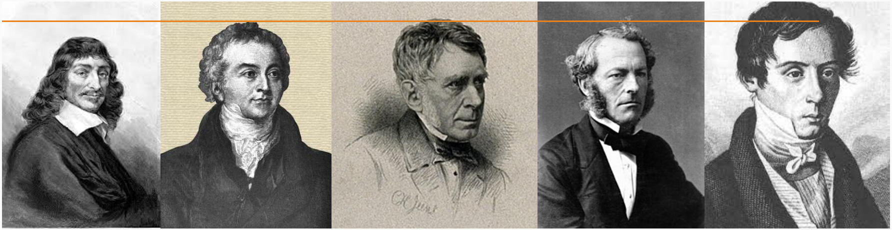
The Mathematics of Rainbows and Caustics II

Asymptotics of the Airy Function

Spring 2022 - R. L. Herman



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A Double Rainbow ...

The Features of a Rainbow

- Angular location.
- Color Order.
- Secondary Bow.
- Alexander’s Dark Band.
- Polarization.
- Supernumaries.

Geometric Optics - History

- Kamāl al-Dīn al-Fārisī (1267-1319)

  Scattering due to raindrops.
- Roger Bacon (1214-1294) - 42°
- Theodoric of Freiberg (1250-1310)
  Due to raindrops not clouds.
- Willebrord Snellius (1580-1626)
  Snell's Law of Refraction
- René Descartes (1596-1650)
  Rediscovered - internal reflections.
- Pierre de Fermat (1607-1665)
  Theory of refraction - Snell's Law.

- Refracted (Bent) towards B.
- Refracted or reflected at B.
- Laws of Optics

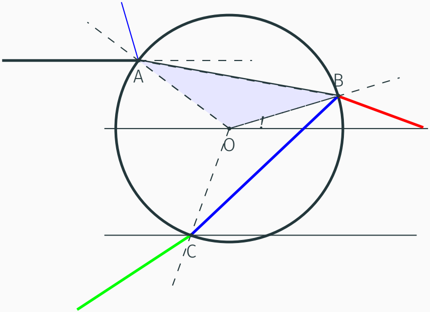
  Incident ∠ = Reflected ∠

  Snell's Law: sin ŋ sin :
- Scattering angles

  A: :

  B: 2 :

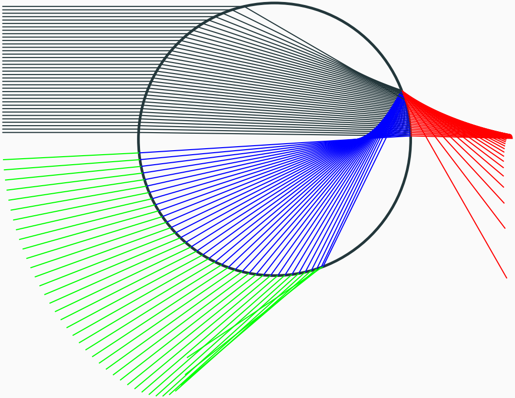
  C: :



The scattering angle is 2p q 2 4 :

Now follow multiple incident rays.

Multiple Incident Rays



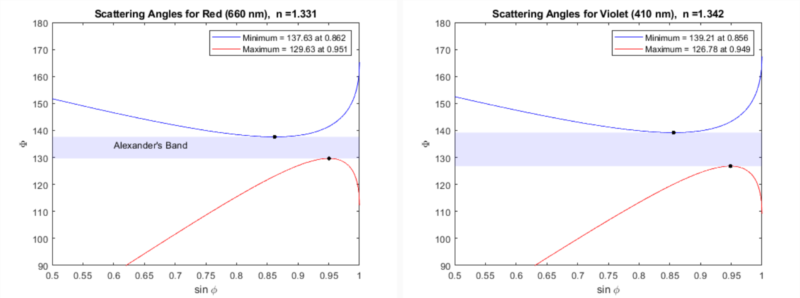
Caustics (focused rays) are formed by the rays.

Seek locations via minimum deviation angle.

