag{11}$$

Inserting F, we have

$$\begin{aligned}\frac{1}{2}\dot{y} - \frac{d}{dt}\left(-\frac{1}{2}y + \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) &= 0,\\ -\frac{1}{2}\dot{x} - \frac{d}{dt}\left(\frac{1}{2}x + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) &= 0.\end{aligned} \tag{12}$$

Each of these equations is a perfect derivative and can be integrated. The results are

$$\begin{aligned}y - \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} &= c_1,\\ x + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} &= c_2,\end{aligned} \tag{13}$$

where c_1 and c_2 are two arbitrary constants. Rearranging, squaring both equations, and adding, we have

$$\begin{aligned}(x-c_2)^2+(y-c_1)^2 &= \frac{\lambda^2\dot{x}^2}{\dot{x}^2+\dot{y}^2}+\frac{\lambda^2\dot{y}^2}{\dot{x}^2+\dot{y}^2}\\ &= \lambda^2. \end{aligned} \tag{14}$$

Therefore, the maximum area is enclosed by a circle.

Geodesics

The shortest distance between two points is a straight line. Well, this is true in a Euclidean plane. But, what is the shortest distance between two points on a sphere? What path should light follow as it passes the sun? Paths that are the shortest distances between points on a curved surface or events in curved spacetime are called geodesics. In general, we can set up an appropriate integral to compute these distances along paths and seek the paths that render the integral stationary. We begin by looking at the geodesics on a sphere.

Example 4. Determine the shortest distance between two points on the surface of a sphere of radius R.

The length of a curve between points A and B is generally given by

$$L=\int_{S_A}^{S_B} ds.$$

We need to represents ds on the sphere. We can do this by recalling from the previous chapter that for orthogonal curvilinear coordinates.

$$ds^2=d\mathbf{r}\cdot d\mathbf{r}=\sum_{i=1}^{3} h_i^2 du_i^2,$$

where h_i are the scale factors and u_i are the generalized coordinates. For spherical coordinates, we have

$$d\mathbf{r}=d\rho d\mathbf{e}_\rho+\rho d\theta d\mathbf{e}_\theta+\rho\sin\theta d\phi d\mathbf{e}_\phi.$$

On the surface of a sphere of radius $\rho=R$, $d\rho=0$ and $h_\phi=R\sin\theta$, $h_\theta=R$. So,

$$ds^2=R^2d\theta^2+R^2\sin^2\theta d\phi^2.$$

Thinking of the path in the form $\phi(\theta)$, the length of the curve then becomes

$$\begin{aligned} L &= \int_{S_A}^{S_B} ds\\ &= R\int_{S_A}^{S_B}\sqrt{d\theta^2+\sin^2\theta d\phi^2}\\ &= R\int_{S_A}^{S_B}\sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\,d\theta. \end{aligned} \tag{15}$$

We can apply Euler's Equation to the integrand $F(\theta,\phi,\phi')=\sqrt{1+\sin^2\theta\phi'^2}$,

$$\frac{\partial F}{\partial\phi}-\frac{d}{d\theta}\left(\frac{\partial F}{\partial\phi'}\right)=0.$$

We first note that F is independent of ϕ. So,

$$\frac{d}{d\theta}\left(\frac{\partial F}{\partial \phi'}\right) = 0,$$

or

$$\frac{\sin^2\theta\phi'}{\sqrt{1+\sin^2\theta\phi'^2}} = c.$$

Solving for ϕ',

$$\frac{d\phi}{d\theta} = \frac{c}{\sin\theta\sqrt{\sin^2\theta - c^2}}.$$

One can solve for $\phi(\theta)$ by integrating:

$$\begin{aligned}\phi &= \int \frac{c d\theta}{\sin\theta\sqrt{\sin^2\theta - c^2}} \\ &= \int \frac{c d\theta}{\sin^2\theta\sqrt{1 - c^2\csc^2\theta}}. \qquad (16)\end{aligned}$$

