# What is $\pi$?

## Mathematics and Statistics Club

3.14159265358979323846264338327950288419716939937510

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Mathematics & Statistics
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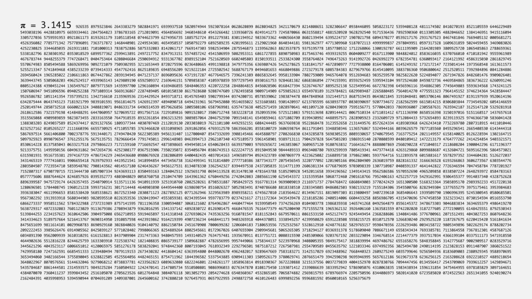


**Figure 1:** 10000 digits of $\pi$.

# Introduction

What is $\pi$?

How do you compute $\pi$?

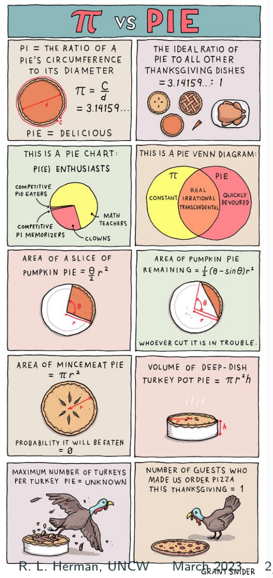
How many digits do we know?

Memorization record?

Who is Akira Haraguchi?

What is your favorite $\pi$ joke?

In the beginning there was ...
Archimedes.



# Eudoxus of Cnidus (c.390 – c. 337 BCE)

- Studied under Plato.
- Taught Aristotle.
- Astronomer, Mathematician.
- Theory of Proportions:

  Circles: $A \propto r^2$,
  Spheres: $V \propto r^3$,
  Volume of a pyramid .
  Volume of a cone.
- Studied reals, continuous quantities.
- Method of Exhaustion:
  Due to Antiphon (480–411 BCE).
  Area from a sequence of inscribed polygons.

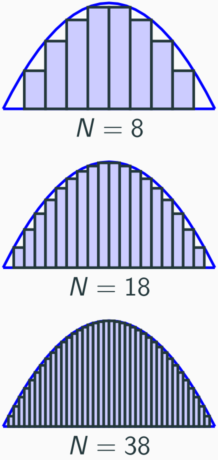
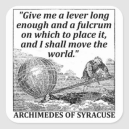
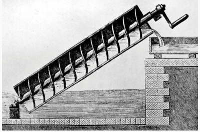



**Figure 2:** Method of Exhaustion.

# Archimedes of Syracuse (287-212 BCE)

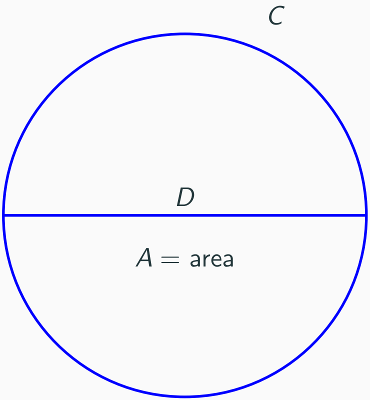
- Went to Alexandria, Egypt
  then, back to Syracuse, Sicily.
- Greatest Mathematician of Antiquity.
- Mathematician, Engineer, Inventor.
  - Archimedean screw, lever, pulley.
- King Heiro II's crown - Eureka.
  Archimedes Principle of Bouyancy.
- According to Plutarch (46-120):
  - Marcellus - Syracuse 212 BCE.
  - Claw of Archimedes.
  - Heat Ray.
  - Prone to intense concentration.
  - Death of Archimedes.



# Archimedes' Mathematics

- Mastered Euclid and Eudoxus' (c. 390-337 BCE) Method of Exhaustion.
- *Measurement of a Circle*
  $\frac{C}{D} = \text{const.}, \quad \frac{A}{D^2} = \text{const.}$



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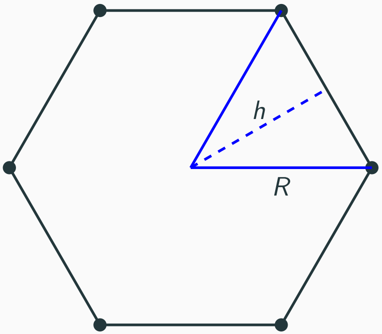
- Regular Polygons ($n$-gon)
  $A_p = \frac{1}{2}hQ$, $Q =$ Perimeter.