Minimum Deviation - Red (660 nm) vs. Violet (410 nm)

Scattering angles showing the extrema for $ȷ$ internal reflections.

$$\sin\ ! \quad \frac{1}{ŋ}\sin\ ! \quad b\frac{pȷ\ 1q^2\ \ ŋ^2}{pȷ\ 1q^2\ \ 1};\ ȷ \quad 1;2:$$



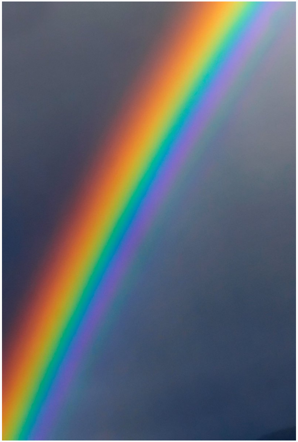
Curves are for the primary (upper) and secondary (lower) rainbows.

Primary: Rays emerge between 40$^{\varphi}$ (violet) and 42$^{\varphi}$ (red).

- Green, pink and purple fringes.
- Thomas Young (1803) - they are due to wave interference.
- R Potter (1835) [illegible]
- George Biddle Airy (1838), [illegible]
- Fringes involve square of

$$\int_0^{[illegible]} \cos \frac{[illegible]}{2} \left([illegible]^3 \; [illegible] \; [illegible] \right)$$

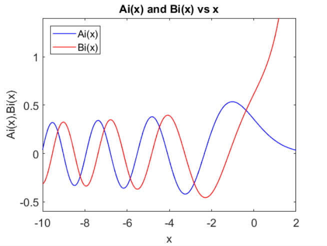
using the location of the caustic.



Airy Functions - $\mathrm{Ai}(x)$, $\mathrm{Bi}(x)$

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt$$

$$\mathrm{Bi}(x) = \frac{1}{\pi}\int_0^\infty \left[e^{-\frac{t^3}{3} + xt} + \sin\left(\frac{t^3}{3} + xt\right)\right] dt$$



Airy obtained the convergent series:

$$\mathrm{Ai}(x) = \left[1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \frac{x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \cdots\right] - \left[x + \frac{x^4}{3\cdot 4} + \frac{x^7}{3\cdot 4\cdot 6\cdot 7} + \frac{x^{10}}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10} + \cdots\right]$$

Can find as solution of $\frac{d^2y}{dx^2} - xy = 0$.

Series Solution of Airy Equation, $\frac{d^2y}{dx^2} + xy = 0$

Solve using power series method: $y(x) = \sum_{n=0}^{\infty} a_n x^n$:

$$2a_2 + \sum_{n=1}^{\infty} \left[(n+1)(n+2)a_{n+2} + a_{n-1}\right] x^n = 0.$$

Then, $a_2 = 0$ and

$$(n+1)(n+2)a_{n+2} = -a_{n-1}, \quad n = 1, 2, \ldots.$$

Solve for a_n's and get two linearly independent solutions:

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n! [2 \cdot 5 \cdots (3n-1)]} x^{3n},$$

$$y_2(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n! [1 \cdot 4 \cdots (3n+1)]} x^{3n+1}$$

Stoke's 1850 Paper

VIII. *On the numerical Calculation of a Class of Definite Integrals and Infinite Series.* *By* G. G. STOKES, M.A., *Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

[Read *March* 11, 1850.]

IN a paper "On the Intensity of Light in the neighbourhood of a Caustic*," Mr. Airy the Astronomer Royal has shown that the undulatory theory leads to an expression for the illumination involving the square of the definite integral $\int_0^\infty \cos\frac{\pi}{2}(w^3 - mw)\,dw$, where m is proportional to the perpendicular distance of the point considered from the caustic, and is reckoned positive towards the illuminated side. Mr. Airy has also given a table of the numerical values of the above integral extending from $m = -4$ to $m = +4$, at intervals of 0.2, which was calculated by the method of quadratures. In a Supplement to the same paper† the table has been re-calculated by means of a series according to ascending powers of m, and extended to $m = \pm 5.6$. The series is convergent for all values of m, however great, but when m is at all large the calculation becomes exceedingly laborious. Thus, for the latter part of the table Mr. Airy was obliged to employ 10-figure logarithms, and even these were not sufficient for carrying the table further. Yet this table gives only the first two roots of the equation $W = 0$, W denoting the definite integral, which answer to the theoretical places of the first two dark bands in a system of spurious rainbows, whereas Professor Miller was able to observe 30 of these bands. To attempt the computation of 30 roots of the equation $W = 0$ by means of the ascending series would be quite out of the question, on account of the enormous length to which the numerical calculation would run.