Letting $u = \cot\theta$ and $du = -\csc^2\theta\, d\theta$, the integral becomes

$$\begin{aligned}\phi &= \int \frac{c d\theta}{\sin^2\theta\sqrt{1 - c^2\csc^2\theta}} \\ &= \int \frac{-c du/\csc^2\theta}{\sin^2\theta\sqrt{1 - c^2(1+u^2)}} \\ &= \int \frac{-c du}{\sqrt{(1-c^2) - u^2}}. \qquad (17)\end{aligned}$$

Letting $u = (1-c^2)\sin t$, $du = (1-c^2)\cos t\, dt$,

$$\begin{aligned}\phi &= \int \frac{-c du}{\sqrt{(1-c^2) - u^2}} \\ &= \frac{-c}{\sqrt{1-c^2}}\int dt \\ &= \frac{-c}{\sqrt{1-c^2}} t + \phi_0 \\ &= \frac{-c}{\sqrt{1-c^2}}\sin^{-1} u + \phi_0 \\ &= \frac{-c}{\sqrt{1-c^2}}\sin^{-1}(\cot\theta) + \phi_0. \qquad (18)\end{aligned}$$

Thus,

$$\cot\theta = a\sin(\phi - \phi_0),$$

where $a = \frac{\sqrt{1-c^2}}{c}$.

While this gives the implicit relation between ϕ and θ, it may not be clear what the geodesics look like. However, writing this relation in Cartesian coordinates is useful. Multiplying the equation by $\sin\theta$ and expanding, we have

$$\cos\theta = a\sin\phi_0\sin\theta\cos\phi - a\cos\phi_0\sin\theta\sin\phi.$$

Recalling the transformation for spherical coordinates from the previous chapter,

$$\begin{aligned}x &= \rho\sin\theta\cos\phi, \\ y &= \rho\sin\theta\sin\phi,\end{aligned}$$

$$z = \rho\cos\theta, \tag{19}$$

we need only multiply the equation by ρ and obtain

$$z = (a\sin\phi_0)x - (a\cos\phi_0)y.$$

This is the equation of a plane in Cartesian coordinates that passes through the origin. Such planes intersect the surface of the sphere in great circles. Thus, geodesics on a sphere lie on great circles. In Figure 3 we plot the geodesic between the points $(\theta,\phi) = (\frac{\pi}{6}, \frac{5\pi}{18})$ and $(\theta,\phi) = (\frac{5\pi}{9}, -\frac{2\pi}{9})$.

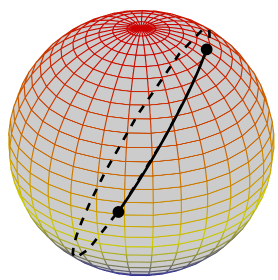

Figure 3: The geodesic between the two points $(\theta,\phi) = (\frac{\pi}{6}, \frac{5\pi}{18})$ and $(\theta,\phi) = (\frac{5\pi}{9}, -\frac{2\pi}{9})$.

We can generalize the search for geodesics in curved space using the general line element. For a three-dimensional space described by the coordinates (u^1, u^2, u^3), the line element is given by

$$ds^2 = g_{ij}du^i du^j, \quad i,j = 1,2,3.$$

Here, repeated indices obey the Einstein summation convention as seen in the previous chapter. Because

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u^i} du^i,$$

then the metric elements are given by

$$g_{ij} = \frac{\partial \mathbf{r}}{\partial u^i} \cdot \frac{\partial \mathbf{r}}{\partial u^j}.$$

Recall that for an orthogonal system,

$$ds^2 = \sum_{i=1}^{3} h_i^2 (du^i)^2.$$

Therefore, the metric g is given by a diagonal matrix,

$$(g_{ij}) = \begin{pmatrix} h1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix}.$$

For example, the line element in spherical coordinates is given by

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2,$$

and the corresponding metric g is given by the matrix

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2 \sin^2\theta \end{pmatrix}.$$

The length of a curve in generalized coordinates is given by the forms

$$\begin{aligned} L &= \int_{s_a}^{s_b} ds \\ &= \int_{s_a}^{s_b} \sqrt{g_{ij} du^i du^j} \\ &= \int_{\sigma_a}^{\sigma_b} \sqrt{g_{ij} \frac{du^i}{d\sigma} \frac{du^j}{d\sigma}}\, d\sigma, \end{aligned} \tag{20}$$

where σ is the parameter used in the parametrization of the path $(u^i(\sigma))$. For an orthogonal system, the expanded form of this would be

$$L = \int_{\sigma_a}^{\sigma_b} \sqrt{g_{11}\left(\frac{du^1}{d\sigma}\right)^2 + g_{22}\left(\frac{du^2}{d\sigma}\right)^2 + g_{33}\left(\frac{du^3}{d\sigma}\right)^2}\, d\sigma.$$

