

The Lane-Emden-Fowler Equation

- An Intriguing ODE

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Dr. R. L. Herman

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Mathematics & Statistics
UNC Wilmington

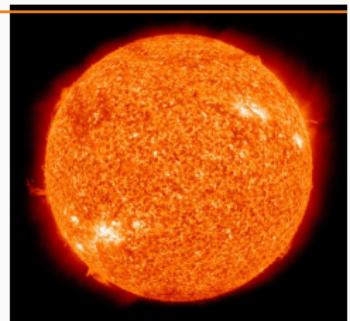


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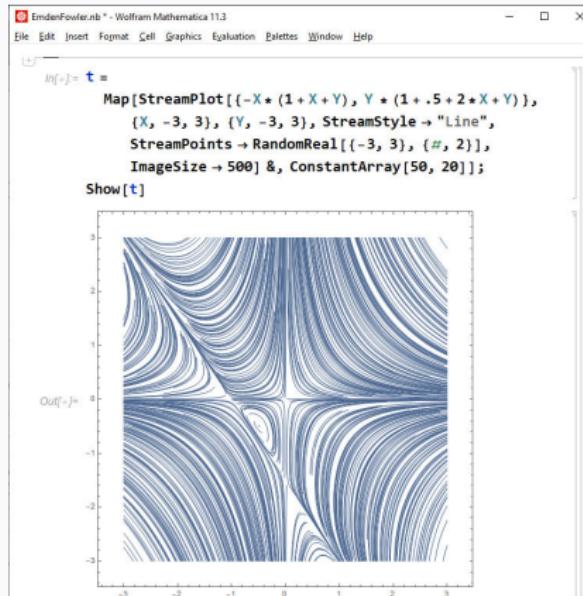
Introduction

The Lane-Emden Equation

In Search of Solutions

Nonlinear Dynamics

Stability Analysis



Student Search of a Capstone Problem

- Interested in applications, physics, nonlinear dynamics, solar energy.
- Mancas and Rosu (2015), *Existence of periodic orbits in nonlinear oscillators of Emden-Fowler form*, **Phys. Lett. A** **380**, 422-428.
- Paper referred to applications and Generalization of Lane-Emden equation,

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n,$$
$$\theta'' + \frac{2}{\xi} \theta' + \theta^n = 0.$$

- Gravitational potential of self-gravitating gas.
- Used by Eddington - internal constitution of stars.
- Thomas-Fermi model of electrons in atoms.
- Authors studied a related dynamical system.

Literature Search

- **Thomson, W.**, 1862 - Convective Equilibrium - Temperature of Earth. Collected Papers, Vol 5, pg 266.
- **Jonathan Homer Lane (1819-1880)**
1870 - *On the Theoretical Temperature of the Sun under the Hypothesis of a Gaseous Mass maintaining its Volume by its Internal Heat, and depending on the Laws of Gases as known to Terrestrial Experiment.*
Independently - A. Ritter (1878), Kelvin (1887).
- **(Jacob) Robert Emden (1862-1940)**
1907 *Gaskugeln* - thermal behavior of a spherical cloud of gas acting under mutual attraction of molecules.
- **Ralph Howard Fowler (1889-1944)**
Published papers in 1914, 1930, 1931 on a generalized equation.
- **Subrahmanyan Chandrasekhar (1910-1995)**
An Introduction to the Study of Stellar Structure, 1939, Univ. Chicago. - Discussed solutions and astrophysical applications.
- **Harold T. Davis**, *Introduction to Nonlinear Differential Equations*, 1962, Dover.
- Emphasizes Emden-like equations.

Stellar Structure

- Hydrostatic Equilibrium

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}.$$

- Mass Conservation

$$\frac{dM}{dr} = -4\pi\rho(r)r^2.$$

- Leads to

$$\begin{aligned}\frac{d}{dr} \left(\frac{r^2}{\rho(r)} \frac{dP}{dr} \right) &= -G \frac{dM(r)}{dr} \\ &= -4\pi G \rho(r) r^2.\end{aligned}$$

Note: Adiabatic $P \propto \rho^\gamma$, $P^{1-\gamma} T^\gamma = \text{const.}$, $T \rho^{1-\gamma} = \text{const.}$

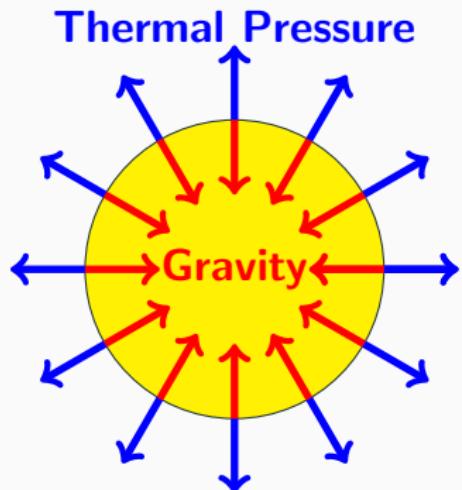


Figure 1: Balance of thermal pressure with gravitational pressure.