After many trials I at last succeeded in putting Mr. Airy's integral under a form from which its numerical value can be calculated with extreme facility when m is large, whether positive or negative, or even moderately large. Moreover the form of the expression points

- For large ’ ; the computation is tedious and convergence is slow .
- Airy only obtained two zeroes (dark bands).
- 30 dark bands measured by W. H. Miller (1841).
- Stoke’s (1850) computed 50 zeroes.

$$\mathring{}p’\,q \qquad G’ \quad 1\{4\,\bar{a} \quad 2’\,3\{2 \quad " \quad 1 \quad \frac{1\;\;5\;\;7\;\;11}{1\;\;144’\,\frac{3}{2}} \quad \frac{1\;\;5}{1\;\;2\;\;144^{2’\,3}} \quad *$$

$$P’ \quad 1\{4\,\bar{a}2’\,3\{2 \quad " \quad 1 \quad \frac{1\;\;5\;\;7\;\;11}{1\;\;144’\,\frac{3}{2}} \quad \frac{1\;\;5}{1\;\;2\;\;144^{2’\,3}} \quad *$$

$$\mathrm{Ai}p\_\,q \quad \frac{\bar{a}^{\frac{2}{3}-\frac{3}{2}}}{2\,?\,\_\,\_^{\frac{1}{4}}} \quad \sum_{\eta\;0}^{8} \frac{p\;\;1q^{\eta}\;\;p\eta\;\;\frac{5}{6}q\;\;p\eta\;\;\frac{1}{6}q\;\;\frac{3}{4}^{\;\eta}}{2\;\;\eta!\_\;\;3\eta\{2} \;; \quad |\hbar\hat{O}p\_\,q| \quad \frac{}{3}:$$

A letter Stokes wrote, London, March 19, 1857.[1]

When the cat's away the mice may play. You are the cat and I am the mouse. I have been doing what I guess you won't let me do when we are married, sitting up till 3 o'clock in the morning fighting hard against a mathematical difficulty. Some years ago I attacked an integral of Airy's, and after a severe trial reduced it to a readily calculable form. But there was one difficulty about it which, though I tried till I almost made myself ill, I could not get over, and at last I had to give it up and profess myself unable to master it. I took it up again a few days ago, and after a two or three days' fight, the last of which I sat up till 3, I at last mastered it. I don't say you won't let me work at such things, but you will keep me to more regular hours. A little out of the way now and then does not signify, but there should not be too much of it. It is not the mere sitting up but the hard thinking combined with it.

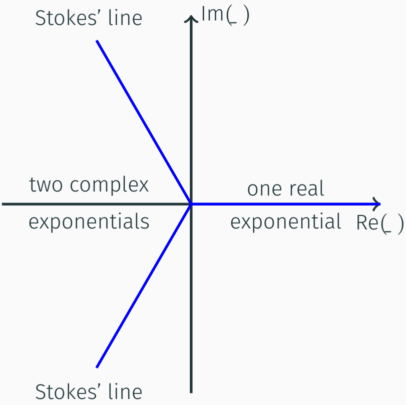
[1] Sir George Gabriel Stokes, *[illegible]*, Cambridge, 1907, Vol. 1, p. 62.

Stokes' Lines

- Stokes (1858) considered

$$I(\lambda)=\int_{-\infty}^{\infty} dt\, e^{i(t^3/3-\lambda t)}, \quad \lambda\in\mathbb{C}.$$

- $\lambda\in\mathbb{R}$ for optics.
- Now called stationary phase method.
- Stationary points $t=\pm(\lambda)^{1/2}$.
- $\lambda\in\mathbb{R}$ and $\mathrm{Re}(\lambda)>0$; one contributes, exponentially small.
- $\lambda\in\mathbb{R}$ and $\mathrm{Re}(\lambda)<0$; both contribute, oscillatory behavior.
- Second exponential has to be born between $\pm t$-axes.



