The Mathematics of Rainbows and Caustics

Fall 2021 - R. L. Herman





What Do You Know About Rainbows?

Questions asked by MIT Professor Walter Lewin:

- What is the radius of the rainbow?
- What is the color sequence in the rainbow?
- What is the sky darkness and brightness?
- Is there a second bow? What is the color sequence?
- Are rainbows polarized?

The Rainbow - What Do You See?

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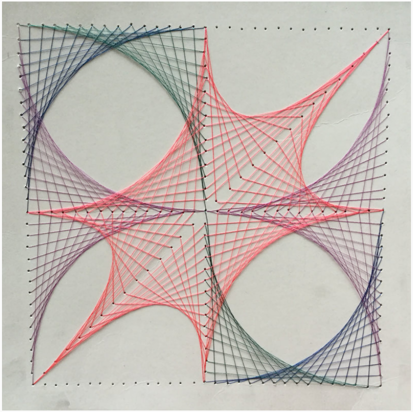


Greeks

- Aristotle (384-322 BCE)
  First to rationally explain
  Reflection of light from clouds
- Alexander of Aphrodisias
  (3rd-2nd century BCE)
  https://plato.stanford.edu/entries/alexander-aphrodisias/
- Alexander's Band

Early Arabic and European Studies

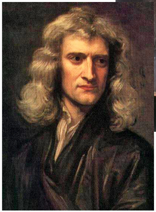
- Kamāl al-Dīn al-Fārisī (1267-1319)

  Pupil of Qutb Al-Din al-Shirāzī (1236–1311), Student of Nasir al-Din al-Tusi (1201-1274) - trigonometry creator.

  Reformed *Book of Optics*, Alhazen [Hasan Ibn al-Haytham] (965-1039) and work of Avicenna (980-1037).

  Scattering due to raindrops.
- Roger Bacon (1214-1294) - 42^o
- Theodoric of Freiberg (1250-1310)
  Due to raindrops not clouds.
- Willebrord Snellius (1580-1626) Refraction
- René Descartes (1596-1650)
  Rediscovered - internal reflections.
- Pierre de Fermat (1607-1665)
  Theory of refraction - Snell's Law.
- Isaac Newton (1642-1727) - Theory of color.



Newton's Opticks

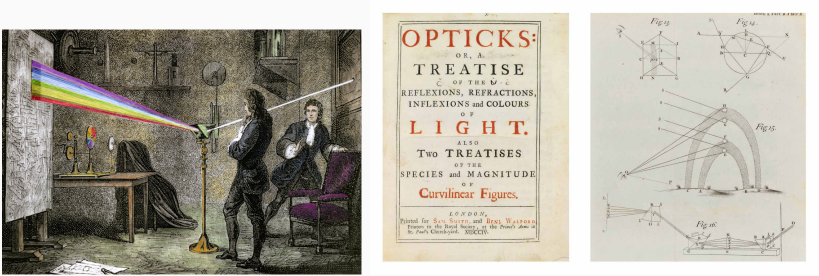


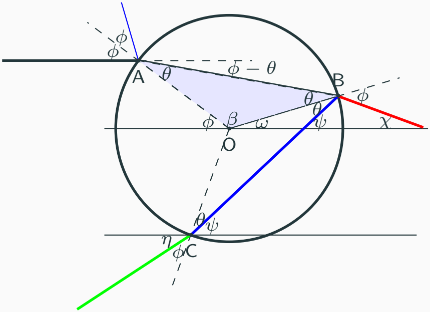
Figure 1: Newton explained dispersion and rainbow optics.

- Index of refraction - $n = \frac{c}{v}$,
- For water, $n = 1.35$ for violet to $n = 1.33$ for red.
- Snell's Law $n_{air} \sin\theta_{air} = n_{prism} \sin\theta_{prism}$.
- Wave Theory: Huygens (1678), Young (1803), Fresnel (1818), Maxwell (1862)

Geometric Optics

- Horizontally incident ray at A.
  $\phi =$ Angle of incidence
- Refracted into the droplet.
  $\theta =$ angle of refraction is
- At B: Either refracted or reflected.
- Laws of Optics

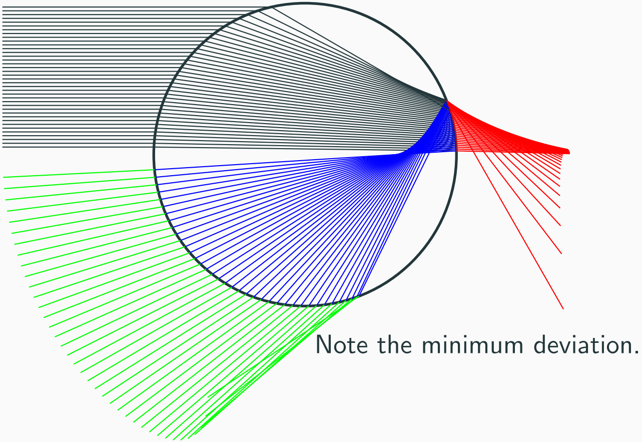
  Incident $\angle =$ Reflected $\angle$
  Snell's Law: $\sin\phi = n\sin\theta$.
- At A, the ray is bent by $\phi - \theta$.
- At B the ray reflects and turns by $\beta = \pi - 2\theta$.
- At C the ray is bent by $\phi - \theta$.
- k reflections: $\Phi = |2(\phi - \theta) + k\beta|$