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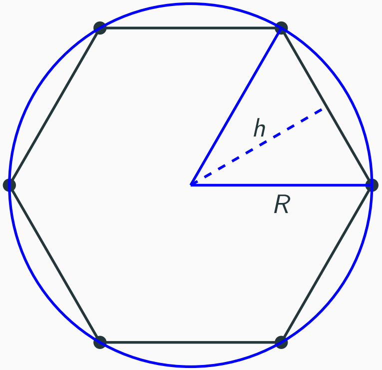

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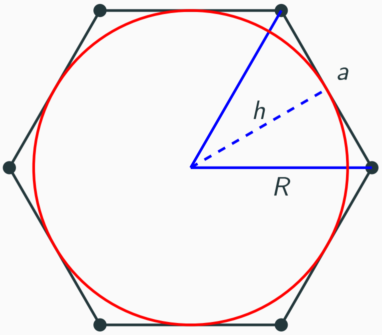
  $a = 2R\sin\frac{180}{n}, h = \sqrt{R^2 - \frac{a^2}{4}}$.



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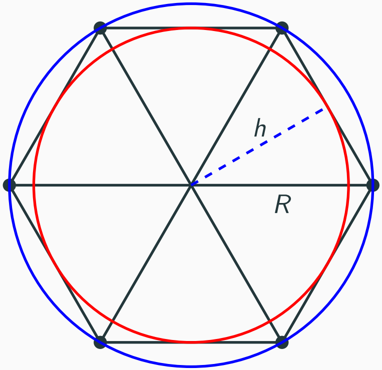
- Approximation of $\pi$,
  $\frac{A_p}{R^2} < \pi < \frac{A_p}{h^2}$.

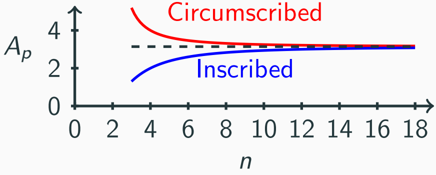


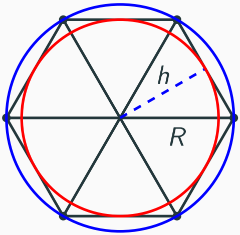
# Estimating $\pi$

- Approximation of $\pi$,
  $\frac{A_p}{R^2} < \pi < \frac{A_p}{h^2}$,
- Recall
  $a = 2R\sin\frac{180}{n}$,
  $h = \sqrt{R^2 - \frac{a^2}{4}} = R\cos\frac{180}{n}$,
  $A_p = \frac{1}{2}anh = nhR\sin\frac{180}{n}$.
- Therefore,

$$\frac{n}{2}\sin\frac{360}{n} < \pi < n\tan\frac{180}{n},$$

- Hexagon $(n = 6)$,
  $2.598 < \pi < 3.464$.
- Archimedes - up to 96-gon
  $3.1394 < \pi < 3.1427$.



# Archimedes' Inscribed and Circumscribed $n$-gons

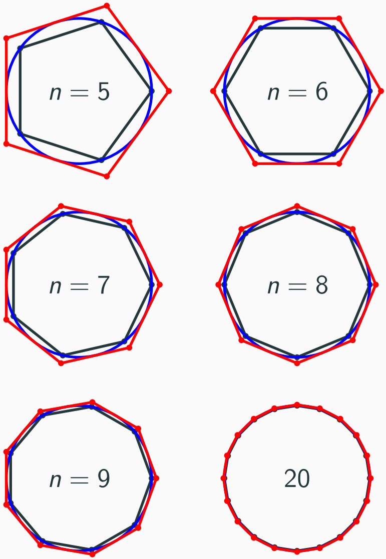
Consider a fixed circle of radius $R$.

- Inscribed $n$-gon: $h = R\cos\frac{180}{n}$, $A_i = nhR\sin\frac{180}{n}$.
- Circumscribed $n$-gon: $r = \frac{R}{\cos\frac{180}{n}}$, $A_c = nHr\sin\frac{180}{n}$.
- Thus,