Polytropic Models

$$\frac{d}{dr} \left(\frac{r^2}{\rho(r)} \frac{dP}{dr} \right) = -4\pi G \rho(r) r^2.$$

Need an Equation of State, $P = P(\rho)$.

$$P = K\rho^\gamma, \quad \gamma = \frac{n+1}{n},$$

where n is the polytropic index.

- $n = 0$, rocky planets.
- $0.5 \leq n \leq 1$, neutron stars.
- $n = 3$, white dwarfs, Sun.

Gives Poisson Equation: $\nabla^2 \rho = f$.

$$\frac{(n+1)K}{4\pi n G} \frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho^{\frac{n-1}{n}}} \frac{d\rho}{dr} \right) = -\rho.$$

Lane-Emden Equation

- Rewrite the Poisson Equation:

$$\frac{(n+1)K}{4\pi nG} \frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho^{\frac{n-1}{n}}} \frac{d\rho}{dr} \right) = -\rho.$$

- Let $\rho = \rho_c \theta^n$, $r = \alpha \xi$, then get Lane-Emden Equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n.$$

- Initial Conditions,
 $\theta(0) = 1$, $\theta'(0) = 0$.
- Fowler generalized to
Emden-Fowler Equation

$$\frac{d}{d\xi} \left(\xi^\rho \frac{d\theta}{d\xi} \right) + b \xi^\sigma \theta^n = 0.$$

$$\alpha = \left(\frac{(n+1)K}{4\pi G} \rho_c^{\frac{1}{n}-1} \right)^{\frac{1}{2}}$$

$n = 3$	Eddington
$n = \frac{3}{2}$	Thomas-Fermi Model
$n = \frac{3}{2}, \frac{5}{2}$	Kelvin, "Home Lane's Function,"

Generalizations: $\frac{d^2y}{dx^2} + \frac{2}{x} \frac{dy}{dx} + f(y) = 0$.

- $f(y) = (y^2 - C)^{\frac{3}{2}}$, Chandrasekhar - white dwarfs
- $f(y) = e^y$, - isothermal gas spheres,
Emden-Chandrasekhar equation.

Known Solutions: $\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n$, $\theta(0) = 1$, $\theta'(0) = 0$.

- $n = 0$, $\theta'' + \frac{2}{\xi}\theta' + 1 = 0$. **Solution:** $\theta = -\frac{1}{6}\xi^2 - \frac{c_1}{\xi} + c_2$.
- $n = 1$, $\theta'' + \frac{2}{\xi}\theta' + \theta = 0$. **Solution:** $\theta = \frac{c_1 \cos \xi + c_2 \sin \xi}{\xi}$.
- Apply conditions: $n = 0 : \theta = 1 - \frac{\xi^2}{6}$, $n = 1 : \theta = \frac{\sin \xi}{\xi}$.
- $n = 5$, $\theta = \frac{1}{\sqrt{1 + \frac{1}{3}\xi^2}}$.
- Series: $\theta = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 - \dots$, $\xi < 1$.
- Sambhunath Srivastava (1962), $n = 5$, $\theta = \frac{\sin(\ln \sqrt{\xi})}{\sqrt{\xi} [3 - 2 \sin^2(\ln \sqrt{\xi})]}$.

Numerical Solutions

Write $\theta'' = -\frac{2}{\xi} \frac{d\theta}{d\xi} - \theta^n$.

For $\theta_i \approx \theta(\xi_i)$, and $\omega_i = \left(\frac{d\theta}{d\xi} \right)_i$,
 $\xi_i = i\Delta\xi, i = 0, 1, \dots, N$,

$$\begin{aligned}\theta_{i+1} &= \theta_i + \Delta\xi \omega_i \\ \omega_{i+1} &= \omega_i + \Delta\xi \left[\frac{2}{\xi_i} \omega_i - \theta_i^n \right].\end{aligned}$$

Set conditions $\theta_0 = 1, \omega_0 = 0$.

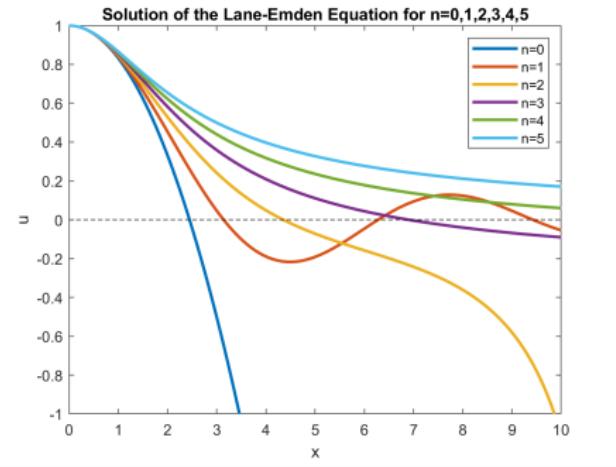


Figure 2: Numerical Solutions for $n = 0, 1, 2, 3, 4, 5$. **Note** - Zeros are important for $n < 5$.