The integrand takes the form

$$F(u^1, \dot{u}^1, u^2, \dot{u}^2, u^3, \dot{u}^3) = \sqrt{g_{ij} \frac{du^i}{d\sigma} \frac{du^j}{d\sigma}}.$$

Thus, the Euler equations are given by

$$\frac{\partial F}{\partial u^k} - \frac{d}{d\sigma}\left(\frac{\partial F}{\partial \dot{u}^k}\right) = 0, \quad k = 1, 2, 3.$$

Before embarking on the general theory of geodesics, we consider applying the Euler-Lagrange equations in order to determine geodesics equations.

Example 5. Geodesics for polar coordinates in the plane.

For polar coordinates, the line element is given by

$$ds^2 = dr^2 + r^2 d\phi^2.$$

The length of the curve is given as

$$L = \int_a^b \sqrt{\left(\frac{dr}{d\sigma}\right)^2 + r^2\left(\frac{d\phi}{d\sigma}\right)^2}\, d\sigma.$$

We apply the Euler equations for

$$F\left(r, \frac{dr}{d\sigma}, \phi, \frac{d\phi}{d\sigma}\right) = \sqrt{\left(\frac{dr}{d\sigma}\right)^2 + r^2\left(\frac{d\phi}{d\sigma}\right)^2}$$

Thus,

$$\begin{aligned} 0 &= \frac{\partial F}{\partial r} - \frac{d}{d\sigma}\left(\frac{\partial F}{\partial(\partial r/\partial \sigma)}\right) \\ &= \frac{r}{F}\left(\frac{d\phi}{d\sigma}\right)^2 - \frac{d}{d\sigma}\left(\frac{1}{F}\frac{dr}{d\sigma}\right) \end{aligned} \tag{21}$$

$$\begin{aligned} 0 &= \frac{\partial F}{\partial \phi} - \frac{d}{d\sigma}\left(\frac{\partial F}{\partial(\partial\phi/\partial\sigma)}\right) \\ &= -\frac{d}{d\sigma}\left(\frac{1}{F} r^2 \frac{d\phi}{d\sigma}\right). \end{aligned} \tag{22}$$

Noting that $\frac{ds}{d\sigma} = F$, and therefore, $\frac{d}{d\sigma} = F\frac{d}{ds}$, these equations become

$$\begin{aligned} r\left(\frac{d\phi}{ds}\right)^2 - \frac{d^2r}{ds^2} &= 0, \\ \frac{d}{ds}\left(r^2\frac{d\phi}{ds}\right) &= 0. \end{aligned} \tag{23}$$

These are the geodesic equations for geodesics in the plane written in polar coordinates. Solving this coupled set of equations leads to the geodesic paths. See the problem set.

Example 6. Determine the geodesic equations for the two-dimensional sphere of radius R.

This problem was actually considered earlier in Example 3. In that problem, the line element was given by

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2.$$

Then, we consider

$$F(\theta, \phi, \phi') = \sqrt{\left(\frac{d\theta}{d\sigma}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\sigma}\right)^2},$$

We apply Euler's equation and find

$$\begin{aligned} \frac{\sin\theta\cos\theta}{F}\left(\frac{d\phi}{d\sigma}\right)^2 - \frac{d}{d\sigma}\left(\frac{1}{F}\frac{d\theta}{d\sigma}\right) &= 0, \\ -\frac{d}{d\sigma}\left(\frac{\sin^2\theta}{F}\frac{d\phi}{d\sigma}\right) &= 0. \end{aligned} \tag{24}$$

Again, using $\frac{ds}{d\sigma} = F$, these give the geodesic equations

$$\begin{aligned} \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 - \frac{d^2\theta}{ds^2} &= 0, \\ \frac{d}{ds}\left(\sin^2\theta\frac{d\phi}{ds}\right) &= 0. \end{aligned} \tag{25}$$

We can extend this process to seeking geodesics in a four-dimensional spacetime. For Minkowski spacetimes, the line element takes the form

$$ds^2 = -(cdt)^2 + dx^2 + dy^2 + dz^2,$$

where c is the speed of light. This line element is related to the proper time, τ, the time measured by a clock carried by the observer. Then,

$$d\tau^2 = -\frac{ds^2}{c^2}.$$

In relativity, one seeks the world line of free test particles moving between two timelike separated points. This particular path extremizes the proper time.