The scattering angle is

$$\Phi = 2(\phi - \theta) + \beta = \pi + 2\phi - 4\theta.$$

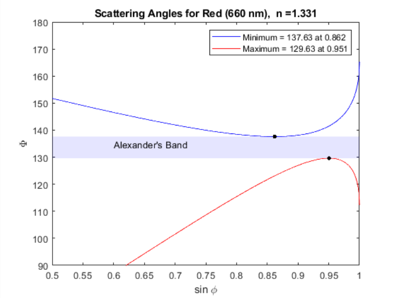
Multiple Incident Rays - Varying Impact Parameters

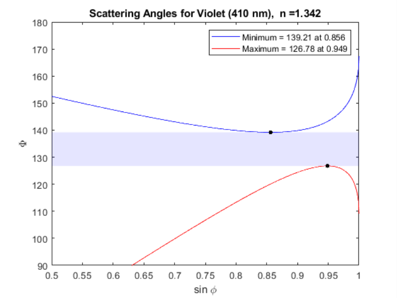


Note that **caustics** are formed by the reflected and refracted rays.

Minimum Deviation

Scattering angles for red (660 nm) and violet (410 nm) showing the extrema of the scattering angles. Extrema: $\sin\theta_c = \sqrt{\frac{(k+1)^2 - n^2}{(k+1)^2 - 1}}$, $k = 1, 2$.

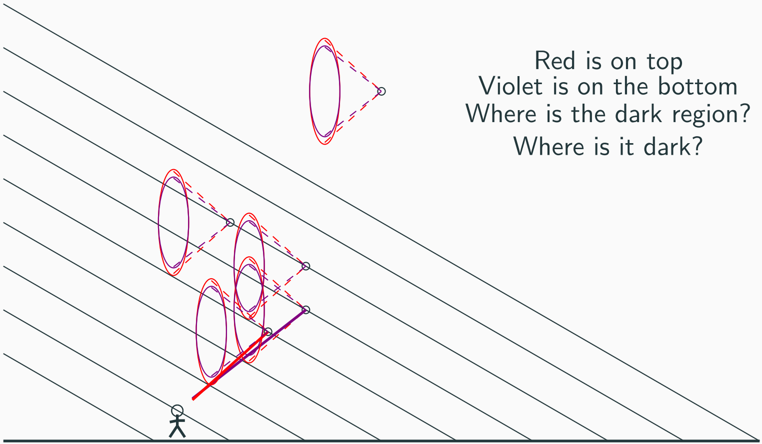




The upper curves are for the primary rainbow and the lower curves are for the secondary rainbow.

Rays emerge between 40^o (violet) and 42^o (red).

Formation of Rainbow



Red is on top
Violet is on the bottom
Where is the dark region?
Where is it dark?

Figure 2: Following the scattering cones from multiple raindrops.

Secondary Rainbow

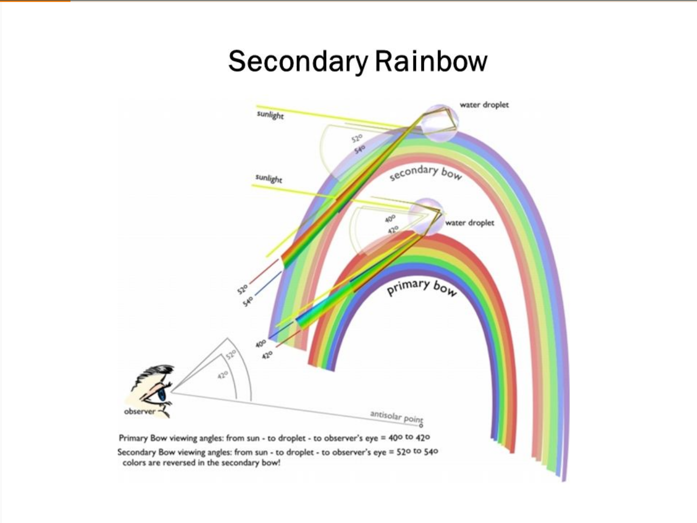
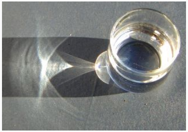


Figure 3: From `https://slideplayer.com/slide/8496591/`.

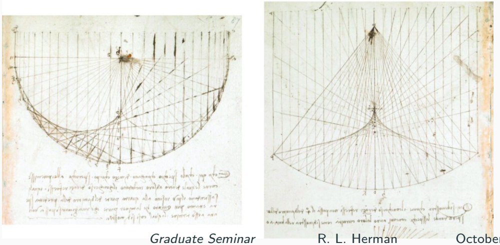
Caustics

Caustic from the Greek $\kappa\alpha\nu\sigma o\zeta$ for burnt.

- Reflection - Catacaustics
- Refraction - Diacaustics
- Leonardo da Vinci sketches.
- MathWorld Catacaustics



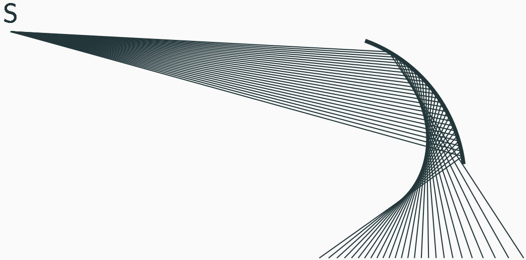
Leonardo Arundel Folio 87



Light rays from a point source

Consider the development of a caustic from a point source S.

Equation of reflected ray: $y - y_P = (x - x_P)\tan\delta$, $y_P = y_P(x_P)$.