Nonlinear Dynamics

Mancas and Rosu [9] studied the Emden-Fowler equation ($\theta \rightarrow \eta$),

$$\frac{d}{d\xi} \left(\xi^2 \frac{d\eta}{d\xi} \right) = \alpha \xi^\lambda \eta^n,$$

Using a homework problem in Jordan and Smith [7], one sets¹

$$X = \frac{\xi \eta'}{\eta} \quad \text{and} \quad Y = \xi^{\lambda-1} \frac{\eta^n}{\eta'}$$

and $\xi = e^t$, to obtain a two-dimensional autonomous system,

$$\begin{aligned}\dot{X} &= -X(1 + X - \alpha Y) \\ \dot{Y} &= Y(1 + \lambda + nX - \alpha Y).\end{aligned}\tag{1}$$

Here the dot represents $\frac{d}{dt}$.

¹Chandrasekhar studied homology invariant functions for Lane-Emden eqn., pg 105,
 $u = -\frac{\xi \eta'^n}{\eta}$, $v = \frac{-\xi \eta'}{\eta}$. Then, $\frac{u}{v} \frac{dv}{du} = \frac{u+v-1}{u+nv-3}$, attributed to E. A. Milne.

Linear Stability Analysis ($n \neq 1$)

$$\begin{aligned}\dot{X} &= -X(1 + X - \alpha Y) \\ \dot{Y} &= Y(1 + \lambda + nX - \alpha Y).\end{aligned}\tag{2}$$

The Jacobian matrix,

$$J(X, Y) = \begin{pmatrix} -1 - 2X + \alpha Y & \alpha X \\ nY & 1 + \lambda + nX - 2\alpha Y \end{pmatrix},$$

with eigenvalues $\mu = \frac{\text{tr} J \pm \sqrt{\text{tr}^2 J - 4 \det J}}{2} \equiv \frac{\text{tr} J \pm \sqrt{\Delta}}{2}$.

(X_0, Y_0)	$\det J$	$\text{tr } J$	Δ
$(0, 0)$	$-1 - \lambda$	λ	$(\lambda + 2)^2$
$(-1, 0)$	$\lambda - n + 1$	$\lambda - n + 2$	$(\lambda - n)^2$
$(0, \frac{1+\lambda}{\alpha})$	$-\lambda(1 + \lambda)$	-1	$(1 + 2\lambda)^2$
$\left(\frac{\lambda}{1-n}, \frac{\lambda-n+1}{\alpha(1-n)}\right)$	$\frac{\lambda(n-\lambda-1)}{n-1}$	$\frac{2\lambda-n+1}{n-1}$	$\frac{4\lambda n(\lambda - n + 1) + (-1 + n)^2}{(n - 1)^2}$

The Behavior of the Equilibrium Points - Standard Diagram

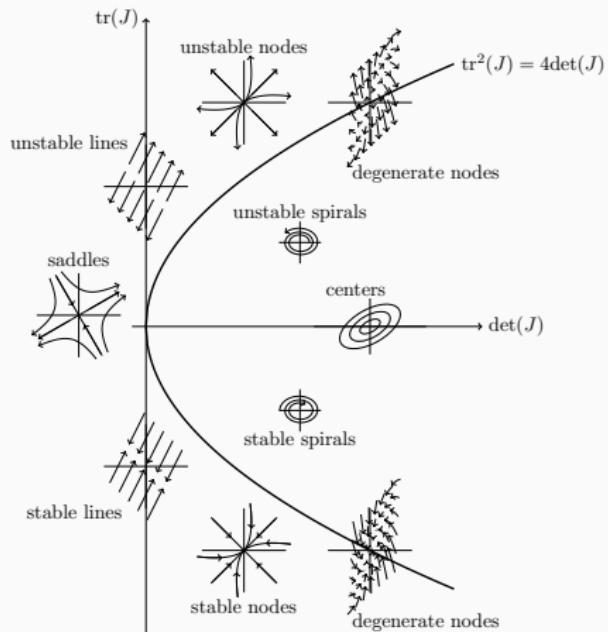
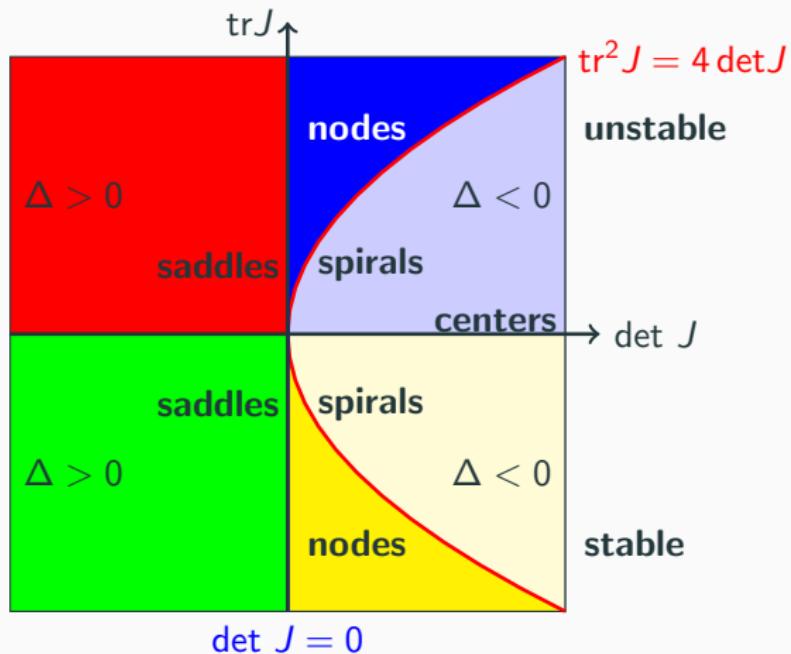