Divergent Series

$$f(\epsilon) = \sum_{n=0}^{N-1} a_n(\epsilon-\epsilon_0)^n + R_N(\epsilon)$$

Convergent: $|R_N(\epsilon)| \to 0$ as $N \to \infty$; ϵ fixed.

Asymptotic: $|R_N(\epsilon)| \ll (\epsilon-\epsilon_0)^N$ as $\epsilon \to \epsilon_0$; N fixed.

Airy functions - factorial divergence.

$$\begin{Bmatrix} 2\,Ai(x) \\ Bi(x) \end{Bmatrix} \sim \frac{e^{\mp\frac{2}{3}x^{3/2}}}{2\pi^{3/2}x^{1/4}} \sum_{n=0}^{\infty} (\mp 1)^n \frac{\Gamma\left(n+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{n!\left(\frac{4}{3}x^{3/2}\right)^n}$$

Connection formula.

$$Ai\left(x e^{\mp\frac{2}{3}\pi i}\right) = \pm\frac{i}{2}e^{\mp\frac{\pi i}{3}}Bi(x) + \frac{1}{2}e^{\mp\frac{\pi i}{3}}Ai(x)$$

The Unreasonable Effectiveness of Divergent Series

Niels Abel (1802–1829) is quoted as saying, “Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever.”

“Divergent series are good at converging because they don’t have to converge”. -

Example
Evaluate for small ϵ:

$$\int_0^\infty \frac{e^{-t/\epsilon}}{1+t}\,dt$$

Main contribution when t small.

$$\int_0^\infty \frac{e^{-t/\epsilon}}{1+t}\,dt \approx \int_0^\infty e^{-t/\epsilon} \sum_{n=0}^\infty (-t)^n\,dt = \epsilon - 1!\,\epsilon^2 + 2!\,\epsilon^3 - 3!\,\epsilon^4 + \cdots \tag{1}$$

The series is divergent! It is also asymptotic.

$$\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

e^{-t^2} is analytic; i.e., $e^{-t^2} = \sum_{n=0}^{\infty} (-t^2)^n / n!$. Therefore,

$$\begin{aligned}\operatorname{Erf}(x) &= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} \\ &= \frac{2}{\sqrt{\pi}} \left(x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 \cdots \right) \qquad (2)\end{aligned}$$

Accuracy of 10^{-5}:[2]

Terms	Up to x
8	1
16	2
31	3
75	5

[2] Computer cannot give correct answer to 10^{-4} at $x = 3$ due to round-off error.

Error Function - Large x

$$\operatorname{Erf}(x) = 1 - \operatorname{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

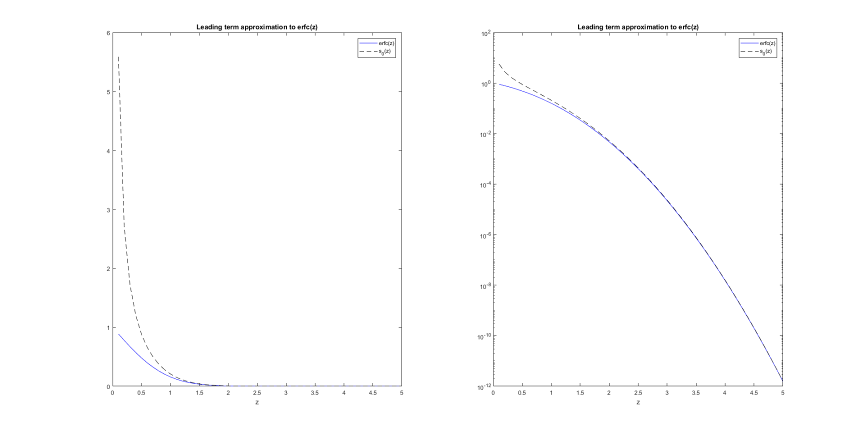
Integrate by parts,

$$\begin{aligned}\int_x^\infty e^{-t^2}\, dt &= \frac{e^{-x^2}}{2x} - \int_x^\infty \frac{e^{-t^2}}{2t^2}\, dt \\ &= \frac{e^{-x^2}}{2x}\left[1 - \frac{1}{2x^2} + \frac{3(1)}{(2x^2)^2} - \cdots\right].\end{aligned} \quad (3)$$