Then, $F(x_P, y(x_P)) = 2y_P'(y - y_P) + (x - x_P)(1 - y_P'^2) = 0$.

For envelopes, set $\dfrac{dF}{dx_P} = 0$, to find $x = x_P - \dfrac{y_P'}{y_P''}$, $y = y_P + \dfrac{1 - y_P'^2}{2y_P''}$.

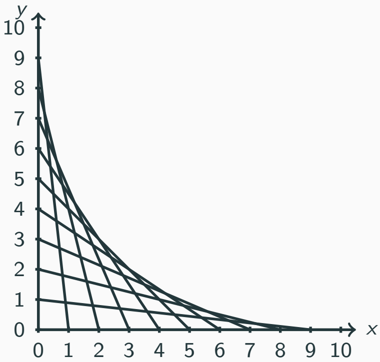
Envelope Example: String Art

- Connect $(0, k)$ with $(10 - k, 0)$.
- Equation of each line

$$y = -\frac{k}{10-k}x + k.$$

- Rewrite the family of curves, $k = t$.

$$\begin{aligned} y &= -\frac{t}{10-t}x + t \\ 1 &= \frac{x}{10-t} + \frac{y}{t} \\ t(10-t) &= xt + (10-t)y \end{aligned}$$



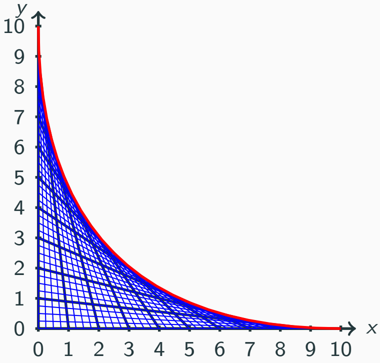
Let $F(x, y; t) \equiv t^2 + (x - y - 10)t + 10y = 0.$

String Art: Envelope Equation

- Solve $F(x, y; t) = 0$, $\dfrac{\partial F(x, y; t)}{\partial t} = 0$.
  $F(x, y; t) \equiv t^2 + (x - y - 10)t + 10y = 0.$
  $\dfrac{\partial F(x, y; t)}{\partial t} = 2t + (x - y - 10) = 0.$
- Eliminating t, we have the envelope

$$(x - y - 10)^2 - 40y = 0.$$

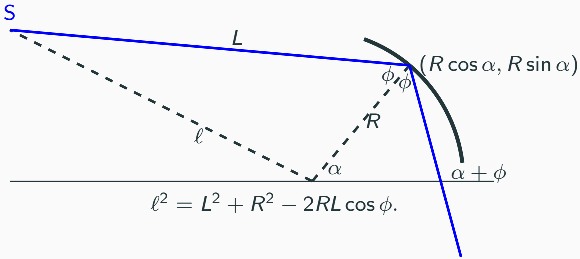
- The red curve is a parabola.
  The general conic is
  $x^2 - 2xy + y^2 - 20x - 20y + 100 = 0.$
  Letting $x = \dfrac{u - v}{\sqrt{2}}$, $y = \dfrac{u + v}{\sqrt{2}}$,
  we obtain $20\sqrt{2}u = 2v^2 + 100.$



Equation of Reflected Ray

Equation:

$$\begin{aligned} y &= (x - R\cos\alpha)\tan(\phi+\alpha) + R\sin\alpha \\ &= \frac{x\sin(\phi+\alpha) - R\cos\alpha\sin(\phi+\alpha) + R\sin\alpha\cos(\phi+\alpha)}{\cos(\phi+\alpha)} \\ &= \frac{x\sin(\phi+\alpha) - R\sin\phi}{\cos(\phi+\alpha)} \qquad (1) \end{aligned}$$



$\ell^2 = L^2 + R^2 - 2RL\cos\phi.$

Seeking Caustic Equation

From $y = \dfrac{x\sin(\phi+\alpha) - R\sin\phi}{\cos(\phi+\alpha)}$, we define
$F(x, y; \alpha) = x\sin(\phi+\alpha) - y\cos(\phi+\alpha) - R\sin\phi = 0.$

Then,
$F_\alpha(x, y; \alpha) = [x\cos(\phi+\alpha) + y\sin(\phi+\alpha)](\phi_\alpha + 1) - R\phi_\alpha\cos\phi = 0.$
Solve the system

$$\begin{aligned} x\sin(\phi+\alpha) - y\cos(\phi+\alpha) &= R\sin\phi, \\ x\cos(\phi+\alpha) + y\sin(\phi+\alpha) &= \frac{R\phi_\alpha\cos\phi}{\phi_\alpha+1}. \end{aligned} \tag{2}$$

$$\begin{aligned} x &= R\sin\phi\sin(\phi+\alpha) + R_2\cos\phi\cos(\phi+\alpha), \\ y &= R_2\cos\phi\sin(\phi+\alpha) - R\sin\phi\cos(\phi+\alpha), \end{aligned} \tag{3}$$

where $R_2 = \frac{\phi_\alpha}{\phi_\alpha+1}R$.

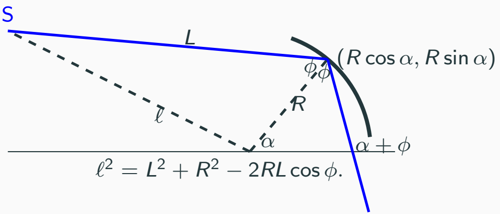
Finding ϕ_α

From $\ell^2 = L^2 + R^2 - 2RL\cos\phi$, we have $L\,dL - R\cos\phi\,dL + RL\sin\phi\,d\phi = 0$, or $d\phi = \dfrac{R\cos\phi - L}{RL\sin\phi}dL$.

