What is π ?

Mathematics and Statistics Club

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Dr. R. L. Herman

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Figure 1: 10000 digits of π .

Introduction

What is π ?

How do you compute π ?

How many digits do we know?

Memorization record?

Who is Akira Haraguchi?

What is your favorite π joke?

In the beginning there was ... Archimedes.



Eudoxus of Cnidus (c.390 - c. 337 BCE)

- Studied under Plato.
- Taught Aristotle.

quantities.

- Astronomer, Mathematician.
- Theory of Proportions:

Circles: $A \propto r^2$, Spheres: $V \propto r^3$, Volume of a pyramid .

Volume of a cone.Studied reals, continuous

Method of Exhaustion:
 Due to Antiphon (480–411 BCE).

 Area from a sequence of inscribed polygons.

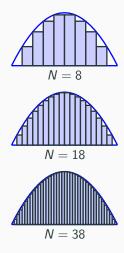


Figure 2: Method of Exhaustion.

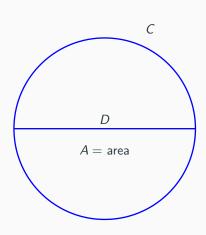
Archimedes of Syracuse (287-212 BCE)

- Went to Alexandria, Egypt then, back to Syracuse, Sicily.
- Greatest Mathematician of Antiquity.
- Mathematician, Engineer, Inventor.
 - Archimedean screw, lever, pulley.
- King Heiro II's crown Eureka.
 Archimedes Principle of Bouyancy.
- According to Plutarch (46-120):
 - Marcellus Syracuse 212 BCE.
 - · Claw of Archimedes.
 - Heat Ray.
 - Prone to intense concentration.
 - Death of Archimedes.

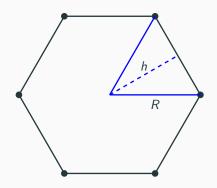




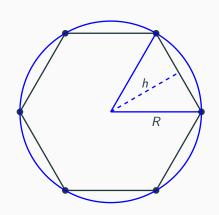
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 Method of Exhaustion.
- Measurement of a Circle $\frac{C}{D} = \text{const.}, \quad \frac{A}{D^2} = \text{const.}$



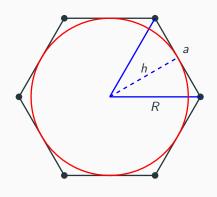
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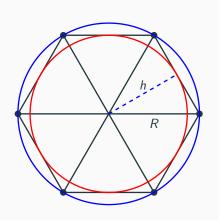
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- Approximation of π , $\frac{A_p}{R^2} < \pi < \frac{A_p}{h^2}$.

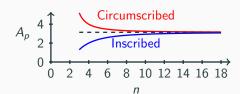


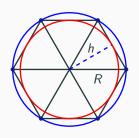
Estimating π

- Approximation of π , $\frac{A_p}{B^2} < \pi < \frac{A_p}{b^2}$,
- Recall $a = 2R \sin \frac{180}{n},$ $h = \sqrt{R^2 \frac{a^2}{4}} = R \cos \frac{180}{n},$ $A_p = \frac{1}{2} anh = nhR \sin \frac{180}{n}.$
- Therefore.

$$\frac{n}{2}\sin\frac{360}{n} < \pi < n\tan\frac{180}{n},$$

- Hexagon (n = 6), $2.598 < \pi < 3.464$.
- Archimedes up to 96-gon $3.1394 < \pi < 3.1427$





Archimedes' Inscribed and Circumscribed *n*-gons

Consider a fixed circle of radius R.

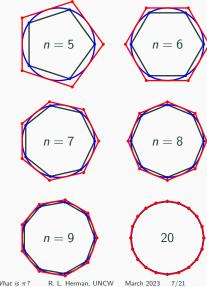
- Inscribed *n*-gon: $h = R \cos \frac{180}{n}$, $A_i = nhR \sin \frac{180}{n}$.
- Circumscribed *n*-gon: $r = \frac{R}{\cos \frac{180}{n}}, A_c = nHr \sin \frac{180}{n}.$
- Thus,

$$A_i = nR^2 \tan \frac{180}{n},$$

$$A_c = \frac{n}{2}R^2 \sin \frac{360}{n}.$$

• This gives

$$\frac{n}{2}\sin\frac{360}{n} < \pi < n\tan\frac{180}{n}$$



Early Approximations of π : Peripherion $\pi\epsilon\rho\iota\phi\epsilon\rho\epsilon\iota\alpha$