Figure 3: The behavior of equilibrium points in the $\det - \text{tr}$ plane. The curve $\text{tr}^2(J) = 4\det(J)$ indicates where the discriminant vanishes.

Stability Analysis - Color Coded

The stability of the equilibrium points depends on the eigenvalues, which in turn depend on the trace and determinant of J .



Stability for $X = 0, Y = 0$

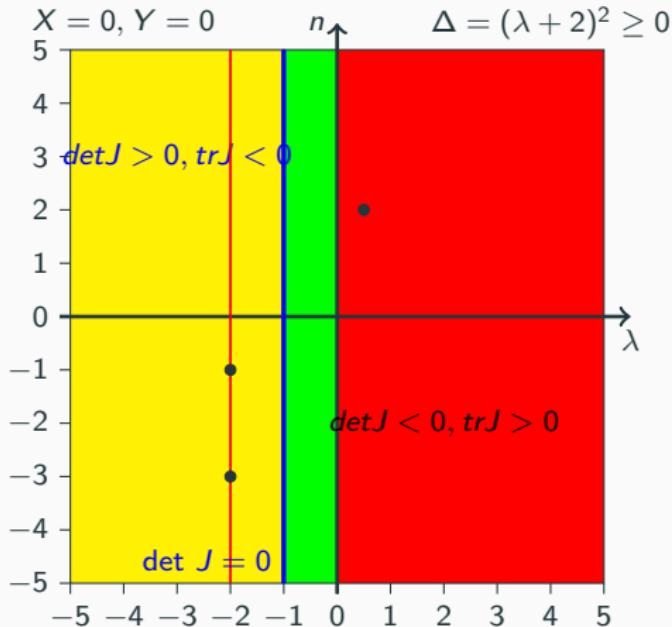


Figure 4: Blue line: $\det J = 0$. Red line: $\Delta = 0$. The equilibrium point is a saddle or a stable node. The black dots are examples in [9] for $\alpha = -1$, and $(n, \lambda) = (-2, -3)$, $(n, \lambda) = (-2, -1)$, and $(n, \lambda) = (\frac{1}{2}, 2)$.

Stability for $X = -1, Y = 0$

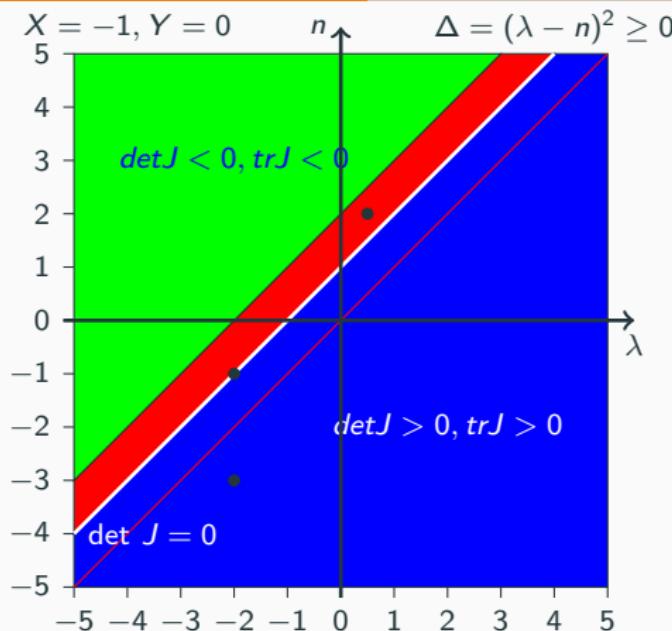


Figure 5: Stability for the equilibrium point $(X, Y) = (-1, 0)$. The white line is where $\det J = 0$. The red line is for $\Delta = 0$. The diagram indicates the equilibrium point can only be a saddle or an unstable node.

Stability for $(X, Y) = \left(\frac{\lambda}{1-n}, \frac{\lambda - n + 1}{\alpha(1-n)} \right)$.