$$A_i = nR^2\tan\frac{180}{n},$$
$$A_c = \frac{n}{2}R^2\sin\frac{360}{n}.$$

- This gives

$$\frac{n}{2}\sin\frac{360}{n} < \pi < n\tan\frac{180}{n},$$



# Early Approximations of $\pi$: Peripherion $\pi\epsilon\rho\iota\phi\epsilon\rho\epsilon\iota\alpha$

- Bible, $\pi \approx 3$.
- Babylonian $3 + \frac{1}{8}$.
- Egyptians, $\left(\frac{4}{3}\right)^4 = \frac{256}{81} \approx 3.1604938$.
- Sulbasutrakaras ($<$ 800 BCE), 3.08.
- Archimedes (250 BCE) $3\frac{10}{71} < \pi < 3\frac{1}{7}$.
- Aryabhata (499), $\frac{62832}{20000}$.
- Ptolemy (150), 360-gon, 3.14166.
- Chinese (430-501) $\frac{355}{113} \approx 3.14159292$.
- Hindu (1100) $\frac{3927}{1250} \approx 3.1416$.
- Viéte, 393,216-gon, $\pi$ to 9 places.
- van Ceulen (1540-1610) Dutch, 35 places.
- William Shanks (1873) 527 digits.
- Lambert (1728-1777) - irrationality proof.
- William Jones (1706) introduced $\pi$.
- Euler popularized notation.
- See Approximations of $\pi$ .
- Leibniz-Madhava $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$
- Euler $$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$
- Ramanujan $$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!}{k!^4}\frac{1103 + 26390k}{396^{4k}}$$
- A Complete Chronology

# Buffon's Needle

1777 essay, Georges-Louis LeClerc, the Comte de Buffon (1707–1788). [See MAA Convergence]

If a needle of length $\ell$ is thrown randomly onto a floor marked with parallel lines, set at distance $d$ apart, what is the probability that the needle will cross one of the lines? $p = \dfrac{2\ell}{\pi d}$.

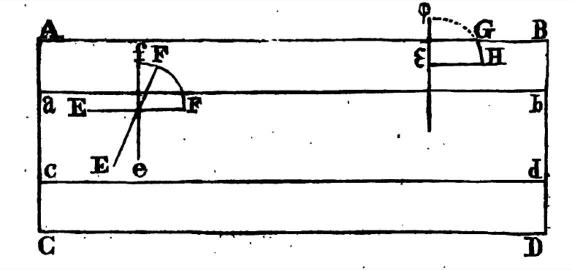


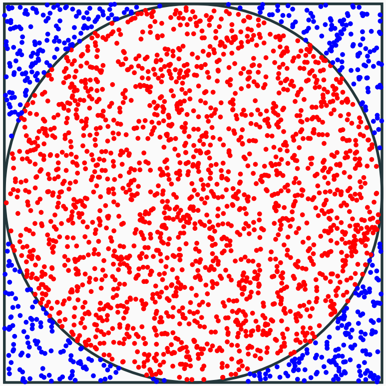

**Figure 3:** Buffon's sketch of the Needle Problem.

# Monte Carlo Approximation

- Simulate random points in the plane with domain as a square of side $2r$ units centered on $(0,0)$.
- Inscribe a circle with radius $r$
- Find the number points that lie inside the circle.
- Then,

$$\frac{\text{Area of circle}}{\text{Area of square}} = \frac{\pi r^2}{4r^2} = \frac{\pi}{4}.$$

$\pi \approx 3.15039$



# Tikz Code for Monte Carlo

```
\begin{tikzpicture}[scale=.5]
\def\r{5}
\def\n{2500}
\pgfmathsetmacro\s{\r*\r}
\edef\k{0}
\draw[line width=1] (0,0) circle (\r);
\draw[line width=1] (-\r,-\r) rectangle (\r,\r);

\foreach \n in {1,2,...,\n}{
\pgfmathsetmacro\myx{\r*rand};
\pgfmathsetmacro\myy{\r*rand};
\pgfmathsetmacro\d{\myx*\myx+\myy*\myy};
\pgfmathsetmacro{\col}{\d<\s ? "red" : "blue" }
\pgfmathsetmacro{\c}{\d<\s ? 1 : 0 }
\pgfmathparse{\k+\c}
\xdef\k{\pgfmathresult}

\draw[fill,color=\col] (\myx,\myy) circle (.05);
}
\pgfmathsetmacro{\p}{\k/\n*4}
\node at (0,6) {$\pi\approx$ \p};
\end{tikzpicture}
```

# More Ways to Compute $\pi$

- Let $a_n$ = length of circumscribed, regular $6 \cdot 2^n$-gon.
- Let $b_n$ =length of inscribed, regular $6 \cdot 2^n$-gon about circle of radius $1/2$.
- $a_0 = 2\sqrt{3},\ b_0 = 3,$

$$\begin{aligned} a_{n+1} &= \frac{2a_n b_n}{a_n + b_n} \\ b_{n+1} &= \sqrt{a_{n+1} b_n} \end{aligned}$$