We begin with the line element

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta, \tag{26}$$

where $g_{\alpha\beta}$ is the metric with $\alpha, \beta = 0, 1, 2, 3$. Also, we are using the Einstein summation convention in that we sum over repeated indices that occur as a subscript and superscript pair. In order to find the geodesic equation, we use the Variational Principle which states that freely falling test particles follow a path between two fixed points in spacetime that extremizes the proper time τ.

The proper time is defined by $d\tau^2 = -ds^2$. (We are assuming that $c = 1$.) So, formally, we have

$$\tau_{AB} = \int_A^B \sqrt{-ds^2} = \int_A^B \sqrt{-g_{\alpha\beta}\, dx^\alpha dx^\beta}.$$

In order to write this as an integral that we can compute, we consider a parametrized worldline, $x^\alpha = x^\alpha(\sigma)$, where the parameter $\sigma = 0$ at point A and $\sigma = 1$ at point B. Then we write

$$\tau_{AB} = \int_0^1 \left[-g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \right] \equiv \int_0^1 L \left[x^\alpha, \frac{dx^\alpha}{d\sigma} \right]. \tag{27}$$

Here we have introduced the Lagrangian $L \left[x^\alpha, \frac{dx^\alpha}{d\sigma} \right]$.

We note also that

$$L = \frac{d\tau}{d\sigma}.$$

Therefore, for functions $f = f(\tau(\sigma))$, we have

$$\frac{df}{d\sigma} = \frac{df}{d\tau} \frac{d\tau}{d\sigma} = L \frac{df}{d\tau}.$$

We will use this later to change derivatives with respect to our arbitrary parameter σ to derivatives with respect to the proper time τ.

Using variational methods as seen in this chapter, we obtain the Euler-Lagrange equations in the form

$$\frac{\partial L}{\partial x^\gamma} - \frac{d}{d\sigma} \left(\frac{\partial L}{\partial (dx^\gamma / d\sigma)} \right) = 0. \tag{28}$$

We carefully compute these derivatives for the general metric. First we find

$$\begin{aligned} \frac{\partial L}{\partial x^\gamma} &= -\frac{1}{2L} \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \\ &= -\frac{L}{2} \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}, \end{aligned} \tag{29}$$

The σ derivatives have been converted to τ derivatives.

Now we compute

$$\begin{aligned} \frac{\partial L}{\partial (dx^\gamma / d\sigma)} &= -\frac{1}{2L} g_{\alpha\beta} \left(\delta^\alpha_\gamma \frac{dx^\beta}{d\sigma} + \frac{dx^\alpha}{d\sigma} \delta^\beta_\gamma \right) \\ &= -\frac{1}{2L} \left(g_{\gamma\beta} \frac{dx^\beta}{d\sigma} + g_{\alpha\gamma} \frac{dx^\alpha}{d\sigma} \right) \\ &= -\frac{1}{L} g_{\alpha\gamma} \frac{dx^\alpha}{d\sigma}. \end{aligned} \tag{30}$$

In the previous step we used the symmetry of the metric and the fact that α and β are dummy indices.

We differentiate the last result to obtain

$$\begin{aligned} -\frac{d}{d\sigma}\left(\frac{\partial L}{\partial(dx^\gamma/d\sigma)}\right) &= \frac{d}{d\sigma}\left(\frac{1}{L}g_{\alpha\gamma}\frac{dx^\alpha}{d\sigma}\right) \\ &= L\frac{d}{d\tau}\left(g_{\alpha\gamma}\frac{dx^\alpha}{d\tau}\right) \\ &= L\left[g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2}+\frac{dg_{\alpha\gamma}}{d\tau}\frac{dx^\alpha}{d\tau}\right] \\ &= L\left[g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2}+\frac{dg_{\alpha\gamma}}{dx^\beta}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\right] \\ &= L\left[g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2}+\frac{1}{2}\left(\frac{dg_{\alpha\gamma}}{dx^\beta}+\frac{dg_{\gamma\beta}}{dx^\alpha}\right)\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\right]. \end{aligned} \tag{31}$$

Again, we have used the symmetry of the metric and the re-indexing of repeated indices. Also, we have eliminated appearances of L by changing to derivatives with respect to the proper time.