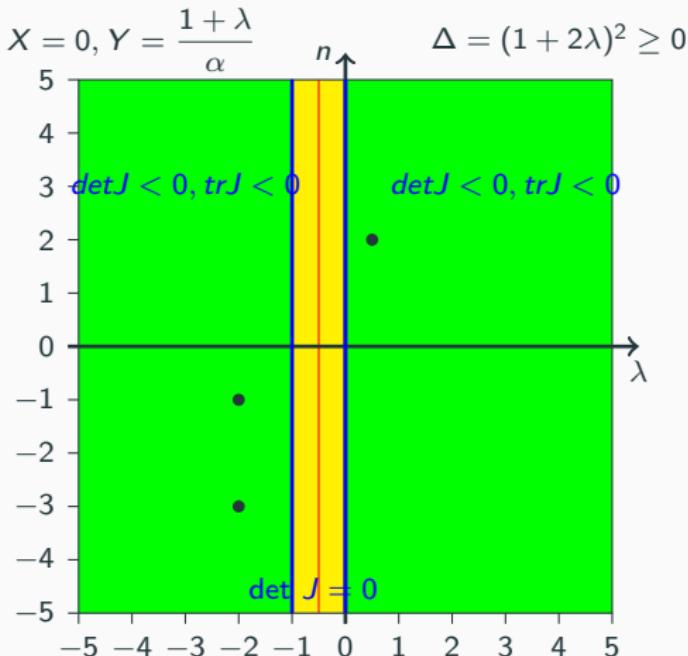


Figure 6: The blue lines show where $\det J = 0$. The red line is where $\Delta = 0$.
The equilibrium point can only be a saddle or a stable node.

Stability for $(X, Y) = \left(\frac{\lambda}{1-n}, \frac{\lambda - n + 1}{\alpha(1-n)} \right)$.

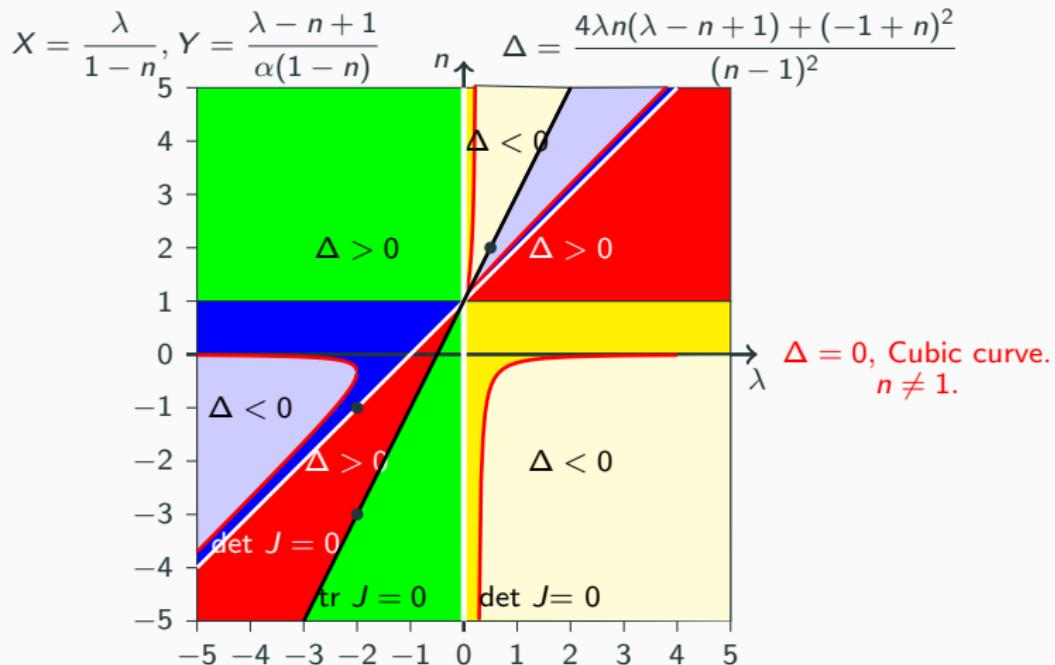
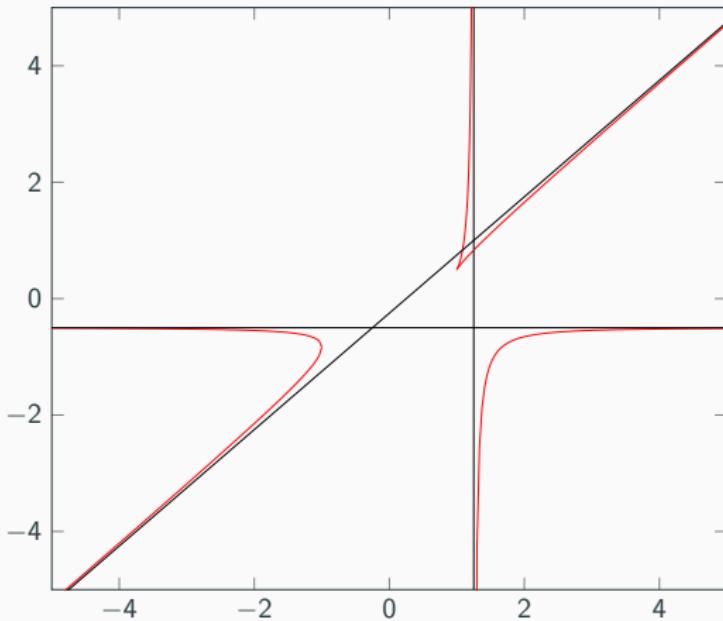