- $a_4 = 3.1427,\ b_n = 3.1410$ Like 96-gon
- François Viéte (1540-1603)

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}$$

- John Wallis (1616-1703)

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdots}$$

- Lord Brouckner (1620-1684)

$$\pi = \cfrac{4}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cdots}}}}$$

- James Gregory (1638-1675)

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} = x - \frac{x^3}{3} + \frac{x^3}{3} - \cdots.$$

- John Machin (1680-1752) - 100 digits

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

# Ramanujan-Type Series

- Ramanujan (1887-1920) used elliptic integral approximations to find

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!}{k!^4} \frac{1103 + 26390k}{396^{4k}}$$

Gives eight correct digits per term. Gosper (1985) used this to find 17 million digits of $\pi$

- David and Gregory Chudnovsky (1989) found

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(13591409 + 545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}}$$

Gives 14 correct digits per term. See The New Yorker, March 1992. The Mountains of Pi

Led to record (1994) 4 billion digits.

# Breaking Records - Digits of $\pi$

- 2016 World record of 22,459,157,718,361 digits using 1,583,677,621,196 terms, Peter Trueb.
- In 2019, 31.4 trillion digits, Emma Haruka Iwao .
- In 2021, scientists at the University of Applied Sciences of the Grisons calculated another 31.4 trillion digits of the constant, bringing the total up to 62.8 trillion decimal places.
- 2022 another record: 100 trillion digits of $\pi$. See here.

# BBP Formula

Compute the $d$th digit of $\pi$ and other transcendental numbers in various bases without computing all of the preceding digits.

Based on identities introduced by D. Bailey, P. Borwein and S. Plouffe [2].

The identity used to obtain the binary and hexidecimal digits of $\pi$ is given by

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left( \frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right). \tag{1}$$

F. Bellard introduced an identity which produced the 40 trillionth bit of $\pi$ [3]. 43% faster than that using the BBP formula.

$$\begin{aligned} \pi &= \frac{1}{2^6} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{10k}} \left( -\frac{2^5}{4k+1} - \frac{1}{4k+3} + \frac{2^8}{10k+1} - \frac{2^6}{10k+3} \right. \\ &- \left. \frac{2^2}{10k+5} - \frac{2^2}{10k+7} + \frac{1}{10k+9} \right). \end{aligned} \tag{2}$$

# Bellard Note

Bellard shows these are easy to find using the series expansion

$$-\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}, \text{ for } |x| < 1.$$

Further noting that $\tan^{-1}\left(\frac{1}{x-1}\right) = \Im\left(\ln(1 - \frac{1+i}{x})\right)$ and setting $x = 2$, the BBP Equation (1) results.

Combining this series expansion for $\tan^{-1}\left(\frac{1}{x-1}\right)$ with the relation

$$\frac{\pi}{4} = 2\tan^{-1}(\frac{1}{2}) - \tan^{-1}(\frac{1}{7}),$$

one can obtain Bellard's Equation (2).

This is just a variation of Machin's 1709 formula

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}.$$

# Integration Formula

These formulae appear to be found through inspired guess work. M. Hirschhorn has shown how formula (1) can be obtained using simple integration formulae [6] using

$$\sum_{k=0}^{\infty} \frac{x^{8k+r}}{8k+r} = \int_0^x \frac{u^{r-1}}{1-u^8}\, du.$$

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{x^{8k+r}}{8k+r} &= \sum_{k=0}^{\infty} x^{8k+r} \int_0^{\infty} e^{-(8k+r)s}\, ds \\
&= \int_0^{\infty} (xe^{-s})^r \sum_{k=0}^{\infty} ((xe^{-s})^8)^k\, ds \\
&= \int_0^x \frac{u^{r-1}}{1-u^8}\, du,
\end{aligned}$$

where $u = xe^{-s}$, $du = -u\, ds$.

# Other Curiosities - Borwein Integrals - 3Blue1Brown

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\,dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\,dx = \frac{\pi}{2}$$

This pattern continues ...

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\cdots\frac{\sin(x/13)}{x/13}\,dx = \frac{\pi}{2}.$$

At the next step the pattern fails,

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\cdots\frac{\sin(x/15)}{x/15}\,dx = ?.$$

# Other Sites

Thanks for listening!!!

- BBP Formula at Wikipedia
- Plouffe's new paper at Simon Plouffe's pages
- John Baez blog

# Some References i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- M. Preston, *The Mountains of Pi*, The New Yorker, 1992, https://www.newyorker.com/magazine/1992/03/02/the-mountains-of-pi.