- Bible, $\pi \approx 3$.
- Babylonian $3 + \frac{1}{8}$.
- Egyptians, $(\frac{4}{3})^4 = \frac{256}{81} \approx 3.1604938$.
- Sulbasutrakaras (< 800 BCE), 3.08.
- \bullet Archimedes (250 BCE) $3\frac{10}{71} < \pi < 3\frac{1}{7}.$
- Aryabhata (499), $\frac{62832}{20000}$.
- Ptolemy (150), 360-gon, 3.14166.
- Chinese (430-501) $\frac{355}{113} \approx 3.14159292$.
- Hindu (1100) $\frac{3927}{1250} \approx 3.1416$.
- $\bullet~$ Viéte, 393,216-gon, π to 9 places.
- van Ceulen (1540-1610) Dutch, 35 places.
- William Shanks (1873) 527 digits.

- Lambert (1728-1777) irrationality proof.
- William Jones (1706) introduced π .
- Euler popularized notation.
- ullet See Approximations of π .
- Leibniz-Madhaya $\frac{\pi}{4} = 1 \frac{1}{3} + \frac{1}{5} \cdots$
- Euler $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$
- Ramanujan $\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!}{k!^4} \frac{1103 + 26390k}{396^{4k}}$
- A Complete Chronology

Buffon's Needle

1777 essay, Georges-Louis LeClerc, the Comte de Buffon (1707–1788). [See MAA Convergence]

If a needle of length ℓ is thrown randomly onto a floor marked with parallel lines, set at distance d apart, what is the probability that the needle will cross one of the lines? $p=\frac{2\ell}{\pi d}$.

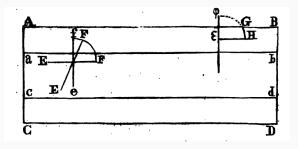


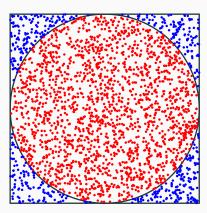
Figure 3: Buffon's sketch of the Needle Problem.

Monte Carlo Approximation

- Simulate random points in the plane with domain as a square of side 2r units centered on (0,0).
- Inscribe a circle with radius r
- Find the number points that lie inside the circle.
- Then,

$$\frac{\text{Area of circle}}{\text{Area of square}} = \frac{\pi r^2}{4r^2} = \frac{\pi}{4}.$$

$\pi \approx 3.15039$



Tikz Code for Monte Carlo

```
\begin{tikzpicture}[scale=.5]
\left( \frac{1}{1} \right)
\def\n{2500}
\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\protect\pro
\edef\k{0}
\draw[line width=1] (0.0) circle (\r):
\draw[line width=1] (-\r.-\r) rectangle (\r.\r):
\foreach \n in \{1,2,..., \n\}
\pgfmathsetmacro\myx{\r*rand};
\pgfmathsetmacro\mvv{\r*rand}:
\pgfmathsetmacro\d{\myx*\myx+\myy*\myy};
\pgfmathsetmacro{\col}{\d<\s ? "red" : "blue" }
\protect{c}{\d<\s ? 1 : 0 }
\pgfmathparse{\k+\c}
\xdef\k{\pgfmathresult}
\draw[fill,color=\col] (\myx,\myy) circle (.05);
\pgfmathsetmacro{\p}{\k/\n*4}
\node at (0,6) {$\pi\approx$ \p};
\end{tikzpicture}
                                                                                                                                                                                What is \pi?
                                                                                                                                                                                                                               R. L. Herman, UNCW March 2023
                                                                                                                                                                                                                                                                                                                                           11/21
```

More Ways to Compute π

- Let $a_n = \text{length of circumscribed, regular } 6 \cdot 2^n \text{gon.}$
- Let b_n =length of inscribed, regular $6 \cdot 2^n$ -gon about circle of radius 1/2.
- $a_0 = 2\sqrt{3}, b_0 = 3,$

$$a_{n+1} = \frac{2a_nb_n}{a_n + b_n}$$
$$b_{n+1} = \sqrt{a_{n+1}b_n}$$

- $a_4 = 3.1427$, $b_n = 3.1410$ Like 96-gon
- François Viéte (1540-1603)

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}}$$

• John Wallis (1616-1703)

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot \cdots}$$

Lord Brouckner (1620-1684)

$$\pi = \frac{4}{1 + \frac{1}{2 + \frac{9}{2 + \frac{25}{2 + \dots}}}}$$

• James Gregory (1638-1675)

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} = x - \frac{x^3}{3} + \frac{x^3}{3} - \cdots.$$

