Gravitational Waves

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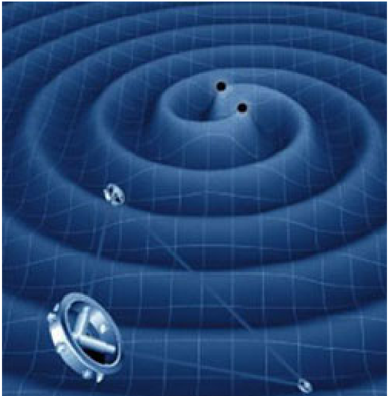

Outline

Einstein's Equation

Einstein's Equation is given by

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Riemann Tensor:

$$R^{\nu}_{\mu\rho\sigma} = \Gamma^{\nu}_{\mu\sigma,\rho} - \Gamma^{\nu}_{\mu\rho,\sigma} + \Gamma^{\nu}_{\lambda\rho}\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\nu}_{\lambda\sigma}\Gamma^{\lambda}_{\mu\rho}$$

Christoffel Symbol:

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\lambda} \left(g_{\lambda\nu,\rho} + g_{\lambda\rho,\nu} - g_{\rho\nu,\lambda} \right)$$

Einstein's Equation (cont'd)

Einstein's Equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

Ricci Tensor:

$$R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu}$$

Curvature Scalar:

$$R = g^{\mu\nu} R_{\mu\nu}$$

- This is a set of nonlinear partial differential equations for $g_{\mu\nu}$.
- There are $4 \times 4 = 16$ equations.
- Symmetry of $G_{\mu\nu}$ and $T_{\mu\nu}$ reduce the number to 10.
- The Bianchi identity, ${G_{\mu\nu}}^{;\nu} = 0$, reduces the number further to 6 equations.

Weak Gravitational Fields

Seek Solutions Near Flat Space

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1.$$

Then

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}\left(h_{\mu\nu,\rho} + h_{\mu\rho,\nu} - h_{\nu\rho,\mu}\right) + O(|h|^2)$$

$$R_{\mu\nu\rho\sigma} = \eta_{\mu\alpha}\Gamma^{\alpha}_{\nu\sigma,\rho} - \eta_{\mu\alpha}\Gamma^{\alpha}_{\nu\rho,\sigma} + O(|h|^2)$$

or,

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(h_{\mu\sigma,\nu\rho} + h_{\nu\rho,\mu\sigma} - h_{\mu\rho,\nu\sigma} - h_{\nu\sigma,\mu\rho}) + O(|h|^2)$$

$$G_{\nu\sigma} \approx \frac{1}{2}\left(h_{\mu\sigma,nu}{}^{\mu} + h_{\mu\nu,\sigma}{}^{\mu} - h_{,\nu\sigma} - h^{\mu}_{\nu\sigma,\mu} - \eta_{\nu\sigma}h_{\mu\alpha,}{}^{\alpha\mu} + \eta_{\nu\sigma}h_{,\alpha}{}^{\alpha}\right)$$

Trace Reverse Tensor $\bar{h}_{\mu\nu}$

Raising Indices:

$$A^{\mu}_{\beta} = g^{\mu\alpha} A_{\alpha\beta}.$$

Linearized version:

$$h^{\mu}_{\beta} = \eta^{\mu\alpha} h_{\alpha\beta}.$$

Trace Reverse Tensor:

$$\bar{h}^{\mu\nu} \equiv h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h.$$

Then

$$G_{\mu\nu} \approx -\frac{1}{2}\left(\bar{h}_{\mu\nu,}{}^{\lambda}_{\lambda} + \eta_{\mu\nu}\bar{h}_{\lambda\rho,}{}^{\lambda\rho} - 2\bar{h}_{\mu\lambda,}{}^{\lambda}_{\nu}\right)$$

Apply Lorentz Gauge: $\bar{h}^{\mu\nu}{}_{,\nu} = 0$. Then,

$$G_{\mu\nu} \approx -\frac{1}{2}\bar{h}_{\mu\nu,}{}^{\lambda}_{\lambda} = 8\pi T^{\mu\nu} \Rightarrow \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{h}^{\mu\nu} = -8\pi T^{\mu\nu}$$

The Lorentz Gauge

Let $x'^{\nu} = x^{\nu} + \xi^{\nu}$ for arbitrary, infinitesimal $\xi^{\mu}(x)$. Then

$$g'_{\mu\nu} = g_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$$

$$\bar{h}'^{\mu\nu} = \bar{h}^{\mu\nu} - \eta^{\mu\rho}\xi^{\nu}{}_{,\rho} - \eta^{\lambda\nu}\xi^{\mu}{}_{,\lambda} + \eta^{\mu\nu}\xi^{\rho}{}_{,\rho}$$

Differentiating,

$$\bar{h}'^{\mu\nu}{}_{,\nu} = \bar{h}^{\mu\nu}{}_{,\nu} - \eta^{\lambda\nu}\xi^{\mu}{}_{,\lambda\nu}$$

Forcing $\bar{h}'^{\mu\nu}{}_{,\nu} = 0$, we have

$$\Box^2\xi^{\mu} = \bar{h}^{\mu\nu}{}_{,\nu}$$

We can add any ψ^{μ} to ξ^{μ} such that $\Box^2\psi^{\mu} = 0$
Note: $\Box^2 \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

Wave Propagation in a Vacuum

1D Wave Equation

$u(x, t)$ satisfies a one dimensional (in space) wave equation:

$$u_{tt} = c^2 u_{xx}$$

General Solution

We seek solutions of the form

$$u(x, t) = Ae^{i(kx-\omega t)}.$$

Then, we obtain the dispersion relation $\omega^2 = c^2 k^2$. A linear combination of solutions yields the general solution

$$u(x, t) = \mathrm{Re} \int_{-\infty}^{\infty} A(k) e^{ik(x-ct)}\, dk.$$

Gravitational Wave Solution

The 3D Wave Equation

$$\Box^2 \bar{h}^{\mu\nu} = \left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \bar{h}^{\mu\nu} = 0$$

where $\frac{\partial}{\partial x^\nu} \bar{h}^{\mu\nu} = 0$.