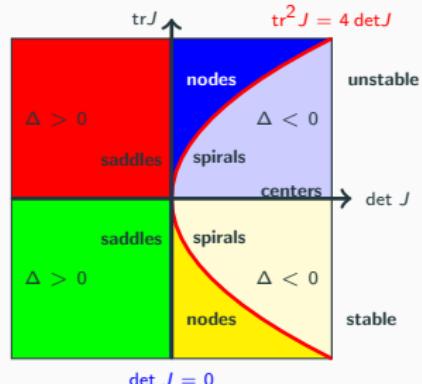
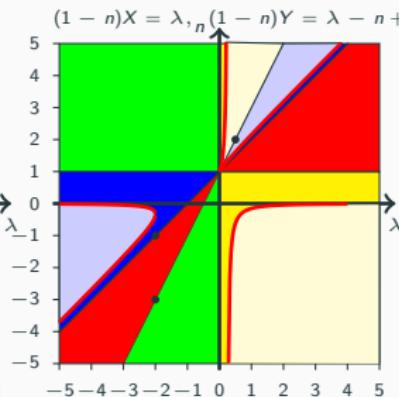
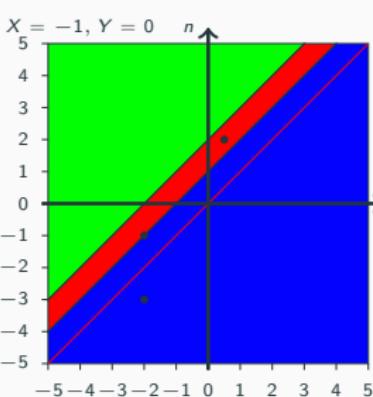
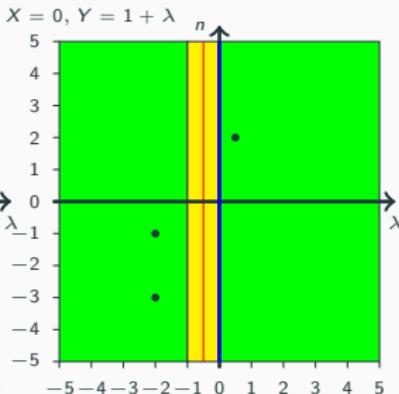
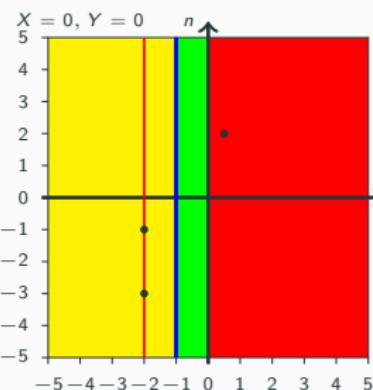
Figure 7: White lines: $\det J = 0$. Red curves: $\Delta = 0$. Equilibrium point is a stable spiral (light yellow), unstable spiral (light blue), or center (black line).

Cubic Curves $x = \lambda + 1, y = n - \frac{1}{2}$

$$p(x, y) = -4 \left(y - x - \frac{1}{4} \right) \left(y + \frac{1}{2} \right) \left(x - \frac{5}{4} \right) - \frac{3}{4}y + \frac{5}{8}.$$