So far, we have found that

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x^\gamma}-\frac{d}{d\sigma}\left(\frac{\partial L}{\partial(dx^\gamma/d\sigma)}\right) \\ &= L\left[g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2}+\frac{1}{2}\left(\frac{dg_{\alpha\gamma}}{dx^\beta}+\frac{dg_{\gamma\beta}}{dx^\alpha}\right)\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}\right]-\frac{L}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}. \end{aligned}$$

Rearranging the terms on the right-hand side and changing the dummy index α to δ , we have

$$\begin{aligned} g_{\alpha\gamma}\frac{d^2x^\alpha}{d\tau^2} &= \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}-\frac{1}{2}\left(\frac{dg_{\alpha\gamma}}{dx^\beta}+\frac{dg_{\gamma\beta}}{dx^\alpha}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \\ &= -\frac{1}{2}\left[\frac{dg_{\alpha\gamma}}{dx^\beta}+\frac{dg_{\gamma\beta}}{dx^\alpha}-\frac{dg_{\alpha\beta}}{dx^\gamma}\right]\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \\ &= -\frac{1}{2}\left[\frac{dg_{\delta\gamma}}{dx^\beta}+\frac{dg_{\gamma\beta}}{dx^\delta}-\frac{dg_{\delta\beta}}{dx^\gamma}\right]\frac{dx^\delta}{d\tau}\frac{dx^\beta}{d\tau} \\ &\equiv -g_{\alpha\gamma}\Gamma^\alpha_{\delta\beta}\frac{dx^\delta}{d\tau}\frac{dx^\beta}{d\tau}. \end{aligned} \tag{32}$$

We have found the geodesic equation,

$$\frac{d^2x^\alpha}{d\tau^2}+\Gamma^\alpha_{\delta\beta}\frac{dx^\delta}{d\tau}\frac{dx^\beta}{d\tau}=0, \tag{33}$$

where the Christoffel symbols satisfy

$$g_{\alpha\gamma}\Gamma^\alpha_{\delta\beta}=\frac{1}{2}\left[\frac{dg_{\delta\gamma}}{dx^\beta}+\frac{dg_{\gamma\beta}}{dx^\delta}-\frac{dg_{\delta\beta}}{dx^\gamma}\right]. \tag{34}$$

This is a linear system of equations for the Christoffel symbols. If the metric is diagonal in the coordinate system, then the computation is relatively simple as there is only one term on the left side of Equation (34). In general, one needs to use the matric inverse of $g_{\alpha\beta}$. Also, we should note that the Christoffel symbol is symmetric in the lower indices,

$$\Gamma^\alpha_{\delta\beta}=\Gamma^\alpha_{\beta\delta}.$$

We can solve for the Christoffel symbols by introducing the inverse of the metric, $g^{\alpha\beta}$, satisfying

$$g_{\mu\gamma}g_{\alpha\gamma} = \delta^{\mu}_{\alpha}.$$

Here, δ^{μ}_{α} is the usual Kronecker delta, which vanishes for $\mu \neq \alpha$ and is one otherwise. Then,

$$g^{\mu\gamma}g_{\alpha\gamma}\Gamma^{\alpha}_{\delta\beta} = \Gamma^{\mu}_{\delta\beta}.$$

Therefore,

$$\Gamma^{\mu}_{\delta\beta} = \frac{1}{2}g^{\mu\gamma}\left[\frac{dg_{\delta\gamma}}{dx^{\beta}} + \frac{dg_{\gamma\beta}}{dx^{\delta}} - \frac{dg_{\delta\beta}}{dx^{\gamma}}\right].$$

Example 7. Find the Christoffel symbols for the surface of a sphere.