$$\operatorname{Erfc}(x) \approx \frac{e^{-x^2}}{\sqrt{\pi}\, x}\left[1 - \frac{1}{2x^2} + \frac{3(1)}{(2x^2)^2} - \frac{5(3)(1)}{(2x^2)^3} + O(x^{-8})\right]$$

At $x = 2.5$, only need three terms for accuracy of 10^{-5} and only need two terms when $x > 3$. However,

Plot of Erfc(z) and Leading Term



Summation of Divergent Series - Example

Euler studied[3]

$$\sum_{n=0}^{\infty} (-1)^n n! = 1 - 1 + 2 - 6 + 24 - 120 + \cdots, \tag{4}$$

obtained from ($x = 1$)

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} (-1)^n n! x^{n+1} \\ &= x - x^2 + 2x^3 - 6x^4 + 24x^5 - 120x^6 + \cdots. \end{aligned} \tag{5}$$

Can show Equation (5) leads to the Exponential integral,

$$E_1(x) = \int_x^{\infty} \frac{e^{-t}}{t}\, dt.$$

[3] Page 220 of Euler's work, `http://eulerarchive.maa.org/docs/originals/E247.pdf`

Let ’ _ 1 and note that

$$ŋ! \quad » \,_0^8 \; {}^{-ŋ} \dot{\bar{a}} \; {}^{-} \breve{a}^{-} \quad L\, \mathrm{r}^{-ŋ}\mathrm{s}_{,} \;\; {}_1 :$$

Then,

$$\overset{8}{\underset{ŋ\;0}{¸}} \; \mathrm{p} \;\; 1\mathrm{q}^{ŋ} ŋ!’ \; ŋ \;\; 1 \qquad \overset{8}{\underset{ŋ\;0}{¸}} \; \mathrm{p} \;\; 1\mathrm{q}^{ŋ} ŋ!\_ \;\; ŋ \;\; 1$$

$$\overset{8}{\underset{ŋ\;0}{¸}} \; \mathrm{p} \;\; 1\mathrm{q}^{ŋ} \;\; »\,_0^8 \; {}^{-ŋ} \dot{\bar{a}} \; {}^{-} \breve{a}^{-} \;\; \_ \;\; ŋ \;\; 1$$

$$\frac{1}{–} \; »\,_0^8 \; \breve{a}^{-} \dot{\bar{a}} \; {}^{-} \;\; \overset{8}{\underset{ŋ\;0}{¸}} \;\; \frac{–}{–} \;\; ŋ$$

$$\frac{1}{–} \; »\,_0^8 \; \frac{\dot{\bar{a}} \;\; {}^{-}}{1 \;\; \frac{-}{-}} \; \breve{a}^{-}; \qquad \frac{–}{–} \qquad 1: \tag{6}$$

Subsitute $\bar{\;}\ \_;\ \breve{a}\quad \breve{a}\bar{\;};$

$$\frac{1}{\_}\int_0^{8}\frac{\bar{\dot{a}}\ \bar{\;}}{1\ \bar{\;}}\breve{a}\bar{\;}\qquad \int_0^{8}\frac{\bar{\dot{a}}\ \bar{\;}}{\bar{\;}\quad \_}\breve{a}\bar{\;}$$

$$\int_{\_}^{8}\bar{\dot{a}}\frac{\bar{\dot{a}}}{\quad}\breve{a}\qquad \bar{\dot{a}}\ \_\_{1}p\_\ q: \tag{7}$$