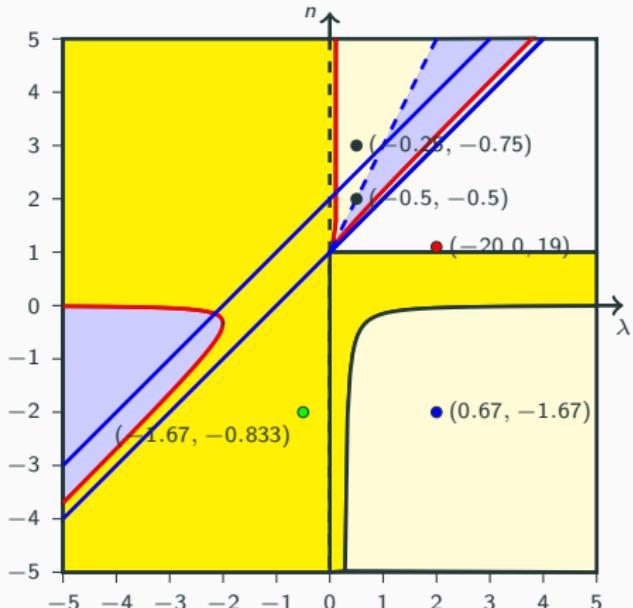


Stability of Equilibria

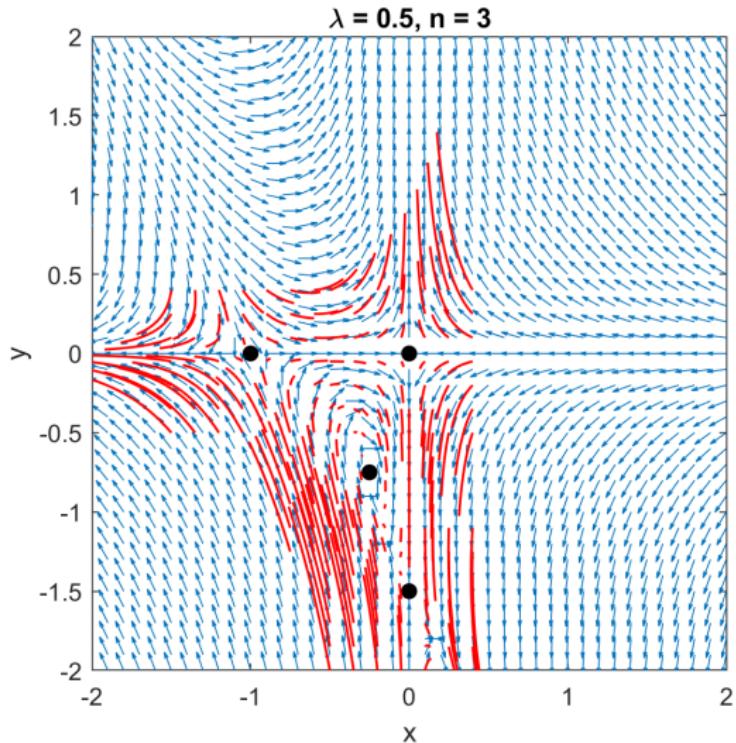


Summary of Analysis

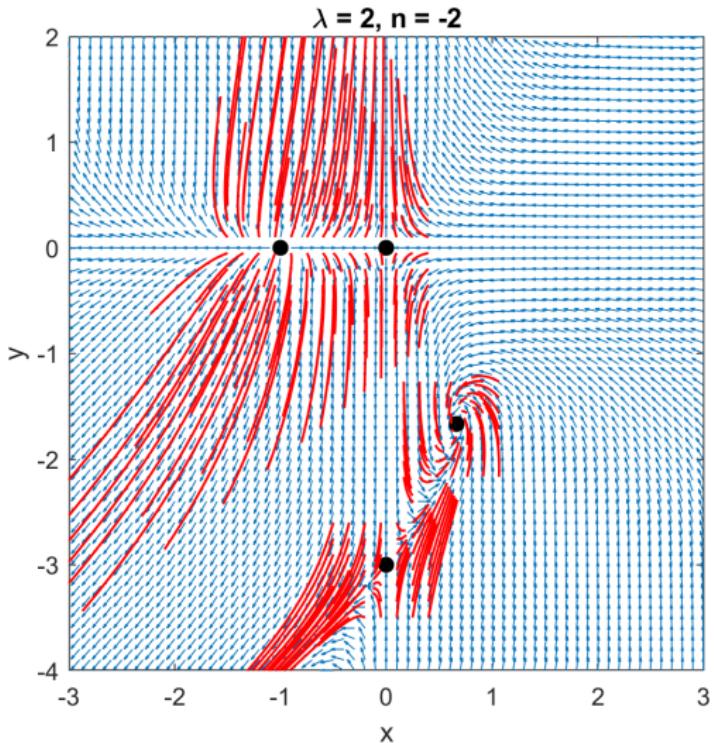
- How does local behavior connect to global behavior?
- Can only be done numerically. Need parameters for interesting behaviors.
- Mancas and Rosu [9] concerned periodic solutions (centers).



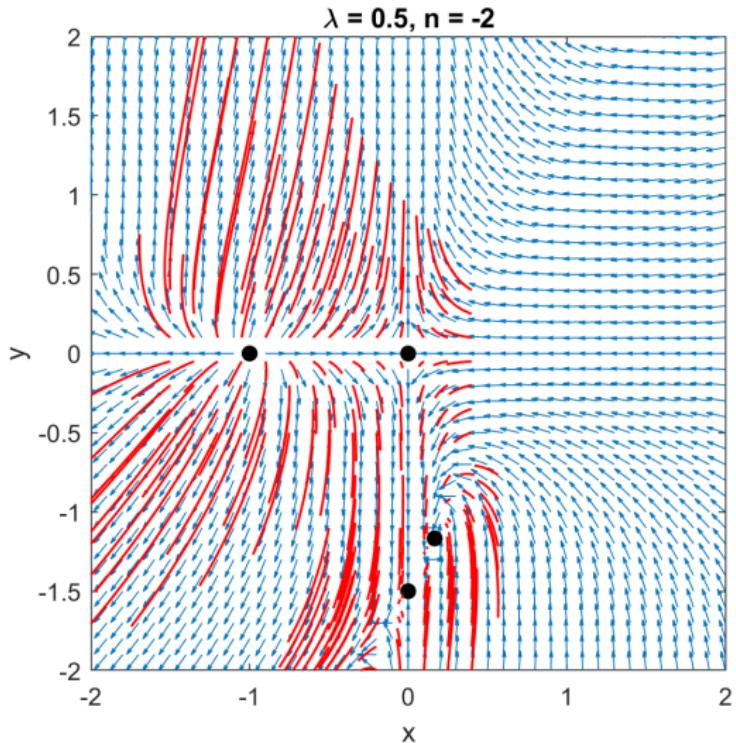
Phase Portrait for $n = 3$, $\lambda = 0.5$ - Stable Spiral