Let S be the point (x_0, y_0). Since $L^2 = (R\cos\alpha - x_0)^2 + (R\sin\alpha - y_0)^2$, $L\,dL = (x_0\sin\alpha - y_0\cos\alpha)R\,d\alpha$.

So,

$$\begin{aligned}\frac{d\phi}{d\alpha} &= \frac{(R\cos\phi - L)}{RL\sin\phi}\frac{(x_0\sin\alpha - y_0\cos\alpha)R}{L}\\ &= \frac{(R\cos\phi - L)(x_0\sin\alpha - y_0\cos\alpha)}{L^2\sin\phi}.\end{aligned} \tag{4}$$



Catacaustic - Mathematics

- Appollonius' *Conica* (c. 200 BCE).
- Catacaustics introduced by Ehrenfried Walther von Tschirnhaus (1682).
- Important for early calculus.
- Jacob Bernoulli (1692) - logarithmic and parabolic spirals.
- Two chapters of Guillaume de l'Hôpital's (1696) calculus text.
- Christiaan Huygens (1673) introduced evolutes in *Horologium Oscillatorum* - cycloiods - ideal pendulum clock.
- Leibniz, Euler, Poisson, Puiseux, etc.

Spherical Mirror

The equations of the reflected rays are given by

$$y = (x - R\cos\phi)\tan 2\phi + R\sin\phi, \tag{5}$$

where ϕ is the angle of incidence. Rearranging, the reflected rays satisfy

$$F(x, y; \phi) = x\sin 2\phi - y\cos 2\phi - R\sin\phi = 0.$$

$$F_\phi(x, y; \phi) = x\cos 2\phi + y\sin 2\phi - R\cos\phi = 0.$$

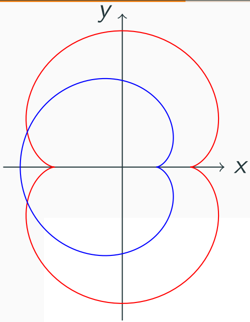
We can solve this system of equations to obtain the caustic envelope:

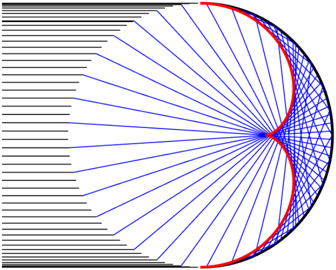
$$\begin{aligned} x &= R\sin\phi\sin 2\phi + \frac{1}{2}\cos\phi\cos 2\phi = \frac{R}{4}\left(3\cos\phi - \cos 3\phi\right). \\ y &= \frac{1}{2}R\cos\phi\sin 2\phi + \sin\phi\cos 2\phi = \frac{R}{4}\left(3\sin\phi - \sin 3\phi\right). \end{aligned} \tag{6}$$

The curve that results is a member of the family of epicycloids.

$$\begin{aligned} x(\phi) &= r(k+1)\cos\phi - r\cos\left((k+1)\phi\right) \\ y(\phi) &= r(k+1)\sin\phi - r\sin\left((k+1)\phi\right). \end{aligned} \tag{7}$$

Nephroid Catacaustics for Spherical Surface





Examples: Cardiod ($k = 1$) and a Nephroid ($k = 2$).

Prism i

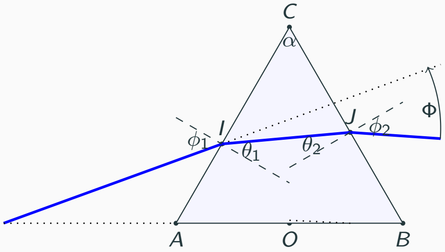
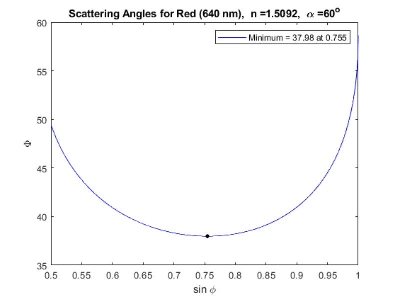


Figure 4: The bending of light through a prism.

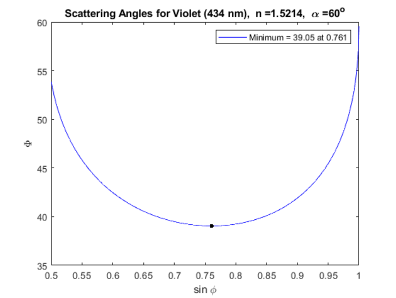
The angle of deviation, or scattering angle, is

$$\Phi = \phi_1 + \sin^{-1}\left(n \sin\left[\alpha - \sin^{-1}\left(\frac{\sin\phi_1}{n}\right)\right]\right) - \alpha.$$

Prism ii

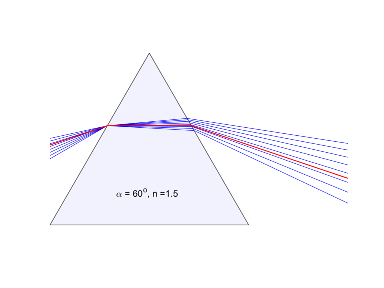


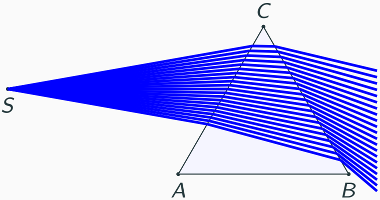
Scattering Angles for Red (640 nm), n =1.5092, α =60°
Minimum = 37.98 at 0.755
Φ
sin ϕ



Scattering Angles for Violet (434 nm), n =1.5214, α =60°
Minimum = 39.05 at 0.761
Φ
sin ϕ

Prism iii



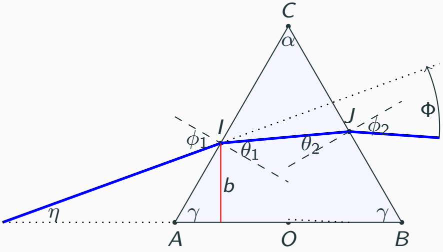


The scattered rays through a prism do not form a real caustic.