This is an example of how the general geodesic computation can be used for Riemannian metrics. First, we look at the geodesics found in Example 6:

$$\begin{aligned} \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 - \frac{d^2\theta}{ds^2} &= 0 \\ \frac{d}{ds}\left(\sin^2\theta\frac{d\phi}{ds}\right) &= 0. \end{aligned} \tag{35}$$

Solving for the second-order derivatives, we have

$$\begin{aligned} \frac{d^2\theta}{ds^2} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 &= 0, \\ \frac{d^2\phi}{ds^2} + 2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} &= 0. \end{aligned} \tag{36}$$

The expanded forms of the geodesic equations for θ and ϕ are

$$\frac{d^2\theta}{ds^2} + \Gamma^{\theta}_{\theta\theta}\frac{d\theta}{ds}\frac{d\theta}{ds} + \Gamma^{\theta}_{\theta\phi}\frac{d\theta}{ds}\frac{d\phi}{ds} + \Gamma^{\theta}_{\phi\theta}\frac{d\phi}{ds}\frac{d\theta}{ds} + \Gamma^{\theta}_{\phi\phi}\frac{d\phi}{ds}\frac{d\phi}{ds} = 0, \tag{37}$$

$$\frac{d^2\phi}{ds^2} + \Gamma^{\phi}_{\theta\theta}\frac{d\theta}{ds}\frac{d\theta}{ds} + \Gamma^{\phi}_{\theta\phi}\frac{d\theta}{ds}\frac{d\phi}{ds} + \Gamma^{\phi}_{\phi\theta}\frac{d\phi}{ds}\frac{d\theta}{ds} + \Gamma^{\phi}_{\phi\phi}\frac{d\phi}{ds}\frac{d\phi}{ds} = 0. \tag{38}$$

Comparing these with the geodesics, we have that

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta,$$

and the rest of the Christoffel symbols vanish.

Now we use Equation (34) to compute the Christoffel symbols. We first note that the metric is given by

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

Because g is diagonal, the coefficients are relatively easy to find. In $g_{\alpha\gamma}\Gamma^{\alpha}_{\delta\beta}$, we note that $\alpha = \gamma$ for nonvanishing contributions from the metric. So, this gives for $\gamma = \theta$

$$g_{\theta\theta}\Gamma^{\theta}_{\delta\beta} = \frac{1}{2}\left[\frac{dg_{\delta\theta}}{dx^{\beta}} + \frac{dg_{\theta\beta}}{dx^{\delta}} - \frac{dg_{\delta\beta}}{dx^{\theta}}\right]. \tag{39}$$

Because g is independent of ϕ, all ϕ derivatives will vanish. Also, $g_{\phi\phi}$ is the only coefficient that depends on θ. So, the only time the right side of the equation does not vanish is for $\delta = \beta = \phi$. This leaves

$$\Gamma^{\theta}_{\phi\phi} = -\frac{dg_{\phi\phi}}{dx^{\theta}} = -\sin\theta\cos\theta$$

Similarly, for $\gamma = \phi$ we have

$$g_{\phi\phi}\Gamma^{\phi}_{\delta\beta} = \frac{1}{2}\left[\frac{dg_{\delta\phi}}{dx^{\beta}} + \frac{dg_{\phi\beta}}{dx^{\delta}} - \frac{dg_{\delta\beta}}{dx^{\phi}}\right]. \tag{40}$$

Using the same arguments about the derivative of the metric elements, we find one of β or δ is ϕ and the other is θ. For example,

$$\begin{aligned} g_{\phi\phi}\Gamma^{\phi}_{\phi\theta} &= \frac{1}{2}\left[\frac{dg_{\phi\phi}}{dx^{\theta}} + \frac{dg_{\phi\theta}}{dx^{\phi}} - \frac{dg_{\phi\theta}}{dx^{\phi}}\right], \\ \sin^2\theta\Gamma^{\phi}_{\phi\theta} &= \frac{1}{2}(2\sin\theta\cos\theta), \\ \Gamma^{\phi}_{\phi\theta} &= \cot\theta. \end{aligned} \tag{41}$$

Because $\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi}$, we have obtained the same results based on reading the geodesic equation.