• John Machin (1680-1752) - 100 digits

$$\frac{\pi}{4} = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}$$

Ramanujan-Type Series

Ramanujan (1887-1920) used elliptic integral approximations to find

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!}{k!^4} \frac{1103 + 26390k}{396^{4k}}$$

Gives eight correct digits per term. Gosper (1985) used this to find 17 million digits of π

David and Gregory Chudnovsky (1989) found

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! (13591409 + 545140134k)}{(3k)! (k!)^3 640320^{3k+3/2}}$$

Gives 14 correct digits per term. See The New Yorker, March 1992. The Mountains of Pi Led to record (1994) 4 billion digits.

Breaking Records - Digits of π

- 2016 World record of 22,459,157,718,361 digits using 1,583,677,621,196 terms, Peter Trueb.
- In 2019, 31.4 trillion digits, Emma Haruka Iwao .
- In 2021, scientists at the University of Applied Sciences of the Grisons calculated another 31.4 trillion digits of the constant, bringing the total up to 62.8 trillion decimal places.
- 2022 another record: 100 trillion digits of π . See here.

BBP Formula

Compute the dth digit of π and other transcendental numbers in various bases without computing all of the preceding digits.

Based on identities introduced by D. Bailey, P. Borwein and S. Plouffe [2].

The identity used to obtain the binary and hexidecimal digits of $\boldsymbol{\pi}$ is given by

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right). \tag{1}$$

F. Bellard introduced an identity which produced the 40 trillionth bit of π [3]. 43% faster than that using the BBP formula.

$$\pi = \frac{1}{2^{6}} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{2^{10k}} \left(-\frac{2^{5}}{4k+1} - \frac{1}{4k+3} + \frac{2^{8}}{10k+1} - \frac{2^{6}}{10k+3} \right)$$

$$- \frac{2^{2}}{10k+5} - \frac{2^{2}}{10k+7} + \frac{1}{10k+9} \right).$$
(2)

Bellard Note

Bellard shows these are easy to find using the series expansion

$$-\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$$
, for $|x| < 1$.

Further noting that $\tan^{-1}\left(\frac{1}{x-1}\right)=\Im\left(\ln\left(1-\frac{1+i}{x}\right)\right)$ and setting x=2, the BBP Equation (1) results.

Combining this series expansion for $\tan^{-1}\left(\frac{1}{x-1}\right)$ with the relation

$$\frac{\pi}{4} = 2 \tan^{-1}(\frac{1}{2}) - \tan^{-1}(\frac{1}{7}),$$

one can obtain Bellard's Equation (2).

This is just a variation of Machin's 1709 formula

$$\frac{\pi}{4} = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}.$$

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Integration Formula

These formulae appear to be found through inspired guess work. M. Hirschhorn has shown how formula (1) can be obtained using simple integration formulae [6] using

$$\sum_{k=0}^{\infty} \frac{x^{8k+r}}{8k+r} = \int_0^x \frac{u^{r-1}}{1-u^8} \ du.$$

$$\sum_{k=0}^{\infty} \frac{x^{8k+r}}{8k+r} = \sum_{k=0}^{\infty} x^{8k+r} \int_{0}^{\infty} e^{-(8k+r)s} ds$$
$$= \int_{0}^{\infty} (xe^{-s})^{r} \sum_{k=0}^{\infty} ((xe^{-s})^{8})^{k} ds$$
$$= \int_{0}^{x} \frac{u^{r-1}}{1-u^{8}} du,$$

where $u = xe^{-s}$, du = -u ds.

Other Curiosities - Borwein Integrals - 3Blue1Brown

$$\int_0^\infty \frac{\sin(x)}{x} dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin(x)}{x} \frac{\sin(x/3)}{x/3} dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin(x)}{x} \frac{\sin(x/3)}{x/3} \frac{\sin(x/5)}{x/5} dx = \frac{\pi}{2}$$

This pattern continues ...

$$\int_0^\infty \frac{\sin(x)}{x} \frac{\sin(x/3)}{x/3} \cdots \frac{\sin(x/13)}{x/13} dx = \frac{\pi}{2}.$$

At the next step the pattern fails,

$$\int_0^\infty \frac{\sin(x)}{x} \frac{\sin(x/3)}{x/3} \cdots \frac{\sin(x/15)}{x/15} dx = ?.$$
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Other Sites

Thanks for listening!!!

- BBP Formula at Wikipedia
- Plouffe's new paper at Simon Plouffe's pages
- John Baez blog

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