We begin with the equation for the emerging rays. The incident light enters at height b at point $I : (x_0, y_0) = (b \tan\frac{\alpha}{2} - a \sin\frac{\alpha}{2}, b)$ along the incident ray $y = (x - x_0)\tan\eta + b$.

We have that $\eta = \phi_1 - \frac{\alpha}{2}$, where ϕ_1 is the incident angle since external angle $\gamma = \eta + \frac{\pi}{2} - \phi_1$. But from the prism, $\gamma = \frac{\pi - \alpha}{2}$.

Prism iv



The ray is internally refracted by $\phi_1 - \theta_1$ from the incident ray.

The resulting angle is $\eta - (\phi_1 - \theta_1) = \theta_1 - \frac{\alpha}{2}$, giving the equation of the line:

$$y = (x - x_0)\tan(\theta_1 - \frac{\alpha}{2}) + b,$$

where $n \sin\theta_1 = \sin\phi_1$.

Prism v

At the second face at $J : (x_1, y_1)$,

$$\begin{aligned} x_1 &= \frac{x_0 \tan(\theta_1 - \frac{\alpha}{2}) + a\cos\frac{\alpha}{2} - b}{\cot\frac{\alpha}{2} + \tan(\theta_1 - \frac{\alpha}{2})} \\ &= \frac{(a\cos\frac{\alpha}{2} - b)\tan\frac{\alpha}{2}\cos\theta_1}{cos(\theta_1 - \alpha)}, \\ y_1 &= (x_1 - x_0)\tan(\theta_1 - \frac{\alpha}{2}) + b \\ &= \frac{a\sin\alpha\sin(\theta_1 - \frac{\alpha}{2}) + b\cos\theta_1}{cos(\theta_1 - \alpha)}. \end{aligned} \tag{8}$$

The second refracted ray starts at J and is bent by $\phi_2 - \theta_2$. Overall,

$$\theta_1 - \frac{\alpha}{2} - (\phi_2 - \theta_2) = \theta_1 + \theta_2 - \frac{\alpha}{2} - \phi_2.$$

Since $\theta_1 + \theta_2 = \alpha$, the emerging rays are

$$y = (x - x_1)\tan(\frac{\alpha}{2} - \phi_2) + y_1.$$

Prism vi

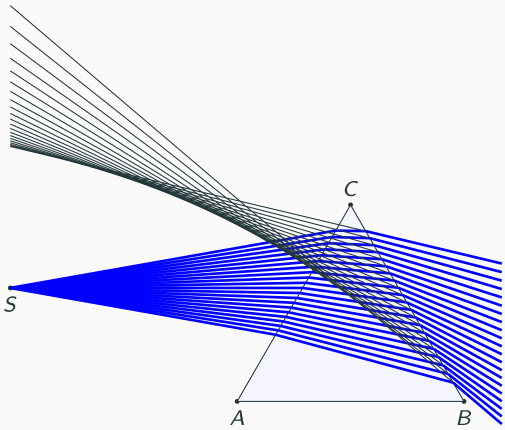


Figure 5: Extending the emerging rays backwards reveals a virtual caustic.

Prism vii

We can seek the equations of the caustic. As before, we define

$$F(x,y;\phi_1) = (x-x_1)\sin(\frac{\alpha}{2}-\phi_2) - y\cos(\frac{\alpha}{2}-\phi_2) + y_1\cos(\frac{\alpha}{2}-\phi_2),$$

where x_1, y_1 are functions of θ_1 and θ_1, ϕ_2 are functions of ϕ_1 [See Equation (8)].Differentiating F with respect to phi_1,

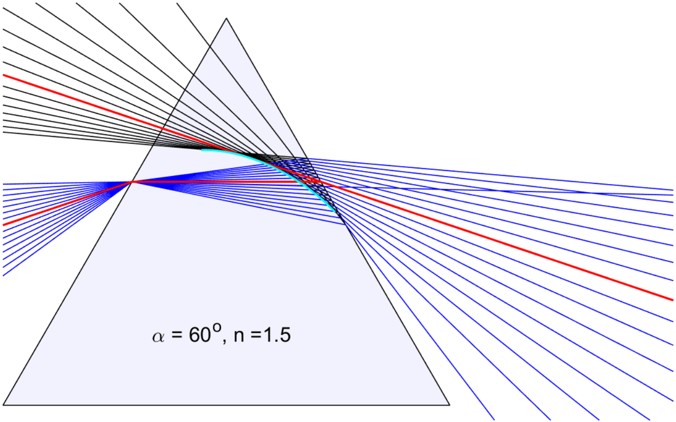
$$\frac{dF}{d\phi_1} = \frac{\partial F}{\partial \phi_1} + \frac{dF}{d\theta_1}\frac{d\theta_1}{d\phi_1} + + \frac{dF}{d\phi_2}\frac{d\phi_2}{d\phi_1} = 0,$$

where

$$\begin{aligned} \frac{d\theta_1}{d\phi_1} &= \frac{\cos\phi_1}{n\cos\theta_1} \\ \frac{d\phi_2}{d\phi_1} &= -\frac{n\cos(\alpha-\theta_1)}{\cos\phi_2}\frac{d\theta_1}{d\phi_1}. \end{aligned} \tag{9}$$

This with $F(x,y;\phi_1) = 0$ leads to parametric equations for the virtual caustic. These are determined using MATLAB and then used to superimpose the solution on Figure figprism4 as shown in Figure 6.