Plane Wave Solutions

$$\bar{h}^{\mu\nu} = \mathrm{Re}\left[A^{\mu\nu} e^{ik_\alpha x^\alpha} \right], \qquad A^{\mu\nu} k_\nu = 0.$$

$A^{\mu\nu}$ is the amplitude matrix, k_ν is the (null) wave 4-vector.
The General solution can be obtained as

$$\hat{h}^{\mu\nu} = \mathrm{Re} \int A^{\mu\nu}(\mathbf{k}) e^{ik_\alpha x^\alpha} \, d^3 k.$$

Polarization

Counting Degrees of Freedom

- $\bar{h}^{\mu\nu} = \bar{h}^{\nu\mu} \Rightarrow A^{\mu\nu}$ has 10 different entries.
- $A^{\mu\nu} k_\nu = 0 \Rightarrow$ 4 conditions, leaving 6 independent entries.
- Gauge freedom, $x^\nu \rightarrow x^\nu + \xi^\nu$, adds 4 more conditions. This leave two possible polarizations.

Consider plane wave propagating in z-direction:
$k^\mu = (k, 0, 0, k)$, $k = \omega/c$.
$A^{\mu\nu} k_\nu = 0 \Rightarrow A^{\mu 0} = A^{\mu 3}$.

$$A^{\mu\nu} = \begin{pmatrix} A^{00} & A^{01} & A^{02} & A^{00} \\ A^{01} & A^{11} & A^{12} & A^{01} \\ A^{02} & A^{11} & A^{22} & A^{02} \\ A^{00} & A^{01} & A^{02} & A^{00} \end{pmatrix}$$

Apply Gauge Freedom

Let $\xi^\mu = -\mathrm{Re}\left[i\varepsilon^\mu e^{ik_\alpha x^\alpha}\right]$ Then

$$A^{\mu'\nu'} = A^{\mu\nu} - \varepsilon^\mu k^\nu - k^\mu \varepsilon^\nu + \eta^{\mu\nu}\varepsilon^\alpha k_\alpha.$$

Therefore,

$$\begin{aligned} A^{0'0'} &= A^{00} - k(\varepsilon^0 + \varepsilon^3), \quad A^{1'1'} = A^{11} - k(\varepsilon^0 - \varepsilon^3) \\ A^{0'1'} &= A^{01} - k\varepsilon^1, \quad A^{1'2'} = A^{12} \\ A^{0'2'} &= A^{02} - k\varepsilon^2, \quad A^{2'2'} = A^{22} - k(\varepsilon^0 - \varepsilon^3) \end{aligned}$$

Choose

$$\begin{aligned} \varepsilon^0 &= (2A^{00} + A^{11} + A^{22})/4k, \quad \varepsilon^1 = A^{01}/k, \\ \varepsilon^2 &= A^{02}/k, \quad \varepsilon^3 = (2A^{00} - A^{11} - A^{22})/4k. \end{aligned}$$

Then $A^{0'0'} = A^{0'1'} = A^{0'2'} = 0$ and $A^{1'1'} = -A^{2'2'}$.

Transverse Traceless Gauge

We chose gauge for waves propagating in z-direction and found

$$A^{\mu\nu}_{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A^{11} & A^{12} & 0 \\ 0 & A^{12} & -A^{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Polarization- $A^{\mu\nu} = \alpha \mathbf{e}_+ + \beta \mathbf{e}_\times$,

$$\mathbf{e}_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{e}_\times = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Example - Circularly Polarized Waves

$\mathbf{e}_R = \frac{1}{\sqrt{2}}(\mathbf{e}_+ + i\mathbf{e}_\times)$ $\mathbf{e}_L = \frac{1}{\sqrt{2}}(\mathbf{e}_+ - i\mathbf{e}_\times)$

Polarization

What happens to free test particles as a gravitational waves passes?
In TT gauge, $\Gamma^{\mu}_{00} = 0$. Therefore,

$$\frac{dx^{\mu}}{d\tau} = c\delta^{\mu}_{0}.$$

Curves having constant spatial coordinates are timelike geodesics - worldlines of a cloud of test particles.
Let $\xi^{\mu} = (0, \xi^1, \xi^2, \xi^3)$ be small spatial separation Then

$$d^2 = \tilde{g}_{ij}\xi^i\xi^j$$

is not constant, where $\tilde{g}_{ij} \equiv -g_{ij} = \delta_{ij} - h_{ij}$.
However

$$\zeta^i \equiv \xi^i + \frac{1}{2}h^i_k\xi^k$$

gives correct spatial separations.

Test Particle Displacements

In TT gauge, $h_i^3 = 0$. Thus, $\zeta^3 = \xi^3 =$const. Showing wave is transverse.
If $A^{\mu\nu} = \alpha e_+^{\mu\nu}$ and $\xi^i = (\xi^1, \xi^2, 0)$,

$$\zeta^i = (\xi^1, \xi^2, 0) - \frac{1}{2}\alpha \cos k(x^0 - x^3)(\xi^1, -\xi^2, 0)$$

So, when particles form circle about test particle when cosine=0, the particles oscillate about center.
Similarly, if $A^{\mu\nu} = \alpha e_\times^{\mu\nu}$

$$\zeta^i = (\xi^1, \xi^2, 0) - \frac{1}{2}\alpha \cos k(x^0 - x^3)(\xi^2, \xi^1, 0)$$