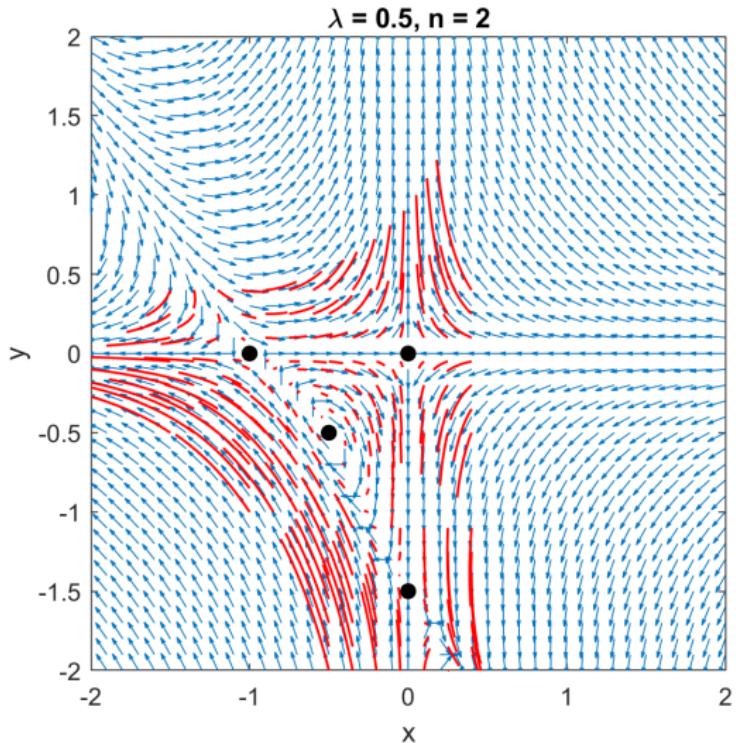
Phase Portrait for $n = -2$, $\lambda = 2$ - Stable Node



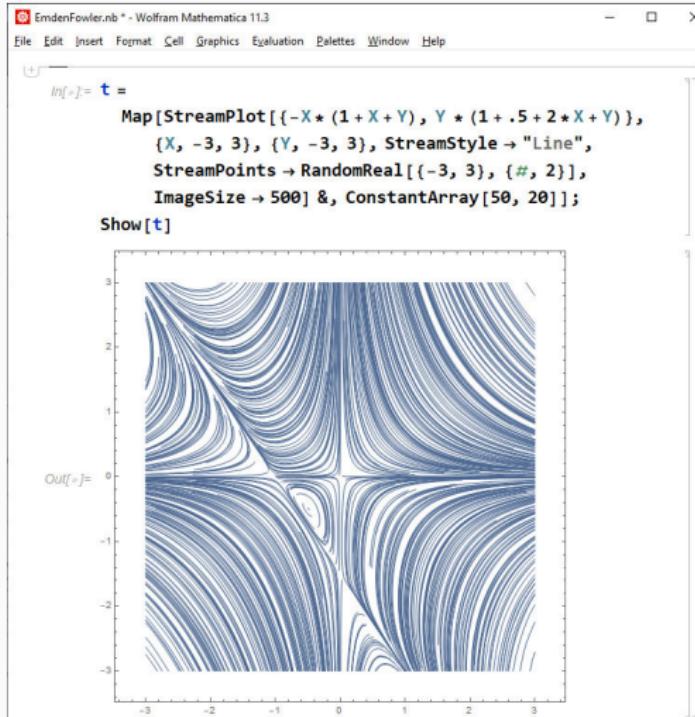
Phase Portrait for $n = -2$, $\lambda = 0.5$ - Stable Node



Phase Portrait for $n = 2$, $\lambda = 0.5$ - Center



Mathematica - Detailed Plot $n = 2, \lambda = 0.5$ - Center



Summary

Convert Emden-Fowler equation $\frac{d}{d\xi} \left(\xi^2 \frac{d\eta}{d\xi} \right) = \alpha \xi^\lambda \eta^n$, using homology invariant functions,

$$X = \frac{\xi \eta'}{\eta} \quad \text{and} \quad Y = \xi^{\lambda-1} \frac{\eta^n}{\eta'}$$

and $\xi = e^t$, to obtain a two-dimensional autonomous system,

$$\begin{aligned}\dot{X} &= -X(1 + X - \alpha Y) \\ \dot{Y} &= Y(1 + \lambda + nX - \alpha Y).\end{aligned}$$

Using color-coding, we can identify interesting dynamics involving nodes, saddles, and spirals or centers.

Thanks for listening!

References i

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-  Thomson, W. **Collected Papers**. Vol 5, p. 266 (1862).