Prism viii



MATLAB Code

```
% Search for Caustic Solution in MATLAB
a = 3.0;    b = 0.5*a;  alpha = 60*pi/180; n = 1.5;

% Determine caustic symbolically
syms p t z x y dtdp dzdp

% f - equation for refracted rays, g = f_phi.
% u=x1, v=y1
u = @(t) cos(t)/cos(t-alpha)*(a*cos(alpha/2)-b)*tan(alpha/2);
v = @(t) (a*sin(alpha)*sin(t-alpha/2)+b*cos(t))/cos(t-alpha);
f = (x-u(t))*sin(alpha/2-z)+v(t)*cos(alpha/2-z) ...
    -y*cos(alpha/2-z);
g = diff(f,p)+diff(f,t)*dtdp+diff(f,z)*dzdp;

% (X,Y) - parametric equations for caustic
[X,Y] = solve([f==0,g==0],[x,y]);
```

Rainbow Caustics i

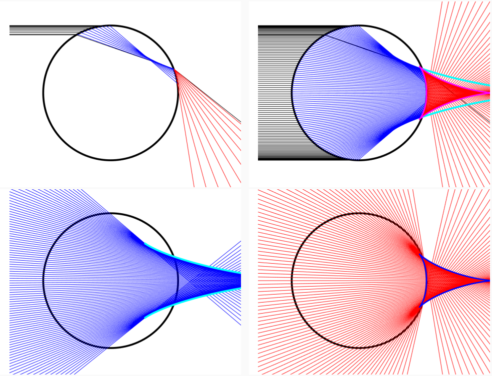


Figure 7: Light rays and caustics after passage with no reflection.

Rainbow Caustics ii

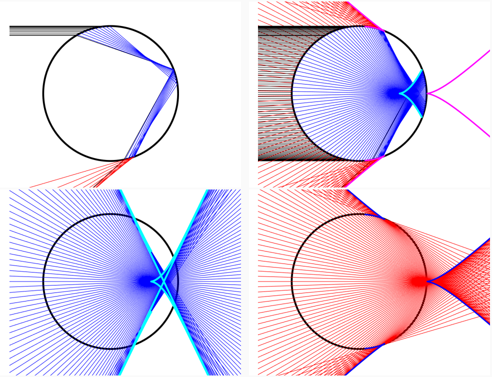


Figure 8: Light rays and caustics after passage with one reflection.

Rainbow Caustics iii

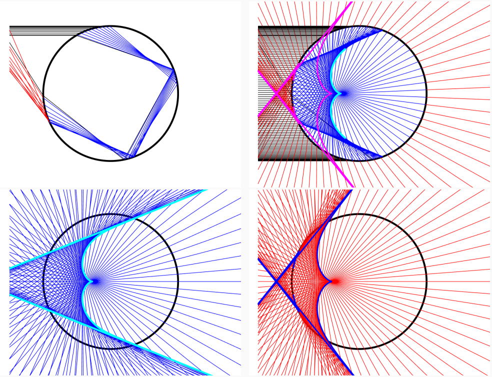


Figure 9: Light rays and caustics after passage with two reflections.

Rainbow Caustics iv

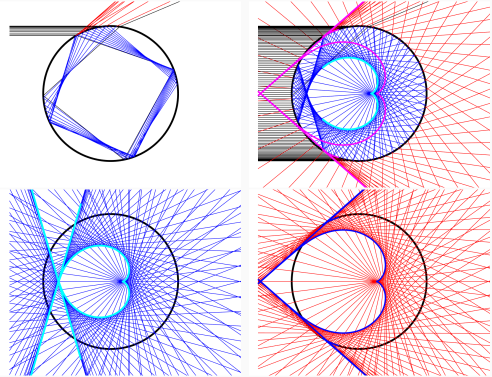


Figure 10: Light rays and caustics after passage with three reflections.

Rainbow Caustics v

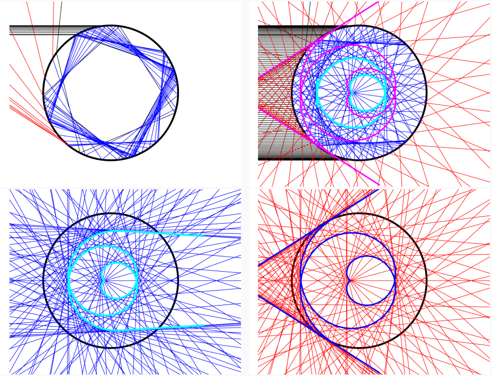
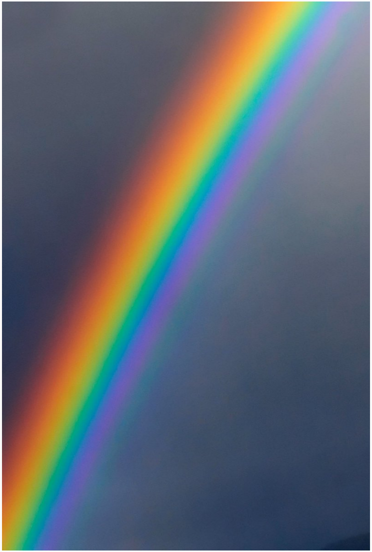