Differential Equation Connection: Define $p\_\ q \quad \circ\int_{\hbar}^{8}\ {}_0p\quad 1q^{\hbar}\hbar!\_\quad \hbar\ ^1:$ Then,

$$^1p\_\ q \qquad p\_\ q \qquad \frac{1}{\_}:$$

General solution:

$$p\_\ q \quad \bar{\dot{a}}\quad p0;\_\ q \quad G\bar{\dot{a}}:$$

$p'\ ;\_\ q$ is the incomplete Gamma function:

$$p'\ ;\_\ q \qquad \int_{\_}^{8}\ \_'\ ^1\bar{a}\ \bar{\;}\breve{a}\bar{\;}\qquad \bar{\dot{a}}\ \_\_{1}\ {}'\ p\_\ q:$$

More generally, we have

$$\begin{aligned}
\sum a_n x^{n+1} &= \sum \frac{a_n}{n!} n! x^{n+1} \\
&= \sum \frac{a_n}{n!} \int_0^\infty t^n e^{-t/x}\, dt \\
&= \int_0^\infty e^{-t/x} \left(\sum \frac{a_n}{n!} t^n \right) dt \\
&= \int_0^\infty e^{-t/x} B[f](t)\, dt. \qquad (8)
\end{aligned}$$

For $f(x) = \sum a_n x^{n+1}$, define the Borel transform as

$$B[f](t) = \sum \frac{a_n}{n!} t^n$$

and the inverse transform as

$$\hat{f}(x) = \int_0^\infty e^{-t/x} B[f](t)\, dt.$$

Scheme for Borel Transform

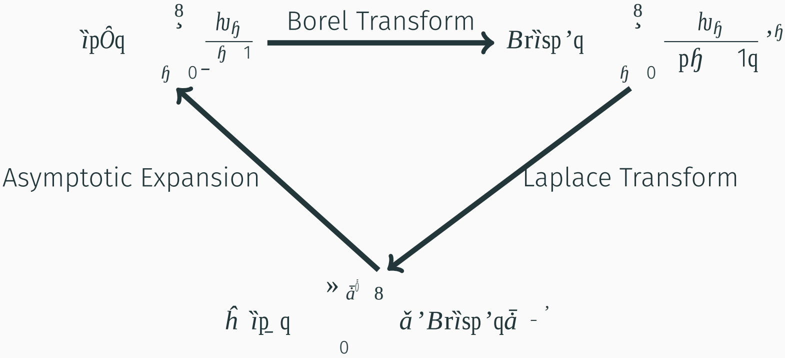


Figure 1: Schematic of Borel Transform and Resummation.

Return to Airy Functions

Recall:

$$\frac{2\ \mathring{\imath}p'q}{@\mathring{\imath}p'q} + \frac{\bar{a}^{\frac{2}{3}\,'\,3\{2}}{2\quad 3\{2'\ 1\{4} \overset{8}{\underset{ɧ\ 0}{¸}} p\quad 1q^{ɧ} \frac{ɧ\quad \frac{1}{6}\quad ɧ\quad \frac{5}{6}}{ɧ!p\frac{4}{3}\,'\ 3\{2q^{ɧ}}$$

Let $\hat{O}\quad \frac{4'\ 3\{2}{3}$; and

$$\hat{W}p\hat{O}q \quad \overset{8}{\underset{ɧ\ 0}{¸}} p\quad 1q^{ɧ} \frac{ɧ\quad \frac{1}{6}\quad ɧ\quad \frac{5}{6}}{2\quad pɧ\quad 1q} \hat{O}^{\ ɧ\quad 1} \tag{9}$$

$$Bp\hat{W}qr\,'s \quad \overset{8}{\underset{ɧ\ 0}{¸}} \frac{ɧ\quad \frac{1}{6}\quad ɧ\quad \frac{5}{6}}{2\quad pɧ\quad 1qɧ!} p\quad 'q^{ɧ} \tag{10}$$

$$_2\{_1\quad \frac{1}{6}; \frac{5}{6}; 1;\quad '\quad : \tag{11}$$

Studying the hypergeometric function for ’ P C leads to connections between $\mathring{\imath}$; @$\mathring{\imath}$; and Stokes’ lines and the richness of divergent series.

Review

- Review of Rainbows
- Geometric Optics
- Airy Functions
- Stokes' Contribution
- Divergent Series
- Borel Resummation
- More found under resurgence, transseries, alien calculus, ...

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