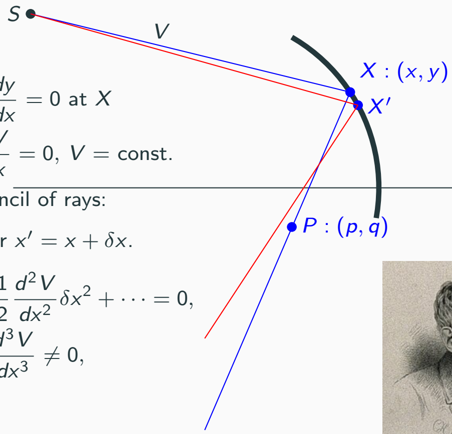
Figure 11: Light rays and caustics after passage through the droplet with six reflections.

Supernumaries

- Green, pink and purple fringes.
- Thomas Young (1803) - suggested due to wave interference.
- R Potter (1835) *Mathematical Considerations on the Problem of the Rainbow.*
- George Biddle Airy, *On the intensity of light in the neighbourhood of a caustic,* 1838.
- Later, improved theory - Mie Scattering and Debye series.
- Rayleigh scattering - why the sky is blue.



Sir George Biddell Airy (1801–1892)



$$\frac{dV}{dx} = \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}\frac{dy}{dx} = 0 \text{ at } X$$

Convergence to focus: $\dfrac{dV}{dx} = 0,\ V = \text{const.}$

Convergence of pencil of rays:

$$\frac{dV}{dx} = 0,\ \frac{dV}{dx'} = 0, \text{ for } x' = x + \delta x.$$

So, $\dfrac{dV}{dx'} = \dfrac{dV}{dx} + \dfrac{dV}{dx}\delta x + \dfrac{1}{2}\dfrac{d^2V}{dx^2}\delta x^2 + \cdots = 0,$

$$\frac{d^2V}{dx^2} = 0, \text{ but } \frac{d^3V}{dx^3} \neq 0,$$

Figure 12: A summary of what lead Airy first to an equation which holds for the convergence of rays based on his Figure I.

Airy Functions

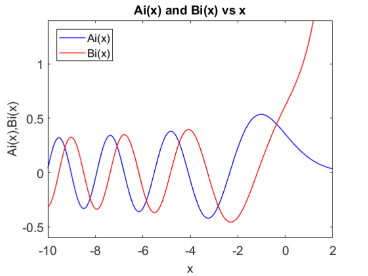
- Airy - Caustic singularity corrected with diffraction.
- Used Thomas Young's 1801 superposition.
- Airy's Integral

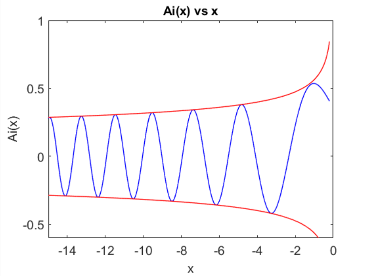
$$Ai(x) = \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right)\, dt$$

- Convergent series

$$\begin{aligned} Ai(x) &= A\left\{1 + \frac{9x+3}{2\cdot 3} + \frac{9^2x^6}{2\cdot 3\cdot 5\cdot 6} + \frac{9^3x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \cdots\right\} \\ &\quad + B\left\{x + \frac{9x^4}{3\cdot 4} + \frac{9^2x^7}{3\cdot 4\cdot 6\cdot 7} + \frac{9^3x^{10}}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10} + \cdots\right\} \end{aligned}$$

Airy Functions





$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right)\, dt$$

$$Bi(x) = \frac{1}{\pi}\int_0^\infty \left[e^{-\frac{t^3}{3}+xt} + \sin\left(\frac{t^3}{3} + xt\right)\right]\, dt$$

Airy Differential Equation

$$\frac{d^2y}{dx^2} - xy = 0,$$

Solve using power series method: $y = \sum_{m=0}^{\infty} a_m y^m$:

$$2a_2 + \sum_{m=1}^{\infty} \left[(m+1)(m+2)a_{m+2} + a_{m-1}\right] x^m = 0.$$

Equate coefficients to zero and Solve for a_m's.

$$\begin{aligned} y_1 &= 1 + \sum_{m=1}^{\infty} \frac{(-1)^n}{3^n n![2 \cdot 5 \cdot \cdots \cdot (3n-1)]} x^{3m}, \\ y_2 &= \sum_{m=1}^{\infty} \frac{(-1)^n}{3^n n![1 \cdot 4 \cdot \cdots \cdot (3n+1)]} x^{3m+1} \end{aligned}$$

Sir George Gabriel Stokes (1819-1903)

- When x large, computation tedious and slow convergence.
- Airy - only two bands.
- But 30 bands measured!
- Stoke's Expansion (1850)

$$
\begin{aligned}
Ai(x) &= Cx^{-1/4}e^{-2x^{3/2}}\left\{1-\frac{1\cdot 5\cdot 7\cdot 11}{1\cdot 144x^{\frac{3}{2}}}+\frac{1\cdot 5}{1\cdot 2\cdot 144^2x^3}+\cdots\right\}\\
&\quad +Dx^{-1/4}e^{2x^{3/2}}\left\{1+\frac{1\cdot 5\cdot 7\cdot 11}{1\cdot 144x^{\frac{3}{2}}}+\frac{1\cdot 5}{1\cdot 2\cdot 144^2x^3}+\cdots\right\}
\end{aligned}
$$

$$
\mathrm{Ai}(z)\sim\frac{e^{-\frac{2}{3}z^{\frac{3}{2}}}}{2\sqrt{\pi}\,z^{\frac{1}{4}}}\left[\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})\left(\frac{3}{4}\right)^n}{2\pi n!z^{3n/2}}\right], |arg(z)|<\frac{\pi}{3}.
$$

A letter Stokes wrote, London, March 19, 1857.[1]

When the cat's away the mice may play. You are the cat and I am the mouse. I have been doing what I guess you won't let me do when we are married, sitting up till 3 o'clock in the morning fighting hard against a mathematical difficulty. Some years ago I attacked an integral of Airy's, and after a severe trial reduced it to a readily calculable form. But there was one difficulty about it which, though I tried till I almost made myself ill, I could not get over, and at last I had to give it up and profess myself unable to master it. I took it up again a few days ago, and after a two or three days' fight, the last of which I sat up till 3, I at last mastered it. I don't say you won't let me work at such things, but you will keep me to more regular hours. A little out of the way now and then does not signify, but there should not be too much of it. It is not the mere sitting up but the hard thinking combined with it.

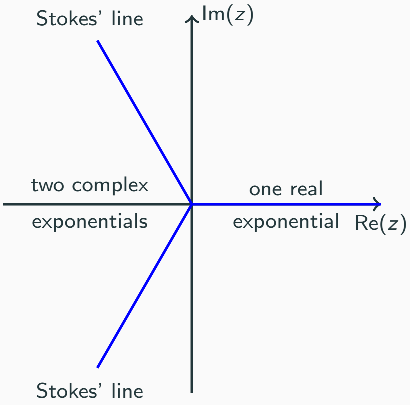
[1] Sir George Gabriel Stokes, *Memoirs and Scientific Correspondence*, Cambridge, 1907, Vol. 1, p. 62.

Stokes' Lines

- Stokes (1858) considered

$$Ai(z) = \int_{-\infty}^{\infty} ds\, e^{i(s^3/3+zs)}, \quad z \in \mathbb{C}.$$

- $z \in \mathbb{R}$ for optics.
- Now called WKB method, or stationary phase method.
- Stationary points $s = \pm(-z)^{1/2}$.
- z real and $\mathrm{Re}(z) \gg 0$, one contributes, exponentially small.
- z real and $\mathrm{Re}(z) \ll 0$, both contribute, oscillatory behavior.
- So, a second exponential has to be born between $\pm x$-axes.



Divergent Series[2]

$$f(x) = \sum_{n=0}^{N-1} a_n (x - x_0)^n + R_N(x)$$

Convergent: $|R_N(x)| \to 0$ as $N \to \infty$, x fixed.

Asymptotic: $|R_N(x)| \ll (x - x_0)^N$ as $x \to x_0$, N fixed.

Airy functions - factorial divergence.

$$\left. \begin{array}{c} 2Ai(x) \\ Bi(x) \end{array} \right\} = \frac{\mp e^{\frac{2}{3}x^{3/2}}}{2\pi^{3/2} x^{1/4}} \sum_{n=0}^{\infty} (\mp 1)^n \frac{\Gamma\left(n + \frac{1}{6}\right)\Gamma\left(n + \frac{5}{6}\right)}{n! (\frac{4}{3} x^{3/2})^n}$$

Connection formula.

$$Ai(e^{\mp \frac{2\pi i}{3}} x) = \pm \frac{i}{2} e^{\mp \frac{\pi i}{3}} Bi(x) + \frac{1}{2} e^{\mp \frac{\pi i}{3}} Ai(x)$$

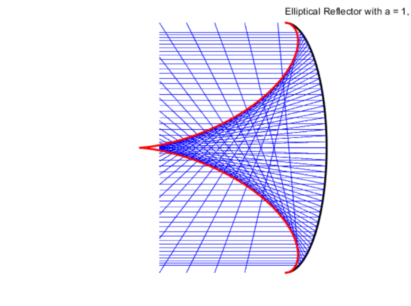
[2] *Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever.* - Niels Abel (1802–1829)

Analytic Continuation Using Borel Transform

$$
\begin{tikzcd}
f(g) = \displaystyle\sum_{n=0}^{\infty} a_n g^{n+1} \arrow[rr, "\text{Borel Transform}"] & & \mathcal{B}[f](s) = \displaystyle\sum_{n=0}^{\infty} \frac{a_n}{n!} s^n \arrow[dl, "\text{Laplace Transform}"] \\
& Sf(g) = \dfrac{1}{g} \displaystyle\int_0^{\infty} e^{-\zeta/g} \mathcal{B}[f](\zeta)\, d\zeta \arrow[ul, "\text{Asymptotic Expansion}"] &
\end{tikzcd}
$$

Review

- Brief History of Rainbows
- Geometric Optics
- Caustics - Other Surfaces
- Airy Functions
- Divergent Series
- Maxwell's Equations (1860's)
- Fresnel Equations, Polarization.
- Scattering Theory - Rayleigh, Mie, Debye
- Take away for students
    - Review sources and history.
    - LaTeX- Figures from Tikz, MATLAB,
    - Presentation in Beamer



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